1 Problem Set 3 - Problem 1

Consider the SV model reported in slide 31 of Lecture 7 and the SV model reported in slide 18 of Lecture 6 . Note that the two models are parameterized in a different way. In the one of Lecture 7 the log volatility follows a zero mean autoregression and you have a parameter σ in the measurement equation. In the one of Lecture 6 you have that the volatility follows a first order autoregression with mean $\omega/(1-\phi)$. Find the mapping between the two parameterizations, i.e. find a way to represent the model in Lecture 6 as the model in Lecture 7 , and viceversa.

A) From slide 31, lecture 7

$$\begin{aligned} y_t &= \sigma \exp\left(\frac{w_t}{2}\right) z_t \\ w_t &= \rho w_{t-1} + \eta_t, \quad \eta_t \overset{iid}{\sim} \mathcal{N}\left(0, \sigma_\eta^2\right) \end{aligned}$$

B) From slide 18, lecture 6

$$\begin{aligned} y_t &= \exp\left(\frac{w_t}{2}\right) u_t \\ w_t &= \omega + \phi w_{t-1} + \eta_t, \quad \eta_t \overset{iid}{\sim} \mathcal{N}\left(0, \sigma_\eta^2\right) \end{aligned}$$

Let $\theta_A = \left(\sigma, \rho, \sigma_\eta^2\right)'$ and $\theta_B = \left(\omega, \phi, \sigma_\eta^2\right)'$

1.1 From (A) to (B)

$$\begin{aligned} & y_t = \sigma \exp\left(\frac{w_t}{2}\right) z_t \\ \underset{\ln(x^2)=2\ln(x),\ \exp[\ln(x)]=x}{\Longrightarrow} \quad & y_t = \exp\left[\frac{1}{2}\ln\left\{\sigma^2\right\}\right] \exp\left(\frac{w_t}{2}\right) z_t \\ & y_t = e^{\frac{1}{2}\ln\left\{\sigma^2\right\}} e^{\frac{w_t}{2}} z_t \\ & y_t = e^{\frac{1}{2}\ln\left\{\sigma^2\right\} + \frac{w_t}{2}} z_t \\ & y_t = \exp\left[\frac{1}{2}\ln\left\{\sigma^2\right\} + \frac{w_t}{2}\right] z_t \\ & y_t = \exp\left[\frac{1}{2}\underbrace{\left(\ln\left\{\sigma^2\right\} + w_t\right)}_{w_t^\star}\right] z_t \\ & y_t = \exp\left(\frac{w_t^\star}{2}\right) z_t \end{aligned}$$

Note that $w_t^\star$ is the solution of an $AR(1)$ process of the kind

$$w_t^\star = \ln\left(\sigma^2\right)(1-\rho) + \rho w_{t-1}^\star + \eta_t$$

Thus we get the mappings

$$\theta_A \to \theta_B = \begin{cases} \omega & = \ln\left(\sigma^2\right)(1-\rho) \\ \phi & = \rho \\ \sigma_\eta^2 & = \sigma_\eta^2 \end{cases}$$

1.2 From (B) to (A)

TODO: make this derivation thoroughly.

Thus we get the mappings

$$\theta_B \to \theta_A = \begin{cases} \sigma & = \exp\left\{\frac{\omega}{2(1-\rho)}\right\} \\ \phi & = \rho \\ \sigma_\eta^2 & = \sigma_\eta^2 \end{cases}$$

Note that,

$$
\begin{aligned}
\sigma &= \exp\left\{\frac{\omega}{2\left(1-\rho\right)}\right\} \\
\ln\left(\sigma\right) &= \frac{\omega}{2\left(1-\rho\right)} \\
\omega &= \left(1-\rho\right)\cdot\ln\left(\sigma^2\right)
\end{aligned}
$$

i.e. the mapping is bijective.