1) Derive the GAS updating....

We have

$$p(y) = \frac{\exp\left\{ - \frac{|(y-\mu)/\varphi|^{\nu}}{2} \right\}}{2^{1+1/\nu}\, \varphi\, \Gamma\left(1+\frac{1}{\nu}\right)}$$

such that

$$\ln p(y) \propto -\ln \varphi - \frac{|y-\mu|^{\nu}}{2\varphi^{\nu}}$$

Note that $\varphi > 0$ such that $|\varphi| = \varphi$

the score with respect to φ is

$$\frac{\partial \ln p(y)}{\partial \varphi} = -\frac{1}{\varphi} + \nu \frac{|y-\mu|^{\nu}}{2\varphi^{\nu+1}} = \frac{1}{\varphi}\left(\frac{\nu}{2} \frac{|y-\mu|^{\nu}}{\varphi^{\nu}} - 1 \right)$$

Note that when $\nu = 2$ we obtain $\frac{1}{\varphi}\left[\left(\frac{y-\mu}{\varphi^2}\right)^2 - 1 \right]$

which is the score of the Gaussian distribution.

The GAS-GED model is defined as

$$\varphi_t = \exp\{\check{\varphi}_t\}$$

$$\check{\varphi}_t = w + \alpha s_{t-1} + \beta \check{\varphi}_{t-1}$$

where $s_t = \dfrac{\partial \ln P(y)}{\partial \check{\varphi}_t} = \dfrac{\partial \varphi_t}{\partial \check{\varphi}_t} \dfrac{\partial \ln P(y)}{\partial \varphi_t} = \varphi_t \cdot \dfrac{1}{\varphi_t}\left[\dfrac{v}{2}\dfrac{|y|^v}{\varphi_t^v} - 1\right]$

$$= \frac{v}{2}\frac{|y_t|^v}{\varphi_t^v} - 1$$

$$\mathbb{E}[s_t \mid \mathcal{F}_{t-1}] = \int s_t \, p(y_t \mid \mathcal{F}_{t-1})\, dy_t =$$

$$\int \frac{\partial \log p(y_t \mid \mathcal{F}_{t-1})}{\partial \tilde{\varphi}_t} p(y_t \mid \mathcal{F}_{t-1})\, dy_t =$$

$$= \int \frac{\partial p(y_t \mid \mathcal{F}_{t-1})}{\partial \check{\varphi}_t} \frac{1}{\cancel{p(y_t \mid \mathcal{F}_{t-1})}} \cancel{p(y_t \mid \mathcal{F}_{t-1})}\, dy_t =$$

$$= \int \frac{\partial p(y_t \mid \mathcal{F}_{t-1})}{\partial \check{\varphi}_t} dy_t =$$

$$= \frac{\partial}{\partial \tilde{\varphi}_t} \int p(y_t \mid \mathcal{F}_{t-1})\, dy_t = \frac{\partial}{\partial \tilde{\varphi}_t} 1 = 0.$$

Also note that

$$\mathbb{E}[s_t] = \mathbb{E}\left[\mathbb{E}[s_t \mid \mathcal{F}_{t-1}]\right] = \mathbb{E}[0] = 0.$$

2) The log likelihood:

$$p(y_t \mid \mathcal{F}_{t-1}) = \frac{\exp\left\{ - \frac{|y_t/\varphi_t|^{\nu}}{2} \right\}}{2^{1+1/\nu}\, \varphi_t\, \Gamma\left(1+\frac{1}{\nu}\right)}$$

$$\ln p(y_t \mid \mathcal{F}_{t-1}) = -\left(1+\frac{1}{\nu}\right)\ln 2 - \ln \varphi_t - \ln \Gamma\left(1+\frac{1}{\nu}\right) + \\ - \frac{|y_t|^{\nu}}{\varphi_t^{\nu} 2}$$

the log likelihood is

$$\mathcal{L}(y_{1:T} \mid \theta) = -T\left(1+\frac{1}{\nu}\right)\ln 2 - T\ln\Gamma\left(1+\frac{1}{\nu}\right) + \\ - \sum_{t=1}^{T} \ln \varphi_t - \frac{1}{2}\sum_{t=1}^{T} \frac{|y_t|^{\nu}}{\varphi_t^{\nu}}$$

where $\varphi_t = \varphi_t(\theta)$, and $\theta = (\omega, \alpha, \beta, \nu)'$

Constraints to impose that $\tilde{\varphi}_t$ is covariance stationary.

$$\tilde{\varphi}_t = \omega + \alpha s_{t-1} + \beta \tilde{\varphi}_{t-1}$$

can be written as

$$\tilde{\varphi}_t = \frac{\omega}{1-\beta} + \alpha \sum_{s=0}^{\infty} \beta^s s_{t-1-s}$$

<u>First moment</u>

$$\mathbb{E}[\tilde{\varphi}_t] = \mathbb{E}\left[\frac{\omega}{1-\beta} + \alpha \sum_{k=0}^{\infty} \beta^k s_{t-1-k}\right] =$$

$$= \frac{\omega}{1-\beta} + \alpha\, \mathbb{E}\left[\sum_{k=0}^{\infty} \beta^k s_{t-1-k}\right] = \frac{\omega}{1-\beta} + \alpha \sum_{k=0}^{\infty} \beta^k\, \mathbb{E}[s_{t-1-k}]$$

and since $\mathbb{E}[s_t] = 0 \quad \forall\, t$

$$\mathbb{E}[\tilde{\varphi}_t] = \frac{\omega}{1-\beta} < \infty \qquad \forall\ |\beta| < 1 \text{ and } |\omega| < \infty$$

<u>Second moment</u>

$$\mathbb{E}[\tilde{\varphi}_t^2] = \mathbb{E}\left[\left(\frac{\omega}{1-\beta} + \alpha \sum_{k=0}^{\infty} \beta^k s_{t-1-k}\right)^2\right]$$

$$= \frac{w^2}{(1-\beta)^2} + \alpha^2 \mathbb{E}\left[\left(\sum_{k=0}^{\infty} \beta^k s_{t-1-k}\right)^2\right] +$$

$$+ \frac{2\alpha w}{1-\beta} \mathbb{E}\left[\sum_{k=0}^{\infty} \beta^k s_{t-1-k}\right] =$$

$$= \frac{w^2}{(1-\beta)^2} + \alpha^2 \mathbb{E}\left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \beta^k \beta^l s_{t-1-k} s_{t-1-l}\right] + \frac{2\alpha w}{1-\beta} \sum_{k=0}^{\infty} \underbrace{\mathbb{E}\left[\beta^k s_{t-1-k}\right]}_{=0}$$

$$= \frac{w^2}{(1-\beta)^2} + \alpha^2 \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \beta^k \beta^l \mathbb{E}\left[s_{t-1-k} s_{t-1-l}\right]$$

Consider

by assumption

$\downarrow$

$$\mathbb{E}\left[s_{t-1-k} s_{t-1-l}\right] = \begin{cases} \mathbb{E}\left[s_{t-1-k}^2\right] = c < \infty & \text{if } k = l \\ \mathbb{E}\left[s_{t-1-k} s_{t-1-l}\right] & \text{if } k \neq 0 \end{cases}$$

Assume $k < l$

$$\mathbb{E}\left[s_{t-1-k} s_{t-1-l}\right] = \mathbb{E}\left[\mathbb{E}\left[s_{t-1-k} s_{t-1-l} \mid \mathcal{F}_{t-1-k}\right]\right] = \mathbb{E}\left[s_{t-1-k}\, \mathbb{E}\left[s_{t-1-l} \mid \mathcal{F}_{t-1-k}\right]\right]$$

but

$$\mathbb{E}\left[s_{t-1-l} \mid \mathcal{F}_{t-1-k}\right] = \mathbb{E}\left[\mathbb{E}\left[s_{t-1-l} \mid \mathcal{F}_{t-2-l}\right] \mid \mathcal{F}_{t-1-k}\right] =$$

$$= \mathbb{E}\left[0 \mid \mathcal{F}_{t-1-k}\right] = 0$$

if $k > l$ the same applies.

so $\mathbb{E}[S_{t-1-k} S_{t-1-l}] = \begin{cases} e < \infty & \text{if } k = l \\ 0 & \text{if } k \neq l \end{cases}$

then

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \beta^k \beta^l \mathbb{E}[S_{t-1-k} S_{t-1-l}] = \sum_{k=0}^{\infty} \beta^{2k} e = \frac{e}{1-\beta^2}$$

and

$$\mathbb{E}[\tilde{\varphi}_t^2] = \frac{w^2}{(1-\beta)^2} + \frac{\alpha^2 e}{1-\beta^2} < \infty \quad \text{if} \quad \begin{aligned} |\alpha| &< \infty \\ |w| &< \infty \\ |\beta| &< 1 \end{aligned}$$

The autocovariance

$$\mathrm{Cov}(\tilde{\varphi}_t, \tilde{\varphi}_{t-k}) = \mathbb{E}[\tilde{\varphi}_t \tilde{\varphi}_{t-k}] - \mathbb{E}[\tilde{\varphi}_t]\mathbb{E}[\tilde{\varphi}_{t-k}]$$

$$= \mathbb{E}[\tilde{\varphi}_t \tilde{\varphi}_{t-k}] - \frac{w^2}{(1-\beta^2)}$$

We have to study $\mathbb{E}[\tilde{\varphi}_t \tilde{\varphi}_{t-k}]$

Consider the case $k=1$

$$\mathbb{E}[\tilde{\varphi}_t \tilde{\varphi}_{t-1}] = \mathbb{E}\left[(\omega + \alpha s_{t-1} + \beta \tilde{\varphi}_{t-1})\tilde{\varphi}_{t-1}\right] =$$

$$= \omega \mathbb{E}[\tilde{\varphi}_{t-1}] + \alpha \underbrace{\mathbb{E}[s_{t-1}\tilde{\varphi}_{t-1}]}_{=0} + \beta \mathbb{E}[\tilde{\varphi}_{t-1}^2]$$

Consider the case $k=2$

$$\mathbb{E}[\tilde{\varphi}_t \tilde{\varphi}_{t-2}] = \mathbb{E}\left[(\omega + \alpha s_{t-1} + \beta \tilde{\varphi}_{t-1})\tilde{\varphi}_{t-2}\right] =$$

$$= \omega \mathbb{E}[\tilde{\varphi}_{t-2}] + \underbrace{\alpha \mathbb{E}[s_{t-1}\tilde{\varphi}_{t-2}]}_{=0} + \beta \mathbb{E}[\tilde{\varphi}_{t-1}\tilde{\varphi}_{t-2}]$$

$$= \omega \mathbb{E}[\tilde{\varphi}_{t-2}] + \beta\left\{\omega \mathbb{E}[\tilde{\varphi}_{t-1}] + \beta \mathbb{E}[\tilde{\varphi}_{t-1}^2]\right\}$$

$$= \omega \mathbb{E}[\tilde{\varphi}_{t-2}] + \beta\omega \mathbb{E}[\tilde{\varphi}_{t-1}] + \beta^2 \mathbb{E}[\tilde{\varphi}_{t-1}^2]$$

However, we know that $\mathbb{E}[\tilde{\varphi}_{t-1}] = \mathbb{E}[\tilde{\varphi}_{t-2}] = \frac{\omega}{1-\beta}$

$$= \omega\left(\mathbb{E}[\tilde{\varphi}_{t-1}] + \beta \mathbb{E}[\tilde{\varphi}_{t-1}]\right) + \beta^2 \mathbb{E}[\tilde{\varphi}_{t-1}^2]$$

By iterative substitutions we obtain

$$\mathbb{E}[\tilde{\varphi}_t \tilde{\varphi}_{t-k}] = w\, \mathbb{E}[\tilde{\varphi}_{t-w}] \sum_{l=0}^{k-1} \beta^l + \beta^k \mathbb{E}[\tilde{\varphi}_{t-1}^2]$$

$$= \frac{w^2}{1-\beta} \sum_{l=0}^{k-1} \beta^l + \beta^k \left[\frac{w^2}{(1-\beta)^2} + \frac{\alpha^2 c}{1-\beta^2} \right]$$

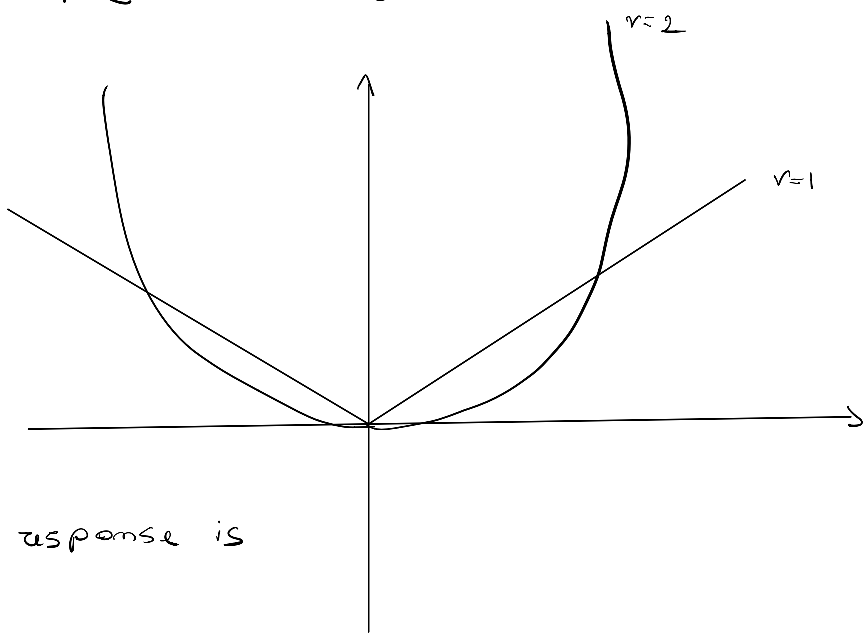
which does not depend from t

So, conditions for covariance stationarity are

$$\boxed{|\beta| < 1, \quad |w| < \infty, \quad |\alpha| < \infty}$$

3) When $v=2$ we have

$$S_t = \frac{y_t^2}{\varphi_t^2} - 1$$



when $v=2$ the response is quadratic

when $v=1$ the response is piecewise linear