DEPARTMENT OF ECONOMICS AND BUSINESS ECONOMICS
AARHUS UNIVERSITY

EXAM ASSIGNMENT

Study Programme and level	MSc Oecon	
Term	V20-21o	
Course name and exam code	4394: Financial Econometrics	461172E002
Exam form and duration	Take-home exam, WHAI	4 days
Date and time	18-22 December 2020	12:00 – 12:00
Anonymity	The exam is **anonymous**. Please add only the flow id number found on your cover sheet in WISEflow on all pages.	
Other	**Avoid being suspected of exam cheating** Remember to use quotation marks and to insert references if you copy text from other sources, incl. indicative exam solutions (plagiarism) or if you re-use parts of a previously submitted (passed) exam paper (self-plagiarism). Avoid sharing your exam paper with others, or communicating about the assignment during the exam. Students must answer the exam assignment **individually**. All submitted exam papers will be checked for plagiarism, so exam cheating (incl. collaboration between students) will be detected.	
Number of pages (excl. front page)	4	

How to contact the lecturer during a take-home exam
In case you have any questions regarding the exam, the lecturer is available to take questions during the first four hours after exam start. Questions should be posed to the lecturer by email to leopoldo.catania@econ.au.dk. Students should not expect instant answers, as a collective answer to all questions received during these four hours will be sent out via Blackboard once the four hours have passed.

How to hand in your exam paper:
Start preparing the hand in well in advance of the exam deadline.
Your exam paper must be handed in as one PDF file in WISEflow. The maximum permitted file size is 200 MB. Additional material/appendices may (if permitted) be uploaded in other file formats. The total maximum permitted file size is 5 GB.

If you experience *problems* uploading and handing in your exam paper in WISEflow, you can send the paper to the following email address: bss.exam@au.dk. You need to ask for permission to hand in your paper for final assessment in WISEflow. Use the formula “Exemption” under “Applications to Study Councils” in the Student Self-Service. You need to apply *as soon as possible* after sending your paper to the email address.
If you need technical assistance during the exam, you can contact BSS IT-support, phone: 8715 0933.
Be aware that exam papers are as a rule only permitted for final assessment if handed in, in the right format/size and within the exam deadline.

Exam questions

Guidelines

General guidelines

The exercises must be done individually. You have to submit two files: one PDF file and one file with the computer code. The computer code should be submitted as "appendix". The PDF file should include the answers to the theoretical part as well as any answer to the computational and empirical part which requires a written comment. The computer code should be organized such that it can reproduce all the computational and empirical answers of your exam. Code must be commented. The exam will be graded on the Danish 7-point grading scale. Good luck!

Guidelines for the theoretical part

Try to be as precise as possible and write your results in a PDF document. If you plan to write a Word document and then transform it to PDF, please be sure that all formulas are well formatted before submitting your exam. When you write your answers, you have to be consistent with the notation used in the exam questions.

Guidelines for the computational part

You are required to comment on your code. For example, state the purpose of the functions you code, their arguments, and what they return as an output. I should be able to run your code without major modifications. You are free to employ your favorite optimization scheme.

Guidelines for writing and presenting your code

You are required to comment on your code. For example, state the purpose of the functions you code, their arguments, and what they return as an output. I should be able to run your code without major modifications. You are free to employ your favorite optimization scheme. Be careful of removing all details that can reveal your identity from the code.

Theoretical part

During the course, we have widely employed the Gaussian and the Student's t distributions to model the conditional distribution of financial returns. However, many other distributions are available, and it is not always easy to choose the best one. For example, Nelson (1991) proposed to model the conditional distribution of financial returns using the Generalized Error Distribution (GED). Specifically, for a generic random variable $Y \in \Re$ with mean $\mu \in \Re$ and scale $\varphi > 0$, the probability density function (pdf) evaluated in y is:

$$p(y; \mu, \varphi, \upsilon) = \left[2^{1+1/\upsilon}\varphi\Gamma(1 + 1/\upsilon)\right]^{-1} \exp\left(-\frac{|(y-\mu)/\varphi|^{\upsilon}}{2}\right), \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function, and $\upsilon > 0$ is a tail-thickness parameter. The standard deviation σ is related to the scale by the formula:

$$\sigma = 2^{1/\upsilon}\sqrt{\frac{\Gamma(3/\upsilon)}{\Gamma(1/\upsilon)}}\varphi. \tag{2}$$

Consider a Generalised Autoregressive Score (GAS) model, where the conditional distribution of financial returns $Y_t|\mathcal{F}_{t-1}$ is GED with location 0 ($\mu = 0$) and time varying scale φ_t:

$$Y_t|\mathcal{F}_{t-1} \sim GED(0, \varphi_t, \upsilon),$$

where $\mathcal{F}_t$ represents the filtration generated by the process $\{Y_s, s \leq t\}$.

- Derive the GAS updating equation for φ_t using an exponential link function, $\varphi_t = \exp(\tilde{\varphi}_t)$, and imposing unit scaling, i.e. do not scale the score ($d = 0$). Use s_t to label the score of $Y_t|\mathcal{F}_{t-1}$ with respect to $\tilde{\varphi}_t$. Evidently, s_t is going to be a function of the realization y_t, the tail-thickness parameter υ, and the reparametrized scale $\tilde{\varphi}_t$, i.e. $s_t = s(y_t, \upsilon, \tilde{\varphi}_t)$. Compute $E[s_t|\mathcal{F}_{t-1}]$ and $E[s_t]$.
- Write the log likelihood of the model and state the model parameter constraints. Which constraint do we need to impose to ensure that the process of $\{\tilde{\varphi}_t, t > 0\}$ is covariance stationary (assume that $E[s_t^2] < \infty$ for all values of $\tilde{\varphi}_t$ and υ)?
- When $\upsilon = 2$, the GED distribution collapses to the Gaussian distribution. Compare the functions $f_1(y_t) = s(y_t, 1, 0)$ and $f_2(y_t) = s(y_t, 2, 0)$. Which conclusion you draw about the response of the scale parameter to different values of y_t in the case $\upsilon = 1$ and $\upsilon = 2$?

Computational part

- Write a function to estimate the GAS–GED model you have derived in the previous exercise. The function should accept the data and return: i) The filtered scale and volatility, i.e. the vectors $(\sigma_1, \sigma_2, \dots, \sigma_T)$ and $(\varphi_1, \varphi_2, \dots, \varphi_T)$, respectively, ii) the estimated model parameters, iii) the value of the likelihood evaluated at its optimum, and iv) the average BIC (the BIC divided by the number of observations).

- Write a function to compute the Value–at–Risk (VaR) at level $\alpha \in (0, 1)$ for the conditional distribution $Y_t|\mathcal{F}_{t-1}$. You have two options: i) write a function that computes the cumulative density function $P(Y_t \leq y|\mathcal{F}_{t-1})$ using the `integrate` function and numerically solve $P(Y_t \leq VaR_t(\alpha)|\mathcal{F}_{t-1}) - \alpha = 0$ (please always use this definition of VaR) by $VaR_t(\alpha)$ using the `uniroot` function, or ii) use the `qdist` function in the **rugarch** package with argument `distribution = "ged"`. If you go for the second option, note that `qdist` is parameterized in terms of the volatility σ_t, and the tail-thickness parameter is labelled as `shape`. The function should accept: i) a vector of volatilities or scales, ii) the tail-thickness parameter, and iii) the α confidence level. The function should return a vector with the VaR computed for each time t at the desired confidence level α.

Empirical part

a) Consider this general dynamic volatility model:

$$y_t = \sigma_t \varepsilon_t,$$

where ε_t is independently and identically distributed as a standardized ($E[\varepsilon_t] = 0$, $E[\varepsilon_t^2] = 1$) GED with tail-thickness parameter υ, and σ_t is a deterministic function of past observations y_{t-s} for $s \geq 1$. Write down the GARCH(1,1), EGARCH(1,1) and GJR-GARCH(1,1) models for σ_t^2.

For the GARCH and GJR-GARCH models, derive the constraints that ensure weak and strong stationarity of y_t.

Download the Dow Jones Industrial Average (DJI) and Standard and Poor's 500 (SP500) indices from Yahoo finance using the **quantmod** package in the time period from = "2007-01-03" to = "2019-01-01" at the daily frequency. The Yahoo ticker for Dow Jones Industrial Average is "^DJI" while the one for Standard and Poor's 500 is "^GSPC". Create a matrix with the percentage logarithmic returns of both series.

For both series:

i) Estimate these three models under the GED assumption for the innovations. You **cannot** use the **rugarch** package.

ii) Compare the filtered volatilities according to these three specifications in a figure.

iii) Select the best model using the BIC criteria.

b) Estimate the GAS-GED model on the DJI and SP500 series. For each series:

i) Compare the filtered volatility of the GAS-GED model with that of the GARCH(1,1) model in a figure.

ii) Compare VaR at levels $\alpha = 1\%$ and $\alpha = 5\%$ estimated with the GAS-GED and GARCH(1,1) models in a figure.

iii) Select the best model between GAS-GED and GARCH(1,1) using BIC.

Try to draw some general conclusion about the difference between the GAS-GED and GARCH(1,1) models.

c) Assume that the correlation matrix (R) between DJI and SP500 is constant over time. Let Σ_t be the covariance matrix between DJI and SP500 at time t. Use the following decomposition of Σ_t:

$$\Sigma_t = D_t R D_t, \tag{3}$$

where D_t is a 2−dimensional diagonal matrix with element $(1,1)$ and $(2,2)$ equal to the standard deviations of DJI and SP500 at time t, respectively.

i) Compute Σ_t for each t, when DJI and SP500 both follow the GAS-GED model.

ii) Compute Σ_t for each t, when DJI and SP500 both follow the GARCH(1,1) model.

iii) Under the two model specifications of previous points i) and ii), compute the weight ω_t associated to the Minimum Variance Portfolio (MVP) constructed using the DJI and SP500 returns at each point in time, i.e. $y_t = \omega_t y_t^{DJI} + (1-\omega_t) y_t^{SP500}$.

iv) Compare the portfolio weights of the two models ω_t in a figure.

References

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica: Journal of the Econometric Society*, pages 347–370.