Non-stationary Time Series

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1 The Big Picture on Time Series

I have to say it takes me a while to grasp the gist of *time series*. Fortunately, I got the big picture, which can be the contour for doing empirical research in macroeconomics. For most students, the starting point of learning time series is the stationary process like AR(1) or MA(1). However, later I realised that lots of time series from real world are not stationary, which make researchers' life difficult and fun. So, what's the big picture of time series? Here we go: at this moment, it's about data-generating process for me.

Follow the idea by Campbell and Perron (1991), we assume that a time series $\{y_t\}$ is a realization of a deterministic trend and a stochastic component:

$$y_t = TD_t + z_t, \tag{1.1}$$

where TD_t assigns a deterministic trend, which can be assumed to have the linear trend, $TD_t = \beta_1 + \beta_2 t$, and z_t represents the stochastic compoment $\phi(L)z_t = \theta(L)\varepsilon_t$ with $\varepsilon_t \sim$i.i.d. Or expand it out, we have:

$$y_t = \beta_0 + \beta_1 t + z_t; \qquad \phi(L)z_t = \theta(L)\varepsilon_t \;\; \varepsilon_t \sim i.i.d$$

The intuition behind this assumption can be summarised by the following table:

Summary of data-generating process

Deterministic Trend		**Stochastic Component**
Fixed effects	Trend	Stochastic pattern with random shocks
β_1	$\beta_2 t$	$\phi(L)z_t = \theta(L)\varepsilon_t$
Reflects Human Being's determination		Law of nature (or unknown process)

For the deterministic trend, things are easy as we can predict it very well if they really follow a certain trend, which is highly unlikely in real data sets. It is common sense that everything is changing. Therefore, we care more about the stochastic part[1].

Now, if all roots of the autoregressive polynomial lie outside the unit circle, then $\{y_t\}$ is stationary around a deterministic trend. In this case, one could remove the trend from the original series $\{y_t\}$ and fit an ARMA(p, q) to the residuals. In R programming, you can do it like this:

[1] or random part, think about normal distribution phenomena, then you will realise this part is the key.

```r
# Detrend the time series
detrended <- residuals(lm(y ~ seq(along = y)))
# fit ARMA model
arma <- arima(detrend, ma=..., ar=...,)
```

The *trend-stationary* model is also called the integrated model of order zero or $I(0)$. But, what if some roots lie on the unit circle[2]? If it happens, things become tricky and that's why I am writing this notes now. For the unit root situation, $\Delta z_t = (1 - L)z_t$ is stationary around a constant mean, the series $\{y_t\}$ is *difference-stationary* because one has to apply the first difference filter with respect to time to obtain a stationary process. This process can be analyzed with *autoregressive integrated moving average* (ARIMA)(p, d, q), where d refers to the order of intergration.

Take away: by decomposing time series $\{y_t\}$, we realise that stochastic part is the key, and unit root situation is the aiming ring.

Before we moving to the next section, let's state the distinction between a trend- and a difference-stationary process by giving the following two processes:

$$y_t = y_{t-1} + \mu = y_0 + \mu t \tag{1.2}$$

$$y_t = y_{t-1} + \varepsilon_t = y_0 + \sum_{s=1}^{t} \varepsilon_s \tag{1.3}$$

2 Unit Root Processes

As we have said that unit root processes are so important, it's worth to devote ourselves to this topic. Let's review the data-generating process:

$$\begin{aligned} y_t &= y_0 + \mu t + z_t && (2.1) \\ &= y_0 + \mu t + (\phi(L)z_t = \theta(L)\varepsilon_t) && (2.2) \end{aligned}$$

Now, let's decompose the stochastic component z_t into a cyclical component c_t and a stochastic trend TS_t. It is assumed that *cyclical component* c_t is a mean-stationary process. Then, equation (2.1) can be rewritten as

$$y_t = y_0 + \mu t + c_t + TS_t \tag{2.3}$$

Equation (2.3) will be the **key function** for almost all our analysis in this document. Now, let's list all situations for y_t in equation (2.3):

- If y_t is trend stationary, then the stochastic trend (TS_t) is zero. Then, y_t can be modeled by ARMA(p, q).
- If y_t is difference stationary, then the autoregressive polynomial contains a unit root that can be factored out by doing the difference, $\phi(L) = (1 - L)\phi^*(L)$. After doing the difference, we get Δz_t which can be modeled as the moving average process.

[2] we don't consider the cases that roots are inside the unit circle as y_t will explore in this case.

Now, if we represent $z_t = TS_t + c_t = \psi(1)S_t + \psi^*(L)\varepsilon_t$, where $\psi(1)$ is the sum of moving average coefficients and S_t is the sume of the past and present of random shocks $\sum_{s=1}^{t} \varepsilon_s$, and $\psi^*(L)$ is the polynomial coefficients. The full equation is[3]:

$$z_t = c_t + TS_t = \psi(1)\sum_{s=1}^{t}\varepsilon_s + \frac{\psi(L)-\psi(1)}{(1-L)}\varepsilon_t \tag{2.4}$$

Final step, substitute (2.4) into (2.3) to get:

$$\begin{aligned} y_t &= y_0 + \mu t + TS_t + c_t \qquad (2.5)\\ &= y_0 + \mu t + \psi(1)\sum_{s=1}^{t}\varepsilon_s + \frac{\psi(L)-\psi(1)}{(1-L)}\varepsilon_t \qquad (2.6)\end{aligned}$$

I am sorry to reclaim that equation (2.6) will be the **key function** for almost all our analysis of time series data. The intuition and summary on (2.6) is given in Figure 2.2.

$$\begin{aligned} y_t &= y_0 + \mu t + TS_t + c_t \\ &= \underbrace{\boxed{y_0 + \mu t}}_{\text{Determinist trend}} + \underbrace{\boxed{\psi(1)\sum_{s=1}^{t}\varepsilon_s}}_{\text{Stochastic compoent: long-run impact of a shock to the level of z}} + \underbrace{\boxed{\frac{\psi(L)-\psi(1)}{(1-L)}\varepsilon_t}}_{\text{Stochastic cyclical compoent: short-run impact on the level of z}} \end{aligned}$$

Figure 2.1: Intuition on y_t decomposition

[3]Check equation (1.3), to see what's going on.

References

Campbell, J. Y. and Perron, P. (1991). Pitfalls and opportunities: what macroeconomists should know about unit roots. *NBER macroeconomics annual*, 6:141–201.