# Exam 2019

**Leopoldo rewritten**

## 1) Derive the GAS updating

We have

$$p\left(y_t \mid y_{1:t-1}; \varphi_t, \upsilon\right) = \left[2^{\left(1+\frac{1}{\upsilon}\right)}\varphi_t \Gamma\left(1+\frac{1}{\upsilon}\right)\right]^{-1} exp\left(-\frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2}\right)$$

Such that

$$\begin{aligned} logp\left(y_t \mid \mathbf{y}_{1:t-1}; \varphi_t, \nu\right) &= -\left[(1+\frac{1}{\nu})log(2) + log\left(\varphi_t\right) + log(\Gamma(1+\frac{1}{\nu}))\right] - \frac{\left|\frac{y_t}{\varphi_t}\right|^{\nu}}{2} \\ logp\left(y_t \mid \mathbf{y}_{1:t-1}; \varphi_t, \nu\right) &\propto - log\left(\varphi\right) - \frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2} \end{aligned}$$

Note that $\varphi > 0$ such that $|\varphi| = \varphi$
The score with respect to $\varphi$ is

$$\frac{\left|\frac{y_t}{\varphi_t}\right|^{\nu}}{2} = \frac{1}{2}y_t^{\upsilon}\varphi_t^{-\nu} = \frac{1}{2} - \nu y_t^{\upsilon}\varphi_t^{-\nu-1} = \frac{1}{2}\left(-\nu\frac{y_t^{\upsilon}}{\varphi_t^{\nu+1}}\right)$$

$$\begin{aligned} \frac{\partial log\left(p\left(y\right)\right)}{\partial\varphi} &= -\frac{1}{\varphi} + \frac{\upsilon\left|y-\mu\right|^{\upsilon}}{2\varphi^{\upsilon+1}} \\ &= \frac{1}{\varphi}\left(\frac{\upsilon}{2}\frac{\left|y-\mu\right|^{\upsilon}}{\varphi^{\upsilon}} - 1\right) \end{aligned}$$

Note that when $\upsilon = 2$ we obtain $\frac{1}{\varphi}\left[\frac{(y-\mu)^2}{\varphi^2} - 1\right]$ which is the score of the gaussian distribution, as on L9S38.
The GAS-GED model is defined as

$$\begin{aligned} \varphi_t &= exp\left(\tilde{\varphi}_t\right) \rightarrow \frac{\partial}{\partial\tilde{\varphi}_t}\left(\varphi_t\right) = \frac{\partial}{\partial\tilde{\varphi}_t}\left(exp\left(\tilde{\varphi}_t\right)\right) \\ \tilde{\varphi}_t &= \omega + \alpha s_{t-1} + \beta\tilde{\varphi}_{t-1} \end{aligned}$$

Where

$$\begin{aligned} s_t &= \frac{\partial log\left(p\left(y_t \mid y_{1:t-1}; \varphi_t, \upsilon\right)\right)}{\partial\tilde{\varphi}_t} \\ &= \frac{\partial\varphi_t}{\partial\tilde{\varphi}_t}\frac{\partial log\left(p\left(y_t \mid y_{1:t-1}; \varphi_t, \upsilon\right)\right)}{\partial\varphi_t} \\ &= \underbrace{\frac{\partial\varphi_t}{\partial\tilde{\varphi}_t}}_{exp(\tilde{\varphi}_t)}\frac{1}{\varphi}\left(\frac{\upsilon}{2}\frac{\left|y-\mu\right|^{\upsilon}}{\varphi^{\upsilon}} - 1\right) \\ &= exp\left(\bar{\varphi}_t\right)\left[-\frac{1}{\varphi_t} + \nu\frac{\frac{|y_t|^2}{|\varphi_t|^{\upsilon+1}}}{2}\right] \\ &= exp\left(\bar{\varphi}_t\right)\left[-\frac{1}{exp\left(\bar{\varphi}_t\right)} + \nu\frac{\frac{|y_t|^{\upsilon}}{exp(\bar{\varphi}_t)^{\upsilon+1}}}{2}\right] \\ &= \left[-\frac{exp\left(\bar{\varphi}_t\right)}{exp\left(\bar{\varphi}_t\right)} + \nu\frac{1}{2}\frac{exp\left(\bar{\varphi}_t\right) * \left|y_t\right|^{\upsilon}}{exp\left(\bar{\varphi}_t\right)^{\upsilon+1}}\right] \\ &= -1 + \nu\frac{1}{2}\frac{\left|y_t\right|^{\upsilon}}{exp\left(\bar{\varphi}_t\right)} \\ s_t\left(y_t, \nu, \bar{\varphi}_t\right) &= \nu\frac{\left|y_t\right|^{\nu}}{2exp\left(\nu\bar{\phi}_t\right)} - 1 \end{aligned}$$

Futher we have

$$
\begin{aligned}
E\left[S_t \mid F_{t-1}\right] &= \int s_t * p\left(y_t \mid F_{t-1}\right) dy_t \\
&= \int \frac{\partial log\left(p\left(y_t \mid F_{t-1}\right)\right)}{\partial \tilde{\varphi}_t} p\left(y_t \mid F_{t-1}\right) dy_t \\
&= \int \frac{\partial\left(p\left(y_t \mid F_{t-1}\right)\right)}{\partial \tilde{\varphi}_t} * \frac{1}{p\left(y_t \mid F_{t-1}\right)} * p\left(y_t \mid F_{t-1}\right) dy \\
&= \int \frac{\partial\left(p\left(y_t \mid F_{t-1}\right)\right)}{\partial \tilde{\varphi}_t} dy \\
&= \frac{\partial}{\partial \tilde{\varphi}_t} \int \left(p\left(y_t \mid F_{t-1}\right)\right) 1 dy \\
&= \frac{\partial}{\partial \tilde{\varphi}_t} 1 = 0
\end{aligned}
$$

Also note that $E\left[s_t\right] = E\left[E\left[s_t \mid F_{t-1}\right]\right] = E\left[0\right] = 0$

## 2) Derive the log likelihood

$$
p\left(y_t \mid y_{1:t-1}; \varphi_t, \upsilon\right) = \left[2^{\left(1+\frac{1}{\upsilon}\right)} \varphi_t \Gamma\left(1+\frac{1}{\upsilon}\right)\right]^{-1} exp\left(-\frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2}\right)
$$

$$
\begin{aligned}
log\left(p\left(y_t \mid y_{1:t-1}; \varphi_t, \upsilon\right)\right) &= log\left(\left[2^{\left(1+\frac{1}{\upsilon}\right)} \varphi_t \Gamma\left(1+\frac{1}{\upsilon}\right)\right]^{-1} exp\left(-\frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2}\right)\right) \\
&= -log\left(2^{\left(1+\frac{1}{\upsilon}\right)}\right) - log\left(\varphi_t\right) - log\left(\Gamma\left(1+\frac{1}{\upsilon}\right)\right) - \frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2} \\
&= -\left(1+\frac{1}{\upsilon}\right) log\left(2\right) - log\left(\varphi_t\right) - log\left(\Gamma\left(1+\frac{1}{\upsilon}\right)\right) - \frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2}
\end{aligned}
$$

The log likelihood is

$$
\begin{aligned}
L\left(y_{1:t} \mid \theta_t\right) &= \sum_{t=1}^{T}\left(-\left(1+\frac{1}{\upsilon}\right) log\left(2\right) - log\left(\varphi_t\right) - log\left(\Gamma\left(1+\frac{1}{\upsilon}\right)\right) - \frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2}\right) \\
&= -\sum_{t=1}^{T}\left(\left(1+\frac{1}{\upsilon}\right) log\left(2\right) + log\left(\varphi_t\right) + log\left(\Gamma\left(1+\frac{1}{\upsilon}\right)\right) + \frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2}\right) \\
&= -T\left(1+\frac{1}{\upsilon}\right) log\left(2\right) - T log\left(\Gamma\left(1+\frac{1}{\upsilon}\right)\right) - \sum_{t=1}^{T}\left(log\left(\varphi_t\right) + \frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2}\right) \\
&= -T\left(1+\frac{1}{\upsilon}\right) log\left(2\right) - T log\left(\Gamma\left(1+\frac{1}{\upsilon}\right)\right) - \sum_{t=1}^{T} log\left(\varphi_t\right) - \sum_{t=1}^{T} \frac{\left|\frac{y_t}{\varphi_t}\right|^{\upsilon}}{2}
\end{aligned}
$$

where $\varphi_t = \varphi_t\left(\theta\right)$ and $\theta = \left(\omega, \alpha, \beta, v\right)'$
Constraints to impose that $\tilde{\varphi}_t$is covariance stationary

$$
\tilde{\varphi}_t = \omega + \alpha s_{t-1} + \beta \tilde{\varphi}_{t-1}
$$

Can be written as

$$
\tilde{\varphi}_t = \frac{\omega}{1-\beta} + \alpha \sum_{s=0}^{\infty} \beta^s s_{t-1-s}
$$

**First moment**

$$
\begin{aligned}
E\left[\tilde{\varphi}_t\right] &= E\left[\frac{\omega}{1-\beta} + \alpha\sum_{k=0}^{\infty}\beta^k s_{t-1-k}\right] \\
&= \frac{\omega}{1-\beta} + \alpha E\left[\sum_{k=0}^{\infty}\beta^k s_{t-1-k}\right] \\
&= \frac{\omega}{1-\beta} + \alpha \sum_{k=0}^{\infty}\beta^k E\left[s_{t-1-k}\right]
\end{aligned}
$$

and sim

$$E\left[s_t\right] = 0$$

$$E\left[\tilde{\varphi}_t\right] = \frac{\omega}{1-\beta} < \infty$$

**Second moment**

$$
\begin{aligned}
E\left[\tilde{\varphi_t^2}\right] &= E\left[\left(\frac{\omega}{1-\beta} + \alpha\sum_{k=0}^{\infty}\beta^k s_{t-1-k}\right)^2\right] \\
&= \left(\frac{\omega}{1-\beta}\right)^2 + \alpha^2 E\left[\left(\sum_{k=0}^{\infty}\beta^k s_{t-1-k}\right)^2\right] + 2*\left(\frac{\omega}{1-\beta}\right)*\left(\alpha E\left[\sum_{k=0}^{\infty}\beta^k s_{t-1-k}\right]\right) \\
&= \frac{\omega^2}{(1-\beta)^2} + \alpha^2 E\left[\left(\sum_{k=0}^{\infty}\beta^k s_{t-1-k}\right)\left(\sum_{l=0}^{\infty}\beta^e s_{t-1-l}\right)\right] + \frac{2\alpha\omega}{1-\beta}\sum_{k=0}^{\infty}\underbrace{E\left[\beta^k s_{t-1-k}\right]}_{=0} \\
&= \frac{\omega^2}{(1-\beta)^2} + \alpha^2 E\left[\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\beta^k\beta^e s_{t-1-k}s_{t-1-e}\right)\right] \\
&= \frac{\omega^2}{(1-\beta)^2} + \alpha^2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\beta^k\beta^e E\left[\left(s_{t-1-k}s_{t-1-e}\right)\right]
\end{aligned}
$$

Conclusion

$$
E\left[\left(s_{t-1-k}s_{t-1-l}\right)\right] = \begin{cases} E\left[\left(s_{t-1-k}^2\right)\right] = e < \infty \; if \; k = e \\ E\left[\left(s_{t-1-k}s_{t-1-e}\right)\right] = 0 \; if \; k \neq 0 \end{cases}
$$

Assume $k < e$

$$
\begin{aligned}
E\left[\left(s_{t-1-k}s_{t-1-l}\right)\right] &= E\left[E\left[\left(s_{t-1-k}s_{t-1-e} \middle| F_{t-1-k}\right)\right]\right] \\
&= E\left[s_{t-1-k}E\left[s_{t-1-e} \middle| F_{t-1-k}\right]\right]
\end{aligned}
$$

But

$$
\begin{aligned}
E\left[\left(s_{t-1-k} \middle| F_{t-1-k}\right)\right] &= E\left[E\left[s_{t-1-e} \middle| F_{t-2-l}\right] \middle| F_{t-1-k}\right] \\
&= E\left[0 \middle| F - t - 1 - k\right] = 0
\end{aligned}
$$

if $k > l$ the same applies

$$
E\left[\left(s_{t-1-k}s_{t-1-e}\right)\right] = \begin{cases} e < \infty \; if \; k = e \\ 0 \; if \; k \neq 0 \end{cases}
$$

Then

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\beta^k\beta^e E\left[s_{t-1-k}s_{t-1-e}\right] = \sum_{k=0}^{\infty}\beta^{2k}e = \frac{e}{1-\beta^2}$$

and

$$E\left[\tilde{\varphi_t}^2\right] = \frac{\omega^2}{(1-\beta)^2} + \frac{\alpha^2 e}{1-\beta^2} < \infty \quad if \;\; |\alpha| < \infty, |\omega| < \infty, |\beta| < 1$$

### The autocovariance

$$\begin{aligned} cov\left(\tilde{\varphi}_t, \tilde{\varphi}_{t-k}\right) &= E\left[\tilde{\varphi}_t \tilde{\varphi}_{t-k}\right] - E\left[\tilde{\varphi}_t\right] E\left[\tilde{\varphi_{t-k}}\right] \\ &= E\left[\tilde{\varphi}_t \tilde{\varphi}_{t-k}\right] - \frac{\omega^2}{(1-\beta^2)} \end{aligned}$$

We have to study $E\left(\tilde{\varphi}_t \tilde{\varphi}_{t-k}\right)$,
Consider the case $k = 1$

$$\begin{aligned} E\left[\tilde{\varphi}_t \tilde{\varphi}_{t-1}\right] &= E\left[\left(\omega + \alpha s_{t-1} + \beta \tilde{\varphi}_{t-1}\right) \tilde{\varphi}_{t-k}\right] \\ &= \omega E\left[\tilde{\varphi}_{t-1}\right] + \alpha \underbrace{E\left[s_{t-1} \tilde{\varphi}_{t-1}\right]}_{=0} + \beta E\left[\tilde{\varphi}_{t-1}^2\right] \end{aligned}$$

Consider the case $k = 2$

$$\begin{aligned} E\left[\tilde{\varphi}_t \tilde{\varphi}_{t-2}\right] &= E\left[\left(\omega + \alpha s_{t-1} + \beta \tilde{\varphi}_{t-1}\right) \tilde{\varphi}_{t-2}\right] \\ &= \omega E\left[\tilde{\varphi}_{t-2}\right] + \alpha \underbrace{E\left[s_{t-1} \tilde{\varphi}_{t-2}\right]}_{=0} + \beta E\left[\tilde{\varphi}_{t-1} \tilde{\varphi}_{t-2}\right] \\ &= \omega E\left[\tilde{\varphi}_{t-2}\right] + \beta\left(\omega E\left[\tilde{\varphi}_{t-1}\right] + \beta E\left[\tilde{\varphi}_{t-1}^2\right]\right) \\ &= \omega E\left[\tilde{\varphi}_{t-2}\right] + \beta \omega E\left[\tilde{\varphi}_{t-1}\right] + \beta^2 E\left[\tilde{\varphi}_{t-1}^2\right] \\ &= \omega E\left[\tilde{\varphi}_{t-2}\right] + \beta \omega E\left[\tilde{\varphi}_{t-1}\right] + \beta^2 E\left[\tilde{\varphi}_{t-1}^2\right] \end{aligned}$$

However we know that $E\left[\tilde{\varphi}_{t-1}\right] = E\left[\tilde{\varphi}_{t-2}\right] = \frac{\omega}{1-\beta}$

$$= \omega\left(E\left[\tilde{\varphi}_{t-1}\right] + \beta E\left[\tilde{\varphi}_{t-1}\right]\right) + \beta^2 E\left[\tilde{\varphi}_{t-1}^2\right]$$

By interative substitutions we obtain

$$\begin{aligned} E\left[\tilde{\varphi}_t \tilde{\varphi}_{t-h}\right] &= \omega E\left[\tilde{\varphi}_{t-1}^2\right] \sum_{e=0}^{k-1} \beta^e + \beta^k E\left[\tilde{\varphi}_{t-1}^2\right] \\ &= \frac{\omega^2}{1-\beta} \sum_{e=0}^{k-1} \beta^e + \beta^k \left[\frac{\omega^2}{(1-\beta)^2} + \frac{\alpha^2 e}{1-\beta^2}\right] \end{aligned}$$

which does not depend from $t$
So conditions for the covariance stationarity are $|\beta| < 1, |\omega| < \infty, |\alpha| < \infty$

**Leopoldo finished**

# 1 Theoretical part

## 1.1 Derive the GAS updating for $\varphi_t$

In general the updating step in GAS model with no scaling is

$$\begin{aligned} \psi_t &= \psi\left(\boldsymbol{y}_{1:t-1}\right) &(1)\\ &= \omega + \alpha u_{t-1} + \beta \psi_{t-1} &(2) \end{aligned}$$

Where $\psi_t$is the variable of interest, and the one we seek to filter out, and where

$$u_t = \nabla_t = \frac{\partial logp\left(y_\tau \mid y_{1:t-1}; \psi\right)}{\partial \psi} \quad (3)$$

which is the unscaled score of the conditional distribution. The general idea is to use this score of the conditional distribution to give the direction update step. In some applications it is convenient to introduce a link function, such that we restrict our variable in some way. In this particular exercise for the GAS-GED model, we want to have the svale parameter $\varphi_t$ which is the variable of interest, is positive. We impose this restriction by introducing a exponential line equation such that:

$$\varphi_t = exp\left(\bar{\varphi}_t\right) \tag{4}$$
$$\tilde{\varphi}_t = \omega + \alpha \tilde{u}_{t-1} + \beta \tilde{\varphi}_{t-1} \tag{5}$$

with $\tilde{u}_t \equiv s_t$ as stated in exercise, where

$$s_t = \frac{\partial logp\left(y_t \mid \boldsymbol{y}_{1:t-1:}; \tilde{\varphi}, \upsilon, \mu\right)}{\partial \tilde{\varphi}_t} = \frac{\partial logp\left(y_t \mid \boldsymbol{y}_{1:t-1:}; \tilde{\varphi}, \upsilon, \mu\right)}{\partial \tilde{\varphi}_t} \frac{\partial \varphi_t}{\partial \tilde{\varphi}_t} \tag{6}$$

is the score of $Y_t | F_{t-1}$with respect to $\tilde{\varphi}$. This means that in this exercise we need to find the logarithm to hte PDF of the conditional distribution of $Y_t$ given infomation up through time $t-1$, $F_{t-1}$which is the genarlized error distribution, i.e. $Y_t | F_{t-1} \sim GED\left(0, \varphi, \upsilon\right)$ (Note that $\mu = 0$ . with $\mu = 0$ the pdf is

$$p\left(y_t \mid \mathbf{y}_{1:t-1}; \varphi_t, \nu\right) = \left[2^{(1+1/v)} \varphi_t \Gamma(1+1/\nu)\right]^{-1} exp\left(-\frac{\left|y_t/\varphi_t\right|^{\nu}}{2}\right) \tag{7}$$

and log-transformed it is:

$$\begin{aligned}
log\left(p\left(y_t \mid \mathbf{y}_{1:t-1}; \varphi_t, \nu\right)\right) &= log\left(\left[2^{(1+1/v)} \varphi_t \Gamma(1+1/\nu)\right]^{-1} exp\left(-\frac{\left|y_t/\varphi_t\right|^{\nu}}{2}\right)\right) \\
&= -log\left(2^{(1+1/v)}\right) - log\left(\varphi_t\right) - log\left(\Gamma(1+1/\nu)\right) - \frac{\left|y_t/\varphi_t\right|^{\nu}}{2} \\
&= -(1+1/v) log\left(2\right) - log\left(\varphi_t\right) - log\left(\Gamma(1+1/\nu)\right) - \frac{\left|y_t/\varphi_t\right|^{\nu}}{2}
\end{aligned} \tag{8}$$

Let us now find $s_t$ by using the chain rule and then insert $\varphi_t = exp\left(\tilde{\varphi}_t\right)$

$$\begin{aligned}
s_t &= \frac{\partial log\left(p\left(y_t \mid y_{1:t-1}; \varphi_t, \upsilon\right)\right)}{\partial \tilde{\varphi}_t} \qquad (9) \\
&= \frac{\partial \varphi_t}{\partial \tilde{\varphi}_t} \frac{\partial log\left(p\left(y_t \mid y_{1:t-1}; \varphi_t, \upsilon\right)\right)}{\partial \varphi_t} \\
&= \underbrace{\frac{\partial \varphi_t}{\partial \tilde{\varphi}_t}}_{exp(\tilde{\varphi}_t)} \frac{1}{\varphi} \left(\frac{\upsilon}{2} \frac{\left|y-\mu\right|^{\upsilon}}{\varphi^{\upsilon}} - 1\right) \\
&= exp\left(\bar{\varphi}_t\right) \left[-\frac{1}{\varphi_t} + \nu \frac{\frac{\left|y_t\right|^2}{\left|\varphi_t\right|^{\upsilon+1}}}{2}\right] \\
&= exp\left(\bar{\varphi}_t\right) \left[-\frac{1}{exp\left(\bar{\varphi}_t\right)} + \nu \frac{\frac{\left|y_t\right|^{\upsilon}}{exp(\bar{\varphi}_t)^{\upsilon+1}}}{2}\right] \\
&= \left[-\frac{exp\left(\bar{\varphi}_t\right)}{exp\left(\bar{\varphi}_t\right)} + \nu \frac{1}{2} \frac{exp\left(\bar{\varphi}_t\right) * \left|y_t\right|^{\upsilon}}{exp\left(\bar{\varphi}_t\right)^{\upsilon+1}}\right] \\
&= -1 + \nu \frac{1}{2} \frac{\left|y_t\right|^{\upsilon}}{exp\left(\bar{\varphi}_t\right)} \\
s_t\left(y_t, \nu, \bar{\varphi}_t\right) &= \nu \frac{\left|y_t\right|^{\nu}}{2exp\left(\nu \bar{\phi}_t\right)} - 1 \qquad (10)
\end{aligned}$$

Now we have everything need in order to formalise the updating step for the GAS-GED model

$$\varphi_t = exp\left(\bar{\varphi}_t\right) \tag{11}$$
$$\tilde{\varphi}_t = \omega + \alpha \tilde{u}_{t-1} + \beta \tilde{\varphi}_{t-1} \tag{12}$$
$$s_t = \nu \frac{\left|y_t\right|^{\nu}}{2exp\left(\nu \bar{\phi}_t\right)} - 1 \tag{13}$$

In order to find the unconditional expected value of $s_t$,$E[s_t]$ we consider $s_t$ in its general expression $s_t = \frac{\partial log(p(y_t|y_{1:t-1};\varphi_t,\upsilon))}{\partial\tilde{\varphi}_t}$

$$E[s_t] = E\left[\frac{\partial log(p(y_t|\, y_{1:t-1};\varphi_t,\upsilon))}{\partial\tilde{\varphi}_t}\right] \tag{14}$$

We compute the expectation by using the PDF of $y_t$ in the conditional distribution, since $s_t$ is the score of the conditional distribution

$$\begin{aligned}
E[s_t] &= \int_y \frac{\partial log(p(y_t|\, y_{1:t-1};\varphi_t,\upsilon))}{\partial\tilde{\varphi}_t} p(y_t|\, y_{1:t-1};\varphi_t,\upsilon)\, dy \\
&= \int_y \frac{\partial(p(y_t|\, y_{1:t-1};\varphi_t,\upsilon))}{\partial\tilde{\varphi}_t} * \frac{1}{p(y_t|\, y_{1:t-1};\varphi_t,\upsilon)} p(y_t|\, y_{1:t-1};\varphi_t,\upsilon)\, dy \\
&= \int_y \frac{\partial(p(y_t|\, y_{1:t-1};\varphi_t,\upsilon))}{\partial\tilde{\varphi}_t} dy \\
&= \frac{\partial \int_y (p(y_t|\, y_{1:t-1};\varphi_t,\upsilon))\, dy}{\partial\tilde{\varphi}_t} \\
&= \frac{\partial 1}{\partial\tilde{\varphi}_t} \\
&= 0
\end{aligned}$$

Similarly, we can find $E[s_t|\, F_{t-1}]$ by using the argument that $s_t$ is the score of the conditional distribution, meaning that conditioning again on the same information set will give us $E(s_t)$ by the law of of interated expectation. That is:

$$E[s_t|\, F_{t-1}] = E(s_t) = 0$$

## 1.2 Log-likelihood and parameter restrictions

In question 1, we have already established the individual log-likelihodd contribution as

$$logp(y_t \mid \mathbf{y}_{1:t-1};\varphi_t,\nu) = -[(1+1/\nu)log(2) + log(\varphi_t) + log(\Gamma(1+1/\nu))] - \frac{|y_t/\varphi_t|^\nu}{2}$$

Adding up, we get the log-likelihood function where we note that $\varphi_t$ is a function of $(\omega,\alpha,\beta)$ such that the log-likelihood is also a function of these parameters

$$\begin{aligned}
logL(\omega,\alpha,\beta,\upsilon) &= \sum_{t=1}^{T}\left(-[(1+1/\nu)log(2) + log(\varphi_t) + log(\Gamma(1+1/\nu))] - \frac{|y_t/\varphi_t|^\nu}{2}\right) \quad (15)\\
&= -T(1+1/\nu)log(2) - Tlog(\Gamma(1+1/\nu)) - \sum_{t=1}^{T}\left(log(\varphi_t) + \frac{|y_t/\varphi_t|^\nu}{2}\right) \quad (16)
\end{aligned}$$

This function must be numerically maximized with respect to the parameters $\omega,\alpha,\beta$ and $\upsilon$. Let us now look at the parameter constraints needed. In order to establish covariance stationarity of the sequence $\tilde{\varphi}_t$ we must have that the mean and variance of $\tilde{\varphi}_t$ is constant over time

$$E[\tilde{\varphi}_t] = E[\tilde{\varphi}_{t+h}] = \mu_{\tilde{\varphi}} < \infty \tag{17}$$
$$Var(\tilde{\varphi}_t) = Var(\tilde{\varphi}_{t+h}) = \sigma^2_{\tilde{\varphi}} \in (0;\infty) \tag{18}$$

The expected value of $\tilde{\varphi}_t$

$$\begin{aligned}
E(\tilde{\varphi}_t) &= E(\omega + \alpha s_{t-1} + \beta\tilde{\varphi}_{t-1}) \\
&= \omega + \beta E(\tilde{\varphi}_{t-1}) \\
E(\tilde{\varphi}) &= \frac{\omega}{1-\beta}
\end{aligned} \tag{19}$$

In order for this to be finite, we need to impose $\beta \neq 1$.The varianxe of $\tilde{\varphi}_t$ is

$$\begin{aligned} Var\left(\tilde{\varphi}_t\right) &= Var\left(\omega + \alpha s_{t-1} + \beta\tilde{\varphi}_{t-1}\right) \\ &= \alpha^2 Var\left(s_t\right) + \beta^2 Var\left(\tilde{\varphi}_{t-1}\right) \\ Var(\tilde{\varphi}) &= \frac{\alpha^2 E\left(s_t^2\right)}{1-\beta^2} \end{aligned} \tag{20}$$

If we assume that $E\left[s_t^2\right] < \infty$, we need to restrict $|\beta| < 1$ in order to have a finite positive variance. This means that the two restrictions we need to make are

$$\upsilon > 0 \ and \ \ |\beta| < 1$$

## 1.3 Scale parameter response to special cases

Let us write the two functions;

$$\begin{aligned} f_1\left(y_t\right) = s\left(y_t, 1, 0\right) = \tfrac{|y_t|}{2} - 1 \\ f_2\left(y_t\right) = s\left(y_t, 2, 0\right) = y_t^2 - 1 \end{aligned} \tag{21}$$

Let us compare the value of two and see when one is bigger than the other. We have $f_1\left(y_t\right) > f_2\left(y_t\right)$ if

$$\begin{aligned} \tfrac{|y_t|}{2} - 1 > y_t^2 - 1 \\ \tfrac{|y_t|}{2} > y_t^2 \\ |y_t| < \tfrac{1}{2} \end{aligned} \tag{22}$$

Intuitively, this makes sense as $\upsilon$ indicates the fat of the tails. For $\upsilon = 1$ the tails are fatter than for $\upsilon = 2$. In a distribution with fatter tails, we will have that the sclae parameter will respond the most(for a given $\alpha$ at least) if we observe $|y_t| < \frac{1}{2}$ since it has less probability mass close to the mean (of zero), which means that it will correct its scale/variance relatively more than the distribution with thinner tails and, hence, more probability mass around its mean. In contrast, we see that for $|y_t| > \frac{1}{2}$ the thinner-tailed distribution will have to correct its scale (and hence variance) the most, in order to make the observation more "likely" to occur.

# 2 Computational part

## 2.1 Estimation of GAS-GED

In order to estimate a GAS-GED model, I have written two central codes. One that filter the scale parameter and, hence, the standard deviation for given parameter values and data. Another that uses maximum likelihood to estimate the parameter values in the filter. I have commented a bit on the input and outputs of each function in the R code. However, I want to comment on a particular decision of mine here. I have initialized the recursions for the scales by setting the standard deviation equal to the sample standard deviation estimated from the first 150 observations of the process. This is done for two reasons: Firstly, in the empirical part, we want to compare the filtered volatilities from a GAS-GED model with those from a GARCH(1,1) estimated by the ugarchfit function in the rugarch package. This function initializes the process, by default, to the sample variance based on the whole sample. However, as mentioned later, I have changed this as well. Secondly, the variance seem to increase significantly towards the financial crises such that initializing the process by the sample variance based on the whole sample, will be a very misleading guess on the initial variance.

$$\sigma = 2^{\frac{1}{\upsilon}} \sqrt{\frac{\Gamma\left(\frac{3}{\upsilon}\right)}{\Gamma\left(\frac{1}{\upsilon}\right)} \varphi} \tag{23}$$

$$\tilde{\psi}_t = \omega + \beta * \tilde{\psi}_{t-1} + \alpha * \underbrace{\left(\frac{|y_{t-1}|^{\upsilon}}{\tilde{\psi}^{\upsilon}{}_{t-1}} - 1\right)}_{s_{t-1}} \tag{24}$$