Suggested Solution to Assignment 1

Analytical Questions

(1) $Y_t = (1+2.4L+0.8L^2)\varepsilon_t$

If z are the roots of the polynomial then we can write

$(1+2.4z+0.8z^2) = (1+0.4z)(1+2z)$

where clearly one of the roots, lies inside the unit circle. $|z|\geq 1$ Therefore; this MA process is not invertible.

The invertible representation

$$Y_t = (1+0.4L)(1+\tfrac{1}{2}L)\tilde{\varepsilon}_t$$

$$= (1+0.4L)(1+0.5L)\tilde{\varepsilon}_t$$

$$Y_t = (1+0.9L+0.2L^2)\tilde{\varepsilon}_t$$

Autocovariances

$$\gamma_0 = E[Y_t^2] = E[\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2})^2$$

$$\Rightarrow \gamma_0 = (1+2.4^2+0.8^2) = 7.4\sigma^2$$

$$\gamma_1 = E[(\varepsilon_t+2.4\varepsilon_{t-1}+0.8\varepsilon_{t-2})(\varepsilon_{t-1}+2.4\varepsilon_{t-2}+0.8\varepsilon_{t-3})]$$

$$= 2.4\sigma^2 + 0.8\times 2.4\sigma^2 = 4.32\sigma^2$$

$$\gamma_2 = 0.8\sigma^2$$

$$\gamma_3, \ldots \gamma_j = 0 \quad \text{for } j \geq 3$$

2) $y_t = 2.5 + 1.1 y_{t-1} - 0.18 y_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim iid(0,1)$

$\Rightarrow (1 - 1.1L - 0.18L^2) Y_t = 2.5 + \varepsilon_t$

The roots of $(1 - 1.1Z - 0.18Z^2) = (1-0.9Z)(1-0.2Z) = 0$

$\lambda_1 = 0.9, \ \lambda_2 = 0.2, \ Z_1 = \frac{1}{0.9}, \ Z_2 = \frac{1}{0.2}$ lie

outside the unit circle. Hence, the process

is stationary and stable.

$Y_t = 2.5 + 1.1 y_{t-1} - 0.18 y_{t-2} + \varepsilon_t$

$E[Y_t] = \dfrac{2.5}{1 - 1.1 + 0.18} = 31.25$

Autocovariances

$\gamma_0 = Var[Y_t] = E[(Y_t - \mu)^2]$

$(Y_t - \mu) = \phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + \varepsilon_t$

where $\phi_1 = 1.1, \ \phi_2 = -0.18$

$\Rightarrow \gamma_j = \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2}$ for $j = 1, 2, \ldots$

$E[Y_t - \mu]^2 = \phi_1 E[(Y_{t-1} - \mu)(Y_t - \mu)] + \phi_2 E[(Y_{t-2} - \mu)(Y_t - \mu)] + E[\varepsilon_t](Y_t - \mu)$

$\Rightarrow \gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2$

where $\sigma^2 = E[\varepsilon_t](Y_t - \mu)] = E[\varepsilon_t][\phi_1 (Y_{t-1} - \mu) + \phi_2 (Y_{t-2} - \mu) + \varepsilon_t]$

$= \sigma^2$

Using the same structure as covariances, autocorrelation have following form

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}, \quad j = 1, 2$$

$$\Rightarrow \rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1 \;\Big|\; \rho_1 = \frac{\phi_1}{1-\phi_2} \;\Big|\; \rho_2 = \phi_1 \rho_1 + \phi_2$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0}, \quad \rho_2 = \frac{\gamma_2}{\gamma_0}$$

since, $\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2$

$$= \phi_1 \rho_1 \gamma_0 + \phi_2 \rho_2 \gamma_0 + \sigma^2$$

$$= \phi_1 \cdot \frac{\phi_1}{1-\phi_2} \cdot \gamma_0 + \phi_2 \cdot \rho_2 \gamma_0 + \sigma^2$$

$$\Rightarrow \gamma_0 = \left[\frac{\phi_1^2}{(1-\phi_2)} + \frac{\phi_2 \phi_1^2}{(1-\phi_2)} + \phi_2^2\right]\gamma_0 + \sigma^2$$

$$\Rightarrow \gamma_0 = 7.89$$

$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_{-1}$$

$$= \phi_1 \gamma_0 + \phi_2 \gamma_1 \qquad \text{since, } \gamma_{-j} = \gamma_j.$$

$$= \phi_1 \gamma_0 + \phi_2 \rho_1 \gamma_0$$

$$\Rightarrow \gamma_1 = \underline{7.355}$$

$$\rho_1 = \frac{\phi_1}{1-\phi_2}, \qquad \rho_2 = \phi_1 \rho_1 + \phi_2$$

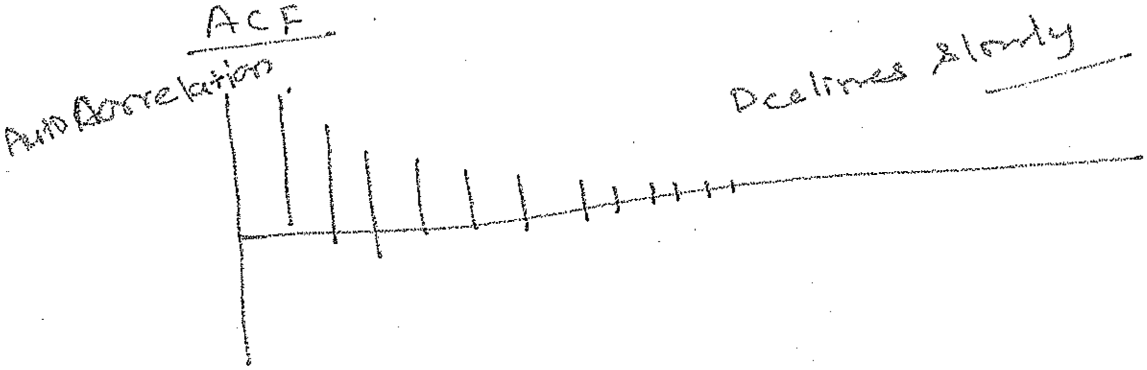
$$\Rightarrow \rho_1 = 0.932$$

$$\rho_2 = 0.845$$

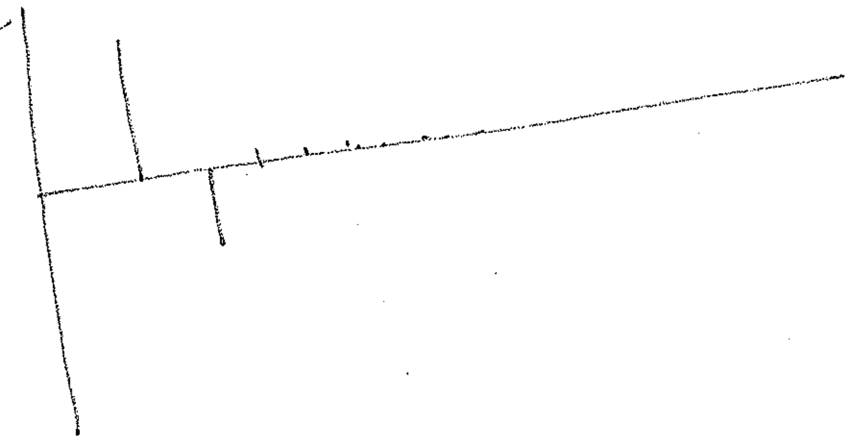
AR(2) model

$\phi_1 = 1.1, \; \phi_2 = -0.18$

ACF



PACF



(b) Wold representation

$(Y_t - 1.1\, Y_{t-1} - 0.18\, Y_{t-2}) = 2.5 + \varepsilon_t$

$\Rightarrow (\underbrace{1 - 1.1L - 0.18L^2}_{\phi(L)})\, Y_t = 2.5 + \varepsilon_t$

$Y_t = \phi(L)^{-1}(2.5 + \varepsilon_t)$

$\phi(L)^{-1} = [1 - 1.1L - 0.18L^2)^{-1}$

$\psi(L) = \phi(L)^{-1}$

$Y_t = \psi(L)\,\varepsilon_t$ where $\psi(L) = \phi(L)^{-1}$

We have

$$\Phi(L)\Psi(L) = 1$$

$$\Rightarrow (1 - 1.1L + 0.18L^2)(1 + \psi_1 L + \psi_2 L^2 + \psi_3 L^3 + \cdots) = 1$$

$$\Rightarrow 1 + (\psi_1 - 1.1)L + (\psi_2 + 0.18 - 1.1\psi_1)L^2 + (\psi_3 + 0.18\psi_1 - 1.1\psi_2)L^3 + (\psi_4 - 1.1\psi_3 + 0.18\psi_2)L^4] + \cdots = 1$$

Equating the coefficients on both sides.

$$\psi_1 = 1.1$$

$$\psi_2 - 1.1\psi_1 = -0.18$$

$$\Rightarrow \psi_2 - 1.1 \cdot 1.1 = -0.18 \Rightarrow \psi_2 = 1.03$$

$$\psi_3 = -0.18\psi_1 + 1.1\psi_2 = 0.935$$

$$\psi_4 = 1.1\psi_3 - 0.18\psi_2$$

$$= 0.8431$$

Impulse response function