1 Problem Set 5 - Problem 1)

1.1 Derive GAS model

We have,

$$Y_t|\mathcal{F}_{t-1} \sim \mathcal{T}(0, \phi, \nu)$$

where the mean is 0, ϕ is the scale parameter and ν is degrees of freedom. Remembering that the $\mathcal{T}$ distribution is defined not by std. deviation but by scale. However the following relation is true

$$\sigma_t^2 = \phi_t^2 \nu(\nu-2), \quad \text{for } \nu > 2$$

The conditional density of $Y_t|\mathcal{F}_{t-1}$ is given by

$$p(y_t|\mathcal{F}_{t-1}) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)\phi_t}\left[1+\frac{y_t^2}{\nu\phi_t^2}\right]^{-\frac{\nu+1}{2}}$$

Derive a GAS model with identity scaling $d = 0$ for the scale parameter ϕ_t. Use an exponential link function to ensure $\phi_t > 0$ for all t.

.

Taking the log of

$$\begin{aligned}
p(y_t|\mathcal{F}_{t-1}) &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)\phi_t}\left[1+\frac{y_t^2}{\nu\phi_t^2}\right]^{-\frac{\nu+1}{2}} \\
\ln\left[p(y_t|\mathcal{F}_{t-1})\right] &= \ln\left[\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\Gamma\left(\frac{\nu}{2}\right)\phi_t}\left[1+\frac{y_t^2}{\nu\phi_t^2}\right]^{-\frac{\nu+1}{2}}\right]
\end{aligned}$$

We know this log density is proportional to. This is due to the fact that the Γ, ν and π can be decomposed into constants. Thus they will not have an effect on the functional form but only on the level of the process.

$$\ln\left[p(y_t|\mathcal{F}_{t-1})\right] \propto -\ln[\phi_t] - \frac{\nu+1}{2}\ln\left[1+\frac{y_t^2}{\nu\phi_t^2}\right]$$

The score with respect to ϕ_t is

$$\begin{aligned}
\ln\left[p(y_t|\mathcal{F}_{t-1})\right] &= -\ln[\phi_t] - \frac{\nu+1}{2}\ln\left[1+\frac{y_t^2}{\nu\phi_t^2}\right] \\
\nabla_t \equiv \frac{\partial\nabla_t}{\partial\phi_t} &= -\ln[\phi_t] - \frac{\nu+1}{2}\ln\left[1+\frac{y_t^2}{\nu\phi_t^2}\right] \\
\nabla_t &= -\frac{1}{\phi_t} - \underbrace{\frac{\nu+1}{2}\frac{1}{1+\frac{y_t^2}{\nu\phi_t^2}}\cdot\left(-\frac{2y_t^2}{\nu\phi_t^3}\right)}_{\text{chain rule}} \\
\nabla_t &= -\frac{1}{\phi_t} + \frac{\nu+1}{\cancel{2}\left(1+\frac{y_t^2}{\nu\phi_t^2}\right)}\cdot\frac{\cancel{2}y_t^2}{\nu\phi_t^3} \\
\nabla_t &= -\frac{1}{\phi_t} + \frac{(\nu+1)\frac{y_t^2}{\nu\phi_t^3}}{1+\frac{y_t^2}{\nu\phi_t^2}} \\
\nabla_t &= -\frac{1}{\phi_t} + \frac{(\nu+1)y_t^2}{\left(1+\frac{y_t^2}{\nu\phi_t^2}\right)\nu\phi_t^3} \\
\nabla_t &= -\frac{1}{\phi_t} + \frac{(\nu+1)y_t^2}{\left(\nu\phi_t^3+\frac{\nu\phi_t^3y_t^2}{\nu\phi_t^2}\right)} \\
\nabla_t &= -\frac{1}{\phi_t} + \frac{(\nu+1)y_t^2}{(\nu\phi_t^3+\phi_t y_t^2)}
\end{aligned}$$

Replicating from Leo's solution

$$
\begin{aligned}
\underset{\text{multiplying } \frac{2}{2}}{\Longrightarrow} \quad & \nabla_t = -\frac{1}{\phi_t} + \frac{(\nu+1)}{2} \frac{2y_t^2}{(\nu\phi_t^3 + \phi_t y_t^2)} \\
& \nabla_t = -\frac{1}{\phi_t} + \frac{(\nu+1)}{2} \frac{2\frac{y_t^2}{\phi_t^2}}{\phi_t\left(\nu + \frac{y_t^2}{\phi_t^2}\right)} \\
\underset{\text{defining}}{\Longrightarrow} \quad & z_t \equiv \frac{y_t}{\phi_t} \\
& \nabla_t = -\frac{1}{\phi_t} + \frac{(\nu+1)}{2} \frac{2z_t^2}{\phi_t\,(\nu + z_t^2)} \\
& \nabla_t = -\frac{1}{\phi_t} + \frac{(\nu+1)\,z_t^2}{\phi_t\,(\nu + z_t^2)} \\
& \nabla_t = \frac{(\nu+1)\,z_t^2}{\phi_t\,(\nu + z_t^2)} - \frac{1}{\phi_t} \\
\underset{\text{factorize}}{\Longrightarrow} \quad & \nabla_t = \frac{1}{\phi_t}\left[\frac{(\nu+1)\,z_t^2}{\nu + z_t^2} - 1\right]
\end{aligned}
$$

We remember that we are to derive a GAS model with identity scaling $d = 0$. Thus we know that

$$
d = 0 \quad \rightarrow \quad \widetilde{u}_t = \frac{\partial \phi_t}{\partial \widetilde{\phi}_t} \nabla_t
$$

The GAS model for ϕ_t is

$$
\begin{aligned}
& \phi_t = \lambda\left(\widetilde{\phi}_t\right) \quad \underset{\text{exponential link function}}{\Longrightarrow} \quad \phi_t = \exp\left(\widetilde{\phi}_t\right) \\
& \widetilde{\phi}_t = \omega + \alpha \widetilde{u}_{t-1} + \beta \widetilde{\phi}_{t-1}
\end{aligned}
$$

Then deriving

$$
\begin{aligned}
\widetilde{u}_t &= \frac{\partial \phi_t}{\partial \widetilde{\phi}_t} \nabla_t \\
\widetilde{u}_t &= \frac{\partial}{\partial \widetilde{\phi}_t}\left(\exp\left(\widetilde{\phi}_t\right)\right) \frac{1}{\phi_t}\left[\frac{(\nu+1)\,z_t^2}{\nu + z_t^2} - 1\right] \\
\widetilde{u}_t &= \underbrace{\exp\left(\widetilde{\phi}_t\right)}_{=\phi_t} \frac{1}{\phi_t}\left[\frac{(\nu+1)\,z_t^2}{\nu + z_t^2} - 1\right] \\
\widetilde{u}_t &= \frac{(\nu+1)\,z_t^2}{\nu + z_t^2} - 1
\end{aligned}
$$

Now we can write up the process and we have the model

$$
\widetilde{\phi}_t = \omega + \alpha\left[\frac{(\nu+1)\,z_{t-1}^2}{\nu + z_{t-1}^2} - 1\right] + \beta \widetilde{\phi}_{t-1} \quad \square
$$

2 Problem Set 5 - Problem 2)

2.1 Derive unconditional mean to initialize ϕ

We keep the process for $\widetilde{\phi}_t$ in mind

$$
\widetilde{\phi}_t = \omega + \alpha \underbrace{\left[\frac{(\nu+1)\,z_{t-1}^2}{\nu + z_{t-1}^2} - 1\right]}_{\widetilde{u}_{t-1}} + \beta \widetilde{\phi}_{t-1}
$$

Using recursive substitution and unfolding the process of $\widetilde{\phi}_t$ we obtain:

$$\begin{aligned}
\widetilde{\phi}_t &= \frac{\omega}{1-\beta} + \alpha \sum_{s=0}^{\infty} \beta^s u_{t-s-1} \\
\mathbb{E}\left[\widetilde{\phi}_t\right] &= \mathbb{E}\left[\frac{\omega}{1-\beta} + \alpha \sum_{s=0}^{\infty} \beta^s u_{t-s-1}\right] \\
\mathbb{E}\left[\widetilde{\phi}_t\right] &= \frac{\omega}{1-\beta} + \underbrace{\mathbb{E}\left[\alpha \sum_{s=0}^{\infty} \beta^s u_{t-s-1}\right]}_{=0,\text{ dont know why...}} \\
\mathbb{E}\left[\widetilde{\phi}_t\right] &= \frac{\omega}{1-\beta}
\end{aligned}$$

This is the **unconditional mean**. We know that we have an exponential link function, whereas

$$\phi_t = \exp\left(\widetilde{\phi}_t\right)$$

Now we can use these values to initialize our GAS log-likelihood function.

2.2 Derive starting parameters

A way of initializing $\theta = (\omega,\ \alpha,\ \beta,\ \nu)$
We can compute:

$$\begin{aligned}
\hat{\sigma}^2 &= Var\left(y_t\right) \\
&= \frac{1}{T}\sum_t y_t^2
\end{aligned}$$

$$\begin{aligned}
\sigma_t^2 &= \frac{\phi_t^2 \nu}{\nu - 2} \\
E\left(\sigma_t^2\right) &= \frac{\nu}{\nu-2} E\left(\exp\left(2\hat{\phi}_t\right)\right)
\end{aligned}$$

This expectation can be derived analytically, but this is very technical. Therefore we use an approximation.

$$\begin{aligned}
E\left(\sigma_t^2\right) &= \frac{\nu}{\nu-2} E\left(\exp\left(2\hat{\phi}_t\right)\right) \\
&\approx \frac{\nu}{\nu-2} \exp\left(2E\left(\hat{\phi}_t\right)\right) \\
&= \frac{\nu}{\nu-2} \exp\left(\frac{2\omega}{1-\beta}\right)
\end{aligned}$$

We initialize ω:

$$\begin{aligned}
\sigma^2 \frac{(\nu-2)}{\nu} &= \exp\left(\frac{2\omega}{1-\beta}\right) \\
\ln\left(\sigma^2 \frac{(\nu-2)}{\nu}\right) \frac{1}{2}(1-\beta) &= \omega
\end{aligned}$$

(Note that σ^2 is here the variance of Y)
So

$$\omega^{\text{initialized}} = \ln\left(\hat{\sigma}^2 \frac{(\nu-2)}{\nu}\right) \frac{1}{2}(1-\beta)$$