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The Fractionally Cointegrated Vector Autoregression Model in R

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Abstract

This article illustrates how to estimate the fractionally cointegrated vector autoregression model in R.

Keywords: cofractional process, cointegration rank, fractional autoregressive model, fractional cointegration, fractional unit root, VAR model, Matlab, R.

1. Introduction: Cointegration and fractional integration in R

The fractionally cointegrated vector autoregression model is an excellent model...

In R (R Core Team 2017), contegration is performed by the `function1()` and `function2()` in the **pscl** package (Jackman 2015).

This R packages is based on the Matlab package **FCVARmodel.m** with documentation in Nielsen and Popiel (2016) and Nielsen and Morin (2014).

The next section describes the FCVAR model and the restricted models that can be estimated with this program. Section 3 describes the functioning of the main program, which is a replication of one of the tables of results in Jones, Nielsen, and Popiel (2014). Section 4 describes another example program, which demonstrates some additional functionality of the software. Importantly, these are the only two files that would need to be changed to apply the program for other empirical analyses.

2. The fractionally cointegrated VAR model

The fractionally cointegrated vector autoregressive (FCVAR) model was proposed in Johansen (2008) and analyzed by, e.g., Johansen and Nielsen (2010, 2012). For a time series X_t of dimension p, the fractionally cointegrated VAR model is given in error correction form as

$$\Delta^d X_t = \alpha\beta'\Delta^{d-b}L_b X_t + \sum_{i=1}^{k}\Gamma_i\Delta^d\ L_b^i X_t + \varepsilon_t, \tag{1}$$

where ε_t is p-dimensional $i.i.d.(0,\Omega)$, Δ^d is the fractional difference operator, and $L_b = 1-\Delta^b$ is the fractional lag operator.[1] Johansen and Nielsen (2012) imposed two restrictions on the parameter space, $d \geq b$ and $d - b < 1/2$, in their asymptotic analysis. However, these restrictions were relaxed in Johansen and Nielsen (2018a,b).

Model (1) includes the Johansen (1995) CVAR model as the special case $d = b = 1$; see Johansen and Nielsen (2018b). Some of the parameters are well-known from the CVAR model and these have the usual interpretations also in the FCVAR model. The most important of these are the long-run parameters α and β, which are $p \times r$ matrices with $0 \leq r \leq p$. The rank r is termed the cointegration, or cofractional, rank. The columns of β constitute the r cointegration (cofractional) vectors such that $\beta' X_t$ are the cointegrating combinations of the variables in the system, i.e. the long-run equilibrium relations. The parameters in α are the adjustment or loading coefficients which represent the speed of adjustment towards equilibrium for each of the variables. The short-run dynamics of the variables are governed by the parameters $\Gamma = (\Gamma_1, \ldots, \Gamma_k)$ in the autoregressive augmentation.

The FCVAR model has two additional parameters compared with the CVAR model, namely the fractional parameters d and b. Here, d denotes the fractional integration order of the observable time series and b determines the degree of fractional cointegration, i.e. the reduction in fractional integration order of $\beta' X_t$ compared to X_t itself. These parameters are estimated jointly with the remaining parameters. This model thus has the same main structure as in the standard CVAR model in that it allows for modeling of both cointegration and adjustment towards equilibrium, but is more general since it accommodates fractional integration and cointegration.

In the next four subsections we briefly describe the accommodation of deterministic terms as well as estimation and testing in the FCVAR model.

2.1. Deterministic terms

There are several ways to accommodate deterministic terms in the FCVAR model (1). The inclusion of the so-called restricted constant was considered in Johansen and Nielsen (2012), and the so-called unrestricted constant term was considered in Dolatabadi, Nielsen, and Xu

[1] Both the fractional difference and fractional lag operators are defined in terms of their binomial expansion in the lag operator, L. Note that the expansion of L_b has no term in L^0 and thus only lagged disequilibrium errors appear in (1).

(2016). A general formulation that encompasses both models is[2]

$$\Delta^d X_t = \alpha \Delta^{d-b} L_b(\beta' X_t + \rho') + \sum_{i=1}^{k} \Gamma_i \Delta^d \; L_b^i X_t + \xi + \varepsilon_t. \tag{2}$$

The parameter ρ is the so-called restricted constant term (since the constant term in the model is restricted to be of the form $\alpha\rho'$), which is interpreted as the mean level of the long-run equilibria when these are stationary, i.e. $E\beta' X_t + \rho' = 0$. The parameter ξ is the unrestricted constant term, which gives rise to a deterministic trend in the levels of the variables. When $d = 1$ this trend is linear. Thus, the model (2) contains both a restricted constant and an unrestricted constant. In the usual CVAR model, i.e. with $d = b = 1$, the former would be absorbed in the latter, but in the fractional model they can both be present and are interpreted differently. For the representation theory related to (2), and in particular for additional interpretation of the two types of constant terms, see Dolatabadi *et al.* (2016).

An alternative formulation of deterministic terms was suggested by Johansen and Nielsen (2016), albeit in a simpler model, with the aim of reducing the impact of pre-sample observations of the process. This model is

$$\Delta^d (X_t - \mu) = \alpha\beta' \Delta^{d-b} L_b (X_t - \mu) + \sum_{i=1}^{k} \Gamma_i \Delta^d \; L_b^i (X_t - \mu) + \varepsilon_t, \tag{3}$$

which can be derived easily from the unobserved components formulation

$$X_t = \mu + X_t^0, \quad \Delta^d X_t^0 = L_b \alpha\beta' X_t^0 + \sum_{i=1}^{k} \Gamma_i \Delta^d \; L_b^i X_t^0 + \varepsilon_t. \tag{4}$$

The formulation (3), or equivalently (4), includes the restricted constant, which may be obtained as $\rho' = \beta'\mu$. More generally, the level parameter μ is meant to accommodate a non-zero starting point for the first observation on the process, i.e., for X_1. It has the added advantage of reducing the bias arising due to pre-sample behavior of X_t, at least in simple models, even when conditioning on no initial values (see below). For details, see Johansen and Nielsen (2016).

2.2. Maximum likelihood estimation

It is assumed that a sample of length $T+N$ is available on X_t, where N denotes the number of observations used for conditioning, for details see Johansen and Nielsen (2016). The models (1), (2), and (3) are estimated by conditional maximum likelihood, conditional on N initial values, by maximizing the function

$$\log L_T(\lambda) = -\frac{Tp}{2}(\log(2\pi) + 1) - \frac{T}{2} \log \det \left\{ T^{-1} \sum_{t=N+1}^{T+N} \varepsilon_t(\lambda) \varepsilon_t(\lambda)' \right\}, \tag{5}$$

where the residuals are defined as

$$\varepsilon_t(\lambda) = \Delta^d X_t - \alpha \Delta^{d-b} L_b(\beta' X_t + \rho') - \sum_{i=1}^{k} \Gamma_i \Delta^d \; L_b^i X_t - \xi, \quad \lambda = (d, b, \alpha, \beta, \Gamma, \rho, \xi), \tag{6}$$

[2]In Dolatabadi *et al.* (2016) the constants are included as $\rho'\pi_t(1)$ and $\xi\pi_t(1)$, where $\pi_t(u)$ denotes coefficients in the binomial expansion of $(1-z)^{-u}$. This is mathematically convenient, but makes no difference in terms of the practical implementation.

for model (2), and hence also for submodels of model (2), such as (1), with the appropriate restrictions imposed on ρ and ξ. For model (3) the residuals are

$$\varepsilon_t(\lambda) = \Delta^d(X_t - \mu) - \alpha\beta'\Delta^{d-b}L_b(X_t - \mu) - \sum_{i=1}^{k}\Gamma_i\Delta^d\ L_b^i(X_t - \mu), \quad \lambda = (d, b, \alpha, \beta, \Gamma, \mu). \tag{7}$$

It is shown in Johansen and Nielsen (2012) and Dolatabadi *et al.* (2016) how, for fixed (d, b), the estimation of model (2) reduces to regression and reduced rank regression as in Johansen (1995). In this way the parameters $(\alpha, \beta, \Gamma, \rho, \xi)$ can be concentrated out of the likelihood function, and numerical optimization is only needed to optimize the profile likelihood function over the two fractional parameters, d and b. In model (3) we can similarly concentrate the parameters (α, β, Γ) out of the likelihood function resulting in numerical optimization over (d, b, μ), making the estimation of model (3) slightly more involved numerically than that of model (2).

For model (2) with $\xi = 0$, Johansen and Nielsen (2012) shows that asymptotic theory is standard when $b < 0.5$, and for the case $b > 0.5$ asymptotic theory is non-standard and involves fractional Brownian motion of type II. Specifically, when $b > 0.5$, Johansen and Nielsen (2012) shows that under i.i.d. errors with suitable moment conditions, the conditional maximum likelihood parameter estimates $(\hat{d}, \hat{b}, \hat{\alpha}, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_k)$ are asymptotically Gaussian, while $(\hat{\beta}, \hat{\rho})$ are locally asymptotically mixed normal. These results allow asymptotically standard (chi-squared) inference on all parameters of the model, including the cointegrating relations and orders of fractionality, using quasi-likelihood ratio tests. As in the CVAR model, see Johansen (1995), the same results hold for the same parameters in the full models (2) and (3), whereas the asymptotic distribution theory for the remaining parameters, ξ and μ, is currently unknown.

2.3. Cointegration rank tests

Letting $\Pi = \alpha\beta'$, the likelihood ratio (LR) test statistic of the hypothesis $\mathcal{H}_r : \text{rank}(\Pi) = r$ against $\mathcal{H}_p : \text{rank}(\Pi) = p$ is of particular interest because it deals with an important empirical question. This statistic is often denoted the "trace" statistic. Let $\theta = (d, b)$ for model (2) and $\theta = (d, b, \mu)$ for model (3) denote the parameters for which the likelihood is numerically maximized. Then let $L(\theta, r)$ be the profile likelihood function given rank r, where (α, β, Γ), and possibly (ρ, ξ) if appropriate, have been concentrated out by regression and reduced rank regression; see Johansen and Nielsen (2012) and Dolatabadi *et al.* (2016) for details.

The profile likelihood function is maximized both under the hypothesis $\mathcal{H}_r$ and under $\mathcal{H}_p$ and the LR test statistic is then $\text{LR}_T(q) = 2\log(L(\hat{\theta}_p, p)/L(\hat{\theta}_r, r))$, where

$$L(\hat{\theta}_p, p) = \max_{\theta} L(\theta, p), \quad L(\hat{\theta}_r, r) = \max_{\theta} L(\theta, r),$$

and $q = p - r$. This problem is qualitatively different from that in Johansen (1995) since the asymptotic distribution of $\text{LR}_T(q)$ depends qualitatively (and quantitatively) on the parameter b. In the case with $0 < b < 1/2$ (sometimes known as "weak cointegration"), $\text{LR}_T(q)$ has a standard asymptotic distribution, see Johansen and Nielsen (2012, Theorem 11(ii)), namely

$$\text{LR}_T(q) \xrightarrow{D} \chi^2(q^2), \quad 0 < b < 1/2. \tag{8}$$

On the other hand, when $1/2 < b \leq d$ ("strong cointegration"), asymptotic theory is non-standard and

$$\mathrm{LR}_T(q) \xrightarrow{D} \mathrm{Tr}\left\{\int_0^1 \mathsf{d}W(s)F(s)'\left(\int_0^1 F(s)F(s)'\mathsf{d}s\right)^{-1}\int_0^1 F(s)\mathsf{d}W(s)'\right\}, \quad b > 1/2, \tag{9}$$

where the vector process $\mathsf{d}W$ is the increment of ordinary (non-fractional) vector standard Brownian motion of dimension $q = p - r$. The vector process F depends on the deterministics in a similar way as in the CVAR model in Johansen (1995), although the fractional orders complicate matters. The following cases have been derived in the literature:

1. When no deterministic term is in the model, $F(u) = W_b(u)$, where $W_b(u) = \Gamma(b)^{-1}\int_0^u (u-s)^{b-1}\mathsf{d}W(s)$ is vector fractional Brownian motion of type II, see Johansen and Nielsen (2012, Theorem 11(i)).

2. When only the restricted constant term is included in model (2), $F(u) = (W_b(u)', u^{-(d-b)})'$, see Johansen and Nielsen (2012, Theorem 11(iv)) for the result with $d = b$ and an earlier working paper version for the general result.

3. In model (3) the same result as in bullet 2. holds because $\beta'\mu = \rho'$ is the restricted constant and $\beta_\perp'\mu$ has no influence on the asymptotic distribution (in a similar way to X_0 in a random walk).

4. When both the restricted and unrestricted constants are included in model (2) with $d = 1$,

$$\begin{aligned} F_i(u) &= W_{b,i}(u) - \int_0^1 W_{b,i}(u)\mathsf{d}u, \ i = 1, ..., q-1, \\ F_q(u) &= u^b - \int_0^1 u^b\mathsf{d}u = u^b - 1/(b+1), \\ F_{q+1}(u) &= u^{b-1} - \int_0^1 u^{b-1}\mathsf{d}u = u^{b-1} - 1/b, \end{aligned}$$

   see Dolatabadi *et al.* (2016).

Importantly, the asymptotic distribution (9) of the test statistic $\mathrm{LR}_T(q)$ depends on both b and $q = p - r$. The dependence on the unknown (true value of the) scalar parameter b complicates empirical analysis compared to the CVAR model. Generally, the distribution (9) would need to be simulated on a case-by-case basis. However, for model (1) and for model (2) with $d = b$ and $\xi = 0$, and hence also for model (3) with $d = b$ in light of bullet 3. above, computer programs for computing asymptotic critical values and asymptotic P values for the LR cointegration rank tests based on numerical distribution functions, are made available by MacKinnon and Nielsen (2014). Their computer programs are incorporated in the present program for the relevant cases/models as discussed and illustrated below.

2.4. Restricted models

Note that a reduced rank restriction has already been imposed on models (1)–(3), where the coefficient matrix $\Pi = \alpha\beta'$ has been restricted to rank $r \leq p$. Other restrictions on the model

parameters can be considered as in Johansen (1995). The most interesting restrictions from an economic theory point of view would likely be restrictions on the adjustment parameters α and cointegration vectors β.

We formulate hypotheses as

$$R_\psi \psi = r_\psi, \tag{10}$$
$$R_\alpha \text{vec}(\alpha) = 0, \tag{11}$$
$$R_\beta \text{vec}(\beta^*) = r_\beta, \tag{12}$$

with $\beta^* = (\beta', \rho')'$, and use the switching algorithm in (Boswijk and Doornik 2004, p. 455) to optimize the likelihood numerically subject to the restrictions. The switching algorithm can be improved by adding a line search, see Doornik (2018). This is done by setting the option `opt.LineSearch = 1`, which is the default setting.

The only limitation on the linear restrictions that can be imposed on (d, b, α, β^*) in (10)–(12) is that only homogenous restrictions can be imposed on vec(α) in (11). Otherwise, any combination of linear restrictions can be imposed on these parameters. For now, the remaining parameters cannot be restricted.

Note that, when the restricted constant term ρ is included in the model, restrictions on β and ρ must be written in the form given by (12). This is without loss of generality.

The restrictions in (10)–(12) above can be implemented individually or simultaneously in the Matlab program. The next section provides an example session illustrating the use of the program with a step-by-step description of a typical empirical analysis, including several restricted models in Section **??**.

2.5. Forecasting from the FCVAR model

Because the FCVAR model is autoregressive, the best linear predictor takes a simple form and is relatively straightforward to calculate. Consider, for example, the model with level parameter in (3). We first note that

$$\Delta^d(X_{t+1} - \mu) = X_{t+1} - \mu - (X_{t+1} - \mu) + \Delta^d(X_{t+1} - \mu) = X_{t+1} - \mu - L_d(X_{t+1} - \mu)$$

and then rearrange (3) as

$$X_{t+1} = \mu + L_d(X_{t+1} - \mu) + \alpha\beta'\Delta^{d-b}L_b(X_{t+1} - \mu) + \sum_{i=1}^{k} \Gamma_i \Delta^d L_b^i(X_{t+1} - \mu) + \varepsilon_{t+1}. \tag{13}$$

Since $L_b = 1 - \Delta^b$ is a lag operator, so that $L_b^i X_{t+1}$ is known at time t for $i \geq 1$, this equation can be used as the basis to calculate forecasts from the model.

We let conditional expectation given the information set at time t be denoted $E_t(\cdot)$, and the best linear predictor forecast of any variable Z_{t+1} given information available at time t be denoted $\hat{Z}_{t+1|t} = E_t(Z_{t+1})$. Clearly, we then have that the forecast of the innovation for period $t+1$ at time t is $\hat{\varepsilon}_{t+1|t} = E_t(\varepsilon_{t+1}) = 0$, and $\hat{X}_{t+1|t}$ is then easily found from (13). Inserting also coefficient estimates based on data available up to time t, denoted[3] $(\hat{d}, \hat{b}, \hat{\mu}, \hat{\alpha}, \hat{\beta}, \hat{\Gamma}_1, \ldots, \hat{\Gamma}_k)$,

[3]To emphasize that these estimates are based on data available at time t, they could be denoted by a subscript t. However, to avoid cluttering the notation we omit this subscript and let it be understood in the sequel.

we have that

$$\hat{X}_{t+1|t} = \hat{\mu} + L_{\hat{d}}(X_{t+1} - \hat{\mu}) + \hat{\alpha}\hat{\beta}'\Delta^{\hat{d}-\hat{b}}L_{\hat{b}}(X_{t+1} - \hat{\mu}) + \sum_{i=1}^{k} \hat{\Gamma}_i \Delta^{\hat{d}} L_{\hat{b}}^{i}(X_{t+1} - \hat{\mu}). \tag{14}$$

This defines the one-step ahead forecast of X_{t+1} given information at time t.

Multi-period ahead forecasts can be generated recursively. That is, to calculate the h-step ahead forecast, we first generalize (14) as

$$\hat{X}_{t+j|t} = \hat{\mu} + L_{\hat{d}}(\hat{X}_{t+j|t} - \hat{\mu}) + \hat{\alpha}\hat{\beta}'\Delta^{\hat{d}-\hat{b}}L_{\hat{b}}(\hat{X}_{t+j|t} - \hat{\mu}) + \sum_{i=1}^{k} \hat{\Gamma}_i \Delta^{\hat{d}} L_{\hat{b}}^{i}(\hat{X}_{t+j|t} - \hat{\mu}), \tag{15}$$

where $\hat{X}_{s|t} = X_s$ for $s \leq t$. Then forecasts are calculated recursively from (15) for $j = 1, 2, \ldots, h$ to generate h-step ahead forecasts, $\hat{X}_{t+h|t}$.

Clearly, one-step ahead and h-step ahead forecasts for the model (2) with a restricted constant term, and possibly also an unrestricted constant term, instead of the level parameter can be calculated entirely analogously.

3. Main Program

Estimating the fractionally cointegrated vector autoregression model works like this...

Here is an example of code:

```
glm(formula, data, subset, na.action, weights, offset,
  family = gaussian, start = NULL, control = glm.control(...),
  model = TRUE, y = TRUE, x = FALSE, ...)
```

4. Illustrations

For a simple illustration of the FCVAR... The data can be loaded by

```
R> data("quine", package = "MASS")
```

and a basic frequency distribution of the response variable is displayed in Figure 1.

As a first model for the `quine` data, we fit the basic Poisson regression model. (Note that JSS prefers when the second line of code is indented by two spaces.)

```
R> m_pois <- glm(Days ~ (Eth + Sex + Age + Lrn)^2, data = quine,
+    family = poisson)
```

Hence, the full summary of that model is shown below.

```
R> summary(m_nbin)
```

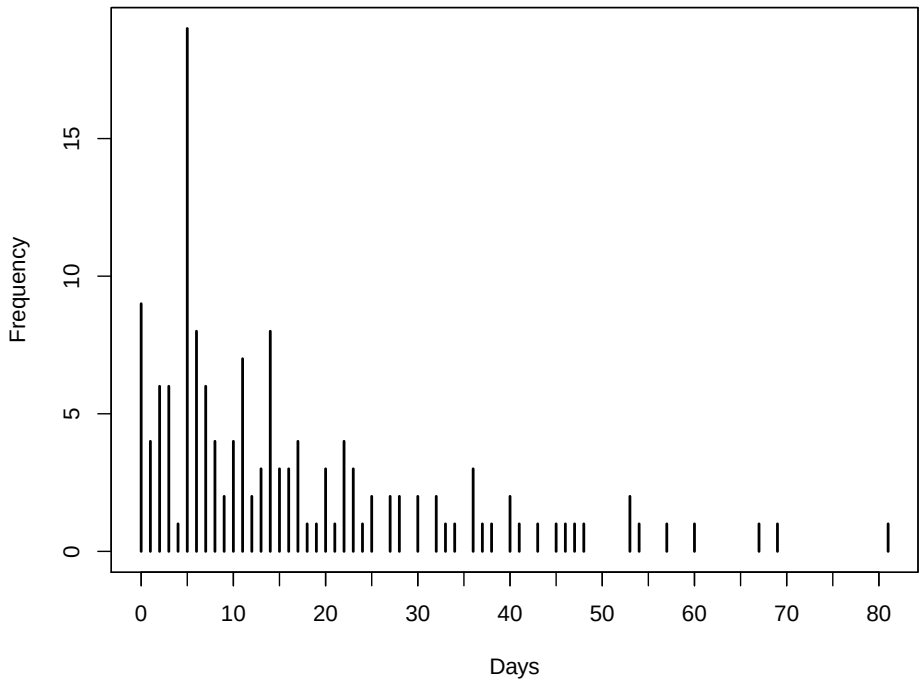

Figure 1: Frequency distribution for number of days absent from school.

```
Call:
glm.nb(formula = Days ~ (Eth + Sex + Age + Lrn)^2, data = quine,
    init.theta = 1.60364105, link = log)

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-3.0857  -0.8306  -0.2620   0.4282   2.0898

Coefficients: (1 not defined because of singularities)
            Estimate Std. Error z value Pr(>|z|)
(Intercept)  3.00155    0.33709   8.904  < 2e-16 ***
SexM        -0.77181    0.38021  -2.030  0.04236 *
EthN:AgeF2  -1.23283    0.42962  -2.870  0.00411 **
SexM:AgeF2   1.55330    0.51325   3.026  0.00247 **
SexM:AgeF3   1.25227    0.45539   2.750  0.00596 **
AgeF3:LrnSL       NA         NA      NA       NA
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for Negative Binomial(1.6036) family taken to be 1)

    Null deviance: 235.23  on 145  degrees of freedom
Residual deviance: 167.53  on 128  degrees of freedom
AIC: 1100.5
```

```
Number of Fisher Scoring iterations: 1


              Theta:  1.604
          Std. Err.:  0.214

 2 x log-likelihood:  -1062.546
```

5. Extensions

5.1. Extension for P-values

Although the Matlab program can run standalone, one of the functions, `RankTests.m`, makes an external system call to a separately installed program, `fdpval`. This external program is the C++ implementation of a Fortran program used to obtain simulated P-values from MacKinnon and Nielsen (2014). If the user would like P-values for the cointegration rank tests to be automatically calculated, we recommend obtaining this companion program, which is made available by Jason Rhinelander and can be downloaded from:

https://github.com/jagerman/fracdist/releases

It can be either installed or downloaded in a compressed folder. It is important to note where the program is stored or installed, because the Matlab program requires the program location as an input in the estimation options. For example, if the program is stored in the folder `/usr/bin/` on a Linux system, the location variable is defined as follows, `progLoc = '"/usr/bin/fdpval"'`. For details see Sections ?? and ??.

5.2. Badly behaved objective function

We also make use of the excellent `extrema.m` and `extrema2.m` functions, which are written by Carlos Adrián Vargas Aguilera and are freely available from the Mathworks website. For simplicity these are included in the Auxiliary subfolder.

6. Summary and discussion

Summary goes here.

Computational details

The results in this paper were obtained using R 3.4.1 with the **MASS** 7.3.47 package. R itself and all packages used are available from the Comprehensive R Archive Network (CRAN) at https://CRAN.R-project.org/.

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A. More technical details

Technical details go here.

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