Copula methods in Financial Econometrics

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Main references: Patton (2006a), Patton (2006b), Jondeau and Rockinger (2006), Chollete et al. (2009), Hafner and Manner (2012), Oh and Patton (2016), Patton (2012), Jondeau et al. (2007, Chapter 6.3)

Introduction

- Often we are not interested in modeling a single random variable;
- Joint probability of several random variables, e.g. returns on several assets;
- Multivariate distributions describe such a joint probability;
- Most of the empirical multivariate distributions deviate from the multivariate Gaussian distribution;
- The copula functions are a statistical tool for a simple treatment of the multivariate distribution of a random vector;

Distribution functions: Univariate

Let us first revise distribution functions.

Definition

A univariate distribution function F is a function from $\mathbb{R} = [-\infty, \infty]$ to [0, 1] such that:

1. F is weakly increasing
2. $F(-\infty) = 0$ and $F(\infty) = 1$

Distribution functions: Bivariate

Definition

A bivariate distribution function F is a function from $\mathbb{R} = [-\infty, \infty]^2$ to [0, 1] such that:

1. F is 2-increasing
2. $F(-\infty, y) = F(x, -\infty) = 0$, and $F(\infty, \infty) = 1$

Hence, F has marginal distributions denoted by

$$F_1(x) = F(x, \infty) \quad \text{and} \quad F_2(y) = F(\infty, y).$$

Multivariate Normal Distribution

We say that the k-dimensional random vector, $\mathbf{x}$ has a multivariate normal distribution with mean vector μ and covariance matrix $\mathbf{\Sigma}$, if its probability density function is

$$f(\mathbf{x}) = \frac{1}{2\pi^{k/2}|\mathbf{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)'\mathbf{\Sigma}^{-1}(x-\mu)\right\}$$

where $|\mathbf{\Sigma}|$ is the determinant of $\mathbf{\Sigma}$. Hence $\mathbf{x} \sim N_k(\mu, \mathbf{\Sigma})$.

Multivariate t-Student Distribution

We say that the k-dimensional random vector, $\mathbf{x}$ has a multivariate Student's t-distribution if

$$\mathbf{x} = \mu + \sqrt{\frac{\nu}{W}}\mathbf{Z}$$

where W is chi-squared distributed random variable with ν degrees of freedom and $\mathbf{Z}$ is $N_k(0, \mathbf{\Lambda})$. Hence $\mathbf{x} \sim t_k(\mu, \mathbf{\Lambda}, \nu)$.

Note that the covariance matrix of $\mathbf{x}$ is

$$\mathbf{\Sigma} = \frac{\nu}{\nu - 2}\mathbf{\Lambda}$$

The parameters of a multivariate t distribution can be estimated by ML.

Elliptical Densities

Both MND and MSTD belong to the elliptical family. A multivariate density is elliptical if it can be expressed as

$$f(x) = kg\left\{(x-\mu)'\Lambda^{-1}(x-\mu)\right\}$$

where g is a non-negative function and k is a normalising constant such that $\int_{\mathcal{R}^k} g(x)dx = 1/k$. As a consequence, the contour of f are concentric ellipses, such that for any $c > 0$,

$$\mathcal{E}(c) = \left\{x : (x-\mu)'\Lambda^{-1}(x-\mu)' = c\right\}$$

is an ellipse centered at μ.

ML estimation

- In most financial problems, a set of unknown parameters θ, must be estimated. θ typically contains:
    - Parameters governing the conditional mean (ARMA terms, Seasonal components, Jumps)
    - Parameters governing the conditional variance, (GARCH effects)
- The parameters governing the shape of the error distribution (Tails, Asymmetry) are contained in the vector, η.
- The full set of unknown parameters is $\zeta = (\theta', \eta')'$.
- The ML estimator, $\hat{\zeta}_{ML}$, is obtained maximizing

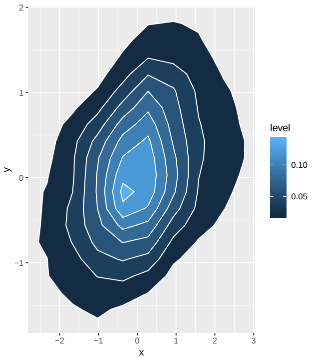
$$\mathcal{L}(\zeta) = \sum_{t=1}^{T} \ell_t(\zeta)$$

- The optimization wrt θ and η must generally be done in one step.
- Exception is the elliptical family (two-steps estimator). QML for θ under Gaussianity, and then η estimated given $\hat{\theta}$.

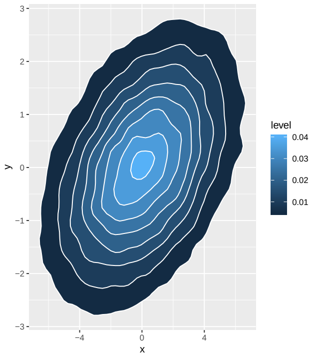
Is this the end of the story?

- Multivariate normal and t distributions (and their skewed extensions) are the only multivariate options available?
- What if the marginals are not Gaussian?
- A multivariate extension of many marginals does not exist.
- What about the dependence between the variables?
- Need for more flexibility!

Contour plot IBM and BAC



(a) Empirical distribution



(b) Estimated Gaussian distribution

Figure: Comparison between the empirical and estimated bivariate distribution between IBM and BAC. Percentage logarithmic returns are computed over the period 2005 – 2018.

Contour plot IBM and BAC

Looking at the scatter plot between the returns of IBM and BAC:

- The joint density does not look Gaussian
- The scatter shows extreme positive and negative dependence (right and left tails)
- Asymmetric dependence (right tail different from the left tail)

How do we fit this joint density with a flexible distribution?

- Fitting a MV Normal with TV Covariance matrix, e.g. multivariate GARCH?
  - Good to capture time-varying correlation and unconditional non-gaussianity
  - What about conditional non-gaussianity?
  - What about extreme dependence?

  We need something else...

Copula models: overview

The ability of econometric models to account for the negative consequences for the overall financial system of extreme events strongly relies on their flexibility to feature the highly nonlinear and asymmetric dependence structures of financial returns.

Over the years, the correlation coefficient has emerged as the most natural measure of dependence. However, despite its widespread use, the correlation fails to capture the important tails behaviour of the joint probability distribution, see, e.g., Embrechts et al. (1999, 2002). Hence, modelling the tail dependence and the asymmetric dependence between pairs of assets have been becoming increasingly more important in nowadays financial markets.

Furthermore, the linear correlation coefficient as measure of dependence is usually associated with the assumption of elliptically contoured distributions.

Copula models: why?

The copula methodology overcomes this limitation allowing also for some flexibility in modelling non–linear dependencies, see, e.g., Joe (2015).

Another advantage of using copulas is the decoupling of marginal distributions from the dependence structure. In many cases, the marginal distributions do not depend on the dependence parameter, such that one can first estimate the parameters of the marginal distributions and then in a second step the copula parameters.

The time–varying feature of the dependence behaviour of stock returns, recognised for example by Engle (2002) and Tse and Tsui (2002a), however, motivates the consideration of dynamic copula models where the dependence parameters evolves smoothly as a function of past assets co–movements.

Copula models: breaks in the dependence structure

The dynamic conditional copula theory has been first developed by Patton (2006a), although the problem of modelling the joint co–movement of stock returns was already present in Bollerslev et al. (1988) and Engle et al. (1990), among others.

Furthermore, occasionally, we observe breaks into the dependence structure, which are more evident during crises periods and other infrequent events, as documented, for example, by Pelletier (2006a).

As regards dependence breaks, Markov switching (MS) models have been proven to effectively capture non–smooth evolutions of the volatility and correlations dynamics. Jondeau and Rockinger (2006), Rodriguez (2007), and Chollete et al. (2009) for example, have firstly adopted MS copula with static regime–dependent parameters to analyse financial contagion.

Copula definition

A d-dimensional copula is a distribution function on $[0,1]^n$ with standard uniform marginal distributions. Hence C is a mapping of the form $C : [0,1]^d \to [0,1]$, i.e. a mapping of the unit hypercube into the unit interval.

The following three properties must hold:

- $C(u_1, \ldots, u_n)$ is increasing in each component u_i.
- $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $i \in \{1, \ldots, n\}, u_i \in [0,1]$.
- For all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0,1]^n$ wit $a_i \leq b_i$ we have:

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{\sum_{l=1}^{n} i_l} C(u_{1,i_1}, \ldots, u_{n,i_n}) \geq 0,$$

where $u_{l1} = a_j$ and $u_{l2} = b_j$ for all $l \in \{1, \ldots, n\}$.

Sklar (1959)'s theorem

Every multivariate cumulative distribution function:

$$H(y_1, \ldots, y_n) = P(Y_1 \leq y_1, Y_n \leq y_n),$$

of a random vector $(Y_1, \ldots, Y_n)$, can be expressed in terms of its marginals $F_i(y_i) = P(Y_i \leq y_i)$ and a copula C:

$$H(y_1, \ldots, y_n) = C(F_1(y_1), \ldots, F_n(y_n)).$$

Provided that a density exists, it is given by:

$$h(y_1, \ldots, y_n) = c(F_1(y_1), \ldots, F_n(y_n)) \prod_{i=1}^{n} f_i(y_i),$$

where $c(\cdot)$ is the density of the copula.

Hence we note that $C : [0, 1]^n \rightarrow [0, 1]$, that is: $C(F_1(y_1), \ldots, F_n(y_n))$ defines a n–dimensional cumulative distribution function.

One Definition and one Proposition

Definition 5.4 of McNeil et al. (2015)

If the random variable $\mathbf{X}$ has joint distribution function F with continuous marginal distributions $F_1, \ldots, F_n$, then the copula of F is the distribution function C of $(F_1(X_1), \ldots, F_n(X_n))$.

Proposition 5.6 of McNeil et al. (2015)

Let $(X_1, \ldots, X_n)$ be a random vector with continuous margins and copula C and let $T_1, \ldots, T_n$ be strictly increasing functions. Then $(T_1(X_1), \ldots, T_n(X_n))$ also has copula C.

For a proof see (McNeil et al., 2015, p. 188)

This proposition is useful when we build *implicit* copulas.

Types of Copula: independence

The independence copula is given by:

$$C(u_1, \ldots, u_n) = \prod_{i=1}^{n} u_i,$$

where $u_i = F_i(y_i)$.

It is clear from Sklar's Theorem, that random variables with continuous distributions are independent if and only if their dependence structure is given by the independence copula.

Types of Copula: implicit copulas

If $\mathbf{Y} \sim \mathcal{N}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a Gaussian random vector, then its copula is a so-called Gaussian copula. In order to build the Gaussian copula consider the transformations:

$$T_i(Y_i) = \frac{Y_i - \mu_i}{\sigma_i}, \quad \text{for} \quad i = 1, \dots, n.$$

This is a series of strictly increasing transformations and thus proposition 5.6 of McNeil et al. (2015) can be applied. It follows that $\mathbf{X} \sim \mathcal{N}_n(\mathbf{0}, \mathbf{R})$, where $\mathbf{X} = (X_1, \dots, X_n)'$, with $X_i = T_i(Y_i)$ and $\mathbf{R} = cor(\mathbf{Y})$, has the same copula of $\mathbf{Y}$.

By definition 5.4 of McNeil et al. (2015), the copula of $\mathbf{X}$ is given by:

$$\begin{aligned} C^{Ga}(\mathbf{u}; \mathbf{R}) &= P(\Phi(X_1) \leq u_1, \dots, \Phi(X_n) \leq u_n) \\ &= \boldsymbol{\Phi}(\Phi(u_1)^{-1}, \dots, \Phi(u_n)^{-1}; \mathbf{R}), \end{aligned}$$

where $\Phi(\cdot)$ is the distribution function of a standardized univariate Gaussian distribution and $\boldsymbol{\Phi}(\cdot)$ denotes the joint distribution function of $\mathbf{X}$.

Types of Copula: implicit copulas

We can build implicit copulas starting from any joint distribution function with continuous marginals.

For example, the n-dimensional t copula takes the form:

$$C^t(\mathbf{u}; \nu, \mathbf{R}) = \mathbf{t}(t(u_1; \nu)^{-1}, \ldots, t(u_n; \nu)^{-1}; \nu, \mathbf{R}),$$

where $t(\cdot)$ is the distribution function of a standard univariate t distribution,$\mathbf{t}(\cdot; \nu, \mathbf{R})$ is the joint distribution function of the vector $\mathbf{X} \sim \mathcal{T}_n(\nu, \mathbf{0}, \mathbf{R})$ and $\mathbf{R}$ is a correlation matrix.

- Implicit copulas do not usually have a closed form formulation.
- Implicit copulas constructed starting from elliptical distributions (such as the Gaussian and Student's t) are called *elliptical*.

Types of Copula: explicit copulas

While the Gaussian and t copulas are copulas implied by well–known multivariate distribution functions and do not themselves have simple closed forms, we can write down a number of copulas which do have simple closed forms.

Gumbel copula

$$C^{Gu}(\mathbf{u};\theta) = \exp\left[-\left(\sum_{i=1}^{n}(-\ln u_i)^\theta\right)^{1/\theta}\right],$$

for $1 \leq \theta < \infty$.

Clayton copula

$$C^{Cl}(\mathbf{u};\theta) = \left(\sum_{i=1}^{n} u_i^{-\theta} - 1\right)^{-1/\theta},$$

for $0 < \theta < \infty$.

The likelihood

Assume to observe T realizations of $\mathbf{Y}_t = (Y_{1,t}, \dots, Y_{n,t})'$. Assume $\mathbf{Y}_t$ is *iid* distributed with density:

$$h(\mathbf{y}_t, \boldsymbol{\theta}) = c(F_1(y_1), \dots, F_n(y_n), \boldsymbol{\theta}_c) \prod_{i=1}^{n} f_i(y_i, \boldsymbol{\theta}_i),$$

where $\boldsymbol{\theta} = (\boldsymbol{\theta}_c', \boldsymbol{\theta}_1', \dots, \boldsymbol{\theta}_n')'$ and $\boldsymbol{\theta}_c$ are copula specific parameters and $\boldsymbol{\theta}_i$ are the parameters of the i-th marginal distribution.

The log likelihood of observing the sequence $\mathbf{Y}_1, \dots, \mathbf{Y}_T$ is:

$$\begin{aligned} \mathcal{L}_{\mathbf{Y}}(\boldsymbol{\theta}) &= \sum_{t=1}^{T} \log h(\mathbf{y}_t, \boldsymbol{\theta}) \\ &= \sum_{t=1}^{T} \sum_{i=1}^{n} \log f_i(y_{i,t}, \boldsymbol{\theta}_i) + \sum_{t=1}^{T} \log c(F_1(y_1, \boldsymbol{\theta}_1), \dots, F_n(y_n, \boldsymbol{\theta}_n), \boldsymbol{\theta}_c) \end{aligned}$$

Multi stage Maximum Likelihood estimator

One of the main appealing characteristics of the copula framework, with respect to standard distributions, relies on its ability to model the marginals' dynamics separately from the joint dependence structure, see, e.g., Nelsen (2006).

Marginals and dependence separability has some additional advantages even from the econometric point of view, since it permits to employ a two–step procedure to estimate the parameters. This two–step procedure is known as Inference Function for margins (IFM) and is usually referred to Godambe (1960) and McLeish and Small (1988).

Patton (2006a) has recently extended the theory for this two step estimation procedure to the case of conditional copulas: $C(F_1(y_1), \dots, F_n(y_n)|\mathcal{F}_{t-1})$, for some filtration $\mathcal{F}_t$.

Multi stage Maximum Likelihood estimator (MSML)

- Estimate $\boldsymbol{\theta}_i$ by:

$$\widehat{\boldsymbol{\theta}}_i = \underset{\boldsymbol{\theta}_i \in \boldsymbol{\Theta}_i \subseteq \Re^{d_i}}{\arg\max} \sum_{t=1}^{T} \log(f_i(y_{i,t}, \boldsymbol{\theta}_i))$$

for $i = 1, \dots, n$.

- Estimate $\boldsymbol{\theta}_c$ by:

$$\widehat{\boldsymbol{\theta}}_c = \underset{\boldsymbol{\theta}_c \in \boldsymbol{\Theta}_c \subseteq \Re^{d_c}}{\arg\max} \sum_{t=1}^{T} \log(c(F_1(y_{1,t}, \widehat{\boldsymbol{\theta}}_1), \dots, F_n(y_{n,t}, \widehat{\boldsymbol{\theta}}_n), \boldsymbol{\theta}_c))$$

Clearly the MSML estimator is asymptotically less efficient than one–stage full MLE (unless $Y_i \perp\!\!\!\perp Y_j$ for all $i \neq j$).

Alternative estimation strategies

- Semiparametric estimation: Employ a nonparametric model for the marginal distributions and a parametric model for the copula. See Genest et al. (1995), Shih and Louis (1995), Chen and Fan (2006a), and Chen and Fan (2006b).
- Nonparametric: Fully nonparametric estimation of the copula. See Genest and Rivest (1993), Genest et al. (2011), Genest et al. (2009), Fermanian et al. (2004), Scaillet and Fermanian (2002), Ibragimov (2009), and Sancetta and Satchell (2004).
- Other estimation procedures are available such as: Method of Moments, Generalized Method of Moments, Simulation methods etc., see Smith (2011) for a review.

Visualizing the dependence between two Random Variables

A nice tool to visualize the dependence between two random variables Y_1 and Y_2 is provided by the so called "quantile dependence" defined as:

$$\begin{aligned} \lambda_L(q) &= P(U_1 \leq q, U_2 \leq q)/q, && \text{for} \quad q \in (0, 0.5] \\ \lambda_U(q) &= P(U_1 > q, U_2 > q)/(1-q), && \text{for} \quad q \in [0.5, 1) \end{aligned}$$

Note that $\lambda_L(0.5) = \lambda_U(0.5)$ by construction since $U_i = F_i(Y_i) \sim \mathcal{U}(0,1)$ for $i = 1, 2$.

The quantities $\lambda_L(q)$ and $\lambda_U(q)$ can be computed for different values of q and represented graphically.

SP100 and SP600

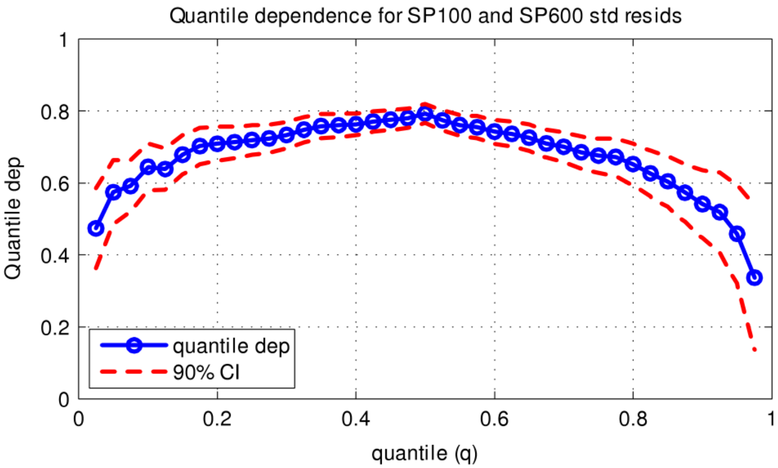


Figure: From Patton (2012): estimated quantile dependence between the standardized residuals for the S&P 100 index and the S&P 600 index along with 90% bootstrap confidence intervals.

SP100 and SP600: estimated copulas

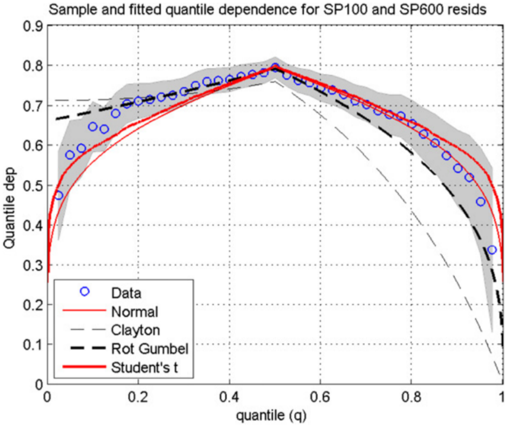


Figure: From Patton (2012): sample quantile dependence between the standardized residuals for the S&P 100 index and the S&P 600 index and 90% bootstrap confidence intervals (shaded), as well as the quantile dependence implied by four copula models.

Others measure of association

Other measures of associations between two random variables X and Y are:

- The Spearman's rank correlation coefficient defined as:

$$r_s = 1 - \frac{6\sum_{i=1}^n d_i^2}{n(n^2-1)},$$

  where $d_i = \mathrm{rank}(x_i) - \mathrm{rank}(y_i)$

- The Kendall's τ coefficient defined as:

$$\tau = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \mathrm{sgn}(x_i - x_j)\mathrm{sgn}(y_i - y_j)$$

SP100 and SP600: rolling rank correlation

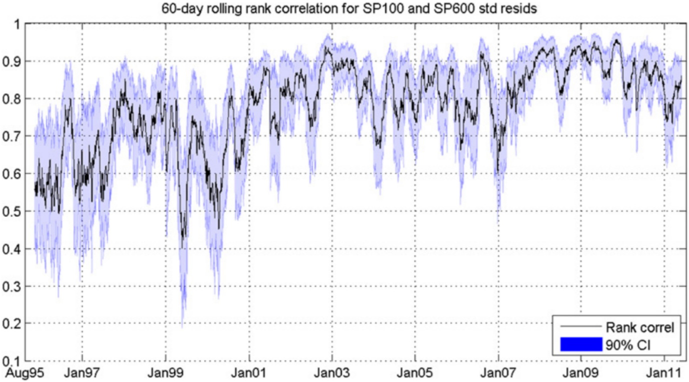


Figure: From Patton (2012): rank correlation between the standardized residuals for the S&P 100 index and the S&P 600 index over a 60-day moving window, along with 90% bootstrap confidence intervals.

The Patton (2006a) model

Patton (2006a,b) initially proposed a time–varying copula model for two random variables $Y_{1,t}$ and $Y_{2,t}$.

As we saw before, the marginals specification can be freely specified such as ARMA–GARCH.

After estimation of the marginals model a careful goodness of fit analysis has to be performed before moving to the analysis of the dependence structure. Indeed, a bad fit of the marginal distributions can heavily affect the estimation of the copula specification. See Patton (2012) for several goodness of fit procedures.

The Patton (2006a,b) model

Let $\rho_t \in (-1, 1)$ be the correlation coefficient of a bivariate elliptical random variable. ρ_t is assumed to follow:

$$
\begin{aligned}
\text{Normal}: \quad & \rho_t = \Lambda\left(\omega + \beta\rho_{t-1} + \alpha\frac{1}{M}\sum_{j=1}^{M}\Phi^{-1}(u_{1,t-j})\Phi^{-1}(u_{2,t-j})\right) \\
\text{Student's} \quad t: \quad & \rho_t = \Lambda\left(\omega + \beta\rho_{t-1} + \alpha\frac{1}{M}\sum_{j=1}^{M}T^{-1}(u_{1,t-j})T^{-1}(u_{2,t-j})\right),
\end{aligned}
$$

where $\Lambda(x) = (1 - e^{-x})/(1 + e^{-x})$ is the modified logistic function and Φ^{-1} and T^{-1} are the inverse cdf of a Gaussian and Student's t standardized random variables, respectively. M is a tuning parameter that can be set equal to 10 for daily observations.

The Patton (2006a,b) model

Let $\theta_t \in [a, b]$ be the dependence parameter of a bivariate archimedian copula. θ_t is assumed to follow:

$$\text{Archimedian}: \qquad \theta_t = \tilde{\Lambda}_{[a,b]}\left(\omega + \beta\theta_{t-1} + \alpha\frac{1}{M}\sum_{j=1}^{M} u_{1,t-j}u_{2,t-j}\right)$$

where $\tilde{\Lambda}_{[a,b]} = a + (b-a)/(1+e^{-x})$ is the modified logistic function in $[a, b]$. M is a tuning parameter that can be set equal to 10 for daily observations.

The forcing variable is of the form:

- $D^{-1}(u_{1,t-s})D^{-1}(u_{2,t-s})$ for elliptical copulas.
- $u_{1,t-s}u_{2,t-s}$.

The choice of the forcing variable

Does this choice of the forcing variable make sense? It is intuitively appealing and builds on our understanding of covariances: if the transformed marginals have the same sign, the correlation should increase. The reverse holds if the transformed marginals are of opposite sign.

Consider the Gaussian case in the situation when $\Phi^{-1}(u_{1,t}) = 1$ and $\Phi^{-1}(u_{2,t}) = 1$. In this case (assume $M = 1$ for simplicity) the forcing variable is $\Phi^{-1}(u_{1,t})\Phi^{-1}(u_{2,t}) = 1$. Alternatively, consider the case $\Phi^{-1}(u_{1,t}) = 0.25$ and $\Phi^{-1}(u_{2,t-j}) = 4$, also in this case $\Phi^{-1}(u_{1,t})\Phi^{-1}(u_{2,t}) = 1$. So the update of ρ will be the same regardless of which of the two scenarios we observe.

A score driven approach

What happens if we assume a GAS model for the ρ_t? Creal et al. (2013) derive the score of a bivariate Gaussian copula as:

$$\nabla_t = \Phi^{-1}(u_{1,t})\Phi^{-1}(u_{2,t}) - \rho_t - \rho_t \frac{(\Phi^{-1}(u_{1,t})^2 + \Phi^{-1}(u_{2,t})^2 - 2)}{(1 + \rho_t^2)}.$$

The most important part is given by the numerator of the last term $(\Phi^{-1}(u_{1,t})^2 + \Phi^{-1}(u_{2,t})^2 - 2)$.

Notice that this term is a Martingale Difference sequence and allows the filter to distinguish between the two scenarios of before. Furthermore, the filter will also account for the current amount of correlation ρ_t. If $\rho_t = 0$, the update is the same as of that of Patton (2006a).

GAS vs Patton (2006a,b)'s model

	$10^3\omega$	A_1	$\ln(B_1/1 - B_1)$	B_1	Log-likelihood
German mark (euro)–US \$, Japanese yen–US \$					
GAS	6.11	0.058	5.30	0.995	1218.16
	(2.48)	(0.009)	(0.37)	(0.990, 0.998)	
Patton	–1.60	0.036	4.27	0.986	1191.51
	(0.85)	(0.003)	(0.10)	(0.983, 0.989)	
German mark (euro)–US \$, British pound–US \$					
GAS	12.55	0.082	4.97	0.993	2218.82
	(3.55)	(0.008)	(0.23)	(0.988, 0.996)	
Patton	–0.97	0.025	4.71	0.991	2090.42
	(0.84)	(0.002)	(0.11)	(0.989, 0.993)	

Figure: From Creal et al. (2013): Parameter estimates for the GAS and Patton models. The data are the marginal AR-GARCH transforms of log exchange rates for the German mark-US dollar and Japanese yen-US dollar (left panel) and for the German mark-US dollar and British pound-US dollar (right panel), January 1986-August 2008. The asymptotic confidence interval is in parentheses for B_1; otherwise the standard error is in parentheses.

The copula–GARCH model of Jondeau and Rockinger (2006)

Jondeau and Rockinger (2006) propose three dynamic copula models which they refer to as "copula–GARCH" models. These are:

1) The "semiparametric" approach.
2) The "Time Varying Copula" approach.
3) The "Markov–Switching" approach.

All models are only defined in the bivariate case of a Student's t copula with time–varying correlation parameter ρ_t.

Jondeau and Rockinger (2006): The "semiparametric" approach

Somehow inspired by Gourieroux and Monfort (1992), The "semiparametric" copula–GARCH model of Jondeau and Rockinger (2006) assumes that:

$$\rho_t = \sum_{j=1}^{16} d_i \mathbb{1}\left\{ \left(\Phi^{-1}(u_{1,t}), \Phi^{-1}(u_{2,t})\right) \in \mathcal{A}_j \right\},$$

where $\mathcal{A}_j$ is the j–th element of the unit square grid and $d_j \in [-1, 1]$. To each parameter d_j, an area $\mathcal{A}_j$ is associated. This model. .

Some features of this model are:

- It has 16 parameters.
- The choice of $\mathcal{A}_j$ for $j = 1, \ldots, 16$ is arbitrary.
- Is not able to capture the persistence in ρ_t.

The "semiparametric" approach

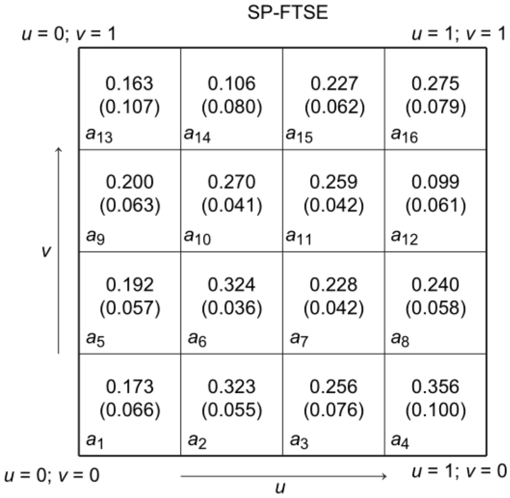


Figure: From Jondeau and Rockinger (2006): The unit square with estimates of parameters d_js and their standard errors, for the SP–FTSE returns.

Jondeau and Rockinger (2006): The "Time Varying Copula" approach

Somehow inspired by the DCC specification of Tse and Tsui (2002b), the "Time Varying Copula" (TVC) model of Jondeau and Rockinger (2006) assumes:

$$\rho_t = (1 - \alpha - \beta)\rho + \alpha\xi_{t-1} + \beta\rho_{t-1},$$

where:

$$\xi_t = \frac{\sum_{i=0}^{m-1} z_{1,t} z_{2,t}}{\sqrt{\sum_{i=0}^{m-1} z_{1,t}^2 \sum_{i=0}^{m-1} z_{2,t}^2}},$$

where $z_{i,t}$ is the residual of series $i = 1, 2$ at time t. Marginal models in Jondeau and Rockinger (2006) belong to the ARMA–GARCH family.

The "Markov–Switching" approach.

Somehow inspired by Ramchand and Susmel (1998), Chesnay and Jondeau (2001), Ang and Bekaert (2002), and Pelletier (2006b). The Markov–Switching copula model of Jondeau and Rockinger (2006) assumes that:

$$\rho_t = \rho_0 S_t + \rho_1 (1 - S_t),$$

where $S_t \in (0, 1)$ denotes the unobserved regime of the system at time t. S_t is assumed to follow a two- state Markov process, with transition probability matrix given by:

$$\begin{pmatrix} p & 1 - p \\ 1 - q & q \end{pmatrix},$$

with:

$$\begin{aligned} p &= P(S_t = 0 | S_{t-1} = 0) \\ q &= P(S_t = 1 | S_{t-1} = 1). \end{aligned}$$

Estimation by direct maximization of the likelihood can be easily implemented.

Comparison between TVC and MS

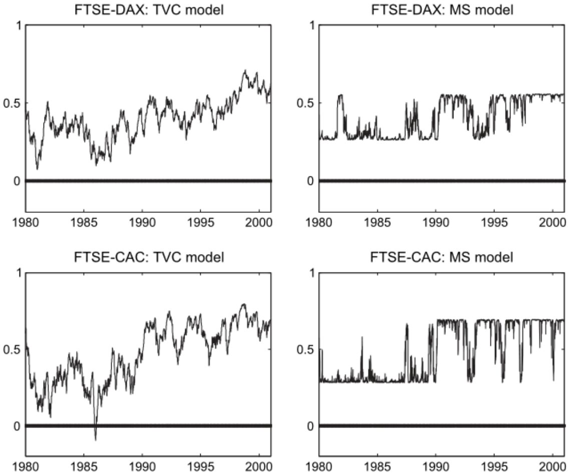


Figure: From Jondeau and Rockinger (2006): The evolution of the parameter r t obtained with the TVC model and with the Markov–switching model, for the FTSE–DAX and the FTSE–CAC pairs, respectively.

Comparison between TVC and MS

	SP–FTSE	SP–DAX	SP–CAC	FTSE–DAX	FTSE–CAC	DAX–CAC
Constant Copula						
ℓL	139.052	227.500	191.420	433.927	667.453	592.095
AIC	−0.060	−0.099	−0.083	−0.189	−0.291	−0.258
SIC	−0.057	−0.096	−0.080	−0.186	−0.288	−0.255
TVC						
ℓL	140.101	239.233	201.314	499.231	826.917	724.092
AIC	−0.060	−0.103	−0.086	−0.217	−0.360	−0.315
SIC	−0.054	−0.097	−0.081	−0.211	−0.354	−0.309
MS						
ℓL	140.291	249.223	204.163	488.740	829.692	699.747
AIC	−0.060	−0.107	−0.087	−0.212	−0.361	−0.304
SIC	−0.054	−0.102	−0.080	−0.206	−0.356	−0.297

Figure: From Jondeau and Rockinger (2006): Log-likelihood, AIC and SIC information criteria for the Constant, TVC Markov–switching copula models.

References I

Ang, A. and Bekaert, G. (2002). International asset allocation with regime shifts. *The Review of Financial Studies*, 15(4):1137–1187.

Bollerslev, T., Engle, R. F., and Wooldridge, J. M. (1988). A capital asset pricing model with time-varying covariances. *Journal of Political Economy*, 96(1):116–131.

Chen, X. and Fan, Y. (2006a). Estimation and model selection of semiparametric copula-based multivariate dynamic models under copula misspecification. *Journal of econometrics*, 135(1-2):125–154.

Chen, X. and Fan, Y. (2006b). Estimation of copula-based semiparametric time series models. *Journal of Econometrics*, 130(2):307–335.

Chesnay, F. and Jondeau, E. (2001). Does correlation between stock returns really increase during turbulent periods? *Economic Notes*, 30(1):53–80.

Chollete, L., Heinen, A., and Valdesogo, A. (2009). Modeling International Financial Returns with a Multivariate Regime-Switching Copula. *Journal of Financial Econometrics*, 7(4):437–480.

References II

Creal, D., Koopman, S. J., and Lucas, A. (2013). Generalized autoregressive score models with applications. *Journal of Applied Econometrics*, 28(5):777–795.

Embrechts, P., McNeil, A., and Straumann, D. (1999). Correlation: pitfalls and alternatives. *RISK-LONDON-RISK MAGAZINE LIMITED*, 12:69–71.

Embrechts, P., McNeil, A. J., and Straumann, D. (2002). Correlation and dependence in risk management: properties and pitfalls. In *Risk management: Value-at-Risk and beyond (Cambridge, 1998)*, pages 176–223. Cambridge Univ. Press, Cambridge.

Engle, R. (2002). Dynamic Conditional Correlation: a simple class of multivariate Generalized Autoregressive Conditional Heteroskedasticity models. *Journal of Business & Economic Statistics*, 20(3):339–350.

Engle, R. F., Ng, V. K., and Rothschild, M. (1990). Asset pricing with a factor-arch covariance structure: Empirical estimates for treasury bills. *Journal of Econometrics*, 45(1):213 – 237.

Fermanian, J.-D., Radulovic, D., Wegkamp, M., et al. (2004). Weak convergence of empirical copula processes. *Bernoulli*, 10(5):847–860.

References III

Genest, C., Ghoudi, K., and Rivest, L.-P. (1995). A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82(3):543–552.

Genest, C., Nešlehová, J., and Ziegel, J. (2011). Inference in multivariate archimedean copula models. *Test*, 20(2):223.

Genest, C. and Rivest, L.-P. (1993). Statistical inference procedures for bivariate archimedean copulas. *Journal of the American statistical Association*, 88(423):1034–1043.

Genest, C., Segers, J., et al. (2009). Rank-based inference for bivariate extreme-value copulas. *The Annals of Statistics*, 37(5B):2990–3022.

Godambe, V. P. (1960). An optimum property of regular maximum likelihood estimation. *Ann. Math. Statist.*, 31(4):1208–1211.

Gourieroux, C. and Monfort, A. (1992). Qualitative threshold arch models. *Journal of econometrics*, 52(1-2):159–199.

References IV

Hafner, C. M. and Manner, H. (2012). Dynamic stochastic copula models: estimation, inference and applications. *Journal of Applied Econometrics*, 27(2):269–295.

Ibragimov, R. (2009). Copula-based characterizations for higher order markov processes. *Econometric Theory*, 25(3):819–846.

Joe, H. (2015). *Dependence modeling with copulas*, volume 134 of *Monographs on Statistics and Applied Probability*. CRC Press, Boca Raton, FL.

Jondeau, E., Poon, S.-H., and Rockinger, M. (2007). *Financial modeling under non-Gaussian distributions*. Springer Science & Business Media.

Jondeau, E. and Rockinger, M. (2006). The Copula-GARCH model of conditional dependencies: An international stock market application. *Journal of International Money and Finance*, 25(5):827 – 853.

McLeish, D. L. and Small, C. G. (1988). *The theory and applications of statistical inference functions*, volume 44 of *Lecture Notes in Statistics*. Springer-Verlag, New York.

References V

McNeil, A. J., Frey, R., and Embrechts, P. (2015). *Quantitative Risk Management: Concepts, Techniques and Tools: Concepts, Techniques and Tools*. Princeton university press.

Nelsen, R. B. (2006). *An introduction to copulas*. Springer Series in Statistics. Springer, New York, second edition.

Oh, D. H. and Patton, A. J. (2016). Time-varying systemic risk: Evidence from adynamic copula model of cds spreads. *Journal of Business & Economic Statistics*, 0(ja):1–47.

Patton, A. J. (2006a). Estimation of multivariate models for time series of possibly different lengths. *Journal of Applied Econometrics*, 21(2):147–173.

Patton, A. J. (2006b). Modelling asymmetric exchange rate dependence. *Internat. Econom. Rev.*, 47(2):527–556.

Patton, A. J. (2012). A review of copula models for economic time series. *Journal of Multivariate Analysis*, 110:4 – 18. Special Issue on Copula Modeling and Dependence.

References VI

Pelletier, D. (2006a). Regime switching for dynamic correlations. *Journal of Econometrics*, 131(2):445 – 473.

Pelletier, D. (2006b). Regime switching for dynamic correlations. *Journal of econometrics*, 131(1-2):445–473.

Ramchand, L. and Susmel, R. (1998). Volatility and cross correlation across major stock markets. *Journal of Empirical Finance*, 5(4):397–416.

Rodriguez, J. C. (2007). Measuring financial contagion: A copula approach. *Journal of Empirical Finance*, 14(3):401 – 423.

Sancetta, A. and Satchell, S. (2004). The bernstein copula and its applications to modeling and approximations of multivariate distributions. *Econometric theory*, 20(3):535–562.

Scaillet, O. and Fermanian, J.-D. (2002). Nonparametric estimation of copulas for time series.

Shih, J. H. and Louis, T. A. (1995). Inferences on the association parameter in copula models for bivariate survival data. *Biometrics*, pages 1384–1399.

References VII

Sklar, M. (1959). Fonctions de repartition an dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, 8:229–231.

Smith, M. S. (2011). Bayesian approaches to copula modelling. *arXiv preprint arXiv:1112.4204*.

Tse, Y. K. and Tsui, A. K. C. (2002a). A multivariate Generalized Autoregressive Conditional Heteroscedasticity model with time-varying correlations. *Journal of Business & Economic Statistics*, 20(3):351–362.

Tse, Y. K. and Tsui, A. K. C. (2002b). A multivariate generalized autoregressive conditional heteroscedasticity model with time-varying correlations. *Journal of Business & Economic Statistics*, 20(3):351–362.