$\boxed{ex\ 1}$

$$Y_t \mid \mathcal{F}_{t-1} \sim T(0, \phi_t, \nu)$$

$$\sigma_t^2 = \frac{\phi_t^2 \nu}{\nu - 2}$$

$$p(Y_t \mid \mathcal{F}_{t-1}) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}\,\phi_t}\left[1 + \frac{y_t^2}{\nu\phi_t^2}\right]^{-\frac{\nu+1}{2}}$$

The log density is proportional to

$$\log p(Y_t \mid \mathcal{F}_{t-1}) \propto -\log \phi_t - \frac{\nu+1}{2}\log\left(1 + \frac{y_t^2}{\nu\phi_t^2}\right)$$

the score of ϕ_t is

$$\frac{\partial \log p(Y_t \mid \mathcal{F}_{t-1})}{\partial \phi_t} = -\frac{1}{\phi_t} + \frac{(\nu+1)\cancel{2}}{\cancel{2}(1 + y_t^2/(\nu\phi_t^2))}\frac{y_t^2}{\nu\phi_t^3} =$$

$$= -\frac{1}{\phi_t} + \frac{(\nu+1)\, y_t^2/\phi_t^2}{\phi_t(\nu + y_t^2/\phi_t^2)}$$

where $z_t = \frac{y_t}{\phi_t}$

$$= \frac{1}{\phi_t}\left(\frac{(\nu+1) z_t^2}{\nu + z_t^2} - 1\right)$$

The GAS model for ϕ_t is

$$\phi_t = \exp\{\tilde{\phi}_t\}$$

$$\tilde{\phi}_t = \omega + \alpha \tilde{v}_{t-1} + \beta \tilde{\phi}_{t-1}$$

where

$$\tilde{v}_t = \frac{\partial \log p(y_t \mid \mathcal{F}_{t-1})}{\partial \tilde{\phi}_t} = \frac{\partial \phi_t}{\partial \tilde{\phi}_t} \frac{\partial \log p(y_t \mid \mathcal{F}_{t-1})}{\partial \phi_t} = \phi_t \frac{\partial \log p(y_t \mid \mathcal{F}_{t-1})}{\partial \phi_t}$$

$$= \frac{(\nu+1) z_t^2}{\nu + z_t^2} - 1$$

so

$$\tilde{\phi}_t = \omega + \alpha \left[\frac{(\nu+1) z_{t-1}^2}{\nu + z_{t-1}^2} - 1 \right] + \beta \tilde{\phi}_{t-1}$$