Cointegration

Cointegration

- The VAR models discussed so far are appropriate for modeling I(0) data, like asset returns or growth rates of macroeconomic time series.
- Economic theory, however, often implies equilibrium relationships between the levels of time series variables that are best described as being I(1).
- Similarly, arbitrage arguments imply that the I(1) prices of certain financial time series are linked.
- The statistical concept of cointegration is required to make sense of regression models and VAR models with I(1) data.
- **Spurious Regression:**
- If some or all of the variables in a regression are I(1) then the usual statistical results may or may not hold. One important case in which the usual statistical results do not hold is spurious regression, when all the regressors are I(1) and not cointegrated. That is, there is no linear combination of the variables that is I(0).

Examples of Cointegration and Common Trends in Economics and Finance

- Cointegration naturally arises in economics and finance. In economics, cointegration is most often associated with economic theories that imply equilibrium relationships between time series variables
- The permanent income model implies cointegration between consumption and income, with consumption being the common trend.
- Money demand models imply cointegration between money, income, prices and interest rates.
- Growth theory models imply cointegration between income, consumption and investment, with productivity being the common trend.
- Purchasing power parity implies cointegration between the nominal exchange rate and foreign and domestic prices.

Examples of Cointegration and Common Trends in Economics and Finance

- Covered interest rate parity implies cointegration between forward and spot exchange rates.
- The Fisher equation implies cointegration between nominal interest rates and inflation.
- The expectations hypothesis of the term structure implies cointegration between nominal interest rates at different maturities.
- The present value model of stock prices states that a stock's price is an expected discounted present value of its expected future dividends or earnings.

Remarks

- The equilibrium relationships implied by these economic theories are referred to as long-run equilibrium relationships, because the economic forces that act in response to deviations from equilibrium may take a long time to restore equilibrium.
- As a result, cointegration is modeled using long spans of low frequency time series data measured monthly, quarterly or annually.
- In finance, cointegration may be a high frequency relationship or a low frequency relationship. Cointegration at a high frequency is motivated by arbitrage arguments.
  - The Law of One Price implies that identical assets must sell for the same price to avoid arbitrage opportunities. This implies cointegration between the prices of the same asset trading on different markets, for example.
  - Similar arbitrage arguments imply cointegration between spot and futures prices, and spot and forward prices, and bid and ask prices.
- Here the terminology long-run equilibrium relationship is somewhat misleading because the economic forces acting to

Cointegration

- A (n×1) vector Y_t of random variables is said to be cointegrated if each of the individual series in the vector y_{it}, is I(1) but there exists a linear combination of series, say $\beta' Y_t$ which is I(0), for a non-zero β' which is called the cointegrating vector.
- Example:

$$y_{1t} = \beta_1 y_{2t} + u_{1t}$$

$$y_{2t} = y_{2,t-1} + u_{2t} \Rightarrow \Delta y_{2t} = u_{2t}$$

- Since first difference of y_{2t} is stationary, the level is non-stationary. u_{1t} and u_{2t} are two uncorrelated white noise processes. Since u_{1t} is a white noise and therefore stationary, in order for the first equation to be balanced, y_{1t} must be I(1).
- Since $y_{1t} - \beta_1 y_{2t} = u_{1t}\tilde{}I(0)$ i.e. stationary. $Y_t = (y_{1t}, y_{2t})'$ is cointegrated with cointegrating vector $\beta = (1, -\beta)'$.
- If Y_t is cointegrated the estimation of the VAR in differences is inconsistent (there is an omitted variable bias in that one needs to include the information contained in the cointegrating relations)

Cointegration

- Cointegration identifies long-run relationships among the cointegrating variables.
- Intuition: I(1) time series with a long-run equilibrium relationship cannot drift too far apart from the equilibrium because economic forces will act to restore the equilibrium relationship.
- Cointegrating vectors are not unique: if $Y_t \tilde{} I(1)$ and $\beta' Y_t \tilde{} I(0)$ then $c\beta' Y_t \tilde{} I(0)$, where c is a scalar.
- Given an (n×1) vector Y_t where each y_{it} is I(1), there can be at most h cointegrating relationships, where h<n. Each cointegrating vector $\beta_{1,} ... \beta_h$ is chosen to be linearly independent of each other (i.e. $\beta_2 \neq c\beta_1$ for any c). Collecting the h vectors in a matrix

$$\beta' = \begin{bmatrix} \beta_1 \\ .. \\ \beta_h \end{bmatrix}'$$

VECM (Vector-Error Correction Model Representation)

- Consider a VAR(2) Model
- $y_t = c + A_1 y_{t-1} + A_2 y_{t-2} + u_t$
- Subtract y_{t-1} from both sides to give
- $\Delta y_t = c - y_{t-1} + A_1 y_{t-1} + A_2 y_{t-2} + u_t$
- Next, add and subtract $A_2 y_{t-1}$ from RHS to give

$$\Delta y_t = c + \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + u_t$$

- where $\Pi = A_1 + A_2 - I_n$ and $\Gamma_1 = -A_2$
- In the above equation Πy_{t-1} is the only term that includes a potential I(1) variable and for Δy_t to be stationary it must be the case that Πy_{t-1} is also I(0). Therefore, Πy_{t-1} must contain the cointegrating relations if they exist. There are two cases

VECM (Vector-Error Correction Model Representation)

- Rank $(\Pi) = 0$. This implies that $\Pi = 0$ and y$_t$˜$I(1)$ and VECM representation above reduces to VAR(1) in first differences.
- Rank $(\Pi) = r < n$. This implies that y$_t$˜$I(1)$ with r cointegrating vectors and n-r common stochastic trends (unit roots).
- $\Pi = \alpha\beta'$ where α and β are (n×r) matrices with rank()=rank()=r.
- $\Delta y_t = c + \alpha\beta' y_{t-1} + \Gamma_1 \Delta y_{t-1} + u_t$
- The elements in the vector α are interpreted as speed of adjustment coefficients.

Testing for Cointegration:

- Testing a Particular Cointegrating Vector
- Step1: Test that y_t is I(1) with a Dickey-Fuller test or a Phillips-Perron test.
- Step2: Construct $z_t = \beta' y_t$ for β known
- Step3: If β is cointegrating vector z_t should be stationary. Perform a unit root test to check this.

Testing and Estimation of the Cointegrating Vector

- 2-Step Engle-Granger Cointegration Test
- Step 1: Estimate by OLS
- $\mathrm{y}_{1t} = \mu + \beta' Y_{-1,t} + u_{1t}$
- Check that u_{1t} is stationary with a unit root test. The estimate of β are superconsistent. Note that this regression can be augmented by the first differences of the leads and lags of $Y_{-1,t}$ to ensure that residuals are white noise and obtain more efficient estimates in short-samples.
- Step2: Construct the z_t from step 1 and estimate the Error Correction Model
- $\Delta y_t = c + \alpha\beta' y_{t-1} + \Gamma_1 \Delta y_{t-1} + ... + \Gamma_p \Delta y_{t-p} u_t$
- The estimated parameters are consistent.

Testing and Estimation of the Cointegrating Vector

- Full Information Maximum Likelihood Analysis of Cointegrated Systems
- Advantages:
  - Directly produces the number of cointegrating relations.
  - Jointly estimates the cointegrating relations and the VAR in ECM form
- The intuition behind the procedure is to test the rank of the matrix Π in $\Delta y_t = c + \Pi y_{t-1} + \Gamma_1 \Delta y_{t-1} + u_t$.
- The rank of Π tells us how many cointegrating relations there are.

- Comment on the specifics of the Johansen test and Eviews
- We have seen that cointegrating vectors are not uniquely identified. Eviews normalizes these vectors by solving for the first h variables in y_t(i.e. assigns a value of 1 to the first variable, then a 1 to the second variable and so on)
- Deterministic Trend Assumptions