Mean - Vol Relationship

VAR

A reduced form VAR of some number of variables, k, and some number of lags, p, is written as,

$$y_t = A_1 y_{t-1} + ... + A_p y_{t-p} + u_t \tag{1}$$

where A_i is a kxk matrix of coefficients and,

$$y_t = \begin{bmatrix} \text{output}_t \\ \text{unemployment}_t \\ \text{inflation}_t \\ ... \\ \text{vfci}_t \end{bmatrix}, \qquad u_t = \begin{bmatrix} u_t^{\text{output}} \\ u_t^{\text{unemployment}} \\ u_t^{\text{inflation}} \\ ... \\ u_t^{\text{vfci}} \end{bmatrix} \tag{2}$$

Each u_t are the reduced form residuals for the accompanying data series.

With this, we can estimate the predicted values, $\widehat{y}_t$, and the residuals, $\widehat{u}_t$.

Volatility

We can then define the log-variance as

$$\text{Var}_t \equiv \log(\widehat{u}_t^2) \tag{3}$$

This can be estimated with the regression,

$$\text{Var}_t = \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + \epsilon_t \tag{4}$$

Then the predicted (conditional) volatility is

$$\widehat{\text{Vol}}_t = \left[\exp(\widehat{\text{Var}_t})\right]^{\frac{1}{2}} \tag{5}$$

Mean-Vol Relationship

The mean-vol relationship is then the relationship between

(1) the predicted means:

$$\widehat{y}_t \tag{6}$$

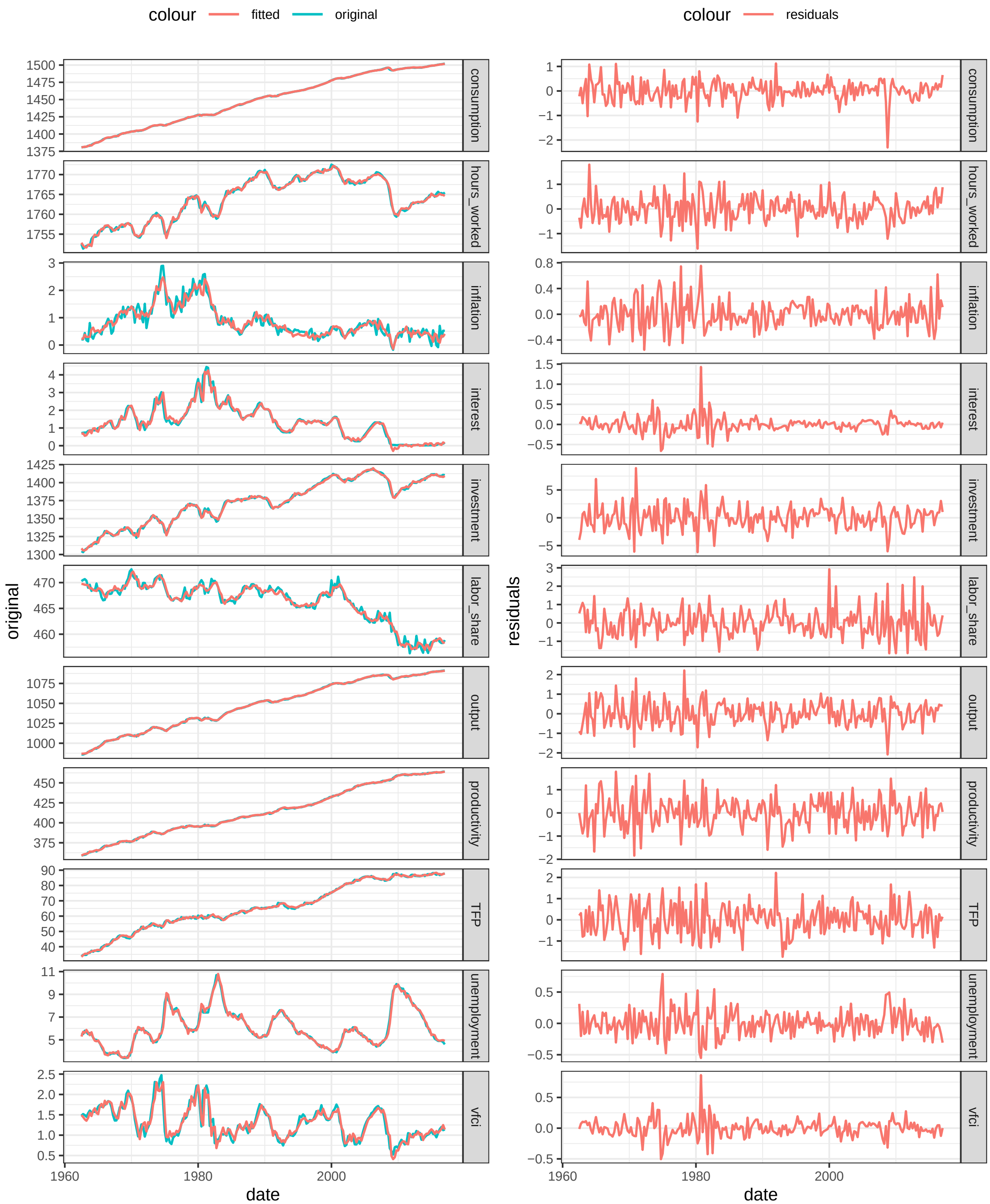
(2) the predicted volatility:

$$\widehat{\text{Vol}}_t \tag{7}$$

Note: There are k mean-vol relationships, one for each variable in the VAR.

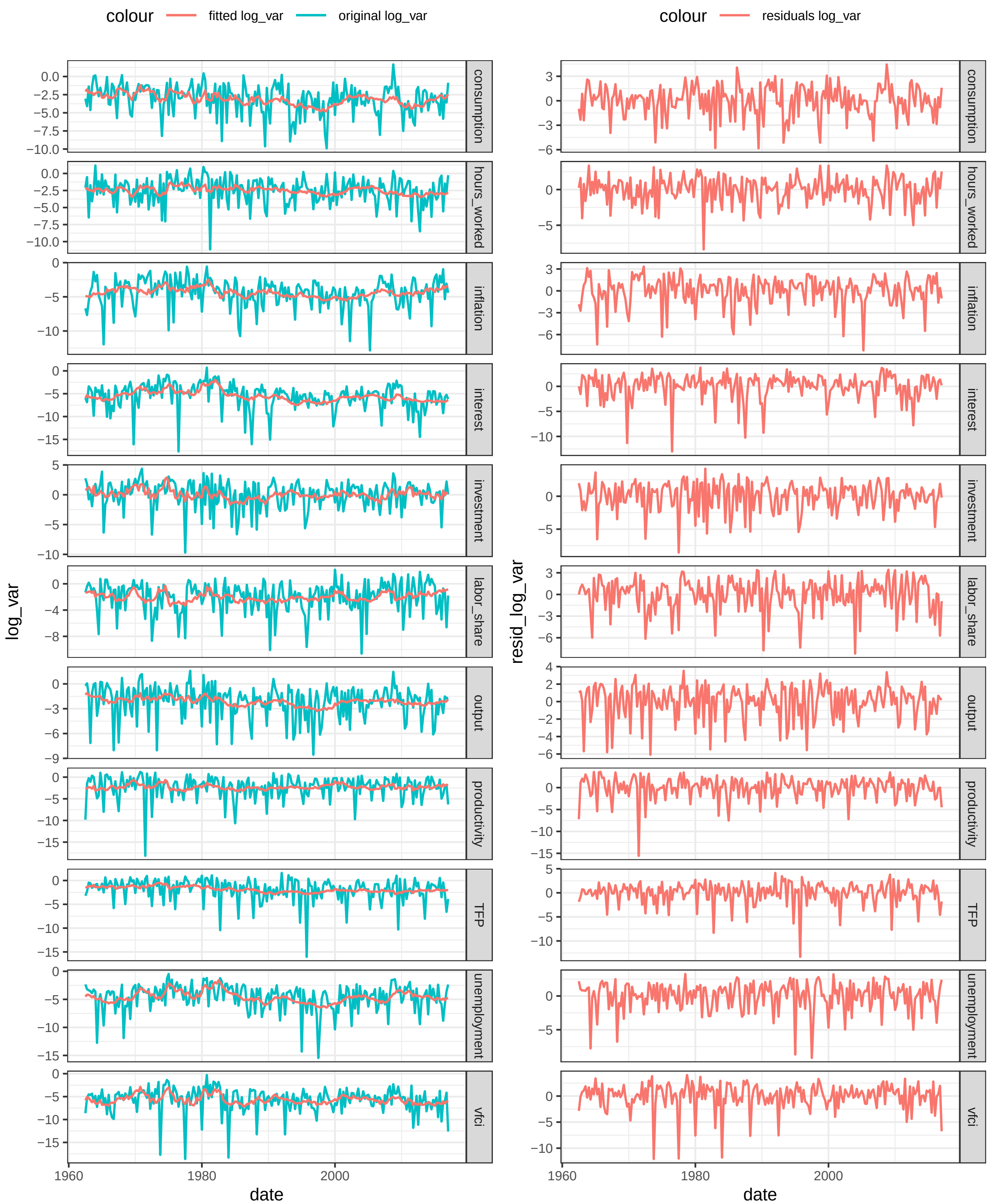
Fitting the VAR

(1) Fit the VAR to the raw (original) data, find the fitted values and the residuals.



Fitting the heteroskedastic variance regression

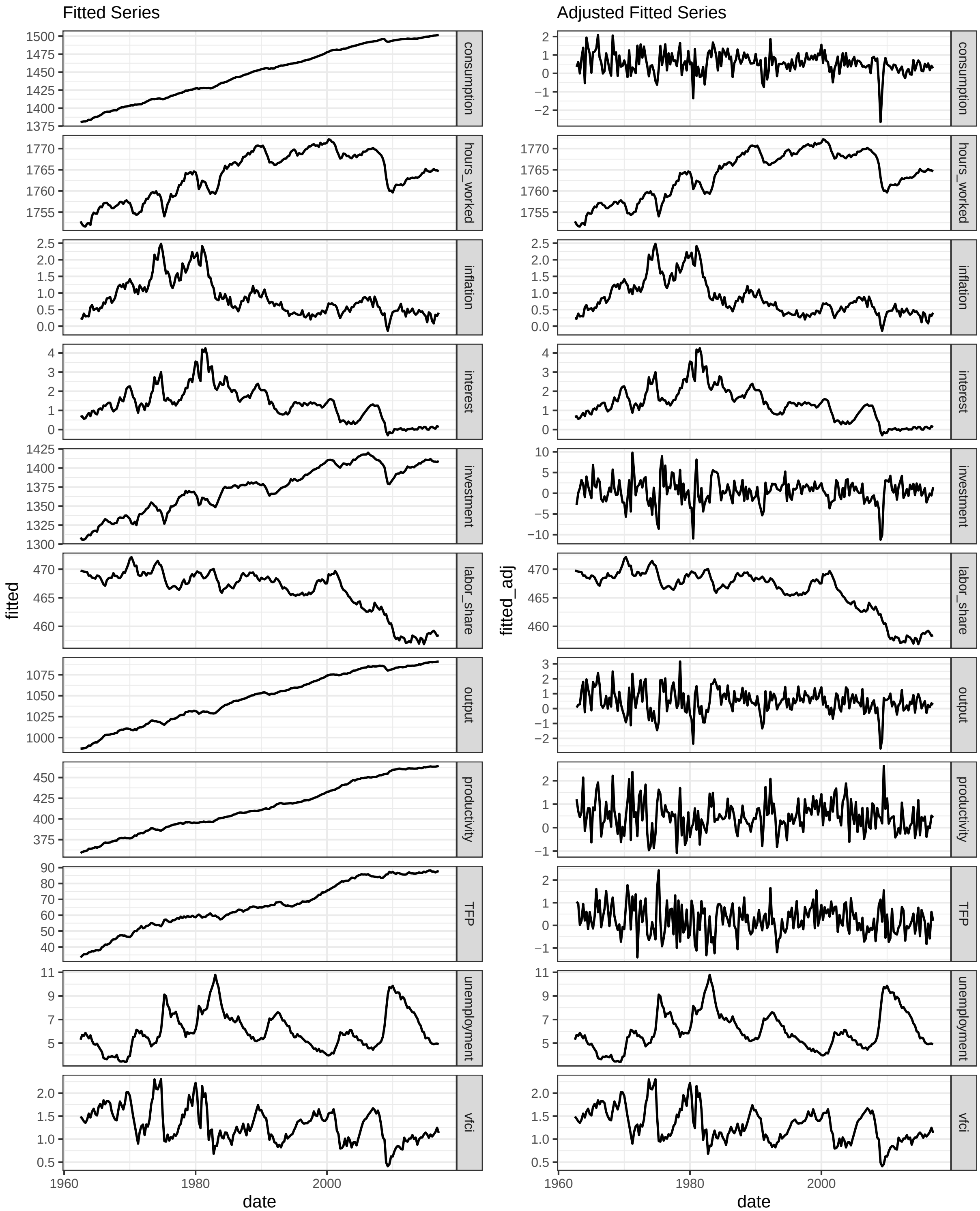
(2) Construct the log-variance for each residual. Fit a regression on the same variables as the VAR, find the fitted value.



Nonstationary Series

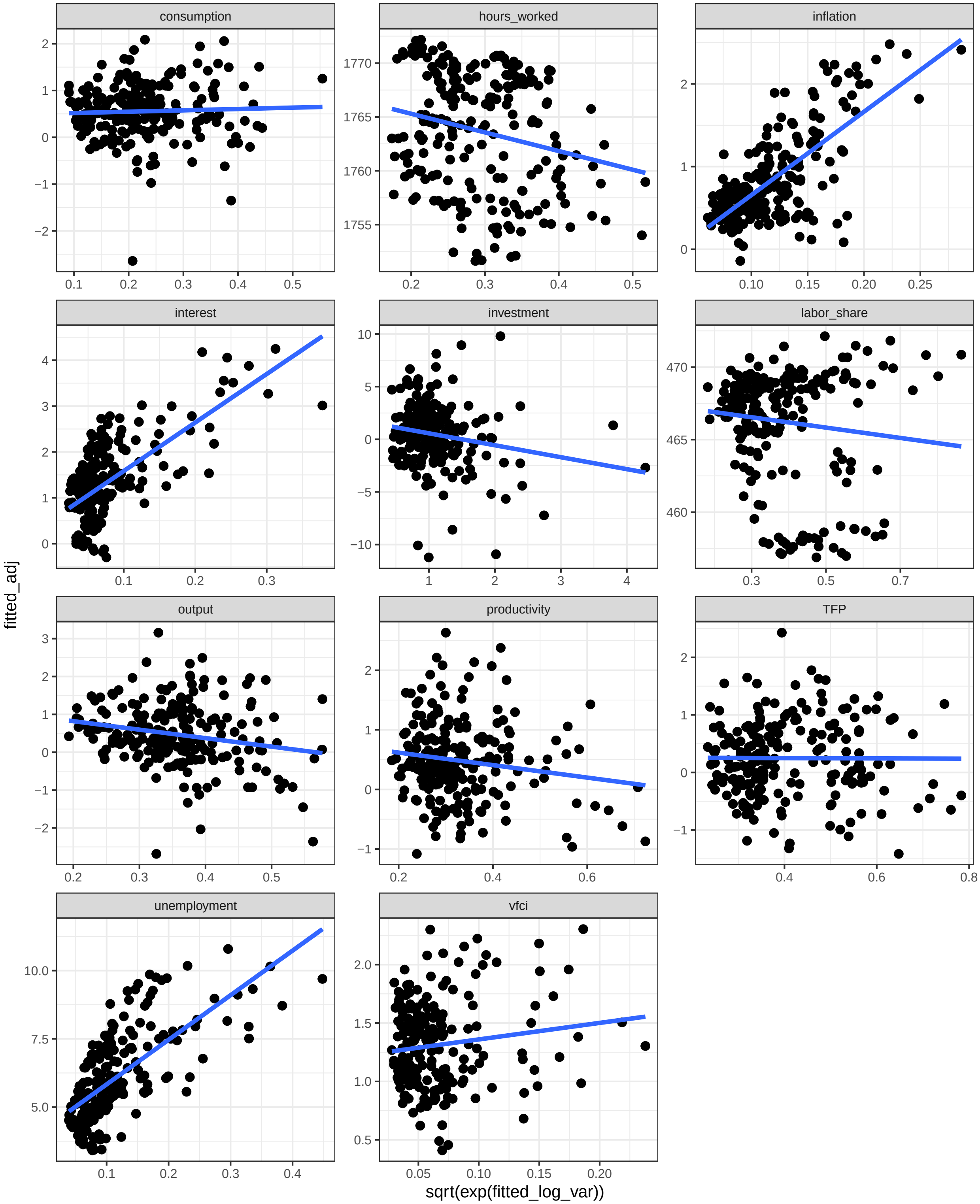
(3) Some of the values in $\widehat{y}_t$ are non-stationary (they have a trend), so we take differences before presenting them in the mean-vol and mean-var figures.

The differences variables are: output, consumption, investment, productivity, and TFP



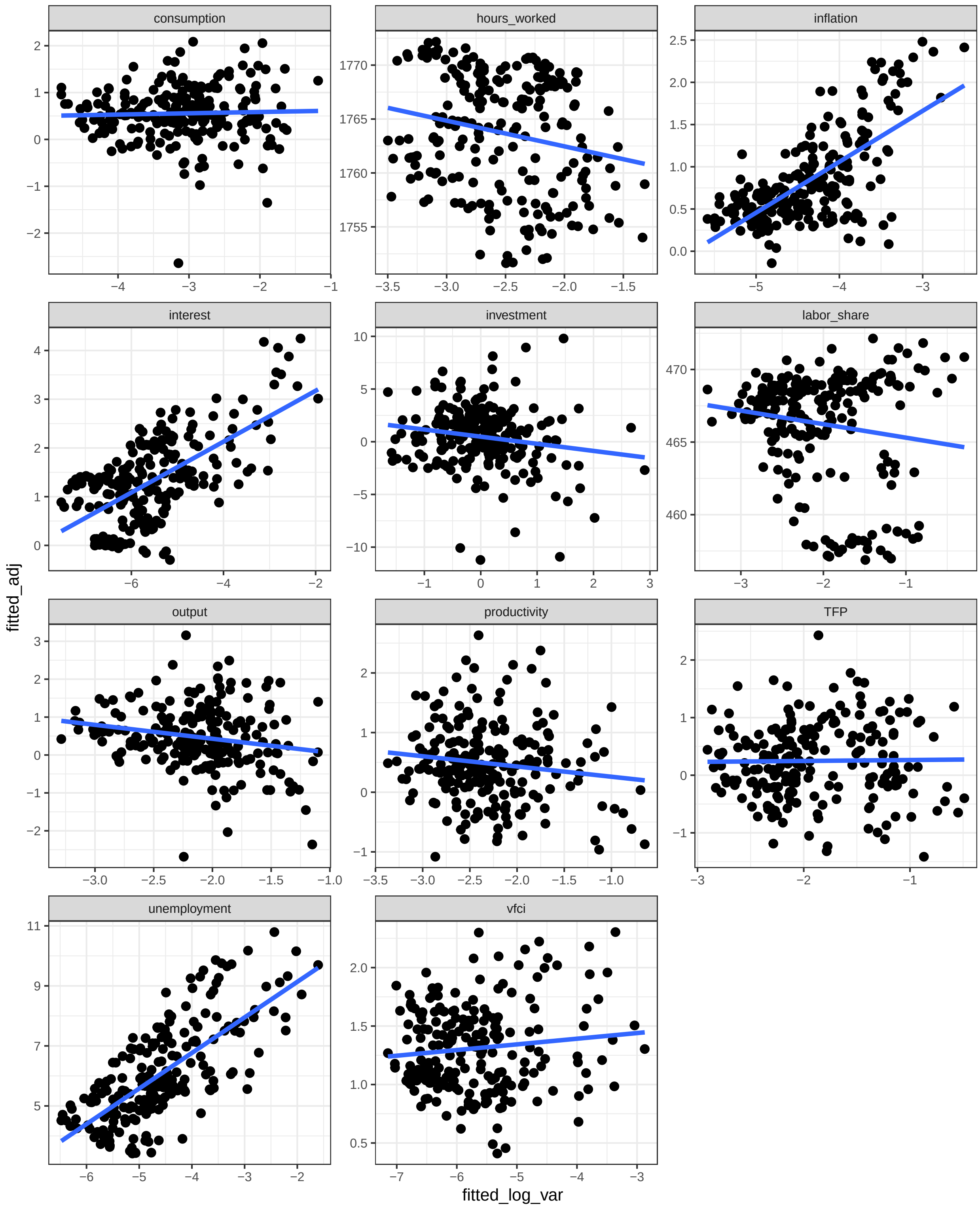
Mean-Vol Relationship

(4) Chart the relationship between the fitted means and the fitted volatilities.



Mean-Var Relationship

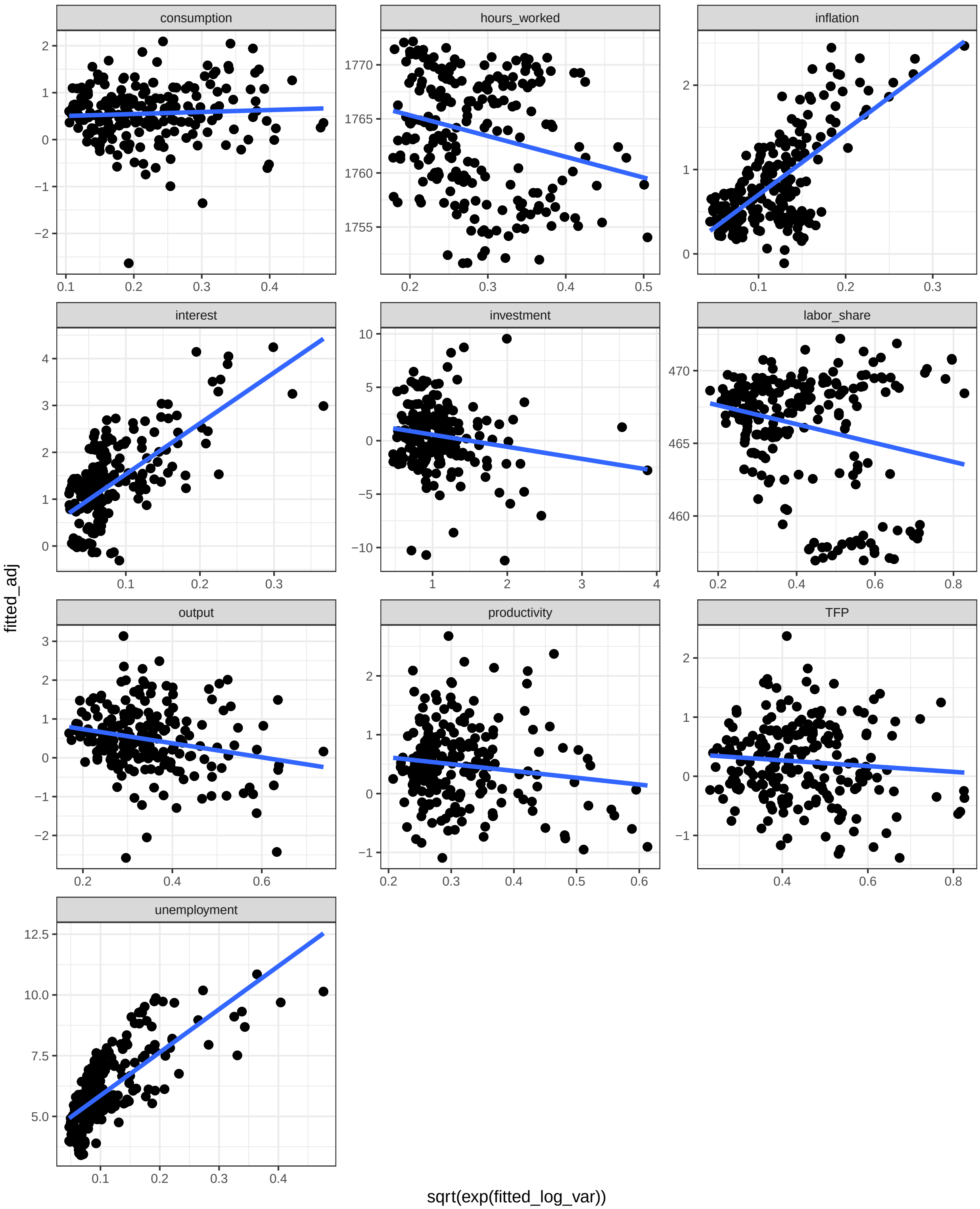
(5) Chart the relationship between the fitted means and the fitted log-variances.



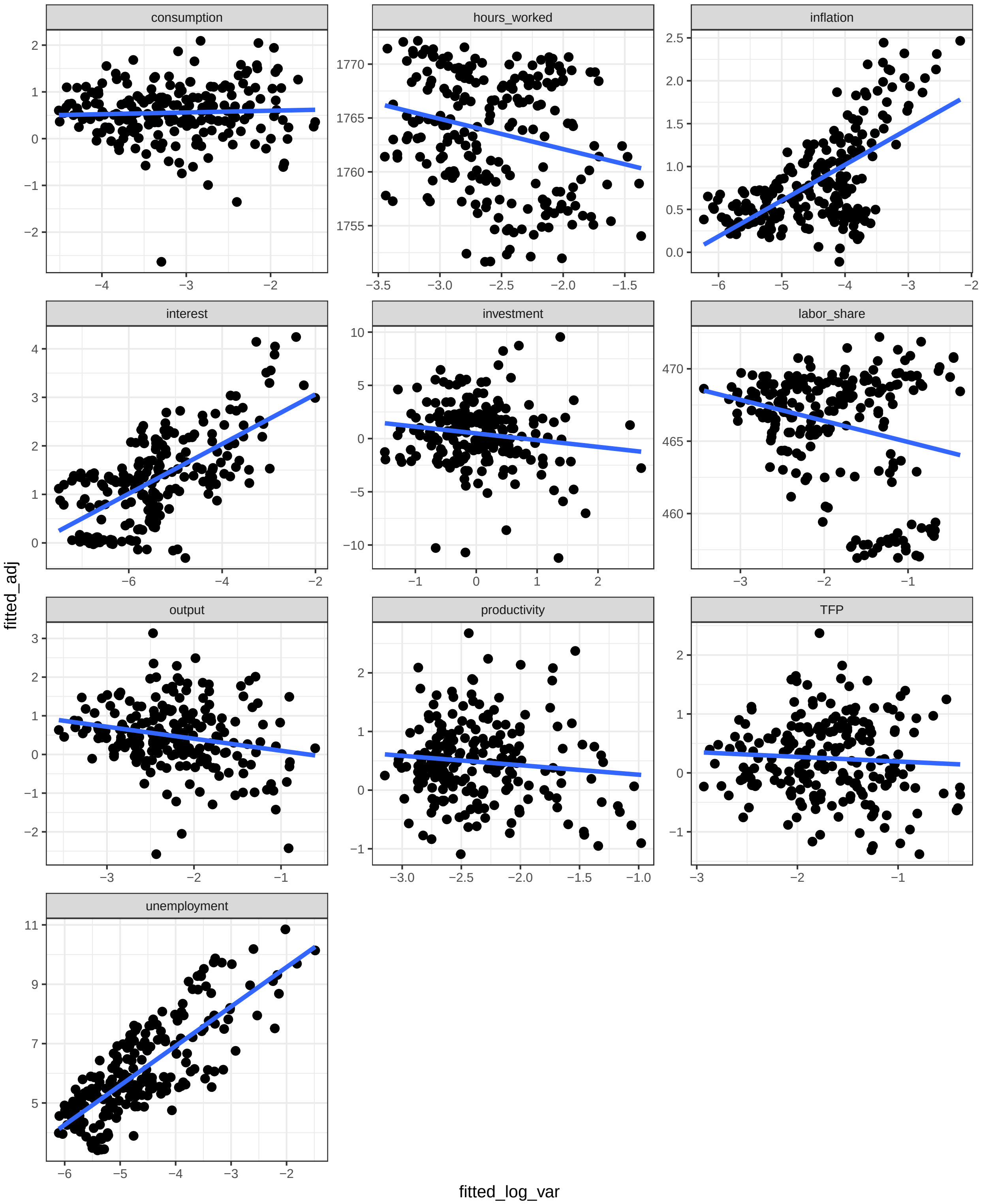
Using 10-variable VAR

We can repeat the above analysis with just 10 variables in the VAR (dropping vfci).

Mean-Vol Relationship - 10 variable VAR

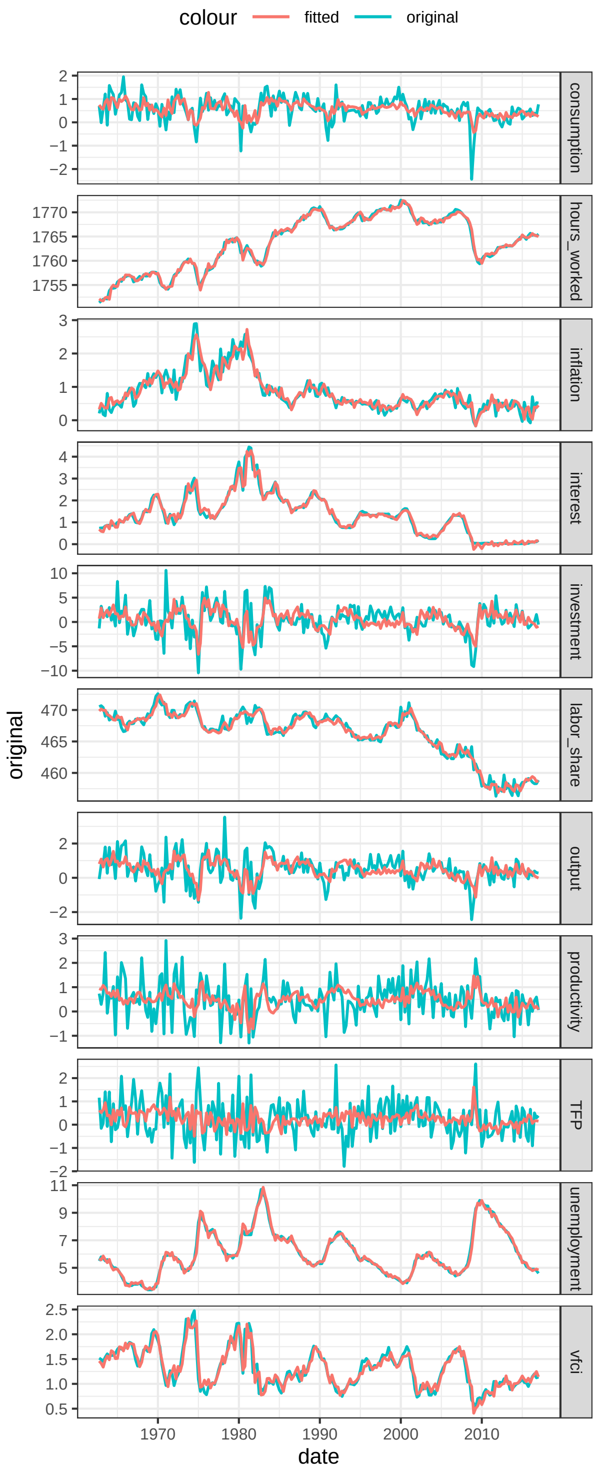


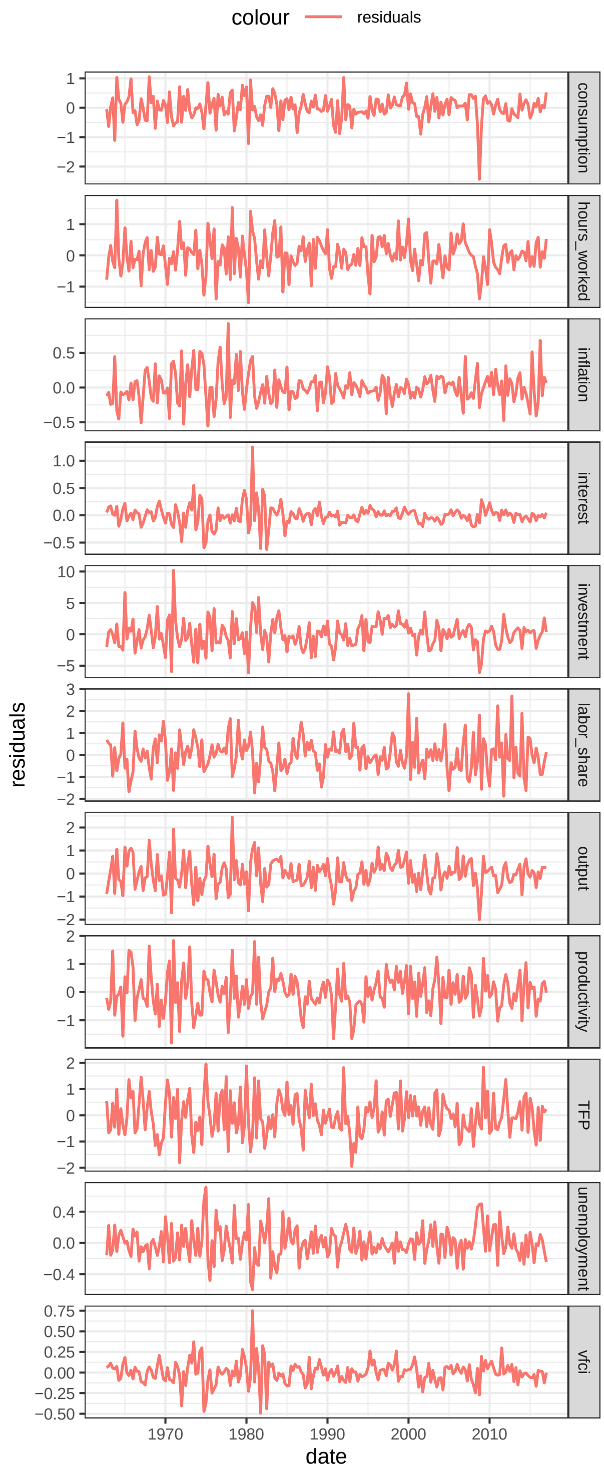
Mean-Var Relationship - 10 variable VAR



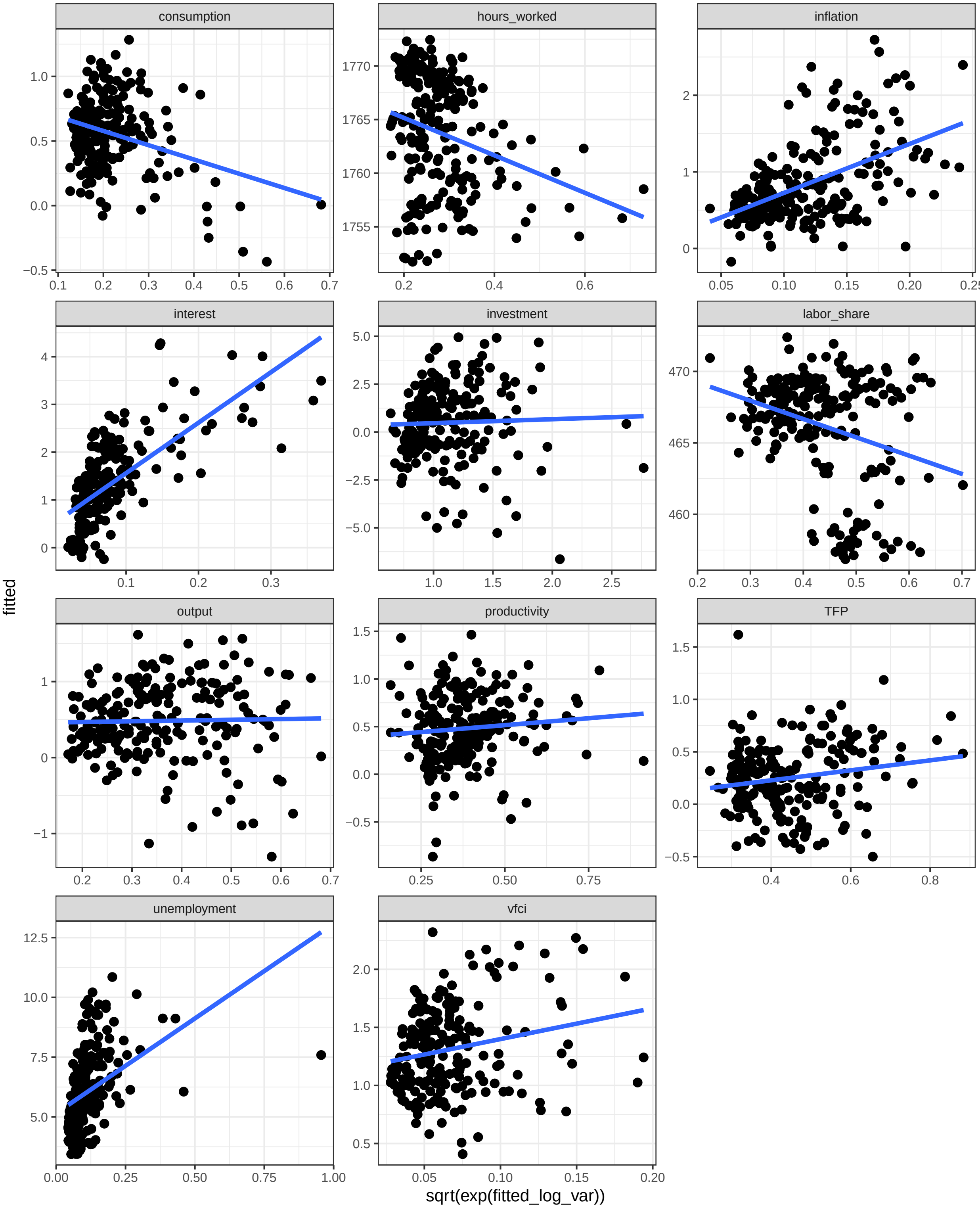
Estimating the VAR in differences

We can estimate the VAR in differences rather than taking differences of the non-stationary variables.

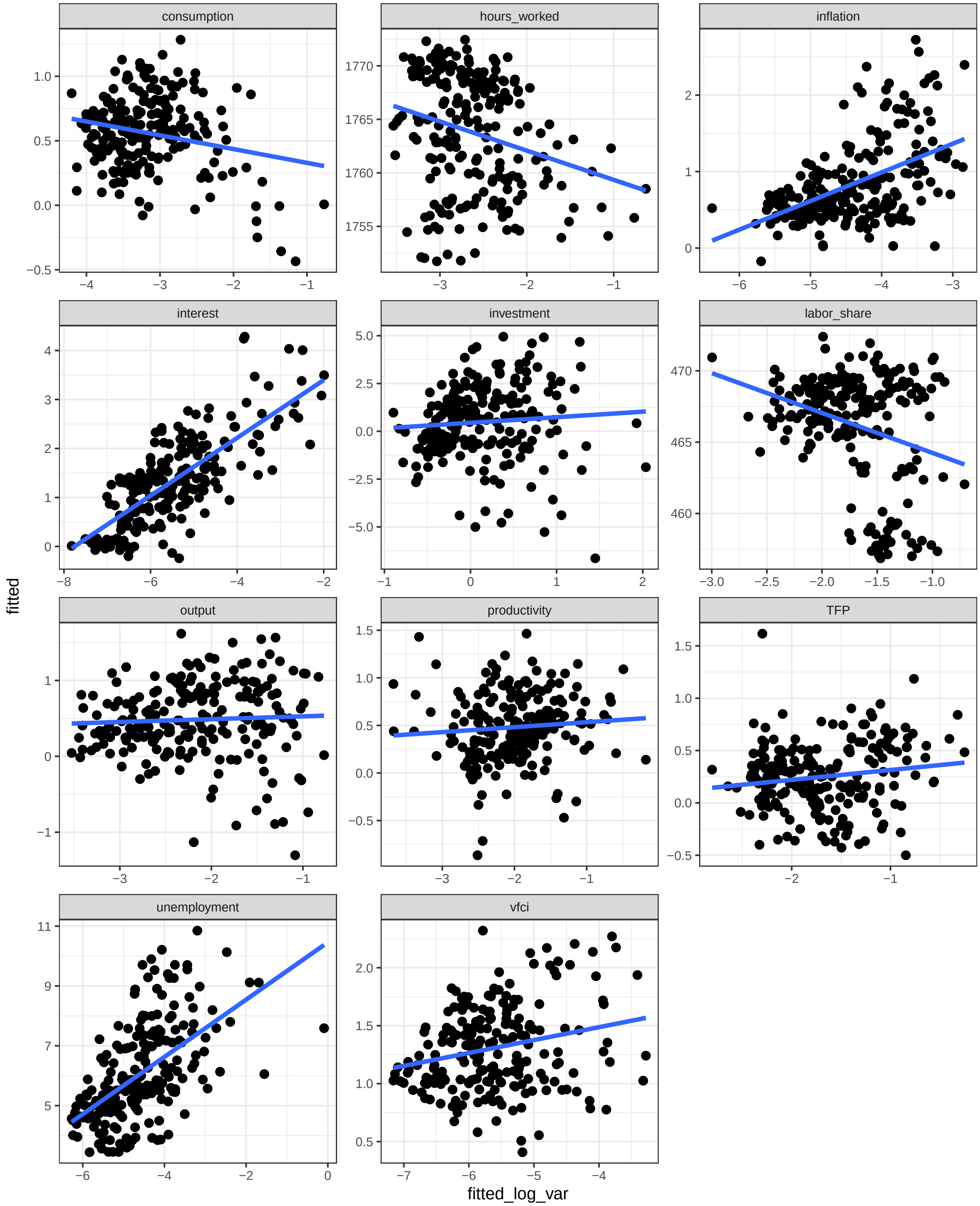




Mean-Vol Relationship - VAR estimated in differences



Mean-Var Relationship - VAR estimated in differences



Business Cycle Shock Mean-Vol

This section attempts to construct the same mean-vol relationship for just the identified business cycle shock.

This requires a change in the definition of the "mean" values.

We would expect (and we find) a null result here, as the business cycle shock should be homoskedastic.

Max Share ID

Using the max share identification method, we identify one structural shock, w_t^{BC}, which drives the business cycle.

$$u_t = PQ^* w_t \tag{8}$$

where P is the choleskey of Σ_u and Q^* is the identified rotation matrix that returns w_t^{BC} in the first column.

Then we can write,

$$y_t = A_1 y_{t-1} + ... + A_p y_{t-p} + PQ^* \begin{bmatrix} w_t^{\text{BC}} \\ w_t^2 \\ w_t^3 \\ ... \\ w_t^k \end{bmatrix} \tag{9}$$

Business Cycle Contribution

Using the identified business cycle shock, we can construct the time series for y_t^{BC} which are the contributions to each variable of the VAR driven by the identified shock.

$$\widehat{y^{\text{BC}}}_t = \begin{bmatrix} \widehat{\text{output}}_t^{\text{BC}} \\ \text{unemployment}_t^{\text{BC}} \\ \text{inflation}_t^{\text{BC}} \\ ... \\ \text{vfci}_t^{\text{BC}} \end{bmatrix} \tag{10}$$

Business Cycle Volatility

The log-variance for the business cycle can be defined as:

$$\text{Var}_t^{\text{BC}} = log(\left[\widehat{w}_t^{\text{BC}}\right]^2) \tag{11}$$

This can also be modeled using the variables of the VAR:

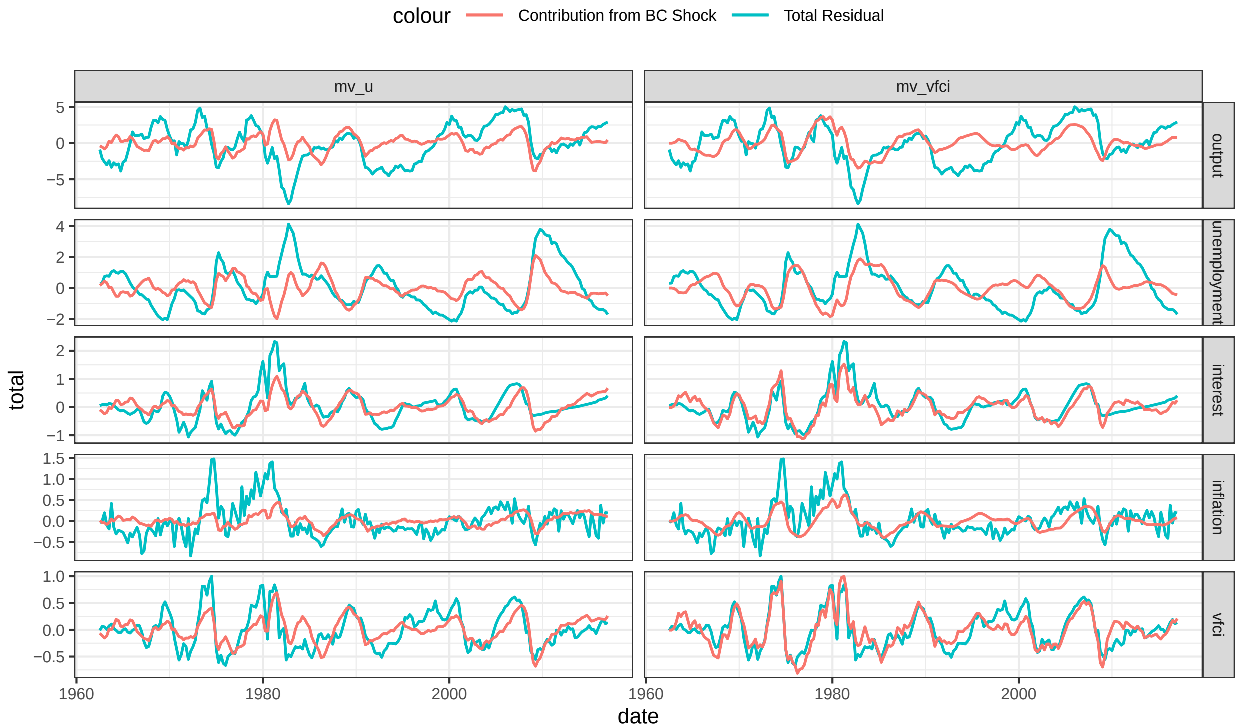
$$\text{Var}_t^{\text{BC}} = \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + \epsilon_t \tag{12}$$

where α_i is a $1xk$ vector of coefficients.

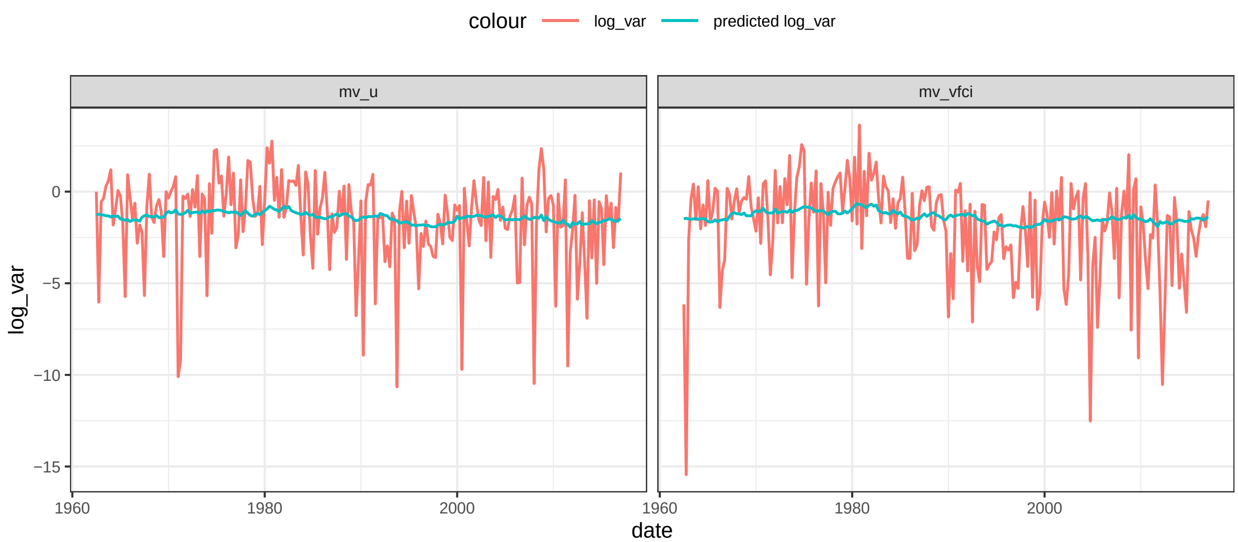
We can then estimate $\widehat{\text{Var}_t}^{\text{BC}}$ as a time series.

This can be done for each model that identifies a structural shock. In our case, we can compare the model targetting unemployment and the one targetting vfci.

Here are the contributions of the Business Shock, $\widehat{y_t^{\mathrm{BC}}}$, to a subset of the variables, compared to the total unexplained varaition in those series.



And here is the log-variance of the business cycle shock, $\mathrm{Var}_t^{\mathrm{BC}}$, and the predicted value, $\widehat{\mathrm{Var}_t}^{\mathrm{BC}}$:



The correlation between the two sets of predicted values are not particularly high. This is just showing a small subsample of the variables in the VAR.