$z_t = \sigma v_t = \exp\left\{\frac{w_t}{2}\right\} v_t \qquad \begin{pmatrix} v_t \\ \eta_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 & 0 \\ 0 & \tau^2 \end{pmatrix}\right)$

$w_t = w + \phi w_{t-1} + \eta_t$

Ex 1a

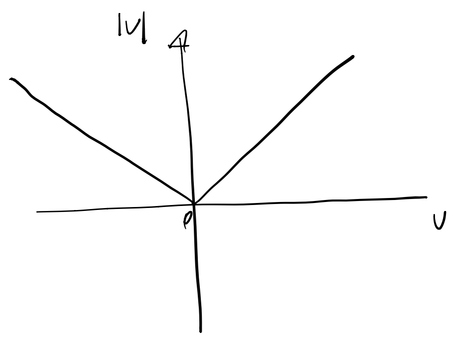
i) $E[|z_t|] = E[|\sigma_t v_t|] = E[\sigma_t |v_t|] = E[\sigma_t]\, E[|v_t|]$

Note that $|v_t|$ is distributed as a folded Gaussian distribution such that $E|v_t| = \sqrt{\frac{2}{\pi}}$

Formal derivation: NOT required, but welcome

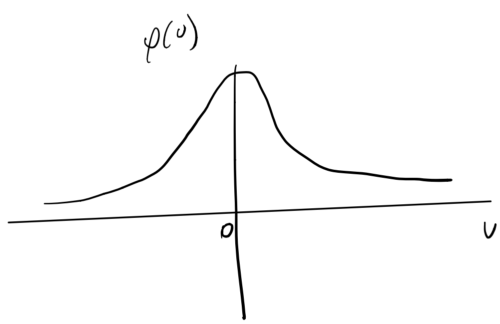
$$E[|v_t|] = \int_{-\infty}^{\infty} |v| \varphi(v)\, dv \quad , \text{where } \varphi(v) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{v^2}{2}\right\}$$

Note that



symmetric around 0

and



symmetric around 0

such that $|v|\varphi(v)$ is also going to be symmetric around 0.

Using this argument we can write

$$\mathbb{E}\big[|v_t|\big] = \int_{-\infty}^{\infty} |v|\varphi(v)\,dv = \int_{-\infty}^{0} |v|\varphi(v)\,dv + \int_{0}^{\infty} |v|\varphi(v)\,dv$$

$$= 2\int_{0}^{\infty} |v|\varphi(v)\,dv = 2\int_{0}^{\infty} v\,\varphi(v)\,dv$$

Note that

$$\int_{0}^{\infty} v\,\varphi(v)\,dv = \int_{0}^{\infty} \frac{v}{\sqrt{2\pi}} \exp\left\{-\frac{v^2}{2}\right\} dv = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} v \exp\left\{-\frac{v^2}{2}\right\} dv$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} -v \exp\left\{-\frac{v^2}{2}\right\} dv = -\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{v^2}{2}\right\}\Big|_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}$$

$$\boxed{\frac{\partial}{\partial v} \exp\left\{-\frac{v^2}{2}\right\} = -v \exp\left\{-\frac{v^2}{2}\right\}}$$

such that

$$2\int_{0}^{\infty} v\,\varphi(v)\,dv = \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}$$

ii) $\mathbb{E}[z_t^2] = \mathbb{E}[v_t^2 \sigma_t^2] = \mathbb{E}[v_t^2]\mathbb{E}[\sigma_t^2] = \mathbb{E}[\sigma_t^2]$

because $\mathbb{E}[v_t^2] = 1$ by assumption

$\boxed{\text{Formal derivation: NOT required, but welcome}}$

$$\mathbb{E}[v_t^2] = \int_{-\infty}^{\infty} v^2 \varphi(v)\, dv = 2 \int_0^{\infty} v^2 \varphi(v)\, dv$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} v^2 \exp\left\{-\frac{v^2}{2}\right\} dv =$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} v \cdot v \exp\left\{-\frac{v^2}{2}\right\} dv$$

remember integration by parts

$$\int_a^b f'(v) g(v)\, dv = f(v) g(v) \Big|_a^b - \int_a^b f(v) g'(v)\, dx$$

Set

$g(v) = v$ $\qquad\qquad g'(v) = 1$

$f'(v) = v \exp\left\{-\frac{v^2}{2}\right\}$ $\qquad\qquad f(v) = -\exp\left\{-\frac{v^2}{2}\right\}$

Such that

$$\frac{2}{\sqrt{2\pi}} \int_0^\infty v \cdot v \exp\left\{-\frac{v^2}{2}\right\} dv =$$

$$= \frac{2}{\sqrt{2\pi}} \left[-v \exp\left\{-\frac{v^2}{2}\right\} \Big|_0^\infty + \int_0^\infty \exp\left\{-\frac{v^2}{2}\right\} dv \right]$$

Note that $-v \exp\left\{-\frac{v^2}{2}\right\} \Big|_0^\infty = 0$ Can you show this?

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \exp\left\{-\frac{v^2}{2}\right\} dv = 2 \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{v^2}{2}\right\} dv$$

$$= 2 \int_0^\infty \varphi(v)\, dv = 2\left(1 - \Phi(0)\right) = 2\left(1 - \frac{1}{2}\right) = 1$$

where $\Phi(x) = \int_{-\infty}^{x} \varphi(v)\, dv$ is the

cumulative density function of a Gaussian distribution.

the same result is obtained by noting that

$$\int_{-\infty}^{\infty} \varphi(v)\, dv = 1 \quad \text{and}$$

$$\int_{-\infty}^{\infty} \varphi(u)\,du = 2\int_{0}^{\infty} \varphi(u)\,du$$

such that $\int_{0}^{\infty} \varphi(u)\,du = \frac{1}{2}$

iii) $\mathbb{E}\left[|\zeta_t|^3\right] = \mathbb{E}\left[\sigma_t^3\right]\mathbb{E}\left[|v_t|^3\right]$

where $\mathbb{E}\left[|v_t|^3\right]$ is the third moment of a folded normal distribution and is equal to $2\sqrt{\frac{2}{\pi}}$

Formal derivation: NOT required, but welcome

$$\mathbb{E}\left(|v_t|^3\right) = \int_{-\infty}^{\infty} |u|^3 \varphi(u)\,du = 2\int_{0}^{\infty} u^3 \varphi(u)\,du$$

$$= 2\int_{0}^{\infty} \frac{u^3}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du =$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} u^2\, u \exp\left\{-\frac{u^2}{2}\right\} du =$$

by parts $\quad f(u) = u^2 \quad f'(u) = 2u$

$g'(u) = u\exp\left\{-\frac{u^2}{2}\right\} \quad g(u) = -\exp\left\{-\frac{u^2}{2}\right\}$

$$= \frac{2}{\sqrt{2\pi}} \left[-u^2 \exp\left\{-\frac{u^2}{2}\right\}\Big|_0^\infty + 2\int_0^\infty u \exp\left\{-\frac{u^2}{2}\right\} du \right]$$

Note that $-u^2 \exp\left\{-\frac{u^2}{2}\right\}\Big|_0^\infty = 0$ $\rightarrow$ show this

$$= \frac{2}{\sqrt{2\pi}} \cdot 2 \int_0^\infty u \exp\left\{-\frac{u^2}{2}\right\} du = 2 \cdot 2 \int_0^\infty \frac{u}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du$$

$$= 2 \cdot 2 \int_0^\infty u \, \varphi(u) \, du = 2\, \mathbb{E}\left[|u|\right] = 2\sqrt{\frac{2}{\pi}}$$

iv) $\mathbb{E}\left[z_t^4\right] = \mathbb{E}\left[\sigma_t^4\right] \mathbb{E}\left[u_t^4\right] = 3\, \mathbb{E}\left[\sigma_t^4\right]$

because the third moment of a standard Gaussian distribution is 3

Formal derivation: NOT required, but welcome

$$\mathbb{E}\left[u_t^4\right] = \int_{-\infty}^{\infty} u^4 \varphi(u) \, du = 2 \int_0^\infty u^4 \varphi(u) \, du$$

$$2 \int_0^\infty \frac{u^4}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du = \frac{2}{\sqrt{2\pi}} \int_0^\infty u^4 \exp\left\{-\frac{u^2}{2}\right\} du$$

by parts

$f(v) = v^3 \qquad f'(v) = 3v^2$

$g'(v) = v \exp\left\{-\frac{v^2}{2}\right\} \qquad g(v) = -\exp\left\{-\frac{v^2}{2}\right\}$

so

$$\frac{2}{\sqrt{2\pi}}\left[-v^3 \exp\left\{-\frac{v^2}{2}\right\}\Big|_0^\infty + 3\int_0^\infty v^2 \exp\left\{-\frac{v^2}{2}\right\} dv\right] =$$

$$-v^3 \exp\left\{-\frac{v^2}{2}\right\}\Big|_0^\infty = 0$$ show this

$$= \frac{2}{\sqrt{2\pi}} \cdot 3\int_0^\infty v^2 \exp\left\{-\frac{v^2}{2}\right\} dv = 3 \cdot 2\int_0^\infty v^2 \varphi(v)\, dv$$

$$= 3\,\mathbb{E}[v^2] = 3$$

v) $\mathbb{E}\left[|z_t z_{t-j}|\right] = \mathbb{E}\left[|v_t v_{t-j} \sigma_t \sigma_{t-j}|\right] =$

$$= \underbrace{\mathbb{E}\left[|v_t v_{t-j}|\right] \mathbb{E}\left[\sigma_t \sigma_{t-j}\right]}_{\text{by independence}} = \mathbb{E}\left[|v_t|\right]\mathbb{E}\left[|v_{t-j}|\right]\mathbb{E}\left[\sigma_t \sigma_{t-j}\right]$$

$$= \underbrace{\mathbb{E}\left[|v_t|\right]^2}_{\text{since identically distributed}} \mathbb{E}\left[\sigma_t \sigma_{t-j}\right] = \frac{2}{\pi}\mathbb{E}\left[\sigma_t \sigma_{t-j}\right]$$

vi) $\mathbb{E}\left[z_t^2 z_{t-j}^2 \right] = \mathbb{E}\left[\sigma_t^2 \sigma_{t-j}^2 \right]$

Exploiting that $\mathbb{E}\left[v_t^2 \right] = \mathbb{E}\left[v_{t-1}^2 \right] = 1$

and $v_t \perp v_{t-j} \quad \forall j > 0$

vii) $\mathbb{E}\left[\sigma_t^p \right]$

$$\sigma_t = \exp\left\{ \frac{W_t}{2} \right\} \Rightarrow \sigma_t^p = \exp\left\{ \frac{p W_t}{2} \right\}$$

$$W_t = \omega + \phi W_{t-1} + \eta_t = \frac{\omega}{1-\phi} + \sum_{s=0}^{\infty} \phi^s \eta_{t-s}$$

$$W_t \sim N\left(\frac{\omega}{1-\phi} ; \frac{\kappa^2}{1-\phi^2} \right)$$

$$\frac{p}{2} W_t = X_t \sim N\left(\frac{p\omega}{2(1-\phi)} ; \frac{p^2 \kappa^2}{4(1-\phi^2)} \right)$$

So

$$\mathbb{E}\left[\sigma_t^p \right] = \mathbb{E}\left[\exp\left\{ X_t \right\} \right]$$

we know that if $z \sim N(a, b^2)$

then $\exp\{z\}$ is Log-Normal with

mean

$$\mathbb{E}\left[\exp\{z\}\right] = \exp\left\{ a + \frac{b^2}{2} \right\}$$

In our case $a = \dfrac{p w}{2(1-\phi)}$; $b^2 = \dfrac{p^2 \kappa^2}{4(1-\phi^2)}$

such that

$$\mathbb{E}\left[\exp\{x_t\}\right] = \mathbb{E}\left[\sigma_t^{p}\right] = \exp\left\{ \frac{p w}{2(1-\phi)} + \frac{p^2 \kappa^2}{8(1-\phi^2)} \right\}$$

Call $\alpha_w = \dfrac{w}{1-\phi}$; $\beta_w^2 = \dfrac{\kappa^2}{1-\phi^2}$

such that $\mathbb{E}\left[\sigma_t^{p}\right] = \exp\left\{ \dfrac{p \alpha_w}{2} + \dfrac{p^2 \beta_w^2}{8} \right\}$ $\blacksquare$

viii) $\mathbb{E}\left[\sigma_t^2 \sigma_{t-j}^2\right]$

Note that $\mathbb{E}\left[\sigma_t^2 \sigma_{t-j}^2\right] = \mathbb{E}\left[\sigma_t^2 \sigma_{t+j}^2\right]$

write $\sigma_{t+j}^2 = \exp\{w_{t+j}\}$ and note that

$$\mathbb{E}\left[\sigma_t^2 \sigma_{t+j}^2\right] = \mathbb{E}\left[\exp\{w_t + w_{t+j}\}\right]$$

by the law of iterated expectation

$$\mathbb{E}\left[\exp\{w_t + w_{t+j}\}\right] = \mathbb{E}\left[\mathbb{E}\left[\exp\{w_t + w_{t+j}\} \mid w_t\right]\right] =$$

$$= \mathbb{E}\left[\exp\{w_t\} \, \mathbb{E}\left[\exp\{w_{t+j}\} \mid w_t\right]\right]$$

To compute $\mathbb{E}\left[\exp\{w_{t+j}\} \mid w_t\right]$ note that

$$w_{t+j} = \omega + \phi w_{t+j-1} + \eta_{t+j}$$

$$w_{t+1} = \omega + \phi w_t + \eta_{t+1}$$

$$w_{t+2} = \omega + \phi w_{t+1} + \eta_{t+2} = \omega + \phi(\omega + \phi w_t + \eta_{t+1}) + \eta_{t+2}$$

$$= \omega(1+\phi) + \phi^2 w_t + \phi \eta_{t+1} + \eta_{t+2}$$

$$w_{t+3} = \omega(1 + \phi + \phi^2) + \phi^3 w_t + \phi^2 \eta_{t+1} + \phi \eta_{t+2} + \eta_{t+3}$$

In general

$$w_{t+j} = \omega \sum_{s=0}^{j-1} \phi^s + \phi^j w_t + \sum_{s=0}^{j-1} \phi^s \eta_{t+j-s}$$

and

$$\exp\{w_{t+j}\} = \exp\{\omega \sum_{s=0}^{j-1} \phi^s\} \exp\{\phi^j w_t\} \exp\{\sum_{s=0}^{j-1} \phi^s \eta_{t+j-s}\}$$

So

$$\mathbb{E}[\exp\{w_{t+j}\} \mid w_t] = \exp\{\omega \sum_{s=0}^{j-1} \phi^s\} \exp\{\phi^j w_t\} \mathbb{E}[\exp\{\sum_{s=0}^{j-1} \phi^s \eta_{t+j-s}\} \mid w_t]$$

Note that

$$\mathbb{E}[\exp\{\sum_{s=0}^{j-1} \phi^s \eta_{t+j-s}\} \mid w_t] = \mathbb{E}[\exp\{\sum_{s=0}^{j-1} \phi^s \eta_{t+j-s}\}] = \mathbb{E}[\prod_{s=0}^{j-1} \exp\{\phi^s \eta_{t+j-s}\}]$$

$$= \prod_{s=0}^{j-1} \mathbb{E}[\exp\{\phi^s \eta_{t+j-1}\}] = \prod_{s=0}^{j-1} \exp\{\frac{\phi^{2s} \kappa^2}{2}\} = \exp\{\frac{\kappa^2}{2} \sum_{s=0}^{j-1} \phi^{2s}\}$$

So

$$= \mathbb{E}[\exp\{w_t\} \mathbb{E}[\exp\{w_{t+j}\} \mid w_t]] =$$

$$= \mathbb{E}[\exp\{w_t\} \exp\{\phi^j w_t\}] \exp\{\omega \sum_{s=0}^{j-1} \phi^s\} \exp\{\frac{\kappa^2}{2} \sum_{s=0}^{j-1} \phi^{2s}\}$$

$$\mathbb{E}[\exp\{w_t\} \exp\{\phi^j w_t\}] = \mathbb{E}[\exp\{w_t(1+\phi^j)\}]$$

$$= \exp\left\{ \frac{\omega(1+\phi^j)}{1-\phi} + \frac{\kappa^2 (1+\phi^j)^2}{2(1-\phi^2)} \right\}$$

Such that

$$\mathbb{E}\left[\exp\left\{w_t + w_{t+j}\right\}\right] =$$

$$= \exp\left\{\frac{w(1+\phi^j)}{1-\phi} + \frac{\kappa^2(1+\phi^j)^2}{2(1-\phi^2)} + w\sum_{s=0}^{j-1}\phi^s + \frac{\kappa^2}{2}\sum_{s=0}^{j-1}\phi^{2s}\right\}$$

Remember that

$$\sum_{s=0}^{n-1} a^n = \frac{1-a^n}{1-a} \quad \text{for } a \neq 1$$

Since $\phi \neq 1$ we have

$$\sum_{s=0}^{j-1}\phi^s = \frac{1-\phi^j}{1-\phi} \quad ; \quad \sum_{s=0}^{j-1}\phi^{2s} = \frac{1-\phi^{2j}}{1-\phi^2}$$

so

$$= \exp\left\{\frac{w(1+\phi^j)}{1-\phi} + \frac{\kappa^2(1+\phi^j)^2}{2(1-\phi^2)} + \frac{w(1-\phi^j)}{1-\phi} + \frac{\kappa^2(1-\phi^{2j})}{2(1-\phi^2)}\right\}$$

$$= \exp\left\{\frac{2w}{1-\phi} + \frac{\kappa^2(1+\phi^j)}{1-\phi^2}\right\} \qquad \text{call } \alpha_w = \frac{w}{1-\phi} \; ; \; \beta_w^2 = \frac{\kappa^2}{1-\phi^2}$$

$$= \exp\left\{2\alpha_w + \beta_w^2(1+\phi^j)\right\} = \exp\left\{2\alpha_w + \beta_w^2\right\}\exp\left\{\beta_w^2\phi^j\right\}$$

$= \mathbb{E}\left[\sigma_t^2\right]^2 \exp\left\{\beta_w^2 \phi^i\right\}$

$\boxed{Ex\ 1b}$

independence

$$\mathbb{E}\left[\sigma_t^2 \varepsilon_{t-j}\right] = \mathbb{E}\left[\sigma_t^2 \sigma_{t-j} v_{t-j}\right] = \mathbb{E}\left[\sigma_t^2 \sigma_{t-j}\right] \mathbb{E}\left[v_{t-j}\right]$$

$= 0$ because $\mathbb{E}\left[v_{t-j}\right] = 0$

Absence of correlation between past returns and current volatility levels implies no leverage effect.

$\boxed{Ex\ 1c}$

i) $\sigma_t^2 = \exp\{w_t\}$; $w_t \sim N(\alpha_w, \beta_w^2)$

such that σ_t^2 is Log Normal

the density of a Log Normal distribution is

$$p(s) = \frac{\exp\left\{-\frac{(\log s - \alpha_w)^2}{2\beta_w^2}\right\}}{s\sqrt{2\pi}\,\beta_w}$$

Formal derivation: NOT required, but welcome

$$P(\sigma_t^2 \le s) = P(\exp\{w_t\} \le s) = P(w_t \le \log(s))$$

$$= P(\alpha_w + \beta_w Z \le \log(s)) \quad \text{where } Z \sim N(0,1)$$

$$= P\left(Z \le \frac{\log(s) - \alpha_w}{\beta_w}\right) = \Phi\left(\frac{\log(s) - \alpha_w}{\beta_w}\right)$$

So

$$p(s) = \frac{\partial P(\sigma_t^2 \le s)}{\partial s} = \frac{\partial}{\partial s} \Phi\left(\frac{\log(s) - \alpha_w}{\beta_w}\right)$$

$$= \varphi\left(\frac{\log(s) - \alpha_w}{\beta_w}\right) \frac{1}{s\beta_w} = \frac{1}{s\beta_w\sqrt{2\pi}} \exp\left\{-\frac{(\log(s) - \alpha_w)^2}{2\beta_w^2}\right\}$$

ii)

$$p(z_t) = \int_0^\infty p(z_t \mid \sigma_t^2)\, p(\sigma_t^2)\, d\sigma_t^2$$

$$= \int_0^\infty \frac{1}{\sigma_t} \varphi\left(\frac{z_t}{\sigma_t}\right) \frac{1}{\sigma_t^2 \beta_w} \varphi\left(\frac{\log \sigma_t^2 - \alpha_w}{\beta_w}\right) d\sigma_t^2$$

iii)

$$p(\sigma_t^2, z_t) = p(z_t \mid \sigma_t^2)\, p(\sigma_t^2) =$$

$$= \frac{1}{\sigma_t}\,\varphi\left(\frac{z_t}{\sigma_t}\right)\frac{1}{\sigma_t^2 \beta_W}\,\varphi\left(\frac{\log \sigma_t^2 - \alpha_W}{\beta_W}\right)$$

iv)

$$p(z_t \mid \sigma_t^2) = \frac{1}{\sigma_t}\,\varphi\left(\frac{z_t}{\sigma_t}\right)$$