Lecture 9: Lucas and Moll (2012)
"Knowledge Growth and the Allocation of Time"
More on Endogenous Growth

ECO 521: Advanced Macroeconomics I

Benjamin Moll

Princeton University

Fall 2012

Endogenous Growth

- What endogenous growth models have you seen?

- Virtually all endogenous growth: what grows is technology (of firms) or human capital (of workers), i.e. "knowledge" (as opposed to physical capital)

- Most existing growth theories: growth = discovery/innovation
  - Lucas (1988)
  - Romer (1990)
  - Grossman and Helpman (1991)
  - Aghion and Howitt (1992)
  - Jones (1995)
  - Survey by Jones (2005)

Endogenous Growth

- **Our paper** (also other papers by Lucas):

  growth = knowledge diffusion

- Similar in spirit to Arrow (1969): "the diffusion of an innovation [is] a process formally akin to the spread of an infectious disease."

- Probably both discovery and diffusion are important.

- Open question: which one is more important?

Lucas and Moll (2012)

- Starting point: individual learning depends on
  - (1) individual effort.
  - (2) learning environment.

- Important: **social interactions** of people with diff. knowledge.

- Existing growth theory: **ignores distinction** betw. (1) and (2) $\Rightarrow$ Silent on a host of interesting questions.

- For example: Suppose learning from others becomes easier. What happens to
  - (i) individuals' effort?
  - (ii) the learning environment?
  - (iii) the economy's growth rate?
  - (iv) the income distribution?

What We Do

- Productivity related knowledge held by large number of agents, each with his own productivity level

(1) Agents allocate time between production and learning (Ben-Porath, 1967; Heckman, 1976; Rosen, 1976).

(2) meet stochastically, learn from others, improve productivity (Kortum, 1997; Eaton and Kortum, 1999; Alvarez, Buera and Lucas, 2008; Lucas, 2009).

- Distinctive feature: simultaneous det. of individual behavior (1), and evolution of agents' learning environment (2).

Plan of Talk

(1) A technology of learning and discovery

(2) Decentralized economy with time-allocation decisions

(3) An optimally planned economy

(4) Alternative learning technologies

- Note: in (1)-(3) we pick one particular technology and go with it.

A Technology of Learning

- Consider economy with continuum of infinitely-lived agents

- All have one unit of labor per unit of time (year). Only factor of production. Produce same, single, non-storable good: GDP.

- Productivity of each is a random variable, $z \sim$ density f, cdf F. Entire distribution is state variable of economy.

A Technology of Learning

- $s(z, t)$: fraction of time searching.

- $1 - s(z, t)$: fraction of time working

- Earnings:

$$y(z, t) = [1 - s(z, t)]z$$

- Per capita GDP:

$$Y(t) = \int_0^\infty [1 - s(z, t)]z f(z, t) dz$$

- How do individual productivities evolve?

Evolution of Productivity Distribution

- Allocate $s(z, t)$ to search.

- Over $(t, t + \Delta)$, meet one other agent with probability

  $\alpha\left[s(z, t)\right]\Delta$.

- Meeting $=$ draw $\tilde{z}$ from $f(\tilde{z}, t)$.

- $z(t + \Delta) = \max\{\tilde{z}, z(t)\}$.

- Meetings completely *asymmetric*.

- Number of agents at z:

$$\frac{\partial f(z, t)}{\partial t} = \left.\frac{\partial f(z, t)}{\partial t}\right|_{\text{out}} + \left.\frac{\partial f(z, t)}{\partial t}\right|_{\text{in}}$$

Evolution of Productivity Distribution

- Outflow: Agent z adopts higher productivity if he meets

  another agent with productivity $y \geq z$.

- Happens w/ prob $\alpha(s(z,t))f(y,t)$. Hence

$$\left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{out}} = -\alpha(s(z,t))f(z,t) \int_z^\infty f(y,t)dy$$

- Inflow: agent $y \leq z$ adopts z if he meets agent at z.

- Happens w/ prob $\alpha(s(y,t))f(z,t)$. Hence

$$\left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{in}} = f(z,t) \int_0^z \alpha(s(y,t))f(y,t)dy$$

Evolution of Productivity Distribution

- Combining ins and outs we have

$$\frac{\partial f(z,t)}{\partial t} = -\alpha(s(z,t))f(z,t)\int_z^\infty f(y,t)dy + f(z,t)\int_0^z \alpha(s(y,t))f(y,t)dy$$

- A Boltzmann equation.

- Many possible generalizations, present two later.

Decentralized Time Allocation Decisions

- Assume that agents maximize expected PV of earnings, discounted at given $\rho > 0$

- No risk aversion, but easy to change this.

- Individual preferences:

$$V(z, t) = \mathbb{E}_t \left\{ \int_t^\infty e^{-\rho(\tau - t)} \left[1 - s(\tilde{z}(\tau), \tau) \right] \tilde{z}(\tau) d\tau \,\middle|\, z(t) = z \right\}$$

- Poisson arrival rate $\alpha(s) \geq 0$ with

$$\alpha'(s) > 0, \quad \alpha''(s) < 0, \quad \text{all } s.$$

Individual's Problem

- **Claim:** Individual's solve the Bellman equation

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} = \max_{s \in [0,1]} \left\{ (1-s)z + \alpha(s) \int_z^\infty [V(y,t) - V(z,t)]f(y,t)dy \right\}.$$

- **Derivation:** from usual discrete approximation

$$\begin{aligned} V(z,t) = \max_{s \in [0,1]} & \Delta(1-s)z + (1 - \Delta\rho) \\ & \times \left\{ \Delta\alpha(s) \int_0^\infty \max\{V(y,t+\Delta), V(z,t+\Delta)\} f(y,t+\Delta)dy \right. \\ & \left. + (1 - \Delta\alpha(s))V(z,t+\Delta) \right\} \end{aligned}$$

- See Appendix A of paper for all steps

Individual's Problem

- **Claim:** Individual's solve the Bellman equation

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} = \max_{s \in [0,1]} \left\{ (1-s)z + \alpha(s) \int_z^\infty [V(y,t) - V(z,t)]f(y,t)dy \right\}.$$

- Note the **global** operator

$$\frac{1}{dt}\mathbb{E}[dV(z,t)] - \frac{\partial V(z,t)}{\partial t} = \alpha(s) \int_z^\infty [V(y,t) - V(z,t)]f(y,t)dy$$

rather than the usual **local** operator

$$\frac{1}{dt}\mathbb{E}[dV(z,t)] - \frac{\partial V(z,t)}{\partial t} = V_z(z,t)\mu(z) + \frac{1}{2}V_{zz}(z,t)\sigma^2(z)$$

Equilibrium

- Associated Bellman equation (BE):

$$\rho V(z,t) - \frac{\partial V(z,t)}{\partial t} = \max_{s \in [0,1]} \left\{ (1-s)z + \alpha(s) \int_z^\infty [V(y,t) - V(z,t)] f(y,t) dy \right\}.$$

- Law of motion for distribution (LM):

$$\frac{\partial f(z,t)}{\partial t} = -\alpha(s(z,t)) f(z,t) \int_z^\infty f(y,t) dy + f(z,t) \int_0^z \alpha(s(y,t)) f(y,t) dy.$$

- *Equilibrium*: triple (f, s, V) of functions on $\mathbb{R}_+^2$ such that given $f(z,0)$, above equations satisfied...

- A "mean field game" (Lasry and Lions, 2007). See also Perla and Tonetti (2012).

Balanced Growth Path

Definition

A *balanced growth path (BGP)* is a number γ and a triple of functions (ϕ, σ, v) on $\mathbb{R}_+$ such that

$$f(z, t) = e^{-\gamma t} \phi(z e^{-\gamma t}),$$

$$V(z, t) = e^{\gamma t} v(z e^{-\gamma t}),$$

$$s(z, t) = \sigma(z e^{-\gamma t})$$

- Intuitively: all z-quantiles grow at γ.
- CDF $F(z, t) = \Phi(z e^{-\gamma t}) \Rightarrow q$th quantile defined by

$$\Phi(z_q(t) e^{-\gamma t}) = q \quad \Leftrightarrow \quad z_q(t) = e^{\gamma t} \Phi^{-1}(q).$$

Balanced Growth Path

- Restate (BE), (LM) for BGP only. Use $x = ze^{-\gamma t}$

$$(\rho - \gamma)\, v(x) + v'(x)\gamma x = \max_{\sigma \in [0,1]} \left\{ (1-\sigma)x + \alpha(\sigma) \int_x^\infty [v(y) - v(x)]\phi(y)\, dy \right\}$$

$$-\phi(x)\gamma - \phi'(x)\gamma x = \phi(x) \int_0^x \alpha(\sigma(y))\phi(y)dy - \alpha(\sigma(x))\phi(x) \int_x^\infty \phi(y)dy.$$

- Growth rate γ?

- **Lemma:** BGP with $\gamma > 0$ iff the initial distribution $F(z,0)$ is unbounded **and** has a Pareto tail. In that case,

$$\gamma = \theta \int_0^\infty \alpha(\sigma(y))\phi(y)dy$$

where $1/\theta$ is the tail parameter of $F(z,0)$.

- Note: **multiple** (continuum of) BGPs – one for each θ.

- First: proof.

- Then: interpretation (which some of you may struggle with)

Proof

- Write BGP (LM) in terms of CDF $F(z, t) = \Phi(ze^{-\gamma t})$

$$-\gamma\Phi'(x)x = -[1 - \Phi(x)]\int_0^x \alpha(\sigma(y))\phi(y)dy \qquad (*)$$

- $F(z, 0)$ has a Pareto tail means: there are $k, \theta > 0$ such that

$$\lim_{z\to\infty}\frac{1 - F(z, 0)}{z^{-1/\theta}} = k \qquad \Leftrightarrow \qquad \lim_{x\to\infty}\frac{1 - \Phi(x)}{x^{-1/\theta}} = k$$

- Therefore, in the tail as $x \to \infty$, $(*)$ is

$$-\gamma\frac{1}{\theta}kx^{-\frac{1}{\theta}} = -kx^{-\frac{1}{\theta}}\int_0^\infty \alpha(\sigma(y))\phi(y)dy$$

$$\Rightarrow \quad \gamma = \theta\int_0^\infty \alpha(\sigma(y))\phi(y)dy. \ \square$$

Balanced Growth Path: Discussion

- Need $F(z, 0)$ to not only be unbounded, but also to have a Pareto tail.

- Taken literally: all knowledge already exists at $t = 0$.

- Alternative interpretation: $F(z, 0)$ is bounded but new knowledge arrives at arbitrarily low frequency ("innovation").

- Observationally equivalent, see section 6.3 in paper ▸

- Second alternative interpretation: add Brownian shocks ("experimentation") as in Luttmer (2012).
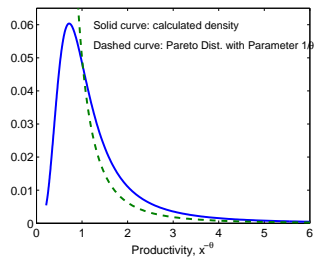
Balanced Growth Path

- Next slide presents results of single simulation of BGP

  - Initial guess for productivity distribution is Frechet(k, θ)

  - Function $\alpha(\sigma)$ given by $\alpha(\sigma) = \alpha_0 \sigma^\eta, \eta < 1$

- Display time allocation function $\sigma(x)$, density $\phi(x)$, and two

  Lorenz curves

  - income Lorenz curve, based on $(1 - \sigma(x)) x$

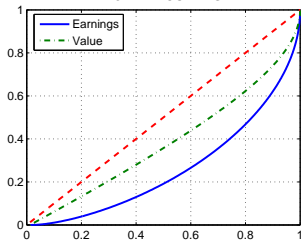 - value Lorenz curve, based on calculated value function, $v(x)$
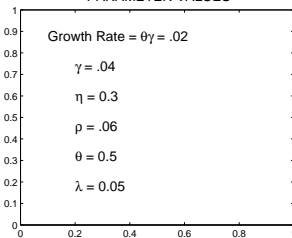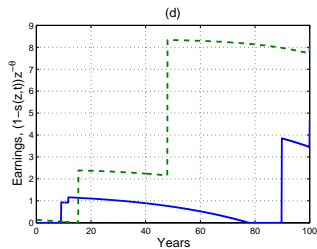
Balanced Growth Path
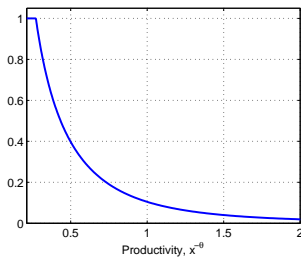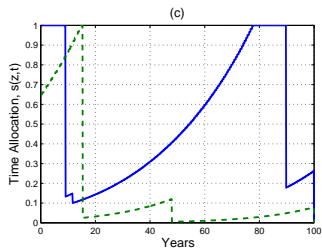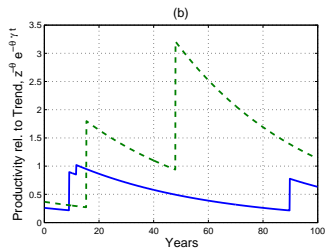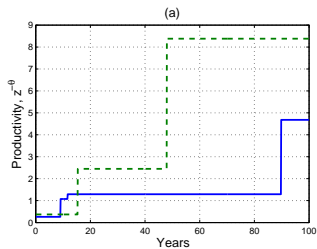
Two Sample Paths

Calibration of Parameters?

- Depends on application and available data.

- But others have looked at data informative about θ and η.

- In US and other OECD countries growth rate fluctuates around $\gamma = .02$

- Information on θ:

  (i) Pareto tail parameter $1/\theta$: $\theta \approx 1$ for firms (Gabaix, 2009; Luttmer, forthcoming).

  (ii) $Var(\log x) \propto \theta^2$: $\theta = .5$ for U.S. census earnings data (Lucas, 2009). $\theta = 1/1.85$ using international relative prices (Eaton and Kortum, 2002).

- Information on η: evidence on on-the-job human capital accumulation as in Ben-Porath (1967); Rosen (1976); Heckman (1976). E.g. Rosen: $\eta = 0.5$.

An Optimally Planned Economy

- Knowledge acquisition – search – in this economy has an external effect

- Decentralized allocation will not be efficient

- Can we work out the economically efficient time allocation? Implement it with taxes/subsidies?

An Optimally Planned Economy

- Formulate planning problem:

$$W[f(z,t)] = \max_{s(\cdot,\cdot)} \int_t^\infty e^{-\rho(\tau-t)} \int_0^\infty [1-s(z,\tau)] \, z f(z,\tau) dz d\tau$$

subject to the law of motion for f :

$$\frac{\partial f(z,\tau)}{\partial \tau} = -\alpha(s(z,\tau))f(z,\tau) \int_z^\infty f(y,\tau)dy + f(z,\tau) \int_0^z \alpha(s(y,\tau))f(y,\tau)dy.$$

and with $f(z,t)$ given.

- Here W maps a set $\mathbb{S}$ of density functions into $\mathbb{R}$

Simplification

- Can reduce dimensionality of problem.

- Define functional derivative of W,

$$\tilde{w}(z, f) = \frac{\delta W(f)}{\delta f(z)}$$

  on $\mathbb{R} \times \mathbb{S}$

- Now define

$$w(z, t) = \tilde{w}(z, f(z, t))$$

  on $\mathbb{R}^2$: derivative valued along an optimal trajectory

- Idea familiar: If planner has an optimal policy, it can't be improved by telling any single individual to deviate

Simplification

- Now write a Bellman equation for the marginal value $w(z, t)$

  to the planner of a single type z agent at date t

$$\rho w(z, t) - \frac{\partial w(z, t)}{\partial t} = \max_{s \in [0,1]} (1 - s) z + \alpha(s) \int_z^\infty [w(y, t) - w(z, t)] f(y, t) dy$$
$$- \int_0^z \alpha(s(y, t)) [w(y, t) - w(z, t)] f(y, t) dy$$

- Have reduced planner's problem from infinite-dimensional to

  two dimensional problem

- Last term in maximand is new: expected value from

  improvements in the productivity of other types $y < z$ to z in

  case they should meet z.

Simplification

- Planner values this external benefit; agent z himself does not

- First order condition is

$$z = \alpha'(s(z, t)) \int_z^\infty [w(y, t) - w(z, t)] f(y, t) dy$$

- Why not third term? Because changing $s(z, t)$ has no *direct* effect on the distribution at $y < z$ which only depends on the search intensities $s(y, t)$ of those with costs $y < z$

Balanced Growth Path

- As in decentralized problem, can restate the equations in

  terms of relative productivities

$$f(z, t) = e^{-\gamma t} \phi(z e^{-\gamma t}), \quad w(z, t) = e^{\gamma t} \omega(z e^{-\gamma t})$$

- Letting $x = z e^{-\gamma t}$ we obtain a BGP Bellman equation

$$(\rho - \gamma) \omega(x) + \omega'(x) \gamma x = \max_{\sigma \in [0,1]} \left\{ (1 - \sigma) x + \alpha(\sigma) \int_x^\infty [\omega(y) - \omega(x)] \phi(y) dy \right\}$$
$$- \int_0^x \alpha(\sigma(y)) [\omega(y) - \omega(x)] \phi(y) dy$$

- Law of motion for density ϕ and equation for γ same as in

  decentralized equilibrium

Balanced Growth Path

- Algorithmic strategy same as in decentralized case, except that (BE) is different

- Next slides compare features of BGP of optimal allocation to the BGP of the decentralized equilibrium studied earlier
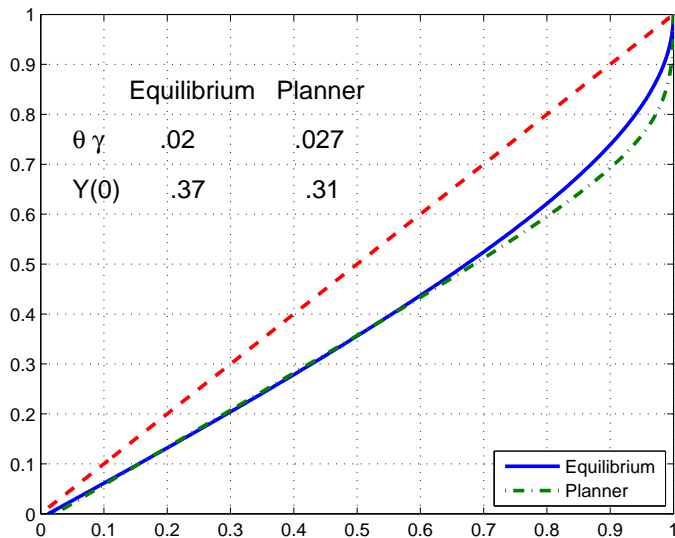
Equilibrium and Optimal Time Allocations
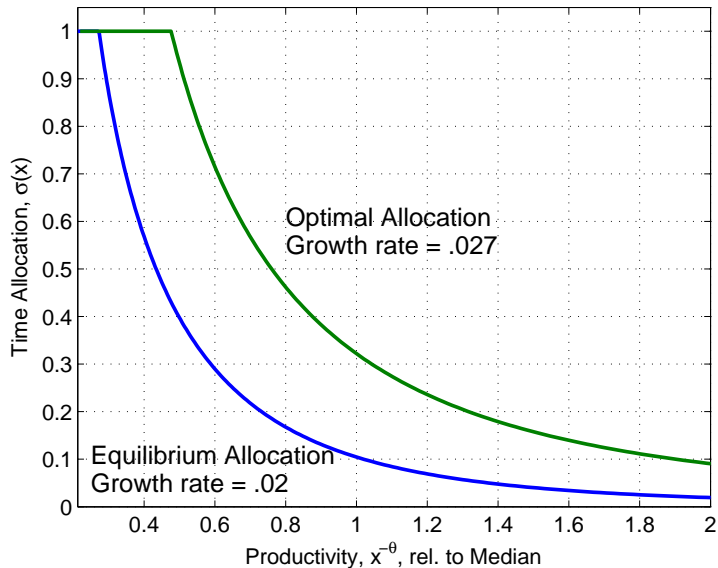
Income Lorenz Curves and Growth Rates

Value Lorenz Curves and Growth Rates

|	Equilibrium	Planner
$\theta\,\gamma$	.02	.027
$Y(0)$	.37	.31

Legend: Equilibrium, Planner
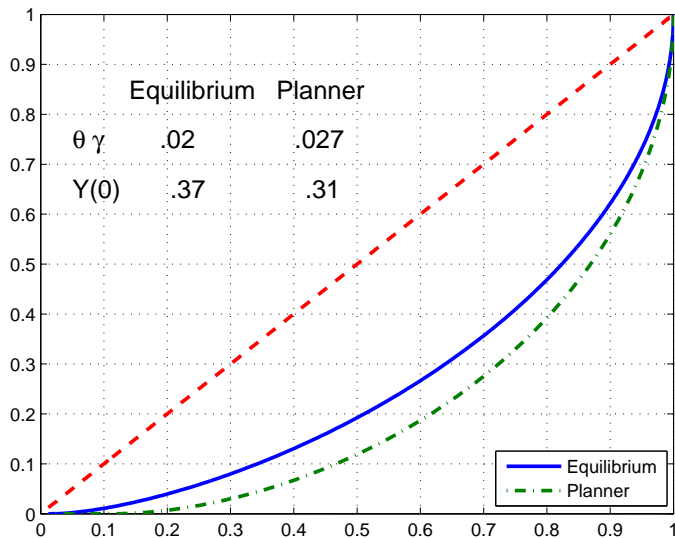
Pigovian Implementation of the Optimal Allocation

- Work out a tax structure that implements the optimal policy

  by setting taxes that equate, on the margin, the private and

  social returns from work and search.

- Simple one is direct subsidy to search, yielding individual (BE)

$$(\rho - \gamma)\, v_n(x) + v_n'(x)\gamma x$$
$$= \max_{\sigma \in [0,1]} \left\{ (1 - \sigma)x + \tau(x)x\sigma + \alpha(\sigma) \int_x^\infty [v_n(y) - v_n(x)]\phi(y)\, dy \right\}$$

- Includes flat rate tax to balance budget: multiplies both sides by a

  constant.

Pigovian Implementation of the Optimal Allocation
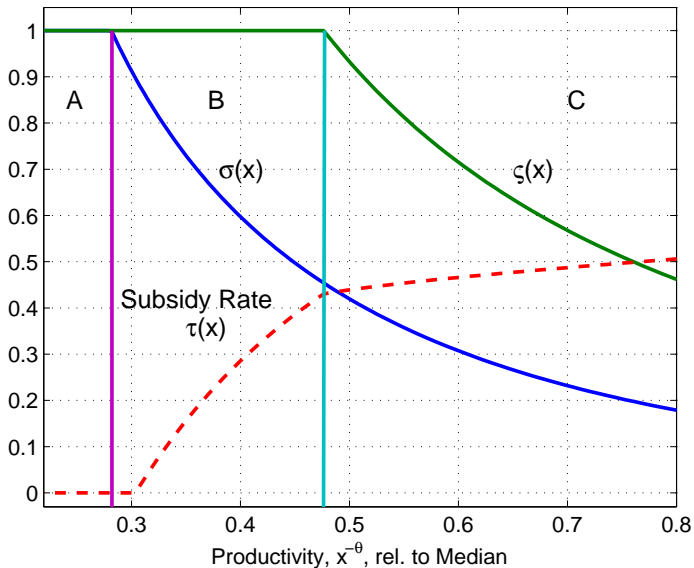
Alternative Learning Technologies

- Learning technology described above involves

  - probabilistic model of agents' meetings

  - description of effects of meetings on agents' knowledge

- Easy to think of modifications, other Boltzman equations

- Algorithm very flexible. Beginning to explore possible

  variations

Limits to Learning

- Impose an order to learning, limits on intellectual range

    - If y meets $z > y$ at t, he can adopt z with given probability
      $k(y, z, t)$

    - w/prob. $1 - k(y, z, t)$ he cannot do this; retains cost y

- Natural assumption: k unchanged if z and y at same
  quantiles. Along BGP this requires

$$k(y, z, t) = k(ye^{-\gamma t}, ze^{-\gamma t}, 0).$$

- Convenient to work with

$$k(y, z, 0) = e^{-\kappa|y-z|}$$

- Alternative interpretation: *meeting* probabilities depend on
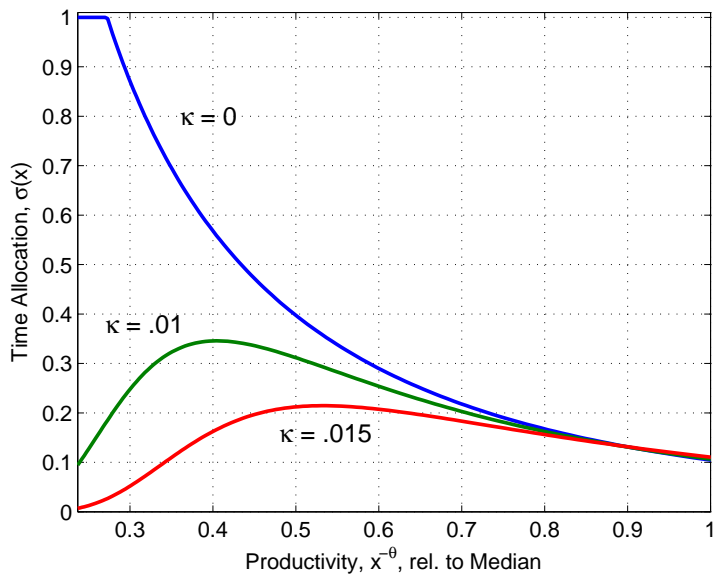 $|y - z|$; socioeconomic stratification or segregation.

Balanced Growth Path

- BGP equations become

$$(\rho - \gamma) \, v(x) + v'(x)\gamma x = \max_{\sigma \in [0,1]} \left\{ (1 - \sigma)x + \alpha(\sigma) \int_x^\infty [v(y) - v(x)]\phi(y)e^{-\kappa(y-x)}dy \right\}$$
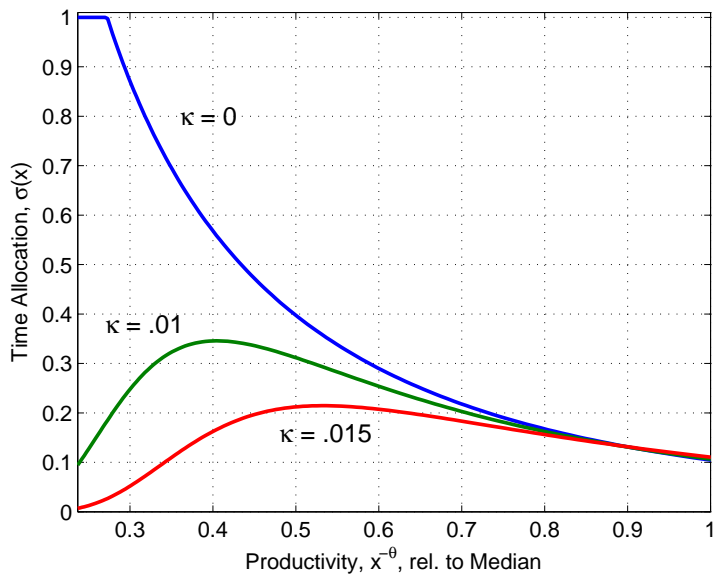
$$\phi(x)\gamma + \phi'(x)\gamma x = \phi(x) \int_0^x \alpha(\sigma(y))\phi(y)e^{-\kappa(x-y)}dy$$

$$- \alpha(\sigma(x))\phi(x) \int_x^\infty \phi(y)e^{-\kappa(y-x)}dy$$

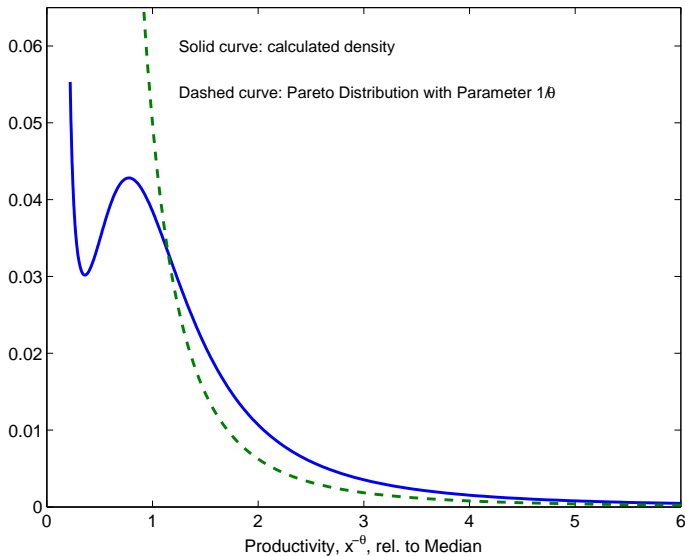$$\gamma = \int_0^\infty \alpha(\sigma(y))\phi(y)e^{-\kappa y}dy$$

Optimal Time Allocation for Various κ Values

Optimal Time Allocation for Various κ Values

Productivity Distribution with Limits to Learning

Solid curve: calculated density

Dashed curve: Pareto Distribution with Parameter $1/\theta$

Productivity, $x^{-\theta}$, rel. to Median

Limits to Learning Reduce Equality and Growth
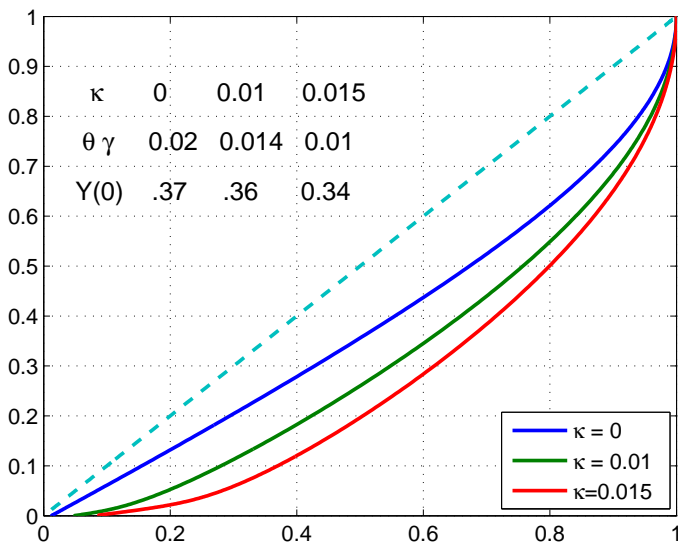
Evidence on κ?

- Prediction 1: high $\kappa \Rightarrow$ slow on-the-job human capital acc.

- Evidence: Experience-earnings profiles flatter in poorer countries (Lagakos et al., 2012).

- Prediction 2: high $\kappa \Rightarrow$ uneven knowledge diff. within country.

- Evidence: diffusion particularly uneven in developing countries with lowest penetration in rural areas (World Bank, 2008).

- Prediction 3: high $\kappa \Rightarrow$ low social mobility.

- Evidence on (i) intra- and (ii) inter-generational mobility:
    - (i) workers' lifetime wage growth relative to initial wages (e.g. Hause, 1980).
    - (ii) intergenerational correlation in income (e.g. Solon, 1992).

Symmetric Meetings

- Assumption of baseline model: can only upgrade knowledge through *active* search.

- In contrast, Arrow (1969): "diffusion of an innovation [is] formally akin to the spread of an infectious disease."

- $\Rightarrow$ New assumption: even if z doesn't search, can meet y with prob. $\beta\alpha(s(y, t))$.

- $\beta = 0$: baseline model; $\beta = 1$: perfectly symmetric meetings.

Symmetric Meetings: Equilibrium

$$\frac{\partial f(z,t)}{\partial t} = - f(z,t) \int_z^\infty [\alpha(s(z,t)) + \beta\alpha(s(y,t))]f(y,t)dy$$

$$+ f(z,t) \int_0^z [\alpha(s(y,t)) + \beta\alpha(s(z,t))]f(y,t)dy.$$

$$\rho V(z,t) = \max_{s\in[0,1]} (1-s)z + \frac{\partial V(z,t)}{\partial t}$$

$$+ \int_z^\infty [\alpha(s) + \beta\alpha(s(y,t))][V(y,t) - V(z,t)]f(y,t)dy$$
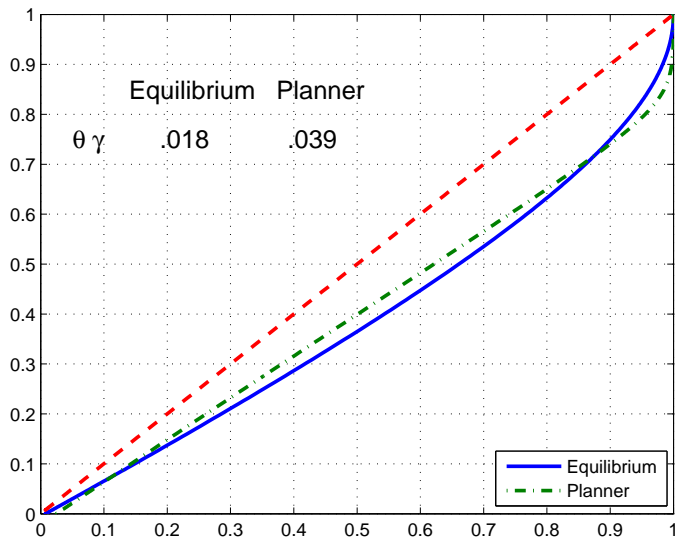
Optimal Time Allocation for Various β Values
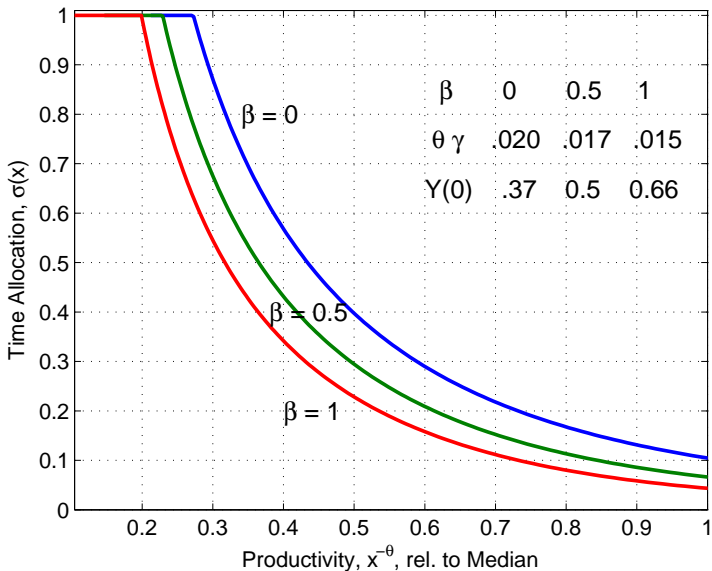
Optimal Time Allocation with Symmetric Meetings
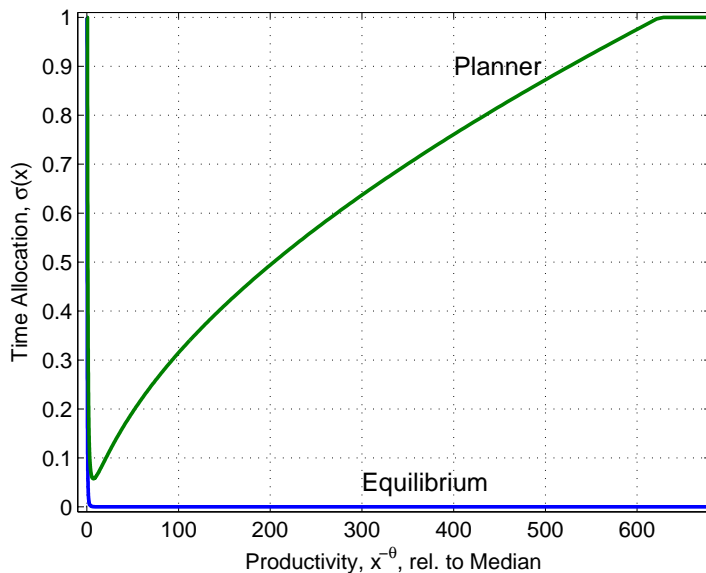
Value Lorenz Curves, Symmetric Meetings

Conclusion

- New model of endogenous growth due to diffusion of ideas.
- Distinctive feature: simultaneous determination of
  (1) individual behavior
  (2) agents' learning environment
- $\Rightarrow$ simultaneous equations problem
  (1) Bellman equation
  (2) law of motion for distribution ("Boltzmann equation")
- BGP: all productivity quantiles grow at constant rate. Stable Lorenz curve.
- Have developed tools to compute both decentralized equilibrium and planned economy: much more general.
- Example: "limits to learning" $\uparrow \Rightarrow$ individual search effort $\downarrow$ (esp. low prod. types), growth $\downarrow$, inequality $\uparrow$, mobility $\downarrow$.

Exogenous Knowledge Shocks (▸ Return)

- Same model as before, but now two sources of ideas

    - **Internal** source: ideas arrive from F at Poisson arrival rate α

    - **External** source: ideas arrive from G at Poisson arrival rate β

- Show: suppose G is bounded above. As long as G is fat-tailed, there is a BGP with positive growth even for arbitrarily small β.

Evolution of Productivity Distribution

- As before:

$$\frac{\partial f(z,t)}{\partial t} = \left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{out}} + \left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{in}}$$

- Outflow:

$$\left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{out}} = -\alpha(1 - F(z,t))f(z,t) - \beta(1 - G(z))f(z,t)$$

- Inflow:

$$\left.\frac{\partial f(z,t)}{\partial t}\right|_{\text{in}} = \alpha F(z,t)f(z,t) + \beta F(z,t)g(z)$$

- Or in terms of $F(z,t)$

$$\frac{\partial F(z,t)}{\partial t} = -\alpha[1 - F(z,t)]F(z,t) - \beta[1 - G(z)]F(z,t)$$

Special Case: Purely Internal Source, $\alpha > 0, \beta = 0$

- **Result 1:** a BPG with $\gamma > 0$ exists iff initial distribution $F(z, 0)$ is unbounded and **fat tailed.**

- Intuition: suppose upper bound $\bar{z}$. Then everyone piles up at $\bar{z}$ over time.

- Growth rate is ($1/\theta$ is tail parameter of $F(z, 0)$)

$$\gamma = \alpha\theta$$

- Interpretation: all knowledge already exists at $t = 0$. Reasonable? Philosophical/metaphysical question.

Evolution of $F(z, t)$ if $F(z, 0)$ is bounded above
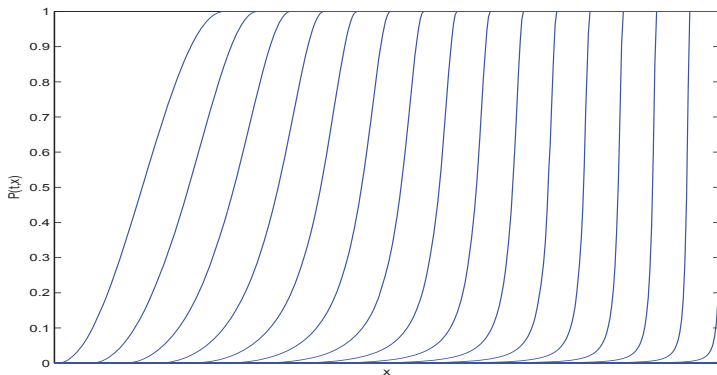
Special Case: Purely External Source, $\alpha = 0, \beta > 0$

- **Result 2:** a BGP with $\gamma > 0$ does not exist for any $G(z)$ (even if fat-tailed)

- Intuition: over time, external source gets worse relative to $F(z, t)$, run out of steam

- **Result 3:** can obtain BGP with $\gamma > 0$ if β grows at rate n: $\beta(t) = \beta_0 e^{nt}$. In this case, $\gamma \propto n$.

- This is a semi-endogenous growth model! (cf. Kortum, 1997)

- Growth built in exogenously through $\beta = \beta_0 e^{nt}$.

Both Internal and External Source, $\alpha > 0, \beta > 0$

- With internal source only: initial distribution needs to be unbounded and fat tailed. All knowledge already exists at $t = 0$.

- With external source only: no BGP, unless build it in with growing β.

- But **combination of internal and external source** avoids both problems!

- **Result 4:** a BPG with $\gamma > 0$ exists even if initial distribution $F(z, 0)$ is bounded, as long as external source distribution $G(z)$ is unbounded and **fat tailed.**

- Growth rate is ($1/\xi$ is tail parameter of $G(z)$)

$$\gamma = \alpha\xi$$

Both Internal and External Source: Implications

- There is no condition on the frequency at which innovations arrive, β, except that it is positive. That is, "innovations" can be very rare without impairing long term growth.

- This is true even if the initial knowledge distribution $F(z, 0)$ is bounded above.

- The frequency at which innovations arrive, β, does not affect the growth rate, γ.

- If there were no diffusion, $\alpha = 0$, there would be no growth!

- In this sense, diffusion rather than innovation is the engine for growth in this economy.

Proof of Results 1,2,4

- Recall PDE for $F(z,t)$:

$$\frac{\partial F(z,t)}{\partial t} = -\alpha[1-F(z,t)]F(z,t) - \beta[1-G(z)]F(z,t) \quad (*)$$

Lemma: The solution to $(*)$ is

$$\frac{1}{F(z,t)} = e^{(\alpha+\beta(1-G(z)))t}\left(\frac{1}{F(z,0)} - \frac{\alpha}{\alpha+\beta(1-G(z))}\right) \\ + \frac{\alpha}{\alpha+\beta(1-G(z))}. \quad (1)$$

Proof: See Appendix C in Lucas and Moll (2013). Idea: Let

$w(t) = F(z,t)$ and $u = 1 - G(z)$ and treat PDE as ODE for each fixed z.

$$\frac{\partial w(t)}{\partial t} = -\alpha w(t)[1-w(t)] - \beta u w(t)$$

Proof of Results 1,2,4

Theorem: Suppose that for some $\eta, \xi \geq 0$ the cdfs $F(z, 0)$ and $G(z)$ satisfy

$$\lim_{z \to \infty} \frac{1 - F(z, 0)}{z^{-1/\eta}} = k \geq 0 \tag{2}$$

and

$$\lim_{z \to \infty} \frac{1 - G(z)}{z^{-1/\xi}} = m \geq 0. \tag{3}$$

Let $\theta = \max(\eta, \xi)$ Then

$$\lim_{t \to \infty} F(e^{\alpha \theta t} z, t) = \frac{1}{1 + \lambda z^{-1/\theta}}. \tag{4}$$

where

$$
\begin{aligned}
\lambda &= 0 && \text{if} \quad k = m = 0 \\
&= k && \text{if} \quad \theta = \eta > \xi \\
&= m && \text{if} \quad \theta = \xi > \eta \\
&= k + (\beta/\alpha) m && \text{if} \quad \theta = \eta = \xi.
\end{aligned}
$$

Proof of Theorem

Proof: In view of (1), (4) is equivalent to

$$\lim_{t\to\infty} [e^{(\alpha+\beta(1-G(e^{\alpha\theta t}z)))t} \left(\frac{1}{F(e^{\alpha\theta t}z,0)} - \frac{\alpha}{\alpha+\beta(1-G(e^{\alpha\theta t}z))} \right)$$
$$+\frac{\alpha}{\alpha+\beta(1-G(e^{\alpha\theta t}z))}] = 1 + \lambda z^{-1/\theta}$$

which since $\lim_{t\to\infty} G(e^{\alpha\theta t}z) = 1$ gives

$$\lim_{t\to\infty} e^{\alpha t} \left(\frac{1}{F(e^{\alpha\theta t}z,0)} - \frac{\alpha}{\alpha+\beta(1-G(e^{\alpha\theta t}z))} \right) = \lambda z^{-1/\theta}. \quad (5)$$

The change of variable $x = e^{\alpha\theta t}z$ so that $z^{-1/\theta} = e^{\alpha t}x^{-1/\theta}$ gives

$$\lim_{x\to\infty} \left(\frac{1}{F(x,0)} - \frac{\alpha}{\alpha+\beta(1-G(x))} \right) x^{1/\theta} = \lambda$$

(proof continued on next slide)

Proof of Theorem (Continued)

Therefore, using (2), (3) and the facts that $F(x, 0) \to 1$ and $G(x) \to 1$ as $x \to \infty$,

$$\lim_{x \to \infty} \frac{1 - F(x, 0)}{x^{-1/\theta}} + (\beta/\alpha) \lim_{x \to \infty} \frac{1 - G(x)}{x^{-1/\theta}} = \lambda.$$

$$\lim_{x \to \infty} \frac{1 - F(x, 0)}{x^{-1/\eta}} \frac{x^{-1/\eta}}{x^{-1/\theta}} + (\beta/\alpha) \lim_{x \to \infty} \frac{1 - G(x)}{x^{-1/\xi}} \frac{x^{-1/\xi}}{x^{-1/\theta}} = \lambda.$$

$$k \lim_{x \to \infty} \frac{x^{-1/\eta}}{x^{-1/\theta}} + (\beta/\alpha) \, m \lim_{x \to \infty} \frac{x^{-1/\xi}}{x^{-1/\theta}} = \lambda. \square$$

Alternative Model of Innovation: Luttmer (2012)

- Suppose no external source, $\beta = 0$.
- But some randomness in z even if not meetings
  (small Brownian shocks with variance ν^2)
- Note: z can either go up or down ("experimentation")
- Can show, law of motion of distribution is:

$$\frac{\partial F(z,t)}{\partial t} = -\alpha F(z,t)(1-F(z,t)) + \frac{\nu^2}{2}\left(z\frac{\partial F(z,t)}{\partial z} + z^2\frac{\partial^2 F(z,t)}{\partial z^2}\right)$$

- **Result 5** If $F(z,0)$ is bounded, economy converges to a BGP
  with growth rate

$$\gamma = \nu\sqrt{2\alpha}$$

- My favorite formula of the year!
- No growth if either no experimentation ($\nu = 0$) or no diffusion
  ($\alpha = 0$). The interplay of the two matters.

Scale Effects?

- Does this model of endogenous growth feature the usual scale effects?

- Growth rate

$$\gamma = \alpha\theta$$

- θ = tail parameter, α = arrival rate.

- Whether there are scale effects depends on how α varies with L.

Scale Effects?

- Two possible assumptions:

(1) α prop. to $L \Rightarrow$ scale effects (in growth rates).

(2) α prop. to no. of industries/locations within a country;

    double pop. $=$ double people within each industry/location $\Rightarrow$

    no scale effects, neither in growth rates nor in levels.

  - Which assumption is more relevant? Not clear.

  - But point: scale effects **can** be avoided.

  - Another difference rel. to semi-endog. growth: policy can

    potentially affect α and hence growth rates.

Calibration/Parameterization?

- To identify parameters θ and α of model, need

    - observations on aggregate growth rate $\alpha\theta$, and

    - observations on variance of earnings (related to θ) or firm size distribution (Zipf's law) etc

- See Lucas, *Economica*, 2009.

Alvarez, Fernando E., Francisco J. Buera, and Robert E. Jr. Lucas. 2008. "Models of Idea Flows." National Bureau of Economic Research, Inc NBER Working Papers 14135.

Ben-Porath, Yoram. 1967. "The Production of Human Capital and the Life Cycle of Earnings." *Journal of Political Economy*, 75: 352.

Eaton, Jonathan, and Samuel Kortum. 1999. "International Technology Diffusion: Theory and Measurement." *International Economic Review*, 40(3): 537–70.

Eaton, Jonathan, and Samuel Kortum. 2002. "Technology, Geography, and Trade." *Econometrica*, 70(5): 1741–1779.

Gabaix, Xavier. 2009. "Power Laws in Economics and Finance." *Annual Review of Economics*, 1(1): 255–293.

Hause, John C. 1980. "The Fine Structure of Earnings and the On-the-Job Training Hypothesis." *Econometrica*, 48(4): 1013–29.

Heckman, James J. 1976. "A Life-Cycle Model of Earnings, Learning, and Consumption." *Journal of Political Economy*, 84(4): S11–44.

Kortum, Samuel S. 1997. "Research, Patenting, and Technological Change." *Econometrica*, 65(6): 1389–1420.

Lagakos, David, Benjamin Moll, Tommaso Porzio, and Nancy Qian. 2012. "Experience Matters: Human Capital and Development Accounting." Princeton mimeo.

Lasry, Jean-Michel, and Pierre-Louis Lions. 2007. "Mean field games." *Japanese Journal of Mathematics*, 2: 229–260.

Lucas, Robert E. 2009. "Ideas and Growth." *Economica*, 76(301): 1–19.

Luttmer, Erzo G.J. forthcoming. "Technology Diffusion and Growth." *Journal of Economic Theory*.

Perla, Jesse, and Christopher Tonetti. 2012. "Equilibrium Imitation and Growth." NYU mimeo.