Suggested Solution to Analytical Exercises

(1) Measurement Equation

$$y_t = (1 \quad 0)\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \varepsilon_t \Rightarrow \theta_t = (y_t, y_{t-1})'$$

Transition Equation

$$\underset{\beta_t}{\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix}} = \underset{F}{\begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}} \underset{\beta_{t-1}}{\begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix}} + \underset{w_t}{\begin{pmatrix} a_t \\ 0 \end{pmatrix}}$$

Initial State distribution

$$\theta_t \sim N(m_0, C_0)$$

$$m_0 = E[\theta_t] = 0$$

$$\mathrm{Vec}(C_0) = (I_4 - F \otimes F)^{-1} \mathrm{vec}(Q)$$

$(I_4 - F \otimes F)^{-1}$ is 4x4 matrix

$$\mathrm{vec}(Q) = \begin{pmatrix} \sigma_\varepsilon^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

[redacted]

2

Measurement Equation

$$\begin{pmatrix}\Delta y_t\\ \Delta c_t\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\Delta y_t\\ \Delta c_t\\ y_t - c_t\end{pmatrix}$$

Transition Equation

$$\begin{pmatrix}\Delta y_t\\ \Delta c_t\\ y_t - c_t\end{pmatrix} = \begin{pmatrix}\phi_{11} & \phi_{12} & \theta_1\\ \phi_{21} & \phi_{22} & \theta_2\\ \phi_{11}-\phi_{21} & \phi_{12}-\phi_{22} & 1+\theta_1-\theta_2\end{pmatrix}\begin{pmatrix}\Delta y_{t-1}\\ \Delta c_{t-1}\\ y_{t-1} - c_{t-1}\end{pmatrix} + \begin{pmatrix}\varepsilon_{1t}\\ \varepsilon_{2t}\\ \varepsilon_{1t}-\varepsilon_{2t}\end{pmatrix}$$

$(1+\theta_1-\theta_2)$

Note that $\Delta y_t = y_t - y_{t-1}$ (1)

$\Delta c_t = c_t - c_{t-1}$ (2)

Subtract (2) from (1) to get the above representation

$\Rightarrow$ ~~[illegible]~~

(3) The model is

$$y_t = T_t + C_t$$

$$T_t = \mu + T_{t-1} + v_t$$

$$C_t = \phi C_{t-1} + e_t \Rightarrow C_t = (1-\phi L)^{-1} e_t$$

Since $Y_t = T_t + C_t$

$$\Rightarrow \Delta Y_t = (1-L) Y_t = \Delta T_t + \Delta C_t$$

$$\Rightarrow \Delta Y_t = \mu + v_t + (1-L)(1-\phi L)^{-1} e_t$$

$\Rightarrow$ ~~(T-ϕ~~

Premultiplying by $(1-\phi L)$

$$(1-\phi L)\Delta Y_t = (1-\phi L)\mu + (1-\phi L) v_t$$

$$+ (1-L) e_t$$

$$\Rightarrow (1-\phi L)\Delta Y_t = \mu^* + v_t - \phi v_{t-1} + e_t - e_{t-1}$$

$$\Rightarrow \underbrace{(1-\phi L)\Delta Y_t = \mu^* + u_t + \theta u_{t-1}}_{\text{Reduced form representation}}$$

Total no of reduced form parameters

$$= 4\ (\mu^*, \phi, \theta, \sigma_u^2)$$

Total number of structural parameters

$$= 5\ (\sigma_v^2, \mu, \phi, \sigma_e^2, \sigma_{ev})$$

The model is not identified unless we impose $\sigma_{ev} = 0$