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WHY ARE THE BEVERIDGE-NELSON AND UNOBSERVED-COMPONENTS DECOMPOSITIONS OF GDP SO DIFFERENT?

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Abstract—This paper reconciles two widely used decompositions of GDP into trend and cycle that yield starkly different results. The Beveridge-Nelson (BN) decomposition implies that a stochastic trend accounts for most of the variation in output, whereas the unobserved-components (UC) implies cyclical variation is dominant. Which is correct has broad implications for the relative importance of real versus nominal shocks. We show the difference arises from the restriction imposed in UC that trend and cycle innovations are uncorrelated. When this restriction is relaxed, the UC decomposition is identical to the BN decomposition. Furthermore, the zero-correlation restriction can be rejected for U.S. quarterly GDP, with the estimated correlation being -0.9.

I. Introduction

THE decomposition of real GDP into trend and cycle is of considerable practical importance, but two widely used methods yield starkly different results. The unobserved-component (UC) approach, introduced by Harvey (1985) and Clark (1987), implies a very smooth trend and a cycle that is large in amplitude and highly persistent. In contrast, the approach of Beveridge and Nelson (1981) (BN) implies that much of the variation in GDP is variation in trend, whereas the cycle component is small and noisy. This contrast is apparent in figures 1 and 2, where the two cycle components are plotted respectively, and it has been widely noted; see Watson (1986) and Stock and Watson (1988), among others.

It should surprise us that the two decompositions are so different, since both are model-based, each letting the data "speak for themselves." Neither imposes smoothness in trend a priori as does a polynomial or the smoother of Hodrick and Prescott (1997). Although it is often stated that BN assumes a perfect negative correlation between trend and cycle innovations, that is a property of the *estimated* trend and cycle, not the unobserved components, and it is a property shared with the UC decomposition. This paper attempts to find out why we do not, after decades of research, have a consistent picture of how variation in a series like real GDP should be allocated between trend and cycle.

Briefly, section II demonstrates the theoretical equivalence between the approaches. Section III investigates the source of the difference observed in practice. Section IV concludes.

II. Theoretical Equivalence of the Beveridge-Nelson and Unobserved-Component Estimates of Trend and Cycle

Trend-cycle decomposition is motivated by the idea that the log of aggregate output is usefully thought of as the sum of a component that accounts for long-term growth and a stationary, transitory deviation from trend. We follow custom in referring to the latter as the *cycle* even if it is not periodic. The UC representation takes the form

$$y_t = \tau_t + c_t, \tag{1a}$$

$$\tau_t = \tau_{t-1} + \mu + \eta_t, \qquad \eta \sim \text{i.i.d. } N(0, \sigma_\eta^2), \tag{1b}$$

$$c_t \text{ is stationary and ergodic,} \tag{1c}$$

where $\{y_t\}$ is the observed series, $\{\tau_t\}$ is the unobserved trend, assumed to be a random walk with mean growth rate μ, and $\{c_t\}$ is the unobserved stationary cycle.[1]

What we refer to as the UC-ARMA model adds the condition that $\{c_t\}$ is a stationary and invertible ARMA(p, q) process with innovations that may be contemporaneously cross-correlated with trend innovations,

$$\begin{gathered}\phi_p(L)c_t = \theta_q(L)\epsilon_t, \quad \epsilon \sim \text{i.i.d. } N(0, \sigma_\epsilon^2), \\ \text{Cov}(\eta_t, \epsilon_{t\pm k}) = \begin{cases} \sigma_{\eta\epsilon} & \text{for } k = 0, \\ 0 & \text{otherwise.}\end{cases}\end{gathered} \tag{1d}$$

[1] Blanchard and Quah (1989) point out that the *structural* trend of output (associated with real shocks) need not follow a random walk. Therefore, a cycle component that represents the deviation from a random-walk trend may reflect transitory effects of both real and nominal shocks. However, it should be noted that under long-run neutrality, the cycle provides an upper-bound estimate of output fluctuations due to nominal shocks.

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In some implementations the rate of drift μ is also allowed to evolve as a random walk and sometimes an additional irregular term is added, although these changes have little influence on the estimated cycle component for U.S. GDP. Harvey (1985), Clark (1987), and Harvey and Jaeger (1993) suggest specifying $p = 2$, which allows the cycle process to be periodic in the sense of having a peak in its spectral density function. They and others then cast the model in state-space form with equation (1a) as the measurement equation along with the cycle (1d) as an error term, while equation (1b) is the state transition equation. This setup implies that trend and cycle innovations are uncorrelated. Thus the model is augmented to include the condition

$$\sigma_{\epsilon\eta} = 0. \tag{1e}$$

We denote this zero-covariance constrained UC-ARMA model as UC-0. This setup remains the standard treatment of trend-cycle decomposition in the state-space framework, as in Proietti (2002), although recent work considers the possibility of nonzero correlation; see Koopman (1997).

In practice, the parameters are estimated from data $(y_1, \ldots, y_n)$ by the maximum likelihood method of Harvey (1981) based on the prediction error decomposition. Given estimated parameters, the Kalman filter generates the expectation of the trend component conditional on data through time t:

$$\hat{\tau}_{t|t} = E[\tau_t|\Omega_t], \qquad \text{where} \quad \Omega_t = (y_1, \ldots, y_t).$$

Smoothed estimates of the components condition on future as well as past data. For U.S. real GDP, smoothed and filtered estimates are qualitatively similar. Harvey and Koopman (2000) show that zero covariance implies symmetry in the weights of the smoother, a property they argue has inherent appeal.

The BN estimate of trend for an I(1) time series $\{y_t\}$ is defined to be the limiting forecast as horizon goes to infinity, adjusted for the mean rate of growth; so

$$BN_t = \lim_{M\to\infty} E[y_{t+M} - M\mu|\Omega_t].$$

BN showed that the time series $\{BN_t\}$ will be a random walk with the same mean growth rate as the observed series, that the deviation from trend is a stationary process, and that the innovations of $\{BN_t\}$ and $\{y_t - BN_t\}$ are perfectly negatively correlated. The series $\{BN_t\}$ is calculated from an estimated ARIMA representation of $\{y_t\}$, which in principle is unique after cancellation of any redundant AR and MA factors.[2]

It is well known that the UC-ARMA model implies an equivalent univariate ARIMA representation for $\{y_t\}$. Two representations are equivalent for our purposes if they have the same autocovariance structure, thereby implying the same joint distribution of the data under normality. Nerlove, Grether, and Carvalho (1979) refer to the ARIMA representation as the canonical form of the UC model, and it may be useful to think of it as the reduced form. It is obtained by substituting equations (1b) and (1d) into equation (1a), taking first differences, and rearranging:

$$\phi_p(L)(1 - L)y_t = \phi_p(1)\mu + \phi_p(L)\eta_t + \theta_q(L)(1 - L)\epsilon_t. \tag{2a}$$

Recognizing that the right-hand side will have nonzero autocovariances through lag max $(p, q + 1)$, Granger's lemma (see Granger & Newbold, 1986, p. 29) implies that the univariate representation will be

$$\phi_p(L)(1 - L)y_t = \mu^* + \theta_{q^*}(L)u_t, \quad u \sim \text{i.i.d. } N(0, \sigma_u^2), \quad q^* = \max(p, q + 1), \tag{2b}$$

where the coefficients of $\theta^*_{q^*}(L)$ and σ_u^2 are obtained by matching the autocovariances of the right-hand sides of equations (2a) and (2b); see Watson (1986). This ARIMA reduced form fully describes the joint distribution of the $\{y_t\}$ and therefore the conditional distribution of future observations given the past, and is unique.

Note that the BN trend for the reduced-form ARIMA model (2b) may be derived from the Wold representation of equation (2b) and expressed as

$$BN_t = BN_{t-1} + \psi(1)u_t = \psi(1)\sum_{j=1}^{t} u_j, \tag{2c}$$

where $\psi(1) = \theta^*_{q^*}(1)/\phi_p(1)$ and $BN_0 = 0$; thus the variance of the innovation to the BN trend is $\psi(1)^2\sigma_u^2$. The BN cycle is obtained by subtracting from y_t the BN trend.

Correspondingly, there is always at least one UC representation of any given ARIMA process; as Cochrane (1988) pointed out, the existence of the BN decomposition guarantees this. In general, however, there will not be a *unique* UC representation, because all the parameters may not be identified. For example, consider the ARIMA(0, 1, 1) process, so that in the notation of equation (2b) the orders are $p = 0$ and $q^* = 1$. By inspection of equation (2a) it is clear that this implies $q = 0$; hence the UC representation is a random walk plus noise. The relations between the two nonzero autocovariances γ_j at lags 0 and 1 (values of which can be inferred from data) and the UC parameters are as follows:

$$\gamma_0 = \sigma_\eta^2 + 2\sigma_\epsilon^2 + 2\sigma_{\eta\epsilon},$$

$$\gamma_1 = -\sigma_\epsilon^2 - \sigma_{\eta\epsilon},$$

$$\gamma_j = 0, \quad j \geq 2.$$

[2] The theoretical justification for the BN decomposition and its relationship to martingale decompositions is given in Phillips and Solo (1992). A corresponding decomposition for seasonal time series is given in Box, Pierce, and Newbold (1987).

Whereas there are three UC parameters, there are only two pieces of information. Note that the variance of the trend innovations may be inferred by adding $2\gamma_1$ to γ_0. However, the variance of the cycle innovations and the covariance are not separately identified; only their sum is. This reflects a basic theme of this paper: the trend process is always identified from the univariate properties of the series, though the cycle process may not be. In the case of random walk plus noise, the zero-covariance restriction is an identifying restriction. If $\gamma_1 > 0$, however, then it is easy to see that there is no UC-0 representation. It is also possible to infer inequalities in this case; see Nelson and Plosser (1982).

More generally, it is easily shown that there will be at least as many nonzero autocovariance relations as parameters if $p \geq q + 2$, a result that we use later to identify the covariance in the UC-ARMA model for GDP.

Given that a time series will not in general have a unique UC representation, the following result may seem surprising:

The BN trend is the conditional expectation of the random walk component for any UC representation of an I(1) process.

As pointed out by Watson (1986), this is true regardless of the covariance structure of the unobserved components. To see why, consider any unrestricted UC representation defined by (1a)–(1c), so cycle and trend innovations may be cross-correlated. The conditional expectation of the trend component at time t is

$$E[\tau_t|\Omega_t] = \lim_{M\to\infty} E[\tau_t + c_{t+M} \mid \Omega_t]$$

since for large enough M, the cycle, by its ergodicity, has an expectation of zero. Further, the expected value of any future innovation in the trend is zero. Thus we have

$$E[\tau_t|\Omega_t] = \lim_{M\to\infty} E\left[\tau_t + \sum_{j=1}^{M} \eta_{t+j} + c_{t+M} \,\middle|\, \Omega_t\right].$$

Recognizing that the terms of the right include all the elements of y_{t+M} except the accumulated drift, we have

$$E[\tau_t|\Omega_t] = \lim_{M\to\infty} E\left[\tau_t + \sum_{j=1}^{M} \eta_{t+j} + c_{t+M} \,\middle|\, \Omega_t\right]$$
$$= \lim_{M\to\infty} E[y_{t+M} - M\mu \mid \Omega_t] = BN_t.$$

Then the conditional expectation of the cycle at time t is simply

$$E[c_t|\Omega_t] = y_t - E[\tau_t|\Omega_t] = y_t - BN_t.$$

Thus, we can always compute the conditional expectation estimates of trend and cycle at any point in time from the ARIMA reduced form. The following two assumptions are sufficient to identify the components: (1) the trend is a random walk, and (2) the cycle is ergodic. This result does not depend on knowing the covariance between trend and cycle innovations, nor does it depend on the existence of a unique UC representation. Intuitively, the forecast at a long enough horizon reflects only the random-walk component. Stronger assumptions may be needed to identify the parameters of a UC representation, but they are irrelevant if the only objective is to estimate trend and cycle at a point in time.

It follows that the Kalman filter estimates of trend and cycle from the Clark-Harvey UC-0 model of GDP must be the same as BN estimates, if the parameters of the ARIMA reduced form are those implied by the UC-0 model. In that case, BN is just an alternative to the Kalman filter for computing $\hat{\tau}_{t|t}$ and $\hat{c}_{t|t}$; the estimates will be numerically identical. UC-0 and BN decompositions thus share an often noted property of the latter, namely that the innovations of the *estimated* trend and cycle series are perfectly correlated. Further, the equivalence holds between UC and BN decompositions in general; the corresponding ARIMA representation will always contain sufficient information to estimate the trend.

The fact that the two approaches have produced such different estimates of trend and cycle in practice implies that they must be based on conflicting representations of the data. Identifying the source of the conflict is the subject of the next section.

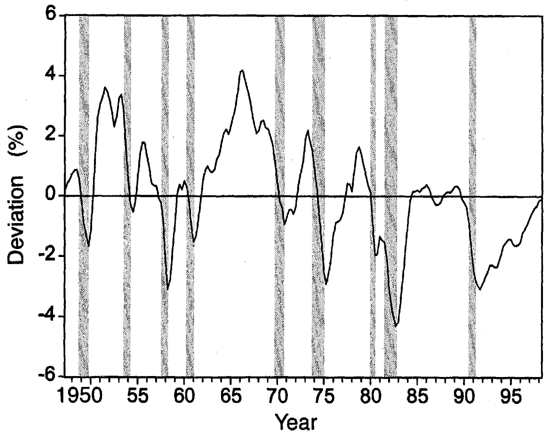
III. In What Way Do UC and ARIMA Models of U.S. Real GDP Conflict?

The results of section II imply that the differing results obtained in practice must be traceable to a conflict between the reduced form ARIMA implied by the Clark-Harvey UC model and the unrestricted ARIMA model used in the BN approach. To isolate the source of conflict we begin by estimating the UC-0 model for U.S. real GDP 1947:1–1998:2 in logs, following Clark (1987) in setting $p = 2$ to allow for cyclical dynamics, and $q = 0$.[3]

In accord with the literature, the estimated UC-0 cycle seen in figure 1 is large in amplitude and very persistent, while the trend is smooth. The scale in all figures is log times 100 and so may be read as the percentage deviation from trend. This is the one-sided or "filtered" estimate; the two-sided or Kalman-smoothing estimate often presented in the literature is even smoother. It is qualitatively similar to the deviation of the log of GDP from the least squares trend line, and both imply that the economy was well below

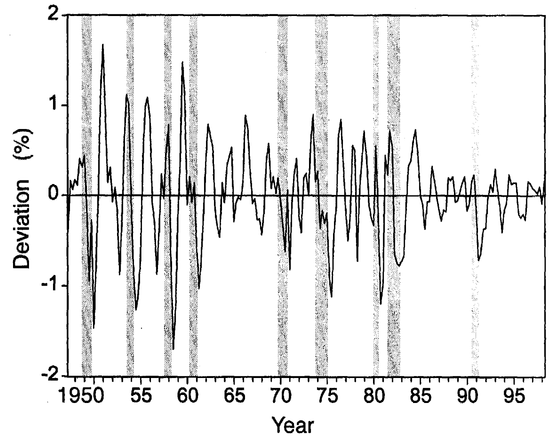
[3] The data series used is *gdpq* from the DRI databank. Clark (1987) allowed the drift parameter to evolve as a random walk, but estimates of the variance are small. We have assumed that this parameter remains constant, implying that output is I(1).

FIGURE 1.—UC-0 CYCLE, U.S. REAL GDP



Percentage deviation from trend. NBER recessions shaded.

FIGURE 2.—BEVERIDGE-NELSON CYCLE, U.S. REAL GDP



Percentage deviation from trend, NBER recessions shaded.

trend during most of the 1990s. Declines in the UC-0 cycle agree reasonably well with the NBER dating of recessions (shaded), though they lead the NBER dating at peaks. We note that the NBER dating procedure draws on a much larger information set and the methodology is largely subjective, in contrast to the model-based univariate decompositions presented here. Although agreement with the NBER dating is not a requirement for a valid decomposition of output into permanent and transitory components, the comparison is illustrative.

Table 1 reports the maximum likelihood estimates of the parameters and their standard errors for the UC-0 model. Detail of the estimation technique used is given in the technical appendix. The roots of the estimated autoregressive polynomial are complex, implying that the business cycle has a period of almost 8 years with a standard deviation of about 3 percentage points around trend, confirming the visual impression of persistence, periodicity, and amplitude from figure 1. By contrast, the trend process innovation has a standard deviation of only about 0.7 percentage points.

The reduced-form ARIMA representation (2b) for this UC-0 model is obtained as follows. Taking first differences gives

TABLE 1.—MAXIMUM LIKELIHOOD ESTIMATES OF UC-0 PARAMETERS

	Estimate	Standard Error
Trend process:		
Drift μ	0.8119	(0.0500)
Innovation σ_η	0.6893	(0.1038)
Cycle process:		
ϕ_1	1.5303	(0.1012)
ϕ_2	-0.6097	(0.1140)
Innovation σ_ϵ	0.6199	(0.1319)
AR roots (inverted): $0.7652 \pm 0.1558i$		
Implied period: 7.7 years; standard deviation: 0.03.		
Log likelihood: -286.6053		

$$\Delta y_t = (1 - L)\tau_t + (1 - L)c_t$$
$$= \mu + \eta_t + (1 - L)(1 - \phi_1 L - \phi_2 L^2)^{-1}\epsilon_t.$$

Next, multiply both sides by $1 - \phi_1 L - \phi_2 L^2$ to obtain

$$(1 - \phi_1 L - \phi_2 L^2)\, \Delta y_t = \mu^* + \eta_t - \phi_1\eta_{t-1} - \phi_2\eta_{t-2} + \epsilon_t - \epsilon_{t-1}$$
$$= \mu^* + u_t + \theta_1^* u_{t-1} + \theta_2^* u_{t-2}. \quad (3)$$

The result in (3) uses the fact that the right-hand side has a representation as an MA(2) by Granger's lemma with the univariate innovations u_t being i.i.d. $N(0, \sigma_u^2)$, and μ^* is $\mu(1 - \phi_1 - \phi_2)$. It is important to recognize that the assumption $\sigma_{\eta\epsilon} = 0$ places complicated nonlinear restrictions on the parameters of the ARIMA(2, 1, 2) model (3). In particular, Lippi and Reichlin (1992) show that the long-run persistence measure, $\psi(1) = \theta^*(1)/\phi(1)$, will be less than or equal to one. Proietti and Harvey (2000) give further restrictions on the autoregressive parameters. These restrictions are testable implications of the UC-0 model but in empirical work they are almost never tested.

While the reduced form of the UC-0 model is a restricted ARIMA(2, 1, 2), when we estimate the unrestricted form of that model by exact maximum likelihood and compute the BN cycle component from it, we get the very different results seen in figure 2.[4] As reported in the literature, the estimated BN cycle is small in amplitude compared to the UC-0 cycle and much less persistent. Table 2 reports the maximum likelihood estimates of the parameters for the unrestricted reduced-form ARIMA(2, 1, 2) model. Confirming one's visual impression, the period of the cycle implied by the AR parameters here is much shorter, only 2.4 years

[4] We follow Morley (2002) in our computation of the BN decomposition.

TABLE 2.—MAXIMUM LIKELIHOOD ESTIMATES FOR ARIMA(2, 1, 2)

	Estimate	Standard Error
Drift μ	0.8156	(0.0864)
ϕ_1	1.3418	(0.1519)
ϕ_2	−0.7059	(0.1730)
θ_1	−1.0543	(0.1959)
θ_2	0.5188	(0.2250)
SE of regression	0.9694	(0.0478)
$\psi(1)$	1.2759	(0.1543)
AR roots (inverted) Implied period = 2.4 years.	$0.6709 \pm 0.5057i$	
MA roots (inverted)	$0.5271 \pm 0.4908i$	
Log likelihood	−284.6507	

instead of nearly 8. The fact that the value of the log likelihood is greater by roughly 2 for the unrestricted ARIMA must reflect restrictions in the UC-0 model not imposed in the reduced form, in particular the zero correlation between trend and cycle innovations. Another indication that the zero-correlation restriction may not be supported by the data is that the estimated value of persistence, $\psi(1)$, is greater than one.

To see what correlation is implied by the ARIMA parameters, we next solve for the parameters of the unrestricted UC-ARMA model of equations (1a)–(1d) that correspond to the estimated unrestricted ARIMA parameters in Table 2. First note that the AR parameters are the same in both the UC and ARIMA reduced form, because the AR polynomial on the left side of equation (2a) is the AR polynomial of the UC cycle. Now the observable moments on the MA side of equation (2a) are the mean, which identifies μ, and the autocovariances:

$$\begin{aligned}\gamma_0 &= (1 + \phi_1^2 + \phi_2^2)\sigma_\eta^2 + 2\sigma_\epsilon^2 + 2(1 + \phi_1)\sigma_{\eta\epsilon},\\ \gamma_1 &= -\phi_1(1 - \phi_2)\sigma_\eta^2 - \sigma_\epsilon^2 - (1 - \phi_2 + \phi_1)\sigma_{\eta\epsilon},\\ \gamma_2 &= -\phi_2\sigma_\eta^2 - \phi_2\sigma_{\eta\epsilon},\\ \gamma_j &= 0, \qquad j \geq 3.\end{aligned} \tag{4}$$

The autocovariances on the left-hand side of equation (4) are

$$\begin{aligned}\gamma_0 &= \sigma_u^2(1 + \theta_1^2 + \theta_2^2),\\ \gamma_1 &= \sigma_u^2\theta_1(1 + \theta_2),\\ \gamma_2 &= \sigma_u^2\theta_2.\end{aligned}$$

The system of equations (4) can be written in matrix form as

$$\begin{pmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix}1 + \phi_1 + \phi_2 & 2 & 2(1 + \phi_1)\\ -\phi_1(1 - \phi_2) & -1 & -(1 - \phi_2 + \phi_1)\\ -\phi_2 & 0 & -\phi_2\end{pmatrix}\begin{pmatrix}\sigma_\eta^2\\ \sigma_\epsilon^2\\ \sigma_{\eta\epsilon}\end{pmatrix}$$

or

$$\gamma = \Phi\sigma.$$

Assuming Φ is invertible, for which a necessary condition is $\phi_2 \neq 0$, we have

$$\sigma = \Phi^{-1}\gamma. \tag{5}$$

Hence, the three nonzero autocovariance from the MA(2) are just sufficient to identify the three remaining parameters of the UC representation, namely σ_η^2, σ_ϵ^2, and $\sigma_{\eta\epsilon}$. We note that in a particular case the solution to (4) might not imply a positive definite covariance matrix for the trend and cycle innovations, in which case there would not exist a corresponding UC-ARMA(2, 0) representation.

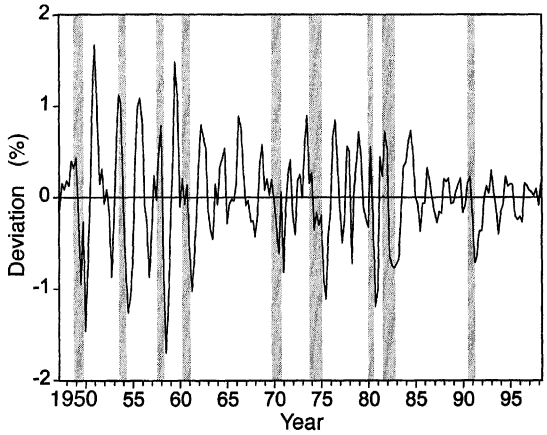
Table 3 compares the estimates from Table 1 for the UC-0 model with the implied estimates from the unrestricted ARIMA(2, 1, 2) reduced form obtained from equation (5). Although the parameters for the cycle component are somewhat similar, the unrestricted reduced form implies a standard deviation for the trend innovation that is almost twice as large as the UC-0 estimate, and a correlation between trend and cycle innovations that is large and negative instead of 0. It is important to note that the correlation presented in Table 3 is the estimated correlation between unobserved innovations, not the correlation between innovations in the observed estimated series $\hat{\tau}_{t|t}$ and $\hat{c}_{t|t}$. The UC-0 model restricts the correlation between unobserved innovations to be 0, whereas the unrestricted ARIMA reduced form estimates that correlation implicitly. Thus, the difference in correlation estimates seen in Table 3 reflects a difference between the two models, not a difference in detrending methods.

In contrast, innovations in the *estimated* components $\hat{\tau}_{t|t}$ and $\hat{c}_{t|t}$ are perfectly negatively correlated in both the UC and BN approaches. Recall from section II that the estimated components obtained by BN and Kalman filter procedures are numerically equivalent if the underlying models have the same reduced form. This implies that the well-known perfect negative correlation of estimated BN components is a property shared by UC filtered estimates. Another way to see why UC estimates have the perfect negative correlation property is to inspect the updating equations for the Kalman filter (see Harvey, 1981, chapter 4), noting that the random-walk property of the trend

TABLE 3.—PARAMETERS OF UC-0 MODEL AND THOSE IMPLIED BY THE UNRESTRICTED ARIMA(2, 1, 2) REDUCED FORM

	UC-0 Model	UC Model Implied by ARIMA
Trend process:		
Drift μ	0.8119	0.8156
Innovation σ_η	0.6893	1.2368
Cycle process:		
ϕ_1	1.5303	1.3418
ϕ_2	−0.6098	−0.7059
Innovation σ_ϵ	0.6199	0.7487
Covariance $\sigma_{\eta\epsilon}$	0 (constrained)	−0.8391
Correlation $\rho_{\eta\epsilon}$	0 (constrained)	−0.9062

FIGURE 3.—UC-UR CYCLE, U.S. REAL GDP



Percentage deviation from trend, NBER recessions shaded.

component implies that the innovation in the estimated trend component is proportional to the forecast error in predicted y_t, just as for the BN estimate of trend. As pointed out by Wallis (1995), the distinction between assumptions (unobserved innovations may be uncorrelated) and properties of estimates (estimated innovations are perfectly correlated) is entirely consistent with least squares estimation, but nevertheless has been a source of frequent confusion in the literature.

The fact that $\sigma_{\eta\epsilon}$ is identified in the UC-ARMA(2, 0) case implies that we can relax the restriction that it is zero in the UC model and estimate it directly by maximum likelihood. We denote the unrestricted UC-ARMA(2, 0) model as UC-UR. It may be cast in state-space form by including the cycle component along with the trend in the state equations as noted by Canova (1998); see the appendix for details of the representation we use. Again, the order condition for identification of the unrestricted UC-ARMA(p, q) model, in the sense of having at least as many moment equations as parameters, is $p \geq q + 2$, and it is just satisfied with $p = 2$, $q = 0$. Intuitively, increasing p increases the number of moment equations corresponding to equation (4) on the MA side of the univariate ARIMA representation without increasing the number of parameters to solve for, since those are always identified by the AR side.

The resulting filtered estimate of the cycle from the UC-UR model is shown in figure 3. This estimated cycle is identical to the estimated cycle from the BN decomposition except for the first observation, that being due to the need to provide the Kalman filter with an initial guess for the value of the random walk. This equivalence verifies that the filtered estimates from the UC model and the BN estimates are the same and does not relate to the particular value of the estimated correlation.

Table 4 reports the maximum likelihood estimates of the parameters for the UC-UR model. We parameterized the model alternatively in terms of the covariance $\sigma_{\eta\epsilon}$ and the correlation $\rho_{\eta\epsilon}$; both estimations produce the same numerical results, and both are strongly negative. A striking feature of these estimates is that they are all essentially the same as the implied estimates from the unrestricted ARIMA model reported in Table 3. Note also that the estimated variance of the permanent component, $\hat{\sigma}_\eta^2 = 1.5296$, is essentially equal to the variance of the innovation to the BN trend from the unrestricted ARIMA(2, 1, 2) model, $\hat{\sigma}_u^2\hat{\psi}(1)^2 = 1.5297$. The estimated value of ϕ_2 is several times its standard error, supporting the $p = 2$ specification, although we note that this is not a standard testing situation, for the model is not identified if the null hypothesis that $\phi_2 = 0$ is true.

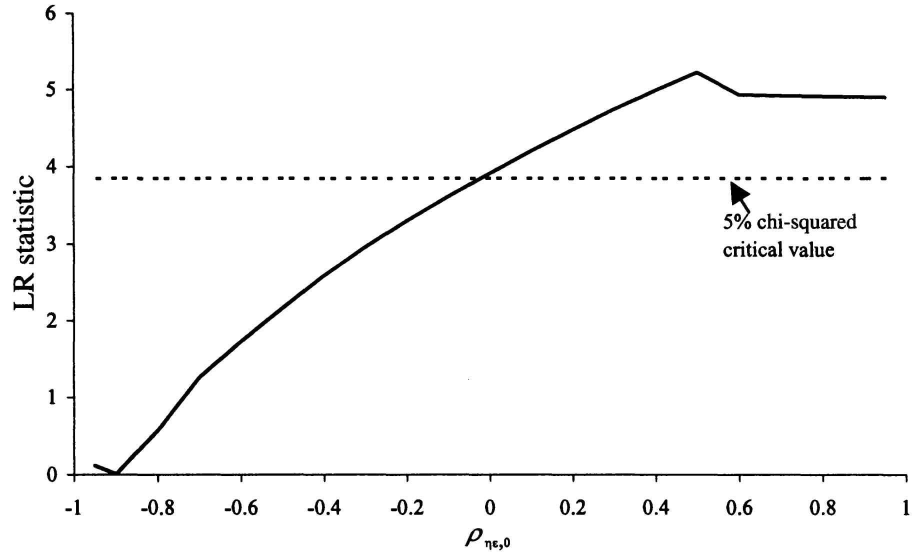
Note that the log likelihood for the UC-UR model is also the same as for the ARIMA model, and significantly larger than for the restricted UC-0 model, thus confirming identification of the covariance between trend and cycle innovations. The likelihood ratio statistic for testing the restriction $\rho_{\eta\epsilon} = 0$, which may be interpreted as a test of overidentification, is 3.909, with a corresponding p-value of 0.048. As a check on the small-sample properties of this test, particularly to determine whether this test rejects zero correlation too often, we generated data from the UC model calibrated to the UC-0 estimates, tested the null hypothesis $\rho_{\eta\epsilon} = 0$, and found the size to be approximately correct. Thus, we can reject the restriction of a zero correlation between permanent and transitory shocks by comparing the results for the UC-0 model with either the results for the reduced-form ARIMA model or the unrestricted UC model.

Note too that the estimate of $\rho_{\eta\epsilon}$ is -0.906 with an estimated standard error of 0.073, so a Wald-type t-test implies a far smaller p-value than the likelihood ratio test reported above. Because the estimated value of $\rho_{\eta\epsilon}$ is near the boundary of admissible values, the small estimated standard error might give a misleading impression of precision. To reconcile the two tests, we estimated the UC-ARIMA model for fixed values of $\rho_{\eta\epsilon} = -0.95, -0.9, \ldots, 0.9, 0.95$ and, for each model computed the likelihood ratio statistic for the hypothesis that $\rho_{\eta\epsilon}$ is equal to the imposed value. Figure 4 is a plot of these likelihood ratio statistics as a function of the hypothetical $\rho_{\eta\epsilon}$. The horizontal line indicates the 95% quantile from the chi-squared distribution

TABLE 4.—MAXIMUM LIKELIHOOD ESTIMATES FOR THE UC-UR MODEL

	Estimate	Standard Error
Trend process:		
Drift μ	0.8156	(0.0865)
Innovation σ_η	1.2368	(0.1518)
Cycle process:		
ϕ_1	1.3419	(0.1456)
ϕ_2	−0.7060	(0.0822)
Innovation σ_ϵ	0.7485	(0.1614)
Roots of AR process	$0.6710 + 0.5058i$	
	$0.6710 - 0.5058i$	
Covariance $\sigma_{\eta\epsilon}$	−0.8389	(0.1096)
Correlation $\rho_{\eta\epsilon}$	−0.9063	(0.0728)
Log likelihood value	−284.6507	

FIGURE 4.—LIKELIHOOD RATIO STATISTICS FOR TESTING $\rho_{\eta\epsilon} = \rho_{\eta\epsilon,0}$



with 1 degree of freedom. The shape of the plot clearly indicates a global maximum of the likelihood at the estimated value of $\rho_{\eta\epsilon}$ and the sharpness of the likelihood around that point is reflected in the small standard error. The implied 95% confidence interval for $\rho_{\eta\epsilon}$ obtained by inverting the likelihood ratio statistic is fairly wide but just barely excludes $\rho_{\eta\epsilon} = 0$. Thus, the difference between the Wald and likelihood ratio test results is traced to local versus global behavior of the likelihood function.

One way of summarizing the information we have about the nature of the trend-cycle decomposition of GDP in the context of these univariate models is to compare results across the range implied by the confidence interval for the innovation correlation. To be conservative, we use the much wider interval implied by the likelihood ratio. At the negative end we use the point estimate -0.906, because it is close to the -1 boundary; at the upper end we use 0, because it is the value assumed in the UC-0 model though it is just outside the 95% critical boundary; and we use two values spaced between, -0.6 and -0.3. This grid gives the range of results for the cycle, trend differences, and trend seen in figure 5. Heuristically, we may think of these as a confidence interval on the decomposition itself. As we move from the lowest value in the positive direction, the variation in GDP associated with the NBER recessions shifts from the cycle component to the trend component. This reflects the contrasting interpretation of recessions in the UC-0 and UC-UR models. In the former they are largely transitory (the trend component only pauses in its growth, but the cycle falls sharply) while in the latter recessions are largely permanent (the trend component falls in accord with NBER recessions, while the cycle is largely noise). Though the data favor the latter interpretation, intermediate outcomes are also well within the confidence interval.

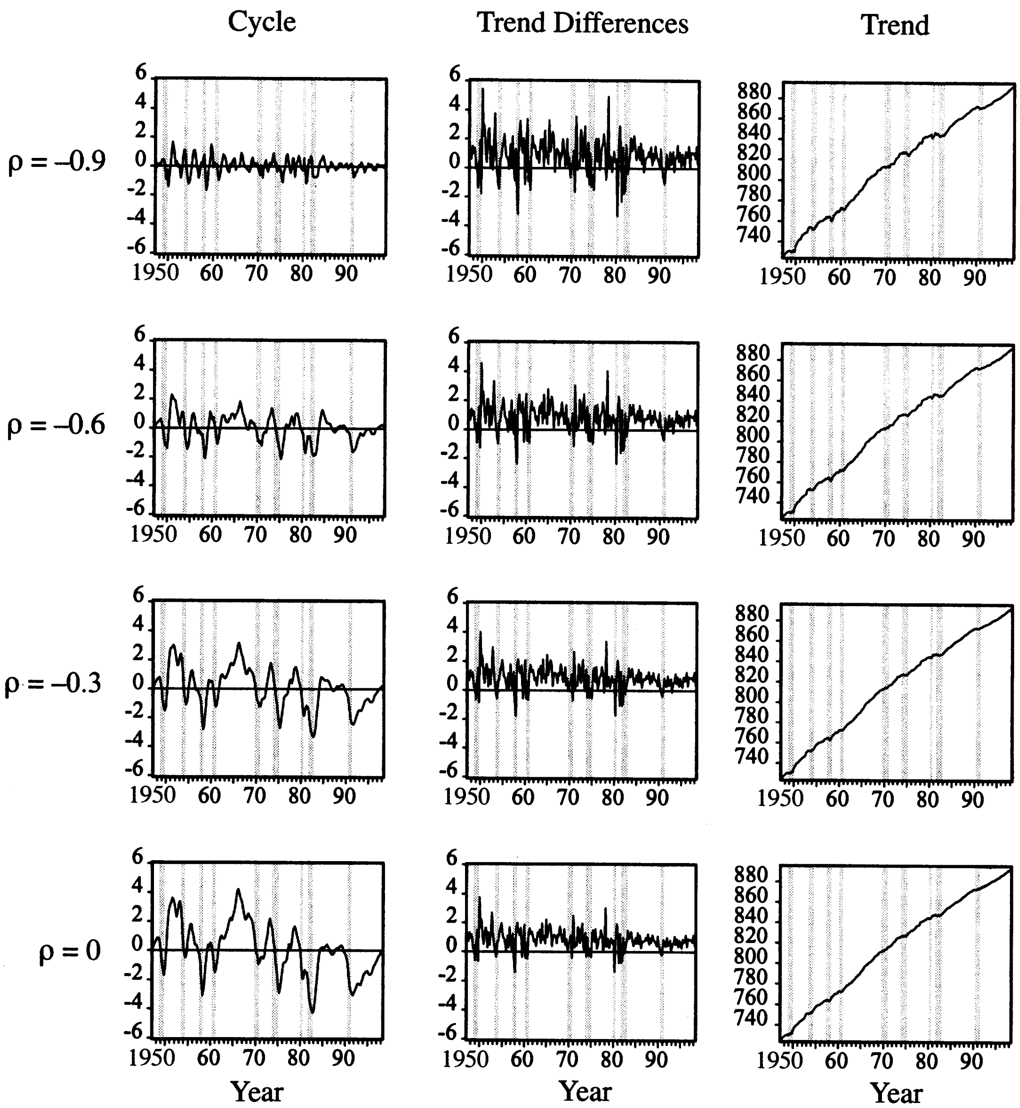
IV. Summary and Conclusions

We have shown that trend-cycle decompositions based on unobserved component models cast in state-space form and on the long-run forecast implied by an ARIMA model are at odds not because they differ in principle but because the underlying empirical models differ. In particular, a testable restriction that innovations in the unobserved trend and cycle are uncorrelated has been imposed in the former, but not in the latter. We note that when this restriction is relaxed in the state-space model, the two approaches lead to identical trend-cycle decompositions and identical univariate representations. Further, the restriction of zero correlation is rejected at the 0.05 level by the data for U.S. real GDP, quarterly 1947–1998.

If we accept the implication that innovations to trend are strongly negatively correlated with innovations to the cycle, then the case for the importance of real shocks in the macro economy is strengthened. As pointed out by Stock and Watson (1988) in their influential paper, real shocks tend to shift the long-run path of output, so short-term fluctuations will largely reflect adjustments toward a shifting trend if real shocks play a dominant role. For example, a positive productivity shock, such as the invention of the Internet, will immediately shift the long-run path of output upward, leaving actual output below trend until it catches up. This implies a negative contemporaneous correlation, for this positive trend shock is associated with a negative shock to the transitory component of output. By contrast, a positive nominal shock, say a shift in Fed policy towards stimulus, will be a positive innovation to the cycle without any impact on trend.

Closing with a few caveats, we note that the decompositions considered here share a common restriction, that the cycle process is symmetric. Recent business cycle research

FIGURE 5.—DIFFERENT CYCLES AND TRENDS, U.S. REAL GDP



Columns correspond to cycles, trend differences, and trends, respectively. Rows correspond to correlations of −0.906, −0.6, −0.3, and 0, respectively. NBER recessions are shaded.

suggests that asymmetry has been an important feature of postwar U.S. experience—recessions being characterized as an occasional sharp drop followed by more gradual recovery; see Neftci (1984), Hamilton (1989), Sichel (1993, 1994), Beaudry and Koop (1993), and Kim and Nelson (1999). The inference that variation in GDP is dominated by variation in trend may reflect primarily the long periods of expansion when actual output is relatively close to potential and any cycle is short-lived and small in amplitude. Finally, the decompositions considered here are univariate with only two sources of shocks. Additional information introduced in a multivariate setting may affect estimates of trend and cycle.

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APPENDIX

State-Space Representation of Unrestricted UC Model

The unrestricted UC model is cast in state-space form by making the observation equation an identity and treating both trend and cycle as state variables:

Observation equation:

$$y_t \equiv [1\ 1\ 0]\begin{bmatrix} \tau_t \\ c_t \\ c_{t-1} \end{bmatrix}. \tag{A1}$$

State equation:

$$\begin{bmatrix} \tau_t \\ c_t \\ c_{t-1} \end{bmatrix} = \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \tau_{t-1} \\ c_{t-1} \\ c_{t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \eta_t \\ \epsilon_t \end{bmatrix}. \tag{A2}$$

Allowing trend and cycle innovations to be correlated, we have

$$\mathbf{Q} \equiv \mathrm{E}[\mathbf{v}_t\mathbf{v}_t'] = \begin{bmatrix} \sigma_\eta^2 & \sigma_{\eta\epsilon} & 0 \\ \sigma_{\eta\epsilon} & \sigma_\epsilon^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{A3}$$

The standard Kalman filter equations are then be applied, given initial values for the expectation of the state vector and its variance. For the random-walk component we use the initial data value, but assign it an extremely large variance. For the transitory component, we use the unconditional mean and variance of the AR(2) process. In maximizing the log-likelihood function we impose stationarity constraints on the autoregressive parameters and a positive definiteness constraint on the innovation covariance matrix. The UC-0 model is estimated as a special case with the restriction $\sigma_{\eta\epsilon} = 0$.