Mean - Vol Relationship

VAR

A reduced form VAR of some number of variables, k, and some number of lags, p, is written as,

$$y_t = A_1 y_{t-1} + ... + A_p y_{t-p} + u_t \tag{1}$$

where A_i is a kxk matrix of coefficients and,

$$y_t = \begin{bmatrix} \text{output}_t \\ \text{unemployment}_t \\ \text{inflation}_t \\ ... \\ \text{vfci}_t \end{bmatrix}, \qquad u_t = \begin{bmatrix} u_t^{\text{output}} \\ u_t^{\text{unemployment}} \\ u_t^{\text{inflation}} \\ ... \\ u_t^{\text{vfci}} \end{bmatrix} \tag{2}$$

Each u_t are the reduced form residuals for the accompanying data series.

Max Share ID

Then, using the max share identification method, we identify one structural shock, w_t^{BC}, which drives the business cycle.

$$u_t = PQ^* w_t \tag{3}$$

where P is the choleskey of Σ_u and Q^* is the identified rotation matrix that returns w_t^{BC} in the first column.

Then we can write,

$$y_t = A_1 y_{t-1} + ... + A_p y_{t-p} + PQ^* \begin{bmatrix} w_t^{\text{BC}} \\ w_t^2 \\ w_t^3 \\ ... \\ w_t^k \end{bmatrix} \tag{4}$$

Business Cycle Contribution

Using the identified business cycle shock, we can construct the time series for y_t^{BC} which are the contributions to each variable of the VAR driven by the identified shock.

$$\widehat{y^{\text{BC}}}_t = \begin{bmatrix} \widehat{\text{output}_t^{\text{BC}}} \\ \text{unemployment}_t^{\text{BC}} \\ \text{inflation}_t^{\text{BC}} \\ ... \\ \text{vfci}_t^{\text{BC}} \end{bmatrix} \tag{5}$$

Business Cycle Volatility

The realized volatility for the business cycle can be defined as:

$$v_t^{\mathrm{BC}} = log(\left[w_t^{\mathrm{BC}}\right]^2) \tag{6}$$

This volatility can also be modeled using the variables of the VAR:

$$v_t^{\mathrm{BC}} = \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + \epsilon_t \tag{7}$$

where α_i is a $1xk$ vector of coefficients.

We can then estimate $\widehat{v_t^{\mathrm{BC}}}$ as a time series.

Mean-Vol Relationship

The mean-vol relationship is then the relationship between

(1) the predicted means:

$$\widehat{y^{\mathrm{BC}}}_t \tag{8}$$

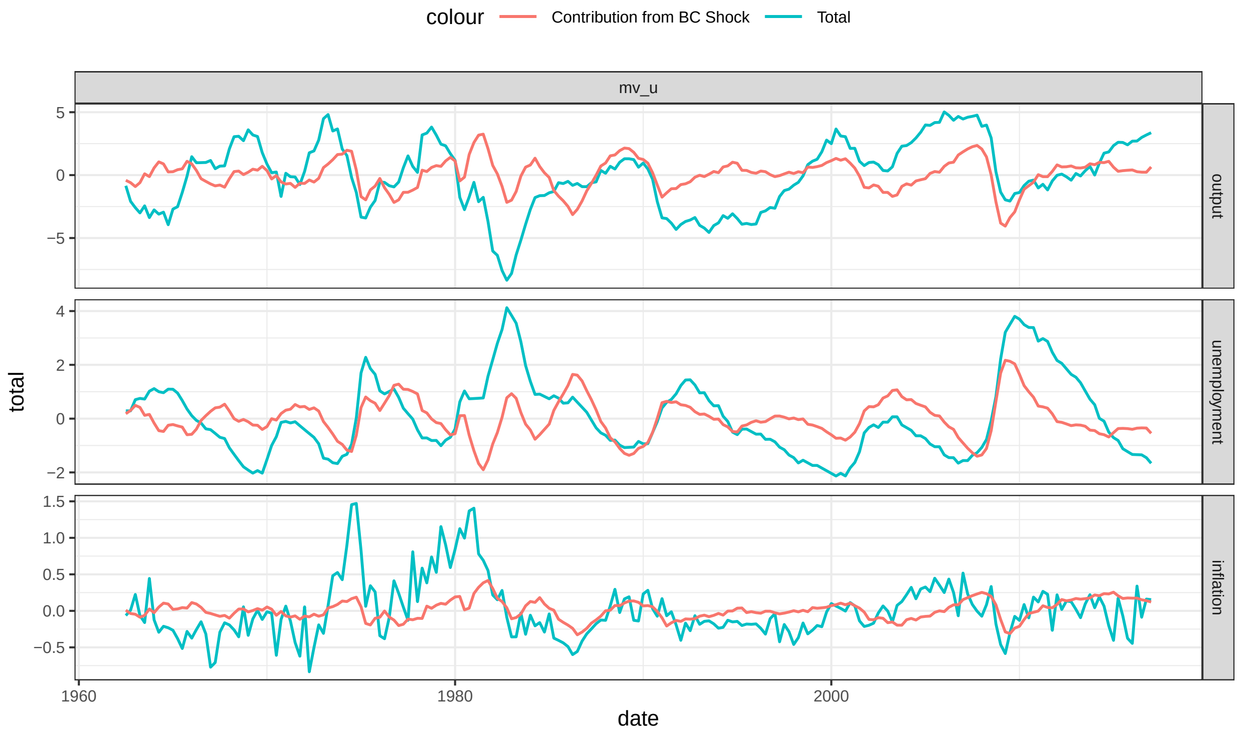
(2) the predicted volatility:

$$\widehat{v_t^{\mathrm{BC}}} \tag{9}$$

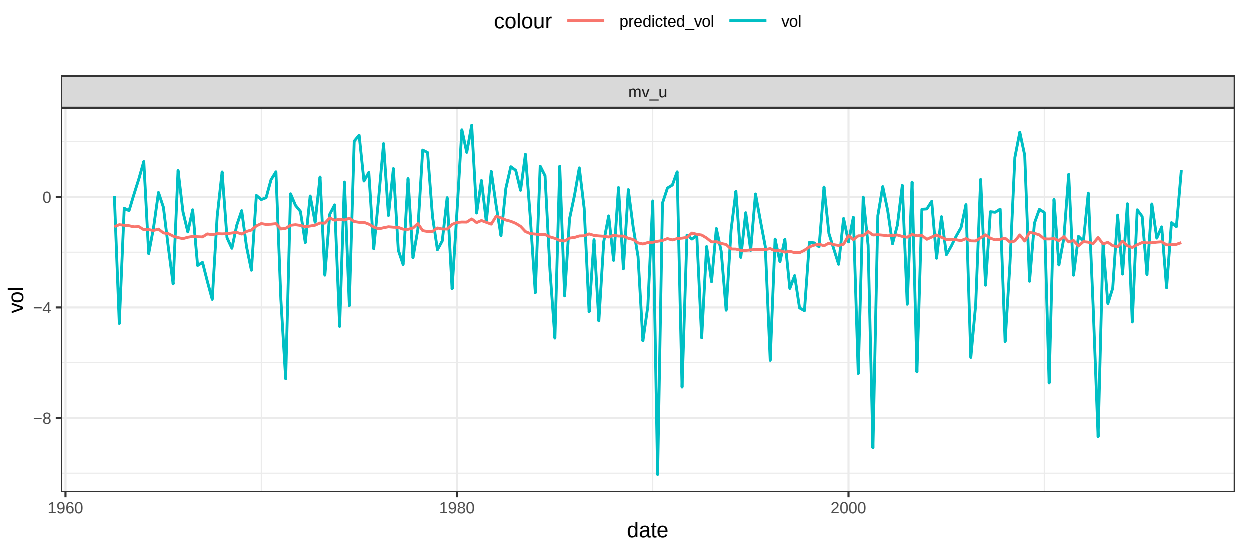
Note: There are multiple predicted means (one for each of the k variables), but only one predicted volatility of the business cycle. Therefore, there will be k mean-vol relationships which will depend upon the relationship between each variable's predicted mean and the business cycle shock.

The Data

Here are the contributions of the Business Shock, $\widehat{y_t^{\text{BC}}}$, to a subset of the variables, compared to the total unexplained varaition in those series.

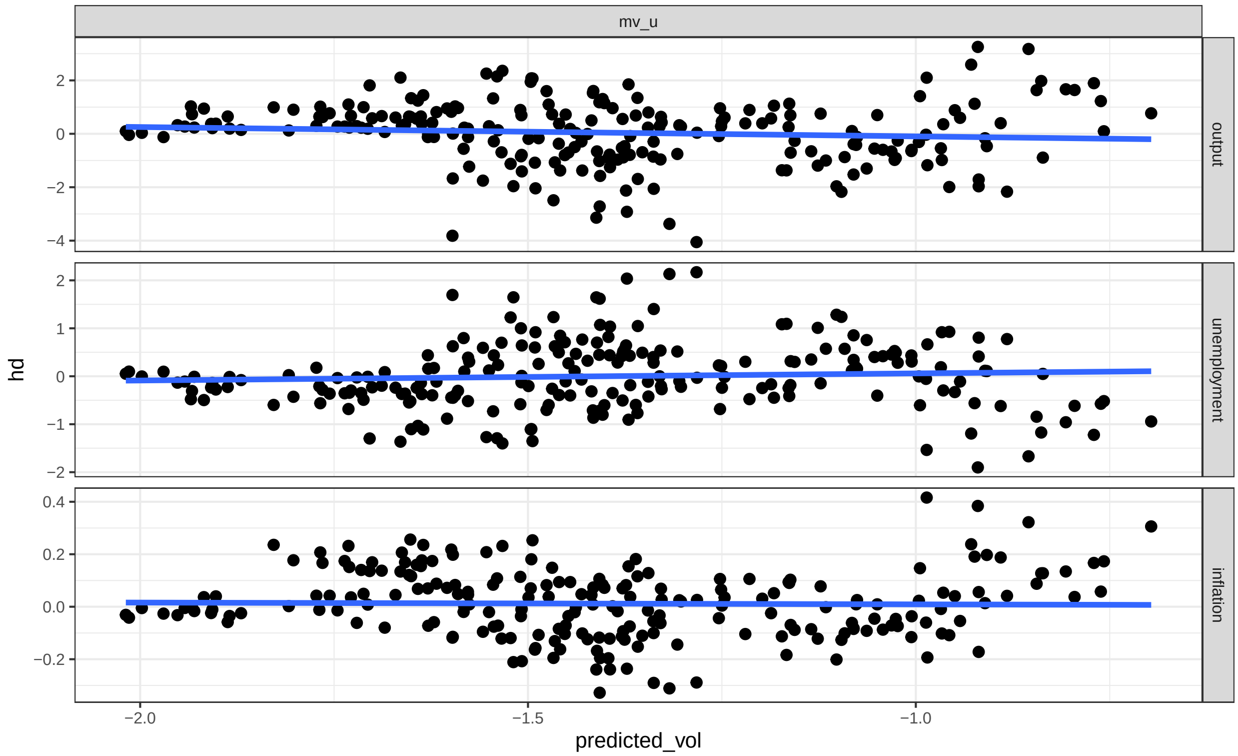


And here is the realized volatility of the business cycle shock, v_t^{BC}, and the predicted value, $\widehat{v_t^{\text{BC}}}$:

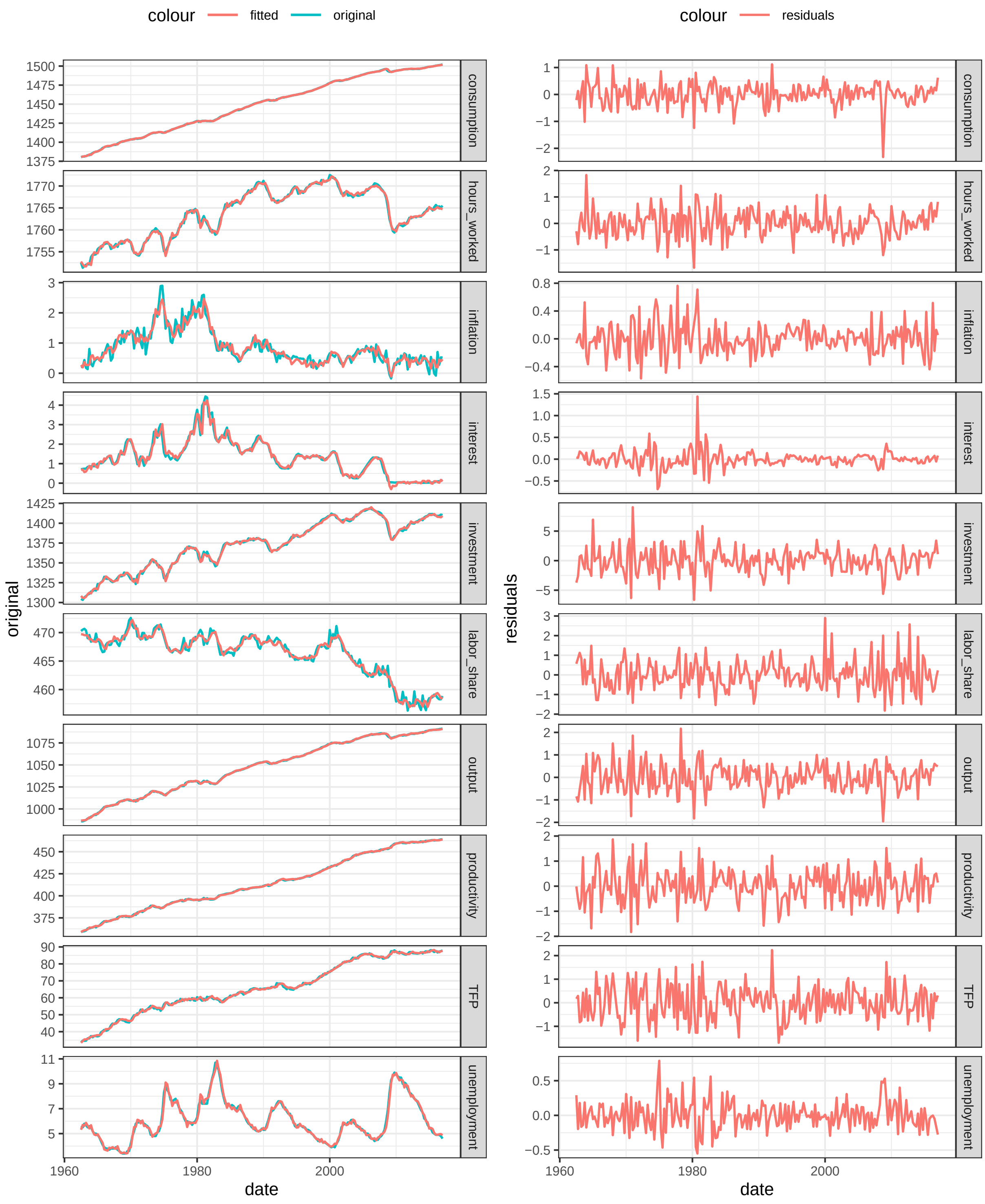


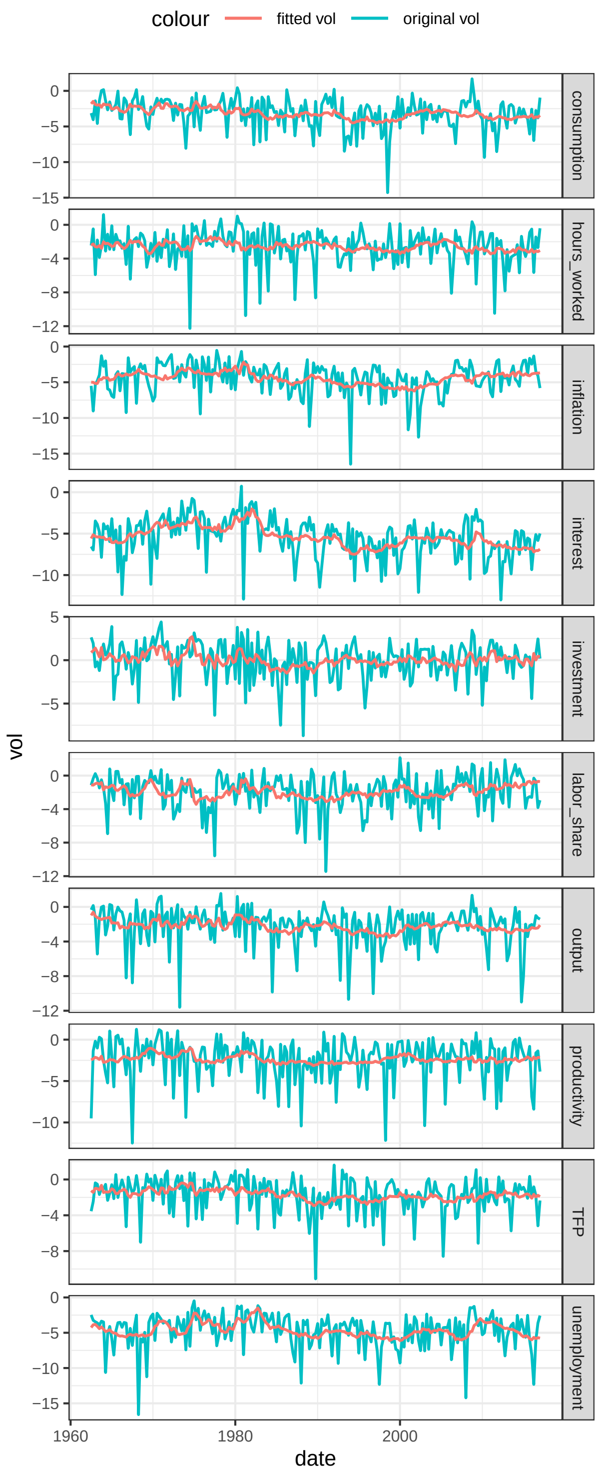
The correlation between the two sets of predicted values are not particularly high:

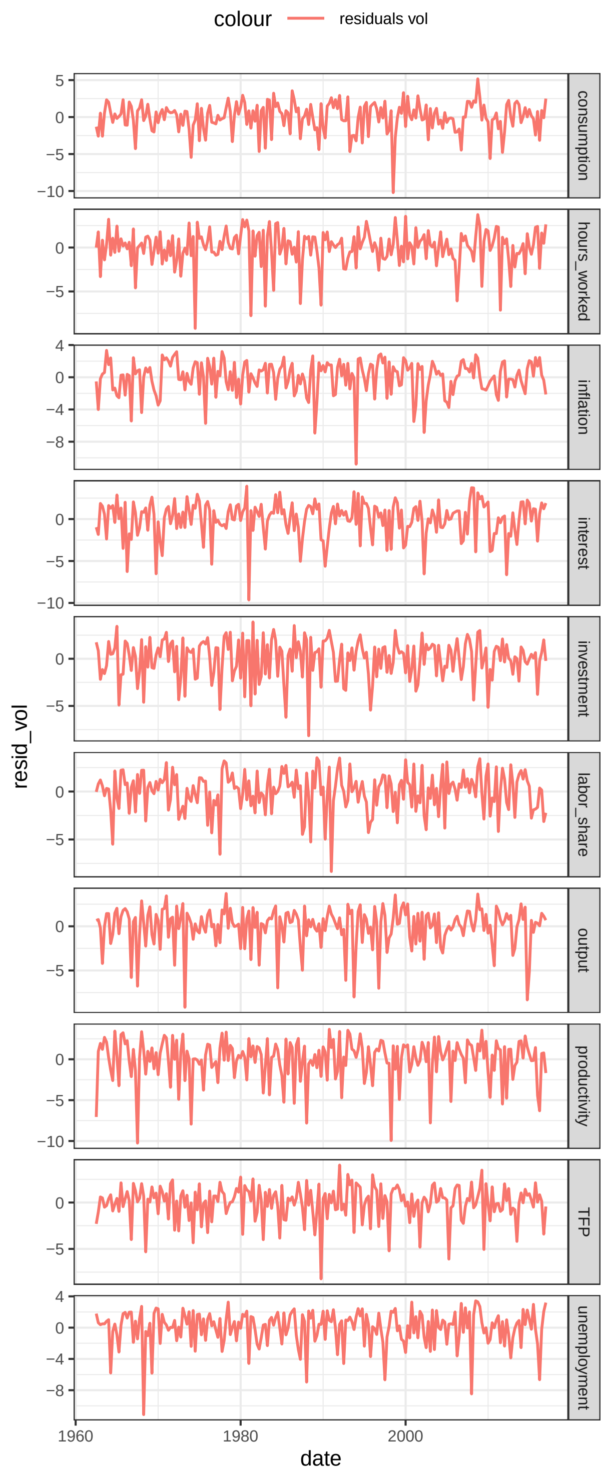
```
## `geom_smooth()` using formula = 'y ~ x'
```



mv_u
output
unemployment
inflation
hd
predicted_vol
−2.0
−1.5
−1.0


colour fitted original
colour residuals
consumption
hours_worked
inflation
interest
investment
labor_share
output
productivity
TFP
unemployment
original
residuals
date
date



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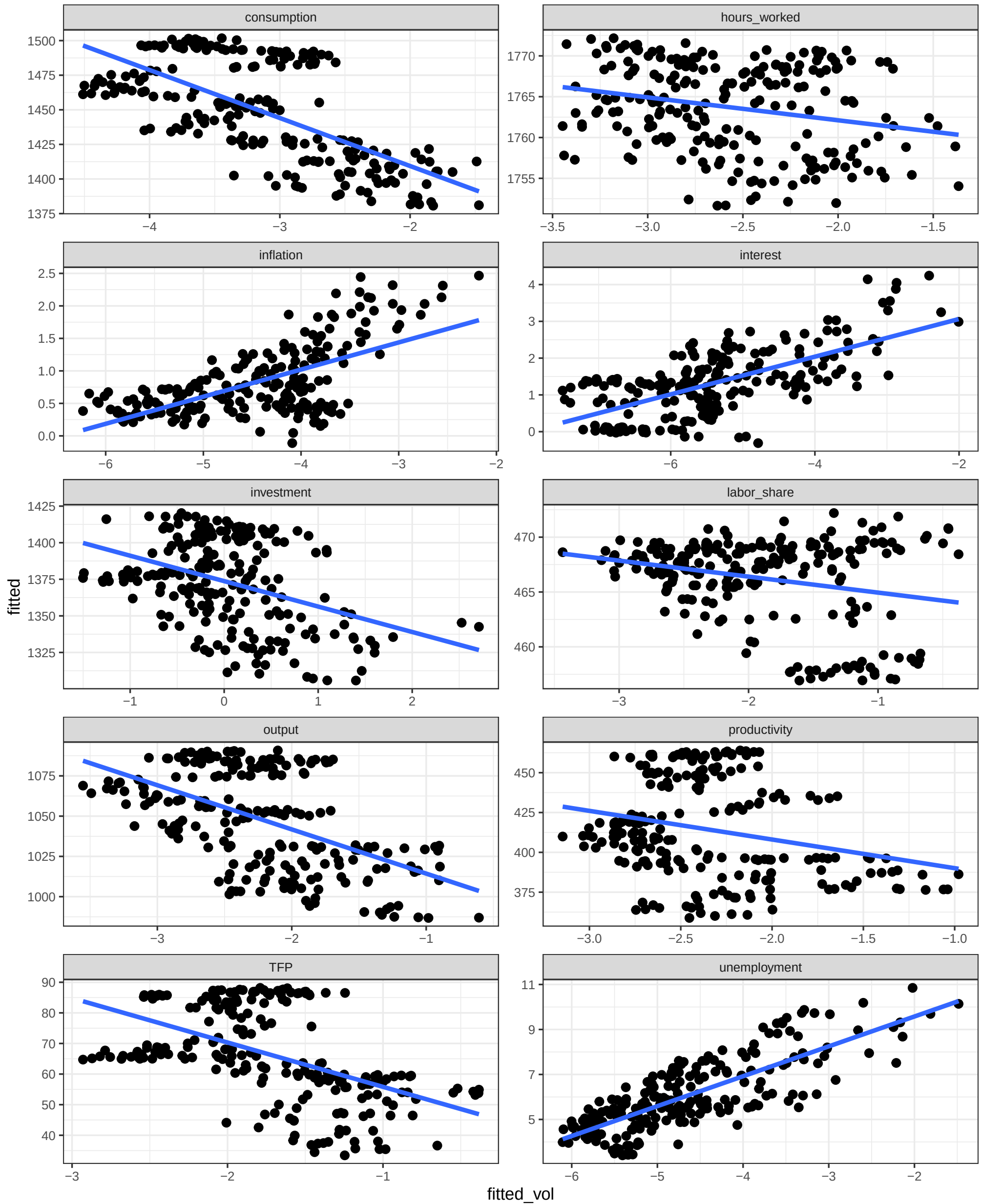
consumption
hours_worked
inflation
interest
investment
labor_share
output
productivity
TFP
unemployment
fitted
fitted_vol