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STATIONARITY AND PERSISTENCE IN THE GARCH(1,1) MODEL

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This paper establishes necessary and sufficient conditions for the stationarity and ergodicity of the GARCH(1,1) process. As a special case, it is shown that the IGARCH(1,1) process with no drift converges almost surely to zero, while IGARCH(1,1) with a positive drift is strictly stationary and ergodic. We examine the persistence of shocks to conditional variance in the GARCH(1,1) model, and show that whether these shocks "persist" or not depends crucially on the definition of persistence. We also develop necessary and sufficient conditions for the finiteness of absolute moments of any (including fractional) order.

1. INTRODUCTION

Since the seminal work of Engle [5], ARCH (autoregressive conditionally heteroskedastic) models have been widely used to model time-varying volatility and the persistence of shocks to volatility. One member of the family of ARCH processes, GARCH(1,1), since its introduction by Bollerslev [2] has been especially popular in econometric modeling. In this paper we investigate the GARCH(1,1) model in depth. Analyzing the properties of GARCH(1,1) not only sheds light on the behavior of this commonly used model, it also provides interesting and natural examples of several important concepts in probability theory, including the distinction between martingales and random walks, between strict and weak stationarity, and between almost sure and L^p convergence.

To define the model, let

$$\{z_t\}_{t=-\infty,\infty} \sim \text{i.i.d.},\ z_t^2 \text{ nondegenerate},\ P[-\infty < z_t < \infty] = 1, \tag{1}$$

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\xi_{t-1}^2, \tag{2}$$

with

$$\xi_t = \sigma_t \cdot z_t \tag{3}$$

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where $\omega \geq 0$, $\beta \geq 0$, and $\alpha > 0$. In most papers using GARCH(1,1), a further restriction has been placed on $\{z_t\}$, namely that

$$E[z_t] = 0, \qquad E[z_t^2] = 1. \tag{4}$$

Under (4), σ_t^2 is the conditional variance of ξ_t given the history of the system. If we assume $E[z_t^2] = 1$ but allow $E[z_t] \neq 0$, then σ_t^2 is the conditional second moment of ξ_t. If we allow the second moment of z_t to be infinite or undefined, then σ_t^2 is a conditional scale parameter. Since the restrictions $E[z_t] = 0$, $E[z_t^2] = 1$ play no role in the main results of this paper, we adopt the less stringent condition (1), along with the requirement

$$E[\ln(\beta + \alpha z_t^2)] \text{ exists.} \tag{5}$$

Note that (5) does not require that $E[\ln(\beta + \alpha z_t^2)]$ be finite, only that the expectations of the positive and negative parts of $\ln(\beta + \alpha z_t^2)$ are not both infinite. For example, (5) holds trivially if $\beta > 0$.

Repeatedly substituting for σ_{t-i}^2 in (2), we have, for $t \geq 2$,

$$\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} (\beta + \alpha z_{t-i}^2) + \omega \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} (\beta + \alpha z_{t-i}^2) \right]. \tag{6}$$

Equation (6) holds for $t = 1$ as well, if we take a sum of the form $\sum_{k=1,0} X_k$ to equal zero. We can close the system in one of two ways: either by defining a probability measure μ_0 for the starting value σ_0^2 or by assuming that the system extends infinitely far into the past. To start the system at time 0, let (7)–(9) hold:

$$P[\sigma_0^2 \in \Gamma] = \mu_0(\Gamma) \qquad \text{for all } \Gamma \in B, \tag{7}$$

where B denotes the Borel sets on $[0,\infty)$,

$$\mu_0((0,\infty)) = 1, \tag{8}$$

that is, σ_0^2 is strictly positive and finite with probability one, and σ_0^2 and

$$\{z_t\}_{t=0,\infty} \text{ are independent.} \tag{9}$$

We call the model for $\{\sigma_t^2, \xi_t\}_{t=0,\infty}$ defined by (1)–(3), (5), and (6)–(9) the *conditional* model. We denote the probability measure for σ_t^2 by μ_t, and the measure for σ_t^2 given $\sigma_0^2 = x$ by $\mu_{t,x}$, that is, for any $\Gamma \in B$, $P[\sigma_t^2 \in \Gamma] = \mu_t(\Gamma)$ and $P[\sigma_t^2 \in \Gamma \mid \sigma_0^2 = x] = \mu_{t,x}(\Gamma)$.

To extend the process infinitely far into the past, we define the *unconditional* process $\{{}_u\sigma_t^2, {}_u\xi_t\}_{t=-\infty,\infty}$ by (1),

$${}_u\sigma_t^2 \equiv \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} (\beta + \alpha z_{t-i}^2) \right], \text{ and} \tag{10}$$

$${}_u\xi_t \equiv {}_u\sigma_t \cdot z_t. \tag{3$'$}$$

If $\omega = 0$, define ${}_u\sigma_t^2 \equiv {}_u\xi_t \equiv 0$ for all t. If ${}_u\sigma_t^2 = \infty$ and $z_t = 0$, define ${}_u\xi_t = \infty \cdot 0 \equiv 0$.

Without further restrictions, there is no guarantee that ${}_u\sigma_t^2$ and ${}_u\xi_t$ are finite. So for the time being, we define these processes on a subset of the extended real plane—that is, $0 \le {}_u\sigma_t^2 \le \infty$, and $-\infty \le {}_u\xi_t \le \infty$. If ${}_u\sigma_t^2$ has a well-defined probability measure on $[0,\infty)$, we call this measure μ_∞.

In the rest of this paper we address the following questions: when is μ_∞ well defined? When do $\mu_t \to \mu_\infty$ and $\mu_{t,x} \to \mu_\infty$ as $t \to \infty$?[1] What is the behavior of $\{\sigma_t^2, \xi_t\}_{t=0,\infty}$ when μ_∞ is not well defined? When does μ_∞ admit finite moments, that is, for any real number q, when is $\int \sigma^{2q}\, d\mu_\infty < \infty$, and when does $\int \sigma^{2q}\, d\mu_t \to \int \sigma^{2q}\, d\mu_\infty$ as $t \to \infty$? Under what circumstances do shocks to ${}_u\sigma_t^2$ and σ_t^2 (defined variously in terms of z_k, ${}_u\xi_k$ or ξ_k for some fixed k) decay as $t \to \infty$? Section 2 presents the results. All proofs are in the appendix.

2. MAIN RESULTS

Throughout the paper, time subscripts t are assumed nonnegative in the conditional model, but may be positive or negative in the unconditional model. All limits are taken as $t \to \infty$, except where otherwise indicated.

The first theorem considers the case $\omega = 0$. Since ${}_u\sigma_t^2 \equiv 0$ for all t in this case, only the conditional model is of interest.

THEOREM 1. *Let* $\omega = 0$. *Then*

$$\sigma_t^2 \to \infty \text{ a.s. } (\textit{almost surely}) \textit{ if and only if } (\textit{iff})\ E[\ln(\beta + \alpha z_t^2)] > 0, \quad \textbf{(11)}$$

$$\sigma_t^2 \to 0 \text{ a.s. } \textit{iff } E[\ln(\beta + \alpha z_t^2)] < 0. \quad \textbf{(12)}$$

If $E[\ln(\beta + \alpha z_t^2)] = 0$, $\ln(\sigma_t^2)$ *is a driftless random walk after time* 0, *with*

$$\limsup_{t\to\infty} \sigma_t^2 = \infty \textit{ and } \liminf_{t\to\infty} \sigma_t^2 = 0 \text{ a.s.} \quad \textbf{(13)}$$

The next theorem considers the more important case of $\omega > 0$.

THEOREM 2. *Let* $\omega > 0$. *If* $E[\ln(\beta + \alpha z_t^2)] \ge 0$, *then*

$$\sigma_t^2 \to \infty \text{ a.s. } \textit{and} \quad \textbf{(14)}$$

$${}_u\sigma_t^2 = \infty \text{ a.s. } \textit{for all } t. \quad \textbf{(15)}$$

If $E[\ln(\beta + \alpha z_t^2)] < 0$, *then* (16)–(20) *hold*:

$$\omega/(1-\beta) \le {}_u\sigma_t^2 < \infty \qquad \textit{for all } t \text{ a.s.}, \quad \textbf{(16)}$$

${}_u\sigma_t^2$ *is strictly stationary*[2] *and ergodic with a well-defined probability measure*

$$\mu_\infty \text{ on } [\omega/(1-\beta),\infty) \qquad \textit{for all } t, \quad \textbf{(17)}$$

$${}_u\sigma_t^2 - \sigma_t^2 \to 0 \text{ a.s.}, \quad \textbf{(18)}$$

$$\mu_t \to \mu_\infty, \textit{ and} \quad \textbf{(19)}$$

$$\mu_\infty \textit{ is nondegenerate.} \quad \textbf{(20)}$$

As an application of Theorems 1 and 2, consider the IGARCH(1,1) model of Engle and Bollerslev [6], in which $E[z_t^2] = 1$ and $\beta + \alpha = 1$. In this model, the conditional expectation of σ_{t+k}^2 $(k \geq 0)$ at time t is given by

$$E(\sigma_{t+k}^2 \mid \sigma_t^2) = \sigma_t^2 + \omega \cdot k. \tag{21}$$

When $\omega = 0$, σ_t^2 is a martingale. In the behavior of its conditional expectation, σ_t^2 with $\omega > 0$ and $\omega = 0$ is analogous to a random walk with and without drift, respectively. Its behavior in other respects is very different from that of a random walk: for example, Geweke [9] and Engle and Bollerslev [7] showed that the structure of the higher moments of σ_t^2 when $\alpha + \beta = 1$ and $\omega = 0$ implies that the distribution of σ_t^2 becomes more and more concentrated around zero with fatter and fatter tails, which is not the case with a random walk. Theorem 1 strengthens this result: by Jensen's inequality and the strict concavity of $\ln(x)$, we have $E[\ln(\beta + \alpha z_t^2)] < \ln(E[\beta + \alpha z_t^2]) = \ln(1) = 0$. In the IGARCH(1,1) model with no drift ($\omega = 0$), therefore, $\sigma_t^2 \to 0$ almost surely. In the IGARCH(1,1) model with $\omega > 0$, ${}_u\sigma_t^2$ is strictly stationary and ergodic, and $\sigma_t^2 \to {}_u\sigma_t^2$ almost surely. This illustrates the important point that the behavior of a martingale can differ very sharply from the behavior of a random walk.

For IGARCH(1,1) with $\omega = 0$, this result could have been obtained by the martingale convergence theorem ([4], Section 10.5). But (12) does not just apply to martingales. Consider, for example, the model

$$\sigma_t^2 = 3 \cdot z_{t-1}^2 \cdot \sigma_{t-1}^2, \tag{22}$$

where $z_t \sim N(0,1)$. In this model, $E[\sigma_t^2] = 3^t \cdot E[\sigma_0^2] \to \infty$ as $t \to \infty$. Nevertheless, $E[\ln(3 \cdot z_{t-1}^2)] < 0$ (see Theorem 6 and Figure 1 below), so $\sigma_t^2 \to 0$ a.s. Theorem 1, therefore, applies not merely when σ_t^2 is a martingale, but often applies when the expectation of σ_t^2 diverges to infinity. If, instead of (22), we have

$${}_u\sigma_t^2 = \omega + 3 \cdot z_{t-1}^2 \cdot {}_u\sigma_{t-1}^2, \tag{23}$$

for $\omega > 0$ and $z_t \sim N(0,1)$, then the results of Theorem 2 hold, and ${}_u\sigma_t^2$ is strictly stationary.

Next we consider the moments of ${}_u\sigma_t^2$ and σ_t^2. Bollerslev [2] evaluates the integer moments of σ_t^2, but the finiteness of the noninteger moments is also of interest. For example, $E[\xi_t^j] = E[z_t^j] \cdot E[(\sigma_t^2)^{j/2}]$, that is, the odd integer moments of ξ_t involve fractional moments of σ_t^2. In addition, many limit theorems (see, e.g., [22]) impose moment conditions of the form $E[|Y_t|^{j+\delta}] < \infty$ for $j = 1$ or 2 and $\delta > 0$. To apply these theorems as broadly as possible requires that δ be as small as possible. The next theorem puts upper and lower bounds on the moments of σ_t^2 and ${}_u\sigma_t^2$. Necessary and sufficient conditions for finiteness follow as a corollary.

THEOREM 3. *Define* $\zeta \equiv E[(\beta + \alpha z_t^2)^p]$. *When* $0 < p \leq 1$,

$$\left[(E(\sigma_0^{2p}))^{1/p}\zeta^{t/p} + \omega \sum_{k=0}^{t-1} \zeta^{k/p}\right]^p \leq E[\sigma_t^{2p}] \leq E(\sigma_0^{2p})\zeta^t + \omega^p \sum_{k=0}^{t-1} \zeta^k, \tag{24}$$

and

$$\omega^p\left[\sum_{k=0}^{\infty} \zeta^{k/p}\right]^p \leq E[{}_u\sigma_t^{2p}] \leq \omega^p \sum_{k=0}^{\infty} \zeta^k. \tag{25}$$

When $1 \leq p$, *the inequalities in* (24) *and* (25) *are reversed.*

COROLLARY. *Let* $\omega > 0$, $p > 0$, *and* $E[\ln(\beta + \alpha z_t^2)] < 0$.

$$E[\sigma_t^{-2p}] < \infty, \qquad t \geq 1, \qquad \textit{and} \tag{26}$$

$$E[{}_u\sigma_t^{-2p}] < \infty \qquad \textit{for all } t. \tag{27}$$

$$E[\sigma_t^{2p}] < \infty \qquad \textit{iff } E[\sigma_0^{2p}] < \infty \quad \textit{and} \quad E[(\beta + \alpha z_t^2)^p] < \infty. \tag{28}$$

$$E[{}_u\sigma_t^{2p}] < \infty \qquad \textit{iff } E[(\beta + \alpha z_t^2)^p] < 1. \tag{29}$$

$$\limsup_{t\to\infty} E[\sigma_t^{2p}] < \infty \qquad \textit{iff } E[\sigma_0^{2p}] < \infty \quad \textit{and} \quad E[(\beta + \alpha z_t^2)^p] < 1. \tag{30}$$

If $E[\sigma_0^{2p}] < \infty$, *then*

$$\lim_{t\to\infty} E[\sigma_t^{2p}] = E[{}_u\sigma_t^{2p}]. \tag{31}$$

The following theorem is useful in verifying the existence of a finite moment of order $j + \delta$, $\delta > 0$:

THEOREM 4. *(a) Let* $\omega > 0$ *and* $E[\ln(\beta + \alpha z_t^2)] < 0$. *If* $E[|z_t|^{2q}] < \infty$ *for some* $q > 0$, *then there exists a* p, $0 < p < q$, *such that* $E[(\beta + \alpha z_t^2)^p] < 1$.
(*b*) *If, in addition,* $E[(\beta + \alpha z_t^2)^r] < 1$ *for* $0 < r < q$, *then there exists a* $\delta > 0$ *such that* $E[(\beta + \alpha z_t^2)^{r+\delta}] < 1$.

Theorem 4(a) says, in essence, that if ${}_u\sigma_t^2$ is strictly stationary and z_t^2 has a finite moment of some (arbitrarily small, possibly fractional) order, then ${}_u\sigma_t^2$ has a finite (possibly fractional) moment as well. The existence of such a finite fractional moment implies, for example, that $E[\ln({}_u\sigma_t^2)] < \infty$ (see [23], Formula 4.1.30).

Part (b) gives a condition for $E[{}_u\sigma_t^{2(p+\delta)}] < \infty$ for some $\delta > 0$, given that $E[{}_u\sigma_t^{2p}] < \infty$. It says, for example, that if $E[(\beta + \alpha z_t^2)^{1/2}] < 1$ and $E[|z_t|^{2p}] < \infty$ for some $p > \frac{1}{2}$, then not only is $E[|{}_u\xi_t|] < \infty$, but there is also a $\delta > 0$ with $E[|{}_u\xi_t|^{1+\delta}] < \infty$.

In many applications, GARCH processes are used to model the persistence of shocks to the conditional variance (see, e.g., [6]). For example, an increase in the conditional variance of stock market returns may cause the market risk premium to rise. However, if the increase is expected to be short-lived, the term structure of market risk premia may move only at the short end, and the valuation of long-lived assets may be affected only slightly [15]. Since, in the GARCH(1,1) model, the changes in σ_t^2 or ${}_u\sigma_t^2$ are driven by past realizations of $\{z_t\}$ (or, equivalently, of $\{\xi_t\}$), it seems desirable to form sensible definitions of "persistence" and develop criteria for the persistence of the effects of ξ_t, ${}_u\xi_t$, or z_t on σ_{t+m}^2 and ${}_u\sigma_{t+m}^2$ for large m.

To define the persistence of z_t in σ_{t+m}^2 and ${}_u\sigma_{t+m}^2$, write the latter two as

$$\begin{aligned}\sigma_{t+m}^2 &= \left[\sigma_0^2 \prod_{i=1}^{t+m} (\beta + \alpha z_{t+m-i}^2) + \omega\left[\sum_{k=m}^{t+m-1} \prod_{i=1}^{k} (\beta + \alpha z_{t+m-i}^2)\right]\right] \\ &\quad + \omega\left[1 + \sum_{k=1}^{m-1} \prod_{i=1}^{k} (\beta + \alpha z_{t+m-i}^2)\right] \\ &\equiv A(\sigma_0^2, z_{t+m-1}, \ldots, z_1, z_0) + B(z_{t+m-1}, \ldots, z_{t+1}) \end{aligned} \tag{32}$$

and

$$\begin{aligned}{}_u\sigma_{t+m}^2 &= \omega\left[\sum_{k=m}^{\infty} \prod_{i=1}^{k} (\beta + \alpha z_{t+m-i}^2)\right] + \omega\left[1 + \sum_{k=1}^{m-1} \prod_{i=1}^{k} (\beta + \alpha z_{t+m-i}^2)\right] \\ &\equiv C(z_{t+m-1}, \ldots, z_1, z_0, z_{-1}, \ldots) + B(z_{t+m-1}, \ldots, z_{t+1}). \end{aligned} \tag{33}$$

By construction, B is independent of z_t, while A and C are not. B, C, and A are all nonnegative a.s.

DEFINITION. *We say that z_t is persistent in σ^2 almost surely unless*

$$A(\sigma_0^2, z_{t+m-1}, \ldots, z_1, z_0) \to 0 \text{ a.s.} \qquad \textit{as } m \to \infty, \tag{34}$$

z_t is persistent in σ^2 in probability unless, for all $b > 0$,

$$P[A(\sigma_0^2, z_{t+m-1}, \ldots, z_1, z_0) > b] \to 0 \qquad \textit{as } m \to \infty, \tag{35}$$

and z_t is persistent in σ^2 in L^p, $p > 0$, unless

$$E[[A(\sigma_0^2, z_{t+m-1}, \ldots, z_1, z_0)]^p] \to 0 \qquad \textit{as } m \to \infty. \tag{36}$$

Similarly, z_t is persistent in ${}_u\sigma^2$ almost surely unless

$$C(z_{t+m-1}, \ldots, z_1, z_0, z_{-1}, \ldots) \to 0 \text{ a.s.} \qquad \textit{as } \mathrm{m} \to \infty, \tag{37}$$

z_t is persistent in ${}_u\sigma^2$ in probability unless, for all $b > 0$,

$$P[C(z_{t+m-1}, \ldots, z_1, z_0, z_{-1}, \ldots) > b] \to 0 \qquad \textit{as } m \to \infty, \tag{38}$$

and z_t is persistent in ${}_u\sigma^2$ in L^p, $p > 0$, unless

$$E[[C(z_{t+m-1}, \ldots, z_1, z_0, z_{-1}, \ldots)]^p] \to 0 \qquad \textit{as } m \to \infty. \tag{39}$$

Alternatively, we can write σ_{t+m}^2 and ${}_u\sigma_{t+m}^2$ as functions of past values of $\{\xi_t\}$ and $\{{}_u\xi_t\}$ and define the persistence of $\{\xi_t\}$ and $\{{}_u\xi_t\}$ in σ^2 and ${}_u\sigma^2$:

$$\sigma_{t+m}^2 = \alpha\cdot\beta^{m-1}\cdot\xi_t^2 + \left[\beta^{m-1}\cdot\omega + \beta^{t+m}\sigma_0^2 + \sum_{i=1,i\neq m}^{t+m}\beta^{i-1}[\alpha\cdot\xi_{t+m-i}^2 + \omega]\right], \tag{40}$$

$${}_u\sigma_{t+m}^2 = \alpha\cdot\beta^{m-1}\cdot{}_u\xi_t^2 + \left[\beta^{m-1}\cdot\omega + \sum_{i=1,i\neq m}^{\infty}\beta^{i-1}[\alpha\cdot{}_u\xi_{t+m-i}^2 + \omega]\right], \tag{41}$$

(40)–(41) break σ_{t+m}^2 and ${}_u\sigma_{t+m}^2$ into two pieces, the first of which depends directly on ξ_t or ${}_u\xi_t$ and the second does not. While $B(z_{t+m-1},\ldots,z_{t+1})$ is independent of z_t, it is easy to verify that the bracketed terms in (40)–(41) are not independent of ξ_t and ${}_u\xi_t$.

DEFINITION. *We say that ξ_t is almost surely persistent unless*

$$\beta^{m-1}\cdot\xi_t^2 \to 0 \text{ a.s.} \qquad \text{as } m\to\infty, \tag{42}$$

ξ_t is persistent in probability unless for every $b > 0$,

$$P[\beta^{m-1}\cdot\xi_t^2 > b] \to 0 \qquad \text{as } m\to\infty, \text{ and} \tag{43}$$

ξ_t is persistent in L^p, $p > 0$, unless

$$E[[\beta^{m-1}\cdot\xi_t^2]^p] \to 0 \qquad \text{as } m\to\infty. \tag{44}$$

We define persistence of ${}_u\xi_t$ similarly, replacing ξ_t with ${}_u\xi_t$ in (42)–(44). *If $\beta = 0$, $m > 1$, and $|{}_u\xi_t| = \infty$, we define $\beta^{m-1}\cdot{}_u\xi_t^2 \equiv 0$.*

THEOREM 5. *z_t is persistent in σ^2, both almost surely and in probability*

$$\text{iff } E[\ln(\beta + \alpha z_t^2)] \geq 0. \tag{45}$$

z_t is persistent in ${}_u\sigma^2$, both almost surely and in probability, iff $\omega > 0$ and

$$E[\ln(\beta + \alpha z_t^2)] \geq 0. \tag{46}$$

z_t is persistent in L^p in σ^2 iff either $E[(\beta + \alpha z_t^2)^p] \geq 1$ or

$$E[\sigma_0^{2p}] = \infty \text{ (or both)}. \tag{47}$$

z_t is persistent in L^p in ${}_u\sigma^2$ iff $E[(\beta + \alpha z_t^2)^p] \geq 1$. **(48)**

ξ_t is persistent, both almost surely and in probability, iff $\beta \geq 1$. **(49)**

${}_u\xi_t$ is persistent, both almost surely and in probability, iff $\beta > 0$, $\omega > 0$, and

$$E[\ln(\beta + \alpha z_t^2)] \geq 0. \tag{50}$$

ξ_t is persistent in L^p iff $\beta > 0$ and either $E[(\beta + \alpha z_t^2)^p] = \infty$, $\beta \geq 1$, or

$$E[\sigma_0^{2p}] = \infty. \tag{51}$$

$$_u\xi_t \textit{ is persistent in } L^p \textit{ iff } \omega > 0,\ \beta > 0, \textit{ and } E[(\beta + \alpha z_t^2)^p] \geq 1. \tag{52}$$

From Theorem 5, it is clear that whether a shock "persists" or not depends very much on our definition of persistence. For example, suppose $\omega > 0$, $E[|z_t|^q] < \infty$ for some $q > 0$, and $E[\ln(\beta + \alpha z_t^2)] < 0$. Then by Theorem 4, $E[(\beta + \alpha z_t^2)^p] < 1$ for some $p > 0$, that is, there are L^p norms in which shocks to $_u\sigma_t^2$ (i.e., z_t and $_u\xi_t$) do not persist. If, however, it is also the case that $P[(\beta + \alpha z_t^2) > 1] > 0$, then by [11] Theorem 193, there exists an $r > 0$ such that $E[(\beta + \alpha z_t^2)^r] > 1$, that is, shocks to $_u\sigma_t^2$ persist in some L^p norms (for all sufficiently large p) but not in others. For example, in the IGARCH model with $\omega > 0$ and $\beta > 0$, $_u\xi_t$ is persistent (and z_t is persistent in $_u\sigma^2$) in L^p, $p \geq 1$, but not almost surely, in probability, or in L^p, $0 < p < 1$.

"Persistence" of shocks also differs in the conditional and unconditional models. (49)–(50) illustrate this: Suppose that $E[\ln(\beta + \alpha z_t^2)] \geq 0$, $0 < \beta < 1$, and $\omega > 0$. In this case, $_u\sigma_t^2 = \infty$ and $\sigma_t^2 \to \infty$ a.s., so that $P[\beta^{m-1} \cdot {_u\xi_t^2} = \infty] > 0$ for all m and t, and $_u\xi_t$ is therefore persistent almost surely. Nevertheless, for any fixed t, ξ_t is almost surely not persistent, since the individual ξ_t terms are finite with probability one and $\beta^{m-1} \cdot \xi_t^2 \to 0$ a.s. as $m \to \infty$. In other words, for fixed t, the effect of an individual ξ_t on σ_{t+m}^2 dies out a.s. as $m \to \infty$, but at a progressively slower rate as $t \to \infty$. This is why it is necessary to define persistence for each t rather than uniformly over all t.

When $E[z_t^2] = 1$, persistence of z_t in L^1 in $_u\sigma_t^2$ corresponds to persistence as defined by Engle and Bollerslev in [6]. In this case, we have

$$E[\sigma_{t+m}^2 | \sigma_t^2] = \sigma_t^2(\beta + \alpha)^m + \omega \cdot \left[\sum_{k=0}^{m-1} (\beta + \alpha)^k\right]. \tag{53}$$

Engle and Bollerslev defined shocks to σ_t^2 to be persistent unless the $\sigma_t^2(\beta + \alpha)^m$ term vanishes as $m \to \infty$, which it does if and only if $(\beta + \alpha) < 1$. Note that according to this definition, when $\beta + \alpha \geq 1$, shocks accumulate (and "persist") not necessarily in the sense that $\sigma_{t+m}^2 \to \infty$ as $m \to \infty$, but rather in the sense that $E[\sigma_{t+m}^2 | \sigma_t^2] \to \infty$, so that if $\omega > 0$, $E[_u\sigma_t^2] = \infty$. Our definitions of persistence in L^p all have similar interpretations: persistence of z_t in L^p in $_u\sigma_t^2$ means that shocks accumulate (and persist) in the sense that $E[\sigma_{t+m}^{2p} | \sigma_t^2] \to \infty$ as $m \to \infty$, implying (when $\omega > 0$) that $E[_u\sigma_t^{2p}] = \infty$. Persistence in L^p therefore corresponds to persistence of shocks in the forecast *moments* of σ_t^2 and $_u\sigma_t^2$, while almost sure persistence corresponds to persistence of shocks in the forecast *distributions* of σ_t^2 and $_u\sigma_t^2$.

In Theorems 1–5, two moment conditions repeatedly arise, namely, $E[\ln(\beta + \alpha z_t^2)] < 0$ and $E[(\beta + \alpha z_t^2)^p] < 1$. In a number of cases of interest, expressions are available for these expectations in terms of widely tabulated functions. The next theorem evaluates these expectations when z_t is either standard normal or Cauchy. Approximations and algorithms for evaluating the functions in Theorem 6 are found in [1], [13], and [20].

THEOREM 6. *If* $z \sim N(0,1)$, *and* $\beta > 0$, *then*

$$E[\ln(\beta + \alpha z^2)] = \ln(2\alpha) + \psi\left(\frac{1}{2}\right) + (2\pi\beta/\alpha)^{1/2}\Phi\left(\frac{1}{2}, 1.5; \beta/2\alpha\right) - (\beta/\alpha)\,{}_2F_2(1,1;2,1.5;\beta/2\alpha), \tag{54}$$

and

$$E[(\beta + \alpha z^2)^p] = (2\alpha)^{-1/2}\beta^{p+1/2}\Psi\left(\frac{1}{2}, p + 1.5; \beta/2\alpha\right), \tag{55}$$

where $\Phi(\cdot,\cdot,\cdot)$ *is a confluent hypergeometric function* ([13], *Section* 9.9), ${}_2F_2(\cdot,\cdot;\cdot,\cdot;\cdot)$ *is a generalized hypergeometric function* ([13], *Section* 9.14), $\Psi(\cdot,\cdot;\cdot)$ *is a confluent hypergeometric function of the second kind* ([13] *Section* 9.10), *and* $\psi(\cdot)$ *is the Euler psi function, with* $\psi(\frac{1}{2}) \approx -1.96351$ ([3]). *Let* $r(\cdot)$ *be the gamma function. When* $\beta = 0$,

$$E[\ln(\alpha z^2)] = \ln(2\alpha) + \psi\left(\frac{1}{2}\right). \tag{56}$$

$$E[(\alpha z^2)^p] = \pi^{-1/2}(2\alpha)^p\Gamma\left(p + \frac{1}{2}\right). \tag{57}$$

If $z \sim$ *standard Cauchy* (*i.e., the probability density of* z *is* $f(z) = [\pi(1 + z^2)]^{-1}$)), *then*

$$E[\ln(\beta + \alpha z^2)] = 2\cdot\ln(\beta^{1/2} + \alpha^{1/2}), \text{ and} \tag{58}$$

$$E[(\beta + \alpha z^2)^p] = \begin{cases} \dfrac{F[-p,\frac{1}{2};\, 1-p;\, 1-(\alpha/\beta)]\cdot\beta^p\cdot\Gamma(\frac{1}{2}-p)}{\pi^{1/2}\cdot\Gamma(1-p)} & p < \dfrac{1}{2}, \\ \infty & p \geq \dfrac{1}{2} \end{cases} \tag{59}$$

where $F(\cdot,\cdot;\cdot;\cdot)$ *is a hypergeometric function* ([13] *Section* 9.1). *When* $p < \frac{1}{2}$ *and* $\beta = 0$, (59) *simplifies to*

$$E[(\alpha z^2)^p] = \alpha^p/\sin\left[\left(p + \frac{1}{2}\right)\pi\right]. \tag{60}$$

Figure 1 summarizes some of the information from Theorem 6 for the case in which $z_t \sim N(0,1)$.[3] Assume that $\omega > 0$. In region 1 and on the boundary between regions 1 and 2, ${}_u\sigma_t^2 = \infty$ a.s. and σ_t^2 is explosively nonstationary. In regions 2, 3, and 4, ${}_u\sigma_t^2$ and ${}_u\xi_t$ are strictly stationary and ergodic. In region 2 and on the boundary between regions 2 and 3, however, $E[{}_u\xi_t]$

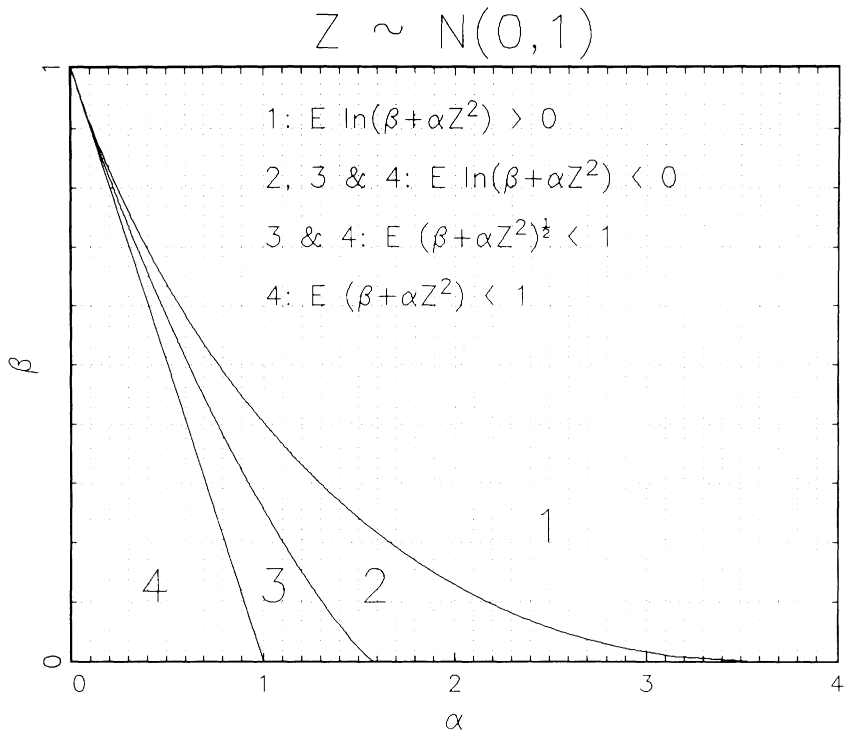

FIGURE 1.

is not defined. In region 3, and on the boundary between regions 3 and 4, $E[{}_u\xi_t] = 0$ but $E[{}_u\xi_t^2] = \infty$. Region 4 contains the classical, covariance stationary GARCH(1,1) models with $E[{}_u\xi_t^2] < \infty$, while in regions 2 and 3, ${}_u\xi_t$ is strictly stationary but not weakly (i.e., covariance) stationary.[4]

Figure 2 is the counterpart of Figure 1 for $z_t \sim$ Cauchy. In this figure there is no counterpart to regions 3 and 4, since $E[{}_u\xi_t]$ is ill-defined and $E[{}_u\xi_t^2] = \infty$ for all parameter values as long as $\omega > 0$. In region 2, however, ${}_u\sigma_t^2$ and ${}_u\xi_t$ are strictly stationary and ergodic. Although ${}_u\sigma_t^2$ has no integer moments, Theorem 4(b) implies that for each α, β combination in region 2 there exists some $q > 0$ with $E[{}_u\sigma_t^{2q}] < \infty$, that is, ${}_u\sigma_t^2$ has *some* finite positive moment.

As we have seen, in the IGARCH(1,1) process, ${}_u\xi_t$ is strictly stationary but not covariance stationary, since $E[{}_u\xi_t^2] = \infty$. Since IGARCH(1,1) has been employed quite often in empirical work (e.g., in [6]), it is of interest to see what the distribution of ${}_u\sigma_t^2$ looks like for this model. Figure 3 plots density estimates for $\ln({}_u\sigma_t^2)$ generated by five IGARCH(1,1) models, with

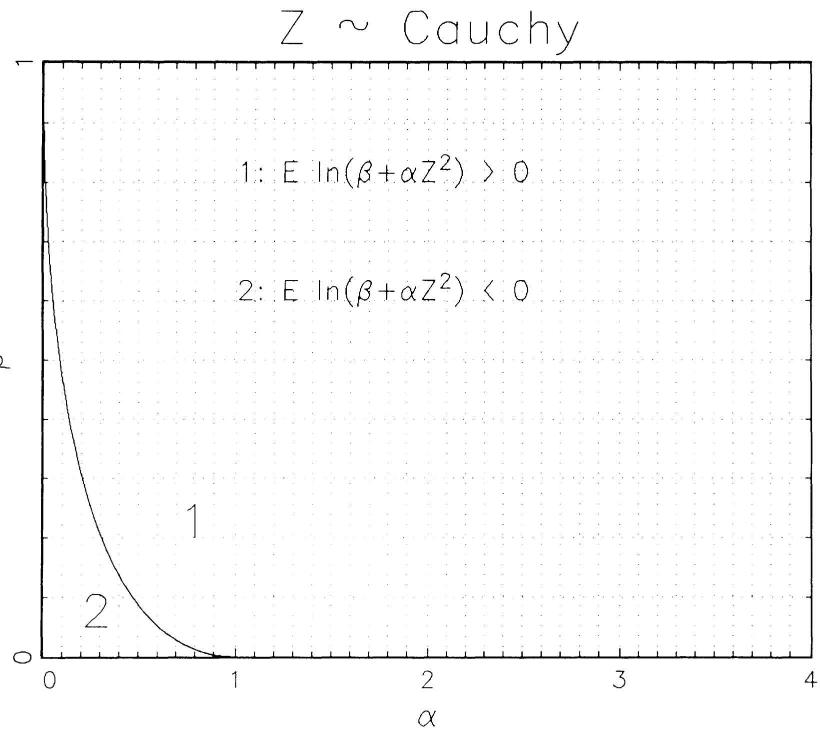

FIGURE 2.

α set equal to 1, 0.75, 0.5, 0.25, and 0.1, respectively. ${}_u\sigma_t^2$ was simulated by σ_T^2 for a suitably large value of T, using the recursion

$$\sigma_t^2 = \omega + (1 - \alpha)\sigma_{t-1}^2 + \alpha\xi_{t-1}^2, \qquad \sigma_0^2 = 1 \tag{61}$$

with $z_t \sim$ i.i.d. $N(0,1)$. ω was set equal to α, giving ${}_u\sigma_t^2$ support on $(1,\infty)$ and $\ln({}_u\sigma_t^2)$ support on $(0,\infty)$. The technical details on how the density estimates were obtained are in the appendix.

What is most striking about Figure 3 is the extreme values that ${}_u\sigma_t^2$ frequently takes. Since ω is only a scale parameter, changing its value would shift the densities of $\ln({}_u\sigma_t^2)$ to the right or left, but would leave the shape of the densities unchanged. Note also that the lower α is, the more strongly skewed to the right the density is. This is not surprising, since the larger α is, the more influence a single draw of z_t^2 has in making σ_{t+1}^2 small or large. Since $z_t^2 \sim \chi^2$ with 1 degree of freedom, its density is infinite at $z = 0$ and is strongly skewed to the right. When $\alpha = 1$, this lends a similar shape to the distribution of $\ln({}_u\sigma_t^2)$.

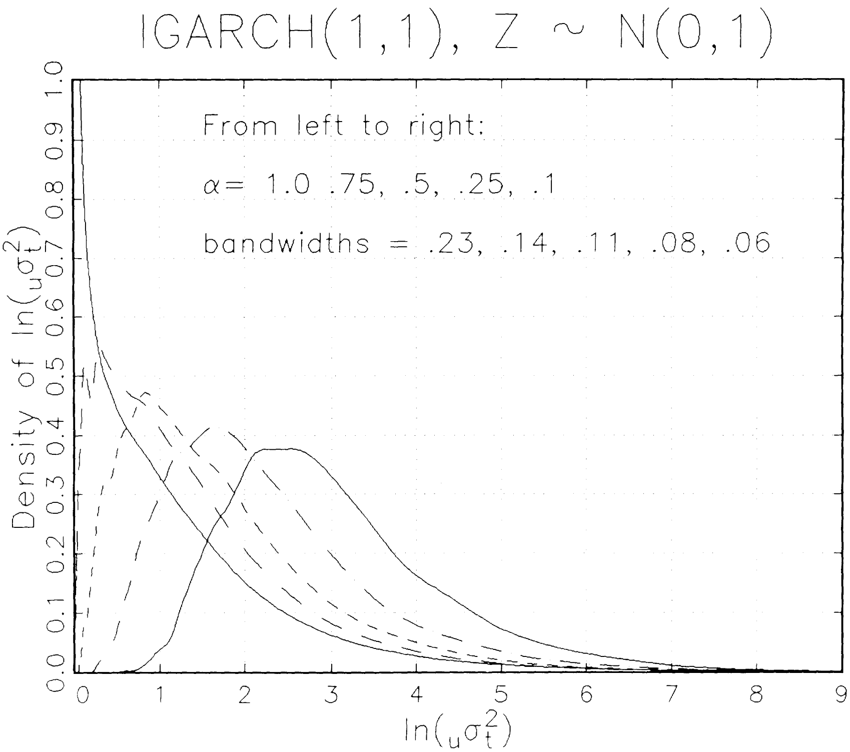

FIGURE 3.

NOTES

1. The convergence is taken to be weak convergence of measures, that is, $\mu_t \to \mu_\infty$ and $\mu_{t,x} \to \mu_\infty$ mean that for any bounded, continuous function f from $[0,\infty)$ into R^1, $\int f\,d\mu_t \to \int f\,d\mu_\infty$ and $\int f\,d\mu_{t,x} \to \int f\,d\mu_\infty$, respectively, as $t \to \infty$.
2. Sampson [18] independently derived the stationarity condition $E[\ln(\beta + \alpha z_t^2)] < 0$.
3. The graphs were produced using GAUSS 2.0 [8].
4. A related paper [14] derives the distribution of ${}_u\sigma_t^2$ as a continuous time limit is approached.
5. A similar argument in a manuscript version of [4] inspired the proof of this theorem.
6. This is one of those rare instances in which the distinction between convergence in probability and almost sure convergence matters, since a weak law of large numbers would not suffice in this proof.

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APPENDIX

Proof of Theorem 1. In the conditional model with $\omega = 0$, we have $\ln(\sigma_t^2) = \ln(\sigma_0^2) + \sum_{i=0,t-1} \ln(\beta + \alpha z_i^2)$, that is, $\ln(\sigma_t^2)$ is a random walk. Its drift is positive, negative, or zero as $E[\ln(\beta + \alpha z_t^2)]$ is positive, negative, or zero. By (8), $|\ln(\sigma_0^2)| < \infty$ a.s. By a direct application of the strong law of large numbers for i.i.d. random variables, $\ln(\sigma_t^2) \to \infty$ if $E[\ln(\beta + \alpha z_t^2)] > 0$ and $\ln(\sigma_t^2) \to -\infty$ if $E[\ln(\beta + \alpha z_t^2)] < 0$ ([21], Chapter 6). By [21] Theorem 6.1.4 and Corollary 6.1.1, when $E[\ln(\beta + \alpha z_t^2)] = 0$ and z_t^2 is nondegenerate (assumed in (1)), $\limsup \ln(\sigma_t^2) = \infty$ and $\liminf \ln(\sigma_t^2) = -\infty$ with probability one.[5] ■

Proof of Theorem 2. (6) implies that

$$\sigma_t^2 \geq \omega\left[\sup_{1 \leq k \leq t-1} \prod_{i=1}^{k} (\beta + \alpha z_{t-i}^2)\right]. \tag{A.1}$$

By Theorem 1, the term on the right diverges to ∞ as $t \to \infty$ if $E[\ln(\beta + \alpha z_t^2)] \geq 0$, proving (14). A similar argument proves (15).

The lower inequality in (16) follows trivially from (10). To prove the upper inequality, we will show that the terms in the sum

$$\sum_{k=1}^{\infty} \prod_{i=1}^{k} (\beta + \alpha z_{t-i}^2)$$

are $O(\exp(-\lambda k))$ for some $\lambda > 0$ with probability one. Define $\gamma \equiv E[\ln(\beta + \alpha z_t^2)] < 0$. By the strong law of large numbers for i.i.d. random variables ([4], Theorem 8.3.5) with probability one there exists a random positive integer $M < \infty$ such that, for all $k > M$,

$$\left| k^{-1}\sum_{i=1}^{k} \ln(\beta + \alpha z_{t-i}^2) - \gamma \right| < |\gamma|/2 \tag{A.2}$$

so, for all $k > M$,

$$\sum_{i=1}^{k} \ln(\beta + \alpha z_{t-i}^2) < \gamma \cdot k/2, \text{ and} \tag{A.3}$$

$$\prod_{i=1}^{k} (\beta + \alpha z_{t-i}^2) < \exp(\gamma \cdot k/2), \tag{A.4}$$

establishing that, with probability one,

$$\prod_{i=1}^{k} (\beta + \alpha z_{t-i}^2) = O(\exp(-\lambda k))$$

with $\lambda \equiv |\gamma|/2 > 0$, so that the series converges a.s., concluding the proof of (16).[6] We next prove (17). Consider the function ${}_u\sigma_t^2 \equiv {}_u\sigma^2(z_{t-1}, z_{t-2}, \ldots)$ from R^∞ into the extended real line $[-\infty,\infty]$ given in (10). By [21] Theorem 3.5.8, (17) follows if we can show that the function ${}_u\sigma^2(z_{t-1}, z_{t-2}, \ldots)$ is measurable. First consider the sequence of functions given by $f_k = \prod_{i=1,k}(\beta + \alpha z_{t-i}^2)$. Since the product (or sum) of a finite number of measurable functions is measurable ([17], Chapter 3, Proposition 5), f_k is measurable for any finite k. By the same reasoning, the functions $r_n \equiv \sum_{k=1,n}\prod_{i=1,k}(\beta + \alpha z_{t-i}^2)$ are measurable for any finite n. Since r_n is increasing in n, we have $\sup_{\{n\}} r_n = r_\infty \equiv \sum_{k=1,\infty}\prod_{i=1,k}(\beta + \alpha z_{t-i}^2) = {}_u\sigma^2(z_{t-1}, z_{t-2}, \ldots)$ which is measurable by [17], Chapter 3, Theorem 20. (17) therefore holds.

Next we prove (18), after which (19) follows by [4], Proposition 9.3.5. We require that

$$\sigma_t^2 - {}_u\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} (\beta + \alpha z_{t-i}^2) - \omega\left[\sum_{k=t}^{\infty} \prod_{i=1}^{k} (\beta + \alpha z_{t-i}^2)\right] \to 0 \tag{A.5}$$

with probability one. The first term vanishes a.s. by Theorem 1: that is, the log of the first term is a random walk with negative drift and therefore diverges to $-\infty$ a.s. The second term also vanishes a.s., since the individual terms in the summation are $O(\exp(-\lambda k))$ a.s. for some $\lambda > 0$.

Finally, suppose that the distribution of ${}_u\sigma_t^2$ is degenerate. Then by (2) and (3)′ we have, with probability one for some constant v and for all t,

$$P[v = \omega + \beta \cdot v + \alpha \cdot v \cdot z_t^2] = 1, \tag{A.6}$$

a contradiction, since $\alpha > 0$, and, by (1), z_t^2 is nondegenerate. ■

Proof of Theorem 3. The left-hand inequalities in (24)–(25) follow by Minkowski's integral inequality ([11], Theorem 198, p. 146), and the right-hand inequalities follow from a companion to Minkowski's inequality, [11], Theorem 199, p. 147. In [11], these inequalities are originally stated using Riemann integrals, but hold for Stieltjes integrals as well—see [11], Section 6.17. They hold in (24)–(25), since the Lebesgue integral $\int \sigma^{2q}\, d\mu_t$ is equivalent to the Stieltjes integral $\int \sigma^{2q}\, dF_t(\sigma^2)$, where the cumulative distribution function $F_t(x) \equiv \mu_t((-\infty, x])$ ([12], Sections 36.1 and 36.2). This holds for μ_∞ and $F_\infty(\sigma^2)$ by the same argument. ■

Proof of the Corollary. (26)–(27) are immediate, since σ_t^2 and ${}_u\sigma_t^2$ are bounded below by ω and $\omega/(1-\beta)$, respectively. (28)–(30) follow directly from (24) and (25). To verify (31) in the case $E[(\beta + \alpha z_t^2)^p] < 1$, define the usual L^p norm by $\|X\|_p \equiv E[|X|^p]$ for $0 < p \le 1$ and $\|X\|_p \equiv E^{1/p}[|X|^p]$ for $p > 1$. By the triangle inequality,

$$\|\sigma_t^2 - {}_u\sigma_t^2\|_p + \|{}_u\sigma_t^2 - 0\|_p \ge \|\sigma_t^2 - 0\|_p, \quad \text{and} \tag{A.7}$$

$$\|{}_u\sigma_t^2 - \sigma_t^2\|_p + \|\sigma_t^2 - 0\|_p \ge \|{}_u\sigma_t^2 - 0\|_p. \tag{A.8}$$

Together, (A.7) and (A.8) imply

$$|\, \|{}_u\sigma_t^2\|_p - \|\sigma_t^2\|_p| \le \|\sigma_t^2 - {}_u\sigma_t^2\|_p, \quad \text{so} \tag{A.9}$$

$$\|\sigma_t^2\|_p - \|{}_u\sigma_t^2\|_p \to 0 \quad \text{and} \quad E[\sigma_t^{2p}] - E[{}_u\sigma_t^{2p}] \to 0 \quad \text{if } E|{}_u\sigma_t^2 - \sigma_t^2|^p \to 0. \tag{A.10}$$

Again, define $\zeta \equiv E[(\beta + \alpha z_t^2)^p]$. When $0 < p \le 1$, [11], Theorem 199 (again with the generalization in [11], Section 6.17) yields

$$E|{}_u\sigma_t^2 - \sigma_t^2|^p \le E(\sigma_0^{2p})\zeta^t + \omega^p \sum_{k=t}^{\infty} \zeta^k, \tag{A.11}$$

and when $p \ge 1$, [11], Theorem 198 yields

$$E|{}_u\sigma_t^2 - \sigma_t^2|^p \le \left[E(\sigma_0^{2p})^{1/p}\zeta^{t/p} + \omega \sum_{k=t}^{\infty} \zeta^{k/p}\right]^p. \tag{A.12}$$

Both bounds collapse to zero if $\zeta < 1$ and $E(\sigma_0^{2p}) < \infty$, proving (31) when $\zeta < 1$. The case $\zeta \ge 1$ is immediate by Theorem 3, since $E[{}_u\sigma_t^{2p}] = \infty$ and $E[\sigma_t^{2p}] \to \infty$. ■

Proof of Theorem 4. By [11], Theorem 194 and the generalization in Section 6.17, if $E^{1/q}[(\beta + \alpha z^2)^q] < \infty$, then $E^{1/p}[(\beta + \alpha z^2)^p]$ is a continuous function of p for

$0 < p < q$. By [11], Theorems 198, 199, and their generalizations in Section 6.17, $E[(\beta + \alpha z^2)^q] < \infty$ if and only if $E[|z|^{2q}] < \infty$. (b) follows immediately. (a) then follows by [11], Theorem 187 and Section 6.17. ∎

Proof of Theorem 5. First consider the proofs of (45) and (46). The case $\omega = 0$ is obvious, so let us take $\omega > 0$. By the reasoning in the proofs of Theorems 1 and 2, if $E[\ln(\beta + \alpha z_t^2)] < 0$, then $\prod_{i=1,k}(\beta + \alpha z_{t+m-i}^2)] = O(\exp(-\lambda k))$ for some $\lambda > 0$, a.s., guaranteeing that A and C vanish almost surely and in probability as $m \to \infty$. Also by the reasoning of Theorems 1 and 2, A and C diverge a.s. if $E[\ln(\beta + \alpha z_t^2)] \geq 0$.

To prove (48), again define $\zeta \equiv E[(\beta + \alpha z_t^2)^p]$. When $0 < p \leq 1$, we have, by the reasoning in the proof of Theorem 3,

$$\omega^p \left[\sum_{k=m,\infty} \zeta^{k/p} \right]^p \leq E[C^p] \leq \omega^p \left[\sum_{k=m,\infty} \zeta^k \right] \tag{A.13}$$

with the inequalities reversed if $1 \leq p$. As $m \to \infty$, both sides collapse to zero if $\zeta < 1$ and diverge to infinity if $\zeta \geq 1$, proving (48). The proof of (47) is similar, and is left to the reader.

By (1) and (6)–(8) we have, for any t, $|\xi_t| < \infty$ a.s. and $P[\xi_t \neq 0] > 0$. (49) follows directly.

Nonpersistence of ${}_u\xi_t$ is obvious if $\beta = 0$ or $\omega = 0$. Suppose $\beta > 0$ and $\omega > 0$ but $E[\ln(\beta + \alpha z_t^2)] < 0$ (which in turn implies $\beta < 1$). Then by Theorem 2, $|{}_u\xi_t| < \infty$ a.s., so that $\beta^{m-1}{}_u\xi_t^2 \to 0$ a.s. as $m \to \infty$, proving sufficiency in (50). Suppose $\beta > 0$ and $\omega > 0$ but $E[\ln(\beta + \alpha z_t^2)] \geq 0$. In the proof of Theorem 1, we saw that $\sum_{i=1,k} \ln(\beta + \alpha z_{t-i}^2)$ is an a.s. nonconvergent random walk if $E[\ln(\beta + \alpha z_t^2)] \geq 0$, which proves necessity in (50).

(52) is trivial when $\omega = 0$ or $\beta = 0$, so take both to be positive. If $\beta < 1$, $E[(\beta^{m-1}{}_u\xi_t^2)^p] \to 0$ as $m \to \infty$ iff $E[({}_u\xi_t^2)^p] < \infty$. By the corollary to Theorem 3, $E[({}_u\xi_t^2)^p] < \infty$ iff $E[(\beta + \alpha z_t^2)^p] < 1$. $E[(\beta + \alpha z_t^2)^p] < 1$ implies $\beta < 1$, so if $E[(\beta + \alpha z_t^2)^p] < 1$, $E[(\beta^{m-1}{}_u\xi_t^2)^p] \to 0$ as $m \to \infty$. Suppose, on the other hand, that $E[(\beta + \alpha z_t^2)^p] \geq 1$ and $\beta > 0$. Since $E[({}_u\xi_t^2)^p] = \infty$ in this case, $E[(\beta^{m-1}{}_u\xi_t^2)^p] = \infty$ for all m, concluding the proof of (52). The proof of (51) is similar, except that now, for any $t < \infty$, the necessary and sufficient condition for $E[(\xi_t^2)^p] < \infty$ is $E[\sigma_0^{2p}] < \infty$ and $E[(\beta + \alpha z_t^2)^p] < \infty$. ∎

Proof of Theorem 6. (54) and (56) are applications of [16], Formula 2.6.23 #4, and [10], Formula 4.352 #1, respectively. (55) follows from the integral representation of $\Psi(\cdot,\cdot,\cdot)$ ([13], Section 9.11). (57), (58), and (59) with $p < \frac{1}{2}$ are applications of [10] Formulas 3.381 #4, 4.295 #7, and 3.227 #1, respectively. When $p \geq \frac{1}{2}$ in (59), $E[(\beta + \alpha z^2)^p] \geq E[(\alpha z^2)^p] = \alpha^p \cdot E[|z|^{2p}] = \infty$. (60) is an application of [10], Formula 3.222 #2. ∎

Obtaining the Density Estimates in Figure 3. By (6) and (10), we have, for any $T > 0$,

$$({}_u\sigma_T^2 - \sigma_T^2) = ({}_u\sigma_0^2 - \sigma_0^2) \prod_{i=1}^{T} [(1 - \alpha) + \alpha z_{T-i}^2]. \tag{A.14}$$

By Theorem 2, ${}_u\sigma_T^2 - \sigma_T^2 \to 0$ a.s. as $T \to \infty$, since the term $\prod_{i=1,T} [(1 - \alpha) + \alpha z_{T-i}^2]$ vanishes at an exponential rate with probability one. After some experimentation, it was found that $T = 5000$ achieved consistently tiny values for $\prod_{i=1,T}[(1-\alpha) + \alpha z_{T-i}^2]$ (typically, on the order of 10^{-40}) for the selected values of α. σ_{5000}^2 was therefore used to simulate ${}_u\sigma_t^2$.

The z_t's were generated by the normal random number generator in GAUSS. Density estimates for $\ln(\ln(\sigma_T^2))$ (which has support on $(-\infty,\infty)$) were formed using the kernel method with a normal kernel and 7000 draws of σ_T^2. The bandwidth was set equal to $0.9 \cdot A \cdot n^{-1/5}$, where n was the sample size (7000) and A was the minimum of the interquartile range divided by 1.34 and the standard deviation of $\ln(\ln(\sigma_T^2))$. This is the bandwidth selection procedure employed in [20], Section 3.4.2; see [20] for details of the statistical properties. By a change of variables, density estimates for $\ln(\sigma_T^2)$ were then obtained. Doubling or halving the bandwidth had little effect on the density estimates.