1 Theoretical part

1.1 Problem 1)

Show that ${}_u\sigma_t^2$ admits the following representation:

$$ {}_u\sigma_t^2 = \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \left(\beta + \alpha g \left(\varepsilon_{t-i} \right) \right) \right] $$

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We apply recursive substitution from the volatility process in Equation **(REF)**.

$$ \begin{aligned} & \sigma_{t+1}^2 = \omega + \left(\beta + \alpha g \left(\varepsilon_t \right) \right) \sigma_t^2 \\ \underset{t-1}{\Longrightarrow} \quad & \sigma_{t+1}^2 = \omega + \left(\beta + \alpha g \left(\varepsilon_t \right) \right) \left(\omega + \sigma_{t-1}^2 \left(\beta + \alpha g \left(\varepsilon_{t-1} \right) \right) \right) \\ \underset{t-2}{\Longrightarrow} \quad & \sigma_{t+1}^2 = \omega + \left(\beta + \alpha g \left(\varepsilon_t \right) \right) \left(\omega + \left(\omega + \sigma_{t-2}^2 \left(\beta + \alpha g \left(\varepsilon_{t-2} \right) \right) \right) \left(\beta + \alpha g \left(\varepsilon_{t-1} \right) \right) \right) \\ \underset{\text{define}}{\Longrightarrow} \quad & \sigma_{t+1}^2 = \omega + \underbrace{\left(\beta + \alpha g \left(\varepsilon_t \right) \right)}_{\phi_t} \left(\omega + \left(\omega + \sigma_{t-2}^2 \left(\beta + \alpha g \left(\varepsilon_{t-2} \right) \right) \right) \left(\beta + \alpha g \left(\varepsilon_{t-1} \right) \right) \right) \\ & \sigma_{t+1}^2 = \omega + \phi_t \left(\omega + \left(\omega + \sigma_{t-2}^2 \phi_{t-2} \right) \phi_{t-1} \right) \\ & \sigma_{t+1}^2 = \omega + \phi_t \omega + \phi_t \phi_{t-1} \left(\omega + \sigma_{t-2}^2 \phi_{t-2} \right) \\ & \sigma_{t+1}^2 = \omega + \phi_t \omega + \phi_t \phi_{t-1} \omega + \phi_t \phi_{t-1} \phi_{t-2} \sigma_{t-2}^2 \end{aligned} $$

We know that there is ω contained in σ_t^2 $\forall t$ thus I am able to factorize ω in the expression above. We're conditioning on all past observations in this case and employ the unconditional notation.

$$ \underset{\text{factorize } \omega \text{ and continue till } -\infty}{\Longrightarrow} \quad {}_u\sigma_{t+1}^2 = \omega \left[1 + \phi_t + \phi_t \phi_{t-1} + \phi_t \phi_{t-1} \phi_{t-2} + \phi_t \phi_{t-1} \phi_{t-2} \phi_{t-3} + \ldots \right] $$

As we're interested in the unconditional process ${}_u\sigma_t^2$ at period t, I lag the entire process one period

$$ {}_u\sigma_t^2 = \omega \left[1 + \phi_{t-1} + \phi_{t-1} \phi_{t-2} + \phi_{t-1} \phi_{t-2} \phi_{t-3} + \phi_{t-1} \phi_{t-2} \phi_{t-3} \phi_{t-4} + \ldots \right] $$

We notice a pattern in this expression and start by writing out the products

$$ \begin{aligned} {}_u\sigma_t^2 &= \omega \left[1 + \phi_{t-1} + \phi_{t-1} \phi_{t-2} + \phi_{t-1} \phi_{t-2} \phi_{t-3} + \phi_{t-1} \phi_{t-2} \phi_{t-3} \phi_{t-4} + \ldots \right] \\ {}_u\sigma_t^2 &= \omega \left[1 + \prod_{i=1}^{1} \phi_{t-i} + \prod_{i=1}^{2} \phi_{t-i} + \prod_{i=1}^{3} \phi_{t-i} + \prod_{i=1}^{4} \phi_{t-i} + \ldots \right] \end{aligned} $$

Now we're getting close - we just need to recognize the sums. We're summing each product for a varying product limit. This limit should be defined by the sum operator. Thus we're able to write

$$
\begin{aligned}
{}_u\sigma_t^2 &= \omega \left[1 + \prod_{i=1}^{1} \phi_{t-i} + \prod_{i=1}^{2} \phi_{t-i} + \prod_{i=1}^{3} \phi_{t-i} + \prod_{i=1}^{4} \phi_{t-i} + \dots \right] \\
{}_u\sigma_t^2 &= \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \phi_{t-i} \right] \\
\underset{\text{substitute } \phi_t}{\Longrightarrow} \quad {}_u\sigma_t^2 &= \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right) \right] \quad \square
\end{aligned}
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1.2 Problem 2)

Show that σ_t^2 admits the following representation:

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\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right) + \omega \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right) \right]
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We're no longer conditioning on all infinite past observations, thus there is a limit for t to the result derived in Problem **(REF)** thus we can write *(copying from before and keeping definition of ϕ_t)*

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\begin{aligned}
\sigma_{t+1}^2 &= \omega + \left(\beta + \alpha g\left(\varepsilon_t\right)\right) \sigma_t^2 \\
&\vdots \\
{}_u\sigma_t^2 &= \omega \left[1 + \prod_{i=1}^{1} \phi_{t-i} + \prod_{i=1}^{2} \phi_{t-i} + \prod_{i=1}^{3} \phi_{t-i} + \prod_{i=1}^{4} \phi_{t-i} + \dots \right]
\end{aligned}
$$

We know that σ_t^2 is initialized at time $t = 0$ where we have some positive value for $\sigma_0^2 > 0$. Thus we're able to write the results from Problem **(REF)** as we know that the process starts at $t = 0$ and not infinite past

$$
\begin{aligned}
\sigma_t^2 &= \omega \left[1 + \prod_{i=1}^{1} \phi_{t-i} + \prod_{i=1}^{2} \phi_{t-i} + \prod_{i=1}^{3} \phi_{t-i} + \prod_{i=1}^{4} \phi_{t-i} + \dots + \prod_{i=1}^{t-1} \phi_{t-i} \right] + \sigma_0^2 \prod_{i=1}^{t} \phi_{t-i} \\
\sigma_t^2 &= \sigma_0^2 \prod_{i=1}^{t} \phi_{t-i} + \omega \left[1 + \prod_{i=1}^{1} \phi_{t-i} + \prod_{i=1}^{2} \phi_{t-i} + \prod_{i=1}^{3} \phi_{t-i} + \prod_{i=1}^{4} \phi_{t-i} + \dots + \prod_{i=1}^{t-1} \phi_{t-i} \right]
\end{aligned}
$$

Now we're writing out the sums again

$$\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} \phi_{t-i} + \omega \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} \phi_{t-i} \right]$$

$$\underset{\text{substitute } \phi_t}{\Longrightarrow} \quad \sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right) \right) + \omega \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right) \right) \right], \quad \square$$

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1.3 Problem 3)

Derive sufficient conditions on (ω, α, β) such that ${}_u\sigma_t^2 > 0$.

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We know that ${}_u\sigma_t^2$ is defined as,

$${}_u\sigma_t^2 = \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right) \right) \right]$$

Investigating this expression for ${}_u\sigma_t^2$ we see that for ${}_u\sigma_t^2 > 0$ we can easily constrain $\omega \neq 0$. We cannot say anything about $\omega < 0$ before investigating conditions on α and β more thoroughly.

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1.4 Problem 4)

Derive the lower and the upper bound of the process $\left\{ {}_u\sigma_t^2 \right\}_{t \in \mathbb{Z}}$, i.e. show that ${}_u\sigma_t^2 \in [l, u]$ where $l < u$. Derive l and u.

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We know from Lecture 4 Slide 16, that \CITE{nelson1990} shows that for $\omega > 0$ we almost surely have $\sigma_t^2 < \infty$. And that the joint process $\left\{ y_t, \sigma_t^2 \right\}$ is strictly stationary iff. $\mathbb{E}\left[\ln\left(\beta + \alpha g\left(\varepsilon_t\right) \right) \right] < 0$

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In the following we need to apply the following assumptions,

$$\begin{aligned}\omega &> 0\\ \alpha &> 0\\ \beta &\geq 0\\ \alpha+\beta &< 1\\ \mathbb{E}\left[\ln\left(\beta+\alpha g\left(\varepsilon_t\right)\right)\right] &< 0\end{aligned}$$

The second assumption implies that $0 < \beta + \alpha g\left(\varepsilon_t\right) < 1$ as $\ln\left(x\right) < 0$ iff. $0 < x < 1$.

We remember that ${}_u\sigma_t^2$ is defined as

$${}_u\sigma_t^2 = \omega\left[1+\sum_{k=1}^{\infty}\prod_{i=1}^{k}\left(\beta+\alpha g\left(\varepsilon_{t-i}\right)\right)\right]$$

I start by finding the lower bound, realizing

$$\min\left[g\left(\varepsilon_{t-i}\right)\right] = \underline{m}$$

inserting this into the expression for ${}_u\sigma_t^2$ yields

$$\begin{aligned} l &= \omega\left[1+\sum_{k=1}^{\infty}\prod_{i=1}^{k}\left(\beta+\alpha\underline{m}\right)\right]\\ l &= \omega\left[1+\sum_{k=1}^{\infty}\left(\beta+\alpha\underline{m}\right)^k\right]\end{aligned}$$

We remember

$$a+ar+ar^2+ar^3+ar^4+\cdots = \sum_{k=0}^{\infty}ar^k = \frac{a}{1-r}, \text{ for } |r|<1 \tag{Geo 1}$$

Thus we can write l as

$$l = \frac{\omega}{1-\left(\beta+\alpha\underline{m}\right)}$$

Analogously we can write the upper bound using the same steps

$$\max\left[g\left(\varepsilon_{t-i}\right)\right] = \overline{m}$$

inserting this into the expression for ${}_u\sigma_t^2$ yields

$$\begin{aligned} u &= \omega\left[1+\sum_{k=1}^{\infty}\prod_{i=1}^{k}\left(\beta+\alpha\overline{m}\right)\right]\\ u &= \omega\left[1+\sum_{k=1}^{\infty}\left(\beta+\alpha\overline{m}\right)^k\right]\end{aligned}$$

Thus we can write u as

$$u = \frac{\omega}{1 - (\beta + \alpha \overline{m})}$$

Thus we have

$$_u\sigma_t^2 \in [l, u] = \begin{cases} l & = \frac{\omega}{1-(\beta+\alpha \underline{m})} \\ u & = \frac{\omega}{1-(\beta+\alpha \overline{m})} \end{cases}$$

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1.5 Problem 5)

Show that if $\omega = 0$:

1.5.1 Problem 5.a

$_u\sigma_t^2 = 0$ for all t.

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$$\begin{aligned} _u\sigma_t^2 &= \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right) \right] \\ _u\sigma_t^2 &= \underbrace{\omega}_{=0} \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right) \right] \\ _u\sigma_t^2 &= 0 \end{aligned}$$

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1.5.2 Problem 5.b

$\sigma_t^2 \to \infty$ as $t \to \infty$ if $E\left[\log\left(\beta + \alpha g\left(\varepsilon_t\right)\right)\right] > 0$.

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We remember that $\omega = 0$ and thus we can write our process for σ_t^2 as

$$\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right) + \underbrace{\omega \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right)\right]}_{=0}$$

$$\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right)$$

$$\underset{\text{log transform}}{\Longrightarrow} \quad \ln\left[\sigma_t^2\right] = \ln\left[\sigma_0^2 \prod_{i=1}^{t} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right)\right]$$

$$\ln\left[\sigma_t^2\right] = \ln\left[\sigma_0^2\right] + \sum_{i=1}^{t} \underbrace{\ln\left[\beta + \alpha g\left(\varepsilon_{t-i}\right)\right]}_{\phi_{t-i}}$$

$$\ln\left[\sigma_t^2\right] = \ln\left[\sigma_0^2\right] + \sum_{i=1}^{t} \phi_{t-i}$$

$$\underset{+/-\ \mathbb{E}(\phi_{t-i})}{\Longrightarrow} \quad \ln\left[\sigma_t^2\right] = \ln\left[\sigma_0^2\right] + \sum_{i=1}^{t} \left\{ \underbrace{\phi_{t-i} - \mathbb{E}\left(\phi_{t-i}\right)}_{\widetilde{\phi}_{t-i}} + \mathbb{E}\left(\phi_{t-i}\right) \right\}$$

$$\ln\left[\sigma_t^2\right] = \ln\left[\sigma_0^2\right] + t \cdot \mathbb{E}\left(\phi_t\right) + \sum_{i=1}^{t} \widetilde{\phi}_{t-i}$$

We recognize this as a random walk with a drift. We remember that the value of a random walk with a drift either diverges to $+\infty$ or $-\infty$ depending on the value of the drift. Thus we can easily see that as $t \to \infty$ then $\ln\left[\sigma_t^2\right] \to \infty$ and $\sigma_t^2 \to \infty$ for all $\mathbb{E}\left(\phi_t\right) = \mathbb{E}\left[\ln\left(\beta + \alpha g\left(\varepsilon_t\right)\right)\right] > 0$.

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1.5.3 Problem 5.c

$\sigma_t^2 \to 0$ as $t \to \infty$ if $E\left[\log\left(\beta + \alpha g\left(\varepsilon_t\right)\right)\right] < 0$.

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Here we see that we have a similar case as in the former problem however the inequality is turned. However similar arguments can be applied. We recognize again a random walk with a drift. We remember that the value of a random walk with a drift either diverges to $+\infty$ or $-\infty$ depending on the value of the drift. Thus we can easily see that as $t \to \infty$ then $\ln\left[\sigma_t^2\right] \to -\infty$ and $\sigma_t^2 \to 0$ for all $\mathbb{E}\left(\phi_t\right) = \mathbb{E}\left[\ln\left(\beta + \alpha g\left(\varepsilon_t\right)\right)\right] < 0$.

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1.6 Problem 6)

Show that if $\omega > 0$ and $\mathbb{E}\left[\log\left(\beta + \alpha g\left(\varepsilon_t\right)\right)\right] > 0$:

1.6.1 Problem 6.a

$\sigma_t^2 \to \infty$ as $t \to \infty$

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We remember our definition of σ_t^2

$$\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right) + \omega \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right)\right]$$

We recognize that

$$\sigma_0^2 \prod_{i=1}^{t} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right) + \omega \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right)\right] \geq \omega \sup_{1 \leq k \leq t-1} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right)$$

taking logs yields

$$\ln\left(\sigma_t^2\right) \geq \ln\left(\omega\right) + \sup_{1 \leq k \leq t-1} \sum_{i=1}^{k} \ln\left(\beta + \alpha g\left(\varepsilon_{t-i}\right)\right)$$

Using the results from Theorem 1 in \CITE{nelson1990} the right term above diverges to $+\infty$ when $t \to \infty$ if $\mathbb{E}\left[\ln\left(\beta + \alpha g\left(\varepsilon_t\right)\right)\right] > 0$. If this expression is smaller than equal to $\ln\left(\sigma_t^2\right)$ this expression must also approach ∞ as $t \to \infty$ thus yielding

$$\sigma_t^2 \to \infty \quad \text{if} \quad \mathbb{E}\left[\ln\left(\beta + \alpha g\left(\varepsilon_t\right)\right)\right] > 0$$

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1.6.2 Problem 6.b

${}_u\sigma_t^2 \to \infty$ for all t

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Remembering we can write ${}_u\sigma_t^2$ as

$$ {}_u\sigma_t^2 = \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right) \right) \right] $$

Again we recognize that

$$ \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right) \right) \right] \geq \omega \sup_{1 \leq k \leq \infty} \prod_{i=1}^{k} \left(\beta + \alpha g\left(\varepsilon_{t-i}\right) \right) $$

Using similar arguments as in REF problem 6.a we see that the right side diverges to $+\infty$ for all t.

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1.7 Problem 7)

Consider the GARCH case where $g\left(\varepsilon_t\right) = \varepsilon_t^2$.

1.7.1 Problem 7.a, i)

Discuss the necessary conditions on (ω, α, β) such that: i) $\{y_t\}$ is weakly stationary,

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To have weak stationarity of y_t we require $\mathbb{E}\left[\sigma_t^2\right] = \sigma^2 < \infty$ for all t. Substituting for $g\left(\varepsilon_t\right) = \varepsilon_t^2$, rewriting, recursively substituting and taking the unconditional expectation of the process to check for this condition.

$$ \begin{aligned} \sigma_{t+1}^2 &= \omega + \sigma_t^2 \left(\beta + \alpha g\left(\varepsilon_t\right) \right) \\ \sigma_{t+1}^2 &= \omega + \sigma_t^2 \left(\beta + \alpha \varepsilon_t^2 \right) \\ \underset{\text{lag 1 period}}{\Longrightarrow} \quad \sigma_t^2 &= \omega + \sigma_{t-1}^2 \left(\beta + \alpha \varepsilon_{t-1}^2 \right) \end{aligned} $$

Recursive substitution

$$
\begin{aligned}
&\sigma_t^2 = \omega + \sigma_{t-1}^2\left(\beta + \alpha\varepsilon_{t-1}^2\right)\\
&\sigma_t^2 = \omega + \left(\omega + \sigma_{t-2}^2\left(\beta + \alpha\varepsilon_{t-2}^2\right)\right)\left(\beta + \alpha\varepsilon_{t-1}^2\right)\\
&\sigma_t^2 = \omega + \left(\omega + \left(\omega + \sigma_{t-3}^2\left(\beta + \alpha\varepsilon_{t-3}^2\right)\right)\left(\beta + \alpha\varepsilon_{t-2}^2\right)\right)\left(\beta + \alpha\varepsilon_{t-1}^2\right)\\
\underset{\text{defining } \phi_t}{\Longrightarrow}\quad &\sigma_t^2 = \omega + \left(\omega + \left(\omega + \sigma_{t-3}^2\left(\beta + \alpha\varepsilon_{t-3}^2\right)\right)\left(\beta + \alpha\varepsilon_{t-2}^2\right)\right)\underbrace{\left(\beta + \alpha\varepsilon_{t-1}^2\right)}_{\phi_{t-1}}\\
&\sigma_t^2 = \omega + \left(\omega + \left(\omega + \sigma_{t-3}^2\left(\phi_{t-3}\right)\right)\left(\phi_{t-2}\right)\right)\phi_{t-1}\\
&\sigma_t^2 = \omega + \omega\phi_{t-1} + \phi_{t-1}\phi_{t-2}\omega + \phi_{t-1}\phi_{t-2}\phi_{t-3}\sigma_{t-3}^2
\end{aligned}
$$

Recognizing the same pattern as earlier and continueing till ∞

$$
\begin{aligned}
&\sigma_t^2 = \omega + \phi_{t-1}\omega + \phi_{t-1}\phi_{t-2}\omega + \phi_{t-1}\phi_{t-2}\phi_{t-3}\omega + \dots\\
&\sigma_t^2 = \omega\left[1 + \phi_{t-1} + \phi_{t-1}\phi_{t-2} + \phi_{t-1}\phi_{t-2}\phi_{t-3} + \dots\right]\\
&\sigma_t^2 = \omega\left[1 + \prod_{i=1}^{1}\phi_{t-i} + \prod_{i=1}^{2}\phi_{t-i} + \prod_{i=1}^{3}\phi_{t-i} + \dots\right]\\
&\sigma_t^2 = \omega\left[1 + \sum_{k=1}^{\infty}\prod_{i=1}^{k}\phi_{t-i}\right]\\
\underset{\text{substitute } \phi_t}{\Longrightarrow}\quad &\sigma_t^2 = \omega\left[1 + \sum_{k=1}^{\infty}\prod_{i=1}^{k}\beta + \alpha\varepsilon_{t-i}^2\right]
\end{aligned}
$$

Taking the conditional expectation

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\begin{aligned}
\mathbb{E}\left[\sigma_t^2\right] &= \mathbb{E}\left[\omega\left[1 + \sum_{k=1}^{\infty}\prod_{i=1}^{k}\beta + \alpha\varepsilon_{t-i}^2\right]\right]\\
\mathbb{E}\left[\sigma_t^2\right] &= \omega\left[1 + \sum_{k=1}^{\infty}\mathbb{E}\left[\prod_{i=1}^{k}\beta + \alpha\varepsilon_{t-i}^2\right]\right]\\
\mathbb{E}\left[\sigma_t^2\right] &= \omega\left[1 + \sum_{k=1}^{\infty}\mathbb{E}\left[\prod_{i=1}^{k}\underbrace{\beta + \alpha\varepsilon_{t-i}^2}_{b_{t-i}}\right]\right]
\end{aligned}
$$

We first show $\mathbb{E}\left[b_t\right]$ and then substitute back into $\mathbb{E}\left[\sigma_t^2\right]$

$$
\mathbb{E}\left[b_t\right] = \mathbb{E}\left[\beta + \alpha\varepsilon_t^2\right] = \beta + \alpha\underbrace{\mathbb{E}\left[\varepsilon_t^2\right]}_{=1} = \beta + \alpha
$$

Now we can derive the unconditional expectation

$$\mathbb{E}\left[\sigma_t^2\right] = \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} (\beta + \alpha)\right]$$
$$\mathbb{E}\left[\sigma_t^2\right] = \omega \underbrace{\left[\sum_{k=0}^{\infty} (\beta + \alpha)^k\right]}_{\text{removing 1 with sum}}$$
$$\mathbb{E}\left[\sigma_t^2\right] = \omega \left[\sum_{k=0}^{\infty} (\beta + \alpha)^k\right]$$

Applying the following geometric series

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \quad \text{for } |r| < 1 \tag{Geo 2}$$

We know that we can only apply **REF** Geo 2, if $|\beta + \alpha| < 1$. Thus we are left with

$$\mathbb{E}\left[\sigma_t^2\right] = \sigma^2 = \frac{\omega}{1 - \alpha - \beta}$$

We know $\sigma^2 > 0$ thus we're left with the following conditions for weak stationarity

$$\alpha + \beta < 1 \implies \alpha < 1, \beta < 1$$
$$\omega > 0$$

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1.7.2 Problem 7.a ii)

ii) $\{y_t\}$ is strongly stationary.
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We know from Lecture 4 Slide 16, that \CITE{nelson1990} shows that for $\omega > 0$ we almost surely have $\sigma_t^2 < \infty$. And that the joint process $\left\{y_t, \sigma_t^2\right\}$ is strictly stationary iff. $\mathbb{E}\left[\ln\left(\beta + \alpha\varepsilon_t^2\right)\right] < 0$. If we apply Jensen's equality we get

$$\mathbb{E}\left[\ln\left(\beta + \alpha\varepsilon_t^2\right)\right] < \ln\left(\mathbb{E}\left[\beta + \alpha\varepsilon_t^2\right]\right)$$

We can reduce

$$\ln\left(\mathbb{E}\left[\beta+\alpha\varepsilon_t^2\right]\right) = \ln\left(\mathbb{E}\left[\beta\right]+\alpha\underbrace{\mathbb{E}\left[\varepsilon_t^2\right]}_{=1}\right)$$
$$\ln\left(\mathbb{E}\left[\beta+\alpha\varepsilon_t^2\right]\right) = \ln\left(\beta+\alpha\right)$$

Inserting

$$\mathbb{E}\left[\ln\left(\beta+\alpha\varepsilon_t^2\right)\right] < \ln\left(\mathbb{E}\left[\beta+\alpha\varepsilon_t^2\right]\right)$$
$$\mathbb{E}\left[\ln\left(\beta+\alpha\varepsilon_t^2\right)\right] < \ln\left(\beta+\alpha\right)$$

We know we have weak stationarity for $\beta+\alpha<1$. However for the expression above we know that $\alpha+\beta=1$ satisfies the condition for strong stationarity as we have

$$\mathbb{E}\left[\ln\left(\beta+\alpha\varepsilon_t^2\right)\right] < \ln\left(\beta+\alpha\right)$$
$$\mathbb{E}\left[\ln\left(\beta+\alpha\varepsilon_t^2\right)\right] < \ln\left(1\right)$$
$$\mathbb{E}\left[\ln\left(\beta+\alpha\varepsilon_t^2\right)\right] < 0$$

Thus we know that for $\alpha+\beta=1$ we have strong, but not weak stationarity.

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1.7.3 Problem 7.b

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Let $\omega=0.1, \alpha=0.041, \beta=0.96$, and assume that ε_t is iid standard Gaussian, i.e. $\varepsilon_t \overset{iid}{\sim} N(0,1)$. Discuss whether $\{y_t\}$ is weakly stationary and/or strongly stationary.

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Investigating whether $\{y_t\}$ is weakly stationary is relatively straightforward. As we have previously shown, the condition for weak stationarity can be stated as $\alpha+\beta<1$. This condition is violated in this case as $0.041+0.96=1.001>1$.

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To study whether we have strong stationarity, I use direct Monte Carlo simulation following example R-code from lectures, filename Code04112021.R.

Here I simulate 1e7 draws from the standard Gaussian distribution and then compute the moment condition. For set.seed(123) I get

$$\mathbb{E}\left[\ln\left(\beta + \alpha\varepsilon_t^2\right)\right] = -0.00053 < 0, \quad \text{(rounded)}$$

Thus the condition for strong stationarity is fulfilled and $\{y_t\}$ is strongly stationary.

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1.7.4 Problem 7.c

Let $\omega = 0.1, \alpha = 0.06, \beta = 0.96$, and assume that ε_t is iid standard Student's t, i.e. $\varepsilon_t \overset{iid}{\sim} T(0, 1, \nu)$, with density

$$p\left(\varepsilon_t\right) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu-2)\pi}}\left[1 + \frac{\varepsilon_t^2}{\nu-2}\right]^{-\frac{\nu+1}{2}}$$

Discuss whether $\{y_t\}$ is weakly stationary and/or strongly stationary in the cases $\nu = 2.5$ and $\nu = 20$. What do you observe here?

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Investigating whether $\{y_t\}$ is weakly stationary is relatively straightforward. As we have previously shown, the condition for weak stationarity can be stated as $\alpha + \beta < 1$. This condition is violated in this case as $0.06 + 0.96 = 1.02 > 1$.

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FIX THIS: https://stats.stackexchange.com/questions/8466/standardized-students-t-distribution

$$p\left(\varepsilon_t\right) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu-2)\pi}}\left[1 + \frac{\varepsilon_t^2}{\nu-2}\right]^{-\frac{\nu+1}{2}}$$

$$\frac{\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}}\left(1 + \frac{\varepsilon_t^2}{\nu}\right)^{-\frac{\nu+1}{2}}}{\sqrt{\frac{\nu}{\nu-2}}} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu-2)\pi}}\left[1 + \frac{\varepsilon_t^2}{\nu-2}\right]^{-\frac{\nu+1}{2}}$$