Technical Companion Note for:
"Local Projections vs. VARs: Lessons From Thousands of DGPs"

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This companion note provides technical details for some of the computations in the replication code for the simulation study of Li et al. (2021). Our calculations rely on several auxiliary ABCD-style forms (Fernández-Villaverde et al., 2007) of the encompassing dynamic factor model, discussed in Section 1. We then in Section 2 describe how these representations can be used to derive various population coefficients of interest (the section "Preliminary Computations: Structural Estimands" in `run_dfm.m`).

1 Auxiliary ABCD forms

We begin with the baseline dynamic factor model of Stock & Watson (2016):

$$
\begin{aligned}
f_t &= \Phi(L) f_{t-1} + H\varepsilon_t && (1)\\
X_t &= \Lambda f_t + v_t && (2)\\
v_{i,t} &= \Delta_i(L) v_{i,t-1} + \Xi_i \xi_{i,t}, && (3)
\end{aligned}
$$

In keeping with our baseline model parameterization, the remainder of this note will provide formulas for the case of two lags in the polynomials $\Phi(L)$ and $\Delta_i(L)$. The extension to arbitrary lag lengths is conceptually straightforward, but somewhat cumbersome notationally.

We map the DFM (1) - (3) into several convenient representations:

1. *General ABCDEF Form.* The first form is a general "ABCDEF" representation, following Anderson et al. (1996). The mapping from dynamic factor model to this representation

is computed in `ABCD_fun_DFM.m`.

The general ABCDEF framework takes the form

$$
\begin{aligned}
s_t &= As_{t-1} + B\epsilon_t && (4)\\
y_t &= Cs_{t-1} + D\epsilon_t + e_t^* && (5)\\
e_t &= Ee_{t-1} + F\omega_t && (6)
\end{aligned}
$$

where $(\epsilon_t, \omega_t) \sim \mathcal{N}(0, I)$ and $e_t = (e_t^{*\prime}, e_{t-1}^{*}{}', \cdots, e_{t-\ell+1}^{*}{}')'$. The mapping from (1) - (3) to (4) - (6) is straightforward:

$$
\begin{aligned}
\begin{pmatrix} f_t \\ f_{t-1} \end{pmatrix} &= \underbrace{\begin{pmatrix} \Phi_1 & \Phi_2 \\ I & 0 \end{pmatrix}}_{A} \begin{pmatrix} f_{t-1} \\ f_{t-2} \end{pmatrix} + \underbrace{\begin{pmatrix} H \\ 0 \end{pmatrix}}_{B} \varepsilon_t && (7)\\
X_t &= \underbrace{\begin{pmatrix} \Lambda & 0 \end{pmatrix} \begin{pmatrix} \Phi_1 & \Phi_2 \\ I & 0 \end{pmatrix}}_{C} \begin{pmatrix} f_{t-1} \\ f_{t-2} \end{pmatrix} + \underbrace{\begin{pmatrix} \Lambda & 0 \end{pmatrix} \begin{pmatrix} H \\ 0 \end{pmatrix}}_{D} \varepsilon_t + v_t && (8)\\
\begin{pmatrix} v_t \\ v_{t-1} \end{pmatrix} &= \underbrace{\begin{pmatrix} \Delta_1 & \Delta_2 \\ I & 0 \end{pmatrix}}_{E} \begin{pmatrix} v_{t-1} \\ v_{t-2} \end{pmatrix} + \underbrace{\begin{pmatrix} \Xi \\ 0 \end{pmatrix}}_{F} \xi_t && (9)
\end{aligned}
$$

2. *VAR ABCD Form.* The second form is an "ABCD" representation that we can use to easily derive VAR(∞)-implied impulse response functions, following Fernández-Villaverde et al. (2007).[1] The mapping is computed in `compute_VARirfs_DFM.m`.

   The general ABCD representation takes the form

$$
\begin{aligned}
s_t &= As_{t-1} + B\varsigma_t && (10)\\
y_t &= Cs_t + D\varsigma_t && (11)
\end{aligned}
$$

   where $\varsigma_t \sim \mathcal{N}(0, I)$. The mapping from (1) - (3) to (10) - (11) given a selected set of

[1]Note that the timing in (10) and (11) is slightly different from that in Fernández-Villaverde et al. (2007). Plugging (10) into (11), and re-defining $\tilde{C} = CA$, $\tilde{D} = CB + D$, we arrive at their representation.

observables $\bar{w}_t$ is

$$\begin{pmatrix} f_t \\ f_{t-1} \\ f_{t-2} \end{pmatrix} = \underbrace{\begin{pmatrix} \Phi_1 & \Phi_2 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}}_{A} \begin{pmatrix} f_{t-1} \\ f_{t-2} \\ f_{t-3} \end{pmatrix} + \underbrace{\begin{pmatrix} H & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{B} \begin{pmatrix} \varepsilon_t \\ \xi_t \end{pmatrix} \tag{12}$$

$$\begin{pmatrix} I & -\bar{\Delta}_1 & -\bar{\Delta}_2 \end{pmatrix} \begin{pmatrix} \bar{w}_t \\ \bar{w}_{t-1} \\ \bar{w}_{t-2} \end{pmatrix} = \underbrace{\begin{pmatrix} I & -\bar{\Delta}_1 & -\bar{\Delta}_2 \end{pmatrix} \begin{pmatrix} \bar{\Lambda} & & \\ & \bar{\Lambda} & \\ & & \bar{\Lambda} \end{pmatrix}}_{C} \begin{pmatrix} f_t \\ f_{t-1} \\ f_{t-2} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & \bar{\Xi} \end{pmatrix}}_{D} \begin{pmatrix} \varepsilon_t \\ \xi_t \end{pmatrix} \tag{13}$$

where bars indicate the selection of matrix rows that correspond to the observables $\bar{w}_t$, as in Li et al. (2021).

3. *Invertibility ABCD Form.* The third form is another "ABCD" representation (again following Fernández-Villaverde et al. (2007)), useful to derive the degree of invertibility of our DGPs. The mapping is computed in `compute_invert_DFM.m`.

   To derive the general invertibility-relevant ABCD framework, note first of all that, from (1) - (3), we get that, for a vector of observables $\bar{w}_t$,

$$\begin{pmatrix} I & -\bar{\Delta}_1 & -\bar{\Delta}_2 \end{pmatrix} \begin{pmatrix} \bar{w}_t \\ \bar{w}_{t-1} \\ \bar{w}_{t-2} \end{pmatrix} = (I - \bar{\Delta}(L))\bar{\Lambda}(I - \Phi(L))^{-1} H\varepsilon_t + \bar{\Xi}\xi_t$$

   We can thus write

$$\zeta_t = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \zeta_{t-1} + \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_t \\ \xi_t \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} I & -\bar{\Delta}_1 & -\bar{\Delta}_2 \end{pmatrix} \begin{pmatrix} \bar{w}_t \\ \bar{w}_{t-1} \\ \bar{w}_{t-2} \end{pmatrix} = C\zeta_t + \begin{pmatrix} 0 & \bar{\Xi} \end{pmatrix} \begin{pmatrix} \varepsilon_t \\ \xi_t \end{pmatrix} \tag{15}$$

   where $\zeta_t = (\varepsilon_t', \varepsilon_{t-1}', \cdots, \varepsilon_{t-H}')'$ with H sufficiently large, and C is set to ensure that, up to lag H, $(I - \bar{\Delta}(L))\bar{\Lambda}(I - \Phi(L))^{-1} H\varepsilon_t = C\zeta_t$.

4. *IV ABCD Form.* The fourth and final form is an "ABCD" representation in observable macro aggregates and IV, useful to derive a natural measure of IV strength. The mapping

is computed in `compute_IVstrength_DFM.m`.

The IV equation is

$$z_t = \rho_z z_{t-1} + \alpha\varepsilon_{1,t} + \sigma_\nu \nu_t \tag{16}$$

We can combine (16) with (1) - (3) to arrive at an augmented dynamic factor model in augmented factors, observables and shocks $(f_t^*, \varepsilon_t^*, X_t^*, \nu_t^*, \xi_t^*)$:

$$\underbrace{\begin{pmatrix} f_t \\ z_t \end{pmatrix}}_{f_t^*} = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \rho_z \end{pmatrix} \begin{pmatrix} f_{t-1} \\ z_{t-1} \end{pmatrix} + \begin{pmatrix} \Phi_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_{t-2} \\ z_{t-2} \end{pmatrix} + \begin{pmatrix} H & 0 \\ \alpha e_1' & \sigma_v \end{pmatrix} \underbrace{\begin{pmatrix} \varepsilon_t \\ \nu_t \end{pmatrix}}_{\varepsilon_t^*} \tag{17}$$

$$\underbrace{\begin{pmatrix} X_t \\ z_t \end{pmatrix}}_{X_t^*} = \begin{pmatrix} \Lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} f_t \\ z_t \end{pmatrix} + \begin{pmatrix} v_t \\ \tilde{v}_t \end{pmatrix} \tag{18}$$

$$\underbrace{\begin{pmatrix} v_t \\ \tilde{v}_t \end{pmatrix}}_{v_t^*} = \begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{t-1} \\ \tilde{v}_{t-1} \end{pmatrix} + \begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{t-2} \\ \tilde{v}_{t-2} \end{pmatrix} + \begin{pmatrix} \Xi & 0 \\ 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \xi_t \\ \tilde{\xi}_t \end{pmatrix}}_{\xi_t^*} \tag{19}$$

From (17) - (19), we can derive an ABCD representation with factors and the IV as states, and the IV and macro aggregates as observables, analogous to (12) - (13).

2 Computing population objects

We now discuss how our codes map the different representations derived above into the various structural population objects that are reported in our analysis.

STRUCTURAL ESTIMANDS. We discuss each of the three structural estimands studied in the main analysis of Li et al. (2021).

1. *Observed shock.* Given the matrix H that defines the structural shock of interest $\varepsilon_{1,t}$ as a function of the reduced-form factor forecast errors η_t, population impulse response functions for all macro observables X_t are readily computed from the ABCDEF representation (7) - (9).[2] All computations are provided in `compute_irfs.m`.

[2] Note that the matrix H is either exogenously fixed or chosen to maximize the impulse response of a certain variable.

2. *IV.* For the target IV impulse responses, we scale the population impulse response of the target variable by the impact response of the IV normalization variable, both computed in `compute_irfs.m`. The normalization is computed in `compute_normalized_irfs.m`.

3. *Recursive shock.* Our computations of the recursive shock estimand rely on the ABCD form (12) - (13). In `compute_VARirfs_DFM.m`, we use this form to derive a VAR(∞) in $\bar{w}_t - \Delta_1 \bar{w}_{t-1} - \Delta_2 \bar{w}_{t-2}$ and so a VAR(∞) in $\bar{w}_t$. Given this VAR(∞), we can readily compute our estimand as the population impulse response function to suitably orthogonalized Wold innovations in $\bar{w}_t$, where the orthogonalization is chosen in line with the proposed (recursive) identification scheme.

Degree of invertibility. For each DGP, we compute the degree of invertibility for the observed shock $\varepsilon_{1,t}$ with respect to the set of observables $\bar{w}_t$. We do so using the ABCD form (14) - (15): letting $\bar{w}_t^* \equiv \bar{w}_t - \bar{\Delta}_1 \bar{w}_{t-1} - \bar{\Delta}_2 \bar{w}_{t-2}$, we run a long Kalman filter algorithm to compute $\text{Var}(\zeta_t \mid \bar{w}_t^*, \bar{w}_{t-1}^*, \bar{w}_{t-2}^*, \dots)$ and so $\text{Var}(\zeta_t \mid \bar{w}_t, \bar{w}_{t-1}, \bar{w}_{t-2}, \dots)$. The degree of invertibility of the true shock is then easily computed as $\text{Var}(\varepsilon_{1,t} \mid \bar{w}_t, \bar{w}_{t-1}, \bar{w}_{t-2}, \dots)$. We provide those computations in `compute_invert_DFM.m`, with the Kalman filter computations in `cond_var_fn_St.m`.

Persistence. It is straightforward to map the ABCDEF representation (7) - (9) into the various measures of DGP persistence reported in the DGP summary statistics in Table 1 (Section 3.4) of Li et al. (2021). We do so in `compute_persist_DFM.m`.

IV strength. Let $i_t \in \bar{w}_t$ denote the normalization variable that we will use to gauge IV strength. From the IV-DFM (17) - (19) we can derive the population VAR(∞) representation of $w_t = (\bar{w}_t', z_t)'$, and so we can easily compute $\text{Var}(i_t \mid w_{t-1}, w_{t-2}, \dots)$ as well as $\text{Var}(i_t \mid z_t, w_{t-1}, w_{t-2}, \dots)$. We then define the IV strength as

$$1 - \frac{\text{Var}(i_t \mid z_t, w_{t-1}, w_{t-2}, \dots)}{\text{Var}(i_t \mid w_{t-1}, w_{t-2}, \dots)} \in [0, 1]$$

The computations are implemented in `compute_IVstrength_DFM.m`, with `cond_var_fn_St_1.m` computing the required residual forecasting variances.

References

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