In Lecture 7 slide 31 we have $(z_t = y_t)$

$$\text{(A)} \quad \begin{cases} y_t = \sigma \exp\left\{\frac{w_t}{2}\right\} z_t \\ w_t = \rho w_{t-1} + \eta_t \end{cases}, \quad \eta_t \overset{iid}{\sim} N(0, \sigma_\eta^2)$$

In Lecture 6 slide 18 we have $(\sigma_\eta^2 = \nu^2)$

$$\text{(B)} \quad \begin{cases} y_t = \exp\left\{\frac{w_t}{2}\right\} v_t \\ w_t = \omega + \phi w_{t-1} + \eta_t \end{cases}, \quad \eta_t \overset{iid}{\sim} N(0, \nu^2)$$

Let $\theta_A = (\sigma, \rho, \sigma_\eta^2)'$ and $\theta_B = (\omega, \phi, \sigma_\eta^2)'$

$\boxed{\text{From (A) to (B)}}$

Note that $y_t = \sigma \exp\left\{\frac{w_t}{2}\right\} z_t = \exp\left\{\frac{1}{2}\log \sigma^2\right\} \exp\left\{\frac{w_t}{2}\right\} z_t$

$$= \exp\left\{\frac{1}{2}\left(\log \sigma^2 + w_t\right)\right\} z_t$$

$$= \exp\left\{\frac{w_t^*}{2}\right\} z_t, \quad w_t^* = \xi + w_t$$

where $\xi = \log \sigma^2$, note that

$$w_t = \sum_{s=0}^{\infty} \rho^s \eta_{t-s}, \quad \text{and} \quad w_t^* = \xi + \sum_{s=0}^{\infty} \rho^s \eta_{t-s}$$

Note that x_t^* is the solution of an AR(1) process of the kind:

$$x_t^* = \xi(1-\rho) + \rho x_{t-1}^* + \eta_t$$

So the mapping from $\theta_A \to \theta_B$ is

$$\begin{cases} \omega = \log \sigma^2 (1-\rho) \\ \phi = \rho \\ \tau^2 = \sigma_\eta^2 \end{cases}$$

$\boxed{\text{FROM (B) to (A)}}$

Note that

$$x_t = \omega + \phi x_{t-1} + \eta_t$$

$$= \frac{\omega}{1-\phi} + \sum_{s=0}^{\infty} \phi^s \eta_{t-s}$$

such that

$$y_t = \exp\left\{\frac{x_t}{2}\right\} v_t = \exp\left\{\frac{1}{2}\left[\frac{\omega}{1-\phi} + \sum_{s=0}^{\infty} \phi^s \eta_{t-s}\right]\right\} v_t$$

$$= \exp\left\{\frac{1}{2}\left[\frac{\omega}{1-\phi}\right]\right\} \exp\left\{\frac{1}{2}\sum_{s=0}^{\infty} \phi^s \eta_{t-s}\right\} v_t$$

set $\sigma = \exp\left\{\frac{\omega}{2(1-\phi)}\right\}$, and note that

$x_t^* = \sum_{s=0}^{\infty} \phi^s \eta_{t-s}$ is the solution of an AR(1) process of the kind

$x_t^* = \phi x_{t-1}^* + \eta_t$, such that, the mapping from ϑ_B to ϑ_A is

$$\begin{cases} \sigma = \exp\left\{\frac{\omega}{2(1-\rho)}\right\} \\ \phi = \rho \\ \sigma_\eta^2 = \tau^2 \end{cases}$$

Note that

$$\sigma = \exp\left\{\frac{\omega}{2(1-\rho)}\right\}$$

$$\Downarrow$$

$$\log(\sigma) = \frac{\omega}{2(1-\rho)}$$

$$\Downarrow$$

$$\boxed{\omega = (1-\rho)\log\sigma^2}$$

i.e. the mapping is bijective.