Univariate Volatility Modeling: Stochastic Volatility

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SV models: a class of models

So far we have discussed models of the form $y_t = \sigma_t \varepsilon_t$, where $\sigma_t = \sigma(\mathcal{F}_{t-1}, \theta)$, that is: models for which the variance at time t given $\mathcal{F}_{t-1}$ is known.

Note that the only source of error in this class of models is ε_t. Indeed, models of this kind are classified as "Single Source of Error" according to Snyder (1985).

However, it might be reasonable to believe that different shocks update models parameters. To this end, Taylor (1986) has introduced the class of Stochastic Volatility (SV) models.

Stochastic volatility

Consider the model

$$y_t = \sigma_t u_t,$$

where

- σ_t is a positive random variable with $Var(\sigma_t|\mathcal{F}_{t-1}) > 0$.
- $\{\sigma_t\}$ is stationary with $E[\sigma_t^4] < \infty$ and $\rho_{\tau,\sigma^2} = cor(\sigma_t^2, \sigma_{t+\tau}^2) > 0$.
- $u_t \overset{iid}{\sim} D(0,1)$.
- $\{u_t\}$ and $\{\sigma_t\}$ are independent.

Note that this model does not nest GARCH models, why?

Stochastic volatility

Since $\{u_t\}$ and $\{\sigma_t\}$ are independent we have that for any function f_1 and f_2

$$E[f_1(\sigma_t, \sigma_{t-1}, \dots)f_2(u_t, u_{t-1}, \dots)] = E[f_1(\sigma_t, \sigma_{t-1}, \dots)]E[f_2(u_t, u_{t-1}, \dots)]$$

Let's evaluate:

- $E[y_t]$
- $var(y_t)$
- $skew(y_t)$
- $kurt(y_t)$

Stochastic volatility: moments

Note that $y_t = \sigma_t u_t$, hence:

$$\begin{aligned} E[\sigma_t u_t] &= E[\sigma_t]E[u_t] = 0 \\ var(\sigma_t u_t) &= E[\sigma_t^2 u_t^2] - (E[\sigma_t u_t])^2 = E[\sigma_t^2]E[u_t^2] = E[\sigma_t^2] \\ E[\sigma_t^3 u_t^3] &= E[\sigma_t^3]E[u_t^3] = E[\sigma_t^3]skew(u_t) \\ E[\sigma_t^4 u_t^4] &= E[\sigma_t^4]E[u_t^4] = E[\sigma_t^4]kurt(u_t) \end{aligned}$$

hence

$$\begin{aligned} skew(\sigma_t u_t) &= \frac{E[\sigma_t^3 u_t^3]}{E[(\sigma_t u_t)^2]^{3/2}} = \frac{E[\sigma_t^3]}{E[\sigma_t^2]^{3/2}} skew(u_t) \\ kurt(\sigma_t u_t) &= kurt(u_t)\left(1 + \frac{var(\sigma_t^2)}{E[\sigma_t^2]^2}\right) > kurt(u_t) \end{aligned}$$

Stochastic volatility: autocovariance

Autocovariance of y_t:

$$
\begin{aligned}
\gamma_{\tau,y} &= cov(y_t, y_{\tau+t}) = cov(\sigma_t u_t, \sigma_{\tau+t} u_{\tau+t}) \\
&= E[\sigma_t u_t \sigma_{\tau+t} u_{\tau+t}] - E[\sigma_t u_t] E[\sigma_{\tau+t} u_{\tau+t}] \\
&= E[\sigma_t \sigma_{\tau+t}] E[u_t u_{\tau+t}] - E[\sigma_t u_t] E[\sigma_{\tau+t} u_{\tau+t}] = 0
\end{aligned}
$$

Autocovariance of $y_t^2 = \sigma_t^2 u_t^2 = s_t$:

$$
\begin{aligned}
\gamma_{\tau,s} &= cov(s_t, s_{t+\tau}) = cov(\sigma_t^2 u_t^2, \sigma_{t+\tau}^2 u_{t+\tau}^2) \\
&= E[\sigma_t^2 u_t^2 \sigma_{t+\tau}^2 u_{t+\tau}^2] - E[\sigma_t^2 u_t^2] E[\sigma_{t+\tau}^2 u_{t+\tau}^2] \\
&= E[\sigma_t^2 \sigma_{t+\tau}^2] E[u_t^2 u_{t+\tau}^2] - E[\sigma_t^2] E[u_t^2] E[\sigma_{t+\tau}^2] E[u_{t+\tau}^2] \\
&= E[\sigma_t^2 \sigma_{t+\tau}^2] - E[\sigma_{t+\tau}^2] E[\sigma_t^2] \\
&= cov(\sigma_t^2, \sigma_{t+\tau}^2) = \gamma_{\tau,\sigma^2}
\end{aligned}
$$

Independent random volatility

Consider the model:

$$
\begin{aligned}
y_t &= \sigma_t u_t \\
&= \exp(w_t/2) u_t
\end{aligned}
$$

where:

$$
\begin{pmatrix} u_t \\ w_t \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ \zeta \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sigma_w^2 \end{pmatrix} \right).
$$

How would you estimate it?

Independent random volatility

We have several alternatives to estimate the model's parameters ζ and σ_w. For instance:

- Maximum Likelihood
- Method of Moments
- Generalized Method of Moments
- Simulation techniques

Consider for example the first two possibilities.

Independent random volatility: ML

In order to perform Maximum Likelihood estimation of the model's parameter we need to derive the log–likelihood function. Assume to observe a sequence of T observations $y_{1:T} = \{y_1, \dots, y_T\}$:

$$\begin{aligned}\mathcal{L}(\zeta, \sigma_w | y_{1:T}) &= \log P(y_{1:T}) \\ &= \sum_{t=1}^{T} \log p(y_t),\end{aligned}$$

since in this model $y_t \perp\!\!\!\perp y_{t+\tau}$ for all $\tau \neq 0$. However:

$$\begin{aligned}p(y_t) &= \int_{\Re} p(y_t | w_t) p(w_t) dw_t \\ &= \frac{1}{2\pi\sigma_w} \int_{\Re} \exp\left[-\frac{1}{2}\left(w_t + \frac{y_t^2}{\exp(w_t)} + \frac{(w_t - \zeta)^2}{\sigma_w^2}\right)\right] dw_t,\end{aligned}$$

does not admit a closed form solution. Let's code it in R!

Intermezzo: Log Normal Distribution

If $X \sim N(a, b^2)$, then $\exp X = Z \sim LN(a, b^2)$ such that:

$$
\begin{aligned}
p(z) &= \frac{1}{zb\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{(\log z - a)^2}{b^2}\right)\right), \quad z > 0 \\
E[Z^n] &= \exp\left(na + \frac{1}{2}n^2 b^2\right) \\
var(Z) &= \exp(2a + b^2)(\exp(b^2) - 1),
\end{aligned}
$$

Independent random volatility: MM

We note that since $w_t \sim N(\zeta, \sigma_w^2)$, then $\sigma_t^2 = exp(w_t) \sim LN(\zeta, \sigma_w^2)$. Also note that $E[\sigma_t^2] = \exp(\zeta + \frac{\sigma_w^2}{2})$

$$E[y_2^2] = E[\sigma_t^2 u_t^2] = E[\sigma_t^2]E[u_t^2] = E[\exp(w_t)] = \exp\left(\zeta + \frac{\sigma_w^2}{2}\right)$$
$$E[y_t^4] = E[\sigma_t^4 u_t^4] = E[\sigma_t^4]E[u_t^4] = 3E[\exp(2w_t)] = 3\exp(2(\zeta + \sigma_w^2))$$

Let $\hat{\mu}_2 = T^{-1}\sum_{t=1}^T y_t^2$ and $\hat{\mu}_4 = T^{-1}\sum_{t=1}^T y_t^4$. By equating empirical and theoretical moments we find:

$$\sigma_w^2 = \log\left(\frac{\hat{\mu}_4}{3\hat{\mu}_2^2}\right), \qquad \zeta = \log\left(\frac{\hat{\mu}_2^2\sqrt{3}}{\sqrt{\hat{\mu}_4}}\right)$$

Let's code it in R!

SV models: a class of models

Stochastic Volatility models belong to the general class of nonlinear non gaussian state space models:

$$y_t = m(\theta_t, \varepsilon_t), \tag{1}$$

$$\theta_{t+1} = q(\theta_t, \eta_t) \tag{2}$$

where ε_t and η_t are (possibly dependent) random variables.

The Log-Normal AR(1) Stochastic Volatility Model

In the original SV model we have:

$$y_t = \sigma_t u_t \tag{3}$$

$$\log \sigma_{t+1} - \alpha = \phi(\log \sigma_t - \alpha) + \zeta_t, \tag{4}$$

where

$$\begin{pmatrix} u_t \\ \zeta_t \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}\right), \tag{5}$$

that is: $\log \sigma_t$ follows a first order autoregressive process. Hence $\log \sigma_t \sim N(\alpha, \beta^2)$, where $\beta^2 = \frac{\sigma_\zeta^2}{1-\phi^2}$.

The Log-Normal AR(1) Stochastic Volatility Model

The model is also written as:

$$y_t = \exp(w_t/2)u_t \tag{6}$$
$$w_{t+1} = \omega + \phi w_t + \eta_t, \tag{7}$$

where $w_t = \log \sigma_t^2$, and

$$\begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sigma_\eta^2 \end{pmatrix} \right), \tag{8}$$

Note that to ensure stationarity of w_t we have to impose $|\phi| < 1$ like in usual AR(1) models. In this case $E[w_t] = \frac{\omega}{1-\phi}$

The Log-Normal AR(1) Stochastic Volatility Model

Basic properties

- $\{y_t\}$ is strictly stationary
- All moments of y_t are finite
- $kurt(y_t) = 3\exp(\beta^2)$, where $\beta^2 = var(w_t)$.
- $cov(y_t, y_{t+\tau}) = 0$
- $cov(s_t, s_{t+\tau}) > 0$, when $\psi > 0$, $s_t = y_t^2$
- ACF of $|y_t|^p$ behaves like ACF of s_t.

SV Model: estimation

Suppose to observe a sequence of T observations from the SV model with zero mean $y_{1:T} = \{y_1, \ldots, y_T\}$. The likelihood is:

$$
\begin{aligned}
L(\alpha, \phi, \sigma_\eta | y_{1:T}) &= \log p(y_{1:T}) \\
&= \log \int p(y_{1:T}|\sigma_{1:T}) p(\sigma_{1:T}) d\sigma_{1:T} \\
&= \log \int \prod_{t=1}^{T} p(y_t|\sigma_t) p(\sigma_t|\sigma_{t-1}) d\sigma_t,
\end{aligned}
$$

where the distribution of σ_1 can be set to the unconditional distribution of the process. Note that direct numerical integration is not feasible. Can you derive a GMM estimator?

SV Model: extensions

- Fat tailed distribution for u_t (E.g. Student's t)
- Dependence between u_t and η_t (Leverage effect)
- Long memory in $\log \sigma_t$
- Multivariate formulations

Not straightforward to implement!

References

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