Math Review

Called "Review" but it's ok if some of these things are new.

Part 1: Non-Calculus Stuff
- growth rates
- intro to random variables
- log rules

Part 2: Calculus Stuff
- derivatives
- log approx
- constrained optimization

Part 1: Non-Calculus Stuff

Growth Rates:
- for a time series of numbers
- describes the percent change from one time period to another

For a list of numbers

$x_1, x_2, x_3, \ldots$

time periods

The growth rate from time period t to time period $t+1$ is $\boxed{\dfrac{x_{t+1} - x_t}{x_t}}$ this is the def.

Doing some algebra gets us:

$$\frac{x_{t+1}-x_t}{x_t} = \frac{x_{t+1}}{x_t} - \frac{x_t}{x_t} = \frac{x_{t+1}}{x_t} - 1.$$

We'll use this one usually. (slightly fewer steps)

Ex: If we produced 207, 234, and 162 apples in 2014, 2015, & 2016 respectively, what is the growth rate of apple production from 2014 to 2016?

$$\%\Delta \text{ apple production} = \frac{\text{apples}_{2016} - \text{apples}_{2014}}{\text{apples}_{2014}}$$

$$= \frac{\text{apples}_{2016}}{\text{apples}_{2014}} - 1 = \frac{162}{207} - 1 = -0.217$$

(Note that if $\text{apples}_{2016} > \text{apples}_{2014}$ this would have been positive.)

Some Statistics:

A random variable is a variable that can take on different values with some probability.

For example: $P[X=x]$

$$X = \begin{cases} 1 & \\ 2 & \\ 3 & \text{w/ probabilities} \\ 4 & \\ 5 & \\ 6 & \end{cases} \qquad \begin{matrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{matrix}$$

RV

values it can take

prob. it takes that value

The <u>expected value</u> of a random variable,

$$E[X] = \sum_{x \in X} x P[X=x]$$

So the expected value of the X above is

$$\begin{aligned} E[X] &= 1(1/6) + 2(1/6) + 3(1/6) + 4(1/6) \\ &\quad + 5(1/6) + 6(1/6) \\ &= 1/6\,[1+2+3+4+5+6] \\ &= \tfrac{1}{6}[21] = \frac{21}{6} \end{aligned}$$

It's called the expected value because it's the value we'd expect X to be if we had to pick only one number.

Some algebra rules for expected values:

- $E[aX] = aE[X]$

- $E[X+Y] = E[X] + E[Y]$
  BUT $E[XY] \neq E[X]E[Y]$
- $E[aX+bY] = E[aX] + E[bY]$
  $= aE[X] + bE[Y]$

Ex: If we roll 2 die, what is the expected value of their sum?

$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases}$ w/p $1/6$ $\qquad Y = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases}$ w/p $1/6$

Let S be the random variable describing this sum.

$S = X+Y$

$$E[S] = E[X+Y] = E[X] + E[Y]$$
$$= \frac{21}{6} + \frac{21}{6} = \frac{42}{6}$$
$$= 7$$

Logarithms :

(*) We'll be working with natural log most of the time

$\log_a x$ = the exponent you put on a to get x.

$\ln(x) = \log_e x$ = the exponent you put on e to get x.

Some useful log algebra :

$\ln(x \cdot y) = \ln(x) + \ln(y)$

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

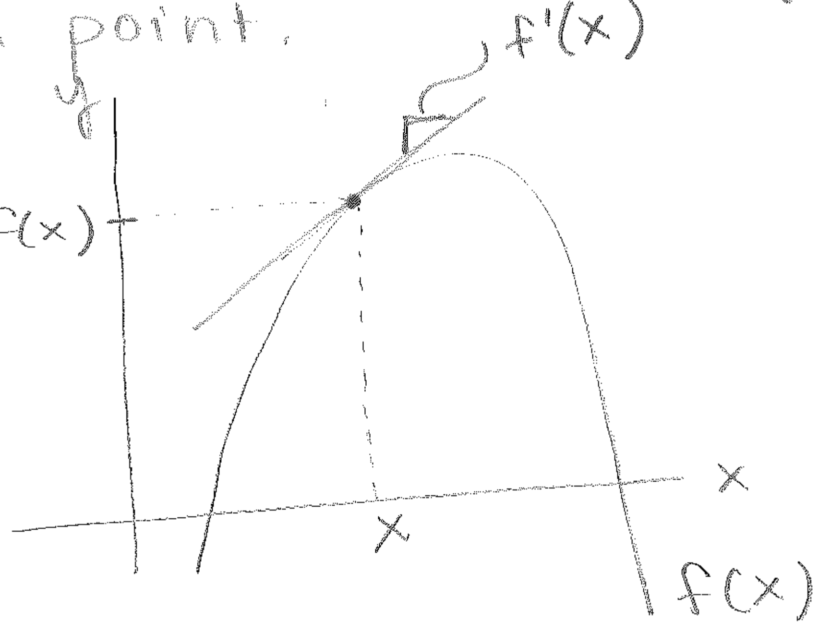
$\ln(x^a) = a\ln(x)$

These are true for log base anything, I just wrote it for the one we'll use

Part 2: Calculus Stuff

Derivatives:

(*) the derivative of a function describes how that function is changing at a given point.



Derivative of a polynomial:

$f(x) = x^n \longleftarrow$ power

$f'(x) = (n)\, x^{n-1}$

Derivative of the natural log.

$f(x) = \ln(x)$

$f'(x) = \frac{}{x}$

Derivative Rules:

$$\left[f(x)+g(x)\right]' = f'(x)+g(x)$$

$$\left[a f(x)\right]' = a f'(x)$$

$$\left[f(x)\cdot g(x)\right]' = f(x)\cdot g'(x) + f'(x)\, g(x)$$
(Product Rule)

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - g'(x)f(x)}{\left[g'(x)\right]^2}$$

$$\left[f(g(x))\right] = f'(g(x))\cdot g'(x) \quad \text{(Chain Rule)}$$

* Using these rules, you only have to know a few derivatives off the top of your head to find other ones.

Ex: Let $f(x) = \dfrac{2\ln(x)}{3x^2}$. Find $f'(x)$.

$$f'(x) = \frac{3x^2\left(\frac{2}{x}\right) - 2\ln(x)\,6x}{(6x)^2}$$

$$= \frac{6x - 2\ln(x)\cdot 6x}{(6x)^2} = \frac{\cancel{6x}(1-2\ln x)}{(6x)^{\cancel{2}}}$$

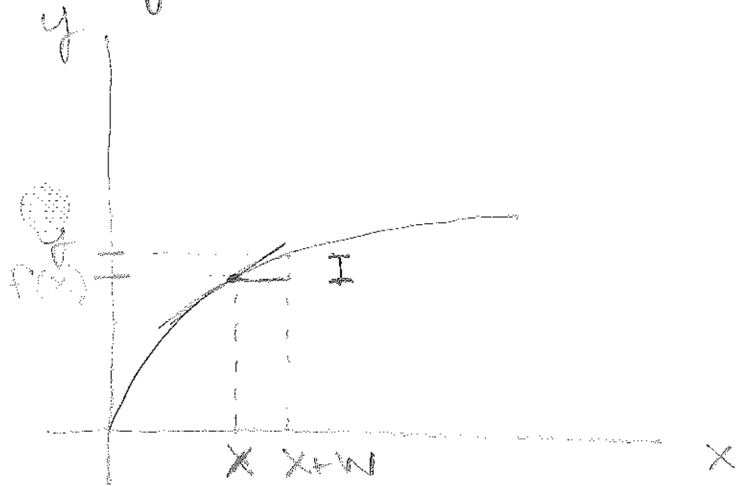
$$= \frac{1-2\ln x}{6x}.$$

Back to logs real quick:

* This is a very useful log identity we'll use often

$$\boxed{\ln(w+1) \approx w}$$

Why is this true?



$$y = f(x+w) \approx f(x) + wf'(x)$$

$$y = \ln(x+w) = \ln(x) + w \cdot \frac{1}{x}$$

What if $x = 1$?

$$y = \ln(1+w) = \ln(1) + w \cdot \frac{1}{1} = 0 + w = w!$$

Optimization:

As you may remember from calc, we "optimize" a function by finding it's minimum or maximum (depending on the context).

→ we do this by taking the derivative of the function and setting it equal to 0. Then figuring out what value of x makes that true.

Ex: If $f(x) = -3x^2 + 4$, what value of x maximizes $f(x)$?

1) Take the derivative:

$$f'(x) = -6x$$

2) Set it $= 0$:

$$f'(x) = -6x = 0$$

3) Solve for x:

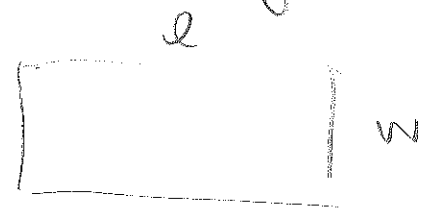
$$-6x = 0$$

$x = 0$ ← this maximizes f.

Economists are terrible at making sure they found a max & not a min but you should check. Most functions we'll work with are set up so you don't need to.

Constrained Optimization.

- Sometimes there is some restriction on the variable youre working with. Like, "I want to build the biggest fenced in garden area I can but I only have 20 ft of fencing."



Then the problem is:

$$\max: \ell \cdot w \text{ by picking } \ell \text{ \& } w.$$
$$\text{s.t. } 2\ell + 2w = 20$$

There are 2 ways to do this.

1) solve the constraint for one of the vars & plug it in.

$$2\ell + 2w = 20$$
$$2w = 20 - 2\ell$$
$$w = 10 - \ell$$

now max $\ell(10-\ell)$ by choosing ℓ

$$f(\ell) = 10\ell - \ell^2$$
$$f'(\ell) = 10 - 2\ell = 0$$
$$10 = 2\ell$$
$$5 = \ell$$

$$w = 10 - \ell$$
$$w = 5$$

2) The Lagrangian

The Lagrangian is a single function that combines the objective function (the function we're maximizing or minimizing) and the constraint.

$$\mathcal{L} = [\text{objective function}] + \lambda\,[\text{bound of constraint} - (\text{constraint function})]$$

$\uparrow$ Lagrange mult.

Then, we optimize the Lagrangian function. But it's just one function now.

Ex. $\mathcal{L} = lw + \lambda[20 - 2l - 2w]$

$$\frac{\partial \mathcal{L}}{\partial w} = l + \lambda(-2) = l - 2\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial l} = w + \lambda(-2) = w - 2\lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 20 - 2l - 2w = 0$$

$2l + 2w = 20$ (original constr.)

$l = 2\lambda$
$w = 2\lambda$ $\longrightarrow$ $l = w$

$2l + 2(l) = 20$
$l = 5 \longrightarrow w = 5.$

Math Practice #2:

Find the derivatives of the following:

(a) $f(x) = 2x^2$ $\qquad f'(x) = 4x$

(b) $f(x) = \ln(2x)$ (don't forget the chain rule)

$f'(x) = \frac{1}{2x} \cdot 2 = \frac{1}{x}$

(c) $f(x) = 2x^2 \ln(2x)$ $\qquad f'(x) = (2x^2)(\frac{1}{x}) + (\ln(2x)4x)$

$= 2x + 4x\ln(2x)$

(d) $f(x) = \dfrac{\ln(2x)}{2x^2}$

$$f'(x) = \frac{2x^2 \cdot \frac{1}{x} - \ln(2x) \cdot 4x}{(2x^2)^2} = \frac{2x - 4x\ln(2x)}{4x^4}$$

I can spend my money on pizza and beer. I have \$30 to spend, pizza costs \$2 per unit and beer costs \$3. If my utility from pizza & beer is:

$u(p, b) = \sqrt{p} + \sqrt{b}$

What is my optimal pizza & beer consumption given my budget constraint.

~~(Do it both ways.)~~

Week 1: GDP & Inflation

Day 3: Definition of GDP

(*) this is macroeconomics so we want to analyze the whole economy. But how?

Gross Domestic Product: dollar value of all output in a country in a year
- nominal GDP: that year's dollar value
- real GDP: base year's dollar value

→ conceptually, the quantity of stuff produced.

Suppose the US produces 2 goods: books & cars.

Let b_{2015} & C_{2015} be the number of each produced in 2015.

The prices are $P_{b,2015}$ & $P_{c,2015}$

Nominal GDP in 2015 =

$$P_{b,2015} \cdot b_{2015} + P_{c,2015} \cdot C_{2015}$$

in 2016 $= P_{b,2016}\, b_{2016} + P_{c,2016}\, C_{2016}$

Why do we need real GDP?
- suppose prices change but quantities don't
- suppose books go up but cars goes down

So we pick a base year to make comp-
isons across years meaningful

Let's make 2015 the base year.

$\longrightarrow nGDP_{2015} = rGDP_{2015}$

Let's calculate real GDP in 2016
- use 2016 quantities
- use base year prices.

$rGDP_{2016} = P_{b,2015} \cdot b_{2016} + P_{c,2015} \cdot c_{2016}$

Note: The level (number) of real GDP doesn't mean a whole lot because it will change based on which base year we pick

$\longrightarrow$ it's the growth rate we're interested in

So lets find that! from 2015 to 2016

$$\%\Delta rGDP_{2016} = \frac{rGDP_{2016}}{rGDP_{2015}} - 1$$

$$= \frac{P_{b2015}\, b_{2016} + P_{c2015}\, c_{2016}}{P_{b2015}\, b_{2015} + P_{c2015}\, c_{2015}} - 1$$

I want to express this in a way that describes the growth rate in terms of growth in car & book production

$$= \frac{P_{b2015}\, b_{2016}}{rGDP_{2015}} + \frac{P_{c2015}\, c_{2016}}{rGDP_{2015}} - 1$$

We can rewrite $b_{2016} = b_{2015} \cdot \frac{b_{2016}}{b_{2015}}$

$$= \frac{P_{b2015}\, b_{2015}}{rGDP_{2015}} \left(\frac{b_{2016}}{b_{2015}}\right) + \frac{P_{c2015} \cdot c_{2015}}{rGDP_{2015}} \left(\frac{c_{2016}}{c_{2015}}\right) - 1$$

Label this $\hat{\phi}_{2015}$

Note that this is

$$\frac{P_{c2015}\, c_{2015}}{P_{b2015}\, b_{2015} + P_{c2015}\, c_{2015}}$$

$$= 1 - \frac{P_{b2015}\, b_{2015}}{P_{b2015}\, b_{2015} + P_{c2015}\, c_{2015}}$$

(*) think about what you would do if you had 3 goods.

$$= 1 - \frac{P_{b2015}\, b_{2015}}{rGDP_{2015}}$$

$$= \hat{\phi}_{2015} \left(\frac{b_{2016}}{b_{2015}}\right) + \left(1 - \hat{\phi}_{2015}\right) \left(\frac{c_{2016}}{c_{2015}}\right) - 1$$

measured expenditure share of GDP on books

growth in book production

m.e.s on cars

growth in car production

In-Class Work:

Ch. 1 HW Problems (p. 38)

2 (a)(b)

4 (don't do the inflation column)

Solutions:

2

(a) nominal GDP in 2000

$= P_{a2000}\, a_{2000} + P_{b2000}\, b_{2000}$

$= (1)(25) + (2.5)(30) = 100$

nominal GDP in 2001

$P_{a2001}\, a_{2001} + P_{b2001}\, b_{2001}$

$= (1.02)(26) + (2.566)(31)$

$= 26.52 + 79.546 = 106.066$

(b) growth rate

$\longrightarrow$ lets pick 2000 as a base year (the levels will be different but the growth rate will be the same)

$rGDP_{2000} = \text{nominal } GDP_{2000} = 100$

$rGDP_{2001} = P_{a2000}\, a_{2001} + P_{b2000}\, b_{2001}$

$= 1(26) + (2.5)(31) = 103.5$

(*) How do changes in GDP relate to overall welfare?

Suppose we citizens have the utility function (in terms of books & cars)

no hat

$$u(b,c) = \phi \ln(b) + (1-\phi)\ln(c)$$

(and $\phi \in (0,1)$)

So our utilities in 2015 & 2016 are:

$$u_{2015} = \phi \ln(b_{2015}) + (1-\phi)\ln(c_{2015})$$

$$u_{2016} = \phi \ln(b_{2016}) + (1-\phi)\ln(c_{2016})$$

How do these compare?

$$u_{2016} - u_{2015} = \phi \ln(b_{2015}) + (1-\phi)\ln(c_{2015}) - [\phi \ln(b_{2016}) + (1-\phi)\ln(c_{2016})]$$

$$= \phi[\ln(b_{2015}) - \ln(b_{2016})] + (1-\phi)[\ln c_{2015} - \ln c_{2016}]$$

$$= \phi\left[\ln\left(\frac{b_{2015}}{b_{2016}}\right)\right] + (1-\phi)\left[\ln\left(\frac{c_{2015}}{c_{2016}}\right)\right]$$

$$= \phi \ln\left[\frac{b_{2016} + b_{2015} - b_{2016}}{b_{2016}}\right] + (1-\phi)\ln\left[\frac{c_{2016} + c_{2015} - c_{2016}}{c_{2016}}\right]$$

+0

$$= \phi \ln\left(1 + \frac{b_{2015} - b_{2016}}{b_{2016}}\right) + (1-\phi)\ln\left(1 + \frac{c_{2015} - c_{2016}}{c_{2016}}\right)$$

$$\approx \phi\left(\frac{b_{2015} - b_{2016}}{b_{2016}}\right) + (1-\phi)\left(\frac{c_{2015} - c_{2016}}{c_{2016}}\right)$$

$$= \phi\left(\frac{b_{2015}}{b_{2016}}\right) + (1-\phi)\left(\frac{c_{2015}}{c_{2016}}\right) - 1$$

In the appendix (& gosh it might make a great test question) it's shown that the optimal amount we citizens should spend on books is ϕ (and on cars is $1-\phi$).

* If utility is such that expenditure shares (ϕ's) are constant over time (& all production consumed), then we can say utility went up when real GDP growth was positive. (when real GDP went up)

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* What has historically been happening to GDP?

(pull up trend real GDP) (p. 13)

- (log) real GDP has increased by a pretty steady rate
- The trend line for log rGDP is the path of rGDP if it had increased by a constant amount every year

Label the slope of that trend line g.

the growth rate of trend real GDP increases by 100 · g units every period.

Why?

Let y_t^* be trend real GDP in period t

$$\ln(y_{t+1}^*) - \ln(y_t^*) = g/1$$

$$\Rightarrow \ln\left(\frac{y_{t+1}^*}{y_t^*}\right) = g$$

$$\ln\left(\frac{y_t^* + y_{t+1}^* - y_t^*}{y_t^*}\right) = g$$

$$\ln\left(1 + \frac{y_{t+1}^* - y_t^*}{y_t^*}\right) = g$$

$$\approx \underbrace{\frac{y_{t+1}^* - y_t^*}{y_t^*}}_{\text{growth rate of RGD}} = \underbrace{g}_{\text{slope of trend line}}$$

So if I tell you that the slope of the log real GDP trend line is 0.036 then you can tell me that trend GDP grew by 3.6%.

Final Notes on today's material:

- GDP doesn't include home-produced goods
  → one reason that the richest countries' per capita GDPs are so much higher than the poorest countries'

- If instead of consuming everything produced in a single period, we set some aside to consume in the next period, our conclusion that positive growth in rGDP implies improved standard of living no longer holds.

Study Time!

(Study Tips)

Day 4: Components of GDP & Inflation

(*) We can't possibly count up all the books & cars (and cups and hair cuts) produced

→ So instead we track how the money in an economy is spent.

GDP = C   Consumption

+ I   Investment

+ G   Government Spending

+ NX   Net Exports

* Expenditure Method

> all the stuff we produce is ~~used~~ bought by households, firms, & the government or (not gov) private consumption, private (not gov) investment, & government spending.

→ NX is there to make sure we only count our stuff

let's talk about each of these components

Consumption

anything that gives the agent utility this period that cannot also give her utility next period.

- in some ways it's easy to measure.
  → money spent on stuff
- but some things are very difficult.
  → consumption related to durable goods (cars, houses, washing machines)

Housing:

We use the rental price of a house as it's consumption value for that period. (even if the "consumer" of the house isn't renting it).

All Other Durable Goods:

- the BEA assumes all other durable goods' values are consumed in the period in which they were bought. (Ya know, the exact opposite of what it means to be a durable good.)

→ for this reason, when we look at consumption data, we often use "consumption excluding durables."

(Look at Consumption trends) (Figure 1.5) p. 19

(*) Consumption is <u>smoother</u> than overall GDP

⟶ it fluctuates up & down less

⟶ therefore, all those fluctuations in GDP need to come from other components, like...

<u>Investment</u>

(*) anything the agent stores away today to get consumption with later.

(As we'll discuss next week) this investment is used by firms to increase their capital stock (the stuff they use to produce output)

We have a formula for this:

$$K_{t+1} = K_t - \delta K_t + I_t$$

"Capital stock tomorrow is the capital stock we have today, minus any that depreciates today plus any investment today."

(Look at Investment) (Figure 1.8) p. 22

(*) Investment is <u>more</u> volatile than overall GDP by a lot.

Government Spending

(*) There are 3 categories:

1. Federal, national defense
2. Federal, non-defense
3. State & Local.

(Government Spending #'s) (Table 1.2) p23

(*) State & Local account for much more than Federal

→ But why? My federal taxes are always much higher!

→ Most of that goes to transfers. (social security, Medicare, etc.)

Net Exports

~~(If we have time in the last week, we'll talk about Trade & NX in more detail)~~

(*) Exports − Imports

Let's look at the GDP equation:

$GDP = C + I + G + NX$

Suppose $G = 0$ and $GDP = C$

$GDP - C = 0$

$= I + NX$

$I = -NX$

$\longrightarrow I = -NX$

So $I = \text{Imports} - \text{Exports}$

If investment is positive (which we know it is), then foreigners own some of our capital stock. Because don't forget:

$$K_{t+1} = K_t - \delta K_t + I_t$$
$$= K_t - \delta K_t + \text{Imports}_t - \text{Exports}_t$$

*) This isn't necessarily bad
⟶ might be because our capital stock is really good and people are willing to pay for it.

In-Class Work 1

Ch. 1 HW Problems
#1 (a).

Explain why government spending in the expenditure method is not related to government tax surpluses/deficits. (See p. 24)

Solutions:
#1 (a):
GDP Stands for Gross Domestic Product
Consumption Government Spending
Investment Net Exports.

Inflation

* it is the rate of change of the price level

$\longrightarrow$ it is not the price level itself

Recall all the way back to yesterday with the books & cars. (Inflation in book prices)

The rate of change in book prices is

$$\frac{P_{b\,2016}}{P_{b\,2015}} - 1$$

Inflation in car prices.

The rate of change in car prices is

$$\frac{P_{c\,2016}}{P_{c\,2015}} - 1$$

The inflation rate on the combined basket of books & cars is defined as

$$\underbrace{\hat{\phi}_{2015}}_{\text{m.e.s on books}} \overbrace{\left(\frac{P_{b\,2016}}{P_{b\,2015}}\right)}^{\text{growth in book prices}} + \underbrace{\left(1-\hat{\phi}_{2015}\right)}_{\text{m.e.s in cars}} \overbrace{\left(\frac{P_{c\,2016}}{P_{c\,2015}}\right)}^{\text{growth in car prices}} - 1$$

~~(We could derive this like we did yesterday but we just did it)~~

* Notice that both price levels don't have go up for the overall price level to go up.

~~(three goods~~)

#4 inflation column

$$\hat{\phi}_{2005} = \frac{Pa_{2005} \cdot a_{2005}}{Pa_{2005} \cdot a_{2005} + Pb_{2005}\, b_{2005}}$$

$$= \frac{2(10)}{2(10) + 1(5)} = \frac{20}{25} = .8$$

$$\text{inflation}_{2006} = \hat{\phi}_{2005}\left(\frac{2.02}{2.00}\right) + (1 - \hat{\phi}_{2005})\left(\frac{1.05}{1.00}\right) - 1$$

$$= .8(1.01) + .2(1.05) - 1$$

$$= .808 + .21 - 1 = .018$$

$$\hat{\phi}_{2006} = \frac{Pa_{2006} \cdot a_{2006}}{Pa_{2006} \cdot a_{2006} + Pb_{2006} \cdot b_{2006}}$$

$$= \frac{2.02(11)}{2.02(11) + 1.05(6)} = \frac{22.22}{28.52}$$

$$= .779$$

$$\text{inflation}_{2007} = \hat{\phi}_{2006}\left(\frac{2.05}{2.02}\right) + (1 - \hat{\phi}_{2006})\left(\frac{1.12}{1.05}\right) -$$

$$= (.779)(1.015) + (.221)(1.067) - 1$$

$$= .791 + .236 - 1 = .027$$

(look at inflation figure) (Figure 1.11) p.33

We often exclude food & energy because they're more volatile

$\longrightarrow$ but recently we've started focusing on it more because those are things that affect consumers directly.

Ch. 1 HW Problems #1(a)

#2 (c) & (d)

#4 inflation column

#5 (a)

Solutions:

#2

(c) $$\text{inflation} = \hat{\phi}_{2000} \frac{P_{a2001}}{P_{a2000}} + (1-\hat{\phi}_{2000})\left(\frac{P_{b2001}}{P_{b2000}}\right) - 1$$

$$\hat{\phi}_{2000} = \frac{P_{a2000} \cdot a_{2000}}{P_{a2000} \cdot a_{2000} + P_{b2000} \cdot b_{2000}}$$

$$= \frac{(1)(25)}{(1)(25) + (2.5)(30)} = \frac{25}{100} = .25$$

$$\text{inflation} = (.25)\left(\frac{1.02}{1.00}\right) + (.75)\left(\frac{2.566}{2.5}\right) -$$

$$= .255 + .7698 - 1 = \boxed{0.0248}$$

Week 2: Firms & Growth

Day 1: Cobb-Douglas Production

(*) Let us all just take a moment to appreciate the quote on p. 45. (I'm thinking of having it printed on business cards to hand out.)

Now were going to start talking about production (but w/ an eye toward macro questions)(specifically growth)

What do we produce?

Real Output, Y_t, in time period t (which is also GDP)

Using what?

Technology, z_t

Real Capital Stock, K_t

Labor, L_t

How?

→ this is described using a production function.

$$Y_t = F(z_t, K_t, L_t) = \underbrace{z_t K_t^{\alpha} L_t^{1-\alpha}}_{\text{Called a Cobb-Douglas Production function}}$$

1

What is α?

$\longrightarrow$ for now, it's just a parameter to allow the function to do more stuff

$\alpha \in (0,1) \longrightarrow$ a quicker way of writing: $0 < \alpha < 1$

(*) Cobb-Douglas Production Functions have two important properties:

1. Constant Returns to Scale
2. Declining Marginal Products

1. Constant Returns to Scale

Mathematically:

$$Z_t (cK_t)^{\alpha} (cL_t)^{1-\alpha} = cY_t$$

Verbally:

Holding z fixed, scaling up (doubling) capital & Labor by some amount, will scale up output (double) by that same amount.

$$Z_t (c K_t)^\alpha (c L_t)^{1-\alpha} = Z_t c^\alpha K_t^\alpha c^{1-\alpha} L_t^{1-\alpha}$$

$$= \left[c^{\alpha} \cdot c^{1-\alpha}\right] Z_t K_t^\alpha L_t^{1-\alpha}$$

$$= c \cdot Z_t K_t^\alpha L_t^{1-\alpha} = c\, Y_t \quad !$$

But another way to write this:

$$2Y_t = \underbrace{Z_t K_t^\alpha L_t^{1-\alpha}}_{\text{firm 1}} + \underbrace{Z_t K_t^\alpha L_t^{1-\alpha}}_{\text{firm 2}}$$

$$= 2\left[Z_t K_t^\alpha L_t^{1-\alpha}\right]$$

$$= \underbrace{Z_t (2K_t)^\alpha (2L_t)^{1-\alpha}}_{\text{just 1 big aggregate firm}}$$

(*) So if we assume Cobb-Douglas Production (& perfect competition), we can act as if there's just one big firm in the economy.

2. Declining Marginal Products

<u>Marginal Products</u>: the change in producti from a single unit increase in an input
$\longrightarrow$ it's the derivative of the production function with respect to that input.

Marginal Product of Capital

$$MP_K = \frac{\partial F(z_t, K_t, L_t)}{\partial K_t}$$

$$= \frac{\partial\left[z_t K_t^{\alpha} L_t^{1-\alpha}\right]}{\partial K_t}$$

$$= z_t \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

$$= \alpha z_t K_t^{-(1-\alpha)} L_t^{1-\alpha}$$

$$= \alpha z_t \frac{1}{K_t^{1-\alpha}} \cdot L_t^{1-\alpha}$$

$$= \alpha z_t \left(\frac{L_t}{K_t}\right)^{1-\alpha}$$

Marginal Product of Labor

$$MP_L = \frac{\partial F(z_t, K_t, L_t)}{\partial L_t}$$

$$= \frac{\partial\left[z_t K_t^{\alpha} L_t^{1-\alpha}\right]}{\partial L_t}$$

$= Z_t K_t (1-\alpha) L_t^{-\alpha}$

$= (1-\alpha) K_t^{\alpha} L_t^{-(\alpha)}$

$= (1-\alpha) Z_t K_t^{\alpha} \dfrac{1}{L_t^{\alpha}}$

$= (1-\alpha) Z_t \left(\dfrac{K_t}{L_t}\right)^{\alpha}$

Marginal Product of Capital,

$= \alpha Z_t \left(\dfrac{L_t}{K_t}\right)^{1-\alpha}$

$\longrightarrow$ as K_t goes up, MP_K goes down

Marginal Product of Labor $\dfrac{\partial F(\,)}{\partial L_t} =$

$= (1-\alpha) Z_t \left(\dfrac{K_t}{L_t}\right)^{\alpha}$

$(1-\alpha) Z_t K_t^{\alpha} L_t^{-\alpha}$
$= (1-\alpha) Z_t \left(\frac{K_t}{L_t}\right)^{\alpha}$

$\longrightarrow$ as L_t goes up, MP_L goes down

* This is intuitively how we think inputs should work.

In Class-Work!

Ch. 2 HW Questions

#5 determine if the production function has CRS & DMP.

#1, #2 (using derivatives), & #3 what you think it is.

Solutions:

#1: The fuel & technology both went in to making the product. Fuel costs money.

#2:

$L_t^{-\alpha}$

$$MP_L = (1-\alpha) z_t \left(\frac{K_t}{L_t}\right)^{\alpha}$$

$$\frac{\partial MP_L}{\partial L} = (1-\alpha) z_t K_t^{\alpha} \left[-\alpha L_t^{-\alpha-1}\right]$$

$-(\alpha+1)$

$$= \frac{-\alpha(1-\alpha) z_t K_t^{\alpha}}{L_t^{\alpha+1}}$$

For any positive z_t, L_t, K_t (which in our contexts they always will be) this is negative because of the negative α.

#3

What do you think the average product of Labor will be?

$$MP_K = \alpha z_t K_t^{\alpha-1} L_t^{1-\alpha}$$

$$\alpha z_t K_t^{-(1-\alpha)} L_t^{1-\alpha}$$

$\alpha(\alpha-1) z_t K_t^{\alpha-2} L_t^{1-\alpha}$

$$\frac{\partial MP_K}{\partial K_t} = \cancel{\alpha(\alpha-1) z_t K_t^{\alpha-2} L_t^{1-\alpha}}$$

$$= \alpha\left[-(1-\alpha)\right] z_t K_t^{-(1-\alpha)-1} L_t^{1-\alpha}$$

$$= \frac{-\alpha(1-\alpha) z_t L_t^{1-\alpha}}{K_t^{1-\alpha+1}}$$

Day 2: Profit Maximization

(*) So far we've described how much output we can make for a given K & L. Now we need to figure out the best choice of K & L

→ We usually take Z_t as given.

(*) The representative big firm ~~Firms~~ choose K & L by maximizing profit

Profit = Total Revenue − Total Cost

$$\pi(K, L) = Y_t - r_t K_t - w_t L_t$$
$$= Z_t K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t$$

(*) Take prices as given (perfect competition)

- the price of output is 1
- the price of capital is r_t
- the price of labor is w_t

(We could make the price of output P_t but then we could just divide everything by P_t & be right back where we started.)

(*) How do we maximize a function?

→ take it's derivative & set it equal to 0.

Specifically, to pick capital we take the partial derivative of profits wrt capital, set it equal to 0 & solve.

$$\frac{\partial \pi(K, L)}{\partial K} = 0 \qquad \text{(Optimal Capital)}$$

$$\frac{\partial \left[z_t K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t \right]}{\partial K_t} = 0$$

$$\alpha z_t K_t^{\alpha-1} L_t^{1-\alpha} - r_t = 0$$

b/c $p_t = 1$

$$\underbrace{\alpha z_t K_t^{\alpha-1} L_t^{1-\alpha}}_{MP_K} = \underbrace{r_t}_{MC_K}$$

$$= MR_K = MC_K$$

Here we see the usual "marginal benefit = marginal cost" optimality condition (maybe you saw it in 301)

Also! If we multiply both sides by K_t . .

$$K_t \alpha z_t K_t^{\alpha-1} L_t^{1-\alpha} = r_t K_t$$

$$\alpha \underbrace{z_t K_t^{\alpha} L_t^{1-\alpha}}_{} = r_t K_t$$

$$\alpha \underbrace{Y_t}_{\text{value of output}} = \underbrace{r_t K_t}_{\text{amount spent on capital}}$$

(in the whole economy)

(*) So the amount spent on capital is a constant share of output. Namely $\underline{\underline{\alpha}}$.

→ α is called the <u>capital</u> <u>share</u> <u>in</u> <u>production</u>.

Next, to find the optimal amount of labor we do the same thing:

$$\frac{\partial \pi(K_t, L_t)}{\partial L_t} = 0$$

$$\frac{\partial\left[z_t K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t\right]}{\partial L_t} = 0$$

$$(1-\alpha) z_t K_t^{\alpha} L_t^{-\alpha} - w_t = 0$$

$$\underbrace{(1-\alpha) z_t K_t^{\alpha} L_t^{-\alpha}}_{MP_L} = \underbrace{w_t}_{MC_L}$$

marginal benefit = marginal cost

Again multiplying both sides by L_t

$$L_t (1-\alpha) z_t K_t^{\alpha} L_t^{-\alpha} = w_t L_t$$

$$(1-\alpha) z_t K_t^{\alpha} L_t^{1-\alpha} = w_t L_t$$

$$(1-\alpha) \underbrace{Y_t}_{\text{value from output}} = \underbrace{w_t L_t}_{\text{amount spent on Labor}}$$

(*) So the amount spent on labor (which is also the total wages earned by workers because this is macro!) is a constant share of the value from output.

⟶ this is called the labor share in production.

(*) So what are profits when firms make their optimizing decicisons?

Let L_t^* & K_t^* be the optimal choices

$w_t L_t^* = (1-\alpha) Y_t$

$r_t K_t^* = \alpha Y_t$

} it's good to get practice making work easier for yourself
⟶ there are ~~two~~ multiple formulas we've written down for L^* & K^* but some of them will make the algebra easier (This is constantly challenging in macro)

Π^* is optimal profit

$\Pi^* = \Pi(K^*, L^*)$

Optimal profits are the value of the profit function with optimal K & L plugged in.

see?

$$\Pi_t^* = \Pi(K_t^*, L_t^*) = Y_t - r_t K_t^* - w_t L_t^*$$

$$= Y_t - \alpha Y_t - (1-\alpha) Y_t$$

$$= Y_t - (\alpha + 1 - \alpha) Y_t = 0$$

(*) So firms make 0 economic profit when they're optimizing.

→ remember economic profits are not accounting profits.

→ any accounting profits are included in rental payments to investors.

In Class Work!

Ch. 2 HW Problems

#5 (a) & find optimal capital & labor

Repeat for the production function in #6.

<u>Solutions:</u>

#5

$$Y_t = K_t^{\alpha}(Z_t L_t)^{1-\alpha} = \underbrace{Z_t^{1-\alpha}}_{\tilde{Z}_t} K_t^{\alpha} L_t^{1-\alpha}$$

(a)

$$\tilde{Z}_t K_t^{\alpha} L_t^{1-\alpha}$$

Firm Profits:

$$\pi(K_t, L_t) = \tilde{Z}_t K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t$$

Optimal Capital:

$$\alpha \tilde{Z}_t K_t^{\alpha} L_t^{1-\alpha} = r_t K_t$$

$$\alpha Y_t = r_t K_t \quad \text{still true}$$

Day 3: Growth!

(*) in order to get meaningful changes in aggregate welfare we need growth in output & that requires growth in productiv

The average product of labor is

$\frac{Y_t}{L_t}$ "the amount of output per unit of labor."

(the average product of capital is

$\frac{Y_t}{K_t}$) "the amount of output per unit of capital."

Recall that:

apl

$(1-\alpha) Y_t = w_t L_t$ or $\longrightarrow (1-\alpha) \frac{Y_t}{L_t} = w_t$

$\longrightarrow$ this means wages only increase if labor productivity changes.
(average product of labor)

(*) The best way to make labor or capital productivity go up is to make Y_t go up w/o increasing K_t & L_t.

$\longrightarrow$ this means increasing z_t.

To get a better handle on all this stuff we need to rewrite the production function in a helpful way.

Today is a great review of all those log rules! ;)

* First, put everything in per-capita terms.

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$$

divide everything by population:

$$Z_t \left(\frac{K_t}{N}\right)^{\alpha}\left(\frac{L_t}{N}\right)^{1-\alpha} = \frac{1}{N} Z_t K_t^{\alpha} L_t^{1-\alpha}$$

$$= \frac{1}{N} Y_t$$

Yay! Constant Returns to Scale!

We typically relabel per-capita variables with lower case letters:

$$y_t = z_t k_t^{\alpha} l_t^{1-\alpha}$$

Next, we log-linearize:

$$\ln(y_t) = \ln(z_t k_t^{\alpha} l_t^{1-\alpha})$$

$$= \ln(z_t) + \ln(k_t^{\alpha}) + \ln(l_t^{1-\alpha})$$

$$\ln(y_t) = \ln(z_t) + \alpha \ln(k_t) + (1-\alpha)\ln(l_t)$$

What we really want is an expression for the rate of change in y_t right? Something that looks like $\frac{y_{t+1} - y_t}{y_t}$.

Write down the t+1 version.

$$\ln(y_{t+1}) = \ln(z_{t+1}) + \alpha \ln(k_{t+1}) + (1-\alpha)\ln(l_{t+1})$$

Subtract $\ln(y_t)$ from $\ln(y_{t+1})$

$$\ln(y_{t+1}) - \ln(y_t) = \ln(z_{t+1}) - \ln(z_t) + \alpha\left[\ln(k_{t+1}) - \ln(k_t)\right] + (1-\alpha)\left[\ln(l_{t+1}) - \ln(l_t)\right]$$

Yay log rules!

$$\ln\left(\frac{y_{t+1}}{y_t}\right) = \ln\left(\frac{z_{t+1}}{z_t}\right) + \alpha \ln\left(\frac{k_{t+1}}{k_t}\right) + (1-\alpha)\ln\left(\frac{l_{t+1}}{l_t}\right)$$

Ok, time for a time-honored math trick: adding and subtracting 1 to get what you want.

$$\ln\left(1 + \frac{y_{t+1}}{y_t} - \frac{y_t}{y_t}\right) = \ln\left(1 + \frac{z_{t+1}}{z_t} - \frac{z_t}{z_t}\right)$$

$$+ \alpha \ln\left(1 + \frac{k_{t+1}}{k_t} - \frac{k_t}{k_t}\right) + (1-\alpha)\ln\left(1 + \frac{l_{t+1}}{l_t} - \frac{l_t}{l_t}\right)$$

$$\ln\left(1+\frac{y_{t+1}-y_t}{y_t}\right)=\ln\left(1+\frac{z_{t+1}-z_t}{z_t}\right)+\alpha\ln\left(1+\frac{k_{t+1}-k_t}{k_t}\right)$$
$$+(1-\alpha)\ln\left(1+\frac{l_{t+1}-l_t}{l_t}\right)$$

OK! Things are looking up! Just a pesky log and a 1 left to deal with!

Thankfully we have this ~~great~~ log approximation.

$$\ln(1+x)\approx x!$$

So now:

$$\underbrace{\frac{y_{t+1}-y_t}{y_t}}_{g_y}=\underbrace{\frac{z_{t+1}-z_t}{z_t}}_{g_z}+\alpha\cdot\overbrace{\frac{k_{t+1}-k_t}{k_t}}^{g_k}+(1-\alpha)\underbrace{\frac{l_{t+1}-l_t}{l_t}}_{g_l}!$$

So we have (mathematically) three potential sources of output growth:

- g_z
- g_k
- g_l

In Class Work:

Ch. 2 HW Problems

#3
#6

Solutions:

#3

Average product of labor: amount of output produced per unit of labor used to produce the output

$$= \frac{Y_t}{L_t}$$

Wages rise with labor productivity because, when optimizing:

$$\alpha Y_t = W_t L_t$$

$$\alpha \left(\frac{Y_t}{L_t} \right) = W_t$$

So when labor productivity goes up, wages go up by a constant amount: α.

#6

Just do the log linearizing & growth rate-ifying parts.

$$\ln(Y_t) = \ln(z_t^{\gamma}) + \ln(K_t^{\alpha}) + \ln(L_t^{\beta})$$

$$= \gamma \ln(z_t) \, \alpha \ln(K_t) + \beta \ln(L_t)$$

$$\ln(Y_{t+1}) = \gamma \ln(z_{t+1}) + \alpha \ln(K_{t+1}) + \beta \ln(L_{t+1})$$

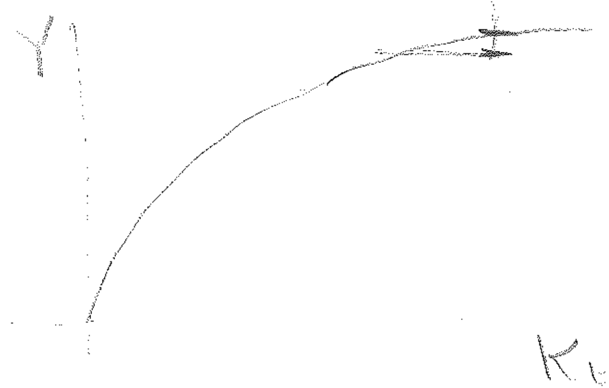
Subtract

$$\ln\left(\frac{Y_{t+1}}{Y_t}\right) = \gamma \ln\left(\frac{z_{t+1}}{z_t}\right) + \alpha \ln\left(\frac{K_{t+1}}{K_t}\right) + \beta \ln\left(\frac{L_{t+1}}{L_t}\right)$$

(*) It turns out that

1) The percapita labor supply is pretty constant (& bounded
→ we only have so many hours to work in a day.)

2) In developed countries like the US the per-capita capital stock is bounded too.



eventually it will be not worth it to invest more.

So:
Sustained growth comes from growth in technology

(*) Historically, the US has been on a balanced growth path.

(*) What is balanced growth?

- the growth rate of output, capital consumption and investment are the same. } sh th
- these growth rates are all tied directly to the growth rate in technology. } show this
- in data { the per-capita labor input is trendless
  - real interest rates are trendless

trendless: they fluctuate around their average pretty evenly; they don't really grow over time.

(*) Let's start with $g_y = g_k$

Using our handy log-linearized rate of change equation

$$g_y = g_z + \alpha g_k + (1-\alpha) g_\ell$$

Let's assume (as mentioned) per capita labor supplied doesn't change: $g_\ell = 0$

$$g_y = g_z + \alpha g_k$$

If we plug in $g_y = g_k$

$$g_k = g_z + \alpha g_k$$

$$(1-\alpha) g_k = g_z \rightarrow g_k = \frac{g_z}{1-\alpha} = g_y$$

I claim in balanced growth C_t/Y_t & I_t/Y_t are constant

Now let's consider the GDP accounting equation (& assume for simplicity that $G = NX = 0$).

$$Y_t = C_t + I_t$$

& remember $K_{t+1} = K_t - \delta K_t + I_t$

$$I_t = K_{t+1} - (1-\delta)K_t$$

Plug that in:

$$Y_t = C_t + K_{t+1} - (1-\delta)K_t$$

Divide everything by Y_t

$$1 = \frac{C_t}{Y_t} + \frac{K_{t+1}}{Y_t} - (1-\delta)\frac{K_t}{Y_t}$$

Get ready for another classic math trick

$$\frac{1}{Y_t} = \frac{1}{Y_{t+1}} \cdot \frac{Y_{t+1}}{Y_t}$$

$$1 = \frac{C_t}{Y_t} + \frac{K_{t+1}}{Y_{t+1}} \cdot \frac{Y_{t+1}}{Y_t} - (1-\delta)\frac{K_t}{Y_t}$$

In balanced growth, these puppies are constant. (b/c $g_y = g_k$)

So we basically have:

$$1 = \frac{C_t}{Y_t} + M$$

$$1 - M = \frac{C_t}{Y_t}$$ so consumption to output ratio is constant.

Go back to

$$Y_t = C_t + I_t$$

(*) because the ratios are constant their growth rates are equal

$$\frac{Y_t}{Y_t} = \frac{C_t}{Y_t} + \frac{I_t}{Y_t}$$

$$\underset{\sim}{1} = \underset{\sim}{\frac{C_t}{Y_t}} + \frac{I_t}{Y_t}$$ So investment to ouput is too.

In Class Work!

#5 (b) & (C)

(b)

$$\alpha Y_t = r_t K_t$$

$$\alpha \frac{Y_t}{K_t} = r_t = r$$

$$\therefore Y_t = \frac{r}{\alpha} K_t$$

$$Y_{t+1} = \frac{r}{\alpha} K_{t+1}$$

$$\frac{Y_{t+1} - Y_t}{Y_t} = \frac{r/\alpha K_{t+1} - r/\alpha K_t}{r/\alpha K_t}$$

$$= \frac{\cancel{r/\alpha}(K_{t+1} - K_t)}{\cancel{r/\alpha} K_t}$$

$$= \frac{K_{t+1} - K_t}{K_t}$$ ✓

$$\ln Y_t = (1-\alpha)\ln z_t + \alpha \ln K_t + (1-\alpha)\ln L_t$$

$$\frac{Y_{t+1} - Y_t}{Y_t} = (1-\alpha)\frac{z_{t+1} - z_t}{z_t} + \alpha \frac{K_{t+1} - K_t}{K_t} + (1-\alpha)\frac{L_{t+1} - L_t}{L_t}$$

$$g_Y = (1-\alpha) g_z + \alpha g_K + (1-\alpha) g_L$$

As we showed in (b) if r_t is constant then $g_Y = g_K$.

$$g_Y = (1-\alpha) g_z + \alpha g_Y + (1-\alpha) g_L$$

$$g_Y - \alpha g_Y = (1-\alpha)[g_z + g_L]$$

$$\cancel{(1-\alpha)} g_Y = \cancel{(1-\alpha)} [g_z + g_L]$$

$$g_Y = g_z + g_L \quad \checkmark$$

Day 4: Measuring K, L, & z:

* In order to say anything about growth, we need some measurements of these key features in the US

Stunningly, let's begin with some algebra:

$$K_{t+1} = K_t - \delta K_t + I_t = (1-\delta)K_t + I_t$$

Rewrite the t one:

$$K_t = (1-\delta)K_{t-1} + I_{t-1}$$

Plug that in for K_t

$$K_{t+1} = (1-\delta)\left[(1-\delta)K_{t-1} + I_{t-1}\right] + I_t$$

$$= I_t + (1-\delta)I_{t-1} + (1-\delta)^2 K_{t-1}$$

you can see where this is going

$$= I_t + (1-\delta)I_{t-1} + (1-\delta)^2 I_{t-2} + (1-\delta)^3 I_{t-3} + \dots$$

$$= \sum_{s=0}^{\infty}(1-\delta)^s I_{t-s}$$

"Perpetual Inventory" Accounting Equation

⟶ this is the equation the BEA uses to calculate the capital stock.

⟶ We can use their data to calculate the capital - output ratio.

Using Figure 2.1 : p.70

Private Fixed Assets:	31,818.5
[illegible] Nom. State & Local Assets:	6,909.4
Nom. Federal Non-Defense:	708.7
(nom. private residential):	17,103.5
Nominal Stock of Non-Defense US Capital	= 22,333.1

Output is going to be measured as nom GDP - "consumption of housing service
= 11,813.4

nominal stock of capital to nominal annual output in 2006 =

22,333.1 / 11,813.4 = 1.89

(Figure 2.4) p. 73

→ the capital-output ratio has been very consistently about 1.8

Balanced Growth!

Next Up is Labor Input

→ now you must begin to tackle questions about labor that you'll probably have to face many times in economics.

What we want: sum of hours worked in the marketplace of all the workers in the economy during the year.

* How do we measure hours worked?

BLS uses 2 surveys:

- payroll survey of big firms (big but skewed)
- household survey (randomly selected) (better distribution but small)

* Should we quality-adjust?

→ probably not although there are some conceivable benefits.

(Figure 2.6) p.78

Ratio of estimated hours worked to relevant population (non-institutionalized civilians over 16):

Two Things to Note:

1. Been trendless for quite sometime
2. ~~Fluctuates Regularly~~
   ↳ gets into the hairy concept of business cycles which we'll discuss later

And finally to technology:

We have:

Output
Capital
Labor
($\alpha \approx .32$)

We also have an equation:

$$\ln(Y_t) = \underbrace{\ln(z_t)} + \alpha \ln(K_t) + (1-\alpha)\ln(L_t)$$

$$\ln(z_t) = \ln(Y_t) - \alpha \ln(K_t) - (1-\alpha)\ln(L_t)$$

(Figure 2.7) p. 81

Things to Note:

- increasing
- by a fairly constant rate
- hence the trend line.

* As you think about these graphs & compare them, think about what's actually being graphed:
    - is it a ratio? Of what?
    - is it logged?

Study Time!

Week 3: Households

Last week we talked about what firms do with capital & labor. This week we talk about:

How households supply labor.

We all face the classic trade off:

I like consuming things, so I need to work to pay for that consumption, but I also like not doing things.

Leisure – Labor Trade-Off

So we need to choose the two things we like: consumption (C) & Leisure (N):

Household Utility:

$$u(C,N) = \theta \ln(C) + (1-\theta)\ln(N)$$

Maximize utility BUT! We have 2 constraints:

$L + N = 16$ (only have 16 hours a day to allocate)

$C = \hat{w} L$ (after tax income all goes to consumption)
(Budget Constraint)

→ no saving or borrowing yet
→ price of cons. $= 1$

How do we maximize a thing subject some other things?

Lagrangian!

Most times we go ahead and plug $N = 16 - L$ into U, but leave the Budget Constraint as a constraint because it will get hairier.

$$\max_{C,L} \; U(C,L) = \theta \ln(C) + (1-\theta)\ln(16-L)$$

$$\text{s.t.} \quad C = \hat{w} L$$

Work On This

$$\mathcal{L} = \theta \ln(c) + (1-\theta)\ln(16-L) + \lambda[\hat{w}L - c]$$

$$\frac{\partial \mathcal{L}}{\partial c} = \frac{\theta}{c} - \lambda = 0 \longrightarrow \frac{\theta}{c} = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial L} = \frac{1-\theta}{-(16-L)} + \lambda \hat{w} = 0 \longrightarrow \frac{1-\theta}{16-L} = \lambda \hat{w}$$

$$c = \hat{w} L$$

$$\lambda = \frac{\theta}{c}$$

$$\lambda = \frac{1-\theta}{\hat{w}(16-L)}$$

$$\frac{\theta}{c} = \frac{1-\theta}{\hat{w}(16-L)}$$

$$c = \hat{w}(16\theta)$$

$$c = 16\hat{w}\theta$$

$$\frac{\theta}{\cancel{\hat{w}} L} = \frac{1-\theta}{\cancel{\hat{w}}(16-L)}$$

$$\theta(16-L) = (1-\theta)L$$

$$16\theta - \cancel{\theta L} = L - \cancel{\theta L}$$

$$16\theta = L$$

And of course $N = 16 - L = 16 - 16\theta$
$= 16(1-\theta)$

3

Notice that $L = 16$ has no $\hat{w}$ in it.
→ the optimal labor choice doesn't depend on $\hat{w}$! (B/c of how we set up the utility function)

Remember: this has been historically true.

A Quick Discussion of Shadow Costs

(*) Our lagrange multiplier, λ, can also be interpreted as the <u>shadow</u> <u>cost</u> of a constraint.

↳ if you relaxed the constraint by one unit by how much would your utility / objective function go up at the optimum.

If that sounds like a derivative that's because it is; it's the derivative of the Lagrangian wrt. the constraint.

$\mathcal{L}$ = (obj. func.) + λ (constraint)

In the example from earlier:

$$\lambda = \frac{\theta}{C} = \frac{\cancel{\theta}}{16\hat{w}\cancel{\theta}} = \frac{1}{16\hat{w}}$$

Our constraint was

$$C = \hat{w}L$$

So if our after tax income goes up by one unit, our utility goes up by $1/16\hat{w}$.

In Class Work:

#4 & #5

Solutions

$$\max_{C,L} \theta \ln(C) + (1-\theta)\ln(105 - L)$$

s.t.

$$C = \hat{w}L$$

(*) redefined our variables to make use of the data we're given.

all that's really changed is now we're thinking about a week instead of a day.

Everywhere there was a 16 now there's a 105

$$L = 105 \cdot \theta \qquad C = 105\hat{w}\theta$$

Since we work about 20 hrs. a week

$$\therefore 20 \longrightarrow 20 = 105 \cdot \theta \longrightarrow \theta = 0.19$$

Day 2: A Two-Period Model

(*) Yesterday, we talked about a model in which households make decisions about variables in one period that only affect utility in that period.

↳ called a <u>Static Model</u>

But of course households make decisions that affect utility next period.

(*) Households can **SAVE** to finance consumption next period.

When they're making a decision, what should they be trying to maximize?

<u>Future</u> <u>Expected</u> <u>Utility</u>.

What would that look like?

Let's make some assumptions to make our lives easier:

- Only two periods
- let's not worry about leisure for now & just focus on consumption

Two Period Utility:

$$U(C_t, C_{t+1}) = \underbrace{\ln(C_t)}_{\text{utility from consumption this period}} + \underbrace{\beta \ln(C_{t+1})}_{\text{utility from consumption next period}}$$

C_t: consumption this period

C_{t+1}: consumption next period

Why do we have to weight next period's utility? from the perspective of today

- Because, right now, consumption tomorrow is not as good as consumptio right now.
- consumption today and consumption tomorrow are two completely differen goods, like books & cars.
  - ↳ utility should be weighted accordingly.

New Budget Constraint: introducing a new variable

(*) Every period we have <u>assets</u> to consume from, A_t (& A_{t+1})

↳ these include: our income
Stuff we've saved previously

In period t, we consume C_t

$$\underbrace{A_t(1+\hat{r}_t)}_{\text{Assets we saved, earned interest on}} + \underbrace{\hat{w}_t}_{\text{income}} - C_t = \underbrace{A_{t+1}}_{\text{What we save for next period.}}$$

money to spend on consumption

* assets are what we have leftover after we buy consumption

$\hat{r}_t$ = after-tax market rate of return on assets in period t

$\hat{w}_t$ after-tax ~~wage~~ income

Also true in period 2:

$$A_{t+1}(1+\hat{r}_{t+1}) + \hat{w}_{t+1} - C_{t+1} = A_{t+2}$$

Oh look! An A_{t+1} in both equations!
Perhaps we should solve this one & plug it in to that one!

$$A_{t+1}(1+\hat{r}_{t+1}) = A_{t+2} - \hat{w}_{t+1} + C_{t+1}$$

$$A_{t+1} = \frac{1}{1+\hat{r}_{t+1}}\left[C_{t+1} + A_{t+2} - \hat{w}_{t+1}\right]$$

$$A_t(1+\hat{r}_t)+\hat{W}_t-C_t=\frac{1}{1+\hat{r}_{t+1}}\left[C_{t+1}+A_{t+2}-\hat{W}_{t+1}\right]$$

So now we have <u>one</u> budget constraint for both time periods!

<u>Intertemporal Budget Constraint</u>

The maximization problem:

$$\max_{C_t,C_{t+1}} \ln(C_t)+\beta\ln(C_{t+1})$$

s.t.

$$A_t(1+\hat{r}_t)+\hat{W}_t-C_t=\frac{1}{1+\hat{r}_{t+1}}\left[C_{t+1}+A_{t+2}-\hat{W}_{t+1}\right]$$

$$\mathcal{L}=\ln(C_t)+\beta\ln(C_{t+1})+\lambda\left[A_t(1+\hat{r}_t)+W_t-C_t-\frac{1}{1+\hat{r}_{t+1}}\left[C_{t+1}+A_{t+2}-\hat{W}_{t+1}\right]\right]$$

$$\frac{\partial\mathcal{L}}{\partial C_t}:\ \frac{1}{C_t}-\lambda=0 \longrightarrow \frac{1}{C_t}=\lambda$$

$$\frac{\partial\mathcal{L}}{\partial C_{t+1}}:\ \frac{\beta}{C_{t+1}}-\frac{\lambda}{1+\hat{r}_{t+1}}=0 \qquad \frac{\beta}{C_{t+1}}(1+\hat{r}_{t+1})=\lambda$$

$$\frac{1}{C_t} = \frac{\beta(1+\hat{r}_{t+1})}{C_{t+1}}$$

Let's rearrang this so that marginal utilities are on one side & prices are on the other

$$\frac{\underbrace{(1/C_t)}_{(MU_{C_t})}}{\underbrace{(\beta/C_{t+1})}_{(MU_{C_{t+1}})}} = \frac{(1+\hat{r}_{t+1})}{1}$$

This sets relative marginal benefit = relative marginal cost

$(1+\hat{r}_{t+1})$ is the price consumption in period t relative to the price in t+1 because if you consume 1 more unit in period you have to forego $(1+\hat{r}_{t+1})$ units in t+1.

Let's rearrange it again:

$$\frac{C_{t+1}}{C_t} = \beta(1+\hat{r}_{t+1})$$

So the ratio of consumptions is the discount factor times the relative price of consumption in t.

In-Class Work:

Consider the equation for C_{t+1}/C_t

What does this imply about consumption this period & consumption next period when the savings interest rate goes up? When it goes down?

6

7

Solutions:

When $\hat{r}_{t+1}$ goes up, we consume more tomorrow & less today. When $\hat{r}_{t+1}$ goes down we consume more today & less tomorrow.

#6

$$A_t + 100 \cancel{+W} - C_t = \frac{A_{t+2} + C_{t+1} - W_{t+1}}{1.10}$$

Today:

$$A_t + 100 \cancel{+W} - C_t = A_{t+1}$$

Tommorrow

$$A_{t+1}(1.10) + W_{t+1} - C_{t+1} = A_{t+2}$$

$$1.10 = 1 + \hat{r}_t$$

$$A_{t+1} = \frac{A_{t+2} + C_{t+1} - W_{t+1}}{1.10}$$

6

Day 3: Discussing the 2 Period Model

Let's talk about that after tax rate of return on capital:

$$\hat{r}_t = \underbrace{(1-\tau_K)}_{\text{capital income taxes}}\underbrace{(r_t - \delta)}_{\text{rental rate on capital paid by firms less depreciation}} = (r_t - \delta) - \tau_K(r_t - \delta)$$

Let's pretend $\tau_K = 0$ for today. $\rightarrow \hat{r}_t = r_t - \delta$

We're going to use this to show that our model of households matches up with our understanding of GDP (single period).

Rewrite the budget constraint:

$$A_t(1+\hat{r}_t) + w_t - C_t = A_{t+1}$$

$$A_t(1+r_t-\delta) + w_t - C_t = A_{t+1}$$

$$A_t + r_t A_t - \delta A_t + w_t - C_t = A_{t+1}$$

$$r_t A_t + w_t - C_t + (A_t - \delta A_t - A_{t+1}) = 0$$

Suppose there's just 1 representative household for the whole economy

→ that household's assets are all the assets

→ that household's assets are aggregate capital!

$$A_t = K_t$$

$$r_t K_t + W_t - C_t + (K_t - \delta K_t - K_{t+1}) = 0$$

Don't Forget:

$$K_{t+1} = K_t - \delta K_t + I_t$$

$$-I_t = K_t - \delta K_t - K_{t+1}$$

$$\underbrace{r_t K_t}_{\text{aggregete capital income}} + \underbrace{W_t}_{\text{aggregate labor income}} - C_t - I_t = 0$$

$= GDP$

$$GDP = C_t + I_t$$

$GDP_1 - C_t - I_t = 0$

$$GDP = C_t + I_t$$

So, not worring about taxes & gov. spending & import balancing, our little two period model is consistent w/ GDP & Wealth accounting!

Intertemporal Elasticity of Substitution

Like we said yesterday, C_t and C_{t+1} are different goods, for which we have different but related demand.

⟶ Naturally, as Economists we are interested in the elasticity of substitution between them.

Intertemporal (between time periods)
Elasticity (how does demand for one change when the price of the other changes?)
of Substitution (they're substitutes.)

The Elasticity of Substitution between 2 goods, a & b is:

$$\mathcal{E}_{a,b} = \frac{\partial \ln(a/b)}{\partial \ln(MU_a/MU_b)}$$

The *Intertemporal* Elasticity of Substitution is:

$$\mathcal{E}_{int} = \frac{\partial[\ln(C_t / C_{t+1})]}{\partial[\ln(MU_{C_t} / MU_{C_{t+1}})]}$$

So let's work on this.

$$u(C_t, C_{t+1}) = \ln(C_t) + \beta \ln(C_{t+1})$$

$$MU_{C_t} = \frac{\partial u(\ )}{\partial C_t} = \frac{1}{C_t}$$

$$MU_{C_{t+1}} = \frac{\partial u(\ )}{\partial C_{t+1}} = \frac{\beta}{C_{t+1}}$$

$$MU_{C_t} / MU_{C_{t+1}} = \frac{1/C_t}{\beta / C_{t+1}} = \frac{C_{t+1}}{\beta C_t}$$

Alright. That's something:

$$\varepsilon_{int} = \frac{\partial [\ln (C_t / C_{t+1})]}{\partial [\ln (C_{t+1} / \beta C_t)]}$$

Let's see what else we can do:

$$\ln (C_{t+1} / \beta C_t) = \ln \left(\frac{C_{t+1}}{C_t} / \beta \right) =$$

$$= \ln \left(\frac{C_{t+1}}{C_t} \right) - \ln \beta$$

$$= - \ln \left(\frac{C_t}{C_{t+1}} \right) - \ln \beta$$

Don't forget: $\ln(x^\alpha) = \alpha \ln(x)$

OK:

$$\varepsilon_{int} = \frac{\partial \left[\ln \left(\frac{C_t}{C_{t+1}} \right) \right]}{\partial \left[- \ln \left(\frac{C_t}{C_{t+1}} \right) - \ln \beta \right]}$$

$$\frac{1}{\varepsilon_{int}} = \frac{\partial \left[- \ln \left(\frac{C_t}{C_{t+1}} \right) - \ln \beta \right]}{\partial \left[\ln \left(\frac{C_t}{C_{t+1}} \right) \right]}$$

Let's call $\ln \left(\frac{C_t}{C_{t+1}} \right) = X$

$$\frac{1}{\varepsilon_{int}} = \frac{2\left[-x - \ln\beta\right]}{2x}$$

We know how to do that!

$$\frac{1}{\varepsilon_{int}} = -1$$

So $\varepsilon_{int} = \frac{1}{1/\varepsilon_{in}} = \frac{1}{-1} = -1$

* So the ~*Intertemporal Elasticity of Substitution*~ for our model is -1

↳ "A 1% <u>increase</u> in the price of consumption tomorrow leads to a 1% <u>decrease</u> in the quantity demanded of consumption today."

In Class Work:

What if

$$u(C_t, C_{t+1}) = \frac{C_t^{1-a}}{1-a} + \beta\frac{C_{t+1}^{1-a}}{1-a} ?$$

What is the Intertemporal Elasticity of Substitution in this case?

Day 4: Assumptions & Uncertainty

Let's talk about some of the key assumptions made in our two period model:

1. In period t, households care about consumption in t+1.
   - to you and me this's probably seems like a natural assumption
   - there's evidence that their are plenty of households for which this is not true.

Welcome to Economics' Elitism

2. They know & recognize how consumption in period t affects consumption in t+1.
   - not only do they care about C_{t+1} but they know how C_{t+1} relates to C_t

   ↳ and can therefore write down an *~ intertemporal budget constraint.*~

3. There are only two-periods

- While it may seem unreasonable to model a person who only lives 2 periods ...

  1) The relationship between C_{t+1} & C_t,

  $$\frac{C_{t+1}}{C_t} = \beta(1+\hat{r}_{t+1})$$

  doesn't change if we model more period.

  2) You can think of it as "early life" & "late life" decisions.

(*) 4. Households know the interest rate for period t+1 <u>in period t</u>.

- We can do something about this (& should)

Before we get our hands dirty we should take a minute to review:

<u>Discrete Probability</u>

In Discrete Probability:

- the random variable can take on a finite number (N) of values (it's a discrete random variable
- E.g. $x \in \{1, 2, 3\}$
- An example of x taking a continuou set of values: $x \in (0, 1)$.
- the probability that x = the ith (x_i) possible value is p_i.

  Eg. $P[x=2] = p_2$

When x is a discrete random variable

$$E[x] = \sum_{i=1}^{N} p_i \cdot x_i$$

Remember, $E[x]$, is the value of x we "expect" to see if we observed draws of x over and over and over...

OK, back to our two-period model.

Households make decisions about C_t & C_{t+1} in period t.

- Some things are known in period t.
- Some things aren't
  ↳ these are now random variables.

IN PERIOD t, what do we know & what's random?

Known:

A_t

$\hat{r}_t$

$\hat{w}_t$

A_{t+1} (b/c determined in t)
$(= A_t(1+\hat{r}_t) + \hat{w}_t - C_t)$

P_{it}'s

Random:

$$\underbrace{\begin{matrix}\hat{r}_{t+1} \\ \hat{w}_{t+1}\end{matrix}}_{\text{can take on N states of the world}}$$

Note: we're choosing C_t & C_{t+1} & A_{t+1}. C_{t+1} will depend on which of the N states of the world occur. $C_{t+1,i}$

(*) $C_{t+1,i}$ is the consumption picked in $t+1$ if the ith State of the world occurs

(*) We can't write an Intertemporal Budget Constraint b/c we don't know the next period vars.

So now the problem is

$$\max_{C_t, C_{t+1}, A_{t+1}} \ln(C_t) + \beta \underbrace{\sum_{i=1}^{N} p_{t,i} \ln(C_{t+1,i})}_{E_t[\ln(C_{t+1})]}$$

= the expected value of $\ln(C_{t+i})$ in period t.

s.t.

$$A_t(1+\hat{r}_t) + \hat{W}_t - C_t = A_{t+1}$$

one for each possible state of the world

$$A_{t+1}(1+\hat{r}_{t+1,1}) + \hat{W}_{t+1,1} - C_{t+1,1} = A_{t+2,1}$$

$$A_{t+1}(1+\hat{r}_{t+1,2}) + \hat{W}_{t+1,2} - C_{t+1,2} = A_{t+2,2}$$

$$\vdots$$

$$A_{t+1}(1+\hat{r}_{t+1,N}) + \hat{W}_{t+1,N} - C_{t+1,N} = A_{t+2,N}$$

→ each occuring w/ probability p_i

$$\mathcal{L} = \ln(C_t) + \beta \sum_{i=1}^{N} P_{t,i} \ln(C_{t+1,i})$$
$$+ \lambda \left[A_t(1+\hat{r}_t) + \hat{W}_t - C_t - A_{t+1}\right]$$
$$+ \sum_{i=1}^{N} \mu_i \left[A_{t+1}(1+\hat{r}_{t+1,i}) + \hat{W}_{t+1,i} - C_{t+1,i} - A_{t+2,i}\right]$$

$$\frac{\partial \mathcal{L}}{\partial C_t} : \frac{1}{C_t} - \lambda = 0 \longrightarrow \frac{1}{C_t} = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1,i}} : \beta \frac{P_{t,i}}{C_{t+1,i}} - \mu_i = 0 \longrightarrow \beta \frac{P_{t,i}}{C_{t+1,i}} = \mu_i \quad \text{for } i = 1, \ldots, N$$

Sure there are N of these
but we can just write
the one.

But we could sit here & write them

$$\frac{\partial \mathcal{L}}{\partial C_{t+1,1}} : \frac{\beta P_{t,1}}{C_{t+1,1}} - \mu_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1,2}} : \frac{\beta P_{t,2}}{C_{t+1,2}} - \mu_2 = 0$$

$$\vdots$$

But the pattern holds true.

$$\frac{\partial \mathcal{L}}{A_{t+1}} : -\lambda + \sum_{i=1}^{N} \mu_i (1+\hat{r}_{t+1,i})$$

$$\hookrightarrow \sum_{i=1}^{N} \underbrace{\mu_i}_{\frac{\beta P_{t,i}}{C_{t+1,i}}} (1+r_{t+1,i}) - \underbrace{\lambda}_{\frac{1}{C_t}}$$

$$\frac{1}{C_t} = \sum_{i=1}^{N} \left(\frac{\beta P_{t,i}}{C_{t+1,i}}\right)(1+\hat{r}_{t+1,i})$$

$$1 = C_t \sum_{i=1}^{N} \left(\frac{\beta P_{t,i}}{C_{t+1,i}}\right)(1+\hat{r}_{t+1,i})$$

$$1 = \sum_{i=1}^{N} P_{t,i} \frac{\beta C_t}{C_{t+1}} (1+\hat{r}_{t+1,i})$$

$$1 = E_t\left[\frac{\beta C_t}{C_{t+1}}(1+r_{t+1,i})\right]$$

In Class Work: Uncertainty Worksheet

$$\frac{\partial \mathcal{L}}{\partial A_{t+1}} \cdot -\lambda + \mu_g(1+\hat{r}_{t+1,g}) + \mu_b(1+\hat{r}_{t+1,b}) = 0$$

$$\lambda = \mu_g(1+\hat{r}_{t+1,g}) + \mu_b(1+\hat{r}_{t+1,b})$$

$$\frac{1}{C_t} = \frac{\beta P_{t,g}}{C_{t+1,g}}(1+\hat{r}_{t+1,g}) + \frac{\beta P_{t,b}}{C_{t+1,b}}(1+\hat{r}_{t+1,b})$$

$$1 = P_{t,g} \cdot \frac{\beta C_t}{C_{t+1,g}}(1+\hat{r}_{t+1,g}) + P_{t,b}\frac{\beta C_t}{C_{t+1,b}}(1+\hat{r}_{t+1,b})$$

$$= E_t\left[\frac{\beta C_t}{C_{t+1}}(1+\hat{r}_{t+1})\right]$$