# Trend-Cycle Decomposition

# Trend-Cycle Decomposition

- Burns and Mitchell (1946); First systematic study of business cycles
- They treated each cycle as a separate episode terminating and starting at a trough and going from trough to peak through an expansion, and from peak to trough through a contraction.
- A typical business cycle was characterized by mean lengths of expansions and contractions.
- Problem with approach
  - Largely judgmental
  - Does not have well desired statistical properties

# Trend-Cycle Decomposition

- Most economic data is non-stationary and often presents seasonal behavior. For example, U.S. GDP exhibits an upward trend over time that is consistent with a growing economy, seasonal behavior characterized by slow winters and summers and strong springs and falls, and a cyclical pattern of expansions and recessions.
- Usually,publicly available time-series data are seasonally adjusted
- Most research in macroeconomics dedicated to the explanation of business cycles therefore relies on pre-filtering methods from which the trend components are isolated from cyclical components.

# Common Detrending Methods

- *Deterministic Time Trends*
- Perhaps the simplest de-trending method consists in specifying the trend as a polynomial in time

$$y_t = y_t^T + y_t^c$$

$$y_t^T = \sum_{i=0}^{k} \alpha_i t^i, y_t^c = u$$

- where $y_t^T$ is the $y_t^c$trend component and  is the cyclical component. K is a finite integer and u is a stationary process.

- By far, the most common choice is to set K=1 so that

$$y_t = \alpha_0 + \alpha_1 t + u_t$$

$$\phi(L) u_t = \theta(L) \varepsilon_t$$

- where  is a white $\varepsilon_t$ noise process and $\phi(L)$ is stationary.

# Deterministic Time Trends

- Deterministic detrending implies that the “trend” and the “cycle” are uncorrelated with each other.
- It is easy to compute: one can use all the data points to consistently estimate the trend via conventional techniques.
- Because its simplicity, it is the least likely to distort the short-run dynamics of the cyclical component for low orders of the deterministic polynomial.
- From a forecasting point of view, it has the rather strong implication that the long-run forecast error variance converges to a fixed value. In practice, we would expect the forecast error variance of a point in the distant future to grow as the forecast horizon increases.

# Hodrick-Prescott Filter

- The H-P filter has become a popular choice among business cycle analysts. The original presentation of the filter in a 1980 working paper did not see its publication debut until 1997. The filter is obtained by solving the minimization problem:
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$$\underset{\{y_t\}_{t=1}^{T}}{Min} \sum_{t=1}^{T} (y_t - y_t^T)^2 + \lambda \sum_{t=2}^{T-1} \left[ (y_{t+1}^T - y_t^T) - (y_t^T - y_{t-1}^T) \right]^2$$

- Where $\lambda$ is an arbitrary constant that penalizes the variability in the smoother so that when $\lambda = 0$ the smooth component is the data itself and no smoothing takes place.
- Conversely, as $\lambda$ grows large the smooth component is a linear trend.

# Hodrick-Prescott Filter

- For quarterly data, Hodrick and Prescott recommended $\lambda\blacksquare = 1600$, although an “optimal” choice is given by the ratio of the variance of the trend to the variance of the cycle. Nelson and Plosser (1982) suggested instead that $\lambda$ should be in the range $[1/6, 1]$.
- One of the virtues and downfalls of the H-P filter is its flexibility: McCallum (2000) showed that the trend produced by the H-P filter applied to GDP data in the early turn of the century suggest the 1930’s depression was only a mild recession.
- This result stems from the inability to calculate an approximately “optimal” $\lambda\blacksquare$ for each series via estimation.
- The filter depends on the choice of $\lambda$ which makes the resulting cyclical component and its statistical properties highly sensitive to this choice.

# Baxter-King Filter

- The B-K filter is a band-pass filter designed to isolate business cycle fluctuations with a period of length ranging between 6 to 32 quarters (the typical ranges for U.S. expansions) although other characterizations are possible. The resulting filter is a centered moving average with symmetric weights

$$y_t^T = \sum_{i=-K}^{K} w_i y_{t-i}$$

- K observations at the beginning and at the end of the sample are lost in the computation of the filter.
- Like many moving average smoothers, the B-K filter has been criticized on the grounds that it induces spurious dynamics in the cyclical component

# Trend-Cycle Decomposition with Stochastic Trends

- Suppose $y_t = TD_t + Z_t$
- where $TD_t$ is a deterministic trend
- Assume $Z_t$~$I(1)$. Then it is possible to decompose $Z_t$ into a stochastic (random walk) trend and a stationary $I(0)$, "cyclical" component:
- $Z_t = TS_t + C_t$
- where $TS_t$~$I(1)$ and $C_t$~$I(0)$
- The stochastic trend, $TS_t$, captures shocks that have a permanent effect on the level of $y_t$. The stationary component, $C_t$, captures shocks that only have a temporary effect on the level of yt.

# Trend-Cycle Decomposition with Stochastic Trends

- The components representation for $y_t$ becomes
- $$y_t = TD_t + TS_t + C_t$$
- $TD_t + TS_t$= overall trend, $C_t$= deviations about trend
- The decomposition of $Z_t$ into $TS_t$ and $C_t$ is not unique. In fact, there are an infinite number of such combinations depending on how $TS_t$ and $C_t$ are defined.
- Two decompositions have been popular in the empirical literature to model stochastic trend: the Beveridge-Nelson (BN) decomposition; and the orthogonal unobserved components (UC) decomposition.
- Both decompositions define $TS_t$ as a pure random walk. They primarily differ in how they model the serial correlation in $\Delta Z_t$.

# Beveridge-Nelson Decomposition

- Beveridge and Nelson (1980) proposed a definition of the permanent component of an I(1) time series $y_t$ with drift $\mu$ as the limiting forecast as horizon goes to infinity, adjusted for the mean rate of growth over the forecast horizon.
- $$TD_t + BN_t = \lim_{h\to\infty}(y_{t+h|t} - TD_{t+h|t}) = \lim_{h\to\infty}(y_{t+h|t} - \mu h)$$
- $BN_t$ is referred to as the BN trend. The implied cycle is $C_t^{BN} = y_t - TD_t - BN_t$
- BN showed that if $\Delta y_t$ has Wold representation
- $$\Delta y_t = \delta + \Psi^*(L)\varepsilon_t$$
- Then, $BN_t$ follows a pure random walk without drift.
- $$BN_t = BN_{t-1} + \Psi^*(1)\varepsilon_t = BN_0 + \Psi^*(1)\sum_{j=1}^{t}\varepsilon_t$$

# Beveridge-Nelson Decomposition

- The derivation of the BN trend relies on the following algebraic result
- Let $\Psi(L) = \sum_{k=0}^{\infty} \psi_k L^k$ with $\psi_0 = 1$
- $\Psi(L) = \Psi(1) + (1 - L)\widetilde{\Psi}(L)$

$$\Psi(1) = \sum_{k=0}^{\infty} \psi_k, \widetilde{\Psi}(L) = \sum_{k=0}^{\infty} \widetilde{\psi}_k L^k, \widetilde{\psi}_J = -\sum_{k=J+1}^{\infty} \psi_k$$

- We can write $y_t$ as
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$$\begin{aligned}
y_t &= y_0 + \delta t + \Psi^*(L) \sum_{j=1}^{t} \varepsilon_j \\
&= y_0 + \delta t + \left(\Psi^*(1) + (1 - L)\widetilde{\Psi^*}(L)\right) \sum_{j=1}^{t} \varepsilon_j \\
&= y_0 + \delta t + \Psi^*(1) \sum_{j=1}^{t} \varepsilon_j + \widetilde{\varepsilon}_t - \widetilde{\varepsilon}_0
\end{aligned}$$

- where $\Psi^*(1)\sum_{j=1}^{t}\varepsilon_j$ is the stochastic trend and $\widetilde{\varepsilon}_t - \widetilde{\varepsilon}_0$ is the cycle.
- where $\widetilde{\varepsilon}_t = \widetilde{\Psi^*}(L)\varepsilon_t$
- To show that $\Psi^*(1)\sum_{j=1}^{t}\varepsilon_j$ is the BN trend, consider the series at time t+h

$$y_{t+h} = y_0 + \delta(t+h) + \Psi^*(1)\sum_{j=1}^{t+h}\varepsilon_j + \widetilde{\varepsilon}_{t+h} - \widetilde{\varepsilon}_0$$

- The limiting forecast as horizon h goes to infinity, adjusted for mean growth, is
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$$\begin{aligned} lim_{h\to\infty} y_{t+h|t} &= y_0 + \delta(t+h) + \Psi^*(1)\sum_{j=1}^{t+h}\varepsilon_j + lim_{h\to\infty}\widetilde{\varepsilon}_{t+h} \\ \lim_{h\to\infty} y_{t+h|t} - \delta h &= y_0 + \delta t + \Psi^*(1)\sum_{j=1}^{t}\varepsilon_j = TD_t + BN_t \end{aligned}$$

# BN Decomposition for MA(1) process

$$\triangle y_t = 0.008 + \varepsilon_t + 0.3\varepsilon_{t-1}, \varepsilon_t \tilde{\ } iid(0, \sigma^2), \widehat{\sigma} = 0.0106$$

$$\triangle y_t = \delta + \Psi^*(L)\varepsilon_t, \Psi^*(L) = 1 + \Psi_1^* L, \Psi_1^* = 0.3$$

$$\Psi^*(1) = 1.3, \widetilde{\Psi}_0 = -\sum_{j=1}^{\infty} \Psi_j^* = -\Psi_1^* = -0.3$$

$$\widetilde{\Psi}_j^* = -\sum_{j=k+1}^{\infty} \Psi_j^* = 0, j = 1, 2$$

- The trend-cycle decomposition of $y_t$ using the BN decomposition becomes

$$y_t = y_0 + \delta t + \Psi^*(1) \sum_{j=1}^{t} \varepsilon_j + \widetilde{\varepsilon}_t = y_0 + 0.008t + 1.3 \sum_{j=1}^{t} \varepsilon_j - 0.3\varepsilon_t$$

# Beveridge-Nelson Decomposition

- The naive computation of the BN decomposition requires the following steps:
  - Estimation of ARMA(p,q) model for $\Delta y_t$
  - Estimation of $\Psi^*(1)$ from estimated ARMA(p,q) model for $\Delta y_t$
  - Estimation of $\sum_{j=1}^{t} \varepsilon_t$ using residuals from estimated ARMA(p,q) model for $\Delta y_t$

# BN Decomposition for AR(1) Model

- AR(1): $\Delta y_t = \delta + \phi \Delta y_{t-1} + \varepsilon_t$
- $\mathrm{TD}_t + BN_t = lim_{h\to\infty}(y_{t+h|t} - h\delta)$
- $\mathrm{TD}_t + BN_t = y_t + \frac{\phi}{1-\phi}(\Delta y_t - \delta)$
- The cyclical component is then
  $\mathrm{C}_t = y_t - TD_t - BN_t$
  $\Rightarrow \mathrm{C}_t = -\frac{\phi}{1-\phi}(\Delta y_t - \delta)$

# The Orthogonal Unobserved Components Model

- The basic idea behind the UC model is to give structural equations for the components on the trend-cycle decomposition.
- For example, Watson (1986) considered UC-ARIMA models of the form
- $$y_t = \mu_t + C_t$$
  $$\mu_t = \alpha + \mu_{t-1} + \varepsilon_t, \varepsilon_t \tilde{} iid(0, \sigma_\varepsilon^2)$$
  $$\phi(L)C_t = \theta(L)\eta_t, \eta_t \tilde{} iid(0, \sigma_\eta^2)$$
  $$\phi(L) = 1 - \phi_1 L - ..... - \phi_P L^P$$
  $$\theta(L) = 1 + \theta_1 L + ... + \theta_q L^q$$

# Identification of UC Model

- The parameters of the UC model are not identified without further restrictions.
- Restrictions commonly used in practice to identify all of the parameters are: (1) the roots of $\phi(\mathrm{z}) = 0$ are outside the unit circle; (2) $\theta(\mathrm{L})=1$, and (3) $\mathrm{cov}(\varepsilon_t, \eta_t)=0$.
- These restrictions identify $\mathrm{C}_t$ as a transitory autoregressive "cyclical" component, and $\mu_t$ as the permanent trend component.