Exercise set #5 – Solutions

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(1): Theoretical part

Assume $\mathbf{y}_t = (y_{1,t}, \ldots, y_{N,t})'$ follows a Gaussian DCC model with AR(1)–GARCH(1,1) volatilities. Let $\boldsymbol{\omega}_t = (\omega_{1,t}, \ldots, \omega_{N,t})'$ with $\sum_{i=1}^N \omega_{i,t} = 1$ be a vector of portfolio weights such that $\omega_{i,t}$ is the proportion of wealth invested in asset i at time t. Let $\mathcal{F}_t$ be the filtration generated by the process up to time t.

a) What is the distribution of $\mathbf{y}_{t+1}|\mathcal{F}_t$?

- The distribution of $\mathbf{y}_{t+1}|\mathcal{F}_t$ is multivariate Gaussian with mean $\boldsymbol{\mu}_{t+1} = (\mu_{1,t+1}, \mu_{2,t+1})$ and covariance matrix $\boldsymbol{\Sigma}_{t+1} = \begin{pmatrix} \sigma^2_{1,t+1} & \sigma_{12,t+1} \\ \sigma_{21,t+1} & \sigma^2_{2,t+1} \end{pmatrix}$ where:

$$\mu_{j,t+1} = \mu_j + \phi_j y_{j,t} \tag{1}$$
$$\sigma^2_{j,t+1} = \omega_j + \alpha_j y_t^2 + \beta_j \sigma^2_{j,t+1}, \tag{2}$$

for $j = 1, 2$ and $\sigma_{12,t+1} = \sigma_{21,t+1} = \sigma_{1,t+1}\sigma_{2,t+1}\rho_{t+1}$ where $\rho_{t+1} = \frac{q_{12,t+1}}{\sqrt{q_{11,t+1}q_{22,t+1}}}$ and:

$$q_{ij,t+1} = \bar{q}_{ij}(1 - a - b) + a\varepsilon_{i,t}\varepsilon_{j,t} + bq_{ij,t}, \tag{3}$$

where $\varepsilon_{i,t} = \frac{y_{i,t} - \mu_{i,t}}{\sigma_{i,t}}$ for $i = 1, 2$.

b) What is the distribution of $\mathbf{y}_{t+h}|\mathcal{F}_t$ for $h > 1$? How would you compute $P(\mathbf{y}_{t+h} \leq \mathbf{y}|\mathcal{F}_t)$ for a generic $\mathbf{y} \in \Re^N$?

- The distribution of $\mathbf{y}_{t+h}|\mathcal{F}_t$ is not available in closed form. To compute $P(\mathbf{y}_{t+h} \leq \mathbf{y}|\mathcal{F}_t)$ we can generate draws from the model and approximate it by its empirical estimate.

c) Assume that $\boldsymbol{\omega}_{t+1}$ is measurable with respect to $\mathcal{F}_t$. What is the distribution of $\boldsymbol{\omega}'_{t+1}\mathbf{y}_{t+1}|\mathcal{F}_t$?

- The distribution of $\boldsymbol{\omega}'_{t+1}\mathbf{y}_{t+1}|\mathcal{F}_t$ is univariate Gaussian with mean $\mu^p_{t+1} = \boldsymbol{\omega}'_{t+1}\boldsymbol{\mu}_{t+1}$ and variance $\sigma^{p^2}_{t+1} = \boldsymbol{\omega}'_{t+1}\boldsymbol{\Sigma}_{t+1}\boldsymbol{\omega}_{t+1}$.

d) Assume that $\boldsymbol{\omega}_{t+h}$ for $h > 1$ is measurable with respect to $\mathcal{F}_t$. What is the distribution of $\boldsymbol{\omega}'_{t+h}\mathbf{y}_{t+h}|\mathcal{F}_t$? How would you compute $P(\boldsymbol{\omega}'_{t+h}\mathbf{y}_{t+h} \leq y|\mathcal{F}_t)$ for a generic $y \in \Re$?

- The distribution of $\boldsymbol{\omega}'_{t+h}\mathbf{y}_{t+h}|\mathcal{F}_t$ is not available in closed form. We can approximate $P(\boldsymbol{\omega}'_{t+h}\mathbf{y}_{t+h} \leq y|\mathcal{F}_t)$ by sampling random draws from the multivariate model (say $\mathbf{y}^b_{t+h}$ for $b = 1, \ldots, 500$) and by computing $y^b_{t+h} = \boldsymbol{\omega}'_{t+h}\mathbf{y}^b_{t+h}$ for $b = 1, \ldots, 500$. The quantity $P(\boldsymbol{\omega}'_{t+h}\mathbf{y}_{t+h} \leq y|\mathcal{F}_t)$ is then computed empirically on the simulated draws.

e) Derive the Value at Risk at level α for the portfolio $y^p_{t+1} = \boldsymbol{\omega}'_{t+1}\mathbf{y}_{t+1}$ distribution conditional on $\mathcal{F}_t$. (Use your favorite definition of VaR, i.e. defined on the returns or on the losses)

- Since $y^p_{t+1}|\mathcal{F}_t$ is Gaussian with mean μ^p_{t+1} and variance $\sigma^{p^2}_{t+1}$ the VaR of $y^p_{t+1}|\mathcal{F}_t$ at level α is simply given by:

$$VaR(\alpha)_{t+1} = \mu^p_{t+1} + \sigma^p_{t+1} q_z(\alpha),$$

where $q_z(\alpha)$ is the α-quantile of a standard Gaussian distribution.

f) Describe a procedure to compute the Value at Risk at level α for the h–step ahead distribution of the portfolio.

- Since the h–step ahead distribution is not available in closed form, we need to simulate from the distribution of $\boldsymbol{\omega}'_{t+h}\mathbf{y}_{t+h}|\mathcal{F}_t$ as in point d) and then compute the VaR as the empirical quantile of the simulated observations.

(2): Computational part

See `Solutions8.R`.

(3): Empirical part

See `Solutions8.R`.