Structural Break Models

Review of Hypothesis Testing

$$Y = X\beta + e, e\tilde{}(0, \sigma^2 I_T)$$
$$\widehat{\beta} = (X'X)^{-1}X'Y$$
$$Var(\widehat{\beta}) = \sigma^2(X'X)^{-1}$$

- Hypothesis to be tested: $\mathrm{H}_0 : \mathrm{R}\beta = r$,$\mathrm{H}_1 : \mathrm{R}\beta \neq r$

$$R\widehat{\beta}\tilde{}(R\beta, R * Var(\widehat{\beta}) * R')$$

$$(R\widehat{\beta} - R\beta)\tilde{}(0, R * Var(\widehat{\beta}) * R')$$

$$\sqrt{T}(R\widehat{\beta} - R\beta) \xrightarrow{d} N(0, T * R * Var(\widehat{\beta}) * R')$$

Review of Hypothesis Testing

- Under the null hypothesis, $R\beta = r$. Thus, we have:

$$\sqrt{T}(R\widehat{\beta} - r) \xrightarrow{d} N(0, T * R * Var(\widehat{\beta}) * R')$$

$$\Rightarrow T(R\widehat{\beta} - r)'[T * R * Var(\widehat{\beta}) * R']^{-1}(R\widehat{\beta} - r) \xrightarrow{d} \chi^2(J),$$

$$\Rightarrow WaldStatistic = (R\widehat{\beta} - r)'[R * Var(\widehat{\beta}) * R']^{-1}(R\widehat{\beta} - r) \xrightarrow{d} \chi^2(J),$$

- Where J is the number of restrictions under the null hypothesis.

Tests of Structural Break

Consider the following simple model of structural break with a break point τ :

$$Y_t = \beta_0 + e_t, t = 1, 2, ...\tau$$

$$Y_t = \beta_1 + e_t, t = \tau + 1, \tau + 2, ... T$$

$$e_t \tilde{} i.i.d.(0, \sigma^2)$$

$$\begin{bmatrix} Y_1 \\ . \\ . \\ Y_\tau \\ Y_{\tau+1} \\ . \\ . \\ Y_T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ . & . \\ . & . \\ 1 & 0 \\ 0 & 1 \\ . & . \\ . & . \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} e_1 \\ . \\ . \\ e_\tau \\ e_{\tau+1} \\ . \\ . \\ e_T \end{bmatrix}$$

$$\Rightarrow Y = X\beta + e$$

- Null Hypothesis:

  $H_0 : \beta_0 = \beta_1 (R\beta = r \Rightarrow \left(\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = 0 \right)$

When the Break point τ is known: [Chow Test]

$$\widehat{\beta}_0 = \frac{1}{\tau} \sum_{t=1}^{\tau} Y_t, Var(\widehat{\beta}_0) = \sigma^2 \frac{1}{\tau}$$

$$\widehat{\beta}_1 = \frac{1}{T-\tau} \sum_{t=\tau+1}^{T} Y_t, Var(\widehat{\beta}_1) = \sigma^2 \frac{1}{T-\tau}$$

$$\Rightarrow Var(\widehat{\beta}) = Var\left(\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} \right) = \sigma^2 \begin{bmatrix} \frac{1}{\tau} & 0 \\ 0 & \frac{1}{T-\tau} \end{bmatrix}$$

$\Rightarrow WaldStatistic = (R\widehat{\beta} - r)'[R * Var(\widehat{\beta}) * R']^{-1}(R\widehat{\beta} - r) \xrightarrow{d} \chi^2(J),$

$J = 1, r = 0$

Break Test When the Break Point is Unknown

- In this case, the breakpoint τ has to estimated as well.
- By the way, the parameter τ is a nuisance parameter that exists only under the alternative hypothesis, but not under the null hypothesis.
- Estimating τ : (i) Calculate the Wald Statistic above for every possible break point (ii) Choose the value of τ that maximizes the Wald statistic, (iii) The maximized Wald statistic is the *Sup Wald* statistic.
- Question: Can we perform the structural break test at the estimated break point, using the standard asymptotic distribution theory? That is, does the *Sup Wald* test have the usual asymptotic chi-square distribution?
- Answer: No. Intuitively, due to the uncertainty associated with the estimated break point, the critical value obtained from the distribution of the *Sup Wald* statistic would be larger than that obtained from a chi-square distribution.

Break Test When the Break Point is Unknown

- Let's first consider the following:

$$\widehat{\beta}_0 = \frac{1}{\tau}\sum_{t=1}^{\tau} Y_t = \beta_0 + \frac{1}{\tau}\sum_{t=1}^{\tau} e_t \tag{1}$$

$$\begin{aligned}\widehat{\beta}_1 &= \frac{1}{T-\tau}\sum_{t=\tau+1}^{T} Y_t = \beta_1 + \frac{1}{T-\tau}\sum_{t=\tau+1}^{T} e_t \\ &= \beta_1 + \frac{1}{T-\tau}\left(\sum_{t=1}^{T} e_t - \sum_{t=1}^{\tau} e_t\right) \end{aligned}\tag{2}$$

Break Test When the Break Point is Unknown

- By denoting $\tau = [\pi T], 0 < \pi < 1$, the above can be represented as:

$$\widehat{\beta}_0 = \beta_0 + \frac{1}{[\pi T]} \sum_{t=1}^{[\pi T]} e_t \tag{1'}$$

$$\widehat{\beta}_1 = \beta_1 + \frac{1}{T - [\pi T]} \left(\sum_{t=1}^{T} e_t - \sum_{t=1}^{[\pi T]} e_t \right) \tag{2'}$$

- By multiplying both sides of (1') and (2') by $\sqrt{T}$, we have:

$$\sqrt{T}(\widehat{\beta}_0 - \beta_0) \xrightarrow{d} \frac{1}{\pi} \sigma W(\pi) \tag{1''}$$

$$\sqrt{T}(\widehat{\beta}_1 - \beta_1) \xrightarrow{d} \frac{1}{1-\pi} \sigma (W(1) - W(\pi)) \tag{2''}$$

- We now consider the Wald statistic as a function of $\tau = [\pi T]$:

$$Wald(\pi) = (R\widehat{\beta} - R\beta)'[R * Var(\widehat{\beta}) * R']^{-1}(R\widehat{\beta} - R\beta)$$

$$\Rightarrow ((\widehat{\beta}_0 - \widehat{\beta}_1) - (\beta_0 - \beta_1))^2 \left(\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sigma^2}{\tau} & 0 \\ 0 & \frac{\sigma^2}{T-\tau} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)^{-1}$$

$$\Rightarrow ((\widehat{\beta}_0 - \widehat{\beta}_1) - (\beta_0 - \beta_1))^2 \left(\frac{\sigma^2}{\tau} + \frac{\sigma^2}{1-\tau} \right)^{-1}$$

$$\Rightarrow (\sqrt{T}(\widehat{\beta}_0 - \beta_0) - \sqrt{T}(\widehat{\beta}_1 - \beta_1))^2 \left(\frac{\sigma^2}{\pi} + \frac{\sigma^2}{(1-\pi)} \right)^{-1}$$

$$\Rightarrow (\sqrt{T}(\widehat{\beta}_0 - \beta_0) - \sqrt{T}(\widehat{\beta}_1 - \beta_1))^2 \frac{\pi(1-\pi)}{\sigma^2}$$

$$\Rightarrow (\sqrt{T}(\widehat{\beta}_0 - \beta_0) - \sqrt{T}(\widehat{\beta}_1 - \beta_1))^2 \frac{\pi(1-\pi)}{\sigma^2}$$

$$\xrightarrow{d} (\frac{1}{\pi}\sigma W(\pi) - \frac{1}{1-\pi}\sigma(W(1) - W(\pi)))^2 \frac{\pi(1-\pi)}{\sigma^2}$$

$$\Rrightarrow \frac{[W(\pi) - \pi W(1)]^2}{\pi(1-\pi)}$$

- Thus, as

$$Wald(\pi) \xrightarrow{d} \frac{[W(\pi) - \pi W(1)]^2}{\pi(1-\pi)}$$

We have:

$$Sup_{\pi \in \Pi} Wald(\pi) \xrightarrow{d} Sup_{\pi \in \Pi} \frac{[W(\pi) - \pi W(1)]^2}{\pi(1-\pi)}$$

- Range of Π
- When estimating the break point (i.e. $\tau = [\pi T]$) and thus $Sup_{\pi\in\Pi} Wald(\pi)$, what range of Π do we use?
- When $\Pi = [0, 1]$;

  Under the null hypothesis, $Sup_{\pi\in[0,1]} Wald(\pi) \xrightarrow{P} \infty$ (Andrews (1993)).[End Point Problem]. Thus, unless Π is bounded away from zero and one, critical values for the test statistics $Sup_{\pi\in\Pi} Wald(\pi)$ must diverge to infinity as T$\rightarrow \infty$. By bounding π away from zero and one, however, a fixed critical value suffices for all T large.
- When $\Pi = [\pi_0, 1 - \pi_0]$;
- For example, when T=100, and π_0=0.15, we search for break point over the interval 15<t<85. [We are assuming that the structural break did not take place in the first and the last 15% of the sample.]

- Range of Π
- Consider the following 3 cases:
- Case $\#1$: $\Pi = [0.15, 0.85]$
- Case $\#2$: $\Pi = [0.3, 0.7]$
- Case $\#3$: $\Pi = [0.5, 0.5]$
- There is no uncertainty about the structural break point for case $\#3$ (Structural break is right in the middle of the sample).
- Case $\#1$ has highest degree of uncertainty about the date of structural break point

Critical Values

- With higher uncertainty, the critical values should be larger. Intuitively, with higher uncertainty you may want to be more conservative in rejecting the null hypothesis. That is, with higher uncertainty yo may not want to reject the null hypothesis unless the evidence is stronger (i.e. unless the test statistic is larger) than with .lower or no uncertainty.
- For case 3, the critical value is the same as that obtained from a chi-square distribution
- $CV_{case3} < CV_{case2} < CV_{case1}$

Test Statistics and Its Distribution Under the Null

$$Sup_{\pi\in[\pi_0,1-\pi_0]} Wald(\pi) \xrightarrow{d} Sup_{\pi\in[\pi_0,1-\pi_0]} Q(\pi)$$

where

$$Q(\pi) = \frac{[W(\pi) - \pi W(1)]^2}{\pi(1-\pi)}$$

- The critical values are obtained from Table 1 of Andrews (1993)
- Approximate the distribution of the supremum of $Q(\pi)$ over $\pi \in [\pi_0, 1-\pi_0]$ by its maximum over a fine grid of points between π_0 and 1-π_0
- Simulate the distribution of $\text{Max}_\pi Q(\pi)$ by Monte Carlo. For details refer to p. 841 of Andrews (1993).
- The choice of π_0 affects the level of uncertainty concerning the break point, and hence, for different choice of π_0 will have different distribution of the test statistics.

Concluding Remarks

- The same logic and the asymptotic distribution applies for the Likelihood ratio test and Lagrange multiplier test.
- We investigated the nature of structural break for the simplest case. For more general models involving K explanatory variables under the Null, the same logic and asymptotic distribution apply.

$$Sup_{\pi \in [\pi_0, 1-\pi_0]} Wald(\pi) \xrightarrow{d} Sup_{\pi \in [\pi_0, 1-\pi_0]} Q(\pi)$$

where

$$Q(\pi) = \frac{[W_J(\pi) - \pi W_J(1)]' [W_J(\pi) - \pi W_J(1)]}{\pi(1-\pi)}$$

where $W_J(\pi)$ refers to J-dimensional Wiener process. Hence, J is the number of restrictions under the null of no structural break.

Concluding Remarks

- What if there exists autocorrelation or heteroscedasticity of unknown form in the disturbance term?
- When calculating the $Var(\widehat{\beta})$, use an autocorrelation-heteroscedastictiy-consistent Variance Covariance matrix