23/03/2021

ARCH(1)

$$\begin{cases} \varepsilon_t = \sigma_t z_t \,, & z_t \overset{iid}{\sim} N(0,1) \\ \sigma_t^2 = w + \alpha\, \varepsilon_{t-1}^2 \end{cases} \Longrightarrow \varepsilon_t \mid \mathcal{F}_{t-1} \sim N(0, \sigma_t^2)$$

$$P(\varepsilon_t \mid \mathcal{F}_{t-1}) \neq P(\varepsilon_t)$$

conditional dist $\neq$ unconditional dist.

$$K(\varepsilon_t \mid \mathcal{F}_{t-1}) = 3 \neq K(\varepsilon_t) = 3\,\frac{1-\alpha^2}{1-3\alpha^2} > 3$$

Kurtosis conditional dist.

Kurt. uncond. dist.

GARCH(1,1)

$$\begin{cases} \varepsilon_t = \sigma_t z_t \,, & z_t \overset{iid}{\sim} N(0,1) \\ \sigma_t^2 = \underline{w + \alpha\, \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2} \end{cases}$$

$$K(\varepsilon_t) = \frac{E[\varepsilon_t^4]}{(E[\varepsilon_t^2])^2}$$

$$E[\varepsilon_t^2] = E[\sigma_t^2] = \frac{w}{1-\alpha-\beta}$$

$$E[\varepsilon_t^4] = E[E[\sigma_t^4 z_t^4 \mid \mathcal{F}_{t-1}]]$$

$$= E[\sigma_t^4 \underbrace{E[z_t^4 \mid \mathcal{F}_{t-1}]}_{3}]$$

$$= 3E[\sigma_t^4]$$

$$\sigma_t^4 = w^2 + \alpha^2 z_{t-1}^4 + \beta^2 \sigma_{t-1}^4 + $$
$$+ 2w\alpha z_{t-1}^2 + 2\alpha\beta z_{t-1}^2 \sigma_{t-1}^2 + 2w\beta\sigma_{t-1}^2$$

$$\mathbb{E}[\sigma_t^4] = w^2 + 3\alpha^2 \mathbb{E}[\sigma_{t-1}^4] + \beta^2 \mathbb{E}[\sigma_{t-1}^4] +$$
$$+ 2w\alpha\, \mathbb{E}[\sigma_{t-1}^2] + 2\alpha\beta\, \mathbb{E}[\sigma_{t-1}^4] + 2w\beta\, \mathbb{E}[\sigma_{t-1}^2]$$

Under w.s. we have $\mathbb{E}[\sigma_t^2] = \mathbb{E}[\sigma_{t-j}^2] \quad \forall j$

$\mathbb{E}[\sigma_t^4] = \mathbb{E}[\sigma_{t-j}^4] \quad \forall j$

$$\mathbb{E}[\sigma_t^4]\left(1 - 3\alpha^2 - \beta^2 - 2\alpha\beta\right) = w^2 + \mathbb{E}[\sigma_{t-1}^2]\left(2w\alpha + 2w\beta\right)$$
$$// \quad = w^2 + \frac{2w^2(\alpha+\beta)}{1-\alpha-\beta}$$
$$// \quad = \frac{w^2 - \cancel{w^2(\alpha+\beta)} + \cancel{2}w^2(\alpha+\beta)}{1-\alpha-\beta}$$
$$// \quad = \frac{w^2(1+\alpha+\beta)}{1-\alpha-\beta}$$

$$\mathbb{E}[\sigma_t^4] = \frac{w^2(1+\alpha+\beta)}{(1-\alpha-\beta)(1-3\alpha^2-\beta^2-2\alpha\beta)}$$

$$\mathbb{E}[z_t^4] = 3\,\mathbb{E}[\sigma_t^4]$$

$\boxed{27/09/2021}$

$$\sigma_t^2 = \omega + \alpha z_{t-1}^2 + \beta \sigma_{t-1}^2 \quad , \quad z_{t-1}^2 = \sigma_{t-1}^2 z_{t-1}^2$$

$$= \omega + \alpha \sigma_{t-1}^2 z_{t-1}^2 + \beta \sigma_{t-1}^2$$

$$= \omega + \sigma_{t-1}^2 (\alpha z_{t-1}^2 + \beta) \Rightarrow \underline{\sigma_{t-s}^2 = \omega + \sigma_{t-s-1}^2 (\alpha z_{t-s-1}^2 + \beta)}$$

$$= \omega + (\omega + \sigma_{t-2}^2 (\alpha z_{t-2}^2 + \beta))(\alpha z_{t-1}^2 + \beta)$$

$$= \omega [1 + (\alpha z_{t-1}^2 + \beta)] + \sigma_{t-2}^2 (\alpha z_{t-2}^2 + \beta)$$

$$\vdots$$

$$= \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} (\alpha z_{t-i}^2 + \beta)\right]$$

Such that

$$E[\sigma_t^2] = E\left[\omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} (\alpha z_{t-i}^2 + \beta)\right]\right]$$

$$= \omega \left[1 + \sum_{i=1}^{\infty} E\left[\prod_{i=1}^{k} (\alpha z_{t-i}^2 + \beta)\right]\right]$$

$$b_t = \alpha z_t^2 + \beta \quad , \quad E[b_t] = \alpha E[z_t^2] + \beta = \alpha + \beta$$

$$= \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} E[\alpha z_{t-i}^2 + \beta]\right]$$

$X \perp Y$

$E[XY] = E[X]E[Y]$

$$= \omega \left[1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} (\alpha + \beta) \right]$$

$$= \omega \left[1 + \sum_{k=1}^{\infty} (\alpha + \beta)^k \right] = \omega \sum_{k=0}^{\infty} (\alpha + \beta)^k = \frac{\omega}{1 - \alpha - \beta}$$

$X \sim$ r.v.

$E[X]$ exists iff

$E[X^+] < \infty$, $E[X^-] = \begin{cases} -\infty \\ c \end{cases}$

$E[X^-] < \infty$, $E[X^+] = \begin{cases} +\infty \\ c \end{cases}$

$E[X]$ does not exist

$E[X^+] = \infty$, $E[X^-] = -\infty$

Thm. 1 Nelson (1990) $(\omega = 0)$

$$\sigma_t^2 = \sigma_0^2 \prod_{i=1}^{t} (\beta + \alpha z_{t-i}^2)$$

$$\ln \sigma_t^2 = \ln \sigma_0^2 + \sum_{i=1}^{t} \ln (\beta + \alpha z_{t-i}^2)$$

$E[\ln b_t] = \bar{b}$

$E[\ln(\alpha z_t^2 + \beta)]$

$$= \ln \sigma_0^2 + \sum_{i=1}^{t} \ln b_{t-i}$$

$$= \ln \sigma_0^2 + \sum_{i=1}^{t} e_{t-i}, \quad e_t = \ln b_t$$

$$= \ln \sigma_0^2 + \sum_{i=1}^{t} \left(\tilde{e}_{t-i} + \mathbb{E}[e_{t-i}] \right) \quad , \quad \tilde{e}_t = e_t - \mathbb{E}[e_t]$$

$$= \ln \sigma_0^2 + t\,\mathbb{E}[e_t] + \sum_{i=1}^{t} \tilde{e}_{t-i}$$

This is a Random walk with

drift $\mathbb{E}[e_t] = \mathbb{E}[\ln(\beta + \alpha z_t^2)]$

$$\begin{cases} \ln \sigma_t^2 \to +\infty & \text{if } \mathbb{E}[e_t] > 0 \\ \ln \sigma_t^2 \to -\infty & \text{if } \mathbb{E}[e_t] < 0 \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_t^2 \to +\infty & \text{if } \mathbb{E}[e_t] > 0 \\ \sigma_t^2 \to 0 & \text{if } \mathbb{E}[e_t] < 0 \end{cases}$$

Riskmetrics recursion

$r_t = \sigma_t z_t \quad , \quad z_t \overset{iid}{\sim} N(0,1)$

$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda) r_{t-1}^2$

$$\boxed{\sigma_t^2 = w + \lambda \sigma_{t-1}^2 + (1-\lambda) r_{t-1}^2}$$

30/09

$r_t = \sigma_t z_t \quad , \quad E[z_t \mid \mathcal{F}_{t-1}] = 0 \quad , \quad E[z_t^2 \mid \mathcal{F}_{t-1}] = 1$

$Var(r_{t+k} \mid \mathcal{F}_t) = E[r_{t+k}^2 \mid \mathcal{F}_t] = E[\sigma_{t+k}^2 \mid \mathcal{F}_t]$

$\sigma_{t+1}^2 = w + \alpha r_t^2 + \beta \sigma_t^2 \quad , \quad E[\sigma_{t+1}^2 \mid \mathcal{F}_t] = \sigma_{t+1}^2$

$\sigma_{t+2}^2 = w + \alpha r_{t+1}^2 + \beta \sigma_{t+1}^2$

$= w + \alpha \sigma_{t+1}^2 z_{t+1}^2 + \beta \sigma_{t+1}^2 = w + \sigma_{t+1}^2 (\beta + \alpha z_{t+1}^2)$

$\sigma_{t+3}^2 = w + \sigma_{t+2}^2 (\beta + \alpha z_{t+2}^2)$

$= w + (\beta + \alpha z_{t+2}^2)[w + \sigma_{t+1}^2 (\beta + \alpha z_{t+1}^2)]$

$\sigma_{t+3}^2 = w[1 + (\beta + \alpha z_{t+2}^2)] + \sigma_{t+1}^2 (\beta + \alpha z_{t+1}^2)(\beta + \alpha z_{t+2}^2)$

$$\sigma_{t+4}^2 = \omega + \sigma_{t+3}^2\left(\beta + \alpha z_{t+3}^2\right)$$

$$= \omega + \left(\beta + \alpha z_{t+3}^2\right)\Big[\omega\left[1 + \left(\beta + \alpha z_{t+2}^2\right)\right] + $$

$$+ \sigma_{t+1}^2\left(\beta + \alpha z_{t+1}^2\right)\left(\beta + \alpha z_{t+2}^2\right)\Big]$$

$$= \omega + \omega\left(\beta + \alpha z_{t+3}^2\right)\left[1 + \left(\beta + \alpha z_{t+2}^2\right)\right] +$$

$$+ \sigma_{t+1}^2\left(\beta + \alpha z_{t+1}^2\right)\left(\beta + \alpha z_{t+2}^2\right)\left(\beta + \alpha z_{t+3}^2\right)$$

$$= \omega\left[1 + \left(\beta + \alpha z_{t+3}^2\right) + \left(\beta + \alpha z_{t+3}^2\right)\left(\beta + \alpha z_{t+2}^2\right)\right] +$$

$$+ \sigma_{t+1}^2\left(\beta + \alpha z_{t+1}^2\right)\left(\beta + \alpha z_{t+2}^2\right)\left(\beta + \alpha z_{t+3}^2\right)$$

$$= \omega\left[1 + \sum_{s=1}^{\overbrace{2}^{k-2}} \prod_{i=1}^{s}\left(\beta + \alpha z_{t+\underset{\uparrow k}{4}-i}^2\right)\right] + \sigma_{t+1}^2 \prod_{i=1}^{\overbrace{3}^{k-1}}\left(\beta + \alpha z_{t+i}^2\right)$$

such that

$$\sigma_{t+k}^2 = \omega\left[1 + \sum_{s=1}^{k-2} \prod_{i=1}^{s}\left(\beta + \alpha z_{t+k-i}^2\right)\right] + \sigma_{t+1}^2 \prod_{i=1}^{k-1}\left(\beta + \alpha z_{t+i}^2\right)$$

$$\mathbb{E}\left[\sigma_{t+k}^2 \mid \mathcal{F}_t\right] = \omega\left[1 + \sum_{s=1}^{k-2}(\beta+\alpha)^s\right] + \sigma_{t+1}^2(\beta+\alpha)^{k-1}$$

$$= \omega \sum_{s=0}^{k-2} (\beta+\alpha)^s + \sigma_{t+1}^2 (\beta+\alpha)^{k-1}$$

$$= \omega \frac{1-(\beta+\alpha)^{k-1}}{1-\alpha-\beta} + \sigma_{t+1}^2 (\beta+\alpha)^{k-1}$$

$$\sigma^2 = E[\sigma_t^2] = \frac{\omega}{1-\alpha-\beta}$$

$$\sum_{s=0}^{m} x^s = \frac{1-x^{m+1}}{1-x}$$

$$= \sigma^2 \left(1-(\beta+\alpha)^{k-1}\right) + \sigma_{t+1}^2 (\beta+\alpha)^{k-1}$$

$\downarrow P_t \rightarrow \uparrow L = \frac{D}{P_t} \longrightarrow \uparrow$ Risk of the Firm $t+1$

05/10

$Z \sim P_Z$

$F_Z(z) = P(Z \leq z)$

$Y = a + bZ$

$$F_Y(y) = P(Y \leq y)$$

$$= P(a + bZ \leq y)$$

$$= P\left(Z \leq \frac{y-a}{b}\right) = F_Z\left(\frac{y-a}{b}\right)$$

$$P_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \frac{\partial}{\partial y} F_Z\left(\frac{y-a}{b}\right) = P_Z\left(\frac{y-a}{b}\right)\frac{1}{b}$$

pdf of Y

$$\frac{\partial}{\partial x} g(f(x)) = g'(f(x)) \frac{\partial f(x)}{\partial x}$$

$$P_Y(y) = P_Z\left(\frac{y-a}{b}\right)\frac{1}{b}$$

For a model

$$\begin{cases} r_t = \sigma_t z_t \ , \quad z_t \overset{iid}{\sim} P_Z(z_t, \eta) \\ \sigma_t^2 = \sigma^2(\psi, r_{1:t-1}) \end{cases} \longrightarrow z_t(\psi) = \frac{r_t}{\sigma_t(\psi)}$$

$$P_r(r_t | \mathcal{F}_{t-1}) = P_Z\left(\frac{r_t}{\sigma_t(\psi)}; \eta\right)\frac{1}{\sigma_t} = P_Z(z_t(\psi); \eta)\frac{1}{\sigma_t(\psi)}$$

$$\ln P_r(r_t | \mathcal{F}_{t-1}) = \ln P_Z(z_t(\psi); \eta) - \ln \sigma_t(\psi)$$

$$= \ln P_Z(z_t(\psi); \eta) - \frac{1}{2}\ln \sigma_t(\psi)^2 = \ell_t(\theta)$$

$$\ln \mathcal{L}_T(\theta | r_{1:T}) = \sum_{t=1}^{T} \ell_t(\theta) \qquad \left(\sigma_1^2 = E[\sigma_t^2] \left(= \frac{\omega}{1-\alpha-\beta} \text{ for GARCH} \right)\right)$$

$$\hat{\theta}_T^{ML} = \underset{\theta \in \Theta}{\arg\max} \ln \mathcal{L}_T(\theta | r_{1:T})$$

FOR GARCH, $\theta = (\omega, \alpha, \beta)'$, $\Theta : \{\theta \in \mathbb{R}^3 : \omega, \alpha, \beta > 0 , \ \alpha + \beta < 1\}$
with $z \sim N(0,1)$

$$\ln \mathcal{L}_T(\theta \mid z_{1:T}) \propto \sum_{t=1}^{T} -\frac{\varepsilon_t^2(\theta)}{2} - \frac{1}{2}\ln \sigma_t^2(\theta)$$

$$\nabla_T(\theta) = \frac{\partial}{\partial\theta} \ln \mathcal{L}_T(\theta \mid z_{1:T}) = \sum_{t=1}^{T} \frac{\partial}{\partial\theta}\Big(\underbrace{-\frac{\varepsilon_t^2(\theta)}{2} - \frac{1}{2}\ln \sigma_t^2(\theta)}\Big)$$

$$= \sum_{t=1}^{T} \frac{\partial}{\partial\theta} l_t(\theta)$$

$$\frac{\partial l_t(\theta)}{\partial\theta} = -\frac{1}{2}\frac{\partial \varepsilon_t^2(\theta)}{\partial\theta} - \frac{1}{2\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial\theta}$$

$$= -\frac{1}{2}\frac{\partial}{\partial\theta}\frac{z_t^2}{\sigma_t^2(\theta)} - \frac{1}{2\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial\theta}$$

$$= +\frac{z_t^2}{2\sigma_t^4(\theta)}\frac{\partial \sigma_t^2(\theta)}{\partial\theta} - \frac{1}{2\sigma_t^2(\theta)} \frac{\partial \sigma_t^2(\theta)}{\partial\theta}$$

$$= \frac{\partial \sigma_t^2(\theta)}{\partial\theta}\frac{1}{2\sigma_t^2(\theta)}\left(\frac{z_t^2}{\sigma_t^2(\theta)} - 1\right)$$

$$\mathbb{E}[l_t(\theta)] = \mathbb{E}\big[\mathbb{E}[l_t(\theta) \mid \mathcal{F}_{t-1}]\big]$$

$$= \mathbb{E}\left[\frac{\partial \sigma_t^2(\theta)}{\partial\theta}\frac{1}{2\sigma_t^2(\theta)}\,\mathbb{E}\left[\left(\frac{z_t^2}{\sigma_t^2(\theta)} - 1\right) \Big| \mathcal{F}_{t-1}\right]\right]$$

$$\mathbb{E}\left[\left(\frac{z_t^2}{\sigma_t^2(\theta)} - 1\right) \Big| \mathcal{F}_{t-1}\right] = \mathbb{E}\left[\left(\frac{\cancel{\sigma_t^2(\theta)}\,\varepsilon_t^2}{\cancel{\sigma_t^2(\theta)}} - 1\right)\right]$$

$$= \mathbb{E}\left[(z_t^2 - 1) \right] = 0$$

Hessian matrix

$$\frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta'} = \left(\frac{z_t^2}{\sigma_t^2(\theta)} - 1\right) \frac{\partial}{\partial\theta'}\left(\frac{\partial\sigma_t^2(\theta)}{\partial\theta} \frac{1}{2\sigma_t^2(\theta)}\right) - \frac{\partial\sigma_t^2(\theta)}{\partial\theta} \frac{1}{2\sigma_t^2(\theta)} \frac{z_t^2}{\sigma_t^4(\theta)} \frac{\partial\sigma_t^2(\theta)}{\partial\theta'}$$

$$= \left(\frac{z_t^2}{\sigma_t^2(\theta)} - 1\right) \frac{\partial}{\partial\theta'}\left(\frac{\partial\sigma_t^2(\theta)}{\partial\theta} \frac{1}{2\sigma_t^2(\theta)}\right) - \frac{z_t^2}{2\sigma_t^6(\theta)} \frac{\partial\sigma_t^2(\theta)}{\partial\theta} \frac{\partial\sigma_t^2(\theta)}{\partial\theta'}$$

$$\mathcal{I}(\theta) = -\mathbb{E}\left[\frac{\partial^2 l_t(\theta)}{\partial\theta\partial\theta'}\right] = \mathbb{E}\left[\frac{z_t^2}{2\sigma_t^6(\theta)} \frac{\partial\sigma_t^2(\theta)}{\partial\theta} \frac{\partial\sigma_t^2(\theta)}{\partial\theta'}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\frac{z_t^2}{2\sigma_t^6(\theta)} \frac{\partial\sigma_t^2(\theta)}{\partial\theta} \frac{\partial\sigma_t^2(\theta)}{\partial\theta'} \,\middle|\, \mathcal{F}_{t-1}\right]\right]$$

$$= \mathbb{E}\left[\frac{\partial\sigma_t^2(\theta)}{\partial\theta} \frac{\partial\sigma_t^2(\theta)}{\partial\theta'} \mathbb{E}\left[\frac{z_t^2 \cancel{\sigma_t^2(\theta)}}{2\sigma_t^{6}(\theta)^{5}} \,\middle|\, \mathcal{F}_{t-1}\right]\right]$$

$$= \mathbb{E}\left[\frac{1}{2\sigma_\epsilon^4(\theta)} \frac{\partial \sigma_\epsilon^2(\theta)}{\partial \theta} \frac{\partial \sigma_\epsilon^2(\theta)}{\partial \theta'} \right] = \mathcal{I}(\theta)$$