

# The Identification of Dynamic Structural Shocks

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# Chapter 1

## Before starting

### WORK IN PROGRESS

The identification and estimation of dynamic responses to structural shocks is one of the principal goals of macroeconometrics. These responses correspond to the effect, over time, of an exogenous intervention that propagates through the economy, as modelled by a system of simultaneous equations.

Over the last decades, several methodologies have been proposed so as to estimate these responses. The objective of this course, developed by Kenza Benhima and Jean-Paul Renne, is to provide an exhaustive view of these methodologies and to provide students with tools enabling them to implement them in various contexts.

Codes associated with this course are part of the **AEC** package, which is available on GitHub. To install it, one need to employ the **devtools** library:

```
library(devtools)
install_github("jrenne/AEC")
library(AEC)
```

### Useful (R) links:

- Download R:
  - R software: <https://cran.r-project.org> (the basic R software)
  - RStudio: <https://www.rstudio.com> (a convenient R editor)
- Tutorials:
  - Rstudio: <https://dss.princeton.edu/training/RStudio101.pdf> (by Oscar Torres-Reyna)
  - R: [https://cran.r-project.org/doc/contrib/Paradis-rdebuts\\_en.pdf](https://cran.r-project.org/doc/contrib/Paradis-rdebuts_en.pdf) (by Emmanuel Paradis)
  - My own tutorial: [https://jrenne.shinyapps.io/Rtuto\\_publiShiny/](https://jrenne.shinyapps.io/Rtuto_publiShiny/)



# Chapter 2

## Time Series

### 2.1 Introduction to time series

A time series is an infinite sequence of random variables indexed by time:  $\{y_t\}_{t=-\infty}^{+\infty} = \{\dots, y_{-2}, y_{-1}, y_0, y_1, \dots, y_t, \dots\}$ ,  $y_i \in \mathbb{R}^k$ . In practice, we only observe samples, typically:  $\{y_1, \dots, y_T\}$ .

Standard time series models are built using **shocks** that we will often denote by  $\varepsilon_t$ . Typically,  $\mathbb{E}(\varepsilon_t) = 0$ . In many models, the shocks are supposed to be i.i.d., but there exist other (less restrictive) notions of shocks. In particular, the definition of many processes is based on white noises:

**Definition 2.1** (White noise). The process  $\{\varepsilon_t\}_{t \in [-\infty, +\infty[}$  is a white noise if, for all  $t$ :

- a.  $\mathbb{E}(\varepsilon_t) = 0$ ,
- b.  $\mathbb{E}(\varepsilon_t^2) = \sigma^2 < \infty$  and
- c. for all  $s \neq t$ ,  $\mathbb{E}(\varepsilon_t \varepsilon_s) = 0$ .

Another type of shocks that are commonly used are Martingale Difference Sequences:

**Definition 2.2** (Martingale Difference Sequence). The process  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  is a martingale difference sequence (MDS) if  $\mathbb{E}(|\varepsilon_t|) < \infty$  and if, for all  $t$ ,

$$\underbrace{\mathbb{E}_{t-1}(\varepsilon_t)}_{\text{Expectation conditional on the past}} = 0.$$

By definition, if  $y_t$  is a martingale, then  $y_t - y_{t-1}$  is a MDS.

**Example 2.1** (ARCH process). The Autoregressive conditional heteroskedasticity (ARCH) process is an example of shock that satisfies the MDS definition but that is not i.i.d.:

$$\varepsilon_t = \sigma_t \times z_t,$$

where  $z_t \sim i.i.d. \mathcal{N}(0, 1)$  and  $\sigma_t^2 = w + \alpha \varepsilon_{t-1}^2$ .

**Example 2.2.** A white noise process is not necessarily a MDS. This is for instance the following process:

$$\varepsilon_t = z_t + z_{t-1} z_{t-2},$$

where  $z_t \sim i.i.d. \mathcal{N}(0, 1)$ .

Let us now introduce the lag operator. The lag operator, denoted by  $L$ , is defined on the time series space and is defined by:

$$L : \{y_t\}_{t=-\infty}^{+\infty} \rightarrow \{w_t\}_{t=-\infty}^{+\infty} \quad \text{with} \quad w_t = y_{t-1}. \quad (2.1)$$

We have:  $L^2 y_t = y_{t-2}$  and, more generally,  $L^k y_t = y_{t-k}$ .

Consider a time series  $y_t$  defined by  $y_t = \mu + \phi y_{t-1} + \varepsilon_t$ , where the  $\varepsilon_t$ 's are i.i.d.  $\mathcal{N}(0, \sigma^2)$ . Using the lag operator, the dynamics of  $y_t$  can be expressed as follows:

$$(1 - \phi L)y_t = \mu + \varepsilon_t.$$

It is easily checked that we have  $L^2 y_t = y_{t-2}$  and, generally,  $L^k y_t = y_{t-k}$ .

If it exists, the **unconditional (or marginal) mean** of the random variable  $y_t$  is given by:

$$\mu_t := \mathbb{E}(y_t) = \int_{-\infty}^{\infty} y_t f_{Y_t}(y_t) dy_t,$$

where  $f_{Y_t}$  is the unconditional (or marginal) density of  $y_t$ . Similarly, if it exists, the **unconditional (or marginal) variance** of the random variable  $y_t$  is:

$$\text{Var}(y_t) = \int_{-\infty}^{\infty} (y_t - \mathbb{E}(y_t))^2 f_{Y_t}(y_t) dy_t.$$

**Definition 2.3** (Autocovariance). The  $j^{\text{th}}$  autocovariance of  $y_t$  is given by:

$$\begin{aligned} \gamma_{j,t} &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} [y_t - \mathbb{E}(y_t)][y_{t-j} - \mathbb{E}(y_{t-j})] \times \\ &\quad f_{Y_t, Y_{t-1}, \dots, Y_{t-j}}(y_t, y_{t-1}, \dots, y_{t-j}) dy_t dy_{t-1} \dots dy_{t-j} \\ &= \mathbb{E}([y_t - \mathbb{E}(y_t)][y_{t-j} - \mathbb{E}(y_{t-j})]), \end{aligned}$$

where  $f_{Y_t, Y_{t-1}, \dots, Y_{t-j}}(y_t, y_{t-1}, \dots, y_{t-j})$  is the joint distribution of  $y_t, y_{t-1}, \dots, y_{t-j}$ .

In particular,  $\gamma_{0,t} = \text{Var}(y_t)$ .

**Definition 2.4** (Covariance stationarity). The process  $y_t$  is covariance stationary—or weakly stationary—if, for all  $t$  and  $j$ ,

$$\mathbb{E}(y_t) = \mu \quad \text{and} \quad \mathbb{E}\{(y_t - \mu)(y_{t-j} - \mu)\} = \gamma_j.$$

Figure 2.1 displays the simulation of a process that is not covariance stationary. This process follows  $y_t = 0.1t + \varepsilon_t$ , where  $\varepsilon_t \sim i.i.d. \mathcal{N}(0, 1)$ . Indeed, for such a process, we have:  $\mathbb{E}(y_t) = 0.1t$ , which depends on  $t$ .

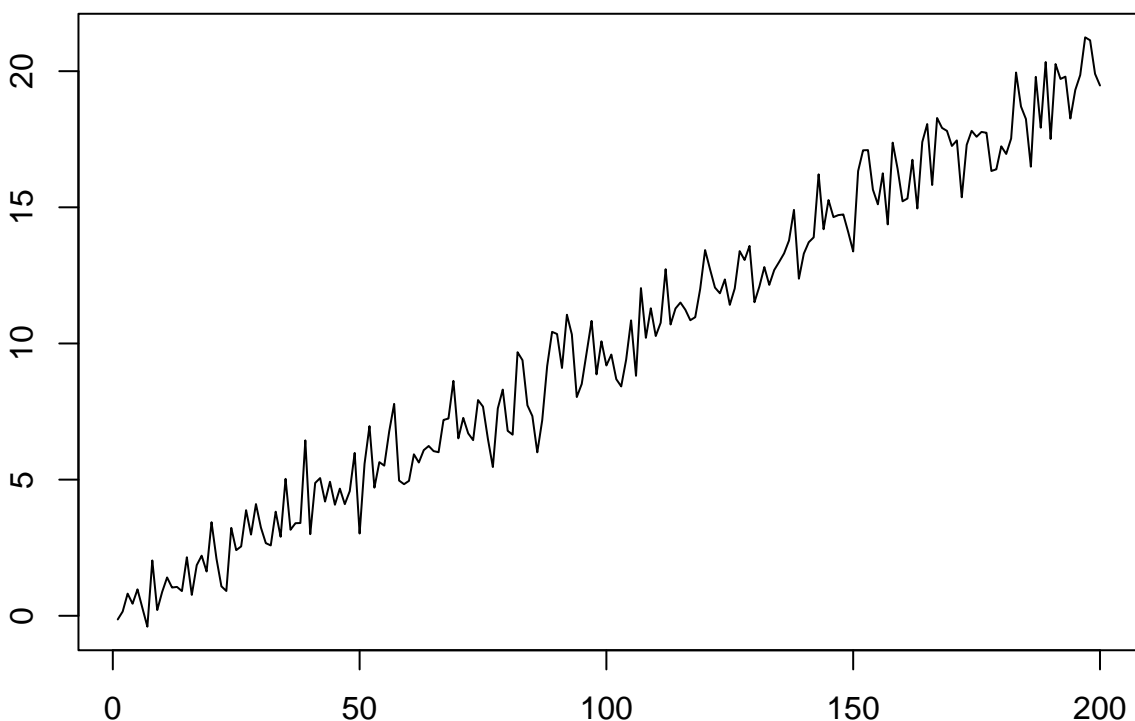


Figure 2.1: Example of a process that is not covariance stationary ( $y_t = 0.1t + \varepsilon_t$ , where  $\varepsilon_t \sim \mathcal{N}(0, 1)$ ).



**Definition 2.5** (Strict stationarity). The process  $y_t$  is strictly stationary if, for all  $t$  and all sets of integers  $J = \{j_1, \dots, j_n\}$ , the distribution of  $(y_t, y_{t+j_1}, \dots, y_{t+j_n})$  depends on  $J$  but not on  $t$ .

The following process is covariance stationary but not strictly stationary:

$$y_t = \mathbb{I}_{\{t < 1000\}} \varepsilon_{1,t} + \mathbb{I}_{\{t \geq 1000\}} \varepsilon_{2,t},$$

where  $\varepsilon_{1,t} \sim \mathcal{N}(0, 1)$  and  $\varepsilon_{2,t} \sim \sqrt{\frac{\nu-2}{\nu}} t(\nu)$  and  $\nu = 4$ .

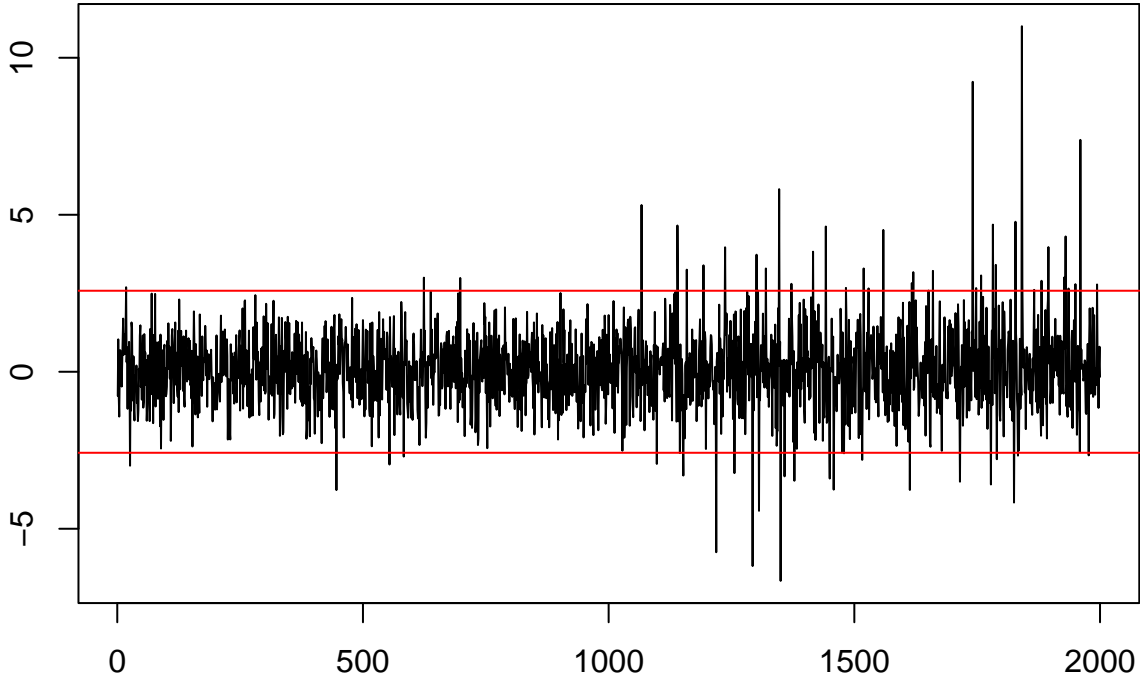


Figure 2.2: Example of a process that is covariance stationary but not strictly stationary. The red lines delineate the 99% confidence interval of the standard normal distribution ( $\pm 2.58$ ).

**Proposition 2.1.** If  $y_t$  is covariance stationary, then  $\gamma_j = \gamma_{-j}$ .

*Proof.* Since  $y_t$  is covariance stationary, the covariance between  $y_t$  and  $y_{t-j}$  (i.e.  $\gamma_j$ ) is the same as that between  $y_{t+j}$  and  $y_{t+j-j}$  (i.e.  $\gamma_{-j}$ ).  $\square$

**Definition 2.6** (Auto-correlation). The  $j^{\text{th}}$  auto-correlation of a covariance-stationary process is:

$$\rho_j = \frac{\gamma_j}{\gamma_0}.$$

Consider a long historical time series of the Swiss GDP growth, taken from the Jordà et al. (2017) dataset.<sup>1</sup>

**Definition 2.7** (Mean ergodicity). The covariance-stationary process  $y_t$  is ergodic for the mean if:

$$\text{plim}_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T y_t = \mathbb{E}(y_t).$$

**Definition 2.8** (Second-moment ergodicity). The covariance-stationary process  $y_t$  is ergodic for second moments if, for all  $j$ :

$$\text{plim}_{T \rightarrow +\infty} \frac{1}{T} \sum_{t=1}^T (y_t - \mu)(y_{t-j} - \mu) = \gamma_j.$$

It should be noted that ergodicity and stationarity are different properties. Typically if the process  $\{x_t\}$  is such that,  $\forall t, x_t \equiv y$ , where  $y \sim \mathcal{N}(0, 1)$  (say), then  $\{x_t\}$  is stationary but not ergodic.

<sup>1</sup>Version 6 of the dataset, available on this website.

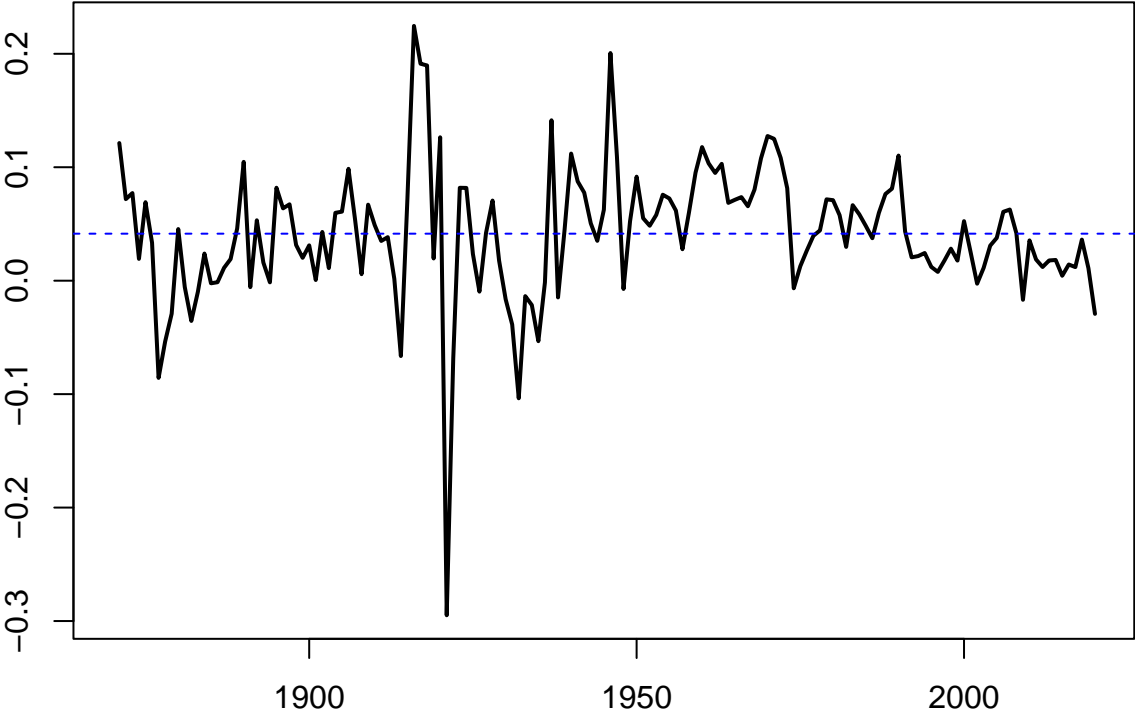


Figure 2.3: Annual growth rate of Swiss GDP, based on the Jorda-Schularick-Taylor Macro-history Database.

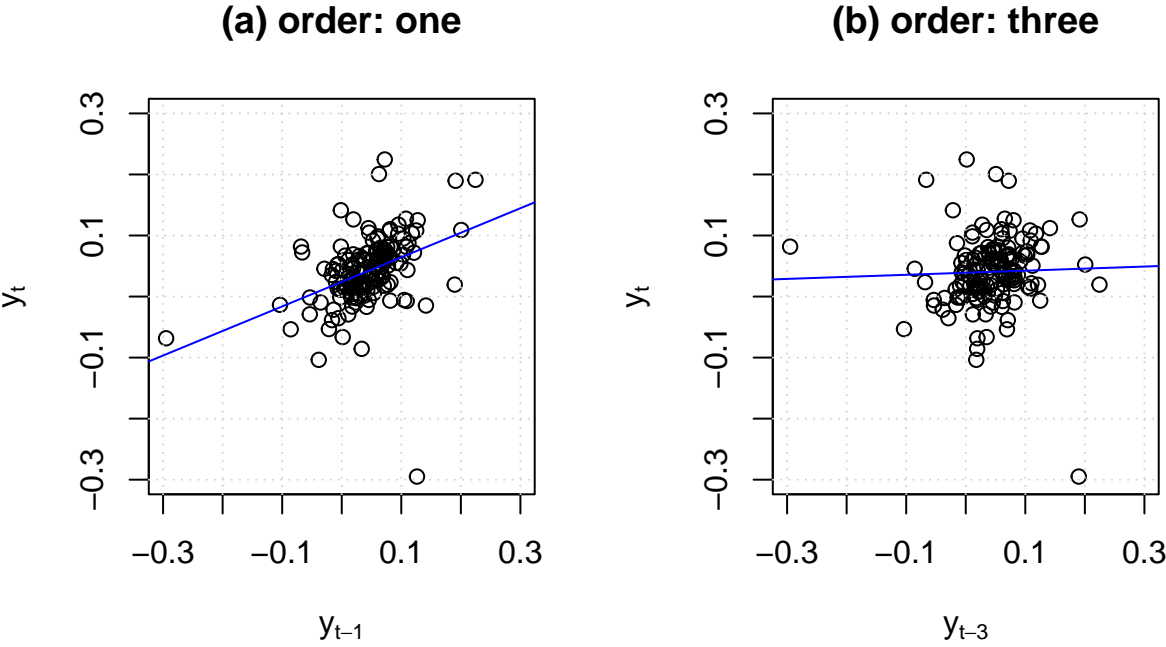


Figure 2.4: For order  $j$ , the slope of the blue line is, approximately,  $\hat{\gamma}_j/\widehat{\text{Var}}(y_t)$ , where hats indicate sample moments.

**Theorem 2.1** (Central Limit Theorem for covariance-stationary processes). *If process  $y_t$  is covariance stationary and if the series of autocovariances is absolutely summable ( $\sum_{j=-\infty}^{+\infty} |\gamma_j| < \infty$ ), then:*

$$\bar{y}_T \xrightarrow{m.s.} \mu = \mathbb{E}(y_t) \quad (2.2)$$

$$\lim_{T \rightarrow +\infty} T \mathbb{E}[(\bar{y}_T - \mu)^2] = \sum_{j=-\infty}^{+\infty} \gamma_j \quad (2.3)$$

$$\sqrt{T}(\bar{y}_T - \mu) \xrightarrow{d} \mathcal{N}\left(0, \sum_{j=-\infty}^{+\infty} \gamma_j\right). \quad (2.4)$$

[Mean square (m.s.) and distribution (d.) convergences: see Definitions 3.18 and 3.16.]

*Proof.* By Proposition 3.8, Eq. (2.3) implies Eq. (2.2). For Eq. (2.3), see Appendix 3.5. For Eq. (2.4), see Anderson (1971), p. 429.  $\square$

**Definition 2.9** (Long-run variance). Under the assumptions of Theorem 2.1, the limit appearing in Eq. (2.3) exists and is called **long-run variance**. It is denoted by  $S$ , i.e.:

$$S = \sum_{j=-\infty}^{+\infty} \gamma_j = \lim_{T \rightarrow +\infty} T \mathbb{E}[(\bar{y}_T - \mu)^2].$$

If  $y_t$  is ergodic for second moments (see Def. 2.8), a natural estimator of  $S$  is:

$$\hat{\gamma}_0 + 2 \sum_{\nu=1}^q \hat{\gamma}_\nu, \quad (2.5)$$

where  $\hat{\gamma}_\nu = \frac{1}{T} \sum_{\nu+1}^T (y_t - \bar{y})(y_{t-\nu} - \bar{y})$ .

However, for small samples, Eq. (2.5) does not necessarily result in a positive definite matrix. Newey and West (1987) have proposed an estimator that does not have this defect. Their estimator is given by:

$$S^{NW} = \hat{\gamma}_0 + 2 \sum_{\nu=1}^q \left(1 - \frac{\nu}{q+1}\right) \hat{\gamma}_\nu. \quad (2.6)$$

Loosely speaking, Theorem 2.1 says that, for a given sample size, the higher the “persistency” of a process, the lower the accuracy of the sample mean as an estimate of the population mean. To illustrate, consider three processes that feature the same marginal variance (equal to one, say), but different autocorrelations: 0%, 70%, and 99.9%. Figure 2.5 displays simulated paths of such three processes. It indeed appears that, the larger the autocorrelation of the process, the further the sample mean (dashed red line) from the population mean (red solid line).

The same type of simulations can be performed using this ShinyApp (use panel “AR(1)”).

## 2.2 Univariate processes

### 2.2.1 Moving Average (MA) processes

**Definition 2.10.** Consider a white noise process  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  (Def. 2.1). Then  $y_t$  is a first-order moving average process if, for all  $t$ :

$$y_t = \mu + \varepsilon_t + \theta \varepsilon_{t-1}.$$

If  $\mathbb{E}(\varepsilon_t^2) = \sigma^2$ , it is easily obtained that the unconditional mean and variances of  $y_t$  are:

$$\mathbb{E}(y_t) = \mu, \quad \text{Var}(y_t) = (1 + \theta^2)\sigma^2.$$

The first auto-covariance is:

$$\gamma_1 = \mathbb{E}\{(y_t - \mu)(y_{t-1} - \mu)\} = \theta\sigma^2.$$

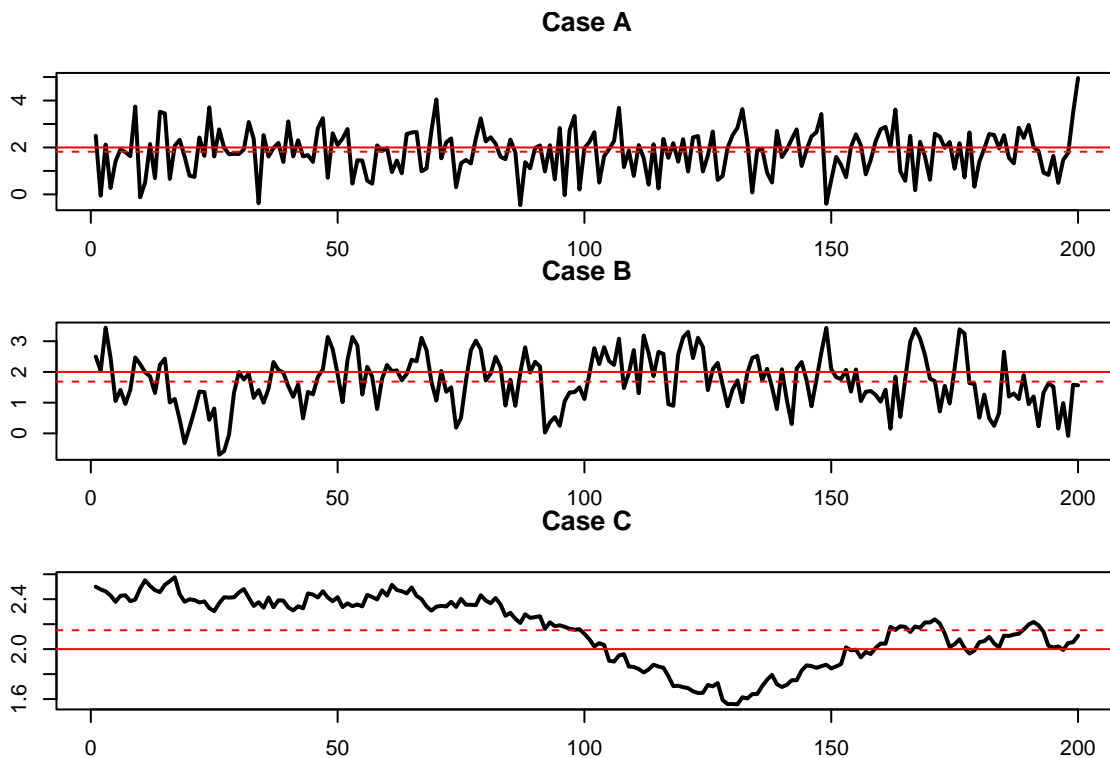


Figure 2.5: The three samples have been simulated using the following data generating process:  $x_t = \mu + \rho(x_{t-1} - \mu) + \sqrt{1 - \rho^2}\varepsilon_t$ , where  $\varepsilon_t \sim \mathcal{N}(0, 1)$ . Case A:  $\rho = 0$ ; Case B:  $\rho = 0.7$ ; Case C:  $\rho = 0.999$ . In the three cases,  $\mathbb{E}(x_t) = \mu = 2$  and  $\text{Var}(x_t) = 1$ .

Higher-order auto-covariances are zero ( $\gamma_j = 0$  for  $j > 1$ ). Therefore: An MA(1) process is covariance-stationary (Def. 2.4).

For a MA(1) process, the autocorrelation of order  $j$  (see Def. 2.6) is given by:

$$\rho_j = \begin{cases} 1 & \text{if } j = 0, \\ \theta/(1 + \theta^2) & \text{if } j = 1 \\ 0 & \text{if } j > 1. \end{cases}$$

Notice that process  $y_t$  defined through:

$$y_t = \mu + \varepsilon_t + \theta\varepsilon_{t-1},$$

where  $\text{Var}(\varepsilon_t) = \sigma^2$ , has the same mean and autocovariances as

$$y_t = \mu + \varepsilon_t^* + \frac{1}{\theta}\varepsilon_{t-1}^*,$$

where  $\text{Var}(\varepsilon_t^*) = \theta^2\sigma^2$ . That is, even if we perfectly know the mean and auto-covariances of this process, it is not possible to identify which specification is the one that has been used to generate the data. Only one of these two specifications is said to be *fundamental*, that is the one that satisfies  $|\theta_1| < 1$  (see Eq. (2.32)).

**Definition 2.11** (MA(q) process). A  $q^{\text{th}}$  order Moving Average process is defined through:

$$y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_q\varepsilon_{t-q}.$$

where  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  is a white noise process (Def. 2.1).

**Proposition 2.2** (Covariance-stationarity of an MA(q) process). *Finite-order Moving Average processes are covariance-stationary.*

Moreover, the autocovariances of an MA(q) process (as defined in Def. 2.11) are given by:

$$\gamma_j = \begin{cases} \sigma^2(\theta_j\theta_0 + \theta_{j+1}\theta_1 + \dots + \theta_q\theta_{q-j}) & \text{for } j \in \{0, \dots, q\} \\ 0 & \text{for } j > q, \end{cases} \quad (2.7)$$

where we use the notation  $\theta_0 = 1$ , and  $\text{Var}(\varepsilon_t) = \sigma^2$ .

*Proof.* The unconditional expectation of  $y_t$  does not depend on time, since  $\mathbb{E}(y_t) = \mu$ . Let's turn to autocovariances. We can extend the series of the  $\theta_j$ 's by setting  $\theta_j = 0$  for  $j > q$ . We then have:

$$\mathbb{E}((y_t - \mu)(y_{t-j} - \mu)) = \mathbb{E}[(\theta_0\varepsilon_t + \theta_1\varepsilon_{t-1} + \dots + \theta_j\varepsilon_{t-j} + \theta_{j+1}\varepsilon_{t-j-1} + \dots) \times (\theta_0\varepsilon_{t-j} + \theta_1\varepsilon_{t-j-1} + \dots)].$$

Then use the fact that  $\mathbb{E}(\varepsilon_t\varepsilon_s) = 0$  if  $t \neq s$  (because  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  is a white noise process).  $\square$

Figure 2.6 displays simulated paths of two MA processes (an MA(1) and an MA(4)). Such simulations can be produced by using panel “ARMA(p,q)” of this web interface.

```
library(AEC)
T <- 100; nb.sim <- 1
y.0 <- c(0)
c <- 1; phi <- c(0); sigma <- 1
theta <- c(1,1) # MA(1) specification
y.sim <- sim.arma(c,phi,theta,sigma,T,y.0,nb.sim)
par(mfrow=c(1,2))
par(plt=c(.2,.9,.2,.85))
plot(y.sim[,1],xlab="",ylab="",type="l",lwd=2,
     main=expression(paste(theta[0], "=1", " ", theta[1], "=1", sep="")))
abline(h=c)
theta <- c(1,1,1,1,1) # MA(4) specification
y.sim <- sim.arma(c,phi,theta,sigma,T,y.0,nb.sim)
plot(y.sim[,1],xlab="",ylab="",type="l",lwd=2,
     main=expression(paste(theta[0], "...=", theta[4], "=1", sep="")))
abline(h=c)
```

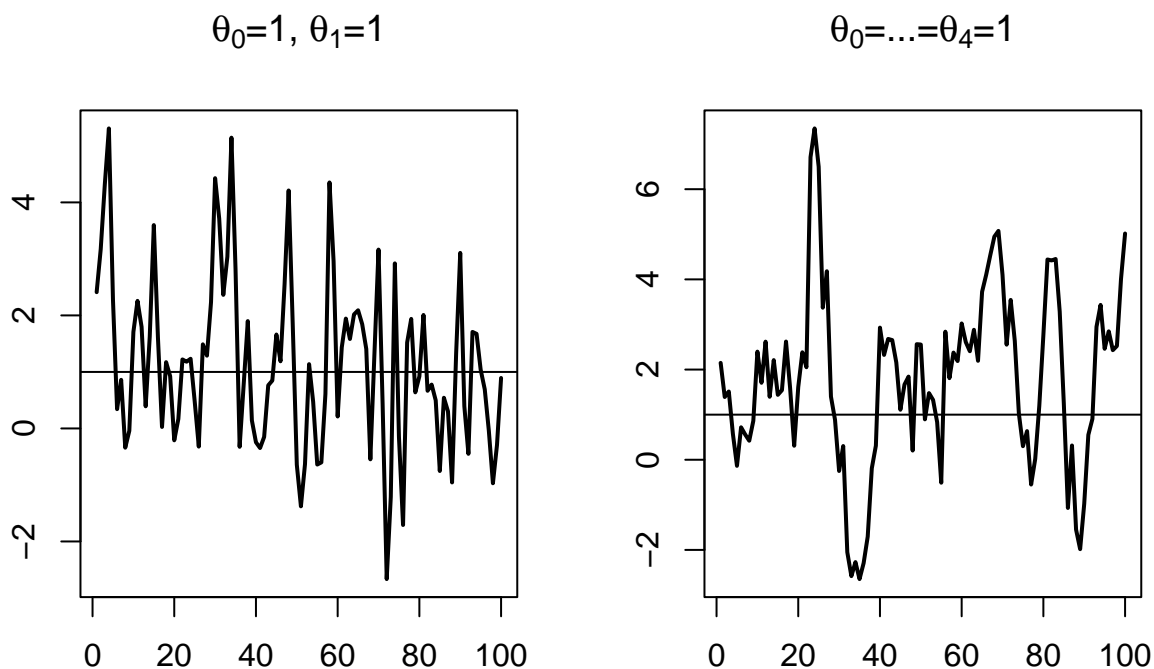


Figure 2.6: Simulation of MA processes.

What if the order  $q$  of an MA( $q$ ) process gets infinite? The notion of **infinite-order Moving Average process** exists and is important in time series analysis. The (infinite) sequence of  $\theta_j$  has to satisfy some conditions for such a process to be well-defined (see Theorem 2.2 below). These conditions relate to the “summability” of  $\{\theta_i\}_{i \in \mathbb{N}}$  (see Definition 2.12).

**Definition 2.12** (Absolute and square summability). The sequence  $\{\theta_i\}_{i \in \mathbb{N}}$  is absolutely summable if  $\sum_{i=0}^{\infty} |\theta_i| < +\infty$ , and it is square summable if  $\sum_{i=0}^{\infty} \theta_i^2 < +\infty$ .

According to Prop. 3.8, absolute summability implies square summability.

**Theorem 2.2** (Existence condition for an infinite MA process). *If  $\{\theta_i\}_{i \in \mathbb{N}}$  is square summable (see Def. 2.12) and if  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  is a white noise process (see Def. 2.1), then*

$$\mu + \sum_{i=0}^{+\infty} \theta_i \varepsilon_{t-i}$$

*defines a well-behaved [covariance-stationary] process, called infinite-order MA process ( $MA(\infty)$ ).*

*Proof.* See Appendix 3.A in Hamilton. “Well behaved” means that  $\sum_{i=0}^T \theta_i \varepsilon_{t-i}$  converges in mean square (Def. 3.16) to some random variable  $Z_t$ . The proof makes use of the fact that:

$$\mathbb{E} \left[ \left( \sum_{i=N}^M \theta_i \varepsilon_{t-i} \right)^2 \right] = \sum_{i=N}^M |\theta_i|^2 \sigma^2,$$

and that, when  $\{\theta_i\}$  is square summable,  $\forall \eta > 0$ ,  $\exists N$  s.t. the right-hand-side term in the last equation is lower than  $\eta$  for all  $M \geq N$  (static Cauchy criterion, Theorem 3.2). This implies that  $\sum_{i=0}^T \theta_i \varepsilon_{t-i}$  converges in mean square (stochastic Cauchy criterion, see Theorem 3.3).  $\square$

**Proposition 2.3** (First two moments of an infinite MA process). *If  $\{\theta_i\}_{i \in \mathbb{N}}$  is absolutely summable, i.e. if  $\sum_{i=0}^{\infty} |\theta_i| < +\infty$ , then*

*i.  $y_t = \mu + \sum_{i=0}^{+\infty} \theta_i \varepsilon_{t-i}$  exists (Theorem 2.2) and is such that:*

$$\begin{aligned} \mathbb{E}(y_t) &= \mu \\ \gamma_0 = \mathbb{E}([y_t - \mu]^2) &= \sigma^2(\theta_0^2 + \theta_1^2 + \dots) \\ \gamma_j = \mathbb{E}([y_t - \mu][y_{t-j} - \mu]) &= \sigma^2(\theta_0 \theta_j + \theta_1 \theta_{j+1} + \dots). \end{aligned}$$

*ii. Process  $y_t$  has absolutely summable auto-covariances, which implies that the results of Theorem 2.1 (Central Limit) apply.*

*Proof.* The absolute summability of  $\{\theta_i\}$  and the fact that  $\mathbb{E}(\varepsilon^2) < \infty$  imply that the order of integration and summation is interchangeable (see Hamilton, 1994, Footnote p. 52), which proves (i). For (ii), see end of Appendix 3.A in Hamilton (1994).  $\square$

## 2.2.2 Auto-Regressive (AR) processes

**Definition 2.13** (First-order AR process (AR(1))). Consider a white noise process  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  (see Def. 2.1). Process  $y_t$  is an AR(1) process if it is defined by the following difference equation:

$$y_t = c + \phi y_{t-1} + \varepsilon_t.$$

If  $|\phi| \geq 1$ ,  $y_t$  is not stationary. Indeed, we have:

$$y_{t+k} = c + \varepsilon_{t+k} + \phi(c + \varepsilon_{t+k-1}) + \phi^2(c + \varepsilon_{t+k-2}) + \dots + \phi^{k-1}(c + \varepsilon_{t+1}) + \phi^k y_t.$$

Therefore, the conditional variance

$$\text{Var}_t(y_{t+k}) = \sigma^2(1 + \phi^2 + \phi^4 + \dots + \phi^{2(k-1)})$$

does not converge for large  $k$ 's. This implies that  $\text{Var}(y_t)$  does not exist.

By contrast, if  $|\phi| < 1$ , one can see that:

$$y_t = c + \varepsilon_t + \phi(c + \varepsilon_{t-1}) + \phi^2(c + \varepsilon_{t-2}) + \dots + \phi^k(c + \varepsilon_{t-k}) + \dots$$

Hence, if  $|\phi| < 1$ , the unconditional mean and variance of  $y_t$  are:

$$\mathbb{E}(y_t) = \frac{c}{1 - \phi} =: \mu \quad \text{and} \quad \text{Var}(y_t) = \frac{\sigma^2}{1 - \phi^2}.$$

Let us compute the  $j^{th}$  autocovariance of the AR(1) process:

$$\begin{aligned}
 \mathbb{E}([y_t - \mu][y_{t-j} - \mu]) &= \mathbb{E}([\varepsilon_t + \phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2} + \dots + \phi^j\varepsilon_{t-j} + \phi^{j+1}\varepsilon_{t-j-1} \dots] \times \\
 &\quad [\varepsilon_{t-j} + \phi\varepsilon_{t-j-1} + \phi^2\varepsilon_{t-j-2} + \dots + \phi^k\varepsilon_{t-j-k} + \dots]) \\
 &= \mathbb{E}(\phi^j\varepsilon_{t-j}^2 + \phi^{j+2}\varepsilon_{t-j-1}^2 + \phi^{j+4}\varepsilon_{t-j-2}^2 + \dots) \\
 &= \frac{\phi^j\sigma^2}{1 - \phi^2}.
 \end{aligned}$$

Therefore  $\rho_j = \phi^j$ .

By what precedes, we have:

**Proposition 2.4** (Covariance-stationarity of an AR(1) process). *The AR(1) process, as defined in Def. 2.13, is covariance-stationary iff  $|\phi| < 1$ .*

**Definition 2.14** (AR(p) process). Consider a white noise process  $\{\varepsilon_t\}_{t=-\infty}^{+\infty}$  (see Def. 2.1). Process  $y_t$  is a  $p^{th}$ -order autoregressive process (AR(p)) if its dynamics is defined by the following difference equation (with  $\phi_p \neq 0$ ):

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t. \quad (2.8)$$

As we will see, the covariance-stationarity of process  $y_t$  hinges on matrix  $F$  defined as:

$$F = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_p \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \quad (2.9)$$

Note that this matrix  $F$  is such that if  $y_t$  follows Eq. (2.8), then process  $\mathbf{y}_t$  follows:

$$\mathbf{y}_t = \mathbf{c} + F\mathbf{y}_{t-1} + \xi_t$$

with

$$\mathbf{c} = \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \xi_t = \begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{y}_t = \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix}.$$

**Proposition 2.5** (The eigenvalues of matrix  $F$ ). *The eigenvalues of  $F$  (defined by Eq. (2.9)) are the solutions of:*

$$\lambda^p - \phi_1 \lambda^{p-1} - \dots - \phi_{p-1} \lambda - \phi_p = 0. \quad (2.10)$$

**Proposition 2.6** (Covariance-stationarity of an AR(p) process). *These four statements are equivalent:*

- i. Process  $\{y_t\}$ , defined in Def. 2.14, is covariance-stationary.
- ii. The eigenvalues of  $F$  (as defined Eq. (2.9)) lie strictly within the unit circle.
- iii. The roots of Eq. (2.11) (below) lie strictly outside the unit circle.

$$1 - \phi_1 z - \dots - \phi_{p-1} z^{p-1} - \phi_p z^p = 0. \quad (2.11)$$

- iv. The roots of Eq. (2.12) (below) lie strictly inside the unit circle.

$$\lambda^p - \phi_1 \lambda^{p-1} - \dots - \phi_{p-1} \lambda - \phi_p = 0. \quad (2.12)$$

*Proof.* We consider the case where the eigenvalues of  $F$  are distinct; Jordan decomposition can be used in the general case. When the eigenvalues of  $F$  are distinct,  $F$  admits the following spectral decomposition:  $F = PDP^{-1}$ , where  $D$  is diagonal. Using the notations introduced in Eq. (2.9), we have:

$$\mathbf{y}_t = \mathbf{c} + F\mathbf{y}_{t-1} + \xi_t.$$

Let's introduce  $\mathbf{d} = P^{-1}\mathbf{c}$ ,  $\mathbf{z}_t = P^{-1}\mathbf{y}_t$  and  $\eta_t = P^{-1}\xi_t$ . We have:

$$\mathbf{z}_t = \mathbf{d} + D\mathbf{z}_{t-1} + \eta_t.$$

Because  $D$  is diagonal, the different component of  $\mathbf{z}_t$ , denoted by  $z_{i,t}$ , follow AR(1) processes. The (scalar) autoregressive parameters of these AR(1) processes are the diagonal entries of  $D$ —which also are the eigenvalues of  $F$ —that we denote by  $\lambda_i$ .

Process  $y_t$  is covariance-stationary iff  $\mathbf{y}_t$  also is covariance-stationary, which is the case iff all  $z_{i,t}$ ,  $i \in [1, p]$ , are covariance-stationary. By Prop. 2.4, process  $z_{i,t}$  is covariance-stationary iff  $|\lambda_i| < 1$ . This proves that (i) is equivalent to (ii). Prop. 2.5 further proves that (ii) is equivalent to (iv). Finally, it is easily seen that (iii) is equivalent to (iv) (as long as  $\phi_p \neq 0$ ).  $\square$

Using the lag operator (see Eq (2.1)), if  $y_t$  is a covariance-stationary AR(p) process (Def. 2.14), we can write:

$$y_t = \mu + \psi(L)\varepsilon_t,$$

where

$$\psi(L) = (1 - \phi_1 L - \dots - \phi_p L^p)^{-1}, \quad (2.13)$$

and

$$\mu = \mathbb{E}(y_t) = \frac{c}{1 - \phi_1 - \dots - \phi_p}. \quad (2.14)$$

In the following lines of codes, we compute the eigenvalues of the  $F$  matrices associated with the following processes (where  $\varepsilon_t$  is a white noise):

$$\begin{aligned} x_t &= 0.9x_{t-1} - 0.2x_{t-2} + \varepsilon_t \\ y_t &= 1.1y_{t-1} - 0.3y_{t-2} + \varepsilon_t \\ w_t &= 1.4w_{t-1} - 0.7w_{t-2} + \varepsilon_t \\ z_t &= 0.9z_{t-1} + 0.2z_{t-2} + \varepsilon_t \end{aligned}$$

```
F <- matrix(c(.9,1,-.2,0),2,2)
lambda_x <- eigen(F)$values
F[1,] <- c(1.1,-.3)
lambda_y <- eigen(F)$values
F[1,] <- c(1.4,-.7)
lambda_w <- eigen(F)$values
F[1,] <- c(.9,.2)
lambda_z <- eigen(F)$values
rbind(lambda_x,lambda_y,lambda_w,lambda_z)
```

```
##                [,1]                [,2]
## lambda_x 0.500000+0.0000000i 0.400000+0.0000000i
## lambda_y 0.600000+0.0000000i 0.500000+0.0000000i
## lambda_w 0.700000+0.4582576i 0.700000-0.4582576i
## lambda_z 1.084429+0.0000000i -0.1844289+0.0000000i
```

The absolute values of the eigenvalues associated with process  $w_t$  are both equal to 0.837. Therefore, according to Proposition 2.6, processes  $x_t$ ,  $y_t$ , and  $w_t$  are covariance-stationary, but not  $z_t$  (because the absolute value of one of the eigenvalues of the  $F$  matrix associated with this process is larger than 1).

The computation of the autocovariances of  $y_t$  is based on the so-called **Yule-Walker equations** (Eq. (2.15)). Let's rewrite Eq. (2.8):

$$(y_t - \mu) = \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \dots + \phi_p(y_{t-p} - \mu) + \varepsilon_t.$$

Multiplying both sides by  $y_{t-j} - \mu$  and taking expectations leads to the (Yule-Walker) equations:

$$\gamma_j = \begin{cases} \phi_1\gamma_{j-1} + \phi_2\gamma_{j-2} + \dots + \phi_p\gamma_{j-p} & \text{if } j > 0 \\ \phi_1\gamma_1 + \phi_2\gamma_2 + \dots + \phi_p\gamma_p + \sigma^2 & \text{for } j = 0. \end{cases} \quad (2.15)$$

Using  $\gamma_j = \gamma_{-j}$  (Prop. 2.1), one can express  $(\gamma_0, \gamma_1, \dots, \gamma_p)$  as functions of  $(\sigma^2, \phi_1, \dots, \phi_p)$ .



### 2.2.3 AR-MA processes

**Definition 2.15** (ARMA(p,q) process).  $\{y_t\}$  is an ARMA(p,q) process if its dynamics is described by the following equation:

$$y_t = c + \underbrace{\phi_1 y_{t-1} + \dots + \phi_p y_{t-p}}_{\text{AR part}} + \underbrace{\varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}}_{\text{MA part}}, \quad (2.16)$$

where  $\{\varepsilon_t\}_{t \in [-\infty, +\infty]}$  is a white noise process (see Def. 2.1).

**Proposition 2.7** (Stationarity of an ARMA(p,q) process). *The ARMA(p,q) process defined in 2.15 is covariance stationary iff the roots of*

$$1 - \phi_1 z - \dots - \phi_p z^p = 0$$

*lie strictly outside the unit circle or, equivalently, iff those of*

$$\lambda^p - \phi_1 \lambda^{p-1} - \dots - \phi_p = 0$$

*lie strictly within the unit circle.*

*Proof.* The proof of Prop. 2.6 can be adapted to the present case. □

We can write:

$$(1 - \phi_1 L - \dots - \phi_p L^p) y_t = c + (1 + \theta_1 L + \dots + \theta_q L^q) \varepsilon_t.$$

If the roots of  $1 - \phi_1 z - \dots - \phi_p z^p = 0$  lie outside the unit circle, we have:

$$y_t = \mu + \psi(L) \varepsilon_t, \quad (2.17)$$

where

$$\psi(L) = \frac{1 + \theta_1 L + \dots + \theta_q L^q}{1 - \phi_1 L - \dots - \phi_p L^p} \quad \text{and} \quad \mu = \frac{c}{1 - \phi_1 - \dots - \phi_p}.$$

Eq. (2.17) is the **Wold representation** of this ARMA process (see Theorem 2.3 below).

The stationarity of the process depends only on the AR specification (or on the eigenvalues of matrix  $F$ , exactly as in Prop. 2.6). If the process is stationary, the weights in  $\psi(L)$  decay at a geometric rate.

### 2.2.4 PACF approach to identify AR/MA processes

We have seen that the  $k^{\text{th}}$ -order auto-correlation of a MA(q) process is null if  $k > q$ . This is exploited, in practice, to determine the order of a MA process. Moreover, since this is not the case for an AR process, this can be used to distinguish an AR from an MA process.

There exists an equivalent approach to determine whether a process can be modeled as an AR process; it is based on partial auto-correlations:

**Definition 2.16** (Partial auto-correlation). In a time series context, the partial auto-correlation  $(\phi_{h,h})$  of process  $\{y_t\}$  is defined as the partial correlation of  $y_{t+h}$  and  $y_t$  given  $y_{t+h-1}, \dots, y_{t+1}$ . (see Def. 3.5 for the definition of partial correlation.)

If  $h > p$ , the regression of  $y_{t+h}$  on  $y_{t+h-1}, \dots, y_{t+1}$  is:

$$y_{t+h} = c + \phi_1 y_{t+h-1} + \dots + \phi_p y_{t+h-p} + \varepsilon_{t+h}.$$

The residuals of the latter regressions  $(\varepsilon_{t+h})$  are uncorrelated to  $y_t$ . Then the partial auto-correlation is zero for  $h > p$ .

Besides, it can be shown that  $\phi_{p,p} = \phi_p$ . Hence  $\phi_{p,p} = \phi_p$  but  $\phi_{h,h} = 0$  for  $h > p$ . This can be used to determine the order of an AR process. By contrast (importantly) if  $y_t$  follows an MA(q) process, then  $\phi_{k,k}$  asymptotically approaches zero instead of cutting off abruptly.

As illustrated below, functions `acf` and `pacf` can be conveniently used to employ the (P)ACF approach. (Note also the use of function `sim.arma` to simulate ARMA processes.)

```

library(AEC)
par(mfrow=c(3,2))
par(plt=c(.2,.9,.2,.95))
theta <- c(1,2,1);phi=0
y.sim <- sim.arma(c=0,phi,theta,sigma=1,T=1000,y.0=0,nb.sim=1)
par(mfg=c(1,1));plot(y.sim,type="l",lwd=2)
par(mfg=c(2,1));acf(y.sim)
par(mfg=c(3,1));pacf(y.sim)
theta <- c(1);phi=0.9
y.sim <- sim.arma(c=0,phi,theta,sigma=1,T=1000,y.0=0,nb.sim=1)
par(mfg=c(1,2));plot(y.sim,type="l",lwd=2)
par(mfg=c(2,2));acf(y.sim)
par(mfg=c(3,2));pacf(y.sim)

```

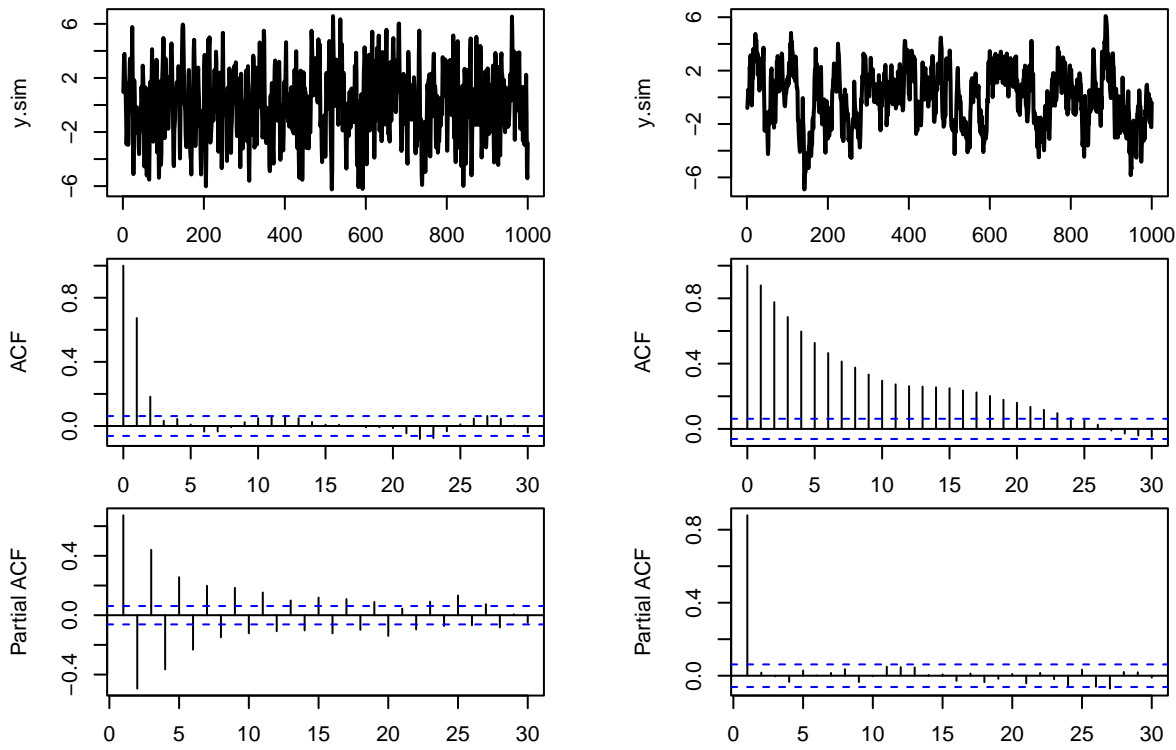


Figure 2.7: ACF/PACF analysis of two processes (MA process on the left, AR on the right).

### 2.2.5 Wold decomposition

The Wold decomposition is an important result in time series analysis:

**Theorem 2.3** (Wold decomposition). *Any covariance-stationary process admits the following representation:*

$$y_t = \mu + \sum_{i=0}^{+\infty} \theta_i \varepsilon_{t-i} + \kappa_t,$$

where

- $\theta_0 = 1$ ,  $\sum_{i=0}^{\infty} \theta_i^2 < +\infty$  (square summability, see Def. 2.12).
- $\{\varepsilon_t\}$  is a white noise (see Def. 2.1);  $\varepsilon_t$  is the error made when forecasting  $y_t$  based on a linear combination of lagged  $y_t$ 's ( $\varepsilon_t = y_t - \hat{\mathbb{E}}[y_t | y_{t-1}, y_{t-2}, \dots]$ ).
- For any  $j \geq 1$ ,  $\kappa_t$  is not correlated with  $\varepsilon_{t-j}$ ; but  $\kappa_t$  can be perfectly forecasted based on a linear combination of lagged  $y_t$ 's (i.e.  $\kappa_t = \hat{\mathbb{E}}(\kappa_t | y_{t-1}, y_{t-2}, \dots)$ ).  $\kappa_t$  is called the **deterministic component** of  $y_t$ .

*Proof.* See Anderson (1971). Partial proof in L. Christiano. □

For an ARMA process, the Wold representation is given by Eq. (2.17). As detailed in Prop. 2.8, it can be computed by recursively replacing the lagged  $y_t$ 's in Eq. (2.16). In this case, the deterministic component ( $\kappa$ ) is null.

### 2.2.6 Impulse Response Functions (IRFs) in ARMA models

Consider the ARMA(p,q) process defined in Def. 2.15, whose associated sequence of white noise is  $\{\varepsilon_t\}$ . Let us construct a novel (counterfactual) sequence of shocks  $\{\tilde{\varepsilon}_t^{(s)}\}$ :

$$\tilde{\varepsilon}_t^{(s)} = \begin{cases} \varepsilon_t & \text{if } t \neq s, \\ \varepsilon_t + \delta & \text{if } t = s. \end{cases}$$

We denote by  $\{\tilde{y}_t^{(s)}\}$  the process following Eq. (2.16) where  $\{\varepsilon_t\}$  is replaced with  $\{\tilde{\varepsilon}_t^{(s)}\}$ . The time series  $\{\tilde{y}_t^{(s)}\}$  is the counterfactual series  $\{y_t\}$  that would have prevailed if  $\varepsilon_t$  had been shifted by  $\delta$  on date  $s$  (and that would be the only change).

The relationship between  $\{y_t\}$  and  $\{\tilde{y}_t^{(s)}\}$  defines the **dynamics multiplier**. The latter is denoted by  $\frac{\partial y_t}{\partial \varepsilon_s}$  and is such that:

$$\tilde{y}_t^{(s)} = y_t + \frac{\partial y_t}{\partial \varepsilon_s} \delta.$$

We will see that the dynamic multipliers are closely related to the infinite MA representation (or **Wold decomposition**, Theorem 2.3) of  $y_t$ :

$$y_t = \mu + \sum_{i=0}^{+\infty} \psi_i \varepsilon_{t-i}.$$

For  $t < s$ , we have  $y_t = \tilde{y}_t^{(s)}$  (because  $\tilde{\varepsilon}_{t-i} = \varepsilon_{t-i}$  for all  $i \geq 0$  if  $t < s$ ).

For  $t \geq s$ :

$$\tilde{y}_t^{(s)} = \mu + \left( \sum_{i=0}^{t-s-1} \psi_i \varepsilon_{t-i} \right) + \psi_{t-s}(\varepsilon_s + \delta) + \left( \sum_{i=t-s+1}^{+\infty} \psi_i \varepsilon_{t-i} \right) = y_t + \frac{\partial y_t}{\partial \varepsilon_s} \delta.$$

Therefore, for  $t \geq s$ , we have:

$$\boxed{\frac{\partial y_t}{\partial \varepsilon_s} = \psi_{t-s}.$$

That is,  $\{y_t\}$ 's dynamic multiplier of order  $k$  is the same object as the  $k^{th}$  loading  $\psi_k$  in the Wold decomposition of  $\{y_t\}$ . The sequence  $\left\{ \frac{\partial y_{t+h}}{\partial \varepsilon_t} \right\}_{h \geq 0} \equiv \{\psi_h\}_{h \geq 0}$  defines the **impulse response function (IRF)** of  $y_t$  to the shock  $\varepsilon_t$ .

For ARMA processes, the computation of the IRFs is easy:

**Proposition 2.8** (IRF of an ARMA(p,q) process). *The coefficients  $\psi_h$ , that define the IRF of process  $y_t$  to  $\varepsilon_t$ , can be computed recursively as follows:*

1. Set  $\psi_{-1} = \dots = \psi_{-p} = 0$ .
2. For  $h \geq 0$ , (recursively) apply:

$$\psi_h = \phi_1 \psi_{h-1} + \dots + \phi_p \psi_{h-p} + \theta_h,$$

where  $\theta_h = 0$  for  $h > q$ .

*Proof.* This is obtained by applying the operator  $\frac{\partial}{\partial \varepsilon_t}$  on both sides of Eq. (2.16):

$$y_{t+h} = c + \phi_1 y_{t+h-1} + \dots + \phi_p y_{t+h-p} + \varepsilon_{t+h} + \theta_1 \varepsilon_{t+h-1} + \dots + \theta_q \varepsilon_{t+h-q}.$$

□

Note that Proposition 2.8 constitutes a simple way to compute the MA( $\infty$ ) representation (or Wold representation) of an ARMA process.

One can use function `sim.arma` of package `AEC` to compute ARMA's IRFs (with the argument `make.IRF = 1`):

```

T <- 21 # number of periods for IRF
theta <- c(1,1,1); phi <- c(0); c <- 0
y.sim <- sim.arma(c,phi,theta,sigma=1,T,y.0=rep(0,length(phi)),
                nb.sim=1,make.IRF = 1)
par(mfrow=c(1,3)); par(plt=c(.25,.95,.2,.85))
plot(0:(T-1),y.sim[,1],type="l",lwd=2,
     main="(a) Process 1",xlab="Time after shock on epsilon",
     ylab="Dynamic multiplier (shock on epsilon at t=0)",col="red")
abline(h=0)
theta <- c(1,.5); phi <- c(0.6)
y.sim <- sim.arma(c,phi,theta,sigma=1,T,y.0=rep(0,length(phi)),
                nb.sim=1,make.IRF = 1)
plot(0:(T-1),y.sim[,1],type="l",lwd=2,
     main="(b) Process 2",xlab="Time after shock on epsilon",
     ylab="",col="red")
theta <- c(1,1,1); phi <- c(0,0,.5,.4)
y.sim <- sim.arma(c,phi,theta,sigma=1,T,y.0=rep(0,length(phi)),
                nb.sim=1,make.IRF = 1)
plot(0:(T-1),y.sim[,1],type="l",lwd=2,
     main="(c) Process 3",xlab="Time after shock on epsilon",
     ylab="",col="red")

```

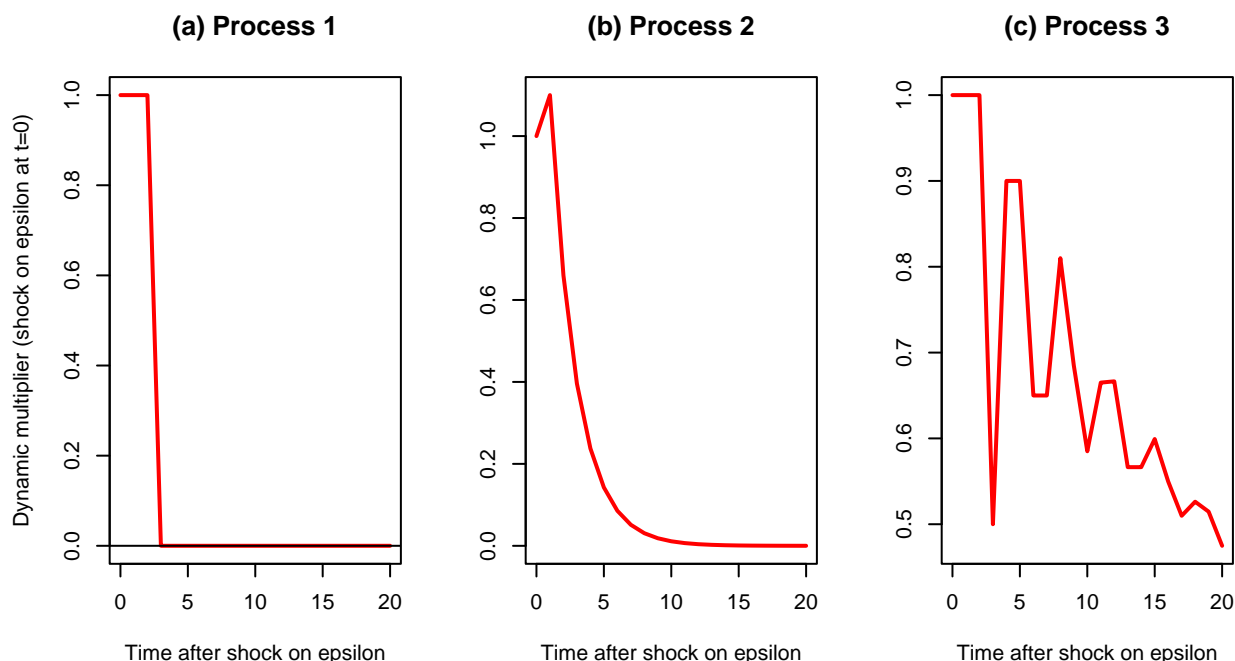


Figure 2.8: IRFs associated with the three processes. Process 1 (MA(2)):  $y_t = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}$ . Process 2 (ARMA(1,1)):  $y_t = 0.6y_{t-1} + \varepsilon_t + 0.5\varepsilon_{t-1}$ . Process 3 (ARMA(4,2)):  $y_t = 0.5y_{t-3} + 0.4y_{t-4} + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}$ .

Consider the annual Swiss GDP growth from the JST macro-history database. Let us first determine relevant orders for AR and MA processes using the (P)ACF approach.

```

library(AEC)
data(JST)
data <- subset(JST,iso=="CHE")
par(plt=c(.1,.95,.1,.95))
T <- dim(data)[1]
growth <- log(data$gdp[2:T]/data$gdp[1:(T-1)])
par(mfrow=c(3,1))
par(plt=c(.1,.95,.15,.95))
plot(data$year[2:T],growth,type="l",xlab="",ylab="",lwd=2)
abline(h=0,lty=2)
acf(growth)
pacf(growth)

```

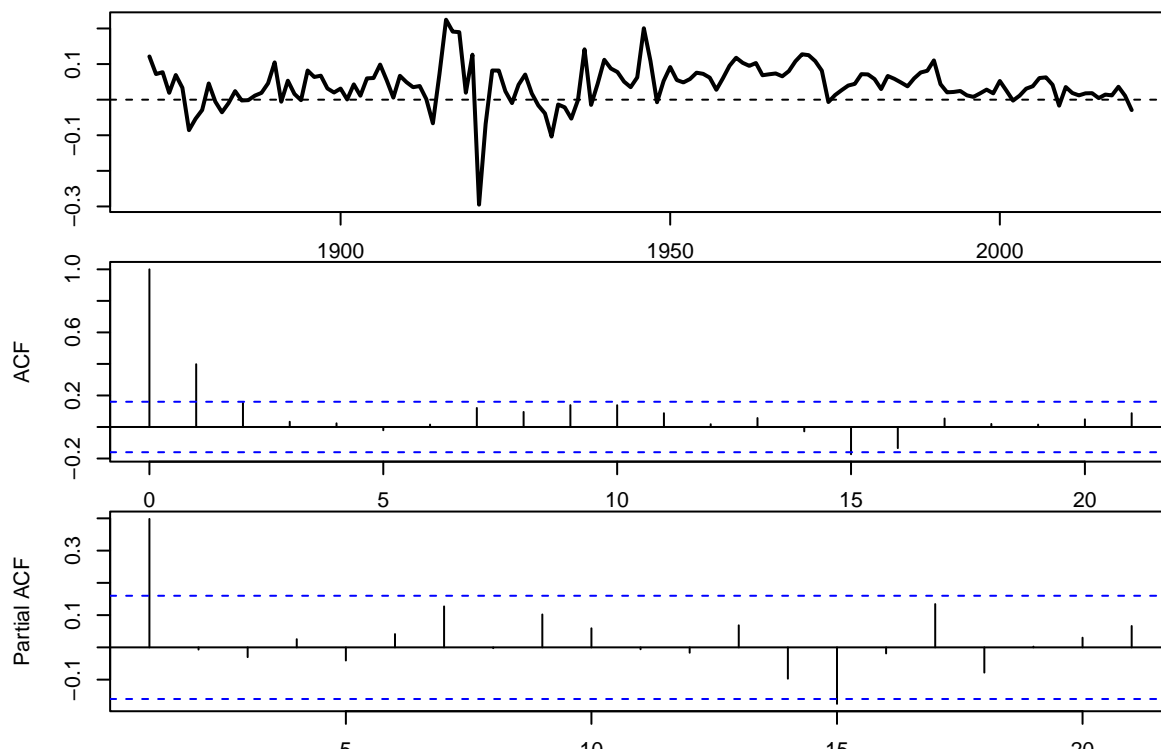


Figure 2.9: (P)ACF analysis of Swiss GDP growth.

The two bottom plots of Figure 2.9 suggest that either an MA(2) or an AR(1) could be used to model the GDP growth rate series. Figure 2.10 shows the IRFs based on these two respective specifications.

```
# Fit an AR process:
res <- arima(growth,order=c(1,0,0))
phi <- res$coef[1]
T <- 11
y.sim <- sim.arma(c=0,phi,theta=1,sigma=1,T,y.0=rep(0,length(phi)),
                 nb.sim=1,make.IRF = 1)
par(plt=c(.15,.95,.25,.95))
plot(0:(T-1),y.sim[,1],type="l",lwd=3,
     xlab="Time after shock on epsilon",
     ylab="Dynamic multiplier (shock on epsilon at t=0)",col="red")
# Fit a MA process:
res <- arima(growth,order=c(0,0,2))
phi <- 0;theta <- c(1,res$coef[1:2])
y.sim <- sim.arma(c=0,phi,theta,sigma=1,T,y.0=rep(0,length(phi)),
                 nb.sim=1,make.IRF = 1)
lines(0:(T-1),y.sim[,1],lwd=3,col="red",lty=2)
abline(h=0)
```

The same kind of algorithm can be used to compute the impact of an increase in an exogenous variable  $x_t$  within an ARMAX(p,q,r) model (see next section).

### 2.2.7 ARMA processes with exogenous variables (ARMA-X)

ARMA processes do not allow to investigate the influence of an exogenous variable (say  $x_t$ ) on the variable of interest (say  $y_t$ ). When  $x_t$  and  $y_t$  have reciprocal influences, the Vector Autoregressive (VAR) model may be used (this tools will be studied later, in Section 2.3). However, when one suspects that  $x_t$  has an “exogenous” influence on  $y_t$ , then a simple extension of the ARMA processes may be considered. Loosely speaking,  $x_t$  has an “exogenous” influence on  $y_t$  if  $y_t$  does not affect  $x_t$ . This extension is called ARMAX(p,q,r).

To begin with, let us formalize this notion of exogeneity. Consider a white noise sequence  $\{\varepsilon_t\}$  (Def. 2.1).

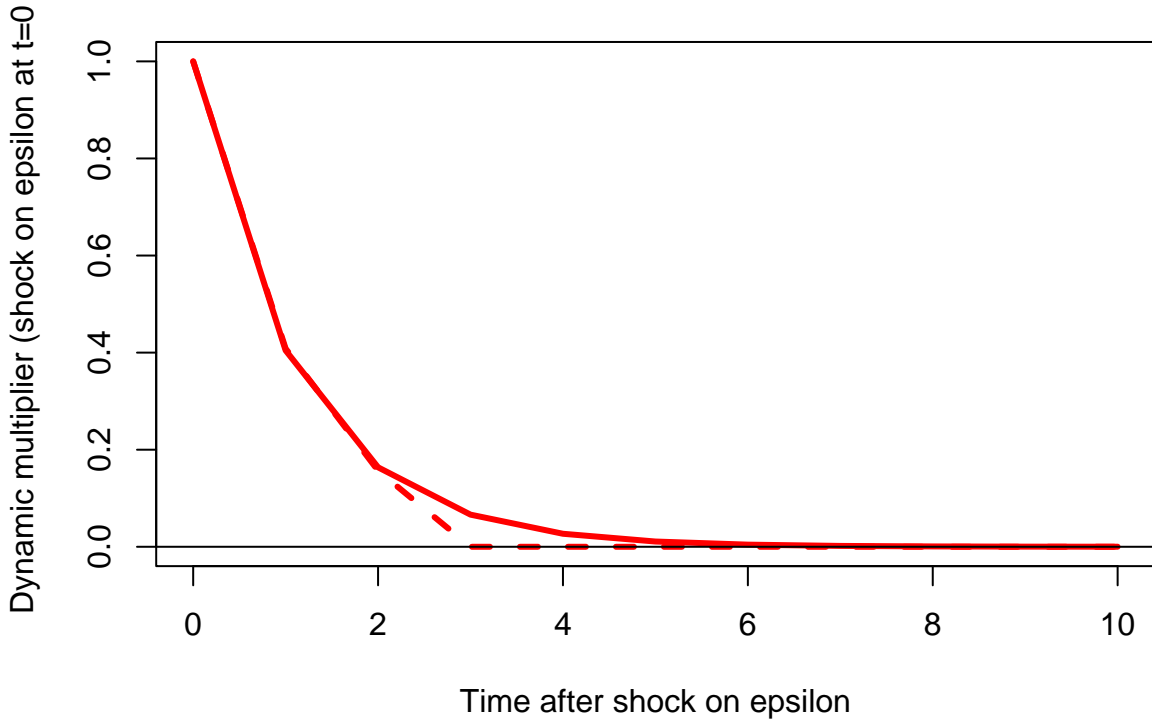


Figure 2.10: Dynamic response of Swiss annual growth to a shock on the innovation  $\varepsilon_t$  at date  $t = 0$ . The solid line corresponds to an AR(1) specification; the dashed line corresponds to a MA(2) specification.

**Definition 2.17** (Exogeneity). We say that  $x_t$  is (strictly) exogenous to  $\{\varepsilon_t\}$  if

$$\mathbb{E}(\varepsilon_t | \underbrace{\dots, x_{t+1}}_{\text{future}}, \underbrace{x_t, x_{t-1}, \dots}_{\text{present and past}}) = 0.$$

Hence, if  $\{x_t\}$  is strictly exogenous to  $\varepsilon_t$ , then past, present and future values of  $x_t$  do not allow to predict the  $\varepsilon_t$ 's.

In the following, we assume that  $\{x_t\}$  is a covariance stationary process.

**Definition 2.18** (ARMAX(p,q,r) model). The process  $\{y_t\}$  is an ARMAX(p,q,r) if it follows a difference equation:

$$y_t = \underbrace{c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p}}_{\text{AR(p) part}} + \underbrace{\beta_0 x_t + \dots + \beta_r x_{t-r}}_{\text{X(r) part}} + \underbrace{\varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}}_{\text{MA(q) part}}. \quad (2.18)$$

where  $\{\varepsilon_t\}$  is an i.i.d. white noise sequence and  $\{x_t\}$  is exogenous to  $y_t$ .

What is the effect of a one-unit increase in  $x_t$  on  $y_t$ ? To address this question, this notion of “effect” has to be formalized. Let us introduce two related sequences of values for  $\{x\}$ . Denote the first by  $\{a\}$  and the second by  $\{\tilde{a}^t\}$ . Further, we posit  $a_s = \tilde{a}_s^t$  for all  $s \neq t$ , and  $\tilde{a}_t^t = a_t + 1$ .

With these notations, we define  $\frac{\partial y_{t+h}}{\partial x_t}$  as follows:

$$\frac{\partial y_{t+h}}{\partial x_t} := \mathbb{E}(y_{t+h} | \{x\} = \{\tilde{a}^t\}) - \mathbb{E}(y_{t+h} | \{x\} = \{a\}). \quad (2.19)$$

Under the exogeneity assumption, it is easily seen that

$$\frac{\partial y_t}{\partial x_t} = \beta_0.$$

Now, since

$$y_{t+1} = c + \phi_1 y_t + \dots + \phi_p y_{t+1-p} + \beta_0 x_{t+1} + \dots + \beta_r x_{t+1-r} + \varepsilon_{t+1} + \theta_1 \varepsilon_t + \dots + \theta_q \varepsilon_{t+1-q},$$

and using the exogeneity assumption, we obtain:

$$\frac{\partial y_{t+1}}{\partial x_t} := \phi_1 \frac{\partial y_t}{\partial x_t} + \beta_1 = \phi_1 \beta_0 + \beta_1.$$

This can be applied recursively to give  $\frac{\partial y_{t+h}}{\partial x_t}$  for any  $h \geq 0$ :

**Proposition 2.9** (Dynamic multipliers in ARMAX models). *One can recursively compute the dynamic multipliers  $\frac{\partial y_{t+h}}{\partial x_t}$  as follows:*

- i. Initialization:  $\frac{\partial y_{t+h}}{\partial x_t} = 0$  for  $h < 0$ .
- ii. For  $h \geq 0$  and assuming that the first  $h - 1$  multipliers have been computed, we have:

$$\frac{\partial y_{t+h}}{\partial x_t} = \phi_1 \frac{\partial y_{t+h-1}}{\partial x_t} + \dots + \phi_p \frac{\partial y_{t+h-p}}{\partial x_t} + \beta_h, \quad (2.20)$$

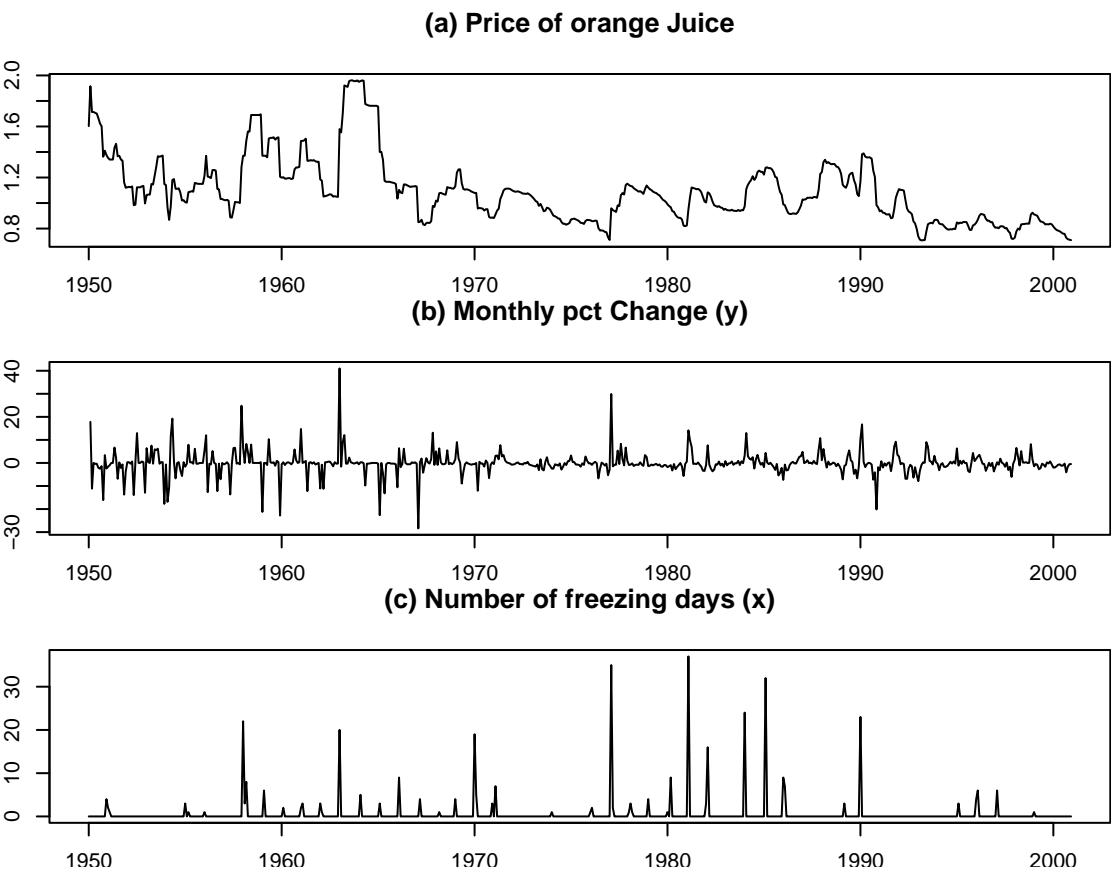
where we use the notation  $\beta_h = 0$  if  $h > r$ .

Remark that the resulting dynamic multipliers are the same as those obtained for an ARMA(p,r) model where the  $\theta_i$ 's are replaced with  $\beta_i$ 's (see Proposition 2.8 in Section 2.2.7).

It has to be stressed that the definition of the dynamic multipliers (Eq. (2.20)) does not reflect a potential persistency of the shock occurring on date  $t$  in process  $\{x\}$  itself. Going in this direction would necessitate to model the joint dynamics of  $x_t$  (for instance using a VAR model, see Section 2.3).

**Example 2.3** (Influence of the number of freezing days on the price of orange juice). This example is based on data used in Stock and Watson (2003) (Chapter 16). The objective is to study the influence of the number of freezing days on the price of orange juice. Let us first estimate a ARMAX(0,0,12) model:

```
library(AEC)
library(AER)
data("FrozenJuice")
FJ <- as.data.frame(FrozenJuice)
date <- time(FrozenJuice)
price <- FJ$price/FJ$ppi
T <- length(price)
k <- 1
dprice <- 100*log(price[(k+1):T]/price[1:(T-k)])
fdd <- FJ$fdd[(k+1):T]
par(mfrow=c(3,1))
par(plt=c(.1,.95,.15,.75))
plot(date,price,type="l",xlab="",ylab="",
      main="(a) Price of orange Juice")
plot(date,c(NaN,dprice),type="l",xlab="",ylab="",
      main="(b) Monthly pct Change (y)")
plot(date,FJ$fdd,type="l",xlab="",ylab="",
      main="(c) Number of freezing days (x)")
```



```
nb.lags <- 12
FDD <- FJ$fdd[(nb.lags+1):T]
names.FDD <- NULL
for(i in 1:nb.lags){
  FDD <- cbind(FDD,FJ$fdd[(nb.lags+1-i):(T-i)])
  names.FDD <- c(names.FDD,paste(" Lag ",toString(i),sep=""))}
colnames(FDD) <- c(" Lag 0",names.FDD)
dprice <- dprice[(length(dprice)-dim(FDD)[1]+1):length(dprice)]
eq <- lm(dprice~FDD)
# Compute the Newey-West std errors:
var.cov.mat <- NeweyWest(eq,lag = 7, prewhite = FALSE)
robust_se <- sqrt(diag(var.cov.mat))
# Stargazer output (with and without Robust SE)
stargazer::stargazer(eq, eq, type = "text",
  column.labels=c("(no HAC)","(HAC)"),keep.stat="n",
  se = list(NULL,robust_se),no.space = TRUE)
```

```
##
## =====
##           Dependent variable:
##           -----
##                dprice
##           (no HAC)      (HAC)
##           (1)          (2)
## -----
## FDD Lag 0      0.496***    0.496***
##                (0.058)    (0.139)
## FDD Lag 1      0.150***    0.150*
##                (0.058)    (0.087)
## FDD Lag 2       0.046       0.046
##                (0.057)    (0.056)
## FDD Lag 3       0.062       0.062
##                (0.057)    (0.046)
## FDD Lag 4       0.024       0.024
##                (0.057)    (0.030)
## FDD Lag 5       0.036       0.036
```



```
##          (0.057)      (0.030)
## FDD Lag 6      0.037      0.037
##          (0.057)      (0.046)
## FDD Lag 7      0.019      0.019
##          (0.057)      (0.015)
## FDD Lag 8     -0.038     -0.038
##          (0.057)      (0.034)
## FDD Lag 9     -0.006     -0.006
##          (0.057)      (0.050)
## FDD Lag 10    -0.112*    -0.112
##          (0.057)      (0.069)
## FDD Lag 11    -0.063     -0.063
##          (0.058)      (0.052)
## FDD Lag 12    -0.140**   -0.140*
##          (0.058)      (0.078)
## Constant    -0.426*    -0.426*
##          (0.238)      (0.243)
## -----
## Observations      600      600
## =====
## Note:              *p<0.1; **p<0.05; ***p<0.01
```

Let us now use function `estim.armax`, from package `AECto` to estimate an ARMA-X(2,0,1) model:

```
nb.lags <- 1
FDD <- FJ$fdd[(nb.lags+1):T]
names.FDD <- NULL
for(i in 1:nb.lags){
  FDD <- cbind(FDD,FJ$fdd[(nb.lags+1-i):(T-i)])
  names.FDD <- c(names.FDD,paste(" Lag ",toString(i),sep=""))
}
colnames(FDD) <- c(" Lag 0",names.FDD)
dprice <- 100*log(price[(k+1):T]/price[1:(T-k)])
dprice <- dprice[(length(dprice)-dim(FDD)[1]+1):length(dprice)]
res.armax <- estim.armax(Y = dprice,p=3,q=0,X=FDD)
```

```
## [1] "=====
## [1] "  ESTIMATING"
## [1] "=====
## [1] "  END OF ESTIMATION"
## [1] "=====
## [1] ""
## [1] "  RESULTS:"
## [1] "  -----"
##          THETA      st.dev    t.ratio
## c          -0.46556249 0.19554352 -2.380864
## phi   t-1    0.09788977 0.04025907  2.431496
## phi   t-2    0.05049849 0.03827488  1.319364
## phi   t-3    0.07155170 0.03764750  1.900570
## sigma          4.64917949 0.13300769 34.954215
## beta   t-0    0.47015552 0.05665344  8.298800
## beta   t-1    0.10015862 0.05972526  1.676989
## [1] "=====
```

Figure 2.11 shows the IRF associated with each of the two models.

```
nb.periods <- 20
IRF1 <- sim.arma(c=0,phi=c(0),theta=req$coefficients[2:13],sigma=1,
```

```

T=nb.periods,y.0=c(0),nb.sim=1,make.IRF=1)
IRF2 <- sim.arma(c=0,phi=res.arma$phi,theta=res.arma$beta,sigma=1,
               T=nb.periods,y.0=rep(0,length(res.arma$phi)),
               nb.sim=1,make.IRF=1)
par(plt=c(.15,.95,.2,.95))
plot(IRF1,type="l",lwd=2,col="red",xlab="months after shock",
     ylab="Chge in price (percent)")
lines(IRF2,lwd=2,col="red",lty=2)
abline(h=0,col="grey")

```

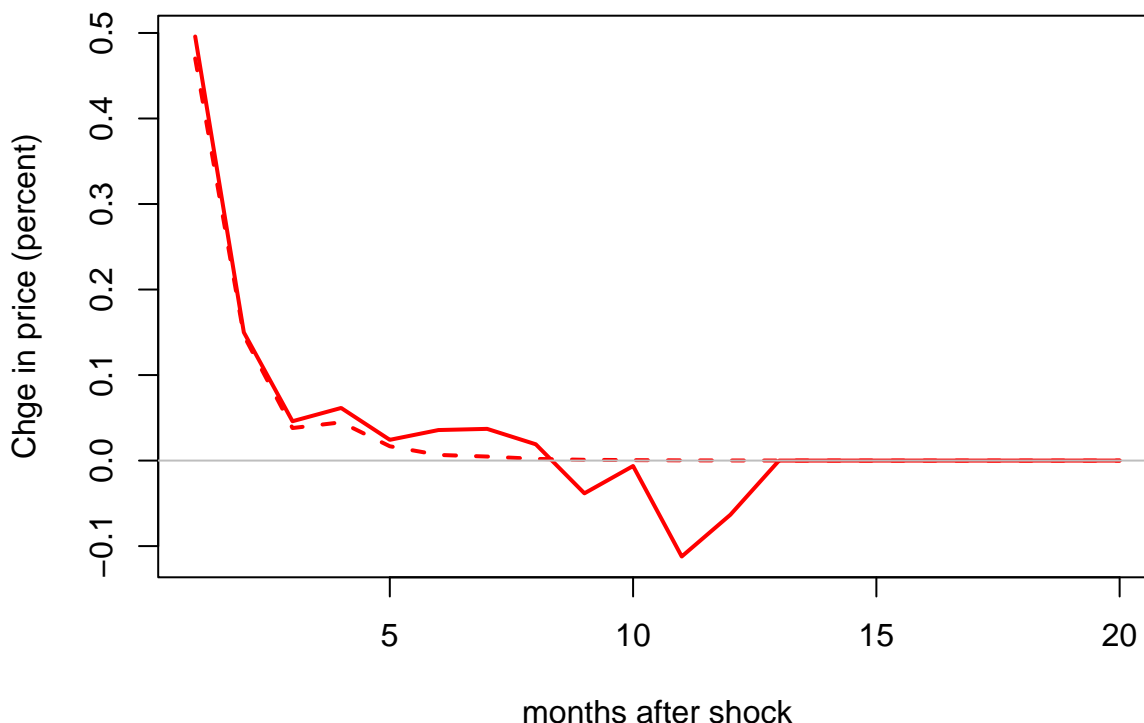


Figure 2.11: Response of changes in orange juice price (in percent) to the number of freezing days. The solid (respectively dashed) line corresponds to the ARMAX(0,0,12) (resp. ARMAX(3,0,1)) model. The first model is estimated by OLS (see above), the second by MLE.

**Example 2.4** (Real effect of a monetary policy shock). In this example, we make use of monetary shocks identified through high-frequency data (see Gertler and Karadi (2015)). This dataset comes from Valerie Ramey's website (see Ramey (2016)).

```

library(AEC)
T <- dim(Ramey)[1]
# Construct growth series:
Ramey$growth <- Ramey$LIP - c(rep(NaN,12),Ramey$LIP[1:(length(Ramey$LIP)-12)])
# Prepare matrix of exogenous variables:
vec.lags <- c(9,12,18)
Matrix.of.Exog <- NULL
shocks <- Ramey$ED2_TC
for(i in 1:length(vec.lags)){Matrix.of.Exog <-
  cbind(Matrix.of.Exog,c(rep(NaN,vec.lags[i]),shocks[1:(T-vec.lags[i])]))}
# Look for dates where data are available:
indic.good.dates <- complete.cases(Matrix.of.Exog)

# Estimate ARMAX:
p <- 1; q <- 0
x <- estim.arma(Ramey$growth[indic.good.dates],p,q,
               X=Matrix.of.Exog[indic.good.dates,])

```

```
## [1] "=====
```

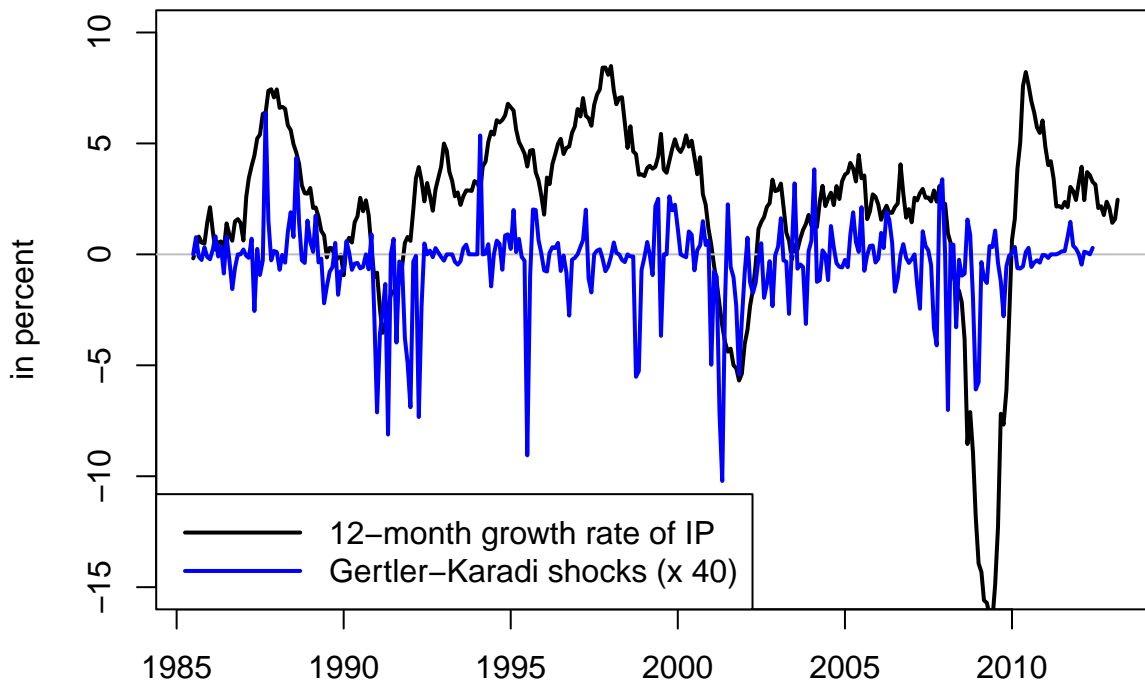


Figure 2.12: The blue line corresponds to monetary-policy shocks identified by means of the Gertler and Karadi (2015)’s approach (high-frequency change in Euro-dollar futures). The black solid line is the year-on-year growth rate of industrial production.

```
## [1] " ESTIMATING"
## [1] "===== "
## [1] " END OF ESTIMATION"
## [1] "===== "
## [1] ""
## [1] " RESULTS:"
## [1] " -----"
##
##          THETA      st.dev   t.ratio
## c      -0.0001716198 0.0005845907 -0.2935726
## phi t-1  0.9825608412 0.0120458531 81.5683897
## sigma  0.0087948724 0.0003211748 27.3834438
## beta t-0 -0.0193570616 0.0087331529 -2.2165032
## beta t-1 -0.0225707935 0.0086750938 -2.6017925
## beta t-2 -0.0070131593 0.0086387440 -0.8118263
## [1] "===== "
```

```
# Compute IRF:
irf <- sim.arma(0,x$phi,x$beta,x$sigma,T=60,y.0=rep(0,length(x$phi)),
               nb.sim=1,make.IRF=1,X=NaN,beta=NaN)
```

Figure 2.13 displays the resulting IRF, with a 95% confidence band. The code used to produce the confidence bands (i.e., to compute the standard deviation of the dynamic multipliers for the different horizons) is based on the Delta method (see Eq. (??)). The codes are available in Appendix 3.6.2.

## 2.2.8 Maximum Likelihood Estimation of ARMA processes

Consider the general case (of any time series); assume we observe a sample  $\mathbf{y} = [y_1, \dots, y_T]'$ . In order to implement ML techniques (see Section ??), we need to evaluate the joint p.d.f. (or “likelihood”) of  $\mathbf{y}$ , i.e.,  $\mathcal{L}(\theta; \mathbf{y})$ , where  $\theta$  is a vector of parameters that characterizes the dynamics of  $y_t$ . The Maximum Likelihood (ML) estimate of  $\theta$  is then given by:

$$\theta_{MLE} = \arg \max_{\theta} \mathcal{L}(\theta; \mathbf{y}) = \arg \max_{\theta} \log \mathcal{L}(\theta; \mathbf{y}).$$

In the time series context, if process  $y_t$  is Markovian, then there exists a useful way to rewrite the likelihood  $\mathcal{L}(\theta; \mathbf{y})$ . Let us first recall the definition of a Markovian process (see also Def. ??):

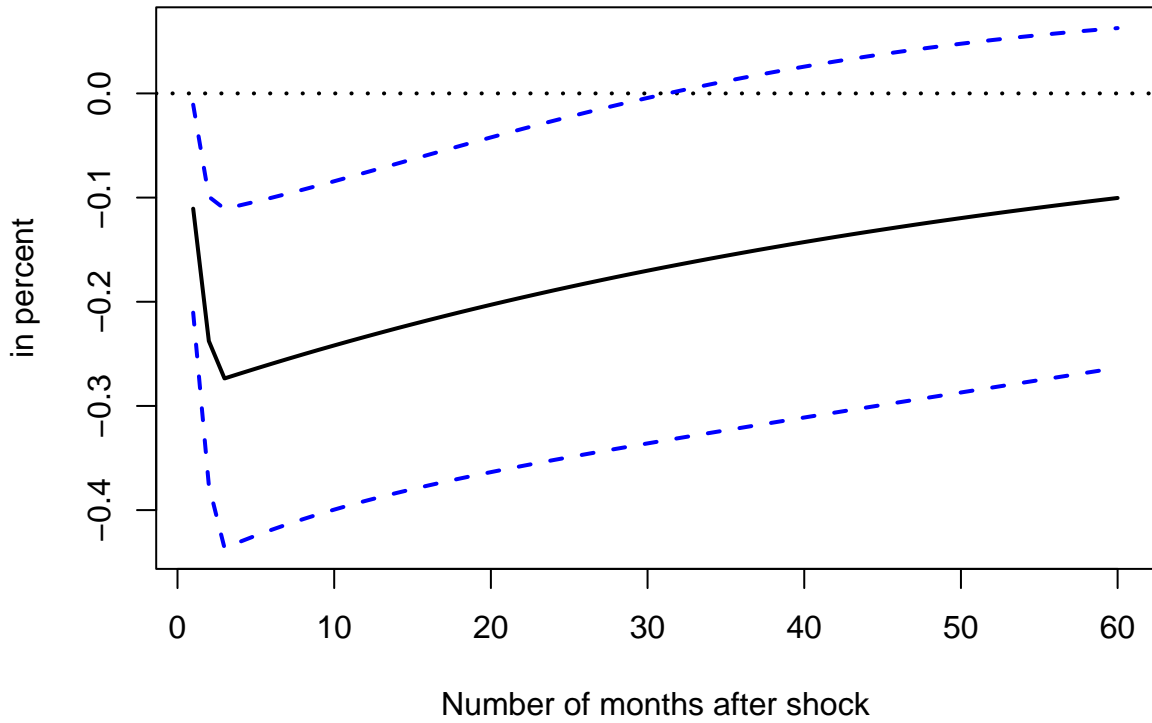


Figure 2.13: Response of industrial-production growth to monetary-policy shocks. Dashed lines correspond to the  $\pm 2$ -standard-deviation bands.

**Definition 2.19** (Markovian process). Process  $y_t$  is Markovian of order one if  $f_{Y_t|Y_{t-1}, Y_{t-2}, \dots} = f_{Y_t|Y_{t-1}}$ . More generally, it is Markovian of order  $k$  if  $f_{Y_t|Y_{t-1}, Y_{t-2}, \dots} = f_{Y_t|Y_{t-1}, \dots, Y_{t-k}}$ .

Now, remember Bayes' formula:

$$\mathbb{P}(X_2 = x, X_1 = y) = \mathbb{P}(X_2 = x|X_1 = y)\mathbb{P}(X_1 = y).$$

Using it leads to the following decomposition of our likelihood function:

$$f_{Y_T, \dots, Y_1}(y_T, \dots, y_1; \theta) = f_{Y_T|Y_{T-1}, \dots, Y_1}(y_T, \dots, y_1; \theta) \times f_{Y_{T-1}, \dots, Y_1}(y_{T-1}, \dots, y_1; \theta).$$

Using the previous expression recursively, one obtains:

$$f_{Y_T, \dots, Y_1}(y_T, \dots, y_1; \theta) = f_{Y_1}(y_1; \theta) \prod_{t=2}^T f_{Y_t|Y_{t-1}, \dots, Y_1}(y_t, \dots, y_1; \theta). \quad (2.21)$$

Let us start with the Gaussian AR(1) process (which is Markovian of order one):

$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2).$$

For  $t > 1$ :

$$f_{Y_t|Y_{t-1}, \dots, Y_1}(y_t, \dots, y_1; \theta) = f_{Y_t|Y_{t-1}}(y_t, y_{t-1}; \theta)$$

and

$$f_{Y_t|Y_{t-1}}(y_t, y_{t-1}; \theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_t - c - \phi_1 y_{t-1})^2}{2\sigma^2}\right).$$

These expressions can be plugged into Eq. (2.21). But what about  $f_{Y_1}(y_1; \theta)$ ? There exist two possibilities:

1. **Case 1:** We use the marginal distribution:  $y_1 \sim \mathcal{N}\left(\frac{c}{1-\phi_1}, \frac{\sigma^2}{1-\phi_1^2}\right)$ .
2. **Case 2:**  $y_1$  is considered to be deterministic. In a way, that means that the first observation is “sacrificed”.

For a Gaussian AR(1) process, we have:

1. **Case 1:** The (exact) log-likelihood is:

$$\begin{aligned} \log \mathcal{L}(\theta; \mathbf{y}) &= -\frac{T}{2} \log(2\pi) - T \log(\sigma) + \frac{1}{2} \log(1 - \phi_1^2) \\ &\quad - \frac{(y_1 - c/(1 - \phi_1))^2}{2\sigma^2/(1 - \phi_1^2)} - \sum_{t=2}^T \left[ \frac{(y_t - c - \phi_1 y_{t-1})^2}{2\sigma^2} \right]. \end{aligned} \quad (2.22)$$

The Maximum Likelihood Estimator of  $\theta = [c, \phi_1, \sigma^2]$  is obtained by numerical optimization.

2. **Case 2:** The (conditional) log-likelihood is:

$$\begin{aligned} \log \mathcal{L}^*(\theta; \mathbf{y}) &= -\frac{T-1}{2} \log(2\pi) - (T-1) \log(\sigma) \\ &\quad - \sum_{t=2}^T \left[ \frac{(y_t - c - \phi_1 y_{t-1})^2}{2\sigma^2} \right]. \end{aligned} \quad (2.23)$$

Exact MLE and conditional MLE have the same asymptotic (i.e. large-sample) distribution. Indeed, when the process is stationary,  $f_{Y_1}(y_1; \theta)$  makes a relatively negligible contribution to  $\log \mathcal{L}(\theta; \mathbf{y})$ .

The conditional MLE has a substantial advantage: in the Gaussian case, the conditional MLE is simply obtained by OLS. Indeed, let us introduce the notations:

$$Y = \begin{bmatrix} y_2 \\ \vdots \\ y_T \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & y_1 \\ \vdots & \vdots \\ 1 & y_{T-1} \end{bmatrix}.$$

Eq. (2.23) then rewrites:

$$\begin{aligned} \log \mathcal{L}^*(\theta; \mathbf{y}) &= -\frac{T-1}{2} \log(2\pi) - (T-1) \log(\sigma) \\ &\quad - \frac{1}{2\sigma^2} (Y - X[c, \phi_1]')' (Y - X[c, \phi_1]'), \end{aligned} \quad (2.24)$$

which is maximised for:

$$[\hat{c}, \hat{\phi}_1]' = (X'X)^{-1} X'Y \quad (2.25)$$

$$\begin{aligned} \hat{\sigma}^2 &= \frac{1}{T-1} \sum_{t=2}^T (y_t - \hat{c} - \hat{\phi}_1 y_{t-1})^2 \\ &= \frac{1}{T-1} Y'(I - X(X'X)^{-1}X')Y. \end{aligned} \quad (2.26)$$

Let us turn to the case of an AR(p) process. We have:

$$\begin{aligned} \log \mathcal{L}(\theta; \mathbf{y}) &= \log f_{Y_p, \dots, Y_1}(y_p, \dots, y_1; \theta) + \\ &\quad \underbrace{\sum_{t=p+1}^T \log f_{Y_t|Y_{t-1}, \dots, Y_{t-p}}(y_t, \dots, y_{t-p}; \theta)}_{\log \mathcal{L}^*(\theta; \mathbf{y})}. \end{aligned}$$

where  $f_{Y_p, \dots, Y_1}(y_p, \dots, y_1; \theta)$  is the marginal distribution of  $\mathbf{y}_{1:p} := [y_p, \dots, y_1]'$ . The marginal distribution of  $\mathbf{y}_{1:p}$  is Gaussian; it is therefore fully characterised by its mean and covariance matrix:

$$\begin{aligned} \mathbb{E}(\mathbf{y}_{1:p}) &= \frac{c}{1 - \phi_1 - \dots - \phi_p} \mathbf{1}_{p \times 1} \\ \text{Var}(\mathbf{y}_{1:p}) &= \begin{bmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_{p-1} \\ \gamma_1 & \gamma_0 & \dots & \gamma_{p-2} \\ \vdots & & \ddots & \vdots \\ \gamma_{p-1} & \gamma_{p-2} & \dots & \gamma_0 \end{bmatrix}, \end{aligned}$$

where the  $\gamma_i$ 's are computed using the Yule-Walker equations (Eq. (2.15)). Note that they depend, in a non-linear way, on the model parameters. Hence, the maximization of the exact log-likelihood necessitates numerical optimization procedures. By contrast, the maximization of the conditional log-likelihood  $\log \mathcal{L}^*(\theta; \mathbf{y})$  only requires OLS, using Eqs. (2.25) and (2.26), with:

$$Y = \begin{bmatrix} y_{p+1} \\ \vdots \\ y_T \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & y_p & \cdots & y_1 \\ \vdots & \vdots & & \vdots \\ 1 & y_{T-1} & \cdots & y_{T-p} \end{bmatrix}.$$

Again, for stationary processes, conditional and exact MLE have the same asymptotic (large-sample) distribution. In small samples, the OLS formula is however biased. Indeed, consider the regression (where  $y_t$  follows an AR(p) process):

$$y_t = \beta' \mathbf{x}_t + \varepsilon_t, \quad (2.27)$$

with  $\mathbf{x}_t = [1, y_{t-1}, \dots, y_{t-p}]'$  and  $\beta = [c, \phi_1, \dots, \phi_p]'$ .

The bias results from the fact that  $\mathbf{x}_t$  correlates to the  $\varepsilon_s$ 's for  $s < t$ . To be sure:

$$\mathbf{b} = \beta + (X'X)^{-1}X'\varepsilon, \quad (2.28)$$

and because of the specific form of  $X$ , we have non-zero correlation between  $\mathbf{x}_t$  and  $\varepsilon_s$  for  $s < t$ , therefore  $\mathbb{E}[(X'X)^{-1}X'\varepsilon] \neq 0$ . Again, asymptotically, the previous expectation goes to zero, and we have:

**Proposition 2.10** (Large-sample properties of the OLS estimator of AR(p) models). *Assume  $\{y_t\}$  follows the AR(p) process:*

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

where  $\{\varepsilon_t\}$  is an i.i.d. white noise process. If  $\mathbf{b}$  is the OLS estimator of  $\beta$  (Eq. (2.27)), we have:

$$\sqrt{T}(\mathbf{b} - \beta) = \underbrace{\left[ \frac{1}{T} \sum_{t=p}^T \mathbf{x}_t \mathbf{x}_t' \right]^{-1}}_{\xrightarrow{p} \mathbf{Q}^{-1}} \underbrace{\sqrt{T} \left[ \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \varepsilon_t \right]}_{\xrightarrow{d} \mathcal{N}(0, \sigma^2 \mathbf{Q})},$$

where  $\mathbf{Q} = \text{plim } \frac{1}{T} \sum_{t=p}^T \mathbf{x}_t \mathbf{x}_t' = \text{plim } \frac{1}{T} \sum_{t=1}^T \mathbf{x}_t \mathbf{x}_t'$  is given by:

$$\mathbf{Q} = \begin{bmatrix} 1 & \mu & \mu & \cdots & \mu \\ \mu & \gamma_0 + \mu^2 & \gamma_1 + \mu^2 & \cdots & \gamma_{p-1} + \mu^2 \\ \mu & \gamma_1 + \mu^2 & \gamma_0 + \mu^2 & \cdots & \gamma_{p-2} + \mu^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mu & \gamma_{p-1} + \mu^2 & \gamma_{p-2} + \mu^2 & \cdots & \gamma_0 + \mu^2 \end{bmatrix}. \quad (2.29)$$

*Proof.* Rearranging Eq. (2.47), we have:

$$\sqrt{T}(\mathbf{b} - \beta) = (X'X/T)^{-1} \sqrt{T}(X'\varepsilon/T).$$

Let us consider the autocovariances of  $\mathbf{v}_t = \mathbf{x}_t \varepsilon_t$ , denoted by  $\gamma_j^v$ . Using the fact that  $\mathbf{x}_t$  is a linear combination of past  $\varepsilon_t$ 's and that  $\varepsilon_t$  is a white noise, we get that  $\mathbb{E}(\varepsilon_t \mathbf{x}_t) = 0$ . Therefore

$$\gamma_j^v = \mathbb{E}(\varepsilon_t \varepsilon_{t-j} \mathbf{x}_t \mathbf{x}_{t-j}').$$

If  $j > 0$ , we have

$$\begin{aligned} \mathbb{E}(\varepsilon_t \varepsilon_{t-j} \mathbf{x}_t \mathbf{x}_{t-j}') &= \mathbb{E}(\mathbb{E}[\varepsilon_t \varepsilon_{t-j} \mathbf{x}_t \mathbf{x}_{t-j}' | \varepsilon_{t-j}, \mathbf{x}_t, \mathbf{x}_{t-j}]) \\ &= \mathbb{E}(\varepsilon_{t-j} \mathbf{x}_t \mathbf{x}_{t-j}' \mathbb{E}[\varepsilon_t | \varepsilon_{t-j}, \mathbf{x}_t, \mathbf{x}_{t-j}]) = 0. \end{aligned}$$

Note that, for  $j > 0$ , we have  $\mathbb{E}[\varepsilon_t | \varepsilon_{t-j}, \mathbf{x}_t, \mathbf{x}_{t-j}] = 0$  because  $\{\varepsilon_t\}$  is an i.i.d. white noise sequence. If  $j = 0$ , we have:

$$\gamma_0^v = \mathbb{E}(\varepsilon_t^2 \mathbf{x}_t \mathbf{x}_t') = \mathbb{E}(\varepsilon_t^2) \mathbb{E}(\mathbf{x}_t \mathbf{x}_t') = \sigma^2 \mathbf{Q}.$$

The convergence in distribution of  $\sqrt{T}(X'\varepsilon/T) = \sqrt{T} \frac{1}{T} \sum_{t=1}^T v_t$  results from Theorem 2.1 (applied on  $\mathbf{v}_t = \mathbf{x}_t \varepsilon_t$ ), using the  $\gamma_j^v$  computed above.  $\square$

These two cases (exact or conditional log-likelihoods) can be implemented when asking R to fit an AR process by means of function `arima`. Let us for instance use the output gap of the US3var dataset (US quarterly data, covering the period 1959:2 to 2015:1, used in Gouriéroux et al. (2017)).

```
library(AEC)
y <- US3var$y.gdp.gap
ar3.Case1 <- arima(y, order = c(3,0,0), method="ML")
ar3.Case2 <- arima(y, order = c(3,0,0), method="CSS")
rbind(ar3.Case1$coef, ar3.Case2$coef)
```

```
##           ar1           ar2           ar3  intercept
## [1,]  1.191267 -0.08934705 -0.1781163 -0.9226007
## [2,]  1.192003 -0.08811150 -0.1787662 -1.0341696
```

The two sets of estimated coefficients appear to be very close to each other.

Let us now turn to Moving-Average processes. Start with the MA(1):

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2).$$

The  $\varepsilon_t$ 's are easily computed recursively, starting with  $\varepsilon_t = y_t - \mu - \theta_1 \varepsilon_{t-1}$ . We obtain:

$$\varepsilon_t = y_t - \theta_1 y_{t-1} + \theta_1^2 y_{t-2} - \dots + (-1)^{t-1} \theta_1^{t-1} y_1 + (-1)^t \theta_1^t \varepsilon_0.$$

Assume that one wants to recover the sequence of  $\{\varepsilon_t\}$ 's based on observed values of  $y_t$  (from date 1 to date  $t$ ). One can use the previous expression, but what value should be used for  $\varepsilon_0$ ? If one does not use the true value of  $\varepsilon_0$  but 0 (say), one does not obtain  $\varepsilon_t$ , but only an estimate of it ( $\hat{\varepsilon}_t$ , say), with:

$$\hat{\varepsilon}_t = \varepsilon_t - (-1)^t \theta_1^t \varepsilon_0.$$

Clearly, if  $|\theta_1| < 1$ , then the error becomes small for large  $t$ . Formally, when  $|\theta_1| < 1$ , we have:

$$\hat{\varepsilon}_t \xrightarrow{p} \varepsilon_t.$$

Hence, when  $|\theta_1| < 1$ , a consistent estimate of the conditional log-likelihood is given by:

$$\log \hat{\mathcal{L}}^*(\theta; \mathbf{y}) = -\frac{T}{2} \log(2\pi) - \frac{T}{2} \log(\sigma^2) - \sum_{t=1}^T \frac{\hat{\varepsilon}_t^2}{2\sigma^2}. \quad (2.30)$$

Loosely speaking, if  $|\theta_1| < 1$  and if  $T$  is sufficiently large:

$$\text{approximate conditional MLE} \approx \text{exact MLE}.$$

Note that  $\hat{\mathcal{L}}^*(\theta; \mathbf{y})$  is a complicated nonlinear function of  $\mu$  and  $\theta$ . Its maximization therefore has to be based on numerical optimization procedures.

Let us not consider the case of a Gaussian MA( $q$ ) process:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2). \quad (2.31)$$

Let us assume that this process is an **invertible MA process**. That is, assume that the roots of:

$$\lambda^q + \theta_1 \lambda^{q-1} + \dots + \theta_{q-1} \lambda + \theta_q = 0 \quad (2.32)$$

lie strictly inside of the unit circle. In this case, the polynomial form  $\Theta(L) = 1 + \theta_1 L + \dots + \theta_q L^q$  is *invertible* and Eq. (2.31) writes:

$$\varepsilon_t = \Theta(L)^{-1}(y_t - \mu),$$

which implies that, if we knew all past values of  $y_t$ , we would also know  $\varepsilon_t$ . In this case, we can consistently estimate the  $\varepsilon_t$ 's by recursively computing the  $\hat{\varepsilon}_t$ 's as follows (for  $t > 0$ ):

$$\hat{\varepsilon}_t = y_t - \mu - \theta_1 \hat{\varepsilon}_{t-1} - \dots - \theta_q \hat{\varepsilon}_{t-q}, \quad (2.33)$$

with

$$\hat{\varepsilon}_0 = \dots = \hat{\varepsilon}_{-q+1} = 0. \quad (2.34)$$

In this context, a consistent estimate of the conditional log-likelihood is still given by Eq. (2.30), using Eqs. (2.33) and (2.34) to recursively compute the  $\hat{\varepsilon}_t$ 's.

Note that we could determine the exact likelihood of an MA process. Indeed, vector  $\mathbf{y} = [y_1, \dots, y_T]'$  is a Gaussian-distributed vector of mean  $\mu = [\mu, \dots, \mu]'$  and of variance:

$$\Omega = \begin{bmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_q & \mathbf{0} & \dots & \mathbf{0} \\ \gamma_1 & \gamma_0 & \gamma_1 & & \ddots & \ddots & \vdots \\ \vdots & \gamma_1 & \ddots & \ddots & & \ddots & \mathbf{0} \\ \gamma_q & & \ddots & & & & \gamma_q \\ \mathbf{0} & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \gamma_1 & \gamma_0 & \gamma_1 \\ \mathbf{0} & \dots & \mathbf{0} & \gamma_q & \dots & \gamma_1 & \gamma_0 \end{bmatrix},$$

where the  $\gamma_j$ 's are given by Eq. (2.7). The p.d.f. of  $\mathbf{y}$  is then given by (see Prop. 3.18):

$$(2\pi)^{-T/2} |\Omega|^{-1/2} \exp \left( -\frac{1}{2} (\mathbf{y} - \mu)' \Omega^{-1} (\mathbf{y} - \mu) \right).$$

For large samples, the computation of this likelihood however becomes numerically demanding.

Finally, let us consider the MLE of an ARMA( $p, q$ ) processes:

$$y_t = c + \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q}, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2).$$

If the MA part of this process is invertible, the log-likelihood function can be consistently approximated by its conditional counterpart (of the form of Eq. (2.30)), using consistent estimates  $\hat{\varepsilon}_t$  of the  $\varepsilon_t$ . The  $\hat{\varepsilon}_t$ 's are computed recursively as:

$$\hat{\varepsilon}_t = y_t - c - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p} - \theta_1 \hat{\varepsilon}_{t-1} - \dots - \theta_q \hat{\varepsilon}_{t-q}, \quad (2.35)$$

given some initial conditions, for instance:

- $\hat{\varepsilon}_0 = \dots = \hat{\varepsilon}_{-q+1} = 0$  and  $y_0 = \dots = y_{-p+1} = \mathbb{E}(y_i) = \mu$ . (Recursions in Eq. (2.35) then start for  $t = 1$ .)
- $\hat{\varepsilon}_p = \dots = \hat{\varepsilon}_{p-q+1} = 0$  and actual values of the  $y_i$ 's for  $i \in [1, p]$ . In that case, the first  $p$  observations of  $y_t$  will not be used. Recursions in Eq. (2.35) then start for  $t = p + 1$ .

## 2.2.9 Specification choice

The previous section explains how to fit a given ARMA specification. But how to choose an appropriate specification? A possibility is to employ the (P)ACF approach (see Figure 2.7). However, the previous approach leads to either an AR or a MA process (and not an ARMA process). If one wants to consider various ARMA( $p, q$ ) specifications, for  $p \in \{1, \dots, P\}$  and  $q \in \{1, \dots, Q\}$ , say, then one can resort to **information criteria**.

In general, when choosing a specification, one faces the following dilemma:

- Too rich a specification may lead to “overfitting”/misspecification, implying additional estimation errors (in out-of-sample forecasts).
- Too simple a specification may lead to potential omission of valuable information (e.g., contained in older lags).

The lag selection approach based on the so-called **information criteria** consists in maximizing the fit of the data, but adding a penalty for the “richness” of the model. More precisely, using this approach amounts to minimizing a loss function that (a) negatively depends on the fitting errors and (b) positively depends on the number of parameters in the model.



**Definition 2.20** (Information Criteria). The Akaike (AIC), Hannan-Quinn (HQ) and Schwarz information (BIC) criteria are of the form

$$c^{(i)}(k) = \underbrace{\frac{-2 \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y})}{T}}_{\text{decreases w.r.t. } k} + \underbrace{\frac{k\phi^{(i)}(T)}{T}}_{\text{increases w.r.t. } k},$$

with  $(i) \in \{AIC, HQ, BIC\}$  and where  $\hat{\theta}_T(k)$  denotes the ML estimate of  $\theta_0(k)$ , which is a vector of parameters of length  $k$ .

Criterion (i)		$\phi^{(i)}(T)$
Akaike	AIC	2
Hannan-Quinn	HQ	$2 \log(\log(T))$
Schwarz	BIC	$\log(T)$

The lag suggested by criterion (i) is then given by:

$$\hat{k}^{(i)} = \underset{k}{\operatorname{argmin}} \quad c^{(i)}(k).$$

In the case of an ARMA(p,q) process,  $k = 2 + p + q$ .

**Proposition 2.11** (Consistency of the criteria-based lag selection). *The lag selection procedure is consistent (see Def. 3.8) if*

$$\lim_{T \rightarrow \infty} \phi(T) = \infty \quad \text{and} \quad \lim_{T \rightarrow \infty} \phi(T)/T = 0.$$

*This is notably the case of the HQ and the BIC criteria.*

*Proof.* The true number of lags is denoted by  $k_0$ . We will show that  $\lim_{T \rightarrow \infty} \mathbb{P}(\hat{k}_T \neq k_0) = 0$ .

- Case  $k < k_0$ : The model with  $k$  parameter is misspecified, therefore:

$$\operatorname{plim}_{T \rightarrow \infty} \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y})/T < \operatorname{plim}_{T \rightarrow \infty} \log \mathcal{L}(\hat{\theta}_T(k_0); \mathbf{y})/T.$$

Hence, if  $\lim_{T \rightarrow \infty} \phi(T)/T = 0$ , we have:  $\lim_{T \rightarrow \infty} \mathbb{P}(c(k_0) \geq c(k)) \rightarrow 0$  and

$$\lim_{T \rightarrow \infty} \mathbb{P}(\hat{k} < k_0) \leq \lim_{T \rightarrow \infty} \mathbb{P}\{c(k_0) \geq c(k) \text{ for some } k < k_0\} = 0.$$

- Case  $k > k_0$ : under the null hypothesis, the likelihood ratio (LR) test statistic (see Def. ??) satisfies:

$$2 \left( \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y}) - \log \mathcal{L}(\hat{\theta}_T(k_0); \mathbf{y}) \right) \sim \chi^2(k - k_0).$$

If  $\lim_{T \rightarrow \infty} \phi(T) = \infty$ , we have:  $\operatorname{plim}_{T \rightarrow \infty} -2 \left( \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y}) - \log \mathcal{L}(\hat{\theta}_T(k_0); \mathbf{y}) \right) / \phi(T) = 0$ . Hence  $\operatorname{plim}_{T \rightarrow \infty} T[c(k_0) - c(k)]/\phi(T) \leq -1$  and  $\lim_{T \rightarrow \infty} \mathbb{P}(c(k_0) \geq c(k)) \rightarrow 0$ , which implies, in the same spirit as before, that  $\lim_{T \rightarrow \infty} \mathbb{P}(\hat{k} > k_0) = 0$ .

Therefore,  $\lim_{T \rightarrow \infty} \mathbb{P}(\hat{k} = k_0) = 1$ . □

**Example 2.5** (Linear regression). Consider a linear regression with normal disturbances:

$$y_t = \mathbf{x}_t' \beta + \varepsilon_t, \quad \varepsilon_t \sim i.i.d. \mathcal{N}(0, \sigma^2).$$

The associated log-likelihood is of the form of Eq. (2.30). In that case, we have:

$$\begin{aligned} c^{(i)}(k) &= \frac{-2 \log \mathcal{L}(\hat{\theta}_T(k); \mathbf{y})}{T} + \frac{k\phi^{(i)}(T)}{T} \\ &\approx \log(2\pi) + \log(\widehat{\sigma^2}) + \frac{1}{T} \sum_{t=1}^T \frac{\varepsilon_t^2}{\widehat{\sigma^2}} + \frac{k\phi^{(i)}(T)}{T}. \end{aligned}$$

For a large  $T$ , for all consistent estimation scheme, we have:

$$\widehat{\sigma^2} \approx \frac{1}{T} \sum_{t=1}^T \varepsilon_t^2 = SSR/T.$$

Hence  $\hat{k}^{(i)} \approx \underset{k}{\operatorname{argmin}} \quad \log(SSR/T) + \frac{k\phi^{(i)}(T)}{T}$ .

**Example 2.6** (Swiss GDP growth). Consider a long historical time series of the Swiss GDP growth (see Figure 2.3), taken from the Jordà et al. (2017) dataset. Let us look for the best ARMA specification using the AIC criteria:

```
library(AEC);data(JST)
data <- subset(JST,iso=="CHE")
T <- dim(data)[1]
y <- c(NaN,log(data$gdp[2:T]/data$gdp[1:(T-1)]))
# Use AIC criteria to look for appropriate specif:
max.p <- 3;max.q <- 3;
all.AIC <- NULL
for(p in 0:max.p){
  for(q in 0:max.q){
    res <- arima(y,order=c(p,0,q))
    if(res$aic<min(all.AIC)){best.p<-p;best.q<-q}
    all.AIC <- c(all.AIC,res$aic)}}
print(c(best.p,best.q))
```

```
## [1] 1 0
```

The best specification therefore is an AR(1) model. That is, although an AR(2) (say) would result in a better fit of the data, the fit improvement is not be large enough to compensate for the additional AIC cost associated with an additional parameter.

## 2.3 Multivariate models

This section presents Vector Auto-Regressive Moving-Average (SVARMA) models. These models are widely used in macroeconomic analysis. While simple and easy to estimate, they make it possible to conveniently capture the dynamics of complex multivariate systems. VAR popularity is notably due to Sims (1980)'s influential work. A nice survey if proposed by Stock and Watson (2016).

In economics, VAR models are often employed in order to identify *structural* shocks, that are independent primitive exogenous forces that drive economic variables (Ramey (2016)). They are often given a specific economic meaning (e.g., demand and supply shocks).

Working with these models (VAR and VARMA models) often is often based on two steps: in a first step, the **reduced-form** version of the model is estimated; in a second step, **structural shocks** are identified and IRFs are produced.

→

### 2.3.1 Definition of VARs (and SVARMA) models

**Definition 2.21** ((S)VAR model). Let  $y_t$  denote a  $n \times 1$  vector of random variables. Process  $y_t$  follows a  $p^{th}$ -order (S)VAR if, for all  $t$ , we have

$$\begin{aligned} VAR: y_t &= c + \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p} + \varepsilon_t, \\ SVAR: y_t &= c + \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p} + B\eta_t, \end{aligned} \quad (2.36)$$

with  $\varepsilon_t = B\eta_t$ , where  $\{\eta_t\}$  is a white noise sequence whose components are mutually and serially independent.

The first line of Eq. (2.36) corresponds to the **reduced-form** of the VAR model (**structural form** for the second line).

While the structural shocks (the components of  $\eta_t$ ) are mutually uncorrelated, this is not the case of the *innovations*, that are the components of  $\varepsilon_t$ . However, in boths cases, vectors  $\eta_t$  and  $\varepsilon_t$  are serially correlated (through time).

As was the case for univariate models, VARs can be extended with MA terms in  $\eta_t$ :

**Definition 2.22** ((S)VARMA model). Let  $y_t$  denote a  $n \times 1$  vector of random variables. Process  $y_t$  follows a VARMA model of order  $(p,q)$  if, for all  $t$ , we have

$$\begin{aligned} VARMA: \quad y_t &= c + \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p} + \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \dots + \Theta_q, \\ SVARMA: \quad y_t &= c + \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p} + B_0 \eta_t + B_1 \eta_{t-1} + \dots + B_q \eta_{t-q}, \end{aligned} \quad (2.37)$$

with  $\varepsilon_t = B_0 \eta_t$  (and  $B_j = \Theta_j B_0$ , for  $j \geq 0$ ), where  $\{\eta_t\}$  is a white noise sequence whose components are mutually and serially independent.

### 2.3.2 IRFs in SVARMA

One of the main objectives of macro-econometrics is to derive IRFs, that represent the dynamic effects of structural shocks (components of  $\eta_t$ ) through the system of variables  $y_t$ .

Formally, an IRF is a difference in conditional expectations:

$$\boxed{\Psi_{i,j,h} = \mathbb{E}(y_{i,t+h} | \eta_{j,t} = 1) - \mathbb{E}(y_{i,t+h})}$$

(effect on  $y_{i,t+h}$  of a one-unit shock on  $\eta_{j,t}$ ).

If the dynamics of process  $y_t$  can be described as a VARMA model, and if  $y_t$  is covariance stationary (see Def. 2.4), then  $y_t$  admits the following infinite MA representation (MA( $\infty$ )):

$$y_t = \mu + \sum_{h=0}^{\infty} \Psi_h \eta_{t-h}. \quad (2.38)$$

This is also the Wold decomposition of process  $\{y_t\}$  (see Theorem 2.3).

Estimating IRFs amounts to estimating the  $\Psi_h$ 's. In general, there exist three main approaches for that:

- Calibrate and solve a (purely structural) Dynamic Stochastic General Equilibrium (DSGE) model at the first order (linearization). The solution takes the form of Eq. (2.38).
- Directly estimate the  $\Psi_h$  based on **projection approaches** (see Section 2.3.11).
- Approximate the infinite MA representation by estimating a parsimonious type of model, e.g. **VAR(MA) models** (see Section 2.3.4). Once a (Structural) VARMA representation is obtained, Eq. (2.38) is easily deduced. For that, one can use the same recursive algorithm as for univariate processes (see Prop. 2.8).

Typically, consider the AR(2) case. The first steps of the algorithm mentioned in the last bullet point are as follows:

$$\begin{aligned} y_t &= \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + B \eta_t \\ &= \Phi_1 (\Phi_1 y_{t-2} + \Phi_2 y_{t-3} + B \eta_{t-1}) + \Phi_2 y_{t-2} + B \eta_t \\ &= B \eta_t + \Phi_1 B \eta_{t-1} + (\Phi_2 + \Phi_1^2) y_{t-2} + \Phi_1 \Phi_2 y_{t-3} \\ &= B \eta_t + \Phi_1 B \eta_{t-1} + (\Phi_2 + \Phi_1^2) (\Phi_1 y_{t-3} + \Phi_2 y_{t-4} + B \eta_{t-2}) + \Phi_1 \Phi_2 y_{t-3} \\ &= \underbrace{B}_{=\Psi_0} \eta_t + \underbrace{\Phi_1 B}_{=\Psi_1} \eta_{t-1} + \underbrace{(\Phi_2 + \Phi_1^2) B}_{=\Psi_2} \eta_{t-2} + f(y_{t-3}, y_{t-4}). \end{aligned}$$

In particular, we have  $B = \Psi_0$ . Matrix  $B$  indeed captures the contemporaneous impact of  $\eta_t$  on  $y_t$ . That is why matrix  $B$  is sometimes called *impulse matrix*.

**Example 2.7** (IRFs of an SVARMA model). Consider the following VARMA(1,1) model:

$$y_t = \underbrace{\begin{bmatrix} 0.5 & 0.3 \\ -0.4 & 0.7 \end{bmatrix}}_{\Phi_1} y_{t-1} + \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}}_B \eta_t + \underbrace{\begin{bmatrix} 2 & 0 \\ 1 & 0.5 \end{bmatrix}}_{\Theta_1} \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}}_B \eta_{t-1}. \quad (2.39)$$

We can use function `simul.VARMA` of package `AEC` to produce IRFs (using `indic.IRF=1` in the list of arguments):

```

library(AEC)
distri <- list(type=c("gaussian","gaussian"),df=c(4,4))
n <- length(distri$type) # dimension of y_t
nb.sim <- 30
eps <- simul.distri(distri,nb.sim)
Phi <- array(NA,c(n,n,1))
Phi[, ,1] <- matrix(c(.5,-.4,.3,.7),2,2)
p <- dim(Phi)[3]
Theta <- array(NA,c(n,n,1))
Theta[, ,1] <- -matrix(c(2,1,0,.5),2,2)
q <- dim(Theta)[3]
Mu <- rep(0,n)
C <- matrix(c(1,-1,2,1),2,2)
Model <- list(
  Mu = Mu,Phi = Phi,Theta = Theta,C = C,distri = distri)
Y0 <- rep(0,n)
eta0 <- c(1,0)
res.sim.1 <- simul.VARMA(Model,nb.sim,Y0,eta0,indic.IRF=1)
eta0 <- c(0,1)
res.sim.2 <- simul.VARMA(Model,nb.sim,Y0,eta0,indic.IRF=1)
par(plt=c(.15,.95,.25,.8))
par(mfrow=c(2,2))
plot(res.sim.1$Y[1,],las=1,
      type="l",lwd=3,xlab="",ylab="",
      main=expression(paste("Response of ",y[1,"*","t"],
                             " to a one-unit increase in ",eta[1],sep="")))
abline(h=0,col="grey",lty=3)
plot(res.sim.2$Y[1,],las=1,
      type="l",lwd=3,xlab="",ylab="",
      main=expression(paste("Response of ",y[1,"*","t"],
                             " to a one-unit increase in ",eta[2],sep="")))
abline(h=0,col="grey",lty=3)
plot(res.sim.1$Y[2,],las=1,
      type="l",lwd=3,xlab="",ylab="",
      main=expression(paste("Response of ",y[2,"*","t"],
                             " to a one-unit increase in ",eta[1],sep="")))
abline(h=0,col="grey",lty=3)
plot(res.sim.2$Y[2,],las=1,
      type="l",lwd=3,xlab="",ylab="",
      main=expression(paste("Response of ",y[2,"*","t"],
                             " to a one-unit increase in ",eta[2],sep="")))
abline(h=0,col="grey",lty=3)

```

### 2.3.3 Covariance-stationary VARMA models

Let's come back to the infinite MA case (Eq. (2.38)):

$$y_t = \mu + \sum_{h=0}^{\infty} \Psi_h \eta_{t-h}.$$

For  $y_t$  to be covariance-stationary (and ergodic for the mean), it has to be the case that

$$\sum_{i=0}^{\infty} \|\Psi_i\| < \infty, \quad (2.40)$$

where  $\|A\|$  denotes a norm of the matrix  $A$  (e.g.  $\|A\| = \sqrt{\text{tr}(AA')}$ ). This notably implies that if  $y_t$  is stationary (and ergodic for the mean), then  $\|\Psi_h\| \rightarrow 0$  when  $h$  gets large.

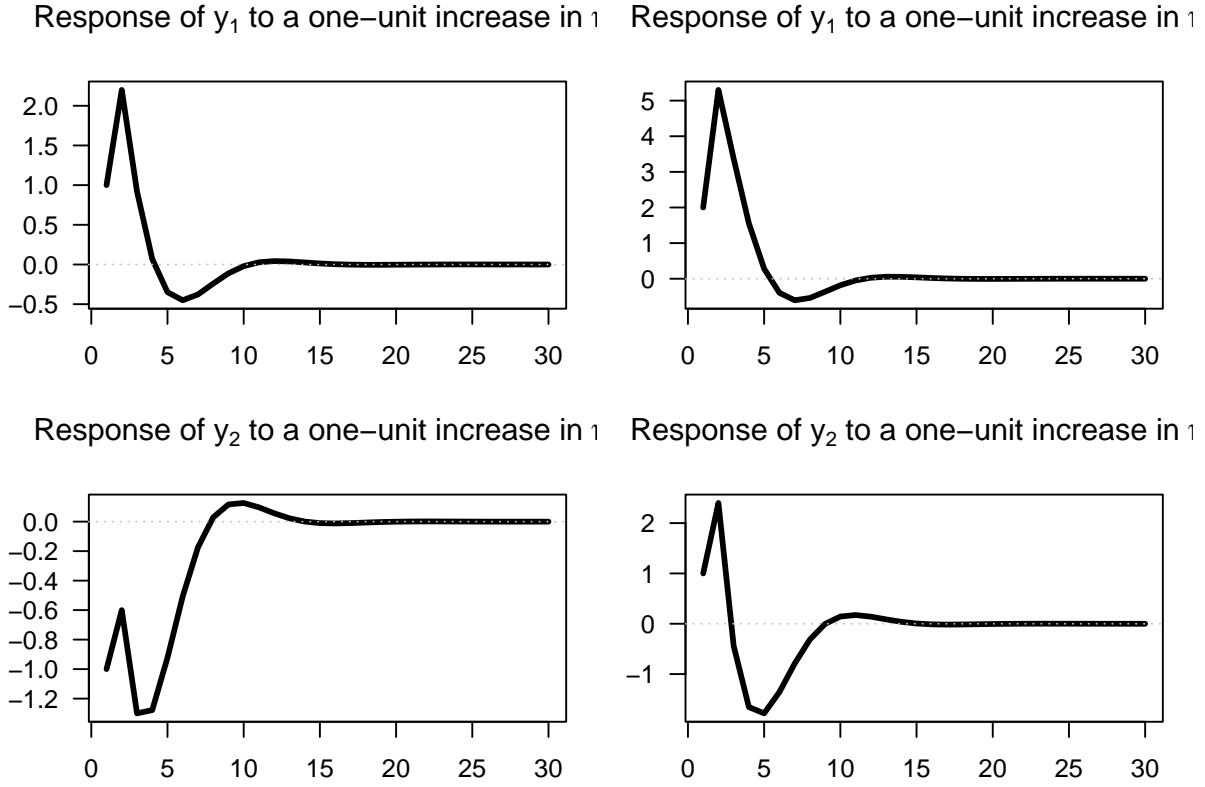


Figure 2.14: Impulse response functions

What should be satisfied by  $\Phi_k$ 's and  $\Theta_k$ 's for a VARMA-based process (Eq. (??)) to be stationary? The conditions will be similar to that we had in the univariate case (see Prop. 2.6). Let us introduce the following notations:

$$\begin{aligned}
 y_t &= c + \underbrace{\Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p}}_{\text{AR component}} + \underbrace{B\eta_t + \Theta_1 B\eta_{t-1} + \dots + \Theta_q B\eta_{t-q}}_{\text{MA component}} \\
 &\Leftrightarrow \underbrace{(I - \Phi_1 L - \dots - \Phi_p L^p)}_{=\Phi(L)} y_t = c + \underbrace{(I - \Theta_1 L - \dots - \Theta_q L^q)}_{=\Theta(L)} B\eta_t.
 \end{aligned} \tag{2.41}$$

Process  $y_t$  is stationary iff the roots of  $\det(\Phi(z)) = 0$  are strictly outside the unit circle or, equivalently, iff the eigenvalues of

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \dots & \Phi_p \\ I & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \tag{2.42}$$

lie strictly within the unit circle. Hence, as was the case for univariate processes, the covariance-stationarity of a VARMA model depends only on the specification of its AR part.

Let's derive the first two unconditional moments of a (covariance-stationary) VARMA process.

Based on Eq. (2.41), we have  $\mathbb{E}(\Phi(L)y_t) = c$ , which gives  $\Phi(1)\mathbb{E}(y_t) = c$ , or::

$$\mathbb{E}(y_t) = (I - \Phi_1 - \dots - \Phi_p)^{-1}c.$$

The autocovariances of  $y_t$  can be deduced from the infinite MA representation (Eq. (2.38)). We have:

$$\gamma_j \equiv \text{Cov}(y_t, y_{t-j}) = \sum_{i=j}^{\infty} \Psi_i \Psi'_{i-j}.$$

(Note that this infinite sum exists as soon as Eq. (2.40) is satisfied.)

Conditional means and autocovariances can also be deduced from Eq. (2.38). For  $0 \leq h$  and  $0 \leq h_1 \leq h_2$ :

$$\begin{aligned}\mathbb{E}_t(y_{t+h}) &= \mu + \sum_{k=0}^{\infty} \Psi_{k+h} \eta_{t-k} \\ \text{Cov}_t(y_{t+1+h_1}, y_{t+1+h_2}) &= \sum_{k=0}^{h_1} \Psi_k \Psi'_{k+h_2-h_1}.\end{aligned}$$

The previous formula implies in particular that the forecasting error  $y_{t+h} - \mathbb{E}_t(y_{t+h})$  has a variance equal to:

$$\text{Var}_t(y_{t+h}) = \sum_{k=1}^h \Psi_k \Psi'_k.$$

Because the  $\eta_t$  are mutually and serially independent (and therefore uncorrelated), we have:

$$\text{Var}(\Psi_k \eta_{t-k}) = \text{Var}\left(\sum_{i=1}^n \psi_{k,i} \eta_{i,t-k}\right) = \sum_{i=1}^n \psi_{k,i} \psi'_{k,i},$$

where  $\psi_{k,i}$  denotes the  $i^{\text{th}}$  column of  $\Psi_k$ .

This suggests the following decomposition of the variance of the forecast error (called **variance decomposition**):

$$\text{Var}_t(y_{t+h}) = \sum_{i=1}^n \underbrace{\sum_{k=1}^h \psi_{k,i} \psi'_{k,i}}_{\text{Contribution of } \eta_{i,t}}.$$

Let us now turn to the estimation of VAR(MA) models.

If there is a MA component, OLS regressions yield biased estimates (even for asymptotically large samples).

Assume  $y_t$  follows a VARMA(1,1) model. We have:

$$y_{i,t} = \phi_i y_{t-1} + \varepsilon_{i,t},$$

where  $\phi_i$  is the  $i^{\text{th}}$  row of  $\Phi_1$ , and where  $\varepsilon_{i,t}$  is a linear combination of  $\eta_t$  and  $\eta_{t-1}$ .

Since  $y_{t-1}$  (the regressor) is correlated to  $\eta_{t-1}$ , it is also correlated to  $\varepsilon_{i,t}$ .

The OLS regression of  $y_{i,t}$  on  $y_{t-1}$  yields a biased estimator of  $\phi_i$ . Hence, SVARMA models cannot be consistently estimated by simple OLS regressions (contrary to VAR models, as we will see in the next section); instrumental-variable approaches can be employed to estimate SVARMA models.

### 2.3.4 VAR estimation

This section discusses the estimation of VAR models. (The estimation of SVARMA models is more challenging, see, e.g., Gouriéroux et al. (2020).) Eq. (2.36) can be written:

$$y_t = c + \Phi(L)y_{t-1} + \varepsilon_t,$$

with  $\Phi(L) = \Phi_1 + \Phi_2 L + \dots + \Phi_p L^{p-1}$ .

Consequently:

$$y_t \mid y_{t-1}, y_{t-2}, \dots, y_{t-p} \sim \mathcal{N}(c + \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p}, \Omega).$$

Using Hamilton (1994)'s notations, denote with  $\Pi$  the matrix  $\begin{bmatrix} c & \Phi_1 & \Phi_2 & \dots & \Phi_p \end{bmatrix}'$  and with  $x_t$  the vector  $\begin{bmatrix} 1 & y'_{t-1} & y'_{t-2} & \dots & y'_{t-p} \end{bmatrix}'$ , we have:

$$y_t = \Pi' x_t + \varepsilon_t. \quad (2.43)$$

The previous representation is convenient to discuss the estimation of the VAR model, as parameters are gathered in two matrices only:  $\Pi$  and  $\Omega$ .

Let us start with the case where the shocks are Gaussian.

**Proposition 2.12** (MLE of a Gaussian VAR). *If  $y_t$  follows a VAR( $p$ ) (see Definition 2.21), and if  $\varepsilon_t \sim i.i.d. \mathcal{N}(0, \Omega)$ , then the ML estimate of  $\Pi$ , denoted by  $\hat{\Pi}$  (see Eq. (2.43)), is given by*

$$\hat{\Pi} = \left[ \sum_{t=1}^T x_t x_t' \right]^{-1} \left[ \sum_{t=1}^T y_t' x_t \right] = (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}, \quad (2.44)$$

where  $\mathbf{X}$  is the  $T \times (np)$  matrix whose  $t^{th}$  row is  $x_t$  and where  $\mathbf{y}$  is the  $T \times n$  matrix whose  $t^{th}$  row is  $y_t'$ .

That is, the  $i^{th}$  column of  $\hat{\Pi}$  ( $b_i$ , say) is the OLS estimate of  $\beta_i$ , where:

$$y_{i,t} = \beta_i' x_t + \varepsilon_{i,t}, \quad (2.45)$$

(i.e.,  $\beta_i' = [c_i, \phi_{i,1}', \dots, \phi_{i,p}']'$ ).

The ML estimate of  $\Omega$ , denoted by  $\hat{\Omega}$ , coincides with the sample covariance matrix of the  $n$  series of the OLS residuals in Eq. (2.45), i.e.:

$$\hat{\Omega} = \frac{1}{T} \sum_{i=1}^T \hat{\varepsilon}_i \hat{\varepsilon}_i', \quad \text{with } \hat{\varepsilon}_i = y_i - \hat{\Pi}' x_i. \quad (2.46)$$

The asymptotic distributions of these estimators are the ones resulting from standard OLS formula.

*Proof.* See Appendix 3.5. □

As stated by Proposition 2.13, when the shocks are not Gaussian, then the OLS regressions still provide consistent estimates of the model parameters. However, since  $x_t$  correlates to  $\varepsilon_s$  for  $s < t$ , the OLS estimator  $\mathbf{b}_i$  of  $\beta_i$  is biased in small sample. (That is also the case for the ML estimator.)

Indeed, denoting by  $\varepsilon_i$  the  $T \times 1$  vector of  $\varepsilon_{i,t}$ 's, and using the notations of  $b_i$  and  $\beta_i$  introduced in Proposition 2.12, we have:

$$\mathbf{b}_i = \beta_i + (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\varepsilon_i. \quad (2.47)$$

We have non-zero correlation between  $x_t$  and  $\varepsilon_{i,s}$  for  $s < t$  and, therefore,  $\mathbb{E}[(\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\varepsilon_i] \neq 0$ .

However, when  $y_t$  is covariance stationary, then  $\frac{1}{n} \mathbf{X}'\mathbf{X}$  converges to a positive definite matrix  $\mathbf{Q}$ , and  $\frac{1}{n} \mathbf{X}'\varepsilon_i$  converges to 0. Hence  $\mathbf{b}_i \xrightarrow{p} \beta_i$ . More precisely:

**Proposition 2.13** (Asymptotic distribution of the OLS estimate of  $\beta_i$ ). *If  $y_t$  follows a VAR model, as defined in Definition 2.21, we have:*

$$\sqrt{T}(\mathbf{b}_i - \beta_i) = \underbrace{\left[ \frac{1}{T} \sum_{t=p}^T x_t x_t' \right]^{-1}}_{\xrightarrow{p} \mathbf{Q}^{-1}} \underbrace{\sqrt{T} \left[ \frac{1}{T} \sum_{t=1}^T x_t \varepsilon_{i,t} \right]}_{\xrightarrow{d} \mathcal{N}(0, \sigma_i^2 \mathbf{Q})},$$

where  $\sigma_i = \text{Var}(\varepsilon_{i,t})$  and where  $\mathbf{Q} = \text{plim } \frac{1}{T} \sum_{t=p}^T x_t x_t'$  is given by:

$$\mathbf{Q} = \begin{bmatrix} 1 & \mu' & \mu' & \dots & \mu' \\ \mu & \gamma_0 + \mu\mu' & \gamma_1 + \mu\mu' & \dots & \gamma_{p-1} + \mu\mu' \\ \mu & \gamma_1 + \mu\mu' & \gamma_0 + \mu\mu' & \dots & \gamma_{p-2} + \mu\mu' \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \mu & \gamma_{p-1} + \mu\mu' & \gamma_{p-2} + \mu\mu' & \dots & \gamma_0 + \mu\mu' \end{bmatrix}. \quad (2.48)$$

*Proof.* See Appendix 3.5. □

The following proposition extends the previous proposition and includes covariances between different  $\beta_i$ 's as well as the asymptotic distribution of the ML estimates of  $\Omega$ .

**Proposition 2.14** (Asymptotic distribution of the OLS estimates). *If  $y_t$  follows a VAR model, as defined in Definition 2.21, we have:*

$$\sqrt{T} \begin{bmatrix} \text{vec}(\hat{\Pi} - \Pi) \\ \text{vec}(\hat{\Omega} - \Omega) \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \Omega \otimes \mathbf{Q}^{-1} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} \right), \quad (2.49)$$

where the component of  $\Sigma_{22}$  corresponding to the covariance between  $\hat{\sigma}_{i,j}$  and  $\hat{\sigma}_{k,l}$  (for  $i, j, l, m \in \{1, \dots, n\}^4$ ) is equal to  $\sigma_{i,l}\sigma_{j,m} + \sigma_{i,m}\sigma_{j,l}$ .

*Proof.* See Hamilton (1994), Appendix of Chapter 11. □

Naturally, in practice,  $\Omega$  is replaced with  $\hat{\Omega}$ ,  $\mathbf{Q}$  is replaced with  $\hat{\mathbf{Q}} = \frac{1}{T} \sum_{t=p}^T x_t x_t'$  and  $\Sigma$  with the matrix whose components are of the form  $\hat{\sigma}_{i,l}\hat{\sigma}_{j,m} + \hat{\sigma}_{i,m}\hat{\sigma}_{j,l}$ , where the  $\hat{\sigma}_{i,l}$ 's are the components of  $\hat{\Omega}$ .

The simplicity of the VAR framework and the tractability of its MLE open the way to convenient econometric testing. Let's illustrate this with the likelihood ratio test (see Def. ??). The maximum value achieved by the MLE is

$$\log \mathcal{L}(Y_T; \hat{\Pi}, \hat{\Omega}) = -\frac{Tn}{2} \log(2\pi) + \frac{T}{2} \log |\hat{\Omega}^{-1}| - \frac{1}{2} \sum_{t=1}^T [\hat{\varepsilon}_t' \hat{\Omega}^{-1} \hat{\varepsilon}_t].$$

The last term is:

$$\begin{aligned} \sum_{t=1}^T \hat{\varepsilon}_t' \hat{\Omega}^{-1} \hat{\varepsilon}_t &= \text{Tr} \left[ \sum_{t=1}^T \hat{\varepsilon}_t' \hat{\Omega}^{-1} \hat{\varepsilon}_t \right] = \text{Tr} \left[ \sum_{t=1}^T \hat{\Omega}^{-1} \hat{\varepsilon}_t \hat{\varepsilon}_t' \right] \\ &= \text{Tr} \left[ \hat{\Omega}^{-1} \sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}_t' \right] = \text{Tr} [\hat{\Omega}^{-1} (T\hat{\Omega})] = Tn. \end{aligned}$$

Therefore, the optimized log-likelihood is simply obtained by:

$$\log \mathcal{L}(Y_T; \hat{\Pi}, \hat{\Omega}) = -(Tn/2) \log(2\pi) + (T/2) \log |\hat{\Omega}^{-1}| - Tn/2. \quad (2.50)$$

Assume that we want to test the null hypothesis that a set of variables follows a  $\text{VAR}(p_0)$  against the alternative specification of  $p_1$  ( $> p_0$ ).

Let us denote by  $\hat{L}_0$  and  $\hat{L}_1$  the maximum log-likelihoods obtained with  $p_0$  and  $p_1$  lags, respectively.

Under the null hypothesis ( $H_0: p = p_0$ ), we have:

$$2(\hat{L}_1 - \hat{L}_0) = T(\log |\hat{\Omega}_1^{-1}| - \log |\hat{\Omega}_0^{-1}|) \sim \chi^2(n^2(p_1 - p_0)).$$

What precedes can be used to help determine the appropriate number of lags to use in the specification. In a VAR, using too many lags consumes numerous degrees of freedom: with  $p$  lags, each of the  $n$  equations in the VAR contains  $n \times p$  coefficients plus the intercept term. Adding lags improve in-sample fit, but is likely to result in over-parameterization and affect the **out-of-sample** prediction performance.

To select appropriate lag length, **selection criteria** can be used (see Definition 2.20). In the context of VAR models, using Eq. (2.50), we have:

$$\begin{aligned} AIC &= cst + \log |\hat{\Omega}| + \frac{2}{T} N \\ BIC &= cst + \log |\hat{\Omega}| + \frac{\log T}{T} N, \end{aligned}$$

where  $N = p \times n^2$ .



### 2.3.5 Block exogeneity and Granger causality

#### Block exogeneity

Let's decompose  $y_t$  into two subvectors  $y_t^{(1)}$  ( $n_1 \times 1$ ) and  $y_t^{(2)}$  ( $n_2 \times 1$ ), with  $y_t' = [y_t^{(1)'} \ y_t^{(2)'}]$  (and therefore  $n = n_1 + n_2$ ), such that:

$$\begin{bmatrix} y_t^{(1)} \\ y_t^{(2)} \end{bmatrix} = \begin{bmatrix} \Phi^{(1,1)} & \Phi^{(1,2)} \\ \Phi^{(2,1)} & \Phi^{(2,2)} \end{bmatrix} \begin{bmatrix} y_{t-1}^{(1)} \\ y_{t-1}^{(2)} \end{bmatrix} + \varepsilon_t.$$

Using, e.g., a likelihood ratio test (see Def. ??), one can easily test for block exogeneity of  $y_t^{(2)}$  (say). The null assumption can be expressed as  $\Phi^{(2,1)} = 0$ .

#### Granger Causality

Granger (1969) developed a method to explore **causal relationships** among variables. The approach consists in determining whether the past values of  $y_{1,t}$  can help explain the current  $y_{2,t}$  (beyond the information already included in the past values of  $y_{2,t}$ ).

Formally, let us denote three information sets:

$$\begin{aligned} \mathcal{J}_{1,t} &= \{y_{1,t}, y_{1,t-1}, \dots\} \\ \mathcal{J}_{2,t} &= \{y_{2,t}, y_{2,t-1}, \dots\} \\ \mathcal{J}_t &= \{y_{1,t}, y_{1,t-1}, \dots, y_{2,t}, y_{2,t-1}, \dots\}. \end{aligned}$$

We say that  $y_{1,t}$  Granger-causes  $y_{2,t}$  if

$$\mathbb{E}[y_{2,t} \mid \mathcal{J}_{2,t-1}] \neq \mathbb{E}[y_{2,t} \mid \mathcal{J}_{t-1}].$$

To get the intuition behind the testing procedure, consider the following bivariate VAR( $p$ ) process:

$$\begin{aligned} y_{1,t} &= c_1 + \sum_{i=1}^p \Phi_i^{(11)} y_{1,t-i} + \sum_{i=1}^p \Phi_i^{(12)} y_{2,t-i} + \varepsilon_{1,t} \\ y_{2,t} &= c_2 + \sum_{i=1}^p \Phi_i^{(21)} y_{1,t-i} + \sum_{i=1}^p \Phi_i^{(22)} y_{2,t-i} + \varepsilon_{2,t}, \end{aligned}$$

where  $\Phi_k^{(ij)}$  denotes the element  $(i, j)$  of  $\Phi_k$ .

Then,  $y_{1,t}$  is said not to Granger-cause  $y_{2,t}$  if

$$\Phi_1^{(21)} = \Phi_2^{(21)} = \dots = \Phi_p^{(21)} = 0.$$

Therefore the hypothesis testing is

$$\begin{cases} H_0 : \Phi_1^{(21)} = \Phi_2^{(21)} = \dots = \Phi_p^{(21)} = 0 \\ H_1 : \Phi_1^{(21)} \neq 0 \text{ or } \Phi_2^{(21)} \neq 0 \text{ or } \dots \Phi_p^{(21)} \neq 0. \end{cases}$$

Loosely speaking, we reject  $H_0$  if some of the coefficients on the lagged  $y_{1,t}$ 's are statistically significant. Formally, this can be tested using the  $F$ -test or asymptotic chi-square test. The  $F$ -statistic is

$$F = \frac{(RSS - USS)/p}{USS/(T - 2p - 1)},$$

where RSS is the Restricted sum of squared residuals and USS is the Unrestricted sum of squared residuals. Under  $H_0$ , the  $F$ -statistic is distributed as  $\mathcal{F}(p, T - 2p - 1)$ . (We have  $pF \xrightarrow{T \rightarrow \infty} \chi^2(p)$ .)

### 2.3.6 Identification problem and standard identification techniques

In Section 2.3.4, we have seen how to estimate  $\text{Var}(\varepsilon_t) = \Omega$  and the  $\Phi_k$  matrices in the context of a VAR model. But the IRFs are functions of  $B$  and the  $\Phi_k$ 's, not of  $\Omega$  the  $\Phi_k$ 's (see Section 2.3.2). We have  $\Omega = BB'$ , but this is not sufficient to recover  $B$ .

Indeed, seen a system of equations whose unknowns are the  $b_{i,j}$ 's (components of  $B$ ), the system  $\Omega = BB'$  contains only  $n(n+1)/2$  linearly independent equations. For instance, for  $n = 2$ :

$$\Leftrightarrow \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = \begin{bmatrix} b_{11}^2 + b_{12}^2 & b_{11}b_{21} + b_{12}b_{22} \\ b_{11}b_{21} + b_{12}b_{22} & b_{21}^2 + b_{22}^2 \end{bmatrix}.$$

We then have 3 linearly independent equations but 4 unknowns. Therefore,  $B$  is not identified based on second-order moments. Additional restrictions are required to identify  $B$ . This section covers two standard identification schemes: **short-run** and **long-run** restrictions:

1. A **short-run restriction (SRR)** prevents a structural shock from affecting an endogenous variable contemporaneously.
  - Easy to implement: the appropriate entries of  $B$  are set to 0.
  - Particular case: **Cholesky, or recursive approach**.
  - Examples: Bernanke (1986), Sims (1986), Galí (1992), Ruibio-Ramírez et al. (2010).
2. A **long-run restriction (LRR)** prevents a structural shock from having a cumulative impact on one of the endogenous variables.
  - Additional computations are required to implement this. One needs to compute the cumulative effect of one of the structural shocks  $u_t$  on one of the endogenous variable.
  - Examples: Blanchard and Quah (1989), Faust and Leeper (1997), Galí (1999), Erceg et al. (2005), Christiano et al. (2007).

The two approaches can be combined (see, e.g., Gerlach and Smets (1995)).

Let us consider a simple example that could motivate short-run restrictions. Consider the following stylized macro model:

$$\begin{aligned} g_t &= \bar{g} - \lambda(i_{t-1} - \mathbb{E}_{t-1}\pi_t) + \underbrace{\sigma_d \eta_{d,t}}_{\text{demand shock}} && \text{(IS curve)} \\ \Delta\pi_t &= \beta(g_t - \bar{g}) + \underbrace{\sigma_\pi \eta_{\pi,t}}_{\text{cost push shock}} && \text{(Phillips curve)} \\ i_t &= \rho i_{t-1} + [\gamma_\pi \mathbb{E}_t \pi_{t+1} + \gamma_g(g_t - \bar{g})] + \underbrace{\sigma_{mp} \eta_{mp,t}}_{\text{Mon. Pol. shock}} && \text{(Taylor rule),} \end{aligned} \tag{2.51}$$

where:

$$\eta_t = \begin{bmatrix} \eta_{\pi,t} \\ \eta_{d,t} \\ \eta_{mp,t} \end{bmatrix} \sim i.i.d. \mathcal{N}(0, I). \tag{2.52}$$

Vector  $\eta_t$  is assumed to be a vector of structural shocks, mutually and serially independent. On date  $t$ :

- $g_t$  is contemporaneously affected by  $\eta_{d,t}$  only;
- $\pi_t$  is contemporaneously affected by  $\eta_{\pi,t}$  and  $\eta_{d,t}$ ;
- $i_t$  is contemporaneously affected by  $\eta_{mp,t}$ ,  $\eta_{\pi,t}$  and  $\eta_{d,t}$ .

System (2.51) could be rewritten in the form:

$$\begin{bmatrix} d_t \\ \pi_t \\ i_t \end{bmatrix} = \Phi(L) \begin{bmatrix} d_{t-1} \\ \pi_{t-1} \\ i_{t-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \bullet & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \end{bmatrix}}_{\substack{=B \\ =\varepsilon_t}} \eta_t \tag{2.53}$$

This is the **reduced-form** of the model. This representation suggests three additional restrictions on the entries of  $B$ ; the latter matrix is therefore identified (up to the signs of its columns) as soon as  $\Omega = BB'$  is known.

There are particular cases in which some well-known matrix decomposition of  $\Omega = \text{Var}(\varepsilon_t)$  can be used to easily estimate some specific SVAR.

Consider the following context:

- A first shock (say,  $\eta_{n_1,t}$ ) can affect instantaneously (i.e., on date  $t$ ) only one of the endogenous variable (say,  $y_{n_1,t}$ );
- A second shock (say,  $\eta_{n_2,t}$ ) can affect instantaneously (i.e., on date  $t$ ) two endogenous variables,  $y_{n_1,t}$  (the same as before) and  $y_{n_2,t}$ ;
- ...

This implies (1) that column  $n_1$  of  $B$  has only 1 non-zero entry (this is the  $n_1^{th}$  entry), (2) that column  $n_2$  of  $B$  has 2 non-zero entries (the  $n_1^{th}$  and the  $n_2^{th}$  ones), etc. Without loss of generality, we can set  $n_1 = n$ ,  $n_2 = n - 1$ , etc. In this context, matrix  $B$  is lower triangular.

The Cholesky decomposition of  $\Omega_\varepsilon$  then provides an appropriate estimate of  $B$ , since this matrix decomposition yields to a lower triangular matrix satisfying:

$$\Omega_\varepsilon = BB'.$$

For instance, Dedola and Lippi (2005) estimate 5 structural VAR models for the US, the UK, Germany, France and Italy to analyse the monetary-policy transmission mechanisms. They estimate SVAR(5) models over the period 1975-1997. The shock-identification scheme is based on Cholesky decompositions, the ordering of the endogenous variables being: the industrial production, the consumer price index, a commodity price index, the short-term rate, monetary aggregate and the effective exchange rate (except for the US). This ordering implies that monetary policy reacts to the shocks affecting the first three variables but that the latter react to monetary policy shocks with a one-period lag only.

Importantly, the Cholesky approach can be useful when one is interested in one specific structural shock. This was the case, e.g., of Christiano et al. (1996). Their identification is based on the following relationship between  $\varepsilon_t$  and  $\eta_t$ :

$$\begin{bmatrix} \varepsilon_{S,t} \\ \varepsilon_{r,t} \\ \varepsilon_{F,t} \end{bmatrix} = \begin{bmatrix} B_{SS} & 0 & 0 \\ B_{rS} & B_{rr} & 0 \\ B_{FS} & B_{Fr} & B_{FF} \end{bmatrix} \begin{bmatrix} \eta_{S,t} \\ \eta_{r,t} \\ \eta_{F,t} \end{bmatrix},$$

where  $S$ ,  $r$  and  $F$  respectively correspond to *slow-moving variables*, the policy variable (short-term rate) and *fast-moving variables*. While  $\eta_{r,t}$  is scalar,  $\eta_{S,t}$  and  $\eta_{F,t}$  may be vectors. The space spanned by  $\varepsilon_{S,t}$  is the same as that spanned by  $\eta_{S,t}$ . As a result, because  $\varepsilon_{r,t}$  is a linear combination of  $\eta_{r,t}$  and  $\eta_{S,t}$  (which are  $\perp$ ), it comes that the  $B_{rr}\eta_{r,t}$ 's are the (population) residuals in the regression of  $\varepsilon_{r,t}$  on  $\varepsilon_{S,t}$ . Because  $\text{Var}(\eta_{r,t}) = 1$ ,  $B_{rr}$  is given by the square root of the variance of  $B_{rr}\eta_{r,t}$ .  $B_{F,r}$  is finally obtained by regressing the components of  $\varepsilon_{F,t}$  on the estimates of  $\eta_{r,t}$ .

An equivalent approach consists in computing the Cholesky decomposition of  $BB'$  and the contemporaneous impacts of the monetary policy shock (on the  $n$  endogenous variables) are the components of the column of  $B$  corresponding to the policy variable.

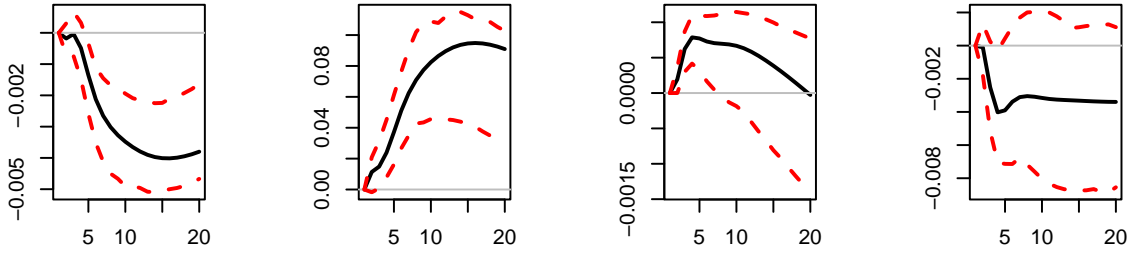
```
library(AEC)
library(vars)
data("USmonthly")
# Select sample period:
First.date <- "1965-01-01"; Last.date <- "1995-06-01"
indic.first <- which(USmonthly$DATES==First.date)
indic.last <- which(USmonthly$DATES==Last.date)
USmonthly <- USmonthly[indic.first:indic.last,]
considered.variables <- c("LIP", "UNEMP", "LCPI", "LPCOM", "FFR", "NBR", "TTR", "M1")
```

```

y <- as.matrix(USmonthly[considered.variables])
res.svar.ordering <- svar.ordering(y,p=3,
                                posit.of.shock = 5,
                                nb.periods.IRF = 20,
                                nb.bootstrap.replications = 100,
                                confidence.interval = 0.90, # expressed in pp.
                                indic.plot = 1 # Plots are displayed if = 1.
)

```

Effect of shock on LIF   Effect of shock on UNEI   Effect of shock on LCI   Effect of shock on LPCI



Effect of shock on FFI   Effect of shock on NB   Effect of shock on TTI   Effect of shock on M1

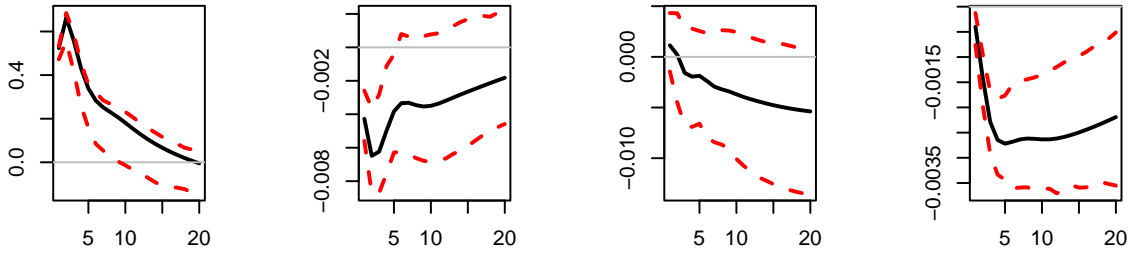


Figure 2.15: Response to a monetary-policy shock. Identification approach of Christiano, Eichenbaum and Evans (1996). Confidence intervals are obtained by bootstrapping the estimated VAR model (see inference section).

Let us now turn to **Long-run restrictions**. Such a restriction concerns the long-run influence of a shock on an endogenous variable. Let us consider for instance a structural shock that is assumed to have no “long-run influence” on GDP. How to express this? The long-run change in GDP can be expressed as  $GDP_{t+h} - GDP_t$ , with  $h$  large. Note further that:

$$GDP_{t+h} - GDP_t = \Delta GDP_{t+h} + \Delta GDP_{t+h-1} + \dots + \Delta GDP_{t+1}.$$

Hence, the fact that a given structural shock ( $\eta_{i,t}$ , say) has no long-run influence on GDP means that

$$\lim_{h \rightarrow \infty} \frac{\partial GDP_{t+h}}{\partial \eta_{i,t}} = \lim_{h \rightarrow \infty} \frac{\partial}{\partial \eta_{i,t}} \left( \sum_{k=1}^h \Delta GDP_{t+k} \right) = 0.$$

This can be easily formulated as a function of  $B$  and of the matrices  $\Phi_i$  when  $y_t$  (including  $\Delta GDP_t$ ) follows a VAR process.

Without loss of generality, we will only consider the VAR(1) case. Indeed, one can always write a VAR( $p$ ) as a VAR(1). To see that, stack the last  $p$  values of vector  $y_t$  in vector  $y_t^* = [y_t', \dots, y_{t-p+1}']'$ ; Eq. (2.36) can then be rewritten in its **companion form**:

$$y_t^* = \underbrace{\begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=c^*} + \underbrace{\begin{bmatrix} \Phi_1 & \Phi_2 & \dots & \Phi_p \\ I & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}}_{=\Phi} y_{t-1}^* + \underbrace{\begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{=\varepsilon_t^*}, \quad (2.54)$$

where matrices  $\Phi$  and  $\Omega^* = \text{Var}(\varepsilon_t^*)$  are of dimension  $np \times np$ ;  $\Omega^*$  is filled with zeros, except the  $n \times n$  upper-left block that is equal to  $\Omega = \text{Var}(\varepsilon_t)$ . (Matrix  $\Phi$  had been introduced in Eq. (2.42).)

Focusing on the VAR(1) case:

$$\begin{aligned}
 y_t &= c + \Phi y_{t-1} + \varepsilon_t \\
 &= c + \varepsilon_t + \Phi(c + \varepsilon_{t-1}) + \dots + \Phi^k(c + \varepsilon_{t-k}) + \dots \\
 &= \mu + \varepsilon_t + \Phi \varepsilon_{t-1} + \dots + \Phi^k \varepsilon_{t-k} + \dots \\
 &= \mu + B\eta_t + \Phi B\eta_{t-1} + \dots + \Phi^k B\eta_{t-k} + \dots,
 \end{aligned}$$

The sequence of shocks  $\{\eta_t\}$  determines the sequence  $\{y_t\}$ . What if  $\{\eta_t\}$  is replaced with  $\{\tilde{\eta}_t\}$ , where  $\tilde{\eta}_t = \eta_t$  if  $t \neq s$  and  $\tilde{\eta}_s = \eta_s + \gamma$ ? Assume  $\{\tilde{y}_t\}$  is the associated “perturbed” sequence. We have  $\tilde{y}_t = y_t$  if  $t < s$ . For  $t \geq s$ , the Wold decomposition of  $\{\tilde{y}_t\}$  implies:

$$\tilde{y}_t = y_t + \Phi^{t-s} B\gamma.$$

Therefore, the cumulative impact of  $\gamma$  on  $\tilde{y}_t$  will be (for  $t \geq s$ ):

$$\begin{aligned}
 (\tilde{y}_t - y_t) + (\tilde{y}_{t-1} - y_{t-1}) + \dots + (\tilde{y}_s - y_s) &= \\
 (Id + \Phi + \Phi^2 + \dots + \Phi^{t-s})B\gamma. &
 \end{aligned} \tag{2.55}$$

Consider a shock on  $\eta_{1,t}$ , with a magnitude of 1. This shock corresponds to  $\gamma = [1, 0, \dots, 0]'$ . Given Eq. (2.55), the long-run cumulative effect of this shock on the endogenous variables is given by:

$$\underbrace{(Id + \Phi + \dots + \Phi^k + \dots)}_{=(Id - \Phi)^{-1}} B \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

that is the first column of  $\Theta \equiv (Id - \Phi)^{-1} B$ .

In this context, consider the following long-run restriction: “ $j^{th}$  structural shock has no cumulative impact on the  $i^{th}$  endogenous variable”. It is equivalent to

$$\Theta_{ij} = 0,$$

where  $\Theta_{ij}$  is the element  $(i, j)$  of  $\Theta$ .

Blanchard and Quah (1989) have implemented such long-run restrictions in a small-scale VAR. Two variables are considered: GDP and unemployment. Consequently, the VAR is affected by two types of shocks. Specifically, authors want to identify **supply shocks** (that can have a permanent effect on output) and **demand shocks** (that cannot have a permanent effect on output).<sup>2</sup>

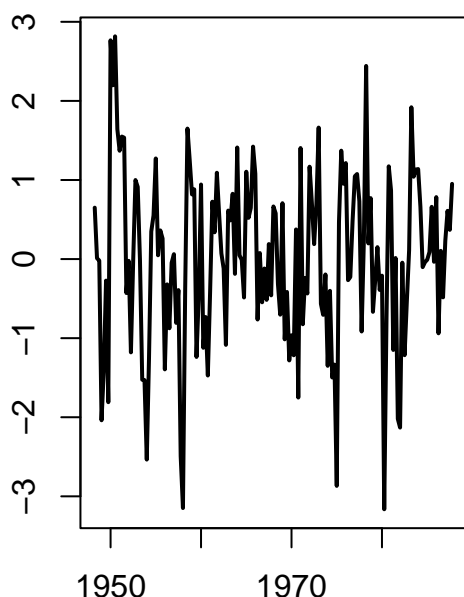
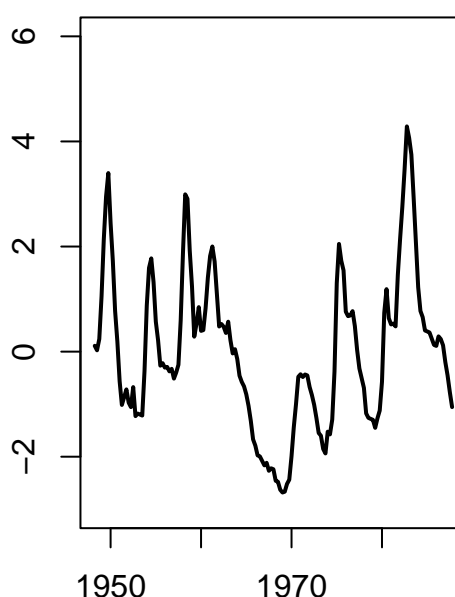
Blanchard and Quah (1989)’s dataset is quarterly, spanning the period from 1950:2 to 1987:4. Their VAR features 8 lags. Here are the data they use:

```

library(AEC)
data(BQ)
par(mfrow=c(1,2))
plot(BQ$Date,BQ$Dgdp,type="l",main="GDP quarterly growth rate",
     xlab="",ylab="",lwd=2)
plot(BQ$Date,BQ$unemp,type="l",ylim=c(-3,6),main="Unemployment rate (gap)",
     xlab="",ylab="",lwd=2)

```

<sup>2</sup>The motivation of the authors regarding their long-run restrictions can be obtained from a traditional Keynesian view of fluctuations. The authors propose a variant of a model from Fischer (1977).

**GDP quarterly growth rate****Unemployment rate (gap)**

Estimate a reduced-form VAR(8) model:

```
library(vars)
y <- BQ[,2:3]
est.VAR <- VAR(y,p=8)
Omega <- var(residuals(est.VAR))
```

Now, let us define a loss function (`loss`) that is equal to zero if (a)  $BB' = \Omega$  and (b) the element (1,1) of  $\Theta B$  is equal to zero:

```
# Compute (Id - Phi)^{-1}:
Phi <- Acoef(est.VAR)
PHI <- make.PHI(Phi)
sum.PHI.k <- solve(diag(dim(PHI)[1]) - PHI)[1:2,1:2]
loss <- function(param){
  B <- matrix(param,2,2)
  X <- Omega - B %*% t(B)
  Theta <- sum.PHI.k[1:2,1:2] %*% B
  loss <- 10000 * ( X[1,1]^2 + X[2,1]^2 + X[2,2]^2 + Theta[1,1]^2 )
  return(loss)
}
res.opt <- optim(c(1,0,0,1),loss,method="BFGS",hessian=FALSE)
print(res.opt$par)
```

```
## [1] 0.8570358 -0.2396345 0.1541395 0.1921221
```

(Note: one can use that type of approach, based on a loss function, to mix short- and long-run restrictions.)

Figure 2.16 displays the resulting IRFs. Note that, for GDP, we cumulate the GDP growth IRF, so as to have the response of the GDP in level.

```
B.hat <- matrix(res.opt$par,2,2)
print(cbind(Omega,B.hat %*% t(B.hat)))
```

```
##           Dgdp      unemp
## Dgdp    0.7582704 -0.17576173 0.7582694 -0.17576173
## unemp -0.1757617 0.09433658 -0.1757617 0.09433558
```

```

nb.sim <- 40
par(mfrow=c(2,2));par(plt=c(.15,.95,.15,.8))
Y <- simul.VAR(c=matrix(0,2,1),Phi,B.hat,nb.sim,y0.star=rep(0,2*8),
               indic.IRF = 1,u.shock = c(1,0))
plot(cumsum(Y[,1]),type="l",lwd=2,xlab="",ylab="",main="Demand shock on GDP")
plot(Y[,2],type="l",lwd=2,xlab="",ylab="",main="Demand shock on UNEMP")
Y <- simul.VAR(c=matrix(0,2,1),Phi,B.hat,nb.sim,y0.star=rep(0,2*8),
               indic.IRF = 1,u.shock = c(0,1))
plot(cumsum(Y[,1]),type="l",lwd=2,xlab="",ylab="",main="Supply shock on GDP")
plot(Y[,2],type="l",lwd=2,xlab="",ylab="",main="Supply shock on UNEMP")

```

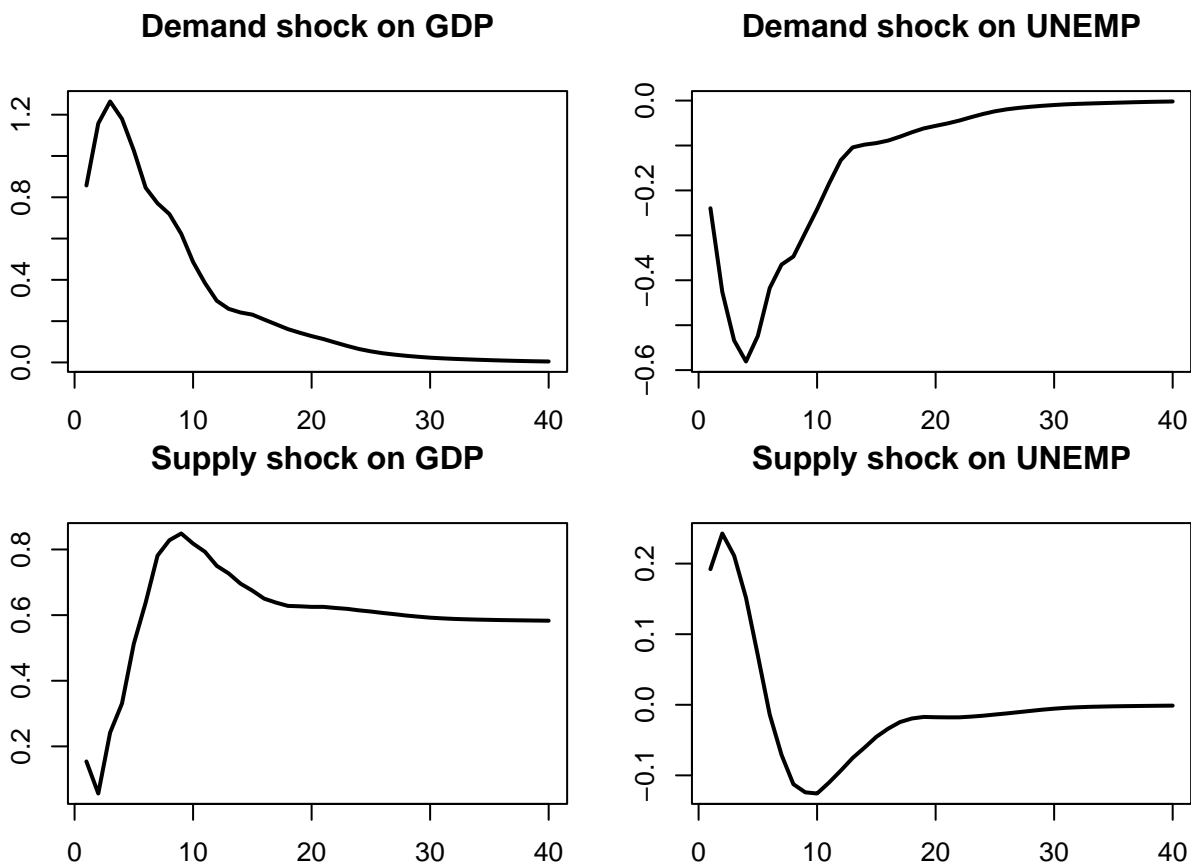


Figure 2.16: IRF of GDP and unemployment to demand and supply shocks.

### 2.3.7 Sign restrictions

To identify the structural shocks, we need to find a matrix  $B$  that satisfies  $\Omega = BB'$  (with  $\Omega = \text{Var}(\varepsilon_t)$ ) and other restrictions. Indeed, as explained above,  $\Omega = BB'$  is not sufficient to identify  $B$  since, if we take any orthogonal matrix  $Q$  (see Def. 2.23), then  $\mathcal{P} = BQ$  also satisfies  $\Omega = \mathcal{P}\mathcal{P}'$ .

**Definition 2.23** (Orthogonal matrix). An orthogonal matrix  $Q$  is a matrix such that  $QQ' = I$ , i.e., all columns (rows) of  $Q$  are orthogonal and unit vectors:

$$q_i'q_j = 0 \text{ if } i \neq j \text{ and } q_i'q_j = 1 \text{ if } i = j,$$

where  $q_i$  is the  $i^{\text{th}}$  column of  $Q$ .

The idea behind the sign-restriction approach is to “draw” random matrices  $\mathcal{P}$  that satisfy  $\Omega = \mathcal{P}\mathcal{P}'$ , and then to constitute a set of admissible matrices, keeping in this set only the simulated  $\mathcal{P}$  matrices that satisfy some predefined sign-based restriction. An example of restriction is “*after one year, a contractionary monetary-policy shocks has a negative impact on inflation*”.

As suggested above, if  $B$  is any matrix that satisfies  $\Omega = BB'$  (for instance,  $B$  can be based on the Cholesky decomposition of  $\Omega$ ), then we also have  $\Omega = \mathcal{P}\mathcal{P}'$  as soon as  $\mathcal{P} = BQ$ ,

where  $Q$  is an orthogonal matrix. Therefore, to draw  $\mathcal{P}$  matrices, it suffices to draw in the set of orthogonal matrices.

To fix ideas, consider dimension 2. In that case, the orthogonal matrices are rotation matrices, and the set of orthogonal matrices can be parameterized by the angle  $x$ , with:

$$Q_x = \begin{pmatrix} \cos(x) & \cos(x + \frac{\pi}{2}) \\ \sin(x) & \sin(x + \frac{\pi}{2}) \end{pmatrix} = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}.$$

(This is an angle- $x$  counter-clockwise rotation.) Hence, in that case, by drawing  $x$  randomly from  $[0, 2\pi]$ , we draw randomly from the set of  $2 \times 2$  rotation matrices. For high-dimensional VAR, we lose this simple geometrical representation, though. It is not always possible to parametrize a rotation matrix (high-dimensional VARs).

How to proceed, then? Arias et al. (2018) provide a procedure. Their approach is based on the so-called  $QR$  decomposition: any square matrix  $X$  may be decomposed as  $X = QR$  where  $Q$  is an orthogonal matrix and  $R$  is an upper diagonal matrix. With this in mind, they propose a two-step approach:

- i. Draw a random matrix  $X$  by drawing each element from independent standard normal distribution.
- ii. Let  $X = QR$  be the  $QR$  decomposition of  $X$  with the diagonal of  $R$  normalized to be positive. The random matrix  $Q$  is orthogonal and is a draw from the uniform distribution over the set of orthogonal matrices.

Equipped with this procedure, the sign-restriction is based on the following algorithm:

1. Draw a random orthogonal matrix  $Q$  (using step i. and ii. described above).
2. Compute  $B = PQ$  where  $P$  is the Cholesky decomposition of the reduced form residuals  $\Omega_\varepsilon$ .
3. Compute the impulse response associated with  $B$   $y_{t,t+k} = \Phi^k B$  or the cumulated response  $\bar{y}_{t,t+k} = \sum_{j=0}^k \Phi^j B$ .
4. Are the sign restrictions satisfied?
  - a. **Yes.** Store the impulse response in the set of admissible response.
  - b. **No.** Discard the impulse response.
5. Perform  $N$  replications and report the median impulse response (and its “confidence” intervals).

Note: to take into account the uncertainty in  $B$  and  $\Phi$ , you can draw  $B$  and  $\Phi$  in Steps 2 and 3 using an inference method (see Section 2.3.12).

The sign-restriction approach method has the advantage of being relatively agnostic. Moreover, it is fairly flexible, as one can impose sign restrictions on any variable, at any horizon. A prominent example is Uhlig (2005). Using US monthly data from 1965.I to 2003.XII, he employs sign restrictions to estimate the effect of monetary policy shocks.

According to conventional wisdom, monetary contractions should:<sup>3</sup>

- Raise the federal funds rate,
- Lower prices,
- Decrease non-borrowed reserves,
- Reduce real output.

The restrictions considered by Uhlig (2005) are as follows: an expansionary monetary policy shock leads to:

---

<sup>3</sup>Standard identification schemes often fail to achieve these 4 points. Two puzzles regularly arise: *liquidity puzzle*: when identifying monetary policy shocks as surprise increases in the stock of money, interest rates tend to go up, not down; *price puzzle*: after a contractionary monetary policy shock, even with interest rates going up and money supply going down, inflation goes up rather than down.



- Increases in prices
- Increase in nonborrowed reserves
- Decreases in the federal funds rate

What about output? Since is the response of interest, we leave it un-restricted.

```
library(AEC);library(vars);library(Matrix)
data("USmonthly")
First.date <- "1965-01-01"
Last.date <- "1995-06-01"
indic.first <- which(USmonthly$DATES==First.date)
indic.last <- which(USmonthly$DATES==Last.date)
USmonthly <- USmonthly[indic.first:indic.last,]
considered.variables<-c("LIP","UNEMP","LCPI","LPCOM","FFR","NBR","TTR","M1")
n <- length(considered.variables)
y <- as.matrix(USmonthly[considered.variables])
sign.restrictions <- list()
horizon <- list()
#Define sign restrictions and horizon for restrictions
for(i in 1:n){
  sign.restrictions[[i]] <- matrix(0,n,n)
  horizon[[i]] <- 1
}
sign.restrictions[[1]][1,3] <- 1
sign.restrictions[[1]][2,5] <- -1
sign.restrictions[[1]][3,6] <- 1
horizon[[1]] <- 1:5
res.svar.signs <-
  svar.signs(y,p=3,
    nb.shocks = 1, #number of identified shocks
    nb.periods.IRF = 20,
    bootstrap.replications = 1, # = 0 if no bootstrap
    confidence.interval = 0.80, # expressed in pp.
    indic.plot = 1, # Plots are displayed if = 1.
    nb.draws = 10000, # number of draws
    sign.restrictions,
    horizon,
    recursive =1 # =0 <- draw Q directly, =1 <- draw q recursively
  )
```

It has to be stressed that the sign restriction approach does not lead to a unique IRF, but to a set of admissible IRFs. Also, we say that this approach is set-identified, not point-identified.

An alternative approach is the so-called **penalty-function approach** (PFA, Uhlig (2005), present in Danne (2015)'s package). This approach relies on a *penalty function*:

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 100.x & \text{if } x > 0 \end{cases}$$

which penalizes positive responses and rewards negative responses.

Let  $\psi_k^j(q)$  be the impulse response of variable  $j$ . The  $\psi_k^j(q)$ 's are the elements of  $\psi_k(q) = \Psi_k q$ .

Let  $\sigma_j$  be the standard deviation of variable  $j$ . Let  $\iota_{j,k} = 1$  if we restrict the response of variable  $j$  at the  $k^{\text{th}}$  horizon to be negative,  $\iota_{j,k} = -1$  if we restrict it to be positive, and  $\iota_{j,k} = 0$  if there is no restriction. The total penalty is given by

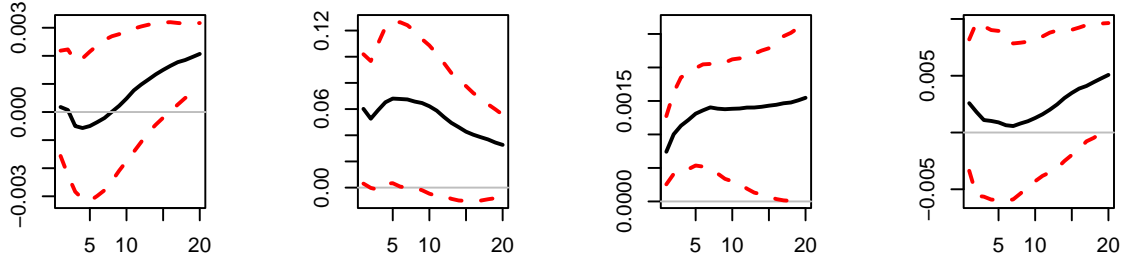
$$\mathbf{P}(q) = \sum_{j=1}^m \sum_{k=0}^K f\left(\iota_{j,k} \frac{\psi_k^j(q)}{\sigma_j}\right).$$

We are looking for a solution to

$$\min_q \mathbf{P}(q)$$

$$\text{s.t. } q'q = 1.$$

Effect of shock 1 on LI Effect of shock 1 on UNE Effect of shock 1 on LC Effect of shock 1 on LPC



Effect of shock 1 on FF Effect of shock 1 on NE Effect of shock 1 on T1 Effect of shock 1 on M

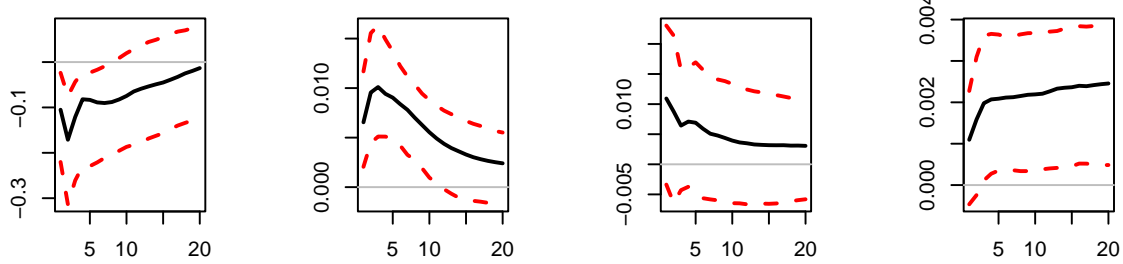


Figure 2.17: IRF associated with a monetary policy shock; sign-restriction approach.

The problem is solved numerically.

### 2.3.8 Forecast error variance maximization

The approach presented in this section exploits the derivations of Uhlig (2004). Barsky and Sims (2011) exploit this approach to identify a TFP news shock, that they define as the shock (a) that is orthogonal to the innovation in current utilization-adjusted TFP and (b) that best explains variation in future TFP.

Consider a process  $\{y_t\}$  that admits the infinite MA representation of Eq. (2.38). Let  $Q$  be an orthogonal matrix, an alternative decomposition is:

$$y_t = \sum_{h=0}^{+\infty} \Psi_h \underbrace{\eta_{t-h}}_{Q\tilde{\eta}_{t-h}} = \sum_{h=0}^{+\infty} \underbrace{\Psi_h Q}_{\tilde{\Psi}_h} \tilde{\eta}_{t-h} = \sum_{h=0}^{+\infty} \tilde{\Psi}_h \tilde{\eta}_{t-h}, \quad (2.56)$$

where  $\tilde{\eta}_{t-h} = Q'\eta_{t-h}$  are the white-noise shocks associated with the new MA representation. (They also satisfy  $\text{Var}(\tilde{\eta}_t) = Id$ .)

The  $h$ -step ahead prediction error of  $y_{t+h}$ , given all the data up to and including  $t-1$  is given by

$$e_{t+h}(h) = y_{t+h} - \mathbb{E}_{t-1}(y_{t+h}) = \sum_{j=0}^h \tilde{\Psi}_h \tilde{\eta}_{t+h-j}.$$

The variance-covariance matrix of  $e_{t+h}(h)$  is

$$\Omega(h) = \sum_{j=0}^h \tilde{\Psi}_j \tilde{\Psi}_j' = \sum_{j=0}^h \Psi_j \Psi_j'.$$

We can decompose  $\Omega(h)$  into the contribution of each shock  $l$  ( $l^{th}$  component of  $\tilde{\eta}_t$ ):

$$\Omega^{(h)} = \sum_{l=1}^n \Omega_l^{(h)}(Q)$$

with

$$\Omega_l^{(h)}(Q) = \sum_{j=0}^h (\Psi_j q_l)(\Psi_j q_l)',$$

where  $q_l$  is the  $l^{th}$  column of  $Q$ .

This decomposition can be used with the objective of finding the **impulse vector**  $b$  that is s.t. that it explains as much as possible of the sum of the  $h$ -step ahead prediction error variance of some variable  $i$ , say, for prediction horizons  $h \in [\underline{h}, \bar{h}]$ .

Formally, the task is to explain as much as possible of the variance

$$\sigma^2(\underline{h}, \bar{h}, q_1) = \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h [(\Psi_j q_1)(\Psi_j q_1)']_{i,i}$$

with a single impulse vector  $q_1$ .

Denote by  $E_{ii}$  the matrix that is filled with zeros, except for its  $(i, i)$  entry, set to 1. We have:

$$\begin{aligned} \sigma^2(\underline{h}, \bar{h}, q_1) &= \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h [(\Psi_j q_1)(\Psi_j q_1)']_{i,i} = \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h \text{Tr} [E_{ii}(\Psi_j q_1)(\Psi_j q_1)'] \\ &= \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h \text{Tr} [q_1' \Psi_j' E_{ii} \Psi_j q_1] \\ &= q_1' S q_1, \end{aligned}$$

where

$$\begin{aligned} S &= \sum_{h=\underline{h}}^{\bar{h}} \sum_{j=0}^h \Psi_j' E_{ii} \Psi_j \\ &= \sum_{j=0}^{\bar{h}} (\bar{h} + 1 - \max(\underline{h}, j)) \Psi_j' E_{ii} \Psi_j \\ &= \sum_{j=0}^{\bar{h}} (\bar{h} + 1 - \max(\underline{h}, j)) \Psi_{j,i}' \Psi_{j,i} \end{aligned}$$

where  $\Psi_{j,i}$  denotes row  $i$  of  $\Psi_j$ , i.e., the response of variable  $i$  at horizon  $j$  (when  $Q = Id$ ).

The maximization problem subject to the side constraint  $q_1' q_1 = 1$  can be written as a Lagrangian:

$$L = q_1' S q_1 - \lambda(q_1' q_1 - 1),$$

with the first-order condition  $S q_1 = \lambda q_1$  (the side constraint is  $q_1' q_1 = 1$ ). From this equation, we see that the solution  $q_1$  is an eigenvector of  $S$ , the one associated with eigenvalue  $\lambda$ . We also see that  $\sigma^2(\underline{h}, \bar{h}, q_1) = \lambda$ . Thus, to maximize this variance, we need to find the eigenvector of  $S$  that is associated with the maximal eigenvalue  $\lambda$ . That defines the first principal component (see Section 3.1). That is, if  $S$  admits the following spectral decomposition:

$$S = \mathcal{P} D \mathcal{P}',$$

where  $D$  is diagonal matrix whose entries are the (ordered) eigenvalues:  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , then  $\sigma^2(\underline{h}, \bar{h}, q_1)$  is maximized for  $q_1 = p_1$ , where  $p_1$  is the first column of  $\mathcal{P}$ .

### 2.3.9 Identification based on non-normality of the shocks

In this section, we show that the non-identification of the structural shocks ( $\eta_t$ ) is specific to the Gaussian case. We propose consistent estimation approaches for SVAR in the context of non-Gaussian shocks.

We have seen in what precedes that we cannot identify  $B$  based on first and second moments only. Since a Gaussian distribution is perfectly determined by the first two moments, it comes that one cannot achieve identification when the structural shocks are Gaussian. That is, even if we observe an infinite number of i.i.d.  $B\eta_t$ , we cannot recover  $B$  if the  $\eta_t$ 's are Gaussian.

Indeed, if  $\eta_t \sim \mathcal{N}(0, Id)$ , then the distribution of  $\varepsilon_t \equiv B\eta_t$  is  $\mathcal{N}(0, BB')$ . Hence  $\Omega = BB'$  is observed (in the population), but for any orthogonal matrix  $Q$  (i.e.  $QQ' = Id$ ), we also have  $BQ\eta_t \sim \mathcal{N}(0, \Omega)$ .

To illustrate, consider the following bivariate Gaussian situations, with  $\Theta_1 = 0$ ):

$$\begin{bmatrix} \eta_{1,t} \\ \eta_{2,t} \end{bmatrix} \sim \mathcal{N}(0, Id), \text{ with } B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} \text{ (rotation).}$$

Figure 2.18 shows that the distributions of  $B\eta_t$  and of  $BQ\eta_t$  are identical. However, the impulse response functions associated with one of the other impulse matrix ( $B$  or  $BQ$ ) are different. This is illustrated by Figure 2.19, that shows the IRFs associated with two identical models (defined by Eq. (2.39)), the only difference being the impulse matrix ( $B$  or  $BQ$ ).

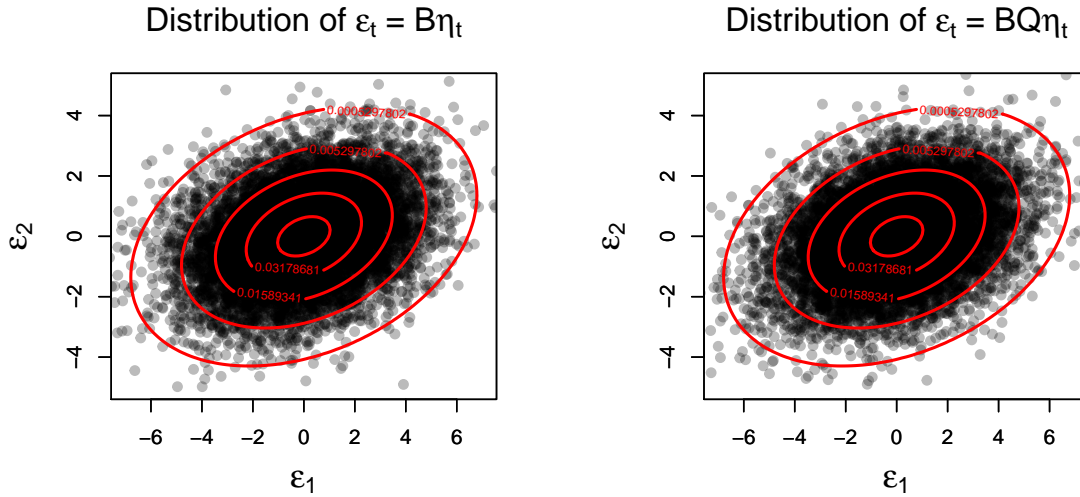


Figure 2.18: This figure compares the distributions of two Gaussian bivariate vectors,  $B\eta_t$  and  $BQ\eta_t$ , where  $\eta_t \sim \mathcal{N}(0, Id)$  (therefore  $\eta_{1,t}$  and  $\eta_{2,t}$  are independent), and  $Q$  is an orthogonal matrix.

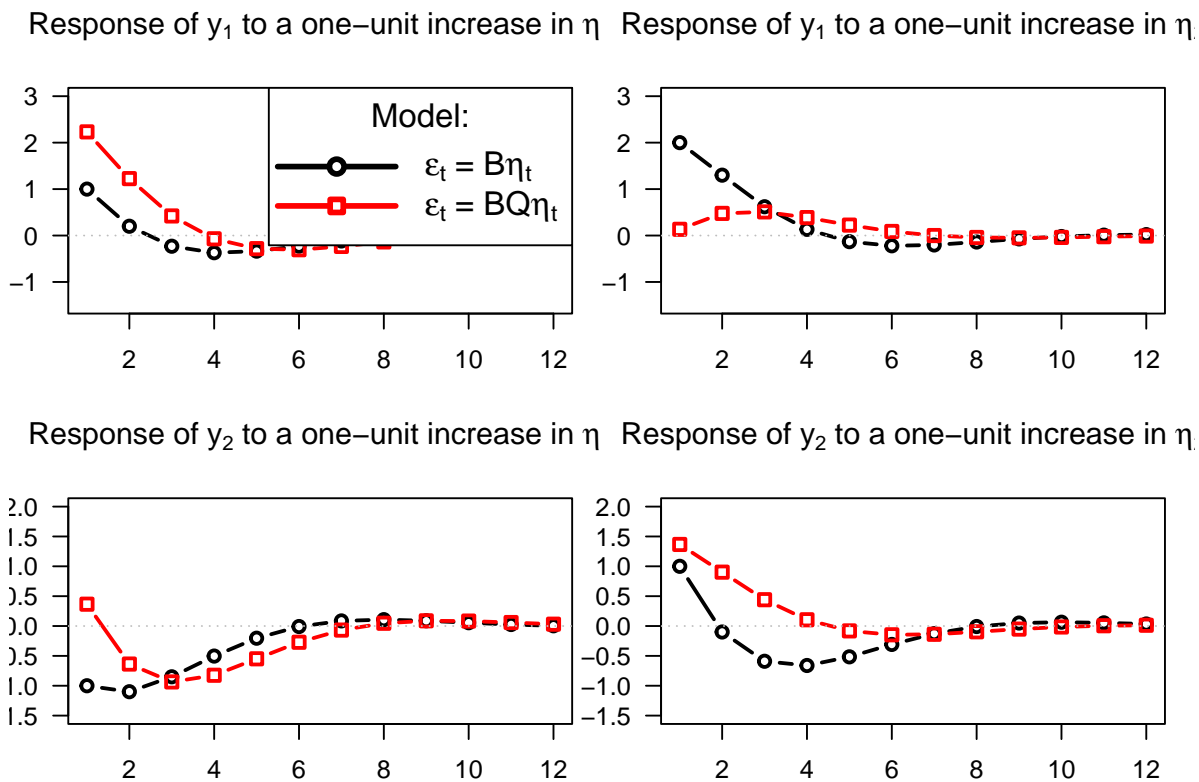


Figure 2.19: This figure shows that the impulse response functions associated with an impulse matrix equal to  $B$  (black line) or  $BQ$  (red line) are different (even if  $BB' = BQ(BQ)'$ ).

Hence, in the Gaussian case, external restrictions (economic hypotheses) are needed to identify  $B$  (see previous sections). But such restrictions may not be necessary if the structural shocks are not Gaussian. That is, the identification problem is very specific to normally-distributed  $\eta_t$ 's (Rigobon (2003), Normandin and Phaneuf (2004), Lanne and Lütkepohl (2008)).

To better see why this can be the case, consider again a bivariate vector of independent structural shocks ( $\eta_{1,t}$  and  $\eta_{2,t}$ ) but, now, assume that one of them is not Gaussian any more. Specifically, assume that  $\eta_{2,t}$  is drawn from a Student distribution with 5 degrees of freedom:  $\eta_{1,t} \sim \mathcal{N}(0, 1)$ ,  $\eta_{2,t} \sim t(5)$ ,  $B = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$  and  $Q = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}$ .

Figure 2.20 shows that, in this case,  $B\eta_t$  and  $BQ\eta_t$  do not have the same distribution any more (in spite of the fact that, in both cases, we have  $\text{Var}(\varepsilon_t) = BB'$ ). This opens the door to the identification of the impulse matrix ( $BQ$ ) in the non-Gaussian case.

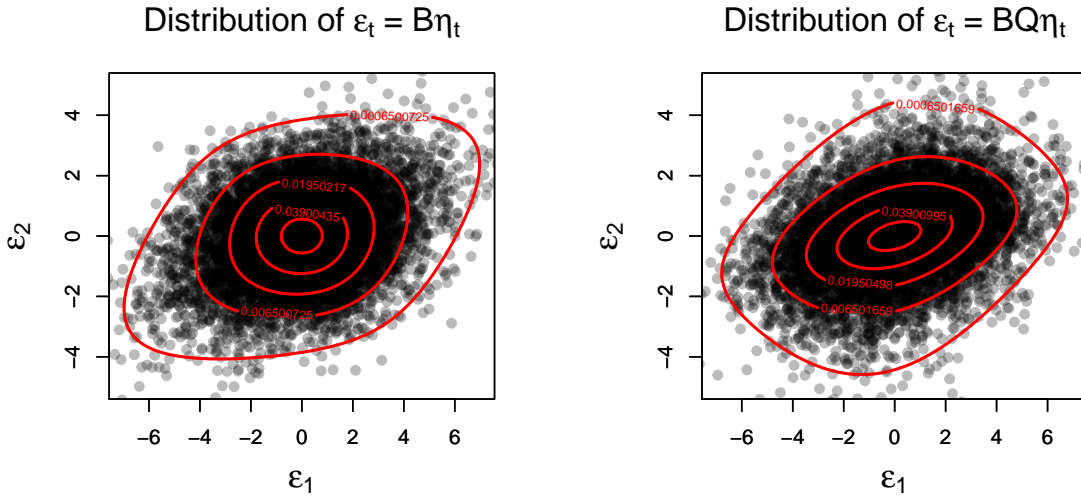


Figure 2.20: This figure compares the distributions of two Gaussian bivariate vectors,  $B\eta_t$  and  $BQ\eta_t$ , where  $\eta_{t1}, t \sim \mathcal{N}(0, 1)$ ,  $\eta_{t2}, t \sim t(5)$ , and  $Q$  is an orthogonal matrix.

The exercise that consists in identifying non-Gaussian independent shocks out of linear combinations of these shocks is a well-known problem of the signal-processing literature, called **independent component analysis (ICA)**. Without loss of generality, we can assume that  $BB' = Id$  (i.e.  $B$  is orthogonal). (If this is not the case, i.e. if  $\text{Var}(\varepsilon_t) = \Omega \neq Id$ , then one can pre-multiply the data by  $\Omega^{-1/2}$ .) The classical ICA problem is as follows: Find  $B$  such that  $\varepsilon_t = B\eta_t$  (or  $\varepsilon_t = B'\eta_t$ ) given that

- i. We observe the  $\varepsilon_t$ 's,
- ii. The components of  $\eta_t$  are independent,
- iii.  $BB' = Id$  (i.e.,  $B$  is orthogonal).

Figure 2.21 represents again some bivariate distributions. The black (red) lines correspond to the distributions of  $\eta_t$  ( $B\eta_t$ ). It is important to note that the two components of vector  $B\eta_t$  are not independent (contrary to those of  $\eta_t$ ).

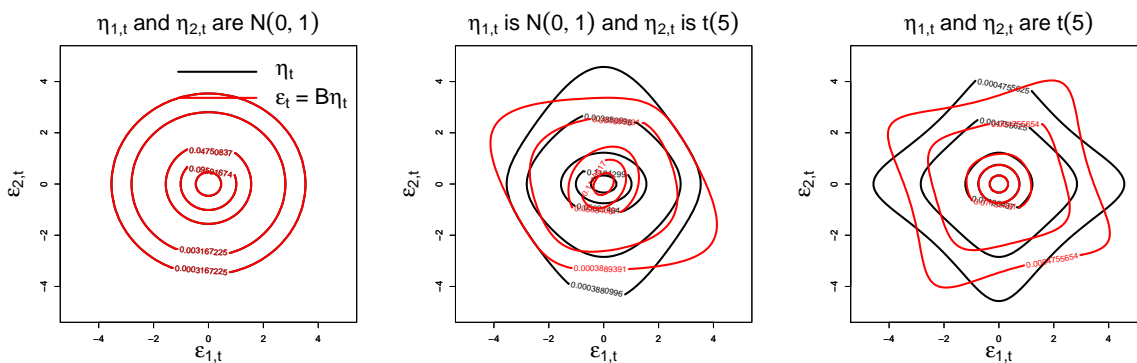


Figure 2.21: The three plots represent the bivariate distributions of  $\eta_t$  (black) and of  $B\eta_t$  (red), where the two components of  $\eta_t$  are independent, of unit variance, and  $B$  is orthogonal. Hence, for each of the three plots,  $\text{Var}(B\eta_t) = Id$ .

In all cases, we have  $\text{Var}(\varepsilon_t) = \text{Var}(\eta_t) = Id$ . But the two components of  $\varepsilon_t$  are not independent. For instance: We have  $\mathbb{E}(\varepsilon_{2,t}|\varepsilon_{1,t} > 4) < 0$  (whereas  $\mathbb{E}(\eta_{2,t}|\eta_{1,t} > 4) = 0$ ). The objective of ICA is to rotate  $\varepsilon_t$  to retrieve independent components ( $\eta_t$ ).

**Hypothesis 2.1.** Process  $\eta_t$  satisfies:

- i. The  $\eta_t$ 's are i.i.d. (across time) with  $\mathbb{E}(\eta_t) = 0$  and  $\text{Var}(\eta_t) = Id$ .

ii. The components  $\eta_{1,t}, \dots, \eta_{n,t}$  are mutually independent. iii We have

$$\varepsilon_t = B_0 \eta_t,$$

with  $\text{Var}(\varepsilon_t) = Id$  (i.e.  $B_0$  is orthogonal).

**Theorem 2.4** (Eriksson, Koivunen (2004)). *If Hypothesis 2.1 is satisfied and if at most one of the components of  $\eta$  is Gaussian, then matrix  $B_0$  is identifiable up to the post multiplication by  $DP$ , where  $P$  is a permutation matrix and  $D$  is a diagonal matrix whose diagonal entries are 1 or  $-1$ .}*

Hence, the structural shocks are identifiable. But how to estimate them based on observations of the  $\varepsilon_t$ 's? Gouriéroux et al. (2017) have proposed a **Pseudo-Maximum Likelihood (PML)** approach. This approach consists in maximizing a so-called **pseudo log-likelihood function**, based on a set of p.d.f.  $g_i(\eta_i), i = 1, \dots, n$  (that may be different from the true p.d.f. of the  $\eta_{i,t}$ 's):

$$\log \mathcal{L}_T(B) = \sum_{t=1}^T \sum_{i=1}^n \log g_i(b'_i Y_t), \quad (2.57)$$

where  $b_i$  is the  $i^{th}$  column of matrix  $B$  (or  $b'_i$  is the  $i^{th}$  row of  $B^{-1}$  since  $B^{-1} = B'$ ).

The log-likelihood function (2.57) is computed as if the errors  $\eta_{i,t}$  had the p.d.f.  $g_i(\eta_i)$ . The PML estimator of matrix  $B$  maximizes the pseudo log-likelihood function:

$$\widehat{B}_T = \arg \max_B \sum_{t=1}^T \sum_{i=1}^n \log g_i(b'_i \varepsilon_t), \quad (2.58)$$

$$s.t. B'B = Id.$$

The restrictions  $B'B = Id$  can be eliminated by parameterizing  $B$  in such a way that, whatever the consider parameters,  $B$  is orthogonal. Gouriéroux et al. (2017) propose to use, for that, the Cayley's representation: any orthogonal matrix with no eigenvalue equal to  $-1$  can be written as

$$B(A) = (Id + A)(Id - A)^{-1}, \quad (2.59)$$

where  $A$  is a skew symmetric (or antisymmetric) matrix, such that  $A' = -A$ . There is a one-to-one relationship with  $A$ , since:

$$A = (B(A) + Id)^{-1}(B(A) - Id). \quad (2.60)$$

Hence, the PML estimator of matrix  $B$  is obtained as  $\widehat{B}_T = B(\hat{A}_T)$ , where:

$$\hat{A}_T = \arg \max_{a_{i,j}, i > j} \sum_{t=1}^T \sum_{i=1}^n \log g_i[b_i(A)' \varepsilon_t]. \quad (2.61)$$

Table 2.1: This table reports usual p.d.f. and their derivatives.

	$\log g(x)$	$\frac{d \log g(x)}{dx}$	$\frac{d^2 \log g(x)}{dx^2}$
Gaussian	$\log \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	$-x$	$-1$
Student $t(\nu > 4)$	$-\frac{1-\nu}{2} \log \left(1 + \frac{x^2}{\nu-2}\right)$	$-\frac{x(1+\nu)}{\nu-2+x^2}$	$-(1+\nu) \frac{\nu-2-x^2}{\nu-2+x^2}$
Hyperbolic secant	$\log \left( \cosh \left\{ \frac{\pi}{2} x \right\} \right)$	$-\frac{\pi}{2} \tanh \left( \frac{\pi}{2} x \right)$	$-\left( \frac{\pi}{2} \frac{1}{\cosh \left( \frac{\pi}{2} x \right)} \right)^2$

$\log g(x)$	$\frac{d \log g(x)}{dx}$	$\frac{d^2 \log g(x)}{dx^2}$
Subgaussian $\log \left( \cosh \left\{ \frac{\pi}{2} x \right\} \right)$	$2\pi x + \frac{\pi}{2} \tanh \left( x \frac{\pi}{2} \right)$	$2\pi + \left( \frac{\pi}{2} \frac{1}{\cosh \left( \frac{\pi}{2} x \right)} \right)^2$

Under assumptions on the  $g_i$  functions (excluding the Gaussian distributions), Gouriéroux et al. (2017) derive the asymptotic properties of the PML estimator. Specifically, the PML estimator  $\widehat{B}_T$  of  $B_0$  is consistent (in  $\mathcal{P}_0$ , the set of matrices obtained by permutation and sign change of the columns of  $B_0$ ) and asymptotically normal, with speed of convergence  $1/\sqrt{T}$ .

The asymptotic variance-covariance matrix of  $\text{vec} \sqrt{T}(\widehat{B}_T - B_0)$  is  $A^{-1} \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} (A')^{-1}$ , where matrices  $A$  and  $\Gamma$  are detailed in Gouriéroux et al. (2017).

Note that the potential misspecification of pseudo-distributions  $g_i$  has no effect on the consistency of these specific PML estimators.

**Example 2.8** (Non-Gaussian monetary-policy shocks). We apply the PML-ICA approach on U.S. data covering the period 1959:IV to 2015:I at the quarterly frequency ( $T = 224$ ). We consider three dependent variables: inflation ( $\pi_t$ ), economic activity ( $z_t$ , the output gap) and the nominal short-term interest rate ( $r_t$ ). Changes in the log of oil prices added as an exogenous variable ( $x_t$ ).

```
library(AEC)
First.date <- "1959-04-01"
Last.date  <- "2015-01-01"
data <- US3var
data <- data[(data$Date>=First.date)&(data$Date<=Last.date),]
Y <- as.matrix(data[c("infl", "y.gdp.gap", "r")])
names.var <- c("inflation", "real activity", "short-term rate")
T <- dim(Y)[1]
n <- dim(Y)[2]
```

Let us denote by  $W_t$  the set of information made of the past values of  $y_t = [\pi_t, z_t, r_t]$ , that is  $\{y_{t-1}, y_{t-2}, \dots\}$ , and of exogenous variables  $\{x_t, x_{t-1}, \dots\}$ . The reduced-form VAR model reads:

$$y_t = \mu + \underbrace{\sum_{i=1}^p \Phi_i y_{t-i}}_{a(W_t; \theta)} + \Theta x_t + u_t$$

where the  $u_t$ 's are assumed to be serially independent, with zero mean and variance-covariance matrix  $\Sigma$ .

Matrices  $\mu$ ,  $\Phi_i$ ,  $\Theta$  and  $\Sigma$  are consistently estimated by OLS. Jarque-Bera tests support the hypothesis of non-normality for all residuals.

```
nb.lags <- 6 # number of lags used in the VAR model
X <- NULL
for(i in 1:nb.lags){
  lagged.Y <- rbind(matrix(NA,n),Y[1:(T-i),])
  X <- cbind(X,lagged.Y)}
X <- cbind(X,data$commo) # add exogenous variables
Phi <- matrix(0,n,n*nb.lags);mu <- rep(0,n)
effect.commo <- rep(0,n)
U <- NULL # Eta is the matrix of OLS residuals
for(i in 1:n){
  eq <- lm(Y[,i] ~ X)
  Phi[i,] <- eq$coef[2:(dim(Phi)[2]+1)]
```



```

mu[i] <- eq$coef[1]
U <- cbind(U,eq$residuals)
effect.commo[i] <- eq$coef[length(eq$coef)]
}
Omega <- var(U) # Covariance matrix of the OLS residuals.
B <- t(chol(Omega)) # Cholesky matrix associated with Omega (lower triang.)
Eps <- U %*% t(solve(B)) # Recover associated structural shocks

```

We want to estimate the orthogonal matrix  $B$  such that  $u_t = SB\eta_t$ , where

- $S$  results from the Cholesky decomposition of  $\Sigma$  and
- the components of  $\eta_t$  are independent, zero-mean with unit variance.

The PML approach is applied on standardized VAR residuals given by:

$$\hat{\varepsilon}_t = \hat{S}_T^{-1} \underbrace{[y_t - a(W_t; \hat{\theta}_T)]}_{\text{VAR residuals}}.$$

By construction of  $\hat{S}_T^{-1}$ , it comes that the covariance matrix of these residuals is  $Id$ .

The pseudo density functions are distinct and asymmetric mixtures of Gaussian distributions.

```

distri <- list(
  type=c("mixt.gaussian","mixt.gaussian","mixt.gaussian"),
  df=c(NaN,NaN,NaN),
  p=c(0.5,.5,.5),mu=c(.1,.1,.1),sigma=c(.5,.7,1.3))
AA.0 <- c(0,0,0)
res.optim <- optim(AA.0,func.2.minimize,
  Y = Eps, distri = distri,
  gr = d.func.2.minimize,
  method="Nelder-Mead",
  control=list(trace=FALSE,maxit=1000))
AA.0 <- res.optim$par
res.optim <- optim(AA.0,func.2.minimize,d.func.2.minimize,
  Y = Eps, distri = distri,
  method="BFGS",
  control=list(trace=FALSE))
AA.est <- res.optim$par
n <- ncol(Y)
M <- make.M(n)
A.est <- matrix(M %*% AA.est,n,n)
C.PML <- (diag(n) + A.est) %*% solve(diag(n) - A.est)

eta.PML <- Eps %*% C.PML # eta.PML are the ICA-estimated structural shocks

A <- make.A.matrix(eta.PML,distri,C.PML)
Omega <- make.Omega(eta.PML,distri)
# Compute asymptotic covariance matrix of C.PML:
V <- make.Asympt.Cov.delta(eta.PML,distri,C.PML)
param <- c(C.PML)
st.dev <- sqrt(diag(V))
t.stat <- c(C.PML)/sqrt(diag(V))
cbind(param,st.dev,t.stat) # print results of PML estimation

```

```

##          param      st.dev      t.stat
## [1,]  0.94417705 0.040848382 23.1141845
## [2,] -0.32711569 0.118802653 -2.7534376
## [3,]  0.03905164 0.074172945  0.5264944
## [4,]  0.32070293 0.119270893  2.6888616

```



```
## [5,] 0.93977707 0.041629110 22.5749976
## [6,] 0.11818924 0.060821400 1.9432179
## [7,] -0.07536139 0.071980455 -1.0469702
## [8,] -0.09906759 0.062185577 -1.5930959
## [9,] 0.99222290 0.007785691 127.4418551
```

(Note: it is always useful to combine two optimization algorithms, such as Nelder-Mead and BFGS.)

We would obtain close results by neglecting commodity prices. In that case, one can simply use the function `estim.SVAR.ICA` of the AEC package. Let us compare the  $C$  matrix obtained in the two cases (with or without commodity prices):

```
ICA.res.no.commo <- estim.SVAR.ICA(Y,distri = distri,p=6)
round(cbind(ICA.res.no.commo$C.PML,NaN,C.PML),3)
```

```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7]
## [1,] 0.956 0.287 -0.059 NaN 0.944 0.321 -0.075
## [2,] -0.292 0.950 -0.108 NaN -0.327 0.940 -0.099
## [3,] 0.025 0.121 0.992 NaN 0.039 0.118 0.992
```

Once  $B$  has been estimated, it remains to label the resulting structural shocks (components of  $\eta_t$ ). Postulated shocks are monetary-policy, supply, and demand shocks. This labelling can be based on the following considerations:

- Contractionary **monetary-policy shocks** have a negative impact on real activity and on inflation.
- **Supply shock** have influences of opposite signs on economic activity and on inflation.
- **Demand shock** have influences of same signs on economic activity and on inflation.

Let us compute the IRFs associated with the three structural shocks. (For the sake of comparison, the first line of plots shows the IRFs to a monetary-policy shock obtained from a Cholesky-based approach where the short-term rate is ordered last.)

```
IRF.Chol <- array(NaN,c(n,41,n))
IRF.ICA <- array(NaN,c(n,41,n))
PHI <- list();for(i in 1:nb.lags){PHI[[i]]<-array(Phi,c(3,3,nb.lags))[, ,i]}
for(jjjj in 1:n){
  u.shock <- rep(0,n)
  u.shock[jjjj] <- 1
  IRF.Chol[, ,jjjj] <-
    t(simul.VAR(c=rep(0,3),Phi=PHI,B=B,nb.sim=41,
                y0.star=rep(0,3*nb.lags),indic.IRF = 1,u.shock = u.shock))
  IRF.ICA[, ,jjjj] <-
    t(simul.VAR(c=rep(0,3),Phi=PHI,B=B%*%C.PML,nb.sim=41,
                y0.star=rep(0,3*nb.lags),indic.IRF = 1,u.shock = u.shock))
}
```

According to Figure 2.22, Shock 1 is a supply shock, Shock 2 is a demand shock, and Shock 3 is a monetary-policy shock. Note that Shock 3 is close to the one resulting from the Cholesky approach.

### Relation with the Heteroskedasticity Identification

In some cases, where the  $\varepsilon_t$ 's are heteroskedastic, the  $B$  matrix can be identified (Rigobon (2003), Lanne et al. (2010)).

Consider the case where we still have  $\varepsilon_t = B\eta_t$  but where  $\eta_t$ 's variance conditionally depends on a regime  $s_t \in \{1, \dots, M\}$ . That is:

$$\text{Var}(\eta_{k,t}|s_t) = \lambda_{s_t,k} \quad \text{for } k \in \{1, \dots, n\}$$

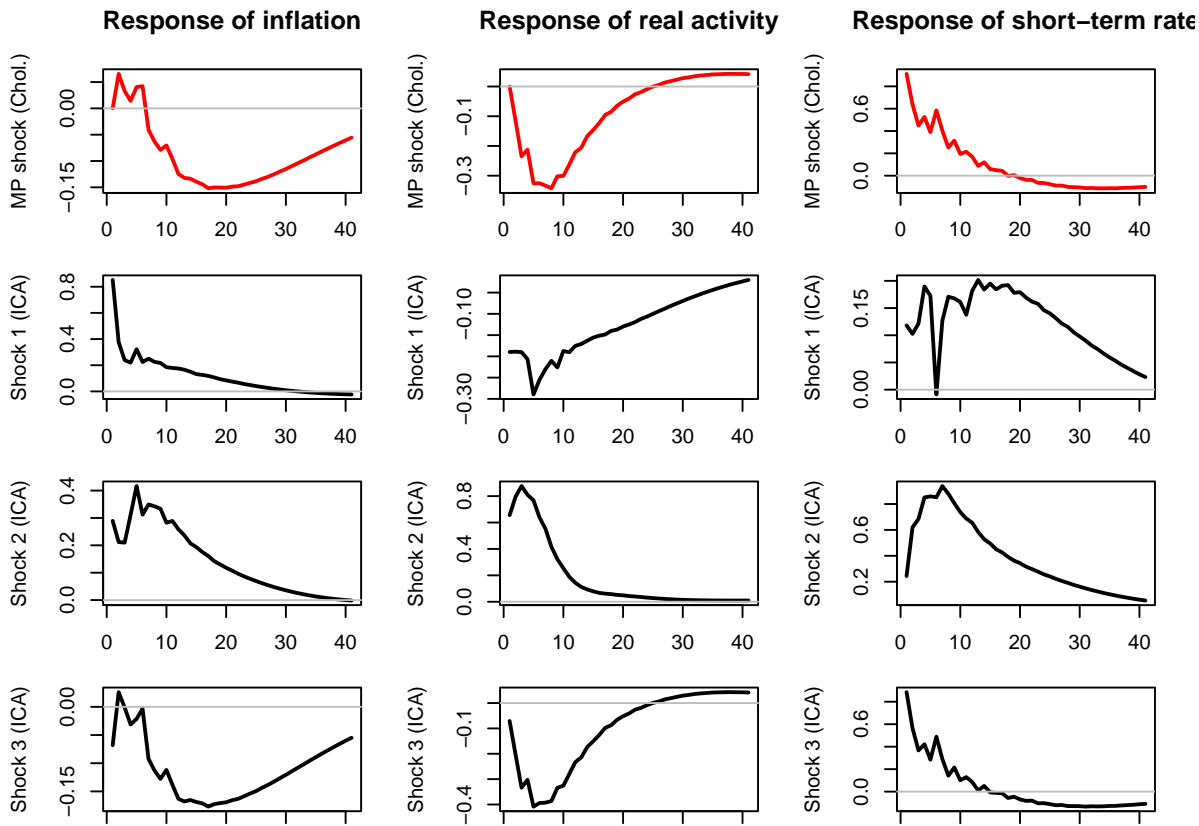


Figure 2.22: The first row of plots shows the responses of the three endogenous variables to the monetary policy shock in the context of a Cholesky-identified SVAR (ordering: inflation, output gap, interest rate). The next three rows of plots show the responses of the endogenous variables to the three structural shocks identified by ICA. The last one (Shock 3) is close to the Cholesky-identified monetary policy shock.

Denoting by  $\Lambda_i$  the diagonal matrix whose diagonal entries are the  $\lambda_{i,k}$ 's, it comes that:

$$\text{Var}(\eta_t | s_t) = \Lambda_{s_t}, \quad \text{and} \quad \text{Var}(\varepsilon_t | s_t) = B \Lambda_{s_t} B'.$$

Without loss of generality, it can be assumed that  $\Lambda_1 = Id$ .

In this context,  $B$  is identified, apart from sign reversal of its columns if for all  $k \neq j \in \{1, \dots, n\}$ , there is a regime  $i$  s.t.  $\lambda_{i,k} \neq \lambda_{i,j}$ . (Prop.1 in @Lanne et al. (2010)).

Bivariate regime case ( $M = 2$ ):  $B$  identified if the  $\lambda_{2,k}$ 's are all different. That is, identification is ensured if “there is sufficient heterogeneity in the volatility changes” (Lütkepohl and Netšunajev (2017)).

If the regimes  $s_t$  are exogenous and serially independent, then this situation is consistent with the “non-Gaussian” situation described above.

### 2.3.10 Factor-Augmented VAR (FAVAR)

VAR models are subject to the curse of dimensionality: If  $n$ , is large, then the number of parameters (in  $n^2$ ) explodes.

In the case where one suspects that the  $y_{i,t}$ 's are mainly driven by a small number of random sources, a factor structure may be imposed, and **principal component analysis** (PCA, see Appendix 3.1) can be employed to estimate the relevant factors (Bernanke et al. (2005)).

Let us denote by  $F_t$  a  $k$ -dimensional vector of latent factors accounting for important shares of the variances of the  $y_{i,t}$ 's (with  $K \ll n$ ) and by  $x_t$  is a small  $M$ -dimensional subset of  $y_t$  (with  $M \ll n$ ). The following factor structure is posited:

$$y_t = \Lambda^f F_t + \Lambda^x x_t + e_t,$$

where the  $e_t$  are “small” serially and mutually i.i.d. error terms. That is  $F_t$  and  $x_t$  are supposed to drive most of the fluctuations of  $y_t$ 's components.

The model is complemented by positing a VAR dynamics for  $[F'_t, x'_t]'$ :

$$\begin{bmatrix} F_t \\ x_t \end{bmatrix} = \Phi(L) \begin{bmatrix} F_{t-1} \\ x_{t-1} \end{bmatrix} + v_t. \quad (2.62)$$

Standard identification techniques of structural shocks can be employed in Eq. (2.62): Cholesky approach can be used for instance if the last component of  $x_t$  is the short-term interest rate and if it is assumed that a MP shock has no contemporaneous impact on other macro-variables (in  $x_t$ ).

In their identification procedure, Bernanke et al. (2005) exploit the fact that macro-finance variables can be decomposed in two sets —fast-moving and slow-moving variables— and that only the former reacts contemporaneously to monetary-policy shocks. Now, how to estimate the (unobserved) factors  $F_t$ ? Bernanke et al. (2005) note that the first  $K + M$  PCA of the whole dataset ( $y_t$ ), that they denote by  $\hat{C}(F_t, x_t)$  should span the same space as  $F_t$  and  $x_t$ . To get an estimate of  $F_t$ , the dependence of  $\hat{C}(F_t, x_t)$  in  $x_t$  has to be removed. This is done by regressing, by OLS,  $\hat{C}(F_t, x_t)$  on  $x_t$  and on  $\hat{C}^*(F_t)$ , where the latter is an estimate of the common components other than  $x_t$ . To proxy for  $\hat{C}^*(F_t)$ , Bernanke et al. (2005) take principal components from the set of slow-moving variables, that are not contemporaneously correlated to  $x_t$ . Vector  $\hat{F}_t$  is then computed as  $\hat{C}(F_t, x_t) - b_x x_t$ , where  $b_x$  are the coefficients coming from the previous OLS regressions.

Note that this approach implies that the vectorial space spanned by  $(\hat{F}_t, x_t)$  is the same as that spanned by  $\hat{C}(F_t, x_t)$ .

Below, we employ this method on the dataset built by McCracken and Ng (2016) —the FRED:MD database— that includes 119 time series.

```
library(BVAR) # contains the fred_md dataset
library(vars)
data <- fred_transform(fred_md, na.rm = FALSE, type = "fred_md")
First.date <- "1959-02-01"
Last.date <- "2020-01-01"
data <- data[(rownames(data) > First.date) & (rownames(data) < Last.date), ]
variables.with.na <- which(is.na(apply(data, 2, sum)))
data <- data[, -variables.with.na]
data.values <- scale(data, center = TRUE, scale = TRUE)
data_scaled <- data
data_scaled[1:dim(data)[1], 1:dim(data)[2]] <- data.values
K <- 3
M <- 1
PCA <- prcomp(data_scaled) # implies that PCA$x %*% t(PCA$rotation) = data
C.hat <- PCA$x[, 1:(K+M)]
fast_moving <- c("HOUST", "HOUSTNE", "HOUSTMW", "HOUSTS", "HOUSTW", "HOUSTS", "AMDNOx",
  "FEDFUNDS", "CP3Mx", "TB3MS", "TB6MS", "GS1", "GS5", "GS10",
  "COMPAPFFx", "TB3SMFFM", "TB6SMFFM", "T1YFFM", "T5YFFM", "T10YFFM",
  "AAAFFM", "EXSZUSx", "EXJPUSx", "EXUSUKx", "EXCAUSx")
data.slow <- data_scaled[, -which(fast_moving %in% names(data))]
PCA.star <- prcomp(data.slow) # implies that PCA$x %*% t(PCA$rotation) = data
C.hat.star <- PCA.star$x[, 1:K]
D <- cbind(data$FEDFUNDS, C.hat.star)
b.x <- solve(t(D) %*% D) %*% t(D) %*% C.hat
F.hat <- C.hat - data$FEDFUNDS %*% matrix(b.x[1,], nrow=1)
data_var <- data.frame(F.hat, FEDFUNDS = data$FEDFUNDS)
p <- 10
var <- VAR(data_var, p)
Omega <- var(residuals(var))
B <- t(chol(Omega))
D <- cbind(F.hat, data$FEDFUNDS)
loadings <- solve(t(D) %*% D) %*% t(D) %*% as.matrix(data_scaled)
irf <- simul.VAR(c=rep(0, (K+M)*p), Phi=Acoef(var), B, nb.sim=120,
```

```

y0.star=rep(0,(K+M)*p),indic.IRF = 1,
u.shock = c(rep(0,K+1),1))
irf.all <- irf %*% loadings
par(mfrow=c(2,2))
variables.2.plot <- c("FEDFUNDS","INDPRO","UNRATE","CPIAUCSL")
par(plt=c(.2,.95,.3,.95))
for(i in 1:length(variables.2.plot)){
  plot(cumsum(irf.all[,which(variables.2.plot[i]==names(data))]),lwd=2,
       type="l",xlab="months after shock",ylab=variables.2.plot[i])
}

```

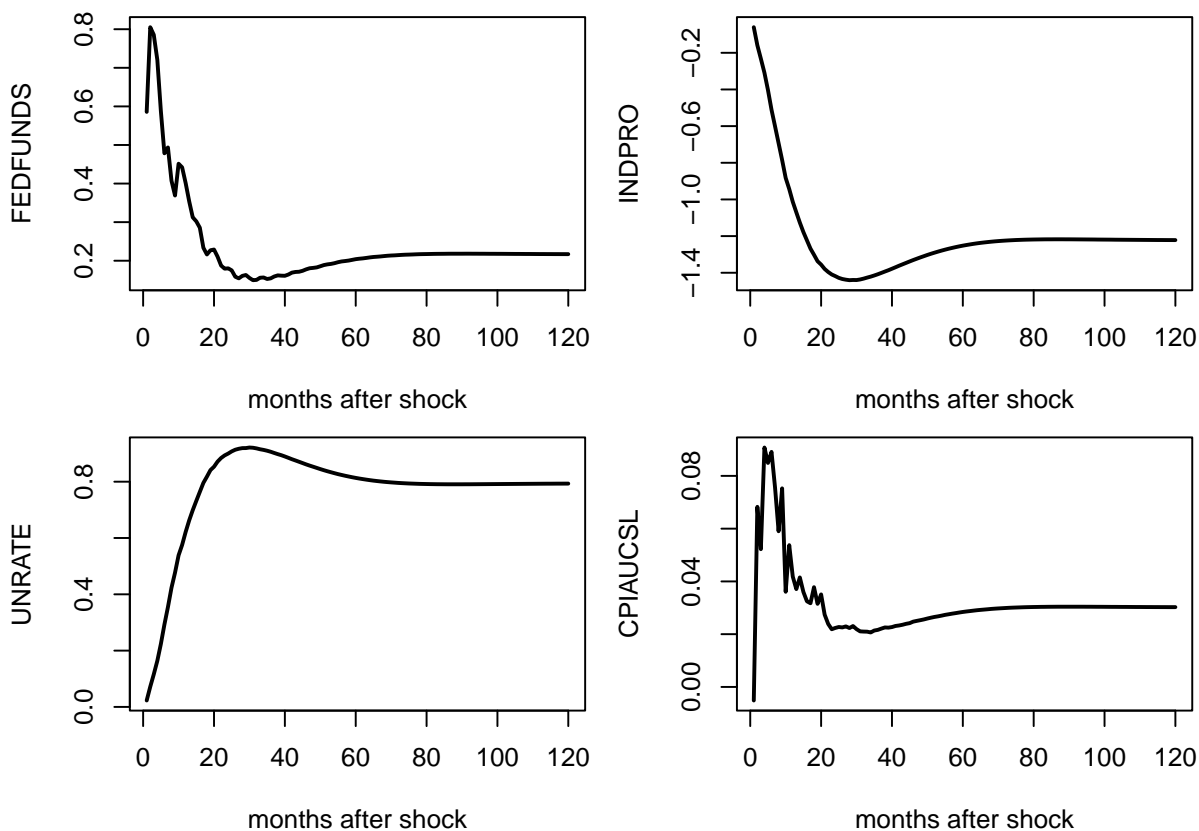


Figure 2.23: Responses of a monetary-policy shock. FAVAR approach of Bernanke, Boivin, and Elias (2005). FRED-MD dataset.

### 2.3.11 Projection Methods

Consider the infinite MA representation of  $y_t$  (Eq. (2.38)):

$$y_t = \mu + \sum_{h=0}^{\infty} \Psi_h \eta_{t-h}.$$

As seen in Section 2.3.2, the entries  $(i, j)$  of the sequence of the  $\Psi_h$  matrices define the IRF of  $\eta_{j,t}$  on  $y_{i,t}$ .

Assume that you observe  $\eta_{j,t}$ , then a consistent estimate of  $\Psi_{i,j,h}$  is simply obtained by the OLS regression of  $y_{i,t+h}$  on  $\eta_{j,t}$ :

$$y_{i,t+h} = \mu_i + \Psi_{i,j,h} \eta_{j,t} + u_{i,j,t+h}. \quad (2.63)$$

Because the residuals  $u_{i,j,t+h}$  are autocorrelated (for  $h > 0$ ), estimates of the covariance of the OLS estimators of the  $\Psi_{i,j,h}$  then have to be based on robust estimators (e.g. Newey-West, see Eq. (??)). This is the core idea of the **local projection approach** proposed by Jordà (2005).

Now, how to proceed in the (usual) case where  $\eta_{j,t}$  is not observed? We consider two situations.

#### Situation A: Without IV

This corresponds to the original Jordà (2005)'s approach.

Assume that the structural shock of interest ( $\eta_{1,t}$ , say) can be consistently obtained as the residual of a regression of a variable  $x_t$  on a set of control variables  $w_t$  independent from  $\eta_{1,t}$ :

$$\eta_{1,t} = x_t - \mathbb{E}(x_t|w_t), \quad (2.64)$$

where  $\mathbb{E}(x_t|w_t)$  is affine in  $w_t$  and where  $w_t$  is an affine transformation of  $\eta_{2:n,t}$  and of past shocks  $\eta_{t-1}, \eta_{t-2}, \dots$ .

Eq. (2.64) implies that, conditional on  $w_t$ , the additional knowledge of  $x_t$  is useful only when it comes to forecast something that depends on  $\eta_{1,t}$ . Hence, given that  $u_{i,1,t+h}$  (see Eq. (2.63)) is independent from  $\eta_{1,t}$  (it depends on  $\eta_{t+h}, \dots, \eta_{t+1}, \eta_{2:n,t}, \eta_{t-1}, \eta_{t-2}, \dots$ ), it comes that

$$\mathbb{E}(u_{i,1,t+h}|x_t, w_t) = \mathbb{E}(u_{i,1,t+h}|w_t).$$

This is the *conditional mean independence* case.

Let's rewrite Eq. (2.63) as follows:

$$\begin{aligned} y_{i,t+h} &= \mu_i + \Psi_{i,1,h}\eta_{1,t} + u_{i,1,t+h} \\ &= \mu_i + \Psi_{i,1,h}x_t - \Psi_{i,1,h}\mathbb{E}(x_t|w_t) + u_{i,1,t+h}, \end{aligned}$$

What precedes implies that the expectation of the blue term, conditional on  $x_t$  and  $w_t$ , is linear in  $w_t$ . Standard results in the conditional mean independence case imply that the regression of  $y_{i,t+h}$  on  $x_t$ , controlling for  $w_t$ , provides a consistent estimate of  $\Psi_{i,1,h}$ :

$$y_{i,t+h} = \alpha_i + \Psi_{i,1,h}x_t + \beta'w_t + v_{i,t+h}. \quad (2.65)$$

This is for instance consistent with the case where  $[\Delta GDP_t, \pi_t, i_t]'$  follows a VAR(1) and the monetary-policy shock do not contemporaneously affect  $\Delta GDP_t$  and  $\pi_t$ .

The IRFs can be estimated by LP, taking  $x_t = i_t$  and  $w_t = [\Delta GDP_t, \pi_t, \Delta GDP_{t-1}, \pi_{t-1}, i_{t-1}]'$ .

This approach closely relates to the SVAR Cholesky-based identification approach. Specifically, if  $w_t = [y_{1,t}, \dots, y_{k-1,t}, y'_{t-1}, \dots, y'_{t-p}]'$ , with  $k \leq n$ , and  $x_t = y_{k,t}$ , then this approach corresponds, for  $h = 0$ , to the SVAR( $p$ ) Cholesky-based IRF (focusing on the responses to the  $k^{th}$  structural shock). However, the two approaches differ for  $h > 0$ , because the LP methodology does not assume a VAR dynamics for  $y_t$ .<sup>4</sup>

### Situation B: IV approach

Consider now that we have a valid instrument  $z_t$  for  $\eta_{1,t}$  (with  $\mathbb{E}(z_t) = 0$ ). That is:

$$\begin{cases} (IV.i) & \mathbb{E}(z_t\eta_{1,t}) \neq 0 & \text{(relevance condition)} \\ (IV.ii) & \mathbb{E}(z_t\eta_{j,t}) = 0 \text{ for } j > 1 & \text{(exogeneity condition)} \end{cases} \quad (2.66)$$

The instrument  $z_t$  can be used to identify the structural shock. Eq. (2.66) implies that there exist  $\rho \neq 0$  and a mean-zero variable  $\xi_t$  such that:

$$\eta_{1,t} = \rho z_t + \xi_t,$$

where  $\xi_t$  is correlated neither to  $z_t$ , nor to  $\eta_{j,t}$ ,  $j \geq 2$ .

*Proof.* Define  $\rho = \frac{\mathbb{E}(\eta_{1,t}z_t)}{\mathbb{E}(z_t^2)}$  and  $\xi_t = \eta_{1,t} - \rho z_t$ . It is easily seen that  $\xi_t$  satisfies the moment restrictions given above.  $\square$

Ramey (2016) reviews the different approaches employed to construct monetary policy-shocks (the two main approaches are presented in 2.9 and 2.10 below). She has also collected time series of such shocks, see her website.

<sup>4</sup>This is reminiscent of the distinction between direct forecasting –based on regressions of  $y_{t+h}$  on  $\{y_t, y_{t-1}, \dots\}$ – and iterated forecasting –based on a recursive model where  $y_{t+1} = g(y_t, y_{t-1}, \dots) + \varepsilon_{t+1}$ , see Marcellino et al. (2006).

**Example 2.9** (Identification of Monetary-Policy Shocks Based on High-Frequency Data). Instruments for monetary-policy shocks can be extracted from high-frequency market data associated with interest-rate products.

The quotes of all interest-rate-related financial products are sensitive to monetary-policy announcements. That is because these quotes mainly depends on investors' expectations regarding future short-term rates:  $\mathbb{E}_t(i_{t+s})$ . Typically, if agents were risk-neutral, the maturity- $h$  interest rate would approximatively be given by:

$$i_{t,h} \approx \mathbb{E}_t \left( \frac{1}{h} \int_0^h i_{t+s} ds \right) = \frac{1}{h} \int_0^h \mathbb{E}_t(i_{t+s}) ds.$$

In general, changes in  $\mathbb{E}_t(i_{t+s})$ , for  $s > 0$ , can be affected by all types of shocks that may trigger a reaction by the central bank.

However, if a MP announcement takes place between  $t$  and  $t + \epsilon$ , then most of  $\mathbb{E}_{t+\epsilon}(i_{t+s}) - \mathbb{E}_t(i_{t+s})$  is to be attributed to the MP shock (see Figure 2.24, from Gürkaynak et al. (2005)). Hence, a monthly time series of MP shocks can be obtained by summing, over each month, the changes  $i_{t+\epsilon,h} - i_{t,h}$  associated with a given interest rate (T-bills, futures, swaps) and a given maturity  $h$ .

See among others: Kuttner (2001), Cochrane and Piazzesi (2002), Gürkaynak et al. (2005), Piazzesi and Swanson (2008), Gertler and Karadi (2015).

**Figure 1. Intraday Trading in Federal Funds Futures Contracts**

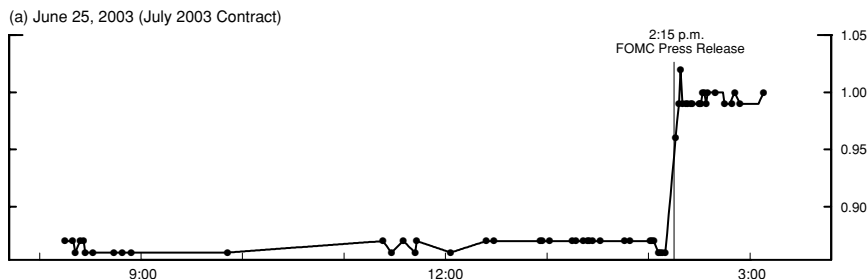


Figure 2.24: Source: Gürkaynak, Sack and Swanson (2005). Transaction rates of Federal funds futures on June 25, 2003, day on which a regularly scheduled FOMC meeting was scheduled. At 2:15 p.m., the FOMC announced that it was lowering its target for the federal funds rate from 1.25% to 1%, while many market participants were expecting a 50 bp cut. This shows that (i) financial markets seem to fully adjust to the policy action within just a few minutes and (ii) the federal funds rate surprise is not necessarily in the same direction as the federal funds rate action itself.

**Example 2.10** (Identification of Monetary-Policy Shocks Based on the Narrative Approach). Romer and Romer (2004) propose a two-step approach:

- a. derive a series for Federal Reserve intentions for the federal funds rate (the explicit target of the Fed) around FOMC meetings,
- b. control for Federal Reserve forecasts.

This gives a measure of intended monetary policy actions not driven by information about future economic developments. a. “intentions” are measured as a combination of narrative and quantitative evidence. Sources: (among others) Minutes of FOMC and “Blue Books”. b. Controls = variables spanning the information the Federal Reserve has about future developments. Data: Federal Reserve’s internal forecasts (inflation, real output and unemployment), “Greenbook’s forecasts” – usually issued 6 days before the FOMC meeting.

The shock measure is the residual series in the linear regression of (a) on (b).

There are two main IV approaches to estimate IRFs see Stock and Watson (2018):

- a. The LP-IV approach, where  $y_t$ 's DGP is left unspecified,

b. The SVAR-IV approach.

The LP-IV approach is based on a set of IV regressions (for each variable of interest, one for each forecast horizon). The SVAR-IV approach is based on IV regressions of VAR innovations only (one for each series of VAR innovations).

If the VAR adequately captures the DGP, then the IV-SVAR is optimal for all horizons. However, if the VAR is misspecified, then specification errors are compounded at each horizon and a local projection method would lead to better results.

### Situation B.1: SVAR-IV approach

Assume you have consistent estimates of  $\varepsilon_t = B\eta_t$ , these estimates ( $\hat{\varepsilon}_t$ ) coming from the estimation of a VAR model. You have, for  $i \in \{1, \dots, n\}$ :

$$\begin{aligned} \varepsilon_{i,t} &= b_{i,1}\eta_{1,t} + u_{i,t} (\#eq:eps_rho) \\ &= b_{i,1}\rho z_t + \underbrace{b_{i,1}\xi_t + u_{i,t}}_{\perp z_t}. \end{aligned} \quad (2.67)$$

( $u_{i,t}$  is a linear combination of the  $\eta_{j,t}$ 's,  $j \geq 2$ ).

Hence, up to a multiplicative factor ( $\rho$ ), the (OLS) regressions of the  $\hat{\varepsilon}_{i,t}$ 's on  $z_t$  provide consistent estimates of the  $b_{i,1}$ 's.

Combined with the estimated VAR (the  $\Phi_k$  matrices), this provides consistent estimates of the IRFs of  $\eta_{1,t}$  on  $y_t$ , though up to a multiplicative factor. This scale ambiguity can be solved by rescaling the structural shock ("unit-effect normalisation", see Stock and Watson (2018)). Let us consider  $\tilde{\eta}_{1,t} = b_{1,1}\eta_{1,t}$ ; by construction,  $\tilde{\eta}_{1,t}$  has a one-unit contemporaneous effect on  $y_{1,t}$ . Denoting by  $\tilde{B}_{i,1}$  the contemporaneous impact of  $\tilde{\eta}_{1,t}$ , we get:

$$\tilde{B}_1 = \frac{1}{b_{1,1}} B_1,$$

where  $B_1$  denotes the 1<sup>st</sup> column of  $B$  and  $\tilde{B}_1 = [1, \tilde{B}_{2,1}, \dots, \tilde{B}_{n,1}]'$ .

Eq. @ref(eq:eps\_rho) gives:

$$\begin{aligned} \varepsilon_{1,t} &= \tilde{\eta}_{1,t} + u_{1,t} \\ \varepsilon_{i,t} &= \tilde{B}_{i,1}\tilde{\eta}_{1,t} + u_{i,t}. \end{aligned}$$

This suggests that  $\tilde{B}_{i,1}$  can be estimated by regressing  $\varepsilon_{i,t}$  on  $\varepsilon_{1,t}$ , using  $z_t$  as an instrument.

What about inference? One cannot use the usual TSLS standard deviations because the  $\varepsilon_{i,t}$ 's are not directly observed. Bootstrap procedures can be resorted to. Stock and Watson (2018) propose, in particular, a Gaussian parametric bootstrap:

Assume you have estimated  $\{\hat{\Phi}_1, \dots, \hat{\Phi}_p, \hat{B}_1\}$  using the SVAR-IV approach based on a size- $T$  sample. Generate  $N$  (where  $N$  is large) size- $T$  samples from the following VAR:

$$\begin{aligned} &\begin{bmatrix} \hat{\Phi}(L) & 0 \\ 0 & \hat{\rho}(L) \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \varepsilon_t \\ e_t \end{bmatrix}, \\ \text{where } &\begin{bmatrix} \varepsilon_t \\ e_t \end{bmatrix} \sim i.i.d. \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \Omega & S'_{\varepsilon,e} \\ S_{\varepsilon,e} & \sigma_e^2 \end{bmatrix} \right), \end{aligned}$$

where  $\hat{\rho}(L)$  and  $\sigma_e^2$  result from the estimation of an AR process for  $z_t$ , and where  $\Omega$  and  $S_{\varepsilon,e}$  are sample covariances for the VAR/AR residuals.

For each simulated sample (of  $\tilde{y}_t$  and  $\tilde{z}_t$ , say), estimate  $\{\tilde{\Phi}_1, \dots, \tilde{\Phi}_p, \tilde{B}_1\}$  and associated  $\tilde{\Psi}_{i,1,h}$ . This provides e.g. a sequence of  $N$  estimates of  $\Psi_{i,1,h}$ , from which quantiles and conf. intervals can be deduced.



```

# Load vars package:
library(vars)
library(AEC)
data("USmonthly")
First.date <- "1990-05-01"
Last.date <- "2012-6-01"
indic.first <- which(USmonthly$DATES==First.date)
indic.last <- which(USmonthly$DATES==Last.date)
USmonthly <- USmonthly[indic.first:indic.last,]
shock.name <- "FF4_TC" #"FF4_TC", "ED2_TC", "ff1_vr", "rrshock83b"
indic.shock.name <- which(names(USmonthly)==shock.name)
Z <- matrix(USmonthly[,indic.shock.name],ncol=1)
par(plt=c(.1,.95,.1,.95))
plot(USmonthly$DATES,Z,type="l",xlab="",ylab="",lwd=2)

```

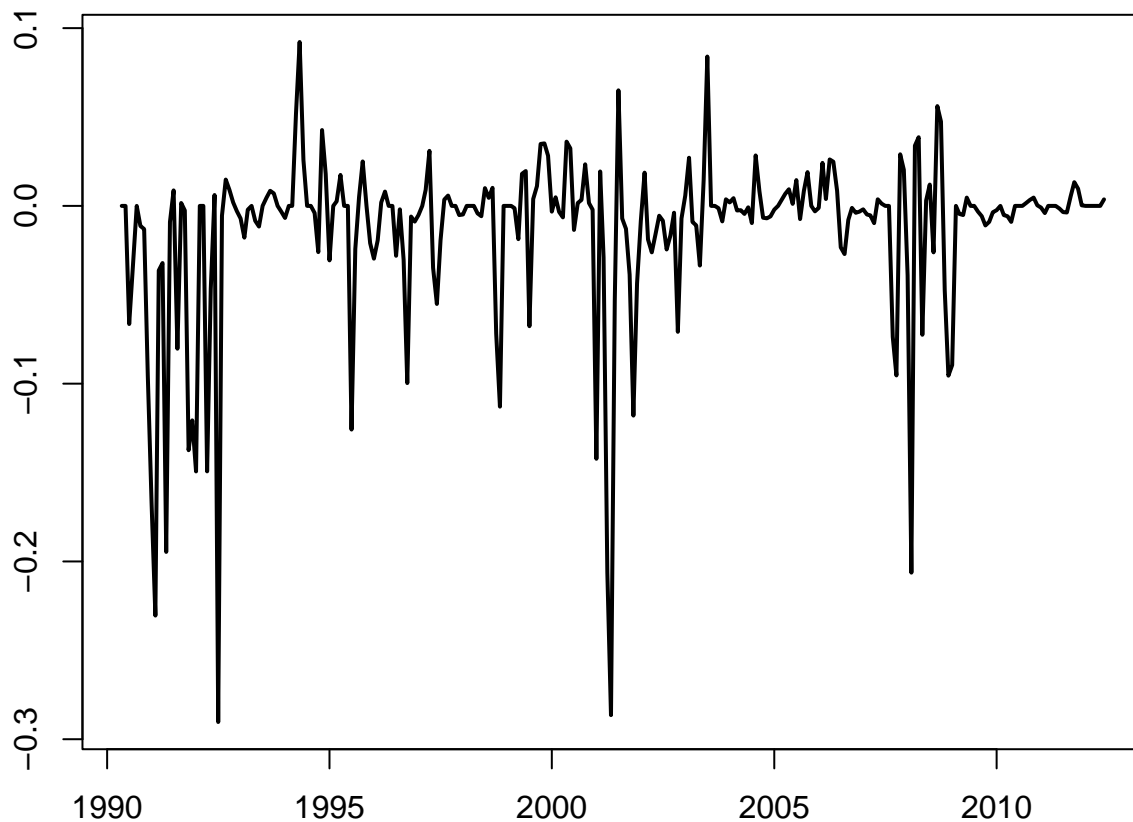


Figure 2.25: Gertler-Karadi monthly shocks, fed funds futures 3 months.

```

considered.variables <- c("GS1","LIP","LCPI","EBP")
y <- as.matrix(USmonthly[,considered.variables])
n <- length(considered.variables)
colnames(y) <- considered.variables
par(plt=c(.15,.95,.15,.8))
res.svar.iv <-
  svar.iv(y,Z,p = 4,names.of.variables=considered.variables,
          nb.periods.IRF = 20,
          z.AR.order=1,
          nb.bootstrap.replications = 100,
          confidence.interval = 0.90,
          indic.plot=1)

```

### Situation B.2: LP-IV

If you do not want to posit a VAR-type dynamics for  $y_t$  –e.g. because you suspect that the true generating model may be a non-invertible VARMA model– you can directly proceed by IV-projection methods to obtain the  $\tilde{\Psi}_{i,1,h} \equiv \Psi_{i,1,h}/b_{1,1}$  (that are the IRFs of  $\tilde{\eta}_{1,t}$  on  $y_{i,t}$ ). However, Assumptions (IV.i) and (IV.ii) (Eq. (2.66)) have to be complemented with (IV.iii):

$$(IV.iii) \quad \mathbb{E}(z_t \eta_{j,t+h}) = 0 \text{ for } h \neq 0 \quad (\text{lead-lag exogeneity})$$



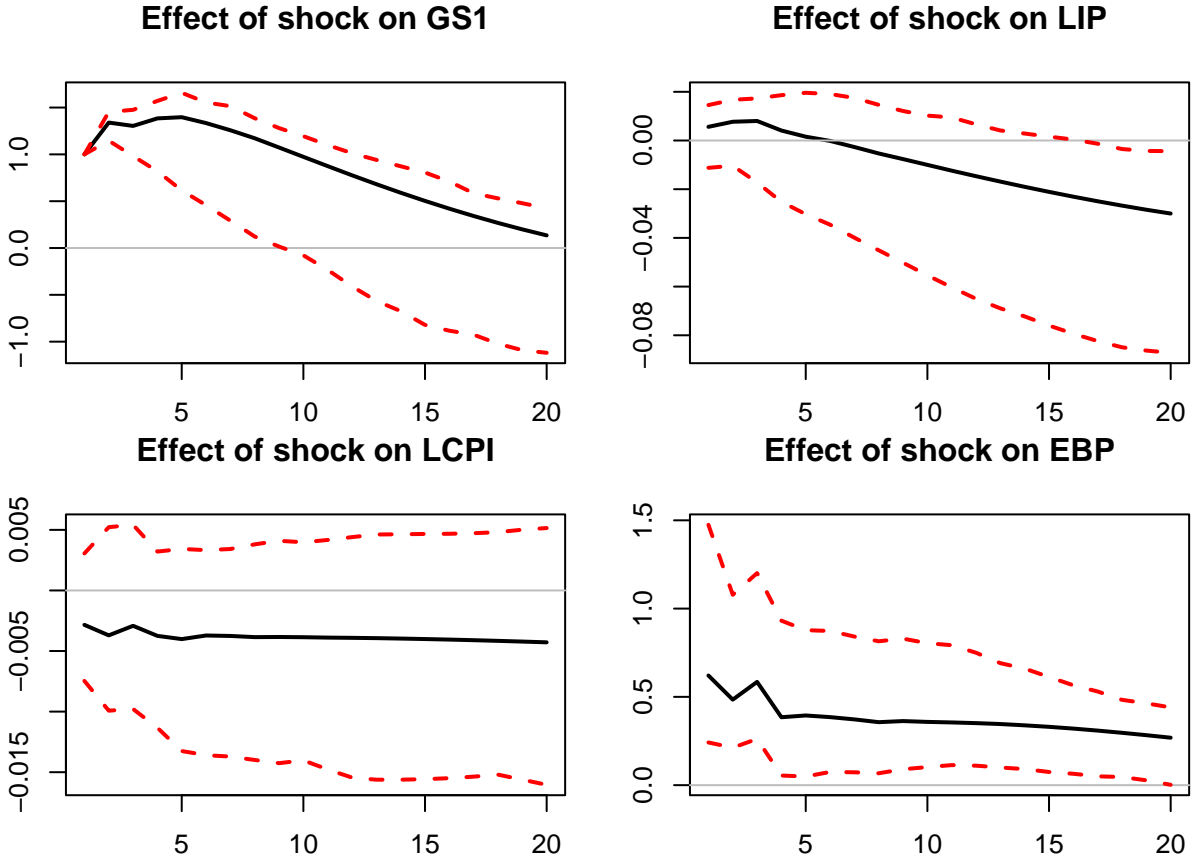


Figure 2.26: Responses to a monetary-policy shock, SVAR-IV approach.

When (IV.i), (IV.ii) and (IV.iii) are satisfied,  $\tilde{\Psi}_{i,1,h}$  can be estimated by regressing  $y_{i,t+h}$  on  $y_{1,t}$ , using  $z_t$  as an instrument, i.e. by considering the TSLS estimation of:

$$y_{i,t+h} = \alpha_i + \tilde{\Psi}_{i,1,h} y_{1,t} + \nu_{i,t+h}, \quad (2.68)$$

where  $\nu_{i,t+h}$  is correlated to  $y_{1,t}$ , but not to  $z_t$ .

We have indeed:

$$\begin{aligned} y_{1,t} &= \alpha_1 + \tilde{\eta}_{1,t} + v_{1,t} \\ y_{i,t+h} &= \alpha_i + \tilde{\Psi}_{i,1,h} \tilde{\eta}_{1,t} + v_{i,t+h}, \end{aligned}$$

where the  $v_{i,t+h}$ 's are uncorrelated to  $z_t$  under (IV.i), (IV.ii) and (IV.iii).

Note again that, for  $h > 0$ , the  $v_{i,t+h}$  (and  $\nu_{i,t+h}$ ) are auto-correlated. Newey-West corrections therefore have to be used to compute std errors of the  $\tilde{\Psi}_{i,1,h}$ 's estimates.

Consider the linear regression:

$$\mathbf{Y} = \mathbf{X}\beta + \varepsilon,$$

where  $\mathbb{E}(\varepsilon) = 0$ , but where the explicative variables  $\mathbf{X}$  are supposed to be correlated to the residuals  $\varepsilon$ .

Moreover, the  $\varepsilon$  are supposed to be possibly heteroskedastic and auto-correlated.

We consider the instruments  $\mathbf{Z}$ , with  $\mathbb{E}(\mathbf{X}'\mathbf{Z}) \neq 0$  but  $\mathbb{E}(\varepsilon'\mathbf{Z}) = 0$ .

The IV estimator of  $\beta$  is obtained by regressing  $\hat{\mathbf{Y}}$  on  $\hat{\mathbf{X}}$ , where  $\hat{\mathbf{Y}}$  and  $\hat{\mathbf{X}}$  are the respective residuals of the regressions of  $\mathbf{Y}$  and  $\mathbf{X}$  on  $\mathbf{Z}$ .

$$\begin{aligned} \mathbf{b}_{iv} &= [\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y} \\ \mathbf{b}_{iv} &= \beta + \underbrace{\frac{1}{\sqrt{T}} T[\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}]^{-1}\mathbf{X}'\mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}}_{=Q(\mathbf{X},\mathbf{Z}) \xrightarrow{p} \mathbf{Q}_{xz}} \underbrace{\sqrt{T} \left( \frac{1}{T} \mathbf{Z}'\varepsilon \right)}_{\xrightarrow{d} \mathcal{N}(0,\mathbf{S})}, \end{aligned}$$

where  $\mathbf{S}$  is the long-run variance of  $\mathbf{z}_t\varepsilon_t$  (see next slide).

The asymptotic covariance matrix of  $\sqrt{T}\mathbf{b}_{iv}$  is  $\mathbf{Q}_{xz}\mathbf{S}\mathbf{Q}_{xz}'$ .

The covariance matrix of  $\mathbf{b}_{iv}$  can be approximated by  $\frac{1}{T}Q(\mathbf{X},\mathbf{Z})\hat{\mathbf{S}}Q(\mathbf{X},\mathbf{Z})'$  where  $\hat{\mathbf{S}}$  is the Newey-West estimator of  $\mathbf{S}$  (see Eq. (??))

(IV.iii) is usually not restrictive for  $h > 0$  ( $z_t$  is usually not affected by future shocks). By contrast, it may be restrictive for  $h < 0$ . This can be solved by adding controls in Regression (2.68). These controls should span the space of  $\{\eta_{t-1}, \eta_{t-2}, \dots\}$ .

If  $z_t$  is suspected to be correlated to past values of  $\eta_{1,t}$  but not to the  $\eta_{j,t}$ 's,  $j > 1$ , then one can add lags of  $z_t$  as controls (method e.g. advocated by Ramey, 2016, p.108, considering the instrument by Gertler and Karadi (2015)).

In the general case, one can use lags of  $y_t$  as controls. Note that, even if (IV.iii) holds, adding controls may reduce the variance of the regression error.

As noted by Stock and Watson (2018), the relevant variance is the long-run variance of the instrument-times-error term. They also recommend (p.926) using leads and lags of  $z_t$  to improve efficiency.

```
res.LP.IV <- make.LPIV.irf(y,Z,
                           nb.periods.IRF = 20,
                           nb.lags.Y.4.control=4,
                           nb.lags.Z.4.control=4,
                           indic.plot = 1, # Plots are displayed if = 1.
                           confidence.interval = 0.90)
```

### 2.3.12 Inference

Consider the following SVAR model:

$$y_t = \Phi_1 y_{t-1} + \dots + \Phi_p y_{t-p} + \varepsilon_t$$

with  $\varepsilon_t = B\eta_t$ ,  $\Omega_\varepsilon = BB'$ .

The corresponding infinite MA representation (Eq. (2.38), or Wold theorem, Theorem 2.3) is:

$$y_t = \sum_{h=0}^{\infty} \Psi_h \eta_{t-h},$$

where  $\Psi_0 = B$  and for  $h = 1, 2, \dots$ :

$$\Psi_h = \sum_{j=1}^h \Psi_{h-j} \Phi_j,$$

with  $\Phi_j = 0$  for  $j > p$  (see Prop. 2.8 for this recursive computation of the  $\Psi_j$ 's).

Inference on the VAR coefficients  $\{\Phi_j\}_{j=1,\dots,p}$  is straightforward (standard OLS inference). But inference is more complicated regarding IRF. Indeed, as shown by the previous equation, the (infinite) MA coefficients  $\{\Psi_j\}_{j=1,\dots}$  are non-linear functions of the  $\{\Phi_j\}_{j=1,\dots,p}$  and of  $\Omega_\varepsilon$ . An other issue pertain to small sample bias: typically, for persistent process, auto-regressive parameters are known to be downward biased.

The main inference methods are the following:

- Monte Carlo method (Hamilton (1994))
- Asymptotic normal approximation (Lütkepohl (1990)), or Delta method
- Bootstrap method (Kilian\_1998)

#### Monte Carlo method

We use Monte Carlo when we need to approximate the distribution of a variable whose distribution is unknown (here: the  $\Psi_j$ 's) but which is a function of another variable whose distribution is known (here, the  $\Phi_j$ 's).

For instance, suppose we know the distribution of a random variable  $X$ , which takes values in  $\mathbb{R}$ , with density function  $p$ . Assume we want to compute the mean of  $\varphi(X)$ . We have:

$$\mathbb{E}(\varphi(X)) = \int_{-\infty}^{+\infty} \varphi(x)p(x)dx$$

Suppose that the above integral does not have a simple expression. We cannot compute  $\mathbb{E}(\varphi(X))$  but, by virtue of the law of large numbers (Theorem ??), we can approximate it as follows:

$$\mathbb{E}(\varphi(X)) \approx \frac{1}{N} \sum_{i=1}^N \varphi(X^{(i)}),$$

where  $\{X^{(i)}\}_{i=1,\dots,N}$  are  $N$  independent draws of  $X$ . More generally, the distribution of  $\varphi(X)$  can be approximated by the empirical distribution of the  $\varphi(X^{(i)})$ 's. Typically, if 10'000 values of  $\varphi(X^{(i)})$  are drawn, the 5<sup>th</sup> percentile of the p.d.f. of  $\varphi(X)$  can be approximated by the 500<sup>th</sup> value of the 10'000 draws of  $\varphi(X^{(i)})$  (after arranging these values in ascending order).

As regards the computation of confidence intervals around IRFs, one has to think of  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}$ , and of  $\widehat{\Omega}$  as  $X$  and  $\{\widehat{\Psi}_j\}_{j=1,\dots,p}$  as  $\varphi(X)$ . (Proposition 2.14 provides us with the asymptotic distribution of the “ $X$ .”)

To summarize, here are the steps one can implement to derive confidence intervals for the IRFs using the Monte-Carlo approach:

For each iteration  $k$ :

1. Draw  $\{\widehat{\Phi}_j^{(k)}\}_{j=1,\dots,p}$  and  $\widehat{\Omega}^{(k)}$  from their asymptotic distribution (using Proposition 2.14).
2. Compute the matrix  $B^{(k)}$  so that  $\widehat{\Omega}^{(k)} = B^{(k)}B^{(k)'}$ , according to your identification strategy.
3. Compute the associated IRFs  $\{\widehat{\Psi}_j\}^{(k)}$ .

Perform  $N$  replications and report the median impulse response (and its confidence intervals).

### Delta method

Suppose  $\beta$  is a vector of parameters and  $\widehat{\beta}$  is an estimator such that

$$\sqrt{T}(\widehat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, \Sigma_\beta),$$

where  $d$  denotes convergence in distribution,  $\mathcal{N}(0, \Sigma_\beta)$  denotes the multivariate normal distribution with mean vector 0 and covariance matrix  $\Sigma_\beta$  and  $T$  is the sample size used for estimation.

Let  $g(\beta) = (g_1(\beta), \dots, g_m(\beta))'$  be a continuously differentiable function with values in  $\mathbb{R}^m$ , and assume that  $\partial g_i / \partial \beta' = (\partial g_i / \partial \beta_j)$  is nonzero at  $\beta$  for  $i = 1, \dots, m$ . Then

$$\sqrt{T}(g(\widehat{\beta}) - g(\beta)) \xrightarrow{d} \mathcal{N}\left(0, \frac{\partial g}{\partial \beta'} \Sigma_\beta \frac{\partial g'}{\partial \beta}\right).$$

(This formula underlies the Delta method, see Eq. (??).)

Using this property, Lütkepohl (1990) provides the asymptotic distributions of the  $\Psi_j$ 's.

A limit of the last two approaches (Monte Carlo and the Delta method) is that they critically rely on asymptotic results. Bootstrapping approaches are more robust in small-sample situations.

### Bootstrap

IRFs' confidence intervals are intervals where 90% (or 95%, 75%, ...) of the IRFs would lie, if we were to repeat the estimation a large number of times in similar conditions ( $T$  observations). We obviously cannot do this, because we have only one sample:  $\{y_t\}_{t=1,\dots,T}$ . But we can try to *construct* such samples.

Bootstrapping consists in:

- re-sampling  $N$  times, i.e., constructing  $N$  samples of  $T$  observations, using the estimated VAR coefficients and
  - a. a sample of residuals from the distribution  $N(0, BB')$  (**parametric approach**), or

- b. a sample of residuals drawn randomly from the set of the actual estimated residuals  $\{\hat{\varepsilon}_t\}_{t=1,\dots,T}$ . (**non-parametric approach**).
- re-estimating the SVAR  $N$  times.

Here is the algorithm:

1. Construct a sample

$$y_t^{(k)} = \widehat{\Phi}_1 y_{t-1}^{(k)} + \dots + \widehat{\Phi}_p y_{t-p}^{(k)} + \hat{\varepsilon}_t^{(k)},$$

with  $\hat{\varepsilon}_t^{(k)} = \hat{\varepsilon}_{s_t^{(k)}}^{(k)}$ , where  $\{s_1^{(k)}, \dots, s_T^{(k)}\}$  is a random set from  $\{1, \dots, T\}^T$ .

2. Re-estimate the SVAR and compute the IRFs  $\{\widehat{\Psi}_j\}^{(k)}$ .

Perform  $N$  replications and report the median impulse response (and its confidence intervals).

### Bootstrap-after-bootstrap (Kilian (1998))

The previous simple bootstrapping procedure deals with non-normality and small sample distribution, since we use the actual residuals. However, it does not deal with the *small sample bias*, stemming, in particular, from small-sample bias associated with OLS coefficient estimates  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}$ . The main idea of the bootstrap-after-bootstrap of Kilian (1998) is to run two consecutive bootstraps: the objective of the first is to compute the bias, which can further be used to correct the initial estimates of the  $\Phi_i$ 's. Further, these corrected estimates are used—in the second bootstrap—to compute a set of IRFs (as in the standard bootstrap).

More formally, the algorithm is as follows:

1. Estimate the SVAR coefficients  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}$  and  $\widehat{\Omega}$

2. **First bootstrap.** For each iteration  $k$ :

- a. Construct a sample

$$y_t^{(k)} = \widehat{\Phi}_1 y_{t-1}^{(k)} + \dots + \widehat{\Phi}_p y_{t-p}^{(k)} + \hat{\varepsilon}_t^{(k)},$$

with  $\hat{\varepsilon}_t^{(k)} = \hat{\varepsilon}_{s_t^{(k)}}^{(k)}$ , where  $\{s_1^{(k)}, \dots, s_T^{(k)}\}$  is a random set from  $\{1, \dots, T\}^T$ .

- b. Re-estimate the VAR and compute the coefficients  $\{\widehat{\Phi}_j\}_{j=1,\dots,p}^{(k)}$ .

3. Perform  $N$  replications and compute the median coefficients  $\{\widehat{\Phi}_j^*\}_{j=1,\dots,p}$ .

4. Approximate the bias terms by  $\widehat{\Theta}_j = \widehat{\Phi}_j^* - \widehat{\Phi}_j$ .

5. Construct the bias-corrected terms  $\widetilde{\Phi}_j = \widehat{\Phi}_j - \widehat{\Theta}_j$ .

6. **Second bootstrap.** For each iteration  $k$ :

- a. Construct a sample now from

$$y_t^{(k)} = \widetilde{\Phi}_1 y_{t-1}^{(k)} + \dots + \widetilde{\Phi}_p y_{t-p}^{(k)} + \hat{\varepsilon}_t^{(k)}.$$

- b. Re-estimate the VAR and compute the coefficients  $\{\widehat{\Phi}_j^*\}_{j=1,\dots,p}^{(k)}$ .

- c. Construct the bias-corrected estimates  $\widetilde{\Phi}_j^{*(k)} = \widehat{\Phi}_j^{*(k)} - \widehat{\Theta}_j$ .

- d. Compute the associated IRFs  $\{\widetilde{\Psi}_j^{*(k)}\}_{j \geq 1}$ .

7. Perform  $N$  replications and compute the median and the confidence interval of the set of IRFs.

It should be noted that correcting for the bias can generate non-stationary results ( $\widetilde{\Phi}$  with eigenvalue with modulus  $> 1$ ). Solution (Kilian (1998)):

In step 5, check if the largest eigenvalue of  $\widetilde{\Phi}$  is of modulus  $< 1$ . If not, shrink the bias: for all  $j$ s, set  $\widehat{\Theta}_j^{(i+1)} = \delta_{i+1} \widehat{\Theta}_j^{(i)}$ , with  $\delta_{i+1} = \delta_i - 0.01$ , starting with  $\delta_1 = 1$  and  $\widehat{\Theta}_j^{(1)} = \widehat{\Theta}_j$ , and compute  $\widetilde{\Phi}_j^{(i+1)} = \widehat{\Phi}_j - \widehat{\Theta}_j^{(i+1)}$  until the largest eigenvalue of  $\widetilde{\Phi}^{(i+1)}$  has modulus  $< 1$ .

Function `VAR.Boot` of package `VAR.etp` (Kim (2022)) can be used to operate the bias-correction approach of Kilian (1998):

```

library(VAR.etc)
library(vars) #standard VAR models
data(dat) # part of VAR.etc package
corrected <- VAR.Boot(dat,p=2,nb=200,type="const")
noncorrec <- VAR(dat,p=2)
rbind(corrected$coef[1,],
      (corrected$coef+corrected$Bias)[1,],
      noncorrec$varresult$inv$coefficients)

##          inv(-1)  inc(-1)  con(-1)  inv(-2)  inc(-2)  con(-2)  const
## [1,] -0.3142137 0.1231657 1.076604 -0.1410342 0.07061601 1.0397597 -0.01983903
## [2,] -0.3196310 0.1459888 0.961219 -0.1605511 0.11460498 0.9343938 -0.01672199
## [3,] -0.3196310 0.1459888 0.961219 -0.1605511 0.11460498 0.9343938 -0.01672199

```

## 2.4 Forecasting

Forecasting has always been an important part of the time series field (De Gooijer and Hyndman (2006)). Macroeconomic forecasts are done in many places: Public Administration (notably Treasuries), Central Banks, International Institutions (e.g. IMF, OECD), banks, big firms. These institutions are interested in the **point estimates** ( $\sim$  most likely value) of the variable of interest. They also sometimes need to measure the **uncertainty** ( $\sim$  dispersion of likely outcomes) associated to the point estimates.<sup>5</sup>

Forecasts produced by professional forecasters are available on these web pages:

- Philly Fed Survey of Professional Forecasters.
- ECB Survey of Professional Forecasters.
- IMF World Economic Outlook.
- OECD Global Economic Outlook.
- European Commission Economic Forecasts.

How to formalize the forecasting problem? Assume the current date is  $t$ . We want to forecast the value that variable  $y_t$  will take on date  $t + 1$  (i.e.,  $y_{t+1}$ ) based on the observation of a set of variables gathered in vector  $x_t$  ( $x_t$  may contain lagged values of  $y_t$ ).

The forecaster aims at minimizing (a function of) the forecast error. It is usual to consider the following (quadratic) loss function:

$$\underbrace{\mathbb{E}([y_{t+1} - y_{t+1}^*]^2)}_{\text{Mean square error (MSE)}}$$

where  $y_{t+1}^*$  is the forecast of  $y_{t+1}$  (function of  $x_t$ ).

**Proposition 2.15** (Smallest MSE). *The smallest MSE is obtained with MSE the expectation of  $y_{t+1}$  conditional on  $x_t$ .*

*Proof.* See Appendix 3.5. □

**Proposition 2.16.** *Among the class of linear forecasts, the smallest MSE is obtained with the linear projection of  $y_{t+1}$  on  $x_t$ . This projection, denoted by  $\hat{P}(y_{t+1}|x_t) := \alpha'x_t$ , satisfies:*

$$\mathbb{E}([y_{t+1} - \alpha'x_t]x_t) = \mathbf{0}. \quad (2.69)$$

*Proof.* Consider the function  $f : \alpha \rightarrow \mathbb{E}([y_{t+1} - \alpha'x_t]^2)$ . We have:

$$f(\alpha) = \mathbb{E}(y_{t+1}^2 - 2y_t x_t' \alpha + \alpha' x_t x_t' \alpha).$$

We have  $\partial f(\alpha)/\partial \alpha = \mathbb{E}(-2y_{t+1}x_t + 2x_t x_t' \alpha)$ . The function is minimised for  $\partial f(\alpha)/\partial \alpha = 0$ . □

---

<sup>5</sup>In its inflation report, the Bank of England displays charts showing the conditional distribution of future inflation, called fan charts. This fan charts show the uncertainty associated with future inflation. See this page.

Eq. (2.69) implies that  $\mathbb{E}(y_{t+1}x_t) = \mathbb{E}(x_t x_t') \alpha$ . (Note that  $x_t x_t' \alpha = x_t(x_t' \alpha) = (\alpha' x_t)x_t'$ .) Hence, if  $\mathbb{E}(x_t x_t')$  is nonsingular,

$$\alpha = [\mathbb{E}(x_t x_t')]^{-1} \mathbb{E}(y_{t+1} x_t). \quad (2.70)$$

The MSE then is:

$$\mathbb{E}([y_{t+1} - \alpha' x_t]^2) = \mathbb{E}(y_{t+1}^2) - \mathbb{E}(y_{t+1} x_t') [\mathbb{E}(x_t x_t')]^{-1} \mathbb{E}(x_t y_{t+1}).$$

Consider the regression  $y_{t+1} = \beta' \mathbf{x}_t + \varepsilon_{t+1}$ . The OLS estimate is:

$$\mathbf{b} = \left[ \underbrace{\frac{1}{T} \sum_{i=1}^T \mathbf{x}_i \mathbf{x}_i'}_{\mathbf{m}_1} \right]^{-1} \left[ \underbrace{\frac{1}{T} \sum_{i=1}^T \mathbf{x}_i' y_{i+1}}_{\mathbf{m}_2} \right].$$

If  $\{x_t, y_t\}$  is covariance-stationary and ergodic for the second moments then the sample moments ( $\mathbf{m}_1$  and  $\mathbf{m}_2$ ) converges in probability to the associated population moments and  $\mathbf{b} \xrightarrow{p} \alpha$  (where  $\alpha$  is defined in Eq. (2.70)).

**Example 2.11** (Forecasting an MA(q) process). Consider the MA(q) process:

$$y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \dots + \theta_q \varepsilon_{t-q},$$

where  $\{\varepsilon_t\}$  is a white noise sequence (Def. 2.1).

We have:

$$\mathbb{E}(y_{t+h} | \varepsilon_t, \varepsilon_{t-1}, \dots) = \begin{cases} \mu + \theta_h \varepsilon_t + \dots + \theta_q \varepsilon_{t-q+h} & \text{for } h \in [1, q] \\ \mu & \text{for } h > q \end{cases}$$

and

$$\text{Var}(y_{t+h} | \varepsilon_t, \varepsilon_{t-1}, \dots) = \mathbb{E}([y_{t+h} - \mathbb{E}(y_{t+h} | \varepsilon_t, \varepsilon_{t-1}, \dots)]^2) = \begin{cases} \sigma^2(1 + \theta_1^2 + \dots + \theta_{h-1}^2) & \text{for } h \in [1, q] \\ \sigma^2(1 + \theta_1^2 + \dots + \theta_q^2) & \text{for } h > q. \end{cases}$$

Remark: The previous reasoning relies on the assumption that the  $\varepsilon_t$ s are observed. But this is generally not the case in practice. Note that consistent estimates are available if the MA process is invertible (see Eq. (2.32)).

**Example 2.12** (Forecasting an AR(p) process). (See this web interface.) Consider the AR(p) process:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_p y_{t-p} + \varepsilon_t,$$

where  $\{\varepsilon_t\}$  is a white noise sequence (Def. 2.1).

Using the notation of Eq. (2.9), we have:

$$\mathbf{y}_t - \mu = F(\mathbf{y}_{t-1} - \mu) + \xi_t,$$

with  $\mu = [\mu, \dots, \mu]'$  ( $\mu$  is defined in Eq. (2.14)). Hence:

$$\mathbf{y}_{t+h} - \mu = \xi_{t+h} + F\xi_{t+h-1} + \dots + F^{h-1}\xi_{t+1} + F^h(\mathbf{y}_t - \mu).$$

Therefore:

$$\begin{aligned} \mathbb{E}(\mathbf{y}_{t+h} | y_t, y_{t-1}, \dots) &= \mu + F^h(\mathbf{y}_t - \mu) \\ \text{Var}([\mathbf{y}_{t+h} - \mathbb{E}(\mathbf{y}_{t+h} | y_t, y_{t-1}, \dots)]) &= \Sigma + F\Sigma F' + \dots + F^{h-1}\Sigma(F^{h-1})', \end{aligned}$$

where:

$$\Sigma = \begin{bmatrix} \sigma^2 & 0 & \dots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix}.$$

Alternative approach: Taking the (conditional) expectations of both sides of

$$y_{t+h} - \mu = \phi_1(y_{t+h-1} - \mu) + \phi_2(y_{t+h-2} - \mu) + \cdots + \phi_p(y_{t-p} - \mu) + \varepsilon_{t+h},$$

we obtain:

$$\begin{aligned} \mathbb{E}(y_{t+h}|y_t, y_{t-1}, \dots) &= \mu + \phi_1(\mathbb{E}[y_{t+h-1}|y_t, y_{t-1}, \dots] - \mu) + \\ &\quad \phi_2(\mathbb{E}[y_{t+h-2}|y_t, y_{t-1}, \dots] - \mu) + \cdots + \\ &\quad \phi_p(\mathbb{E}[y_{t+h-p}|y_t, y_{t-1}, \dots] - \mu), \end{aligned}$$

which can be exploited recursively.

The recursion begins with  $\mathbb{E}(y_{t-k}|y_t, y_{t-1}, \dots) = y_{t-k}$  (for any  $k \geq 0$ ).

**Example 2.13** (Forecasting an ARMA(p,q) process). Consider the process:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \quad (2.71)$$

where  $\{\varepsilon_t\}$  is a white noise sequence (Def. 2.1). We assume that the MA part of the process is invertible (see Eq. (2.32)), which implies that the information contained in  $\{y_t, y_{t-1}, y_{t-2}, \dots\}$  is identical to that in  $\{\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \dots\}$ .

While one could use a recursive algorithm to compute the conditional mean (as in Example 2.12), it is convenient to employ the Wold decomposition of this process (see Theorem 2.3 and Prop. 2.8 for the computation of the  $\psi_i$ 's in the context of ARMA processes):

$$y_t = \mu + \sum_{i=0}^{+\infty} \psi_i \varepsilon_{t-i}.$$

This implies:

$$\begin{aligned} y_{t+h} &= \mu + \sum_{i=0}^{h-1} \psi_i \varepsilon_{t+h-i} + \sum_{i=h}^{+\infty} \psi_i \varepsilon_{t+h-i} \\ &= \mu + \sum_{i=0}^{h-1} \psi_i \varepsilon_{t+h-i} + \sum_{i=0}^{+\infty} \psi_{i+h} \varepsilon_{t-i}. \end{aligned}$$

Since  $\mathbb{E}(y_{t+h}|y_t, y_{t-1}, \dots) = \mu + \sum_{i=0}^{+\infty} \psi_{i+h} \varepsilon_{t-i}$ , we get:

$$\mathbb{V}ar(y_{t+h}|y_t, y_{t-1}, \dots) = \mathbb{V}ar\left(\sum_{i=0}^{h-1} \psi_i \varepsilon_{t+h-i}\right) = \sigma^2 \sum_{i=0}^{h-1} \psi_i^2.$$

How to use the previous formulas in practice?

One has first to select a specification and to estimate the model. Two methods to determine relevant specifications:

- a. Information criteria (see Definition 2.20).
- b. Box-Jenkins approach.

Box and Jenkins (1976) have proposed an approach that is now widely used.

1. Data transformation. The data should be transformed to “make them stationary”. To do so, one can e.g. take logarithms, take changes in the considered series, remove (deterministic) trends.
2. Select  $p$  and  $q$ . This can be based on the PACF approach (see Section 2.2.4), or on selection criteria (see Definition 2.20).
3. Estimate the model parameters. See Section 2.2.8.
4. Check that the estimated model is consistent with the data. See below.

### Assessing the performances of a forecasting model

Once one has fitted a model on a given dataset (of length  $T$ , say), one compute MSE (mean square errors) to evaluate the performance of the model. But this MSE is the **in-sample** one. It is easy to reduce in-sample MSE. Typically, if the model is estimated by OLS, adding covariates mechanically reduces the MSE (see Props. ?? and ??). That is, even if additional data are irrelevant, the  $R^2$  of the regression increases. Adding irrelevant variables increases the (in-sample)  $R^2$  but is bound to increase the **out-of-sample** MSE.

Therefore, it is important to analyse **out-of-sample** performances of the forecasting model:

- Estimate a model on a sample of reduced size  $(1, \dots, T^*, \text{ with } T^* < T)$
- Use the remaining available periods  $(T^* + 1, \dots, T)$  to compute **out-of-sample** forecasting errors (and compute their MSE). In an out-of-sample exercise, it is important to make sure that the data used to produce a forecasts (as of date  $T^*$ ) where indeed available on date  $T^*$ .

### Diebold-Mariano test

How to compare different forecasting approaches? Diebold and Mariano (1995) have proposed a simple test to address this question.

Assume that you want to compare approaches A and B. You have historical data sets and you have implemented both approaches in the past, providing you with two sets of forecasting errors:  $\{e_t^A\}_{t=1,\dots,T}$  and  $\{e_t^B\}_{t=1,\dots,T}$ .

It may be the case that your forecasts serve a specific purpose and that, for instance, you dislike positive forecasting errors and you care less about negative errors. We assume you are able to formalise this by means of a **loss function**  $L(e)$ . For instance:

- If you dislike large positive errors, you may set  $L(e) = \exp(e)$ .
- If you are concerned about both positive and negative errors (indifferently), you may set  $L(e) = e^2$  (standard approach).

Let us define the sequence  $\{d_t\}_{t=1,\dots,T} \equiv \{L(e_t^A) - L(e_t^B)\}_{t=1,\dots,T}$  and assume that this sequence is covariance stationary. We consider the following null hypothesis:  $H_0 : \bar{d} = 0$ , where  $\bar{d}$  denotes the population mean of the  $d_t$ s. Under  $H_0$  and under the assumption of covariance-stationarity of  $d_t$ , we have (Theorem @ref{(hm:CLTcovstat)}):

$$\sqrt{T}\bar{d}_T \xrightarrow{d} \mathcal{N}\left(0, \sum_{j=-\infty}^{+\infty} \gamma_j\right),$$

where the  $\gamma_j$ s are the autocovariances of  $d_t$ .

Hence, assuming that  $\hat{\sigma}^2$  is a consistent estimate of  $\sum_{j=-\infty}^{+\infty} \gamma_j$  (for instance the one given by the Newey-West formula, see Def. ??), we have, under  $H_0$ :

$$DM_T := \sqrt{T} \frac{\bar{d}_T}{\sqrt{\hat{\sigma}^2}} \xrightarrow{d} \mathcal{N}(0, 1).$$

$DM_T$  is the test statistics. For a test of size  $\alpha$ , the critical region is:<sup>6</sup>

$$]-\infty, -\Phi^{-1}(1 - \alpha/2)] \cup [\Phi^{-1}(1 - \alpha/2), +\infty[,$$

where  $\Phi$  is the c.d.f. of the standard normal distribution.

**Example 2.14** (Forecasting Swiss GDP growth). We use a long historical time series of the Swiss GDP growth taken from the Jordà et al. (2017) dataset (see Figure 2.3, and Example 2.6).

We want to forecast this GDP growth. We envision two specifications : an AR(1) specification (the one advocated by the AIC criteria, see Example 2.6), and an ARMA(2,2) specification. We are interested in 2-year-ahead forecasts (i.e.,  $h = 2$  since the data are yearly).

---

<sup>6</sup>This ShinyApp application illustrates the notion of statistical test (illustrating the p-value and the critical region, in particular).



```

library(AEC)
library(forecast)

## Registered S3 method overwritten by 'quantmod':
##   method      from
##   as.zoo.data.frame zoo

data <- subset(JST,iso=="CHE")
T <- dim(data)[1]
y <- c(NaN,log(data$gdp[2:T]/data$gdp[1:(T-1)]))
first.date <- T-50
e1 <- NULL; e2 <- NULL;h<-2
for(T.star in first.date:(T-h)){
  estim.model.1 <- arima(y[1:T.star],order=c(1,0,0))
  estim.model.2 <- arima(y[1:T.star],order=c(2,0,2))
  e1 <- c(e1,y[T.star+h] - predict(estim.model.1,n.ahead=h)$pred[h])
  e2 <- c(e2,y[T.star+h] - predict(estim.model.2,n.ahead=h)$pred[h])
}
res.DM <- dm.test(e1,e2,h = h,alternative = "greater")
res.DM

##
## Diebold-Mariano Test
##
## data:  e1e2
## DM = -0.82989, Forecast horizon = 2, Loss function power = 2, p-value =
## 0.7946
## alternative hypothesis: greater

```

With `alternative = "greater"` The alternative hypothesis is that method 2 is more accurate than method 1. Since we do not reject the null (the p-value being of 0.795), we are not led to use the more sophisticated model (ARMA(2,2)) and we keep the simple AR(1) model.

Assume now that we want to compare the AR(1) process to a VAR model (see Def. 2.21). We consider a bivariate VAR, where GDP growth is complemented with CPI-based inflation rate.

```

library(vars)
infl <- c(NaN,log(data$cpi[2:T]/data$cpi[1:(T-1)]))
y_var <- cbind(y,infl)
e3 <- NULL
for(T.star in first.date:(T-h)){
  estim.model.3 <- VAR(y_var[2:T.star,],p=1)
  e3 <- c(e3,y[T.star+h] - predict(estim.model.3,n.ahead=h)$fcst$y[h,1])
}
res.DM <- dm.test(e1,e2,h = h,alternative = "greater")
res.DM

##
## Diebold-Mariano Test
##
## data:  e1e2
## DM = -0.82989, Forecast horizon = 2, Loss function power = 2, p-value =
## 0.7946
## alternative hypothesis: greater

```

Again, we do not find that the alternative model (here the VAR(1) model) is better than the AR(1) model to forecast GDP growth.



# Chapter 3

## Appendix

### 3.1 Principal component analysis (PCA)

**Principal component analysis (PCA)** is a classical and easy-to-use statistical method to reduce the dimension of large datasets containing variables that are linearly driven by a relatively small number of factors. This approach is widely used in data analysis and image compression.

Suppose that we have  $T$  observations of a  $n$ -dimensional random vector  $x$ , denoted by  $x_1, x_2, \dots, x_T$ . We suppose that each component of  $x$  is of mean zero.

Let denote with  $X$  the matrix given by  $\begin{bmatrix} x_1 & x_2 & \dots & x_T \end{bmatrix}'$ . Denote the  $j^{th}$  column of  $X$  by  $X_j$ .

We want to find the linear combination of the  $x_i$ 's ( $x \cdot u$ ), with  $\|u\| = 1$ , with “maximum variance.” That is, we want to solve:

$$\begin{aligned} \arg \max_u \quad & u' X' X u. \\ \text{s.t.} \quad & |u| = 1 \end{aligned} \tag{3.1}$$

Since  $X'X$  is a positive definite matrix, it admits the following decomposition:

$$\begin{aligned} X'X &= PDP' \\ &= P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P', \end{aligned}$$

where  $P$  is an orthogonal matrix whose columns are the eigenvectors of  $X'X$ .

We can order the eigenvalues such that  $\lambda_1 \geq \dots \geq \lambda_n$ . (Since  $X'X$  is positive definite, all these eigenvalues are positive.)

Since  $P$  is orthogonal, we have  $u' X' X u = u' P D P' u = y' D y$  where  $\|y\| = 1$ . Therefore, we have  $y_i^2 \leq 1$  for any  $i \leq n$ .

As a consequence:

$$y' D y = \sum_{i=1}^n y_i^2 \lambda_i \leq \lambda_1 \sum_{i=1}^n y_i^2 = \lambda_1.$$

It is easily seen that the maximum is reached for  $y = [1, 0, \dots, 0]'$ . Therefore, the maximum of the optimization program (Eq. (3.1)) is obtained for  $u = P [1, 0, \dots, 0]'$ . That is,  $u$  is the eigenvector of  $X'X$  that is associated with its larger eigenvalue (first column of  $P$ ).

Let us denote with  $F$  the vector that is given by the matrix product  $XP$  (note that its last column is equal to  $Xu$ ). The columns of  $F$ , denoted by  $F_j$ , are called **factors**. We have:

$$F'F = P'X'XP = D.$$

Therefore, in particular, the  $F_j$ 's are orthogonal.

Since  $X = FP'$ , the  $X_j$ 's are linear combinations of the factors. Let us then denote with  $\hat{X}_{i,j}$  the part of  $X_i$  that is explained by factor  $F_j$ , we have:

$$\begin{aligned}\hat{X}_{i,j} &= p_{ij}F_j \\ X_i &= \sum_j \hat{X}_{i,j} = \sum_j p_{ij}F_j.\end{aligned}$$

Consider the share of variance that is explained –through the  $n$  variables  $(X_1, \dots, X_n)$ – by the first factor  $F_1$ :

$$\frac{\sum_i \hat{X}_{i,1} \hat{X}'_{i,1}}{\sum_i X_i X'_i} = \frac{\sum_i p_{i1} F_1 F'_1 p_{i1}}{\text{tr}(X'X)} = \frac{\sum_i p_{i1}^2 \lambda_1}{\text{tr}(X'X)} = \frac{\lambda_1}{\sum_i \lambda_i}.$$

Intuitively, if the first eigenvalue is large, it means that the first factor embed a large share of the fluctuations of the  $n$   $X_i$ 's.

Let us illustrate PCA on the term structure of yields. The term structure of yields (or yield curve) is known to be driven by only a small number of factors (e.g., Litterman and Scheinkman (1991)). One can typically employ PCA to recover such factors. The data used in the example below are taken from the Fred database (tickers: “DGS6MO”, “DGS1”, ...). The second plot shows the factor loadings, that indicate that the first factor is a level factor (loadings = black line), the second factor is a slope factor (loadings = blue line), the third factor is a curvature factor (loadings = red line).

To run a PCA, one simply has to apply function `prcomp` to a matrix of data:

```
library(AEC)
USyields <- USyields[complete.cases(USyields),]
yds <- USyields[c("Y1", "Y2", "Y3", "Y5", "Y7", "Y10", "Y20", "Y30")]
PCA.yds <- prcomp(yds, center=TRUE, scale. = TRUE)
```

Let us now visualize some results. The first plot of Figure 3.1 shows the share of total variance explained by the different principal components (PCs). The second plot shows the factor loadings. The two bottom plots show how yields (in black) are fitted by linear combinations of the first two PCs only.

```
par(mfrow=c(2,2))
par(plt=c(.1,.95,.2,.8))
barplot(PCA.yds$sdev^2/sum(PCA.yds$sdev^2),
        main="Share of variance expl. by PC's")
axis(1, at=1:dim(yds)[2], labels=colnames(PCA.yds$x))
nb.PC <- 2
plot(-PCA.yds$rotation[,1], type="l", lwd=2, ylim=c(-1,1),
     main="Factor loadings (1st 3 PCs)", xaxt="n", xlab="")
axis(1, at=1:dim(yds)[2], labels=colnames(yds))
lines(PCA.yds$rotation[,2], type="l", lwd=2, col="blue")
lines(PCA.yds$rotation[,3], type="l", lwd=2, col="red")
Y1.hat <- PCA.yds$x[,1:nb.PC] %*% PCA.yds$rotation["Y1",1:2]
Y1.hat <- mean(USyields$Y1) + sd(USyields$Y1) * Y1.hat
plot(USyields$date, USyields$Y1, type="l", lwd=2,
     main="Fit of 1-year yields (2 PCs)",
     ylab="Obs (black) / Fitted by 2PCs (dashed blue)")
lines(USyields$date, Y1.hat, col="blue", lty=2, lwd=2)
Y10.hat <- PCA.yds$x[,1:nb.PC] %*% PCA.yds$rotation["Y10",1:2]
Y10.hat <- mean(USyields$Y10) + sd(USyields$Y10) * Y10.hat
plot(USyields$date, USyields$Y10, type="l", lwd=2,
     main="Fit of 10-year yields (2 PCs)",
     ylab="Obs (black) / Fitted by 2PCs (dashed blue)")
lines(USyields$date, Y10.hat, col="blue", lty=2, lwd=2)
```

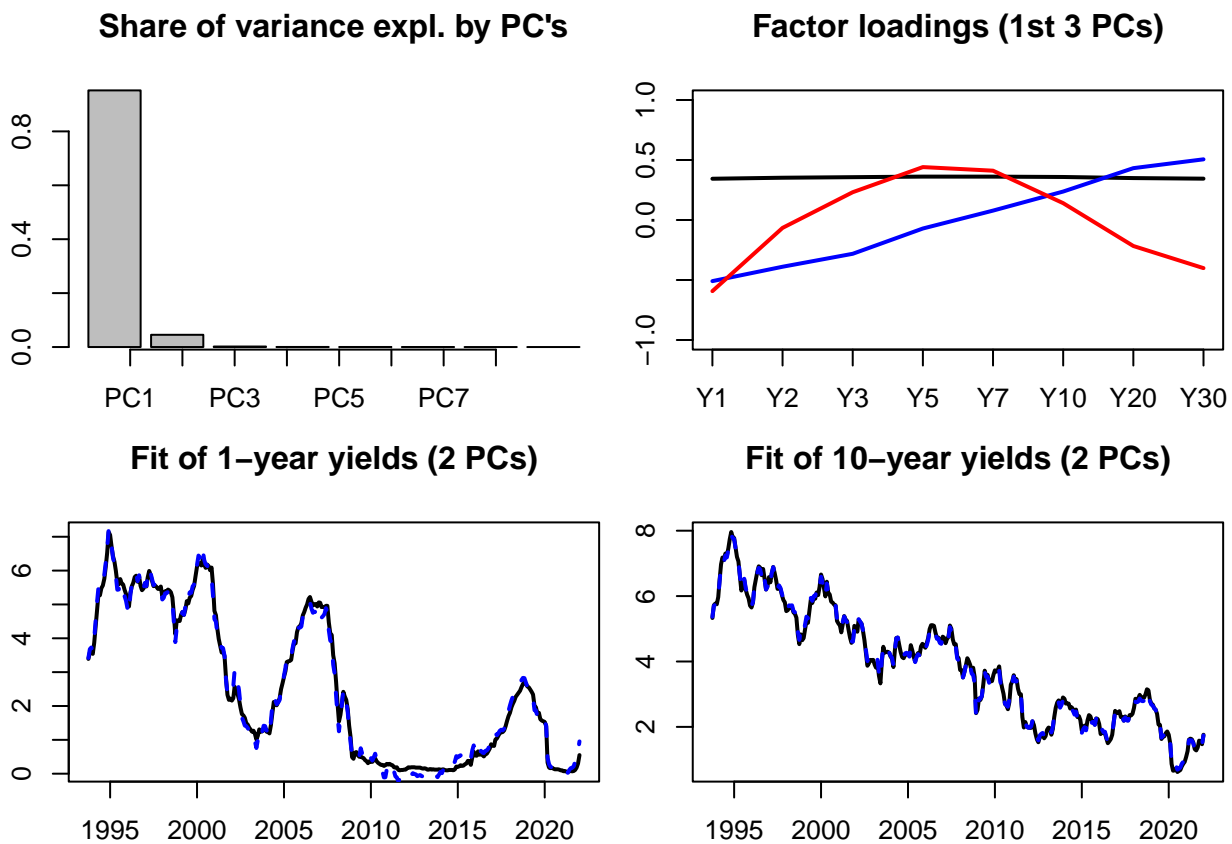


Figure 3.1: Some PCA results. The dataset contains 8 time series of U.S. interest rates of different maturities.

## 3.2 Linear algebra: definitions and results

**Definition 3.1** (Eigenvalues). The eigenvalues of a matrix  $M$  are the numbers  $\lambda$  for which:

$$|M - \lambda I| = 0,$$

where  $|\bullet|$  is the determinant operator.

**Proposition 3.1** (Properties of the determinant). *We have:*

- $|MN| = |M| \times |N|$ .
- $|M^{-1}| = |M|^{-1}$ .
- If  $M$  admits the diagonal representation  $M = TDT^{-1}$ , where  $D$  is a diagonal matrix whose diagonal entries are  $\{\lambda_i\}_{i=1,\dots,n}$ , then:

$$|M - \lambda I| = \prod_{i=1}^n (\lambda_i - \lambda).$$

**Definition 3.2** (Moore-Penrose inverse). If  $M \in \mathbb{R}^{m \times n}$ , then its Moore-Penrose pseudo inverse (exists and) is the unique matrix  $M^* \in \mathbb{R}^{n \times m}$  that satisfies:

- i.  $MM^*M = M$
- ii.  $M^*MM^* = M^*$
- iii.  $(MM^*)' = MM^*$  .iv  $(M^*M)' = M^*M$ .

**Proposition 3.2** (Properties of the Moore-Penrose inverse). • If  $M$  is invertible then  $M^* = M^{-1}$ .

- The pseudo-inverse of a zero matrix is its transpose. \*
- \*

The pseudo-inverse of the pseudo-inverse is the original matrix.

**Definition 3.3** (Idempotent matrix). Matrix  $M$  is idempotent if  $M^2 = M$ .

If  $M$  is a symmetric idempotent matrix, then  $M'M = M$ .

**Proposition 3.3** (Roots of an idempotent matrix). The eigenvalues of an idempotent matrix are either 1 or 0.

*Proof.* If  $\lambda$  is an eigenvalue of an idempotent matrix  $M$  then  $\exists x \neq 0$  s.t.  $Mx = \lambda x$ . Hence  $M^2x = \lambda Mx \Rightarrow (1 - \lambda)Mx = 0$ . Either all element of  $Mx$  are zero, in which case  $\lambda = 0$  or at least one element of  $Mx$  is nonzero, in which case  $\lambda = 1$ .  $\square$

**Proposition 3.4** (Idempotent matrix and chi-square distribution). *The rank of a symmetric idempotent matrix is equal to its trace.*

*Proof.* The result follows from Prop. 3.3, combined with the fact that the rank of a symmetric matrix is equal to the number of its nonzero eigenvalues.  $\square$

**Proposition 3.5** (Constrained least squares). *The solution of the following optimisation problem:*

$$\begin{aligned} \min_{\beta} \quad & \|\mathbf{y} - \mathbf{X}\beta\|^2 \\ \text{subject to} \quad & \mathbf{R}\beta = \mathbf{q} \end{aligned}$$

is given by:

$$\beta^r = \beta_0 - (\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\{\mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\}^{-1}(\mathbf{R}\beta_0 - \mathbf{q}),$$

where  $\beta_0 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$ .

*Proof.* See for instance Jackman, 2007.  $\square$

**Proposition 3.6** (Inverse of a partitioned matrix). *We have:*

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{A}_{12}(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1} \\ -(\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1}\mathbf{A}_{21}\mathbf{A}_{11}^{-1} & (\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1} \end{bmatrix}.$$

**Definition 3.4** (Matrix derivatives). Consider a fonction  $f : \mathbb{R}^K \rightarrow \mathbb{R}$ . Its first-order derivative is:

$$\frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) = \begin{bmatrix} \frac{\partial f}{\partial b_1}(\mathbf{b}) \\ \vdots \\ \frac{\partial f}{\partial b_K}(\mathbf{b}) \end{bmatrix}.$$

We use the notation:

$$\frac{\partial f}{\partial \mathbf{b}'}(\mathbf{b}) = \left( \frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) \right)'.$$

**Proposition 3.7.** *We have:*

- If  $f(\mathbf{b}) = \mathbf{A}'\mathbf{b}$  where  $\mathbf{A}$  is a  $K \times 1$  vector then  $\frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) = \mathbf{A}$ .
- If  $f(\mathbf{b}) = \mathbf{b}'\mathbf{A}\mathbf{b}$  where  $\mathbf{A}$  is a  $K \times K$  matrix, then  $\frac{\partial f}{\partial \mathbf{b}}(\mathbf{b}) = 2\mathbf{A}\mathbf{b}$ .

**Proposition 3.8** (Square and absolute summability). *We have:*

$$\underbrace{\sum_{i=0}^{\infty} |\theta_i| < +\infty}_{\text{Absolute summability}} \quad \Rightarrow \quad \underbrace{\sum_{i=0}^{\infty} \theta_i^2 < +\infty}_{\text{Square summability}}.$$

*Proof.* See Appendix 3.A in Hamilton. Idea: Absolute summability implies that there exist  $N$  such that, for  $j > N$ ,  $|\theta_j| < 1$  (deduced from Cauchy criterion, Theorem 3.2 and therefore  $\theta_j^2 < |\theta_j|$ ).  $\square$

### 3.3 Statistical analysis: definitions and results

#### 3.3.1 Moments and statistics

**Definition 3.5** (Partial correlation). The **partial correlation** between  $y$  and  $z$ , controlling for some variables  $\mathbf{X}$  is the sample correlation between  $y^*$  and  $z^*$ , where the latter two variables are the residuals in regressions of  $y$  on  $\mathbf{X}$  and of  $z$  on  $\mathbf{X}$ , respectively.

This correlation is denoted by  $r_{yz}^{\mathbf{X}}$ . By definition, we have:

$$r_{yz}^{\mathbf{X}} = \frac{\mathbf{z}^{*'} \mathbf{y}^*}{\sqrt{(\mathbf{z}^{*'} \mathbf{z}^*)(\mathbf{y}^{*'} \mathbf{y}^*)}}. \quad (3.2)$$

**Definition 3.6** (Skewness and kurtosis). Let  $Y$  be a random variable whose fourth moment exists. The expectation of  $Y$  is denoted by  $\mu$ .

- The skewness of  $Y$  is given by:

$$\frac{\mathbb{E}[(Y - \mu)^3]}{\{\mathbb{E}[(Y - \mu)^2]\}^{3/2}}.$$

- The kurtosis of  $Y$  is given by:

$$\frac{\mathbb{E}[(Y - \mu)^4]}{\{\mathbb{E}[(Y - \mu)^2]\}^2}.$$

**Theorem 3.1** (Cauchy-Schwarz inequality). *We have:*

$$|\text{Cov}(X, Y)| \leq \sqrt{\text{Var}(X)\text{Var}(Y)}$$

and, if  $X \neq 0$  and  $Y \neq 0$ , the equality holds iff  $X$  and  $Y$  are the same up to an affine transformation.

*Proof.* If  $\text{Var}(X) = 0$ , this is trivial. If this is not the case, then let's define  $Z$  as  $Z = Y - \frac{\text{Cov}(X, Y)}{\text{Var}(X)}X$ . It is easily seen that  $\text{Cov}(X, Z) = 0$ . Then, the variance of  $Y = Z + \frac{\text{Cov}(X, Y)}{\text{Var}(X)}X$  is equal to the sum of the variance of  $Z$  and of the variance of  $\frac{\text{Cov}(X, Y)}{\text{Var}(X)}X$ , that is:

$$\text{Var}(Y) = \text{Var}(Z) + \left( \frac{\text{Cov}(X, Y)}{\text{Var}(X)} \right)^2 \text{Var}(X) \geq \left( \frac{\text{Cov}(X, Y)}{\text{Var}(X)} \right)^2 \text{Var}(X).$$

The equality holds iff  $\text{Var}(Z) = 0$ , i.e. iff  $Y = \frac{\text{Cov}(X, Y)}{\text{Var}(X)}X + \text{cst}$ . □

**Definition 3.7** (Asymptotic level). An asymptotic test with critical region  $\Omega_n$  has an asymptotic level equal to  $\alpha$  if:

$$\sup_{\theta \in \Theta} \lim_{n \rightarrow \infty} \mathbb{P}_{\theta}(S_n \in \Omega_n) = \alpha,$$

where  $S_n$  is the test statistic and  $\Theta$  is such that the null hypothesis  $H_0$  is equivalent to  $\theta \in \Theta$ .

**Definition 3.8** (Asymptotically consistent test). An asymptotic test with critical region  $\Omega_n$  is consistent if:

$$\forall \theta \in \Theta^c, \quad \mathbb{P}_{\theta}(S_n \in \Omega_n) \rightarrow 1,$$

where  $S_n$  is the test statistic and  $\Theta^c$  is such that the null hypothesis  $H_0$  is equivalent to  $\theta \notin \Theta$ .

**Definition 3.9** (Kullback discrepancy). Given two p.d.f.  $f$  and  $f^*$ , the Kullback discrepancy is defined by:

$$I(f, f^*) = \mathbb{E}^* \left( \log \frac{f^*(Y)}{f(Y)} \right) = \int \log \frac{f^*(y)}{f(y)} f^*(y) dy.$$

**Proposition 3.9** (Properties of the Kullback discrepancy). *We have:*

- i.  $I(f, f^*) \geq 0$

ii.  $I(f, f^*) = 0$  iff  $f \equiv f^*$ .

*Proof.*  $x \rightarrow -\log(x)$  is a convex function. Therefore  $\mathbb{E}^*(-\log f(Y)/f^*(Y)) \geq -\log \mathbb{E}^*(f(Y)/f^*(Y)) = 0$  (proves (i)). Since  $x \rightarrow -\log(x)$  is strictly convex, equality in (i) holds if and only if  $f(Y)/f^*(Y)$  is constant (proves (ii)).  $\square$

**Definition 3.10** (Characteristic function). For any real-valued random variable  $X$ , the characteristic function is defined by:

$$\phi_X : u \rightarrow \mathbb{E}[\exp(iuX)].$$

### 3.3.2 Standard distributions

**Definition 3.11** (F distribution). Consider  $n = n_1 + n_2$  i.i.d.  $\mathcal{N}(0, 1)$  r.v.  $X_i$ . If the r.v.  $F$  is defined by:

$$F = \frac{\sum_{i=1}^{n_1} X_i^2}{\sum_{j=n_1+1}^{n_1+n_2} X_j^2} \frac{n_2}{n_1}$$

then  $F \sim \mathcal{F}(n_1, n_2)$ . (See Table 3.4 for quantiles.)

**Definition 3.12** (Student-t distribution).  $Z$  follows a Student-t (or  $t$ ) distribution with  $\nu$  degrees of freedom (d.f.) if:

$$Z = X_0 / \sqrt{\frac{\sum_{i=1}^{\nu} X_i^2}{\nu}}, \quad X_i \sim i.i.d. \mathcal{N}(0, 1).$$

We have  $\mathbb{E}(Z) = 0$ , and  $\text{Var}(Z) = \frac{\nu}{\nu-2}$  if  $\nu > 2$ . (See Table 3.2 for quantiles.)

**Definition 3.13** (Chi-square distribution).  $Z$  follows a  $\chi^2$  distribution with  $\nu$  d.f. if  $Z = \sum_{i=1}^{\nu} X_i^2$  where  $X_i \sim i.i.d. \mathcal{N}(0, 1)$ . We have  $\mathbb{E}(Z) = \nu$ . (See Table 3.3 for quantiles.)

**Proposition 3.10** (Inner product of a multivariate Gaussian variable). Let  $X$  be a  $n$ -dimensional multivariate Gaussian variable:  $X \sim \mathcal{N}(0, \Sigma)$ . We have:

$$X' \Sigma^{-1} X \sim \chi^2(n).$$

*Proof.* Because  $\Sigma$  is a symmetrical definite positive matrix, it admits the spectral decomposition  $PDP'$  where  $P$  is an orthogonal matrix (i.e.  $PP' = Id$ ) and  $D$  is a diagonal matrix with non-negative entries. Denoting by  $\sqrt{D^{-1}}$  the diagonal matrix whose diagonal entries are the inverse of those of  $D$ , it is easily checked that the covariance matrix of  $Y := \sqrt{D^{-1}}P'X$  is  $Id$ . Therefore  $Y$  is a vector of uncorrelated Gaussian variables. The properties of Gaussian variables imply that the components of  $Y$  are then also independent. Hence  $Y'Y = \sum_i Y_i^2 \sim \chi^2(n)$ .

It remains to note that  $Y'Y = X'PD^{-1}P'X = X'\text{Var}(X)^{-1}X$  to conclude.  $\square$

**Definition 3.14** (Generalized Extreme Value (GEV) distribution). The vector of disturbances  $\varepsilon = [\varepsilon_{1,1}, \dots, \varepsilon_{1,K_1}, \dots, \varepsilon_{J,1}, \dots, \varepsilon_{J,K_J}]'$  follows the Generalized Extreme Value (GEV) distribution if its c.d.f. is:

$$F(\varepsilon, \rho) = \exp(-G(e^{-\varepsilon_{1,1}}, \dots, e^{-\varepsilon_{J,K_J}}; \rho))$$

with

$$\begin{aligned} G(\mathbf{Y}; \rho) &\equiv G(Y_{1,1}, \dots, Y_{1,K_1}, \dots, Y_{J,1}, \dots, Y_{J,K_J}; \rho) \\ &= \sum_{j=1}^J \left( \sum_{k=1}^{K_j} Y_{jk}^{1/\rho_j} \right)^{\rho_j} \end{aligned}$$



### 3.3.3 Stochastic convergences

**Proposition 3.11** (Chebychev's inequality). *If  $\mathbb{E}(|X|^r)$  is finite for some  $r > 0$  then:*

$$\forall \varepsilon > 0, \quad \mathbb{P}(|X - c| > \varepsilon) \leq \frac{\mathbb{E}[|X - c|^r]}{\varepsilon^r}.$$

*In particular, for  $r = 2$ :*

$$\forall \varepsilon > 0, \quad \mathbb{P}(|X - c| > \varepsilon) \leq \frac{\mathbb{E}[(X - c)^2]}{\varepsilon^2}.$$

*Proof.* Remark that  $\varepsilon^r \mathbb{1}_{\{|X| \geq \varepsilon\}} \leq |X|^r$  and take the expectation of both sides.  $\square$

**Definition 3.15** (Convergence in probability). The random variable sequence  $x_n$  converges in probability to a constant  $c$  if  $\forall \varepsilon, \lim_{n \rightarrow \infty} \mathbb{P}(|x_n - c| > \varepsilon) = 0$ .

It is denoted as:  $\text{plim } x_n = c$ .

**Definition 3.16** (Convergence in the  $L^r$  norm).  $x_n$  converges in the  $r$ -th mean (or in the  $L^r$ -norm) towards  $x$ , if  $\mathbb{E}(|x_n|^r)$  and  $\mathbb{E}(|x|^r)$  exist and if

$$\lim_{n \rightarrow \infty} \mathbb{E}(|x_n - x|^r) = 0.$$

It is denoted as:  $x_n \xrightarrow{L^r} c$ .

For  $r = 2$ , this convergence is called **mean square convergence**.

**Definition 3.17** (Almost sure convergence). The random variable sequence  $x_n$  converges almost surely to  $c$  if  $\mathbb{P}(\lim_{n \rightarrow \infty} x_n = c) = 1$ .

It is denoted as:  $x_n \xrightarrow{a.s.} c$ .

**Definition 3.18** (Convergence in distribution).  $x_n$  is said to converge in distribution (or in law) to  $x$  if

$$\lim_{n \rightarrow \infty} F_{x_n}(s) = F_x(s)$$

for all  $s$  at which  $F_X$  –the cumulative distribution of  $X$ – is continuous.

It is denoted as:  $x_n \xrightarrow{d} x$ .

**Proposition 3.12** (Rules for limiting distributions (Slutsky)). *We have:*

i. **Slutsky's theorem:** *If  $x_n \xrightarrow{d} x$  and  $y_n \xrightarrow{p} c$  then*

$$\begin{aligned} x_n y_n &\xrightarrow{d} xc \\ x_n + y_n &\xrightarrow{d} x + c \\ x_n / y_n &\xrightarrow{d} x/c \quad (\text{if } c \neq 0) \end{aligned}$$

ii. **Continuous mapping theorem:** *If  $x_n \xrightarrow{d} x$  and  $g$  is a continuous function then  $g(x_n) \xrightarrow{d} g(x)$ .*

**Proposition 3.13** (Implications of stochastic convergences). *We have:*

$$\begin{array}{ccccc} \boxed{\begin{array}{c} L^s \\ \rightarrow \end{array}} & \xRightarrow{1 \leq r \leq s} & \boxed{\begin{array}{c} L^r \\ \rightarrow \end{array}} & & \\ & & \Downarrow & & \\ \boxed{\begin{array}{c} a.s. \\ \rightarrow \end{array}} & \Rightarrow & \boxed{\begin{array}{c} p \\ \rightarrow \end{array}} & \Rightarrow & \boxed{\begin{array}{c} d \\ \rightarrow \end{array}}. \end{array}$$

*Proof.* (of the fact that  $\left(\xrightarrow{p}\right) \Rightarrow \left(\xrightarrow{d}\right)$ ). Assume that  $X_n \xrightarrow{p} X$ . Denoting by  $F$  and  $F_n$  the c.d.f. of  $X$  and  $X_n$ , respectively:

$$F_n(x) = \mathbb{P}(X_n \leq x, X \leq x + \varepsilon) + \mathbb{P}(X_n \leq x, X > x + \varepsilon) \leq F(x + \varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon). \quad (3.3)$$

Besides,

$$F(x - \varepsilon) = \mathbb{P}(X \leq x - \varepsilon, X_n \leq x) + \mathbb{P}(X \leq x - \varepsilon, X_n > x) \leq F_n(x) + \mathbb{P}(|X_n - X| > \varepsilon),$$

which implies:

$$F(x - \varepsilon) - \mathbb{P}(|X_n - X| > \varepsilon) \leq F_n(x). \quad (3.4)$$

Eqs. (3.3) and (3.4) imply:

$$F(x - \varepsilon) - \mathbb{P}(|X_n - X| > \varepsilon) \leq F_n(x) \leq F(x + \varepsilon) + \mathbb{P}(|X_n - X| > \varepsilon).$$

Taking limits as  $n \rightarrow \infty$  yields

$$F(x - \varepsilon) \leq \liminf_{n \rightarrow \infty} F_n(x) \leq \limsup_{n \rightarrow \infty} F_n(x) \leq F(x + \varepsilon).$$

The result is then obtained by taking limits as  $\varepsilon \rightarrow 0$  (if  $F$  is continuous at  $x$ ).  $\square$

**Proposition 3.14** (Convergence in distribution to a constant). *If  $X_n$  converges in distribution to a constant  $c$ , then  $X_n$  converges in probability to  $c$ .*

*Proof.* If  $\varepsilon > 0$ , we have  $\mathbb{P}(X_n < c - \varepsilon) \xrightarrow{n \rightarrow \infty} 0$  i.e.  $\mathbb{P}(X_n \geq c - \varepsilon) \xrightarrow{n \rightarrow \infty} 1$  and  $\mathbb{P}(X_n < c + \varepsilon) \xrightarrow{n \rightarrow \infty} 1$ . Therefore  $\mathbb{P}(c - \varepsilon \leq X_n < c + \varepsilon) \xrightarrow{n \rightarrow \infty} 1$ , which gives the result.  $\square$

**Example 3.1** (Convergence in probability but not  $L^r$ ). Let  $\{x_n\}_{n \in \mathbb{N}}$  be a series of random variables defined by:

$$x_n = nu_n,$$

where  $u_n$  are independent random variables s.t.  $u_n \sim \mathcal{B}(1/n)$ .

We have  $x_n \xrightarrow{p} 0$  but  $x_n \not\xrightarrow{L^r} 0$  because  $\mathbb{E}(|X_n - 0|) = \mathbb{E}(X_n) = 1$ .

**Theorem 3.2** (Cauchy criterion (non-stochastic case)). *We have that  $\sum_{i=0}^T a_i$  converges ( $T \rightarrow \infty$ ) iff, for any  $\eta > 0$ , there exists an integer  $N$  such that, for all  $M \geq N$ ,*

$$\left| \sum_{i=N+1}^M a_i \right| < \eta.$$

**Theorem 3.3** (Cauchy criterion (stochastic case)). *We have that  $\sum_{i=0}^T \theta_i \varepsilon_{t-i}$  converges in mean square ( $T \rightarrow \infty$ ) to a random variable iff, for any  $\eta > 0$ , there exists an integer  $N$  such that, for all  $M \geq N$ ,*

$$\mathbb{E} \left[ \left( \sum_{i=N+1}^M \theta_i \varepsilon_{t-i} \right)^2 \right] < \eta.$$

### 3.3.4 Central limit theorem

**Theorem 3.4** (Law of large numbers). *The sample mean is a consistent estimator of the population mean.*

*Proof.* Let's denote by  $\phi_{X_i}$  the characteristic function of a r.v.  $X_i$ . If the mean of  $X_i$  is  $\mu$  then the Talyor expansion of the characteristic function is:

$$\phi_{X_i}(u) = \mathbb{E}(\exp(iuX)) = 1 + iu\mu + o(u).$$

The properties of the characteristic function (see Def. 3.10) imply that:

$$\phi_{\frac{1}{n}(X_1 + \dots + X_n)}(u) = \prod_{i=1}^n \left( 1 + i\frac{u}{n}\mu + o\left(\frac{u}{n}\right) \right) \rightarrow e^{iu\mu}.$$

The facts that (a)  $e^{iu\mu}$  is the characteristic function of the constant  $\mu$  and (b) that a characteristic function uniquely characterises a distribution imply that the sample mean converges in distribution to the constant  $\mu$ , which further implies that it converges in probability to  $\mu$ .  $\square$

**Theorem 3.5** (Lindberg-Levy Central limit theorem, CLT). *If  $x_n$  is an i.i.d. sequence of random variables with mean  $\mu$  and variance  $\sigma^2 \in ]0, +\infty[$ , then:*

$$\boxed{\sqrt{n}(\bar{x}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \quad \text{where} \quad \bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i.}$$

*Proof.* Let us introduce the r.v.  $Y_n := \sqrt{n}(\bar{X}_n - \mu)$ . We have  $\phi_{Y_n}(u) = \left[ \mathbb{E} \left( \exp(i \frac{1}{\sqrt{n}} u (X_1 - \mu)) \right) \right]^n$ . We have:

$$\begin{aligned} \left[ \mathbb{E} \left( \exp \left( i \frac{1}{\sqrt{n}} u (X_1 - \mu) \right) \right) \right]^n &= \left[ \mathbb{E} \left( 1 + i \frac{1}{\sqrt{n}} u (X_1 - \mu) - \frac{1}{2n} u^2 (X_1 - \mu)^2 + o(u^2) \right) \right]^n \\ &= \left( 1 - \frac{1}{2n} u^2 \sigma^2 + o(u^2) \right)^n. \end{aligned}$$

Therefore  $\phi_{Y_n}(u) \xrightarrow{n \rightarrow \infty} \exp(-\frac{1}{2} u^2 \sigma^2)$ , which is the characteristic function of  $\mathcal{N}(0, \sigma^2)$ .  $\square$

### 3.4 Some properties of Gaussian variables

**Proposition 3.15.** *If  $\mathbf{A}$  is idempotent and if  $\mathbf{x}$  is Gaussian,  $\mathbf{Lx}$  and  $\mathbf{x}'\mathbf{Ax}$  are independent if  $\mathbf{LA} = \mathbf{0}$ .*

*Proof.* If  $\mathbf{LA} = \mathbf{0}$ , then the two Gaussian vectors  $\mathbf{Lx}$  and  $\mathbf{Ax}$  are independent. This implies the independence of any function of  $\mathbf{Lx}$  and any function of  $\mathbf{Ax}$ . The results then follows from the observation that  $\mathbf{x}'\mathbf{Ax} = (\mathbf{Ax})'(\mathbf{Ax})$ , which is a function of  $\mathbf{Ax}$ .  $\square$

**Proposition 3.16** (Bayesian update in a vector of Gaussian variables). *If*

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \right),$$

*then*

$$\begin{aligned} Y_2 | Y_1 &\sim \mathcal{N}(\Omega_{21} \Omega_{11}^{-1} Y_1, \Omega_{22} - \Omega_{21} \Omega_{11}^{-1} \Omega_{12}). \\ Y_1 | Y_2 &\sim \mathcal{N}(\Omega_{12} \Omega_{22}^{-1} Y_2, \Omega_{11} - \Omega_{12} \Omega_{22}^{-1} \Omega_{21}). \end{aligned}$$

**Proposition 3.17** (Truncated distributions). *If  $X$  is a random variable distributed according to some p.d.f.  $f$ , with c.d.f.  $F$ , with infinite support. Then the p.d.f. of  $X|a \leq X < b$  is*

$$g(x) = \frac{f(x)}{F(b) - F(a)} \mathbb{1}_{\{a \leq x < b\}},$$

*for any  $a < b$ .*

*In partiucular, for a Gaussian variable  $X \sim \mathcal{N}(\mu, \sigma^2)$ , we have*

$$f(X = x | a \leq X < b) = \frac{\frac{1}{\sigma} \phi\left(\frac{x - \mu}{\sigma}\right)}{Z}.$$

*with  $Z = \Phi(\beta) - \Phi(\alpha)$ , where  $\alpha = \frac{a - \mu}{\sigma}$  and  $\beta = \frac{b - \mu}{\sigma}$ .*

*Moreover:*

$$\mathbb{E}(X | a \leq X < b) = \mu - \frac{\phi(\beta) - \phi(\alpha)}{Z} \sigma. \quad (3.5)$$

*We also have:*

$$\begin{aligned} &\text{Var}(X | a \leq X < b) \\ &= \sigma^2 \left[ 1 - \frac{\beta \phi(\beta) - \alpha \phi(\alpha)}{Z} - \left( \frac{\phi(\beta) - \phi(\alpha)}{Z} \right)^2 \right] \end{aligned} \quad (3.6)$$

*In particular, for  $b \rightarrow \infty$ , we get:*

$$\text{Var}(X | a < X) = \sigma^2 [1 + \alpha \lambda(-\alpha) - \lambda(-\alpha)^2], \quad (3.7)$$

*with  $\lambda(x) = \frac{\phi(x)}{\Phi(x)}$  is called the **inverse Mills ratio**.*

Consider the case where  $a \rightarrow -\infty$  (i.e. the conditioning set is  $X < b$ ) and  $\mu = 0$ ,  $\sigma = 1$ . Then Eq. (3.5) gives  $\mathbb{E}(X|X < b) = -\lambda(b) = -\frac{\phi(b)}{\Phi(b)}$ , where  $\lambda$  is the function computing the inverse Mills ratio.

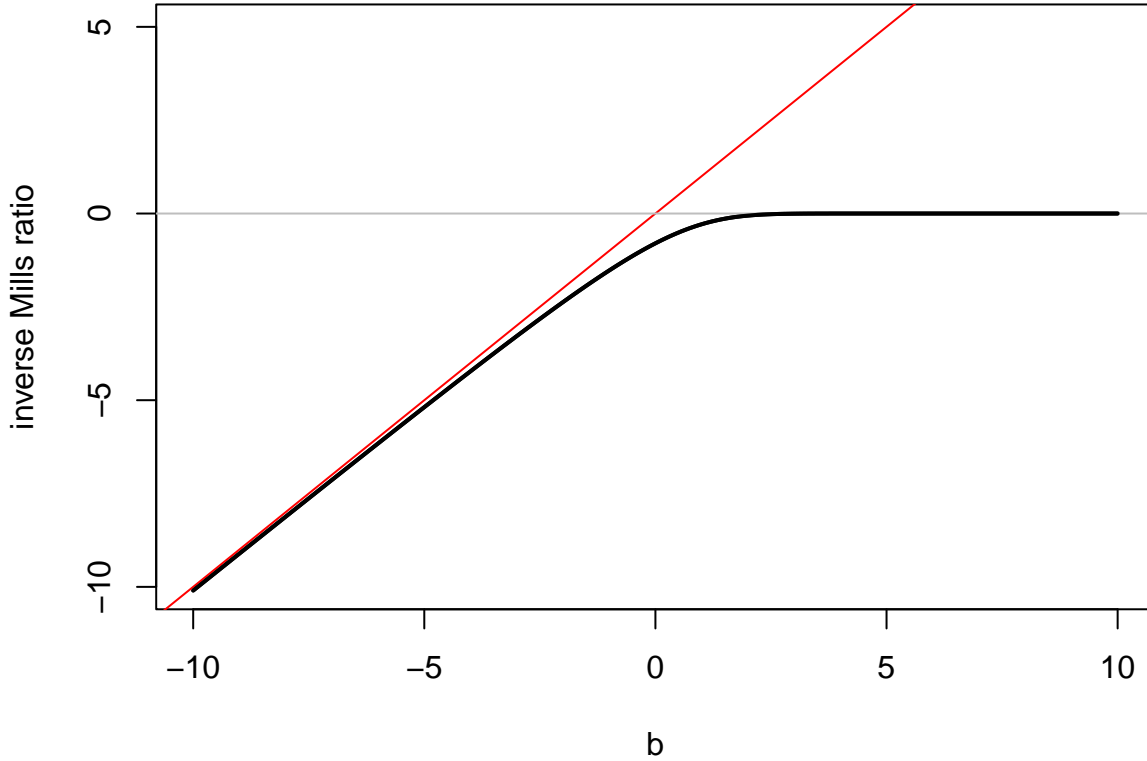


Figure 3.2:  $\mathbb{E}(X|X < b)$  as a function of  $b$  when  $X \sim \mathcal{N}(0, 1)$  (in black).

**Proposition 3.18** (p.d.f. of a multivariate Gaussian variable). *If  $Y \sim \mathcal{N}(\mu, \Omega)$  and if  $Y$  is a  $n$ -dimensional vector, then the density function of  $Y$  is:*

$$\frac{1}{(2\pi)^{n/2}|\Omega|^{1/2}} \exp \left[ -\frac{1}{2} (Y - \mu)' \Omega^{-1} (Y - \mu) \right].$$

## 3.5 Proofs

### Proof of Proposition ??

*Proof.* Assumptions (i) and (ii) (in the set of Assumptions ??) imply that  $\theta_{MLE}$  exists ( $= \operatorname{argmax}_{\theta} (1/n) \log \mathcal{L}(\theta; \mathbf{y})$ ).

$(1/n) \log \mathcal{L}(\theta; \mathbf{y})$  can be interpreted as the sample mean of the r.v.  $\log f(Y_i; \theta)$  that are i.i.d. Therefore  $(1/n) \log \mathcal{L}(\theta; \mathbf{y})$  converges to  $\mathbb{E}_{\theta_0}(\log f(Y; \theta))$  – which exists (Assumption iv).

Because the latter convergence is uniform (Assumption v), the solution  $\theta_{MLE}$  almost surely converges to the solution to the limit problem:

$$\operatorname{argmax}_{\theta} \mathbb{E}_{\theta_0}(\log f(Y; \theta)) = \operatorname{argmax}_{\theta} \int_{\mathbf{y}} \log f(y; \theta) f(y; \theta_0) dy.$$

Properties of the Kullback information measure (see Prop. 3.9), together with the identifiability assumption (ii) implies that the solution to the limit problem is unique and equal to  $\theta_0$ .

Consider a r.v. sequence  $\theta$  that converges to  $\theta_0$ . The Taylor expansion of the score in a neighborhood of  $\theta_0$  yields to:

$$\frac{\partial \log \mathcal{L}(\theta; \mathbf{y})}{\partial \theta} = \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} + \frac{\partial^2 \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta \partial \theta'} (\theta - \theta_0) + o_p(\theta - \theta_0)$$

$\theta_{MLE}$  converges to  $\theta_0$  and satisfies the likelihood equation  $\frac{\partial \log \mathcal{L}(\theta; \mathbf{y})}{\partial \theta} = \mathbf{0}$ . Therefore:

$$\frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} \approx -\frac{\partial^2 \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta \partial \theta'} (\theta_{MLE} - \theta_0),$$

or equivalently:

$$\frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} \approx \left( -\frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \log f(y_i; \theta_0)}{\partial \theta \partial \theta'} \right) \sqrt{n}(\theta_{MLE} - \theta_0),$$

By the law of large numbers, we have:  $\left( -\frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \log f(y_i; \theta_0)}{\partial \theta \partial \theta'} \right) \rightarrow \frac{1}{n} \mathbf{I}(\theta_0) = \mathcal{J}_Y(\theta_0)$ .

Besides, we have:

$$\begin{aligned} \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} &= \sqrt{n} \left( \frac{1}{n} \sum_i \frac{\partial \log f(y_i; \theta_0)}{\partial \theta} \right) \\ &= \sqrt{n} \left( \frac{1}{n} \sum_i \left\{ \frac{\partial \log f(y_i; \theta_0)}{\partial \theta} - \mathbb{E}_{\theta_0} \frac{\partial \log f(Y_i; \theta_0)}{\partial \theta} \right\} \right) \end{aligned}$$

which converges to  $\mathcal{N}(0, \mathcal{J}_Y(\theta_0))$  by the CLT.

Collecting the preceding results leads to (b). The fact that  $\theta_{MLE}$  achieves the FDCR bound proves (c).  $\square$

### Proof of Proposition ??

*Proof.* We have  $\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, \mathcal{J}(\theta_0)^{-1})$  (Eq. ??eq:normMLE)). A Taylor expansion around  $\theta_0$  yields to:

$$\sqrt{n}(h(\hat{\theta}_n) - h(\theta_0)) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h(\theta_0)'}{\partial \theta} \right). \quad (3.8)$$

Under  $H_0$ ,  $h(\theta_0) = 0$  therefore:

$$\sqrt{n}h(\hat{\theta}_n) \xrightarrow{d} \mathcal{N} \left( 0, \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h(\theta_0)'}{\partial \theta} \right). \quad (3.9)$$

Hence

$$\sqrt{n} \left( \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h(\theta_0)'}{\partial \theta} \right)^{-1/2} h(\hat{\theta}_n) \xrightarrow{d} \mathcal{N}(0, Id).$$

Taking the quadratic form, we obtain:

$$nh(\hat{\theta}_n)' \left( \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h(\theta_0)'}{\partial \theta} \right)^{-1} h(\hat{\theta}_n) \xrightarrow{d} \chi^2(r).$$

The fact that the test has asymptotic level  $\alpha$  directly stems from what precedes. **Consistency of the test:** Consider  $\theta_0 \in \Theta$ . Because the MLE is consistent,  $h(\hat{\theta}_n)$  converges to  $h(\theta_0) \neq 0$ . Eq. (3.8) is still valid. It implies that  $\xi_n^W$  converges to  $+\infty$  and therefore that  $\mathbb{P}_\theta(\xi_n^W \geq \chi_{1-\alpha}^2(r)) \rightarrow 1$ .  $\square$

### Proof of Proposition ??

*Proof.* Notations: “ $\approx$ ” means “equal up to a term that converges to 0 in probability”. We are under  $H_0$ .  $\hat{\theta}^0$  is the constrained ML estimator;  $\hat{\theta}$  denotes the unconstrained one.

We combine the two Taylor expansion:  $h(\hat{\theta}_n) \approx \frac{\partial h(\theta_0)}{\partial \theta'} (\hat{\theta}_n - \theta_0)$  and  $h(\hat{\theta}_n^0) \approx \frac{\partial h(\theta_0)}{\partial \theta'} (\hat{\theta}_n^0 - \theta_0)$  and we use  $h(\hat{\theta}_n^0) = 0$  (by definition) to get:

$$\sqrt{n}h(\hat{\theta}_n) \approx \frac{\partial h(\theta_0)}{\partial \theta'} \sqrt{n}(\hat{\theta}_n - \hat{\theta}_n^0). \quad (3.10)$$

Besides, we have (using the definition of the information matrix):

$$\frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \approx \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} - \mathcal{J}(\theta_0) \sqrt{n}(\hat{\theta}_n^0 - \theta_0) \quad (3.11)$$

and:

$$0 = \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n; \mathbf{y})}{\partial \theta} \approx \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} - \mathcal{J}(\theta_0) \sqrt{n}(\hat{\theta}_n - \theta_0). \quad (3.12)$$

Taking the difference and multiplying by  $\mathcal{J}(\theta_0)^{-1}$ :

$$\sqrt{n}(\hat{\theta}_n - \hat{\theta}_n^0) \approx \mathcal{J}(\theta_0)^{-1} \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \mathcal{J}(\theta_0). \quad (3.13)$$

Eqs. (3.10) and (3.13) yield to:

$$\sqrt{n}h(\hat{\theta}_n) \approx \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta}. \quad (3.14)$$

Recall that  $\hat{\theta}_n^0$  is the MLE of  $\theta_0$  under the constraint  $h(\theta) = 0$ . The vector of Lagrange multipliers  $\hat{\lambda}_n$  associated to this program satisfies:

$$\frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} + \frac{\partial h'(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \hat{\lambda}_n = 0. \quad (3.15)$$

Substituting the latter equation in Eq. (3.14) gives:

$$\sqrt{n}h(\hat{\theta}_n) \approx -\frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h'(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \frac{\hat{\lambda}_n}{\sqrt{n}} \approx -\frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h'(\theta_0; \mathbf{y})}{\partial \theta} \frac{\hat{\lambda}_n}{\sqrt{n}}$$

which yields:

$$\frac{\hat{\lambda}_n}{\sqrt{n}} \approx -\left( \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h'(\theta_0; \mathbf{y})}{\partial \theta} \right)^{-1} \sqrt{n}h(\hat{\theta}_n). \quad (3.16)$$

It follows, from Eq. (3.9), that:

$$\frac{\hat{\lambda}_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N} \left( 0, \left( \frac{\partial h(\theta_0)}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial h'(\theta_0; \mathbf{y})}{\partial \theta} \right)^{-1} \right).$$

Taking the quadratic form of the last equation gives:

$$\frac{1}{n} \hat{\lambda}_n' \frac{\partial h(\hat{\theta}_n^0)}{\partial \theta'} \mathcal{J}(\hat{\theta}_n^0)^{-1} \frac{\partial h'(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \hat{\lambda}_n \xrightarrow{d} \chi^2(r).$$

Using Eq. (3.15), it appears that the left-hand side term of the last equation is  $\xi^{LM}$  as defined in Eq. (??). Consistency: see Remark 17.3 in Gouriéroux and Monfort (1995).  $\square$

### Proof of Proposition ??

*Proof.* Let us first demonstrate the asymptotic equivalence of  $\xi^{LM}$  and  $\xi^{LR}$ .

The second-order Taylor expansions of  $\log \mathcal{L}(\hat{\theta}_n, \mathbf{y})$  and  $\log \mathcal{L}(\hat{\theta}_n^0, \mathbf{y})$  are:

$$\begin{aligned} \log \mathcal{L}(\hat{\theta}_n, \mathbf{y}) &\approx \log \mathcal{L}(\theta_0, \mathbf{y}) + \frac{\partial \log \mathcal{L}(\theta_0, \mathbf{y})}{\partial \theta'} (\hat{\theta}_n - \theta_0) - \frac{n}{2} (\hat{\theta}_n - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n - \theta_0) \\ \log \mathcal{L}(\hat{\theta}_n^0, \mathbf{y}) &\approx \log \mathcal{L}(\theta_0, \mathbf{y}) + \frac{\partial \log \mathcal{L}(\theta_0, \mathbf{y})}{\partial \theta'} (\hat{\theta}_n^0 - \theta_0) - \frac{n}{2} (\hat{\theta}_n^0 - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n^0 - \theta_0). \end{aligned}$$

Taking the difference, we obtain:

$$\xi_n^{LR} \approx 2 \frac{\partial \log \mathcal{L}(\theta_0, \mathbf{y})}{\partial \theta'} (\hat{\theta}_n - \hat{\theta}_n^0) + n(\hat{\theta}_n^0 - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n^0 - \theta_0) - n(\hat{\theta}_n - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n - \theta_0).$$

Using  $\frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\theta_0; \mathbf{y})}{\partial \theta} \approx \mathcal{J}(\theta_0) \sqrt{n}(\hat{\theta}_n - \theta_0)$  (Eq. (3.12)), we have:

$$\xi_n^{LR} \approx 2n(\hat{\theta}_n - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n - \hat{\theta}_n^0) + n(\hat{\theta}_n^0 - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n^0 - \theta_0) - n(\hat{\theta}_n - \theta_0)' \mathcal{J}(\theta_0) (\hat{\theta}_n - \theta_0).$$

In the second of the three terms in the sum, we replace  $(\hat{\theta}_n^0 - \theta_0)$  by  $(\hat{\theta}_n^0 - \hat{\theta}_n + \hat{\theta}_n - \theta_0)$  and we develop the associated product. This leads to:

$$\xi_n^{LR} \approx n(\hat{\theta}_n^0 - \hat{\theta}_n)' \mathcal{J}(\theta_0)^{-1}(\hat{\theta}_n^0 - \hat{\theta}_n). \quad (3.17)$$

The difference between Eqs. (3.11) and (3.12) implies:

$$\frac{1}{\sqrt{n}} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \approx \mathcal{J}(\theta_0) \sqrt{n}(\hat{\theta}_n - \hat{\theta}_n^0),$$

which, associated to Eq. (3.10), gives:

$$\xi_n^{LR} \approx \frac{1}{n} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta'} \mathcal{J}(\theta_0)^{-1} \frac{\partial \log \mathcal{L}(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \approx \xi_n^{LM}.$$

Hence  $\xi_n^{LR}$  has the same asymptotic distribution as  $\xi_n^{LM}$ .

Let's show that the LR test is consistent. For this, note that:

$$\frac{\log \mathcal{L}(\hat{\theta}, \mathbf{y}) - \log \mathcal{L}(\hat{\theta}^0, \mathbf{y})}{n} = \frac{1}{n} \sum_{i=1}^n [\log f(y_i; \hat{\theta}_n) - \log f(y_i; \hat{\theta}_n^0)] \rightarrow \mathbb{E}_0[\log f(Y; \theta_0) - \log f(Y; \theta_\infty)],$$

where  $\theta_\infty$ , the pseudo true value, is such that  $h(\theta_\infty) \neq 0$  (by definition of  $H_1$ ). From the Kullback inequality and the asymptotic identifiability of  $\theta_0$ , it follows that  $\mathbb{E}_0[\log f(Y; \theta_0) - \log f(Y; \theta_\infty)] > 0$ . Therefore  $\xi_n^{LR} \rightarrow +\infty$  under  $H_1$ .

Let us now demonstrate the equivalence of  $\xi^{LM}$  and  $\xi^W$ .

We have (using Eq. (3.10)):

$$\xi_n^{LM} = \frac{1}{n} \hat{\lambda}'_n \frac{\partial h(\hat{\theta}_n^0)}{\partial \theta'} \mathcal{J}(\hat{\theta}_n^0)^{-1} \frac{\partial h'(\hat{\theta}_n^0; \mathbf{y})}{\partial \theta} \hat{\lambda}_n.$$

Since, under  $H_0$ ,  $\hat{\theta}_n^0 \approx \hat{\theta}_n \approx \theta_0$ , Eq. (3.16) therefore implies that:

$$\xi^{LM} \approx n h(\hat{\theta}_n)' \left( \frac{\partial h(\hat{\theta}_n)}{\partial \theta'} \mathcal{J}(\hat{\theta}_n)^{-1} \frac{\partial h'(\hat{\theta}_n; \mathbf{y})}{\partial \theta} \right)^{-1} h(\hat{\theta}_n) = \xi^W,$$

which gives the result. □

### Proof of Eq. (2.3)

*Proof.* We have:

$$\begin{aligned} & T \mathbb{E} [(\bar{y}_T - \mu)^2] \\ &= T \mathbb{E} \left[ \left( \frac{1}{T} \sum_{t=1}^T (y_t - \mu) \right)^2 \right] = \frac{1}{T} \mathbb{E} \left[ \sum_{t=1}^T (y_t - \mu)^2 + 2 \sum_{s < t \leq T} (y_t - \mu)(y_s - \mu) \right] \\ &= \gamma_0 + \frac{2}{T} \left( \sum_{t=2}^T \mathbb{E} [(y_t - \mu)(y_{t-1} - \mu)] \right) + \frac{2}{T} \left( \sum_{t=3}^T \mathbb{E} [(y_t - \mu)(y_{t-2} - \mu)] \right) + \dots \\ &\quad + \frac{2}{T} \left( \sum_{t=T-1}^T \mathbb{E} [(y_t - \mu)(y_{t-(T-2)} - \mu)] \right) + \frac{2}{T} \mathbb{E} [(y_T - \mu)(y_{T-(T-1)} - \mu)] \\ &= \gamma_0 + 2 \frac{T-1}{T} \gamma_1 + \dots + 2 \frac{1}{T} \gamma_{T-1}. \end{aligned}$$

Therefore:

$$T \mathbb{E} [(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j = -2 \frac{1}{T} \gamma_1 - 2 \frac{2}{T} \gamma_2 - \dots - 2 \frac{T-1}{T} \gamma_{T-1} - 2 \gamma_T - 2 \gamma_{T+1} + \dots$$

And then:

$$\left| T\mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j \right| \leq 2\frac{1}{T}|\gamma_1| + 2\frac{2}{T}|\gamma_2| + \dots + 2\frac{T-1}{T}|\gamma_{T-1}| + 2|\gamma_T| + 2|\gamma_{T+1}| + \dots$$

For any  $q \leq T$ , we have:

$$\begin{aligned} \left| T\mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j \right| &\leq 2\frac{1}{T}|\gamma_1| + 2\frac{2}{T}|\gamma_2| + \dots + 2\frac{q-1}{T}|\gamma_{q-1}| + 2\frac{q}{T}|\gamma_q| + \\ &\quad 2\frac{q+1}{T}|\gamma_{q+1}| + \dots + 2\frac{T-1}{T}|\gamma_{T-1}| + 2|\gamma_T| + 2|\gamma_{T+1}| + \dots \\ &\leq \frac{2}{T}(|\gamma_1| + 2|\gamma_2| + \dots + (q-1)|\gamma_{q-1}| + q|\gamma_q|) + \\ &\quad 2|\gamma_{q+1}| + \dots + 2|\gamma_{T-1}| + 2|\gamma_T| + 2|\gamma_{T+1}| + \dots \end{aligned}$$

Consider  $\varepsilon > 0$ . The fact that the autocovariances are absolutely summable implies that there exists  $q_0$  such that (Cauchy criterion, Theorem 3.2):

$$2|\gamma_{q_0+1}| + 2|\gamma_{q_0+2}| + 2|\gamma_{q_0+3}| + \dots < \varepsilon/2.$$

Then, if  $T > q_0$ , it comes that:

$$\left| T\mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j \right| \leq \frac{2}{T}(|\gamma_1| + 2|\gamma_2| + \dots + (q_0-1)|\gamma_{q_0-1}| + q_0|\gamma_{q_0}|) + \varepsilon/2.$$

If  $T \geq 2(|\gamma_1| + 2|\gamma_2| + \dots + (q_0-1)|\gamma_{q_0-1}| + q_0|\gamma_{q_0}|) / (\varepsilon/2)$  ( $= f(q_0)$ , say) then

$$\frac{2}{T}(|\gamma_1| + 2|\gamma_2| + \dots + (q_0-1)|\gamma_{q_0-1}| + q_0|\gamma_{q_0}|) \leq \varepsilon/2.$$

Then, if  $T > f(q_0)$  and  $T > q_0$ , i.e. if  $T > \max(f(q_0), q_0)$ , we have:

$$\left| T\mathbb{E}[(\bar{y}_T - \mu)^2] - \sum_{j=-\infty}^{+\infty} \gamma_j \right| \leq \varepsilon.$$

□

### Proof of Proposition 2.15

*Proof.* We have:

$$\begin{aligned} \mathbb{E}([y_{t+1} - y_{t+1}^*]^2) &= \mathbb{E}([\{y_{t+1} - \mathbb{E}(y_{t+1}|x_t)\} + \{\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*\}]^2) \\ &= \mathbb{E}([y_{t+1} - \mathbb{E}(y_{t+1}|x_t)]^2) + \mathbb{E}([\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*]^2) \\ &\quad + 2\mathbb{E}([y_{t+1} - \mathbb{E}(y_{t+1}|x_t)][\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*]). \end{aligned} \quad (3.18)$$

Let us focus on the last term. We have:

$$\begin{aligned} &\mathbb{E}([y_{t+1} - \mathbb{E}(y_{t+1}|x_t)][\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*]) \\ &= \mathbb{E}(\mathbb{E}([y_{t+1} - \mathbb{E}(y_{t+1}|x_t)][\underbrace{\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*}_{\text{function of } x_t}])|x_t)) \\ &= \mathbb{E}([\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*]\mathbb{E}([y_{t+1} - \mathbb{E}(y_{t+1}|x_t)]|x_t)) \\ &= \mathbb{E}([\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*]\underbrace{\mathbb{E}(y_{t+1}|x_t) - \mathbb{E}(y_{t+1}|x_t)}_{=0}) = 0. \end{aligned}$$

Therefore, Eq. (3.18) becomes:

$$\begin{aligned} &\mathbb{E}([y_{t+1} - y_{t+1}^*]^2) \\ &= \underbrace{\mathbb{E}([y_{t+1} - \mathbb{E}(y_{t+1}|x_t)]^2)}_{\geq 0 \text{ and does not depend on } y_{t+1}^*} + \underbrace{\mathbb{E}([\mathbb{E}(y_{t+1}|x_t) - y_{t+1}^*]^2)}_{\geq 0 \text{ and depends on } y_{t+1}^*}. \end{aligned}$$

This implies that  $\mathbb{E}([y_{t+1} - y_{t+1}^*]^2)$  is always larger than  $\mathbb{E}([y_{t+1} - \mathbb{E}(y_{t+1}|x_t)]^2)$ , and is therefore minimized if the second term is equal to zero, that is if  $\mathbb{E}(y_{t+1}|x_t) = y_{t+1}^*$ . □



**Proof of Proposition 2.12**

*Proof.* Using Proposition ?? (in Appendix ??), we obtain that, conditionally on  $x_1$ , the log-likelihood is given by

$$\begin{aligned} \log \mathcal{L}(Y_T; \theta) &= -(Tn/2) \log(2\pi) + (T/2) \log |\Omega^{-1}| \\ &\quad - \frac{1}{2} \sum_{t=1}^T [(y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)]. \end{aligned}$$

Let's rewrite the last term of the log-likelihood:

$$\begin{aligned} \sum_{t=1}^T [(y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)] &= \\ \sum_{t=1}^T [(y_t - \hat{\Pi}' x_t + \hat{\Pi}' x_t - \Pi' x_t)' \Omega^{-1} (y_t - \hat{\Pi}' x_t + \hat{\Pi}' x_t - \Pi' x_t)] &= \\ \sum_{t=1}^T [(\hat{\varepsilon}_t + (\hat{\Pi} - \Pi)' x_t)' \Omega^{-1} (\hat{\varepsilon}_t + (\hat{\Pi} - \Pi)' x_t)], \end{aligned}$$

where the  $j^{th}$  element of the  $(n \times 1)$  vector  $\hat{\varepsilon}_t$  is the sample residual, for observation  $t$ , from an OLS regression of  $y_{j,t}$  on  $x_t$ . Expanding the previous equation, we get:

$$\begin{aligned} \sum_{t=1}^T [(y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)] &= \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} \hat{\varepsilon}_t \\ &\quad + 2 \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} (\hat{\Pi} - \Pi)' x_t + \sum_{t=1}^T x_t' (\hat{\Pi} - \Pi) \Omega^{-1} (\hat{\Pi} - \Pi)' x_t. \end{aligned}$$

Let's apply the trace operator on the second term (that is a scalar):

$$\begin{aligned} \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} (\hat{\Pi} - \Pi)' x_t &= Tr \left( \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} (\hat{\Pi} - \Pi)' x_t \right) \\ &= Tr \left( \sum_{t=1}^T \Omega^{-1} (\hat{\Pi} - \Pi)' x_t \hat{\varepsilon}_t' \right) = Tr \left( \Omega^{-1} (\hat{\Pi} - \Pi)' \sum_{t=1}^T x_t \hat{\varepsilon}_t' \right). \end{aligned}$$

Given that, by construction (property of OLS estimates), the sample residuals are orthogonal to the explanatory variables, this term is zero. Introducing  $\tilde{x}_t = (\hat{\Pi} - \Pi)' x_t$ , we have

$$\sum_{t=1}^T [(y_t - \Pi' x_t)' \Omega^{-1} (y_t - \Pi' x_t)] = \sum_{t=1}^T \hat{\varepsilon}_t' \Omega^{-1} \hat{\varepsilon}_t + \sum_{t=1}^T \tilde{x}_t' \Omega^{-1} \tilde{x}_t.$$

Since  $\Omega$  is a positive definite matrix,  $\Omega^{-1}$  is as well. Consequently, the smallest value that the last term can take is obtained for  $\tilde{x}_t = 0$ , i.e. when  $\Pi = \hat{\Pi}$ .

The MLE of  $\Omega$  is the matrix  $\hat{\Omega}$  that maximizes  $\Omega \xrightarrow{\ell} L(Y_T; \hat{\Pi}, \Omega)$ . We have:

$$\log \mathcal{L}(Y_T; \hat{\Pi}, \Omega) = -(Tn/2) \log(2\pi) + (T/2) \log |\Omega^{-1}| - \frac{1}{2} \sum_{t=1}^T [\hat{\varepsilon}_t' \Omega^{-1} \hat{\varepsilon}_t].$$

Matrix  $\hat{\Omega}$  is a symmetric positive definite. It is easily checked that the (unrestricted) matrix that maximizes the latter expression is symmetric positive definite matrix. Indeed:

$$\frac{\partial \log \mathcal{L}(Y_T; \hat{\Pi}, \Omega)}{\partial \Omega} = \frac{T}{2} \Omega' - \frac{1}{2} \sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}_t' \Rightarrow \hat{\Omega}' = \frac{1}{T} \sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}_t',$$

which leads to the result. □

**Proof of Proposition 2.13**

*Proof.* Let us drop the  $i$  subscript. Rearranging Eq. (2.47), we have:

$$\sqrt{T}(\mathbf{b} - \beta) = (X'X/T)^{-1}\sqrt{T}(X'\varepsilon/T).$$

Let us consider the autocovariances of  $\mathbf{v}_t = x_t\varepsilon_t$ , denoted by  $\gamma_j^v$ . Using the fact that  $x_t$  is a linear combination of past  $\varepsilon_t$ s and that  $\varepsilon_t$  is a white noise, we get that  $\mathbb{E}(\varepsilon_t x_t) = 0$ . Therefore

$$\gamma_j^v = \mathbb{E}(\varepsilon_t \varepsilon_{t-j} x_t x'_{t-j}).$$

If  $j > 0$ , we have  $\mathbb{E}(\varepsilon_t \varepsilon_{t-j} x_t x'_{t-j}) = \mathbb{E}(\mathbb{E}[\varepsilon_t \varepsilon_{t-j} x_t x'_{t-j} | \varepsilon_{t-j}, x_t, x_{t-j}]) = \mathbb{E}(\varepsilon_{t-j} x_t x'_{t-j} \mathbb{E}[\varepsilon_t | \varepsilon_{t-j}, x_t, x_{t-j}]) = 0$ . Note that we have  $\mathbb{E}[\varepsilon_t | \varepsilon_{t-j}, x_t, x_{t-j}] = 0$  because  $\{\varepsilon_t\}$  is an i.i.d. white noise sequence.

If  $j = 0$ , we have:

$$\gamma_0^v = \mathbb{E}(\varepsilon_t^2 x_t x'_t) = \mathbb{E}(\varepsilon_t^2) \mathbb{E}(x_t x'_t) = \sigma^2 \mathbf{Q}.$$

The convergence in distribution of  $\sqrt{T}(X'\varepsilon/T) = \sqrt{T}\frac{1}{T}\sum_{t=1}^T v_t$  results from the Central Limit Theorem for covariance-stationary processes, using the  $\gamma_j^v$  computed above.  $\square$

## 3.6 Additional codes

### 3.6.1 Simulating GEV distributions

The following lines of code have been used to generate Figure ??.

```
n.sim <- 4000
par(mfrow=c(1,3),
    plt=c(.2,.95,.2,.85))
all.rhos <- c(.3,.6,.95)
for(j in 1:length(all.rhos)){
  theta <- 1/all.rhos[j]
  v1 <- runif(n.sim)
  v2 <- runif(n.sim)
  w <- rep(.000001,n.sim)
  # solve for f(w) = w*(1 - log(w)/theta) - v2 = 0
  for(i in 1:20){
    f.i <- w * (1 - log(w)/theta) - v2
    f.prime <- 1 - log(w)/theta - 1/theta
    w <- w - f.i/f.prime
  }
  u1 <- exp(v1^(1/theta) * log(w))
  u2 <- exp((1-v1)^(1/theta) * log(w))

  # Get eps1 and eps2 using the inverse of
  # the Gumbel distribution's cdf:
  eps1 <- -log(-log(u1))
  eps2 <- -log(-log(u2))
  cbind(cor(eps1,eps2),1-all.rhos[j]^2)
  plot(eps1,eps2,pch=19,col="#FF000044",
       main=paste("rho = ",toString(all.rhos[j]),sep=""),
       xlab=expression(epsilon[1]),
       ylab=expression(epsilon[2]),
       cex.lab=2,cex.main=1.5)
}
```

### 3.6.2 Computing the covariance matrix of IRF using the delta method

```

irf.function <- function(THETA){
  c <- THETA[1]
  phi <- THETA[2:(p+1)]
  if(q>0){
    theta <- c(1,THETA[(1+p+1):(1+p+q)])
  }else{
    theta <- 1
  }
  sigma <- THETA[1+p+q+1]
  r <- dim(Matrix.of.Exog)[2] - 1
  beta <- THETA[(1+p+q+1+1):(1+p+q+1+(r+1))]

  irf <- sim.arma(0,phi,beta,sigma=sd(Ramey$ED3_TC,na.rm=TRUE),T=60,
                 y.0=rep(0,length(x$phi)),nb.sim=1,make.IRF=1,
                 X=NaN,beta=NaN)

  return(irf)
}

IRF.0 <- 100*irf.function(x$THETA)
eps <- .00000001
d.IRF <- NULL
for(i in 1:length(x$THETA)){
  THETA.i <- x$THETA
  THETA.i[i] <- THETA.i[i] + eps
  IRF.i <- 100*irf.function(THETA.i)
  d.IRF <- cbind(d.IRF,
                 (IRF.i - IRF.0)/eps
                 )
}
mat.var.cov.IRF <- d.IRF %*% x$I %*% t(d.IRF)

```

## 3.7 Statistical Tables

Table 3.1: Quantiles of the  $\mathcal{N}(0, 1)$  distribution. If  $a$  and  $b$  are respectively the row and column number; then the corresponding cell gives  $\mathbb{P}(0 < X \leq a + b)$ , where  $X \sim \mathcal{N}(0, 1)$ .

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	0.5000	0.6179	0.7257	0.8159	0.8849	0.9332	0.9641	0.9821	0.9918	0.9965
0.1	0.5040	0.6217	0.7291	0.8186	0.8869	0.9345	0.9649	0.9826	0.9920	0.9966
0.2	0.5080	0.6255	0.7324	0.8212	0.8888	0.9357	0.9656	0.9830	0.9922	0.9967
0.3	0.5120	0.6293	0.7357	0.8238	0.8907	0.9370	0.9664	0.9834	0.9925	0.9968
0.4	0.5160	0.6331	0.7389	0.8264	0.8925	0.9382	0.9671	0.9838	0.9927	0.9969
0.5	0.5199	0.6368	0.7422	0.8289	0.8944	0.9394	0.9678	0.9842	0.9929	0.9970
0.6	0.5239	0.6406	0.7454	0.8315	0.8962	0.9406	0.9686	0.9846	0.9931	0.9971
0.7	0.5279	0.6443	0.7486	0.8340	0.8980	0.9418	0.9693	0.9850	0.9932	0.9972
0.8	0.5319	0.6480	0.7517	0.8365	0.8997	0.9429	0.9699	0.9854	0.9934	0.9973
0.9	0.5359	0.6517	0.7549	0.8389	0.9015	0.9441	0.9706	0.9857	0.9936	0.9974
1	0.5398	0.6554	0.7580	0.8413	0.9032	0.9452	0.9713	0.9861	0.9938	0.9974
1.1	0.5438	0.6591	0.7611	0.8438	0.9049	0.9463	0.9719	0.9864	0.9940	0.9975
1.2	0.5478	0.6628	0.7642	0.8461	0.9066	0.9474	0.9726	0.9868	0.9941	0.9976
1.3	0.5517	0.6664	0.7673	0.8485	0.9082	0.9484	0.9732	0.9871	0.9943	0.9977
1.4	0.5557	0.6700	0.7704	0.8508	0.9099	0.9495	0.9738	0.9875	0.9945	0.9977
1.5	0.5596	0.6736	0.7734	0.8531	0.9115	0.9505	0.9744	0.9878	0.9946	0.9978
1.6	0.5636	0.6772	0.7764	0.8554	0.9131	0.9515	0.9750	0.9881	0.9948	0.9979
1.7	0.5675	0.6808	0.7794	0.8577	0.9147	0.9525	0.9756	0.9884	0.9949	0.9979
1.8	0.5714	0.6844	0.7823	0.8599	0.9162	0.9535	0.9761	0.9887	0.9951	0.9980
1.9	0.5753	0.6879	0.7852	0.8621	0.9177	0.9545	0.9767	0.9890	0.9952	0.9981
2	0.5793	0.6915	0.7881	0.8643	0.9192	0.9554	0.9772	0.9893	0.9953	0.9981
2.1	0.5832	0.6950	0.7910	0.8665	0.9207	0.9564	0.9778	0.9896	0.9955	0.9982
2.2	0.5871	0.6985	0.7939	0.8686	0.9222	0.9573	0.9783	0.9898	0.9956	0.9982
2.3	0.5910	0.7019	0.7967	0.8708	0.9236	0.9582	0.9788	0.9901	0.9957	0.9983
2.4	0.5948	0.7054	0.7995	0.8729	0.9251	0.9591	0.9793	0.9904	0.9959	0.9984
2.5	0.5987	0.7088	0.8023	0.8749	0.9265	0.9599	0.9798	0.9906	0.9960	0.9984
2.6	0.6026	0.7123	0.8051	0.8770	0.9279	0.9608	0.9803	0.9909	0.9961	0.9985
2.7	0.6064	0.7157	0.8078	0.8790	0.9292	0.9616	0.9808	0.9911	0.9962	0.9985
2.8	0.6103	0.7190	0.8106	0.8810	0.9306	0.9625	0.9812	0.9913	0.9963	0.9986
2.9	0.6141	0.7224	0.8133	0.8830	0.9319	0.9633	0.9817	0.9916	0.9964	0.9986

Table 3.2: Quantiles of the Student- $t$  distribution. The rows correspond to different degrees of freedom ( $\nu$ , say); the columns correspond to different probabilities ( $z$ , say). The cell gives  $q$  that is s.t.  $\mathbb{P}(-q < X < q) = z$ , with  $X \sim t(\nu)$ .

	0.05	0.1	0.75	0.9	0.95	0.975	0.99	0.999
1	0.079	0.158	2.414	6.314	12.706	25.452	63.657	636.619
2	0.071	0.142	1.604	2.920	4.303	6.205	9.925	31.599
3	0.068	0.137	1.423	2.353	3.182	4.177	5.841	12.924
4	0.067	0.134	1.344	2.132	2.776	3.495	4.604	8.610
5	0.066	0.132	1.301	2.015	2.571	3.163	4.032	6.869
6	0.065	0.131	1.273	1.943	2.447	2.969	3.707	5.959
7	0.065	0.130	1.254	1.895	2.365	2.841	3.499	5.408
8	0.065	0.130	1.240	1.860	2.306	2.752	3.355	5.041
9	0.064	0.129	1.230	1.833	2.262	2.685	3.250	4.781
10	0.064	0.129	1.221	1.812	2.228	2.634	3.169	4.587
20	0.063	0.127	1.185	1.725	2.086	2.423	2.845	3.850
30	0.063	0.127	1.173	1.697	2.042	2.360	2.750	3.646
40	0.063	0.126	1.167	1.684	2.021	2.329	2.704	3.551
50	0.063	0.126	1.164	1.676	2.009	2.311	2.678	3.496
60	0.063	0.126	1.162	1.671	2.000	2.299	2.660	3.460
70	0.063	0.126	1.160	1.667	1.994	2.291	2.648	3.435
80	0.063	0.126	1.159	1.664	1.990	2.284	2.639	3.416
90	0.063	0.126	1.158	1.662	1.987	2.280	2.632	3.402
100	0.063	0.126	1.157	1.660	1.984	2.276	2.626	3.390
200	0.063	0.126	1.154	1.653	1.972	2.258	2.601	3.340
500	0.063	0.126	1.152	1.648	1.965	2.248	2.586	3.310

Table 3.3: Quantiles of the  $\chi^2$  distribution. The rows correspond to different degrees of freedom; the columns correspond to different probabilities.

	0.05	0.1	0.75	0.9	0.95	0.975	0.99	0.999
1	0.004	0.016	1.323	2.706	3.841	5.024	6.635	10.828
2	0.103	0.211	2.773	4.605	5.991	7.378	9.210	13.816
3	0.352	0.584	4.108	6.251	7.815	9.348	11.345	16.266
4	0.711	1.064	5.385	7.779	9.488	11.143	13.277	18.467
5	1.145	1.610	6.626	9.236	11.070	12.833	15.086	20.515
6	1.635	2.204	7.841	10.645	12.592	14.449	16.812	22.458
7	2.167	2.833	9.037	12.017	14.067	16.013	18.475	24.322
8	2.733	3.490	10.219	13.362	15.507	17.535	20.090	26.124
9	3.325	4.168	11.389	14.684	16.919	19.023	21.666	27.877
10	3.940	4.865	12.549	15.987	18.307	20.483	23.209	29.588
20	10.851	12.443	23.828	28.412	31.410	34.170	37.566	45.315
30	18.493	20.599	34.800	40.256	43.773	46.979	50.892	59.703
40	26.509	29.051	45.616	51.805	55.758	59.342	63.691	73.402
50	34.764	37.689	56.334	63.167	67.505	71.420	76.154	86.661
60	43.188	46.459	66.981	74.397	79.082	83.298	88.379	99.607
70	51.739	55.329	77.577	85.527	90.531	95.023	100.425	112.317
80	60.391	64.278	88.130	96.578	101.879	106.629	112.329	124.839
90	69.126	73.291	98.650	107.565	113.145	118.136	124.116	137.208
100	77.929	82.358	109.141	118.498	124.342	129.561	135.807	149.449
200	168.279	174.835	213.102	226.021	233.994	241.058	249.445	267.541
500	449.147	459.926	520.950	540.930	553.127	563.852	576.493	603.446

Table 3.4: Quantiles of the  $\mathcal{F}$  distribution. The columns and rows correspond to different degrees of freedom (resp.  $n_1$  and  $n_2$ ). The different panels correspond to different probabilities ( $\alpha$ ) The corresponding cell gives  $z$  that is s.t.  $\mathbb{P}(X \leq z) = \alpha$ , with  $X \sim \mathcal{F}(n_1, n_2)$ .

	1	2	3	4	5	6	7	8	9	10
alpha = 0.9										
5	4.060	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.316	3.297
10	3.285	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.347	2.323
15	3.073	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.086	2.059
20	2.975	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.965	1.937
50	2.809	2.412	2.197	2.061	1.966	1.895	1.840	1.796	1.760	1.729
100	2.756	2.356	2.139	2.002	1.906	1.834	1.778	1.732	1.695	1.663
500	2.716	2.313	2.095	1.956	1.859	1.786	1.729	1.683	1.644	1.612
alpha = 0.95										
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026
100	3.936	3.087	2.696	2.463	2.305	2.191	2.103	2.032	1.975	1.927
500	3.860	3.014	2.623	2.390	2.232	2.117	2.028	1.957	1.899	1.850
alpha = 0.99										
5	16.258	13.274	12.060	11.392	10.967	10.672	10.456	10.289	10.158	10.051
10	10.044	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368
50	7.171	5.057	4.199	3.720	3.408	3.186	3.020	2.890	2.785	2.698
100	6.895	4.824	3.984	3.513	3.206	2.988	2.823	2.694	2.590	2.503
500	6.686	4.648	3.821	3.357	3.054	2.838	2.675	2.547	2.443	2.356

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