Econ 493 B1 - Winter 2023

Homework 3 - Solution

Exercise 1

Consider the stationary AR(2) process:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2).$$

a. Derive the autocorrelation function (ACF) for the AR(2) process.

First note that $E[y_t] = 0$.

$$\begin{aligned} E[y_t] &= \phi_1 E[y_{t-1}] + \phi_2 E[y_{t-2}] + E[\varepsilon_t] \\ &= 0 \end{aligned}$$

To compute $V[y_t] = \gamma_0$, square y_t and take expectations.

$$\begin{aligned} \gamma_0 = E[y_t^2] &= E[y_t(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t)] \\ &= \phi_1 E[y_t y_{t-1}] + \phi_2 E[y_t y_{t-2}] + E[y_t \varepsilon_t] \\ &\Rightarrow \gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2 \end{aligned}$$

Similarly, to compute γ_1 multiply by y_{t-1} and take expectations.

$$\begin{aligned} \gamma_1 = E[y_{t-1} y_t] &= E[y_{t-1}(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t)] \\ &= \phi_1 E[y_{t-1}^2] + \phi_2 E[y_{t-1} y_{t-2}] + E[y_{t-1} \varepsilon_t] \\ &\Rightarrow \gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1 \end{aligned}$$

Similarly, to compute γ_2 multiply by y_{t-2} and take expectations.

$$\begin{aligned} \gamma_2 = E[y_{t-2} y_t] &= E[y_{t-2}(\phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t)] \\ &= \phi_1 E[y_{t-2} y_{t-1}] + \phi_2 E[y_{t-2}^2] + E[y_{t-2} \varepsilon_t] \\ &\Rightarrow \gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0 \end{aligned}$$

The autocovariance are:

$$\begin{aligned} \gamma_0 &= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2 \\ \gamma_1 &= \phi_1 \gamma_0 + \phi_2 \gamma_1 \\ \gamma_2 &= \phi_1 \gamma_1 + \phi_2 \gamma_0 \\ \gamma_k &= \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \text{ for } k > 2 \end{aligned}$$

The autocorrelations are:

$$\begin{aligned}
\rho_1 &= \gamma_1/\gamma_0 = \frac{\phi_1}{1-\phi_2} \\
\rho_2 &= \gamma_2/\gamma_0 = \frac{\phi_1^2}{1-\phi_2} + \phi_2 \\
\rho_k &= \gamma_k/\gamma_0 = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} \text{ for } k > 2
\end{aligned}$$

b. Find the optimal forecasts made at time T, $y_{T+h|T}$, for $h = 1, 2, 3$.

For $h = 1$:

$$\begin{aligned}
y_{T+1} &= \phi_1 y_T + \phi_2 y_{T-1} + \varepsilon_{T+1} \\
y_{T+1|T} &= \phi_1 y_T + \phi_2 y_{T-1}
\end{aligned}$$

For $h = 2$:

$$\begin{aligned}
y_{T+2} &= \phi_1 y_{T+1} + \phi_2 y_T + \varepsilon_{T+2} \\
y_{T+2|T} &= \phi_1 y_{T+1|T} + \phi_2 y_T \\
y_{T+2|T} &= \phi_1(\phi_1 y_T + \phi_2 y_{T-1}) + \phi_2 y_T
\end{aligned}$$

For $h = 3$:

$$\begin{aligned}
y_{T+3} &= \phi_1 y_{T+2} + \phi_2 y_{T+1} + \varepsilon_{T+3} \\
y_{T+3|T} &= \phi_1 y_{T+2|T} + \phi_2 y_{T+1|T} \\
y_{T+3|T} &= \phi_1(\phi_1(\phi_1 y_T + \phi_2 y_{T-1}) + \phi_2 y_T) + \phi_2(\phi_1 y_T + \phi_2 y_{T-1})
\end{aligned}$$

c. Find the corresponding forecast errors $\varepsilon_{T+h|T}$ for $h = 1, 2, 3$.

For $h = 1$:

$$\begin{aligned}
\varepsilon_{T+1|T} &= y_{T+1} - y_{T+1|T} \\
&= \varepsilon_{T+1}
\end{aligned}$$

For $h = 2$:

$$\begin{aligned}
\varepsilon_{T+2|T} &= y_{T+2} - y_{T+2|T} \\
&= \phi_1\varepsilon_{T+1|T} + \varepsilon_{T+2} \\
&= \phi_1\varepsilon_{T+1} + \varepsilon_{T+2}
\end{aligned}$$

For $h = 3$:

$$\begin{aligned}
\varepsilon_{T+3|T} &= y_{T+3} - y_{T+3|T} \\
&= \phi_1\varepsilon_{T+2|T} + \phi_2\varepsilon_{T+1|T} + \varepsilon_{T+3} \\
&= \phi_1(\phi_1\varepsilon_{T+1} + \varepsilon_{T+2}) + \phi_2\varepsilon_{T+1} + \varepsilon_{T+3}
\end{aligned}$$

d. Find the forecast error variances for $h = 1, 2, 3$.

For $h = 1$:

$$V[\varepsilon_{T+1|T}] = \sigma^2$$

For $h = 2$:

$$\begin{aligned} V[\varepsilon_{T+2|T}] &= V[\phi_1 \varepsilon_{T+1} + \varepsilon_{T+2}] \\ &= \phi_1^2 \sigma^2 + \sigma^2 \\ &= (1 + \phi_1^2)\sigma^2 \end{aligned}$$

For $h = 3$:

$$\begin{aligned} V[\varepsilon_{T+2|T}] &= V[\phi_1(\phi_1 \varepsilon_{T+1} + \varepsilon_{T+2}) + \phi_2 \varepsilon_{T+1} + \varepsilon_{T+3}] \\ &= (\phi_1^2 + \phi_2)^2 \sigma^2 + \phi_1^2 \sigma^2 + \sigma^2 \\ &= [1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2]\sigma^2 \end{aligned}$$

Exercise 2

Consider the simple trend stationary process:

$$y_t = \delta_0 + \delta_1 t + \varepsilon_t, \quad \varepsilon_t \sim WN(0, \sigma^2).$$

Its first difference can be written as

$$y_t' = \delta_1 + \varepsilon_t',$$

with $y_t' = y_t - y_{t-1}$ and $\varepsilon_t' = \varepsilon_t - \varepsilon_{t-1}$.

a. What process does ε_t' follow? Is this process inveritble?

Since $\varepsilon_t' = \varepsilon_t - \varepsilon_{t-1}$, then it follows a non-invertible MA(1) process with $\theta_1 = -1$.

b. Derive the mean, variance, and autocorrelation function of ε_t'.

The mean, variance, and autocovariances are:

$$\begin{aligned}
E[\varepsilon_t'] &= E[\varepsilon_t - \varepsilon_{t-1}] = 0 \\
V[\varepsilon_t'] &= V[\varepsilon_t - \varepsilon_{t-1}] = \sigma^2 + \sigma^2 \\
\gamma_1 &= E[(\varepsilon_t - \varepsilon_{t-1})(\varepsilon_{t-1} - \varepsilon_{t-2})] = -\sigma^2 \\
\gamma_2 &= E[(\varepsilon_t - \varepsilon_{t-1})(\varepsilon_{t-2} - \varepsilon_{t-3})] = 0 \\
\gamma_k &= 0 \text{ for } k > 2
\end{aligned}$$

The autocorrelations are:

$$\begin{aligned}
\rho_1 &= \gamma_1/\gamma_0 = \frac{-\sigma^2}{2\sigma^2} = -1/2 \\
\rho_k &= \gamma_k/\gamma_0 = 0 \text{ for } k > 1
\end{aligned}$$

c. Plot the autocorrelation function.

The ACF has a significant spike at lag 1 with $r_1 = -1/2$, but none at higher lags.

Exercise 3 (R)

The annual bituminous coal production in the US from 1920 to 1968 is in data set `bicoal`.

a. You decide to fit the following model to the series:

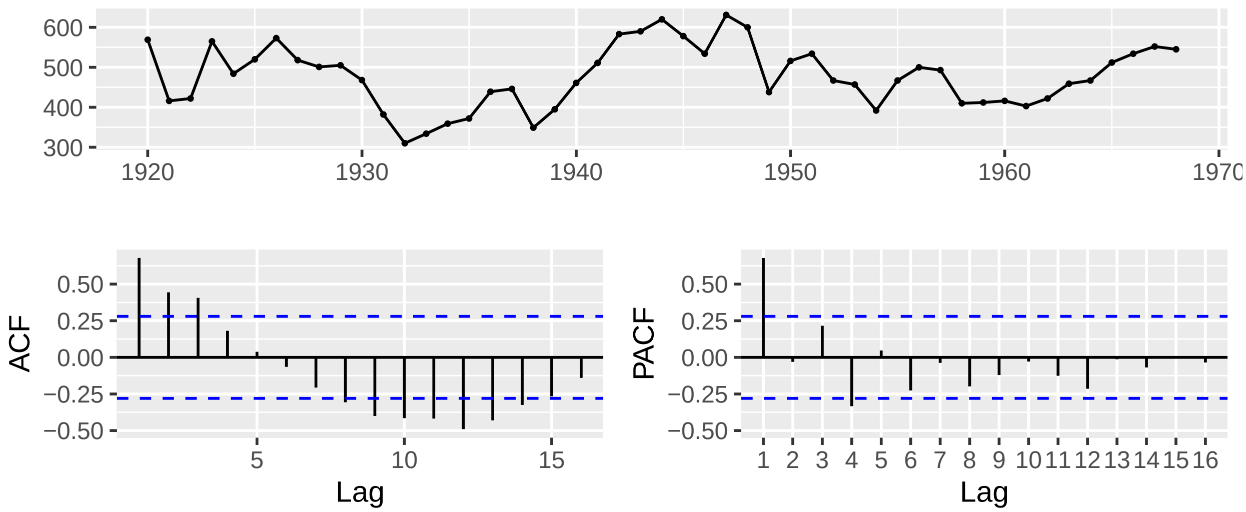
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \phi_4 y_{t-4} + \varepsilon_t$$

where y_t is the coal production in year t and ε_t is a white noise series. What sort of ARIMA model is this (that is, what are p, d, q)?

ARIMA(4,0,0). $p = 4$, $d = 0$, $q = 0$.

b. Explain why this model was chosen using the ACF and PACF.

```
ggtsdisplay(bicoal)
```



The PACF has a signifcant spike at lag 4, but none at higher lags.

c. The estimated parameters are $c = 162.00$, $\phi_1 = 0.83$, $\phi_2 = -0.34$, $\phi_3 = 0.55$, and $\phi_4 = -0.38$. The last five values of the series are given below. Without using the `forecast` function, calculate forecasts for the next three years (1969–1971).

>	Year	1964	1965	1966	1967
Millions of tons	467	512	534	552	545

$$
\begin{aligned}
\hat{y}_{T+1|T} &= c + \phi_1 y_T + \phi_2 y_{T-1} + \phi_3 y_{T-2} + \phi_4 y_{T-3} \\
&= 162 + 0.83 * 545 - 0.34 * 552 + 0.55 * 534 - 0.38 * 512 \\
&= 525.81 \\
\hat{y}_{T+2|T} &= c + \phi_1 \hat{y}_{T+1|T} + \phi_2 y_T + \phi_3 y_{T-1} + \phi_4 y_{T-2} \\
&= 162 + 0.83 * 525.81 - 0.34 * 545 + 0.55 * 552 - 0.38 * 534 \\
&= 513.80 \\
\hat{y}_{T+3|T} &= c + \phi_1 \hat{y}_{T+2|T} + \phi_2 \hat{y}_{T+1|T} + \phi_3 y_T + \phi_4 y_{T-1} \\
&= 162 + 0.83 * 513.80 - 0.34 * 525.81 + 0.55 * 545 - 0.38 * 552 \\
&= 499.67
\end{aligned}
$$

d. Now fit the model in R and obtain the forecasts from the same model. How are they different from yours? Why?

```
fit <- Arima(bicoal, order=c(4,0,0))
forecast(fit, h=3)
```

```
##      Point Forecast  Lo 80  Hi 80  Lo 95  Hi 95
## 1969         527.63 459.88 595.38 424.02 631.24
## 1970         517.19 429.00 605.38 382.32 652.07
## 1971         503.81 412.48 595.13 364.13 643.48
```

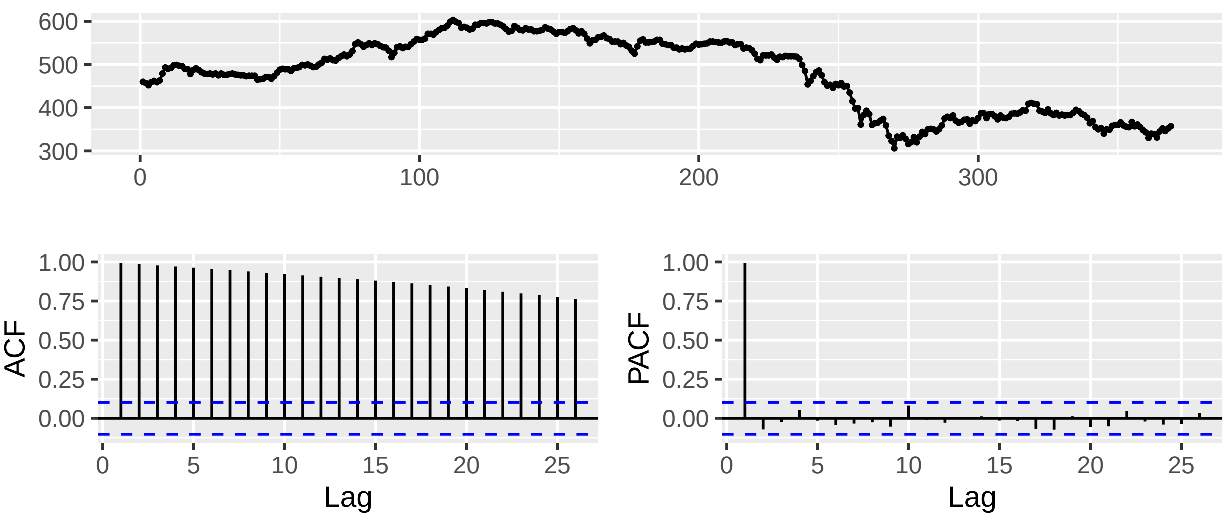
Any differences between these values and those computed "by hand" are probably due to rounding errors.

Exercise 4 (R)

A classic example of a non-stationary series is the daily closing IBM stock price series (data set `ibmclose`).

a. Use R to plot the series, the ACF, and PACF. Explain how each plot shows that the series is non-stationary and should be differenced.

```
ggtsdisplay(ibmclose)
```



The time plot shows the series "wandering around", which is a typical indication of non-stationarity. Differencing the series should remove this feature.

ACF does not drop quickly to zero, moreover the value r_1 is large and positive (almost 1 in this case). All these are signs of a non-stationary time series. Therefore it should be differenced to obtain a stationary series.

PACF value r_1 is almost 1. This is a sign of a non-stationary process that should be differenced in order to obtain a stationary series.

b. Fit a stationary AR(1) process to the series. What is the estimated coefficient? How do the results relate to your answers in a?

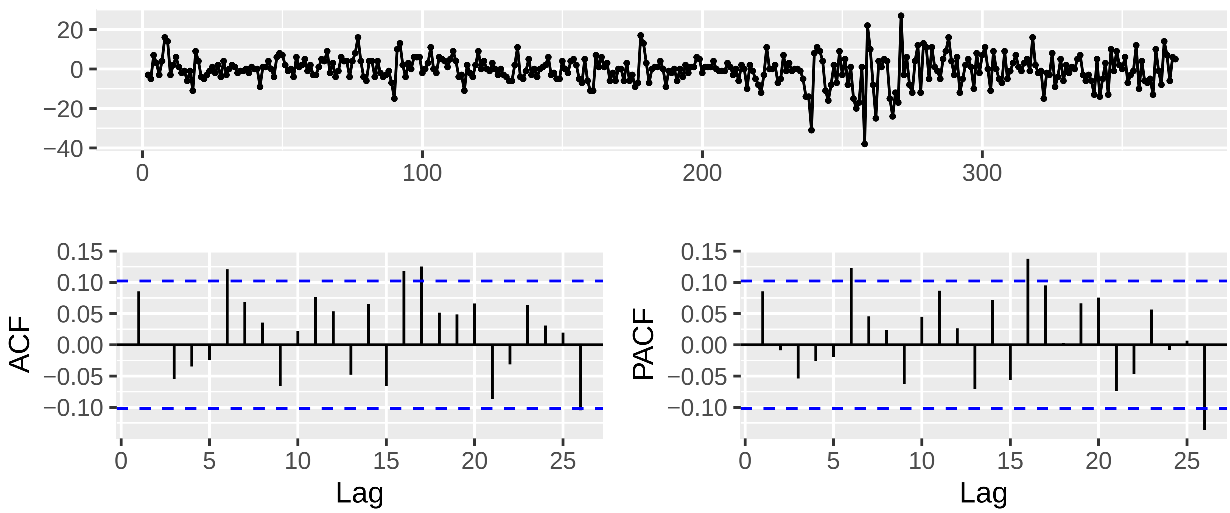
```
(fit <- Arima(ibmclose,order=c(1,0,0)))
```

```
## Series: ibmclose
## ARIMA(1,0,0) with non-zero mean
##
## Coefficients:
##          ar1     mean
##        0.996  439.038
## s.e.   0.003   63.136
##
## sigma^2 = 52.8:  log likelihood = -1256.7
## AIC=2519.4   AICc=2519.5   BIC=2531.1
```

The estimated coefficient is *very* close to one, again a sign of non-stationarity.

c. Use R to plot the first difference of the series, the ACF and PACF. Explain how each plot shows that the differenced series is stationary.

```
ggtsdisplay(diff(ibmclose))
```



The plots now show patterns typically associated with stationarity: not much persistance, small ACF and PACF values.

d. Fit a stationary AR(1) process to the differenced series. What is the estimated coefficient? How do the results relate to your answers to c?

```
(fit <- Arima(ibmclose,order=c(1,1,0)))
```

```
## Series: ibmclose
## ARIMA(1,1,0)
##
## Coefficients:
##          ar1
##        0.087
## s.e.  0.052
##
## sigma^2 = 52.4:  log likelihood = -1250
## AIC=2503.9   AICc=2504   BIC=2511.8
```

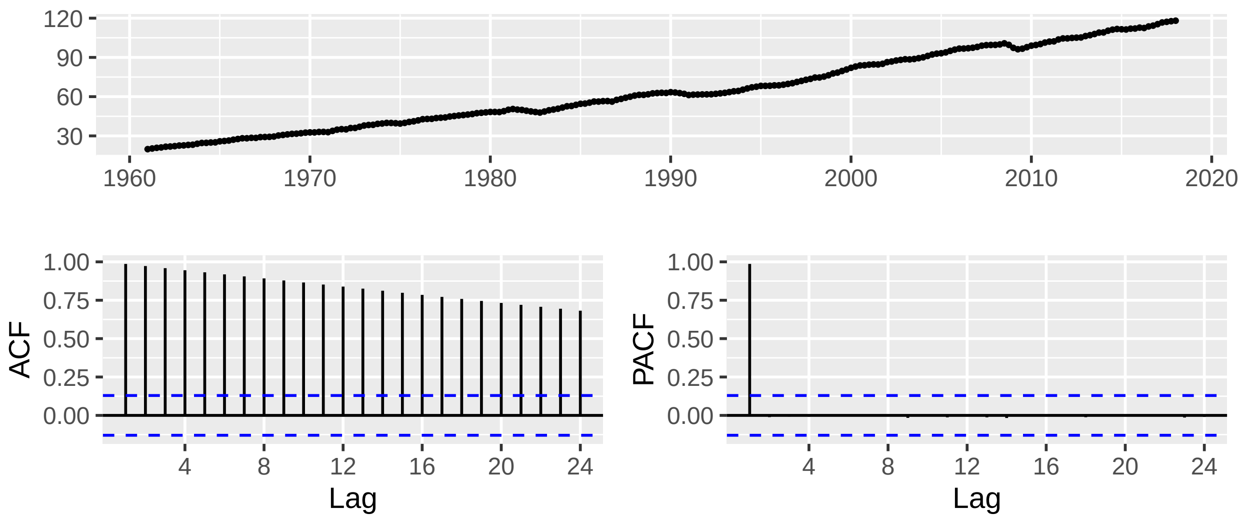
The estimated coefficient is well below one, in absolute value.

Exercise 5 (R)

The file `NAEXKP01CAQ661S.csv` contains the series of quarterly real gross domestic product (RGDP) for Canada for the quarters 1961:Q1 to 2018:Q1, measured in millions of 2010 Canadian dollars and seasonally adjusted.

a. Use R to plot the series, the ACF, and PACF. Does the series appear to be stationary?

```
# read canada quarterly real gdp data
data <- read.table("NAEXKP01CAQ661S.csv", sep=",", header=TRUE)
cangdp <- ts(data$NAEXKP01CAQ661S,start=1961,frequency=4)
ggtsdisplay(cangdp)
```

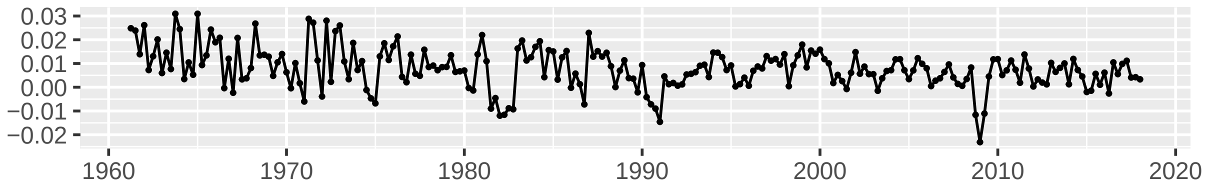


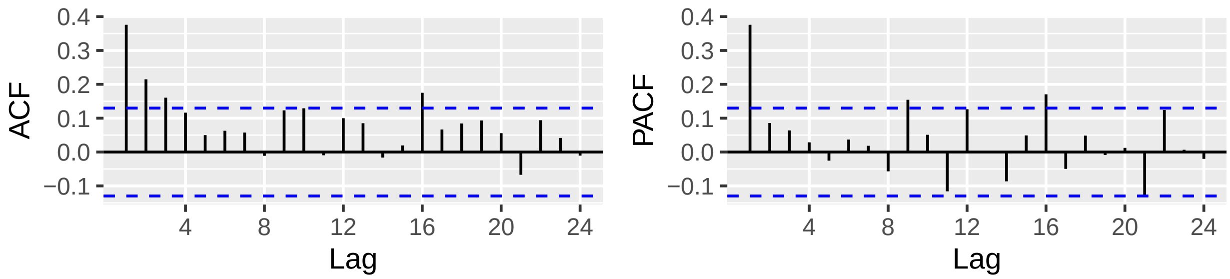
The time plot shows the series exhibits a time trend, which is a typical indication of non-stationarity. Differencing the series should remove this feature.

ACF does not drop quickly to zero, moreover the value r_1 is large and positive (almost 1 in this case). PACF value r_1 is almost 1. All these are signs of a non-stationary time series. Therefore it should be differenced to obtain a stationary series.

b. Use R to plot the change in log RGDP (that is, growth rate of the series), the ACF, and PACF. Does the differenced series appear to be stationary?

```
lcangdp <- log(cangdp)
ggtsdisplay(diff(lcangdp))
```





Differencing removes all the signs of non-stationarity noticed above.

c. What model do the plots in (b) suggest?

The PACF has a signifcant spike at lag 1, but none at higher lags. This suggests an ARIMA(1,1,0) model.

d. Using the BIC, find the AR model that adequately describes the change in log RGDP. Motivate the steps that you take. Make sure your model includes a drift to capture the time trend observed in the series.

```
BIC(Arima(lcangdp,order=c(0,1,0), include.drift = TRUE))
```

```
## [1] -1515
```

```
BIC(Arima(lcangdp,order=c(1,1,0), include.drift = TRUE))
```

```
## [1] -1544.9
```

```
BIC(Arima(lcangdp,order=c(2,1,0), include.drift = TRUE))
```

```
## [1] -1541.3
```

```
BIC(Arima(lcangdp,order=c(3,1,0), include.drift = TRUE))
```

```
## [1] -1536.9
```

```
BIC(Arima(lcangdp,order=c(4,1,0), include.drift = TRUE))
```

```
## [1] -1531.8
```

The BIC selects an ARIMA(1,1,0) model.

```
(Arima(lcangdp,order=c(1,1,0), include.drift = TRUE))
```

```
## Series: lcangdp
## ARIMA(1,1,0) with drift
```

```
##
## Coefficients:
##          ar1   drift
##        0.381   0.008
## s.e.   0.062   0.001
##
## sigma^2 = 6.28e-05:  log likelihood = 780.57
## AIC=-1555.1   AICc=-1555   BIC=-1544.8
```

e. Using the AIC, find the AR model that adequately describes the change in log RGDP. Motivate the steps that you take. Make sure your model includes a drift to capture the time trend observed in the series.

```
AIC(Arima(lcangdp,order=c(0,1,0), include.drift = TRUE))
```

```
## [1] -1521.9
```

```
AIC(Arima(lcangdp,order=c(1,1,0), include.drift = TRUE))
```

```
## [1] -1555.1
```

```
AIC(Arima(lcangdp,order=c(2,1,0), include.drift = TRUE))
```

```
## [1] -1555.1
```

```
AIC(Arima(lcangdp,order=c(3,1,0), include.drift = TRUE))
```

```
## [1] -1554
```

```
AIC(Arima(lcangdp,order=c(4,1,0), include.drift = TRUE))
```

```
## [1] -1552.4
```

The AIC also selects an ARIMA(1,1,0) model or an ARIMA(2,1,0) model.

```
(Arima(lcangdp,order=c(2,1,0), include.drift = TRUE))
```

```
## Series: lcangdp
## ARIMA(2,1,0) with drift
##
## Coefficients:
##          ar1     ar2   drift
##        0.347   0.092   0.008
## s.e.   0.066   0.067   0.001
##
## sigma^2 = 6.25e-05:  log likelihood = 781.53
## AIC=-1555.1   AICc=-1554.9   BIC=-1541.3
```

f. Use the models selected in parts (d) and (e) to forecast the quarterly log RGDP in 2018:Q2, 2018:Q3, and 2018:Q4. Compare your results.

```r
fit1 <- Arima(lcangdp,order=c(1,1,0), include.drift = TRUE)
forecast(fit1, h = 3)
```

```
##         Point Forecast  Lo 80  Hi 80  Lo 95  Hi 95
## 2018 Q2         4.7775 4.7673 4.7876 4.7619 4.7930
## 2018 Q3         4.7847 4.7674 4.8020 4.7582 4.8111
## 2018 Q4         4.7923 4.7690 4.8155 4.7567 4.8278
```

```r
fit2 <- Arima(lcangdp,order=c(2,1,0), include.drift = TRUE)
forecast(fit2, h = 3)
```

```
##         Point Forecast  Lo 80  Hi 80  Lo 95  Hi 95
## 2018 Q2         4.7773 4.7672 4.7874 4.7618 4.7928
## 2018 Q3         4.7841 4.7671 4.8011 4.7581 4.8101
## 2018 Q4         4.7914 4.7682 4.8146 4.7559 4.8269
```

The forecasts are almost the same.