DEPARTMENT OF ECONOMICS
AND BUSINESS ECONOMICS
AARHUS UNIVERSITY

EXAM ASSIGNMENT

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<tr><td>Date and time</td><td colspan="4">6-10 January 2022</td><td colspan="2">12:00 noon – 12:00 noon</td></tr>
<tr><td>Anonymous exam?</td><td>Yes</td><td>X</td><td>No</td><td></td><td colspan="2">Comments: Please do not write your name or student ID number anywhere. Use your flow ID number (find this on the cover sheet).</td></tr>
<tr><td>Other relevant information</td><td colspan="6">Avoid being suspected of exam cheating
Remember to state references and use quotation marks if you copy text from other sources or re-use parts of a previously submitted exam paper (plagiarism and self-plagiarism). Students must answer the exam assignment individually. All submitted exam papers are checked for plagiarism, so cheating and collaboration between students will be detected.</td></tr>
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How to contact the lecturer during a take-home exam
Questions to any problems in the assignment must be posed during the first 4 hours to Leopoldo Catania via email at leopoldo.catania@econ.au.dk. Students should not expect instant answers as a collective answer to all questions received during these 4 hours will be sent out via Brightspace once the 4 hours have passed.

Exam questions for 4394:Financial Econometrics

Leopoldo Catania

Guidelines

General guidelines

The exercises must be done individually. You have to submit two files: one PDF file and one file with the computer code. The computer code should be submitted as "appendix". The PDF file should include the answers to the theoretical part as well as any answer to the computational and empirical part which requires a written comment. The computer code should be organized such that it can reproduce all the computational and empirical answers of your exam. Code must be commented. The exam will be graded on the Danish 7-point grading scale. Good luck!

Guidelines for the theoretical part

Try to be as precise as possible and write your results in a PDF document. If you plan to write a Word document and then transform it to PDF, please be sure that all formulas are well formatted before submitting your exam. When you write your answers, you have to be consistent with the notation used in the exam questions.

Guidelines for writing and presenting your code

You are required to comment on your code. For example, state the purpose of the functions you code, their arguments, and what they return as an output. I should be able to run your code without major modifications. You are free to employ your favorite optimization scheme. Make sure to remove all details that can reveal your identity from the code.

Theoretical part

Consider the general volatility model:

$$\begin{aligned} y_t &= \sigma_t \varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} (0,1) \\ \sigma_{t+1}^2 &= \omega + \sigma_t^2(\beta + \alpha g(\varepsilon_t)), \end{aligned} \tag{1}$$

where $\varepsilon_t \overset{iid}{\sim} (0,1)$ means that the random variable ε_t is identically and independently distributed over time with $\mathbb{E}[\varepsilon_t] = 0$ and $\mathbb{E}[\varepsilon_t^2] = 1$. Also assume that ε_t is symmetrically distributed around zero.

The real-valued function $g()$ maps the real line, $\mathbb{R}$, to the interval $[\underline{m}, \overline{m}]$ ($g : \mathbb{R} \to [\underline{m}, \overline{m}]$) where $0 \leq \underline{m} < \overline{m}$, it is even ($g(x) = g(-x)$), and such that $\mathbb{E}[g(\varepsilon_t)] = \bar{g}$ with $\underline{m} < \bar{g} < \overline{m}$, for all t. Model (1) nests many volatility models such as the GARCH for $g(\varepsilon_t) = \varepsilon_t^2$ and the GJRGARCH for $g(\varepsilon_t) = \varepsilon_t^2 + \gamma \mathbb{1}(\varepsilon_t < 0)$, where $\mathbb{1}(A)$ is the indicator function for the event A.

Indicate by $\{{}_u\sigma_t^2\}_{t\in\mathbb{Z}}$ the squared volatility process initialized at the infinite past, and by $\{\sigma_t^2\}_{t\in\mathbb{N}}$ the squared volatility process initialized at time 0 ($t = 0$) at the value $\sigma_0^2 > 0$. Exploiting the results of Nelson (1991), solve:

1) Show that ${}_u\sigma_t^2$ admits the following representation:

$${}_u\sigma_t^2 = \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} (\beta + \alpha g(\varepsilon_{t-i}))\right]$$

2) Show that σ_t^2 admits the following representation:

$$\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} (\beta + \alpha g(\varepsilon_{t-i})) + \omega \left[1 + \sum_{k=1}^{t-1} \prod_{i=1}^{k} (\beta + \alpha g(\varepsilon_{t-i}))\right]$$

3) Derive sufficient conditions on (ω, α, β) such that ${}_u\sigma_t^2 > 0$.

4) Derive the lower and the upper bound of the process $\{{}_u\sigma_t^2\}_{t\in\mathbb{Z}}$, i.e. show that ${}_u\sigma_t^2 \in [l, u]$ where $l < u$. Derive l and u.

5) Show that if $\omega = 0$:

   a) ${}_u\sigma_t^2 = 0$ for all t.

   b) $\sigma_t^2 \to \infty$ as $t \to \infty$ if $E[\log(\beta + \alpha g(\varepsilon_t))] > 0$.

   c) $\sigma_t^2 \to 0$ as $t \to \infty$ if $E[\log(\beta + \alpha g(\varepsilon_t))] < 0$.

6) Show that if $\omega > 0$ and $E[\log(\beta + \alpha g(\varepsilon_t))] > 0$:

a) $\sigma_t^2 \to \infty$ as $t \to \infty$

b) ${}_u\sigma_t^2 \to \infty$ for all t

7) Consider the GARCH case where $g(\varepsilon_t) = \varepsilon_t^2$.

a) Discuss the necessary conditions on (ω, α, β) such that: i) $\{y_t\}$ is weakly stationary, ii) $\{y_t\}$ is strongly stationary.

b) Let $\omega = 0.1$, $\alpha = 0.041$, $\beta = 0.96$, and assume that ε_t is iid standard Gaussian, i.e. $\varepsilon_t \overset{iid}{\sim} N(0,1)$. Discuss whether $\{y_t\}$ is weakly stationary and/or strongly stationary.

c) Let $\omega = 0.1$, $\alpha = 0.06$, $\beta = 0.96$, and assume that ε_t is iid standard Student's t, i.e. $\varepsilon_t \overset{iid}{\sim} T(0,1,\nu)$, with density

$$p(\varepsilon_t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\nu-2)\pi}}\left[1 + \frac{\varepsilon_t^2}{\nu-2}\right]^{-\frac{\nu+1}{2}}.$$

Discuss whether $\{y_t\}$ is weakly stationary and/or strongly stationary in the cases $\nu = 2.5$ and $\nu = 20$. What do you observe here?

Hint for points 5b)-5c): i) Compute $\log \sigma_t^2$, ii) show that this is a Random Walk (RW) with drift $\mathbb{E}[\log(\beta + \alpha g(\varepsilon_t))]$, iii) Remember that a RW with a drift either diverges to $+\infty$ or $-\infty$, depending on the value of the drift.

Hint for points 7b)-7c): You need to use R to compute the moment condition required to check strong stationarity. You can compute it by Monte Carlo simulation (use a large number of draws, like `1e7`) or by numerical integration using the function `integrate()`.

Computational part

Let $\mathbf{y}_t = (y_{1,t}, \ldots, y_{p,t})'$ be a vector of p financial returns with $Var(\mathbf{y}_t|\mathcal{F}_{t-1}) = \mathbf{\Sigma}_t$, where $\mathcal{F}_{t-1}$ indicates the past information of the process $\{\mathbf{y}_t\}$. Tse and Tsui (2002) developed the following multivariate volatility model:

$$\mathbf{y}_t = \mathbf{\Sigma}_t^{1/2}\boldsymbol{\varepsilon}_t, \tag{2}$$

where $\mathbf{\Sigma}_t = \mathbf{\Sigma}_t^{1/2}\mathbf{\Sigma}_t^{1/2'}$, and it is assumed that $\mathbb{E}[\boldsymbol{\varepsilon}_t|\mathcal{F}_{t-1}] = \mathbf{0}$ and $Var(\boldsymbol{\varepsilon}_t|\mathcal{F}_{t-1}) = \mathbb{I}_p$ where $\mathbb{I}_p$ is the $p \times p$ identity matrix. The following factorization of the covariance matrix is considered: $\mathbf{\Sigma}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t$, where $\mathbf{D}_t$ is a diagonal matrix with generic element $\sigma_{i,t} = \sqrt{Var(y_{i,t}|\mathcal{F}_{t-1})}$, i.e. the volatility of asset i at time t given past information, which is assumed to follow a GARCH(1,1) process:

$$\sigma_{i,t}^2 = \omega_i + \alpha_i y_{i,t-1}^2 + \beta_i \sigma_{i,t-1}^2, \tag{3}$$

with $\omega_i > 0$, $\alpha_i > 0$, $\beta_i \geq 0$ and $\alpha_i + \beta_i < 1$ for all $i = 1, \ldots, p$. $\mathbf{R}_t$ is the time-varying correlation matrix with generic element $\rho_{ij,t} = Cor(y_{i,t}, y_{j,t}|\mathcal{F}_{t-1})$, where $i, j = 1, \ldots, p$. The peculiar feature of the model by Tse and Tsui (2002) is the updating equation for $\mathbf{R}_t$ which is defined as:

$$\mathbf{R}_t = (1 - a - b)\mathbf{R} + a\mathbf{\Psi}_{t-1} + b\mathbf{R}_{t-1},$$

where $a > 0$, $b > 0$ are scalars such that $a + b < 1$, $\mathbf{R}$ is a correlation matrix (usually set to the empirical correlation matrix estimated from the data), and $\mathbf{\Psi}_{t-1}$ is a $p \times p$ matrix with generic element:

$$\psi_{ij,t-1} = \frac{\sum_{h=1}^{M} \eta_{i,t-h}\eta_{j,t-h}}{\sqrt{\left(\sum_{h=1}^{M} \eta_{i,t-h}^2\right)\left(\sum_{h=1}^{M} \eta_{j,t-h}^2\right)}},$$

where $\eta_{j,t} = y_{j,t}/\sigma_{j,t}$, for $j = 1, \ldots, p$, and $M \geq 1$ is a constant that must be selected. Note that, the fact that $\mathbf{R}$ is a correlation matrix together with the constraints on a and b, ensures that $\mathbf{R}_t$ is a correlation matrix at each t.

1) Write a code to simulate T observations from the model of Tse and Tsui (2002) under the assumption that $\boldsymbol{\varepsilon}_t$ is iid multivariate Gaussian with mean $\mathbf{0}$ and covariance matrix $\mathbb{I}_p$. It should be possible to choose any (reasonable) M. Set $\mathbf{R}_t = \mathbf{R}$ for $t \leq M$.

2) Write a code to estimate a and b using the two steps Quasi Maximum Likelihood estimator for multivariate volatility models discussed in Engle (2002). You can estimate the univariate models using the **rugarch** package. Set $\mathbf{R}$ to the empirical correlation matrix estimated from the data.

3) Set the seed to 123 (`set.seed(123)`), $p = 2$, $M = 5$, $\omega_i = 0.01$, $\alpha_i = 0.04$, $\beta = 0.95$ for $i = 1, 2$, $a = 0.003$, $b = 0.995$ and $\mathbf{R} = \begin{pmatrix} 1 & 0.8 \\ 0.8 & 1 \end{pmatrix}$.

   a) Simulate $T = 1000$ observations from the model using the function you created at point 1). Plot the series of simulated returns, time-varying variances, and time-varying correlation.

   b) Estimate the parameters of the model using the function you created at point 2). Compare the estimated parameters with the true ones. (It is a good idea to initialize the optimizer at the true values of a and b).

Hint for point 2): It is easier to use the `solnp()` function of the **Rsolnp** package to perform the numerical optimization and impose the constraints $a > 0$, $b > 0$, $a + b < 1$.

Empirical part

Consider $p = 3$ series of your choice from the `dji30ret` dataset in the **rugarch** package. Multiply the series by 100 and remove the mean.

1) Report descriptive statistics of your choice for the three series. Discuss the unconditional distribution of these series and provide evidence for the presence of heteroscedasticity.

2) For $M = 1, \ldots, 5$ estimate the model of Tse and Tsui (2002) on the three series. Select the value of M that resulted in the highest log likelihood value at its maximum.

3) Estimate the DCC model of Engle (2002) using the **rmgarch** (i.e., the returns have the stochastic representation reported in (2), with GARCH variances as in (3), but $\mathbf{R}_t$ follows the DCC updating equation). Which model between the one of Tse and Tsui (2002) and the one of Engle (2002) provides the highest likelihood value?

4) In a plot, compare the estimated correlations of the two models.

5) Draw some general conclusion about the evolution of the estimated volatilities and correlations for the series you considered.

References

Engle, R. (2002). Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, 20(3):339–350.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica: Journal of the Econometric Society*, pages 347–370.

Tse, Y. K. and Tsui, A. K. C. (2002). A multivariate generalized autoregressive conditional heteroscedasticity model with time-varying correlations. *Journal of Business & Economic Statistics*, 20(3):351–362.