

Suggested Solution to Analytical Exercises

(1) Measurement Equation

$$y_t = (1 \ 0) \begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} + \varepsilon_t \Rightarrow \theta_t = (y_t, y_{t-1})'$$

Transition Equation

$$\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}}_F \underbrace{\begin{pmatrix} y_{t-1} \\ y_{t-2} \end{pmatrix}}_{\beta_{t-1}} + \underbrace{\begin{pmatrix} a_t \\ 0 \end{pmatrix}}_{w_t}$$

β_t Initial state distribution

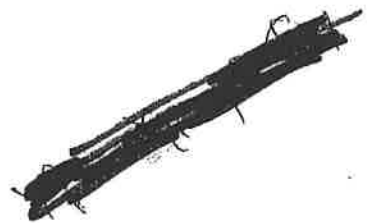
$$\theta_t \sim N(m_0, C_0)$$

$$m_0 = E[\theta_t] = 0$$

$$\text{vec}(C_0) = (I_4 - F \otimes F)^{-1} \text{vec}(Q)$$

$(I_4 - F \otimes F)^{-1}$ is 4x4 matrix

$$\text{vec}(Q) = \begin{pmatrix} 2 \\ \sigma_\varepsilon^2 \\ 0 \\ 0 \end{pmatrix}$$



②

Measurement Equation

$$\begin{pmatrix} \Delta y_t \\ \Delta c_t \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta y_t \\ \Delta c_t \\ y_t - c_t \end{pmatrix}$$

Transition Equation

$$\begin{pmatrix} \Delta y_t \\ \Delta c_t \\ y_t - c_t \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} & \theta_1 \\ \phi_{21} & \phi_{22} & \theta_2 \\ \phi_{11} - \phi_{21} & \phi_{12} - \phi_{22} & 1 + \theta_1 - \theta_2 \end{pmatrix} \begin{pmatrix} \Delta y_{t-1} \\ \Delta c_{t-1} \\ y_{t-1} - c_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{1t} - \varepsilon_{2t} \end{pmatrix}$$

\swarrow
 $(1 + \theta_1 - \theta_2)$

Note that $\Delta y_t = y_t - y_{t-1}$ ①

$\Delta c_t = c_t - c_{t-1}$ ②

Subtract ② from ① to get the

\Rightarrow ~~$y_t - c_t = (y_{t-1} - c_{t-1})$~~ above representation

(3)

The model is

$$y_t = T_t + C_t$$

$$T_t = \mu + T_{t-1} + v_t$$

$$C_t = \phi C_{t-1} + e_t \Rightarrow C_t = (1 - \phi L)^{-1} e_t$$

Since $y_t = T_t + C_t$

$$\Rightarrow \Delta y_t = (1 - L) y_t = \Delta T_t + \Delta C_t$$

$$\Rightarrow \Delta y_t = \mu + v_t + (1 - L) (1 - \phi L)^{-1} e_t$$

$$\Rightarrow \Delta y_t = \mu + v_t + (1 - L) (1 - \phi L)^{-1} e_t$$

Pre multiplying by $(1 - \phi L)$

$$(1 - \phi L) \Delta y_t = (1 - \phi L) \mu + (1 - \phi L) v_t + (1 - L) e_t$$

$$\Rightarrow (1 - \phi L) \Delta y_t = \mu^* + v_t - \phi v_{t-1} + e_t - e_{t-1}$$

$$\Rightarrow (1 - \phi L) \Delta y_t = \mu^* + u_t + \theta u_{t-1}$$

Reduced form

Total no of reduced form parameters representation
 $= 4 (\mu^*, \phi, \theta, \sigma_u^2)$

Total number of structural parameters
 $= 5 (\sigma_v^2, \mu, \phi, \sigma_e^2, \sigma_{ev})$

The model is not identified unless we
impose $\sigma_{ev} = 0$