1 Problem Set 4 - Problem 1, point 1)

1.1 Derivation of density for weights

In SV model:

$$y_t = \exp\left(\frac{\alpha_t}{2}\right)\eta$$

Factorize joint density:

$$p\left(Y_{1:t} \mid \alpha_{1:t}\right) = \prod_{s=1}^{t} p\left(y_t \mid \alpha_t\right)$$

α_t is only dependent on the previosly value of α_{t-1}. The y 's are only dependent through the values of α.

$$p\left(Y_{1:t} \mid \alpha_{1:t}\right) = \prod_{s=1}^{t} \exp\left(-\frac{\alpha_t}{2}\right) p\left(y_t \exp\left(\frac{\alpha_t}{2}\right)\right)$$

When coding this in R, then it is a matter of foreloops, evaluating normal densities and taking their products. Slide 12: Choice of function x_t :

$$x_t\left(\alpha_{1:t}\right) = \exp\left(\frac{\alpha_t}{2}\right)$$

2 Problem Set 4 - Problem 1, point 3)

2.1 Derive unconditional mean:

$$\begin{aligned}
\alpha_t &= \omega + \phi\alpha_{t-1} + \tau\eta_t \\
\underset{\text{Unconditional mean}}{\Longrightarrow} \quad \mathbb{E}\left[\alpha_t\right] &= \mathbb{E}\left[\omega + \phi\alpha_{t-1} + \tau\eta_t\right] \\
\mathbb{E}\left[\alpha\right] &= \mathbb{E}\left[\omega + \phi\alpha + \tau\eta_t\right] \\
\mathbb{E}\left[\alpha\right] &= \mathbb{E}\left[\omega\right] + \mathbb{E}\left[\phi\alpha\right] + \mathbb{E}\left[\tau\eta_t\right] \\
\mathbb{E}\left[\alpha\right] &= \mathbb{E}\left[\omega\right] + \mathbb{E}\left[\phi\alpha\right] + \underbrace{\mathbb{E}\left[\tau\eta_t\right]}_{=0} \\
\mathbb{E}\left[\alpha\right] &= \omega + \phi\mathbb{E}\left[\alpha\right] \\
\mathbb{E}\left[\alpha\right] - \phi\mathbb{E}\left[\alpha\right] &= \omega \\
\mathbb{E}\left[\alpha\right]\left(1-\phi\right) &= \omega \\
\mathbb{E}\left[\alpha\right] &= \frac{\omega}{\left(1-\phi\right)}
\end{aligned}$$

2.2 Derive unconditional variance:

$$\begin{aligned}
\alpha_t &= \omega + \phi\alpha_{t-1} + \tau\eta_t \\
V\left[\alpha_t\right] &= V\left[\omega + \phi\alpha_{t-1} + \tau\eta_t\right] \\
\underset{\text{Unconditional variance}}{\Longrightarrow} \quad V\left[\alpha\right] &= V\left[\omega + \phi\alpha + \tau\eta_t\right] \\
V\left[\alpha\right] &= V\left[\omega\right] + \phi^2 V\left[\alpha\right] + \tau^2 V\left[\eta_t\right] \\
V\left[\alpha\right] &= V\left[\omega\right] + \phi^2 V\left[\alpha\right] + \tau^2 \underbrace{V\left[\eta_t\right]}_{=1,\text{ by assump.}} \\
V\left[\alpha\right] - \phi^2 V\left[\alpha\right] &= V\left[\omega\right] + \tau^2 \\
V\left[\alpha\right]\left(1-\phi^2\right) &= \underbrace{V\left[\omega\right]}_{=0,\text{ constant}} + \tau^2 \\
V\left[\alpha\right] &= \frac{\tau^2}{\left(1-\phi^2\right)}
\end{aligned}$$