Real Business Cycle Model with Investment Risk

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Abstract

In the notes - *Investment Risk*, I have examined the time series characteristics of US real investment and found that investment should be modeled with autoregressive conditional heteroskedasticity method. In this short collection notes, I will try to embed investment risk into the real business cycles (RBC) model.

1 The Decentralized Model without Investment Risk

I will assume that firms own the capital stock and find the equilibrium conditions for the model. Let's start it from the household problem:

$$\max_{C_t, N_t, B_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t [\ln C_t + \theta \ln(1 - N_t)] \tag{1.1}$$

$$s.t.\ C_t + B_{t+1} \leq w_t N_t + \Pi_t + (1 + r_{t-1}) B_t \tag{1.2}$$

To solve this problem, we form the lagrangian:

$$\mathcal{L} = \beta^t \{[\ln C_t + \theta \ln(1 - N_t)] + \lambda_t [w_t N_t + \Pi_t + (1 + r_{t-1}) B_t - C_t - B_{t+1}]\} +$$
$$\beta^{t+1} \{[\ln C_{t+1} + \theta \ln(1 - N_{t+1})] + \lambda_{t+1} [w_{t+1} N_{t+1} + \Pi_{t+1} + (1 + r_t) B_{t+1} - C_{t+1} - B_{t+2}]\}$$

The first order conditions characterizing an interior solution are:

$$\frac{\partial L}{\partial C_t} = \frac{1}{C_t} - \lambda_t = 0 \Leftrightarrow \frac{1}{C_t} = \lambda_t \tag{1.3}$$

$$\frac{\partial L}{\partial N_t} = \frac{-\theta}{1 - N_t} + \lambda_t w_t \Leftrightarrow \frac{\theta}{1 - N_t} = \lambda_t w_t \tag{1.4}$$

$$\frac{\partial L}{\partial B_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1 + r_t) = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1} (1 + r_t) \tag{1.5}$$

Combine them, we can have:

$$\theta C_t = w_t (1 - N_t) \tag{1.6}$$

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} (1 + r_t) \tag{1.7}$$

In addition, there is the transversality condition:

$$\lim_{t \to \infty} \beta^t B_{t+1} \frac{1}{C_t} = 0 \tag{1.8}$$

Now, we solve the firms' problem by define the discount factor first:

$$M_t = \beta^t \frac{E_0 u'(C_t)}{u'(C_0)} \tag{1.9}$$

Firms are trying to solve the following problem:

$$\max_{N_t, I_t, D_{t+1}} V_0 = E_0 \sum_{t=0}^{\infty} M_t [A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - I_t + D_{t+1} - (1 + r_{t-1}) D_t] \tag{1.10}$$

$$s.t.\ K_{t+1} = I_t + (1 - \delta) K_t \tag{1.11}$$

Again, we can form the lagrangian to solve the problem and first order conditions are

$$\frac{\partial V_0}{\partial N_t} = 0 \Leftrightarrow (1 - \alpha) A_t K_t^{\alpha} N_t^{-\alpha} = w_t \tag{1.12}$$

$$\frac{\partial V_0}{\partial K_{t+1}} = 0 \Leftrightarrow M_t = E_t M_{t+1} [\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + (1 - \delta)] \tag{1.13}$$

$$\frac{\partial V_0}{D_{t+1}} = 0 \Leftrightarrow M_t = E_t M_{t+1} (1 + r_t) \tag{1.14}$$

To close the model, we assume the financial market is frictionless and the exogenous technology follows the following process:

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t \tag{1.15}$$

Collect all first order conditions, we can get:

$$\frac{1}{C_t} = \beta E_t \big(\frac{1}{C_{t+1}} [\alpha A_{t+1} K_{t+1}^{\alpha} N_{t+1}^{1-\alpha} + (1 - \delta)] \big) \tag{1.16}$$

$$\frac{\theta}{1 - N_t} = \frac{1}{C_t} (1 - \alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{1.17}$$

$$K_{t+1} = A_t K_t^{\alpha} N_t^{1-\alpha} - C_t + (1 - \delta) K_t \tag{1.18}$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t \tag{1.19}$$

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{1.20}$$

$$Y_t = C_t + I_t \tag{1.21}$$

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} (1 + r_t) \tag{1.22}$$

$$w_t = (1 - \alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{1.23}$$

$$r_t = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} \tag{1.24}$$

Later, we will discuss the calibrations for this model, and put the code into Dynare to see the IRFs.

2 The Decentralized Model with Investment Risk

Now, we embed the RBC in section 1 with investment risk. Let's start a simplified model:

$$\max_{N_t, I_t, D_{t+1}} V_0 = E_0 \sum_{t=0}^{\infty} M_t [A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - I_t + D_{t+1} - (1 + r_{t-1}) D_t] \tag{2.1}$$

$$s.t.\ K_{t+1} = I_t + (1 - \delta) K_t \tag{2.2}$$

$$I_t = (1 + \sigma_t \varepsilon_t) I_{t-1} \tag{2.3}$$

Again, we form the lagrangian to solve this problem

$$\begin{gathered}
\mathcal{L} = M_t \{A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - I_t + D_{t+1} - (1 + r_{t-1}) D_t + \\
\lambda_t^1 [I_t + (1 - \delta) K_t - K_{t+1}] + \\
\lambda_t^2 \big[\sqrt{\omega + \phi \big(\frac{I_t}{I_{t-1}} - 1\big)^2 + \varphi \sigma_{t-1}^2} \varepsilon_t - \frac{I_t}{I_{t-1}} + 1\big]\} + \\
M_{t+1} \{A_{t+1} K_{t+1}^{\alpha} N_{t+1}^{1-\alpha} - w_{t+1} N_{t+1} - I_{t+1} + D_{t+2} - (1 + r_t) D_{t+1} + \\
\lambda_{t+1}^1 [I_{t+1} + (1 - \delta) K_{t+1} - K_{t+2}] + \\
\lambda_{t+1}^2 \big[\sqrt{\omega + \phi \big(\frac{I_{t+1}}{I_t} - 1\big)^2 + \varphi \sigma_t^2} \varepsilon_{t+1} - \frac{I_{t+1}}{I_t} + 1\big]\}
\end{gathered}$$

Remark: I realise that I should focus on those two equatinos:

$$\begin{aligned}
\ln A_t &= \ln A_{t-1} + \sigma_t \varepsilon_t \\
\sigma_t^2 &= \omega + \phi \sigma_{t-1}^2 + \varphi \varepsilon_{t-1}^2
\end{aligned}$$

References