Aggregate Output Changes in Production Networks

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Abstract

Shocks to the global production network, such as firms losing access to a supply line, affect aggregate output. This paper describes a model of production network formation and uses this model to analyze the effect on aggregate output of a firm losing access to an input. This model includes a refinement of pairwise-stable equilibrium in which the equilibrium network is stable to deviations by any combination of two-member coalitions. I then simulate this model to characterize the types of networks that lead to larger changes in output. Contrary to economic intuition and existing results, when a supply line is removed, aggregate output may increase rather than decrease. This result is similar to Braess paradox, the result that additional routes in a transportation network may increase total travel time. This supply chain paradox is more likely to occur when the re-formed production network is more interconnected and when the firm that loses access to its input supplier has more alternative suppliers to choose from. When production networks re-form endogenously, the aggregate output that the network produces can increase rather than decrease.

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1. Introduction

Producers often experience surprise supply line disruptions. When Hurricane Harvey hit the Gulf Coast of Texas in 2017, the oil refineries located there were shut down, forcing American oil manufacturers to switch to refineries elsewhere in the US. [19] In 2018, Toyota factories in the United States were forced to temporarily halt production because an earthquake in Japan had damaged factories there that produce parts needed in the American factories. [9] Early in the coronavirus pandemic in 2020, meat packing plants were closed, preventing restaurants and grocery stores from obtaining meat, especially beef and pork. As a result of this, restaurants were forced to redesign their menus. [17] Grocery stores also reported spikes in Tofu sales, as customers substituted away from the scarce meat. [14] The global production network is complex and critical to the health of the macroeconomy. It is particularly important to understand how this network responds when production circumstances change. How will firms choose new inputs and thereby rebuild the network if some link in the network is removed?

The production network is an interwoven structure of firms and their input relationships. Firms buy products made by other firms and use these to make their own products. This interwoven structure of inputs and outputs creates a complex web that connects firms all over the world. The decisions of one firm can affect firms throughout the network, as well as the output produced by the network as a whole.

When a firm loses a supply line, they need to make a new input decision. This decision creates new circumstances for the other firms in the network, changing the entire network. When the network rebuilds itself after the loss of a supply line, output can be higher or lower. The result depends on the initial structure of the network and on the position within the network of the firm that loses its input supplier. Simulations of the model show how network characteristics and firm characteristics affect changes in output.

When a supply line is removed, the production network may rebuild itself in a way that *increases* aggregate output, contrary to previous models. In the solution to the planner's problem, output will always decrease when a firm loses an input supplier. However, there exist circumstances such that when a supply line is deleted from a pairwise-stable equilibrium, it is possible for output to be greater in the new pairwise-stable equilibrium. In fact, this result survives in the coordination-proof equilibrium, which is a stronger equilibrium definition. In most cases, when an edge is deleted, the set of new equilibrium networks is a strict subset of the previous set of equilibrium networks and thus output is lower. However, there are situations in which the edge that is deleted was the only available deviation preventing a new network from being an equilibrium. When that edge is deleted, the resulting equilibrium network may have higher output. This occurs when one buyer-supplier pair is made better off at the expense of lower output in the economy as a whole. When this buyer-supplier relationship is no longer possible, the higher-output network is now stable to deviations and therefore an equilibrium.The increase in output is more likely to occur when the production network is less interconnected before the supply line disappears and more interconnected afterwards. The increase is also more likely when the firm that loses its supply line has more alternative suppliers to choose from.

This result - that output can increase - is similar to Braess Paradox: a result from the transportation network literature that adding a new route to an existing network may actually increase total travel time. [6] Acemoglu et al. (2018) extend this result to an informational context in which travelers learning about additional routes may increase travel time, as well. [3] This paper extends the concept of Braess Paradox to the context of production networks. This *supply chain paradox* occurs when a supply line is removed and total aggregate output falls. This is, to my knowledge, the only extension of Braess Paradox in which the addition of a link leads to a positive externality, rather than a negative one.

To my knowledge, this is the first paper to use a model of endogenous network formation to simulate the effect on aggregate output of a firm losing an input supplier. This model allows for all firms to reevaluate their input decisions after the input shock, rather than only the firm that loses its input supplier. This is why this supply chain paradox occurs in this paper and does not occur in previous work. Because production networks are inherently discrete and discontinuous objects, they often behave in discrete and discontinuous ways. The equilibrium production networks are defined by a lack of profitable deviations. If, with the removal of a supply line, a profitable deviation is removed, it may create a new equilibrium network with higher output.

Model simulations suggest network characteristics that are correlated with output increasing when an input is removed. Acemoglu and Azar (2020) show that the level of connectivity in an economic network affects the aggregate outcomes of that network. [1] To measure the connectivity of a given network, I use the average distance of the shortest paths between each node. The results of the simulations indicate that the probability that output increases is higher when the production network is less connected before the supply line is removed and more connected after the supply line is removed. In addition to investigating the role of the connectivity of the network, this paper also investigates the role of the centrality of the individual firm that loses its input supplier. The simulation results indicate that aggregate output is more likely to increase when the firm that loses its input has more alternative suppliers from which to choose a new input. The increase in output is also more likely when the firm has few customers before the input is removed and many customers after the input is removed.

This paper expands on several different models of economic network formation. Jackson and Wolinsky (1996) present a model of stable social and economic networks in which agents choose to form (or not form) links with one another. [13] They are the first to define the concept of a pairwise-stable network, the equilibrium network definition used widely in the economic network literature. Jackson and van den Nouweland (2005) expand this concept and define a network stable to deviations of any size coalition of agents. [12] In this paper, I build on these definitions and define a refinement of pairwise stability that is not as stringent as Jackson and van den Nouweland's "strong stability." [12] The equilibrium definition defined in this paper is that of a network stable to any combination of pairs. The definition only considers pairs of agents deviating together, rather than coalitions of three or four, but considers all possible sets of pairs. The literature on matching models also addresses deviations by pairs, but this paper differs from that literature in that the agents in the model do not form two distinct groups and that, while each firm has exactly one supplier, they may have multiple (or zero) customers.[1]

This paper also contributes to the growing literature on the role that production networks play in aggregate output. It is most closely related to the work by Oberfield (2018) and the working paper by Taschereau-Dumouchel (2020). [16] [20] Oberfield (2018) describes a model of endogenous production network formation in which a continuous mass of firms form an equilibrium network that is stable to deviations by any group of firms. [16] This paper describes a special case of the model described in Oberfield (2018) in which the set of firms is finite. The finiteness of the set of firms create different model specifics, as well as different model applications, from those discussed in Oberfield (2018). Additionally, the refinement of pairwise-stability described in this paper, coordination-proof equilibrium, is a special case of the N-stability described in Oberfield (2018). A coordination-proof network is a network that is stable to deviations from any two-member coalition. The model described in this paper also features a simpler pricing structure than that in Oberfield (2018), so as to focus more on the role that input choices, rather than prices, play in equilibrium network formation. While Oberfield (2018) uses the model to characterize the emergence of star

[1]See Roth (1992) and Demenge and Gale (1995), for example. [18] [10]

suppliers and how this affects aggregate output, the focus of this paper is instead on the application of the removal of a supply line. On the other hand, Taschereau-Dumouchel (2020) explores a network with a finite set of firms, but in which the entry and exit decisions of firms drive the network structure, rather than changes to firms' input decisions. [20] Many authors analyze how production network formation affects output. [1] [7] [2] [5] Like these papers, this paper analyzes how firms' choices affect output, but differs in that it considers how firms behave when an input choice is no longer available and the network must re-form.[2]

2. Model of Production Network Formation

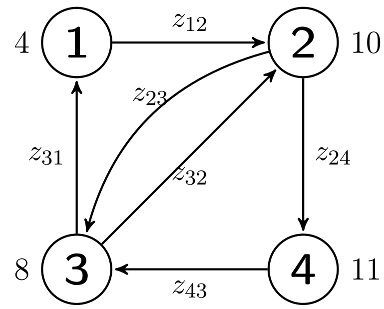
In this section, I describe a model of production network formation that is a refinement of the method of formation most commonly used in the literature, pairwise-stable equilibrium network formation.[3] I describe the model in terms of payoffs to each firm and go into the detail of the pricing that determines these payoffs in the next section. The utility of a given economic agent in an economic network depends on which edges exist in the network. Furthermore, this utility depends not only on the edges connected to a given agent, but the edges connected to the other agents in the network as well.

In a production network, the nodes represent firms. These firms choose other firms' products to use as inputs in their own production process. The edges, or links, between them represent these input relationships, or supply lines, and the weights on these edges represent productivity match values. The higher the match value, the better suited the input is for the production of that good. The profits of each firm depend on their choice of inputs, as well as their inputs' choice of inputs, and so on. In Figure 1 each node represents a firm, and an edge from one firm to another indicates that the latter firm is buying the former firm's input to use in production. Figure 1 compares two different production networks, N_1 and N_2. The numbers to the side of the nodes represent profits. In N_1, Firm 3 is using two inputs - Firm 2's product and Firm 4's product - and earns a profit of \$8. In N_2, Firm 3 switches to using only one input - Firm 1's product - and earns a higher profit of \$10.

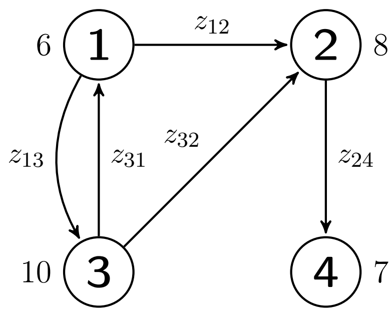
[2] For interesting examples of how network structure contributes to the provision of public goods, rather than aggregate output, see Kurosaka (2020) and Allouch (2019). [15] [4]

[3] A *network* consists of a set, S, of nodes and a set, E, of directed, weighted, edges between the nodes. Specifically, the elements that compose the set E consist of ordered pairs of nodes, and associated edge weights: $E = \{e = ((s_1, s_2), w_{s_1 s_2})\}$ where $s_1, s_2 \in S$ and $w_{s_1,s_2} \in \mathbb{R}$. [11]

Figure 1: Production Networks



(a) N_1



(b) N_2

Figure Note: *This is an example of two different production networks, each consisting of four firms.*

2.1. Equilibrium Networks

I describe a refinement of pairwise stability that is not as strong as Jackson and van den Nouweland's "strong stability." [12] The set of networks which could be equilibrium networks depends on the set of edges, or relationships, available to each economic agent. The *potential production network* includes every supply line available to each agent. This delimits all of the possible equilibrium networks.

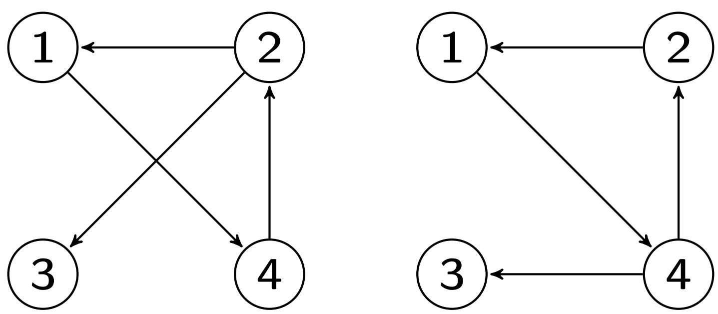
The set of feasible networks, $\mathcal{F}$, is the set of all networks that could be equilibrium networks. Which subnetworks are feasible networks depends on the economic context and on the utility generating process. For example, using notation similar to that of Oberfield (2018), consider a production function for a firm j defined by the edge, $e = ((i, j), z(e))$, that points to that firm, with the form $y_j = \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(e)x(e)^\alpha l_j^{1-\alpha}$. [16] Here the edge weight, $z(e)$, is an edge-specific match value, $x(e)$ is the amount of the input that firm j uses, l_j is the amount of labor that firm j uses, and α is common to all firms. Then, in equilibrium, each firm will use only one input. Thus, the set of feasible networks will consist of subnetworks of the potential network such that every node has exactly one edge pointing to it. This is discussed further in Section 3.

Note that the potential network could be the complete network, that is, the network such that there is an edge from every node to every node. However, by specifying a potential network, this model allows for contexts in which the complete network is not available. For example, in a production network, the complete network would not be a reasonable potential network. We would not expect to see an edge from a firm that produces green beans to a firm that produces truck tires.

If there are no utility-improving deviations from a given network, such a network is an equilibrium network. The deviations which must be ruled out determine the equilibrium network definition. For a given network, $N \in \mathcal{F}$, an i-adjacent network is another network, $\widetilde{N} \in \mathcal{F}$ that differs by exactly i edges. Such a network is a deviation from N. Note that the set of feasible networks is by definition closed under deviations; deviations can be made only to networks in the set of feasible networks.

Figure 2 depicts two feasible production networks, N_1 and N_2, that are 1-adjacent to each other. Firm 3 is using Firm 2 as a supplier in network N_1 and switches to using Firm 4 as a supplier in N_2.

Figure 2: 1-Adjacent Production Networks



(a) N_1 (b) N_2

Figure Note: *This figure depicts two networks that are 1-adjacent.*

For a given feasible network, N, let A be a subset of nodes in N. Define the ordered set $S_A = \{S(f) : f \in A\}$ to be a set of alternative nodes to which each node in A could be related to but are not related to in N. That is, S_A defines a specific set of deviations from N for the nodes in A. Let $\widetilde{N}^{A,S_A}$ denote the $|A|$-adjacent network to N associated with the agents in set A switching from the relationships in use in N to the relationships specified by the agents in the ordered set S_A.

Let $I_j = \{s \in S : \{(s,j), w_{sj}\} \in E\}$ be the set of nodes which have edges pointing to node j in the potential network. That is, I_j is the set of economic agents with which j may form a relationship. Let $\{u_j^N\}_{j \in S}$ be the set of utilities for each agent in S for a given network N. A pairwise-stable network is a network, $N^* \in \mathcal{F}$, such that no agent j, along with any potential relative $i \in I_j$, would be made better off by moving to the 1-adjacent network defined by j and i. A pairwise-stable network is any network with no utility-improving pairwise deviations. A pairwise deviation from a network N is any network that differs from N by one edge (formed by a pair of agents). The set of deviations that must be considered and ruled out is the set of all 1-adjacent networks to N^*. A pairwise-stable network is any network that is stable to deviations by one edge only.

Consider a network that is pairwise-stable. After checking every network that differs by exactly one edge, and comparing all of the appropriate utilities, there exist no pair of agents that would be made better off by switching exactly one edge. But suppose that by switching two edges, that is, if two agents change their relationships simultaneously, those two agents and their new relatives saw higher utilities. This is a reasonable deviation to consider in many contexts. However, in checking for a pairwise-stable network, this deviation is not considered. The definition of pairwise stability restricts the deviations that must be ruled out to only those that differ by one edge. It is reasonable to assume that sets of agents would independently choose to simultaneously switch input suppliers. Next I describe a new definition of an equilibrium network that is stable to deviations by all possible numbers of edges (that is, deviations from all existing relationships).

Let C_S^m be the set of all combinations of size m of the nodes in S. E.g., if $S = \{1, 2, 3, 4\}$, then $C_S^2 = \big\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\big\}$. Let $C_S = \big\{C_S^m\big\}_{m=1}^{|S|}$. For a given feasible network, $F \in \mathcal{F}$, let I_j^F be the set of edges pointing to j in F. A coordination-proof network, N^*, is a network such that no set of any size - from 1 to the number of firms - of firms can be made better off by jointly deviating to another feasible network. Formally, coordination-proof networks are $\{N^* \in \mathcal{F} : \forall C_S^m \in C_S, \forall j \in C_S^m, \forall I \in \mathcal{P}(I_j \setminus I_J^{N^*}), \neg\exists(j,k)k \in I, s.t. u_j^{\widetilde{N}_{jk}} > u_j^N$ and $u_k^{\widetilde{N}_{jk}} > u_k^N\}$. That is, it is a feasible network such that for every possible combination of agents and possible related agents, no such combination would see higher utility by deviating to the associated adjacent network. A coordination-proof network is a feasible network that does not have any utility-improving group deviations, where a group can consist of any combination of nodes. To find a coordination-proof network, all feasible networks that differ by any number of edges must be ruled out as utility-improving. This requires checking networks that differ by 1 edge, 2 edges, 3 edges, and so on. The set of deviations that must be considered is the set of all i-adjacent networks for $i = 1, 2, 3, \ldots M$, where M is the maximum number of edges that can be changed.

Because these deviations are independent, a coordination-proof network need not be in the core. These are not coalitions of agents jointly choosing to deviate, they pairs of agents deviating simultaneously. A coordination-proof network is not necessarily coalition-proof.

As with pairwise-stable networks, for a given potential network, set of feasible networks, and associated utilities, there may be zero or multiple coordination-proof networks. Any coordination-proof network is also pairwise-stable but not necessarily vice-versa. Therefore, the number of coordination-proof networks will be less than or equal to the number of pairwise-stable networks. In the case of the production network example discussed throughout this section, about one third of the pairwise-stable networks are coordination-proof. The Appendix contains the description of an algorithm for computing pairwise-stable and coordination proof networks.

3. Supply Line Removal and Aggregate Output

In this section, I use the model described in Section 2 to analyze the effect on the aggregate output produced by the firms in the production network of the removal of a supply line. Following notation similar to that in Oberfield (2018), let there be a finite set, J, of firms, represented by the nodes of the network. Each firm produces a single distinct good. [16] All of the firms' output is consumed by either other firms as inputs or by a representative consumer. In the potential network, there is an edge from node i to node j if firm j can use firm i's product as an input. Each edge, e, from i to j has a weight, $z(e)$, which describes the technological match of that input in firm j's production. Each input available to firm j, that is, each edge pointing to j in the potential networks, defines a specific production function that depends on the edge weight, $z(e)$. The amount of firm j's product that it can produce using the input specified by edge e is

$$y_j = \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(e)x(e)^\alpha l_j^{1-\alpha}$$

where $x(e)$ is the amount of the input that firm j uses and l_j is the amount of labor that firm j uses. The parameter, α, is common across all of the firms and input options.

Because these production functions exhibit constant returns to scale, in equilibrium each firm will choose only one input supplier. If a firm were using two inputs, one would offer the larger marginal return and the firm would substitute entirely over to that input. As a result, the set of feasible networks is the set of all subnetworks such that each node has exactly one edge pointing to it. Note that there are no restrictions on the number of edges pointing away from a given node. This will be determined by the equilibrium network. See the Appendix for a discussion of the effects of using a production function that leads to each firm only using one input.

Let y_j^0 be the amount of y_j that is consumed by the representative consumer. The representative consumer has preferences over the goods produced by the firms in J,

$$U(y_1, \dots, y_{|J|}) = \Big(\sum_{j\in J}(y_j^0)^{\frac{\epsilon-1}{\epsilon}}\Big)^{\frac{\epsilon}{\epsilon-1}}$$

and supplies L units of labor, inelastically.

3.1. The Planner's Problem

Here I describe the planner's problem to both provide a basis for comparison for other equilibrium outcomes and to build intuition for the model. The planner considers all of the feasible networks and for each solves a standard consumer utility maximization problem.

$$\max_{\left\{\{y_j^0, x(e_j), l_j\}_{j\in J}\right\}_{N\in\mathcal{N}}} \Big(\sum_{j\in J}(y_j^0)^{\frac{\epsilon-1}{\epsilon}}\Big)^{\frac{\epsilon}{\epsilon-1}} \equiv Y^0$$

s.t.

$$y_j^0 + \sum_{e\in\hat{D}_j} x(e) \le \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(e_j)x(e_j)^\alpha l_j^{1-\alpha} \qquad \forall j \in J$$

$$\sum_{j\in J} l_j = L$$

Each $N \in \mathcal{N}$ defines a different edge pointing to each firm j, labeled e_j, and a different set of edges pointing away from firm j to each of its customers, labeled $\hat{D}_j$. Thus, the output of each firm is split into the amount consumed by the representative consumer and the amount consumed as inputs by other firms: $y_j = y_j^0 + \sum_{e\in\hat{D}_j} x(e)$. The first constraint is the technology constraint: the consumer and the customers of firm j cannot consume more than firm j produces using the input defined by N. The second constraint is the labor constraint: all of the firms in J use only the labor supplied by the representative consumer.

Each of the feasible networks in $\mathcal{N}$ defines a different maximization problem across $\{y_j^0, x(e_j), l_j\}_{j\in J}$, each of which the planner solves. The planner then selects the network and choice variables corresponding to the largest Y^0.

3.2. Prices in Equilibrium Production Networks

Firms set prices for the portion of their output consumed by the representative consumer and the portion consumed by each of their network customers - the other firms which use their good as an input. Label the price of y_j^0, the amount of firm j's output that is consumed by the consumer, as p_j^0. For each network customer of firm j, assume that j operates in a perfectly competitive market so that j charges a per-unit price equal to their marginal cost. That is, j sets $p(e)_{e \in \hat{D}_j}$, for each $\hat{D}_j$ defined by each $N \in \mathcal{N}$ such that $p(e) = MC_j$. As a result of this, the per-unit price firm j charges is a function of the marginal cost of the input supplier used by firm j, $p(e) = MC_j = \frac{1}{z(e_j)} MC_{s(e_j)} w^{1-\alpha}$, where w is the price of labor to all firms. Similarly, the price charged by that supplier is a function of the marginal cost of *their* supplier, and so on.

When exactly one edge is pointing to each firm, there are only two possible network shapes that can make up each connected component of the entire network. These are cycles with branches.[4] Any connected component must contain exactly one cycle and any branch in that connected component must have its root on the cycle.[5] These two shapes are critical in calculating the prices.

Because the price charged by each firm can be written in terms of the supplier's marginal cost, all of these prices can be calculated using only the network structure and $z(e_j)$'s. The price charged by any firm on a cycle can be traced back through each supplier until it is expressed in terms of itself, thus there is a closed form solution for any price on a cycle. The price charged by any firm on a branch can be traced up to the root node of the cycle, which must be on a cycle, thus any such price can be calculated.

For a cycle of length c, there are c firms and c edges. Label these firms $1, \ldots, c$. Without loss of generality, we find the price of firm c and label the supplier c uses as 1, the supplier 1 uses as 2 and so on.

The price that c pays for its input is $p(e_c) = \frac{1}{z(e_1)} p(e_1)^\alpha w^{1-\alpha}$, where $p(e_1)$ is the price firm 1 pays for its input. This price is $p(e_1) = \frac{1}{z(e_2)} p(e_2)^\alpha w^{1-\alpha}$, where $p(e_2)$ is the price firm 2 pays for its input. Continuing in this way I can write the price that firm $c-1$ pays as $p(e_{c-1}) = \frac{1}{z(e_c)} p(e_c)^\alpha w^{1-\alpha}$.

[4] A cycle is a set of nodes such that the in-degree and out-degree of each node are both exactly one. A branch is a set of nodes such that the in-degree of each node is one but the out-degree is unrestricted.

[5] If it had no cycle then there would need to exist one node with no supplier and if there was more than one cycle then there would exist some node with more that one supplier.

Substituting each price expression into the previous one gives:

$$p(e_c) = p(e_c)^{\alpha^c}\Big[\frac{1}{z(e_1)}\frac{1}{z(e_2)^{\alpha}}\cdots\frac{1}{z(e_c)^{\alpha^{c-1}}}\Big]w^{(1-\alpha)+\alpha(1-\alpha)+\ldots\alpha^{c-1}(1-\alpha)}.$$

Solving for $p(e_c)$ gives:

$$p(e_c) = \Big(\Big[\frac{1}{z(e_1)}\frac{1}{z(e_2)^{\alpha}}\cdots\frac{1}{z(e_c)^{\alpha^{c-1}}}\Big]w^{(1-\alpha)+\alpha(1-\alpha)+\ldots\alpha^{c-1}(1-\alpha)}\Big)^{\frac{1}{1-\alpha^c}}$$

$$= w^{\frac{1-\alpha}{1-\alpha^c}\sum_{k=1}^{c}\alpha^{k-1}}\prod_{i=1}^{c}\Big(\frac{1}{z(e_{i+1})}\Big)^{\frac{\alpha^{i-1}}{1-\alpha^c}}$$

$$= w^{\frac{1-\alpha}{1-\alpha^c}\frac{\alpha^c-1}{\alpha-1}}\prod_{i=1}^{c}\Big(\frac{1}{z(e_{i+1})}\Big)^{\frac{\alpha^{i-1}}{1-\alpha^c}}$$

$$= w\prod_{i=1}^{c}\Big(\frac{1}{z(e_{i+1})}\Big)^{\frac{\alpha^{i-1}}{1-\alpha^c}}.$$

Each connected component of the network has one cycle, and potentially many branches emanating from that cycle. Thus, each branch has a root node on the cycle. Because each firm pays the marginal cost of its supplier, the cost of any branch firm can be traced back and written in terms of the price of this root node. Let firm d be d edges down the branch from the node where $d > 1$.

The price that d pays for its input is the marginal cost of its supplier. The price the supplier pays is the marginal cost of his supplier and so on up to the root node, whose price was found in the above derivation. Label $MC_r = \frac{1}{z(e_r)}p_r^{\alpha}w^{1-\alpha}$. Then the price that firm d pays is

$$p(e_d) = MC_r^{\alpha^{d-1}}\Big[\prod_{i=1}^{d-1}\frac{1}{z(e_{b_i})^{\alpha^{d-i-1}}}\Big]w^{(1-\alpha)\sum_{k=0}^{d-2}\alpha^k}$$

$$= MC_r^{\alpha^{d-1}}\Big[\prod_{i=1}^{d-1}\frac{1}{z(e_{b_i})^{\alpha^{d-i-1}}}\Big]w^{(1-\alpha)\frac{\alpha^d-1}{\alpha-1}}$$

$$= MC_r^{\alpha^{d-1}}\Big[\prod_{i=1}^{d-1}\frac{1}{z(e_{b_i})^{\alpha^{d-i-1}}}\Big]w^{1-\alpha^d}.$$

For a firm that is only one edge away, the price that firm pays is the marginal cost of the root node, MC_r.

With these prices taken as given, the profit maximization problem each firm j solves is

$$\max_{p_j^0, y_j^0, x(e_j), l_j} p_j^0 y_j^0 + \sum_{e \in \hat{D}_j} p(e)x(e) - p(e_j)x(e_j) - wl_j$$

s.t.

$$y_j^0 + \sum_{e \in \hat{D}_j} x(e) \leq \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} z(e_j)x(e_j)^\alpha l_j^{1-\alpha}.$$

Each $N \in \mathcal{N}$ defines a different profit maximization problem for each firm and the solutions to these produce a set of profits for each firm in each feasible network. These profits are the payoffs that determine the pairwise-stable and the coordination-proof equilibria, as defined in the previous section.

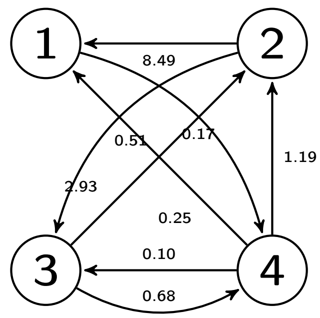
3.3. Supply Line Removal and Increasing Output

I compare the aggregate output generated by an equilibrium network before and after a supply line in the network is removed. Let e^* be the edge of the production network representing the supply line and j^* be the identity of the firm to which e^* points, that is, the buyer using the supply line. When this edge is deleted, it creates a new potential network, and thus a new set of feasible networks. This new set of feasible networks is a strict subset of the original set. The new equilibrium network is determined from among this new set of feasible networks. I find that it is possible for the aggregate output produced collectively by all the firms in the production network to increase after this supply line removal.

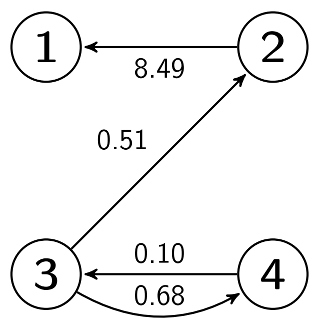
Without allowing for the endogenous re-formation of the production network, we might expect the new network to be the same as the old network with only one change: the firm that lost its input chooses its next lowest marginal cost input. However, such a new network is not necessarily an equilibrium. See Figure 3 for an example. Figure 3 (a) shows the potential network and Figure 3 (b) shows the original coordination-proof equilibrium. Note that this is also then a pairwise-stable equilibrium. If the edge from firm 2 to firm 1 is deleted and the other edges are held fixed, while firm 1 chooses the lowest marginal cost supplier available to it - firm 4 - then the network will be as shown in Figure 3 (c). However, this network is neither pairwise-stable nor coordination-proof.

The coordination-proof equilibrium that results from deleting the edge from firm 2 to firm 1 is shown in Figure 3 (d).

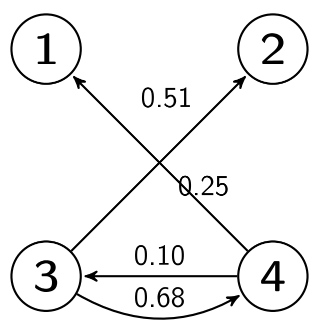
Figure 3: Deleting an edge and holding the other edges fixed is not necessarily an equilibrium



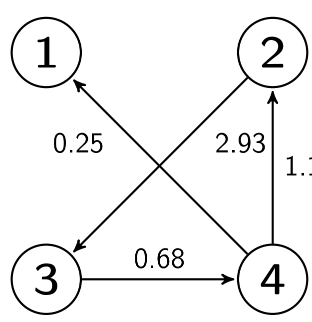
(a) Potential Network



(b) Original Equilibrium Network



(c) Holding other edges fixed



(d) The equilibrium network when the edge is deleted

Figure Note: *This figure shows an example of a network that is not allowed to entirely re-form endogenously is not an equilibrium.*

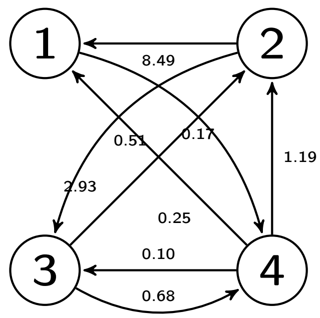
While in the case of a new solution to the planner's problem, the new output will be lower than the original output, this is not necessarily true in the case of a new pairwise-stable equilibrium, and, in fact, this result survives the equilibrium refinement of the coordination-proof equilibrium.

This supply chain paradox will occur when the edge that is deleted is the only edge preventing a higher-output network from being pairwise stable or coordination proof. If, when an edge is deleted the set of new equilibrium networks is a strict subset of the original set of equilibrium networks, then certainly output will decrease. However, this need not be the case. It is possible that deleting an edge makes it possible for a new network to be pairwise stable or coordination proof.

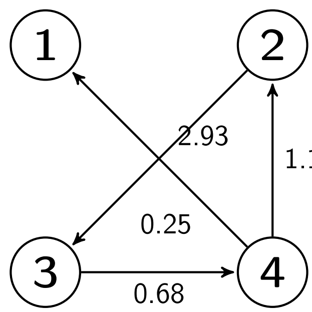
Removing an edge means removing deviations that could otherwise prevent a network from being an equilibrium network. Suppose that there are two networks, N_1, which is an equilibrium network, and N_2 which is not. Additionally, suppose that N_2 has higher aggregate output than N_1 but for two firms in the network, the payoffs for these two firms (only) are higher in N_1 than in N_2. Finally, suppose that the only deviation from N_2 that prevents it from being an equilibrium network is an edge between the two firms. If this edge is removed, that deviation is no longer available and N_2 is now an equilibrium network.

Figure 4 describes an example of this occurring in a potential network using the model parameterization that I use for the simulations in Section 4. There are three coordination-proof (and pairwise-stable) networks corresponding to the potential network shown in Figure 4 (a). Of those, the one that offers the highest output, 0.1430, is depicted in Figure 4 (b). When the edge from firm 4 to firm 1 is deleted, the new set of coordination-proof equilibria consists of five networks. From those, the highest possible output is now 0.1904. The network that produces this output is depicted in Figure 4 (c).

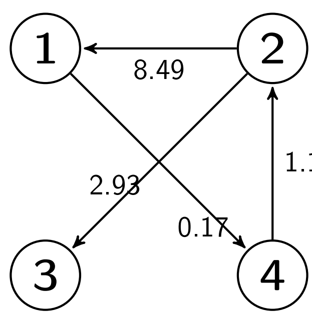
Figure 4: Output increases when the edge from firm 4 to firm 1 is deleted



(a) Potential Network



(b) Original Equilibrium Network, Output = 0.1430



(c) New Equilibrium, Output= 0.1904

Figure Note: *This figure shows an example of output increasing after a supply line is removed.*

4. Simulation Results

I simulate this model to understand what network characteristics play a role in aggregate output, and in particular what characteristics play a role in an increase in output when a supply line is removed. I do this by generating potential production networks and then, for each, finding a pairwise-stable equilibrium, a coordination proof equilibrium, and a solution to the planner's problem. I create the potential network by drawing a number of possible input suppliers for each firm from a Poisson(3) distribution. The identity of each supplier is drawn uniformly with replacement from the other firms. The productivity parameter, $z(e)$, for each link e is drawn from a Pareto$(0.2, -1.8)$ distribution. See Table 4 in the Appendix for the details of the parameterization of the model used in this simulation. This parameterization is motivated by the Carvalho (2012) survey on Input-Output analysis. [8] Note that drawing the supplier identities with replacement allows for multiple edges from a given firm. However, because the productivity parameters are realizations of continuous random variables, the edge with the higher z will always be chosen. This simulation consisted of the creation of $1,000$ potential networks, each of which consisted of five firms. Enough simulation repetitions were conducted to ensure statistical significance of as many of the coefficient estimates as possible; exceptions are noted below. In 554 of these potential networks, a solution to all three problems was found.

This simulation process builds a random sample of potential production networks with five firms from the universe of potential production networks consisting of five firms. The number of available input suppliers, the identities of the input suppliers, and the pairwise match values (edge weights) are created using a Sobol Set quasi-random number generator. As noted in the previous section, there may be multiple pairwise-stable or coordination-proof equilibrium networks. When this occurs, I select the network that creates the largest aggregate output. By doing this, I ensure that any increase in output after an input relationship is removed is not the result of multiple equilibria.

To investigate the removal of an input supplier, I run supply line removal experiments on three different sets of networks: solutions to the planner's problem, pairwise-stable networks, and coordination-proof networks. I find one of each of these for each potential network created. Then, for each network, I remove each supply line, one at a time. I do this by removing the corresponding link from the potential network and then finding a new solution of each type. This resulted in $2,770$ supply line removal experiments. In $2,691$ of these supply line removal experiments, a new solution to all three allocations is found.

In most of these experiments, aggregate output fell after the removal of a supply line. However, in a small number of experiments, aggregate output rose. In all of the supply line removal experiments, the output produced by the solution to the planner's problem falls after a supply line is removed. In $2,534$ of the supply line removal experiments, the output produced by the new pairwise-stable production network fell, but in the remaining 157, output increased. Similarly, in $2,355$ of the experiments, the output produced by the new coordination proof network was lower than in the original network, but in 336, the output rose.

I measure the connectivity of each network using the average shortest path distance. I do this by calculating the length of, that is, the number of links along, the shortest undirected path from each node to every other node and taking the average across all such paths. I find these path distances using Dijkstra's algorithm. A longer average shortest path distance corresponds to a less connected network and a shorter average shortest path distance corresponds to a more connected network.

To understand the role that production network connectivity plays in output after a supply line is removed, I regress the resulting aggregate output after the average shortest path length of the original equilibrium production network. I do this for each method of determining the original production network: solving the planner's problem, finding a pairwise-stable equilibrium network, and finding a coordination proof network. I separate these regressions into two categories: when aggregate output decreases after a supply line is removed and when aggregate output increases after a supply line is removed. Note that output never increases in the solution to the planner's problem. For each supply line removal, i, I regress the output that results from the new production network on the average shortest path distance of the corresponding original production network. That is, I estimate the following regression equation.

$$\widehat{\text{output}}^{E} = \beta(\text{avg. shortest path distance})$$

for each $E \in$ {Planner's Solution, Pairwise-Stable, Coordination-Proof} using ordinary least squares. The coefficients and their 95% confidence intervals are reported in Table 1.

Table 1: Output and Connectivity		
(a) When Output Falls		
E	$\hat{\beta}$	95% CI
Planner's Solution	0.0424	$[0.0407, 0.0442]$
Pairwise-Stable	0.0689	$[0.0669, 0.0709]$
Coordination-Proof	0.0567	$[0.0548, 0.0587]$
(b) When Output Rises		
E	$\hat{\beta}$	95% CI
Pairwise-Stable	0.1620	$[0.1401, 0.1838]$
Coordination-Proof	0.1344	$[0.1213, 0.1474]$

First, note that all of these estimated coefficients are positive. This means that, on average, when the average shortest path distance of the original production network is longer, the aggregate output after a supply line is removed is larger. That is, when production networks are more spread out and less interconnected, the aggregate output that results after a supply line is removed from that network is higher. These coefficients are positive both in the typical case that output falls and in the unusual case that output rises. When the removal of a supply line leads to a drop in output, a more dispersed production network helps output and a more connected production network hurts output. Additionally, this remains true in the more unusual case that output increases after the removal of a supply line. A less connected original production network will, in general, have a higher increase in output.

Next, to understand the role of buyer centrality in output, I regress the aggregate output that results after a supply line is removed on the number of alternative suppliers available to the buyer who loses the supply line, as well as the number of customers that buyer has in the original production network. I again separate these regressions into two cases, based on whether output falls or rises after the supply line is removed.

Each supply line removal experiment, i, defines a buyer, j^*, that loses its supply line. Label the number of input suppliers available to j^* in the potential network as $\#\text{sup}_i$ and the number of customers j^* has in the original equilibrium production network $\#\text{cust}_i$. For each equilibrium type,

$E \in \{$Planner's Solution, Pairwise-Stable, Coordination-Proof$\}$, I estimate

$$\widehat{\text{output}}^E = \gamma_1(\#\text{sup}) + \gamma_2(\#\text{cust})$$

using ordinary least squares. The coefficients and their 95% confidence internvals are reported in Table 2.

Table 2: Output and Buyer Centrality

(a) When Output Falls

E	$\hat{\gamma_1}$	$\hat{\gamma_1}$ 95% CI	$\hat{\gamma_2}$	$\hat{\gamma_2}$ 95% CI
Planner's Solution	0.0296	$[0.0278, 0.0324]$	0.0016	$[-0.0022, 0.0054]$
Pairwise-Stable	0.0516	$[0.0.0498, 0.0583]$	-0.0087	$[-0.0125, -0.0050]$
Coordination-Proof	0.0430	$[0.0410, 0.0449]$	-0.0069	$[-0.0109, -0.0030]$

(b) When Output Rises

E	$\hat{\gamma_1}$	$\hat{\gamma_1}$ 95% CI	$\hat{\gamma_2}$	$\hat{\gamma_2}$ 95% CI
Pairwise-Stable	0.1055	$[0.0.0850, 0.1261]$	0.0377	$[-0.0132, 0.0886]$
Coordination-Proof	0.0842	$[0.0704, 0.0980]$	0.0345	$[0.0033, 0.0657]$

All coefficients are significant at the 5% level except for those on the number of customers in the solution to the planner's problem and the pairwise-stable equilibrium network.

All of the coefficients on the number of suppliers available to the buyer who loses its supply line are positive. That is, if the buyer has more alternative suppliers to choose from, aggregate output tends to be higher, regardless of whether output rises or falls. If, as most often happens, the removal of a supply line leads to a drop in aggregate output, more alternative suppliers to choose from blunts this impact. If, on the other hand, output rises, it rises by more when there are more suppliers available to the buyer.

Focusing on the coordination proof results, when output falls, the coefficient on the number of customers is negative but when output rises, it is positive. When a firm loses it's supply line, the number of customers it sells to plays a role in the resulting aggregate output when the production network rebuilds itself. If the new network produces a lower aggregate output, the number of customers hurts this output. But, if in the unusual instance that the network rebuilds itself to produce a higher amount of output, the more customers to whom the firm sells, the higher this resulting output.

These connectivity and centrality results can also be used to understand the externalities created by the equilibrium formation of production networks, that is, the difference between the equilibrium output and the output that would be produced by the solution to the planner's problem. When the output produced by an equilibrium network increases after a supply line is removed, this is an example of a decrease in that externality. The connectivity of the production network and the centrality of the buyer who loses the supply line play a role in this externality. Focusing on the coordination proof network results, the positive coefficient on average shortest path distance indicates that a less connected network mitigates this externality more than a more connected network. The positive coefficient on the number of suppliers and on the number of customers of the buyer in panel (b) of Table 2 indicate that a larger number of suppliers and a larger number of customers also make this externality smaller.

I investigate the role that network connectivity and buyer centrality play in the probability that output increases after an supply line is removed from the production network. In doing this, I restrict my focus to the coordination-proof supply line removal experiments. I do this because any coordination-proof equilibrium is also pairwise stable, and because the output always decreases when a supply line is removed in the planner's problem. To understand the role of network connectivity, I consider the average shortest path distance of the potential network, the original equilibrium production network, and the new equilibrium production network that results after the input is removed. I also include characteristics of j^*, the buyer who loses his supply line. Specifically, I consider the number of suppliers available to j^* in the potential production network, the number of firms that buy j^*'s product in the original equilibrium production network (j^*'s original customers), and the number of firms that buy j^*'s product in the new equilibrium production network ($j^{*\prime}$'s new customers). These six characteristics are the explanatory variables in a binary Logistic regression for which the dependent variable is the probability that output increased after the edge was removed. The coefficients on each of these explanatory variables, their standard errors, and their marginal effects are reported in Table 3.

Table 3: Logit Regression Results			
Characteristic	Coeff.	SE	ME
Potential Network Average Shortest Path Distance	−0.2643	0.5949	−0.0280
Original Equilibrium Average Shortest Path Distance	0.6338	0.1466	0.0671
New Equilibrium Average Shortest Path Distance	−0.2457	0.1713	−0.0260
Number of Possible Suppliers	0.0686	0.0937	0.0073
Original Number of Customers	−0.2993	0.0873	−0.0317
New Number of Customers	0.2391	0.0839	0.0253

The results regarding the connectivity of the production network indicate the following. First, the more connected the potential network is, the higher the likelihood that output will increase after a supply line is removed. A one-link increase in the average shortest path distance of the potential network is associated with a 2.8 percentage point decrease in the likelihood that output increases. Second, the less connected the original equilibrium production network and the more connected the new equilibrium production network, the higher the likelihood that output will increase. A one-link increase in the average shortest path distance of the original equilibrium is associated with a 6.71 percentage point increase in the likelihood that output increases, while a one-link increase in the average shortest path distance of the new equilibrium is associated with a 2.6 percentage point decrease in the likelihood that output increases.

The firm characteristic results indicate the following. First, the more suppliers available to j^*, the higher the likelihood that output will increase. One more supplier is associated with a 0.73 percentage point increase in the probability that output increases. Second, the fewer firms buying from j^* in the original equilibrium production network and the more firms buying from j^* in the new equilibrium production network, the higher the likelihood that output increases. One more customer in the original equilibrium is associated with a 3.17 percentage point decrease in the probability that output will increase, while one more customer in the new equilibrium is associated with a 2.53 percentage point increase in the probability that output increase.

5. Conclusion

The key contributions of this paper are a network model which features a finite number of firms and endogenous network determination, a refinement of the standard network equilibrium, and a better understanding of the role network connectivity and firm centrality play in the determination of aggregate outcomes. I apply the model to investigate the effect of a firm losing an input supplier in the production network and find that when this happens the resulting aggregate output can be higher. Simulation results indicate the following. First, on average, the more connected a production network is, the smaller the decrease in aggregate output will be when a firm loses an input supplier. Second, on average, the more alternative suppliers that firm has, the smaller the drop in output will be when that firm loses its input supplier. Finally, it is more likely output will increase when (1) the firm that loses its supplier goes from having fewer customers before it lost its supplier to having many customers afterwards and (2) the network as a whole goes from less connected before this input is removed to more connected afterward this input is removed.

The model, application, and results discussed in this paper suggest several avenues for future work. First, the externality created by these equilibrium networks presents an opportunity to explore how taxes or subsidies applied to key firms could be used to mitigate the difference between the planner's output and the equilibrium network output. Secondly, the algorithms that I created to compute pairwise-stable and coordination proof equilibrium networks are the only such algorithms that I know of. The efficiency of these algorithms and potential improvements to that efficiency could be interesting to explore.

Acknowledgements

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6. Appendix

6.1. Model Parameterization in Simulations

Table 4: Model Parameterization

Parameter	Value
Number of Firms, $\|J\|$	5
Production function capital parameter, α	0.33
Consumer preference parameter, ϵ	0.10
Labor supply, L	1
Number of possible suppliers drawn from	Poisson(3)
Productivity parameters drawn from	Pareto(0.2, −1.8)

6.2. Robustness to Multiple Inputs: A Case Study

Consider a production function that specifies two inputs for firm j, e_j^1 and e_j^2,

$$y_j = z(e_j^1)z(e_j^2)\Big[\big(\frac{x(e_j^1)}{\gamma}\big)^{\gamma}\big(\frac{x(e_j^2)}{1-\gamma}\big)^{1-\gamma}\Big]^{\alpha} l_j^{1-\alpha}.$$

The set of feasible networks expands to include not just networks in which every firm has one input but networks in which every firm has either one or two inputs. If in a given feasible network, firm j has one input, it solves the profit maximization problem described in the main body of the paper. If instead, a firm has two inputs, it solves the following profit maximization problem.

$$\max_{p_j^0, y_j^0, x(e_j^1), x(e_j^2), l_j} p_j^0 y_j^0 + \sum_{e\in\hat{D}_j}[p(e)x(e)+\tau(e)] - [p(e_j^1)x(e_j^1)+\tau(e_j^1)] - [p(e_j^2)x(e_j^2)+\tau(e_j^2)] - wl_j$$

s.t.

$$y_j^0 + \sum_{e \in \hat{D}_j} x(e) \leq z(e_j^1) z(e_j^2) \Big[\big(\frac{x(e_j^1)}{\gamma}\big)^{\gamma} \big(\frac{x(e_j^2)}{1-\gamma}\big)^{1-\gamma} \Big]^{\alpha} l_j^{1-\alpha}$$

I run an edge deletion experiment on a specific potential network as a case study in the robustness of this model with respect to the assumption of constant returns to scale. I find initial equilibrium networks for the two different production functions, and then compare the outcomes when edges are deleted. The potential network is depicted below, in Figure 5, with the edge weights specified in matrix W, below.

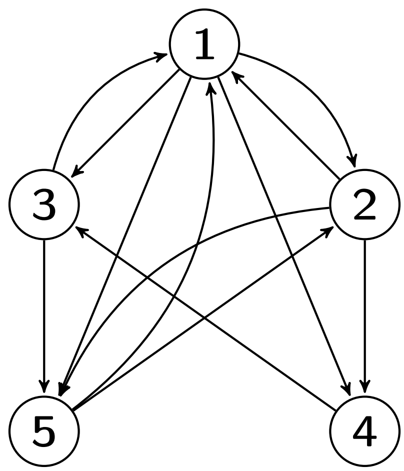

Figure 5: Potential Network

$$W = \begin{bmatrix} 0 & 0.16 & 2.84 & 0 & 0.08 \\ 0.33 & 0 & 0 & 0 & 0.98 \\ 0.21 & 0 & 0 & 17.31 & 0 \\ 0.10 & 1.26 & 0 & 0 & 0 \\ 5.03 & 0.25 & 0.07 & 0 & 0 \end{bmatrix}$$

Firms 1 and 5 have three available suppliers while the rest of the firms have two available suppliers. There are $3 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 72$ feasible networks in which every firm has only one input supplier. There are $[3 + \binom{3}{2}] \cdot [2 + \binom{2}{2}] \cdot [2 + \binom{2}{2}] \cdot [2 + \binom{2}{2}] \cdot [3 + \binom{3}{2}] = 972$ feasible networks in which firms have either one or two input suppliers.

When firms are restricted to one input, there are four coordination-proof networks, shown in Figure 6. These four constitute 5.6% of the 72 feasible networks. The mean output across these is

0.1586. The network with the highest output, 0.3117, is depicted in Figure 6(b).

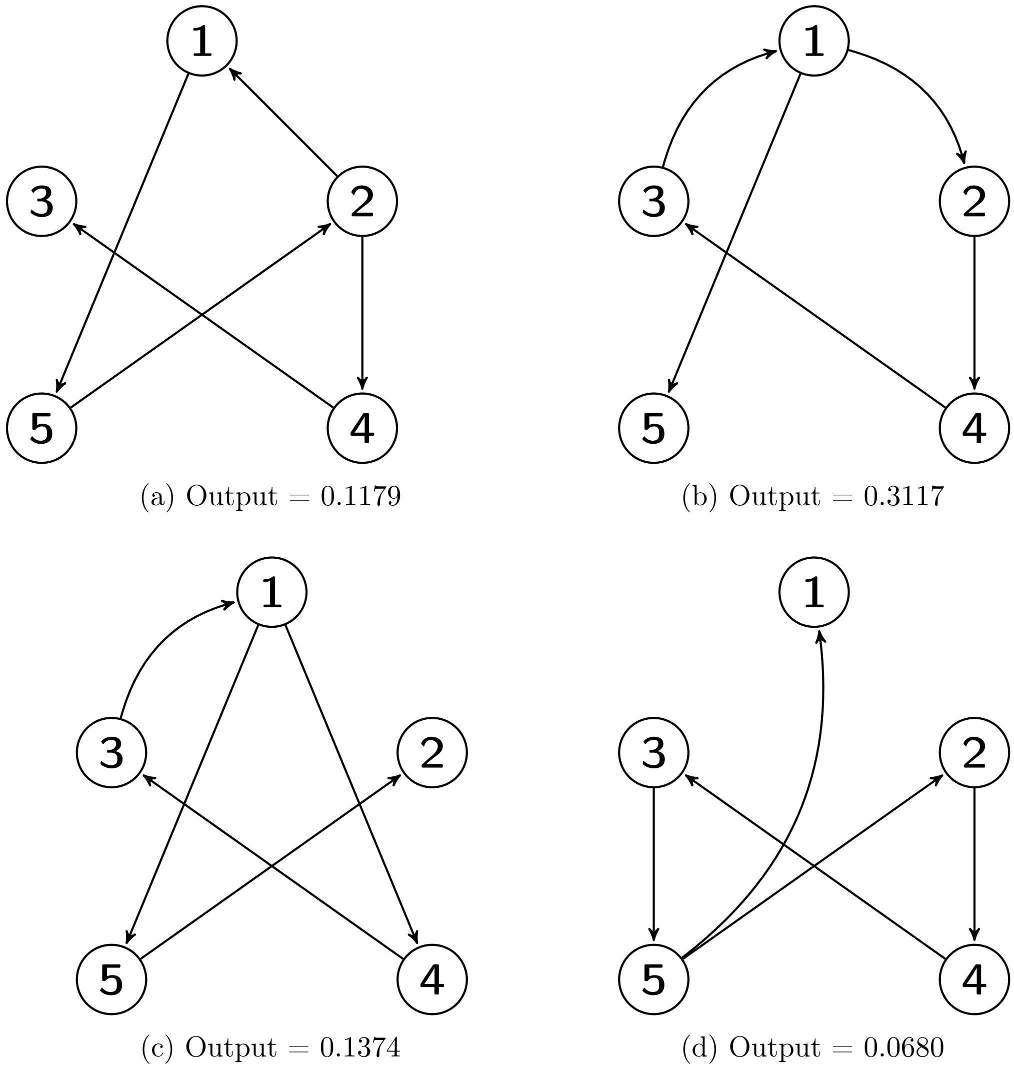

Figure 6: Coordination-Proof Networks for the 1 - Input Model

When firms can have either one or two inputs, there are 53 coordination-proof networks, constituting 5.5% of the 972 feasible networks. This percentage is very similar to the one-input case. Furthermore, the network with the highest output in the multiple-input case is the same network as in the one-input case, the network shown in Figure 6 (b), with an output of 0.3117. The mean output across the 53 coordination-proof networks is 0.0895, lower than the mean output in the one-input case.

I delete two edges from this equilibrium - using both the one-input model and the two-input model - and compare the outcomes. I delete Firm 1 and Firm 5's input suppliers, one at a time. Both Firm 1 and Firm 5 have two more suppliers to choose from, but they differ in that Firm 1 has two network customers in the original equilibrium and Firm 5 has none.

First, I delete the edge from Firm 3 to Firm 1 from the original potential network and find the coordination-proof networks under both the one-input assumption and allowing for two inputs. When only one input is possible, the new set of coordination-proof networks consists of only two networks, both of which are in the original set of one-input, coordination-proof networks. They are the networks depicted in Figure 6 (a) and (d). No new one-input equilibrium networks are created by deleting this edge. The mean output across the two fell to 0.0930. The highest output given by either of these is 0.1179. The network depicted in Figure 6 (a) produces this output. In this case, when an edge is deleted, the resulting equilibrium network is the same whether firms are restricted to one input or whether they can have two inputs.

When two inputs are possible, the new set of equilibrium networks consists of nine networks, more than one of which has firms using two inputs. The mean across these fell to 0.0384. However, the network that produces the highest output in this case is the same network as in the one-input case, the network depicted in Figure 6 (a), with an output of 0.1179.

Next, I delete the edge from Firm 1 to Firm 5 from the original potential network and find the coordination-proof networks for the one-input case and two-input case. When only one input is possible, there are three coordination proof networks. The mean across these is 0.0543. Two new coordination-proof networks are created when this edge is removed, but the network that produces the highest output is one of the four original coordination-proof networks, depicted in Figure 6 (d). This network produces an output of 0.0680.

When two inputs are possible, the new set of coordination-proof networks consists of 41 networks. The mean output across these is 0.0383, which is very similar to the mean output when the previous edge is deleted and the two-input equilibrium is found, 0.0384. The maximum output created by any of these 41 networks is 0.1349. Note that this is a smaller decrease from the initial equilibrium than when firms are restricted to one input. The network that produces this output is depicted in Figure 7.

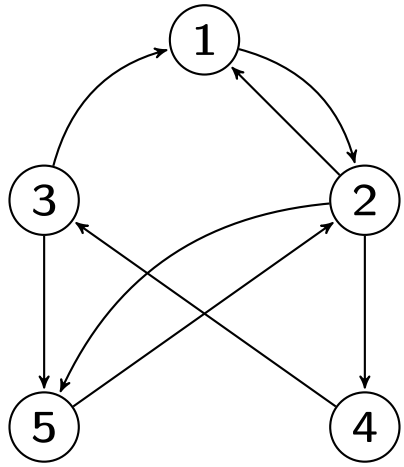

Figure 7: Highest Output Network, 2nd Edge Deletion, Two Inputs Possible

While further research is needed to understand the distributional effects of allowing for more than one input, this case study illuminates several facts. First, even when multiple inputs are allowed, the maximal output network may still have every firm using only one input. Second, despite dramatically increasing the number of feasible networks, the percentage of feasible networks that are coordination-proof does not necessarily dramatically change. Finally, allowing for two inputs may lead to a smaller drop in output when an edge is removed but the possibility of output increasing remains.

6.3. An Algorithm for Computing Pairwise-Stable Networks

1. Initialize the set of pairwise-stable networks as empty.

2. Enumerate the set of feasible networks.

3. For each feasible network:

   (a) Enumerate the pairwise deviations.

   (b) For each deviation:

      i. Check if both firms in the pair are made better off.

      ii. If they are, stop.

      iii. If they are not, check the next deviation.

   (c) If any of the deviations are profitable, this feasible network is not pairwise-stable; stop.

(d) If there are no profitable deviations, this feasible network is pairwise-stable; add it to the set of pairwise-stable networks.

4. Return the set of pairwise-stable networks.

6.4. An Algorithm for Computing Coordination-Proof Networks

1. Initialize the set of coordination-proof networks as empty.

2. Enumerate the set of feasible networks.

3. For each feasible network:

   (a) Enumerate the group deviations.

   (b) For each deviation:

      i. Check if all of the firms in the group are made better off.

      ii. If they are, stop.

      iii. If they are not, check the next deviation.

   (c) If any of the deviations are profitable, this feasible network is not coordination-proof; stop.

   (d) If there are no profitable deviations, this feasible network is coordination-proof; add it to the set of coordination-proof networks.

4. Return the set of coordination-proof networks.

The Matlab code that runs the simulations in this paper and computes the equilibria as described here can be found on Dr. Luedtke's website at `http://allisonoldhamluedtke.com/research`.