# Mean - Vol Relationship

## VAR

A reduced form VAR of some number of variables, $k$, and some number of lags, $p$, is written as,

$$y_t = A_1 y_{t-1} + ... + A_p y_{t-p} + u_t \tag{1}$$

where $A_i$ is a $kxk$ matrix of coefficients and,

$$y_t = \begin{bmatrix} \text{output}_t \\ \text{unemployment}_t \\ \text{inflation}_t \\ ... \\ \text{vfci}_t \end{bmatrix}, \qquad u_t = \begin{bmatrix} u_t^{\text{output}} \\ u_t^{\text{unemployment}} \\ u_t^{\text{inflation}} \\ ... \\ u_t^{\text{vfci}} \end{bmatrix} \tag{2}$$

Each $u_t$ are the reduced form residuals for the accompanying data series.

With this, we can estimate the predicted values, $\widehat{y}_t$, and the residuals, $\widehat{u}_t$.

### Volatility

We can then define the log-variance as

$$\text{Var}_t \equiv \log(\widehat{u}_t^2) \tag{3}$$

This can be estimated with the regression,

$$\text{Var}_t = \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + \epsilon_t \tag{4}$$

Then the predicted (conditional) volatility is

$$\widehat{\text{Vol}}_t = \left[\exp(\widehat{\text{Var}}_t)\right]^{\frac{1}{2}} \tag{5}$$

### Mean-Vol Relationship

The mean-vol relationship is then the relationship between

(1) the predicted means:

$$\widehat{y}_t \tag{6}$$

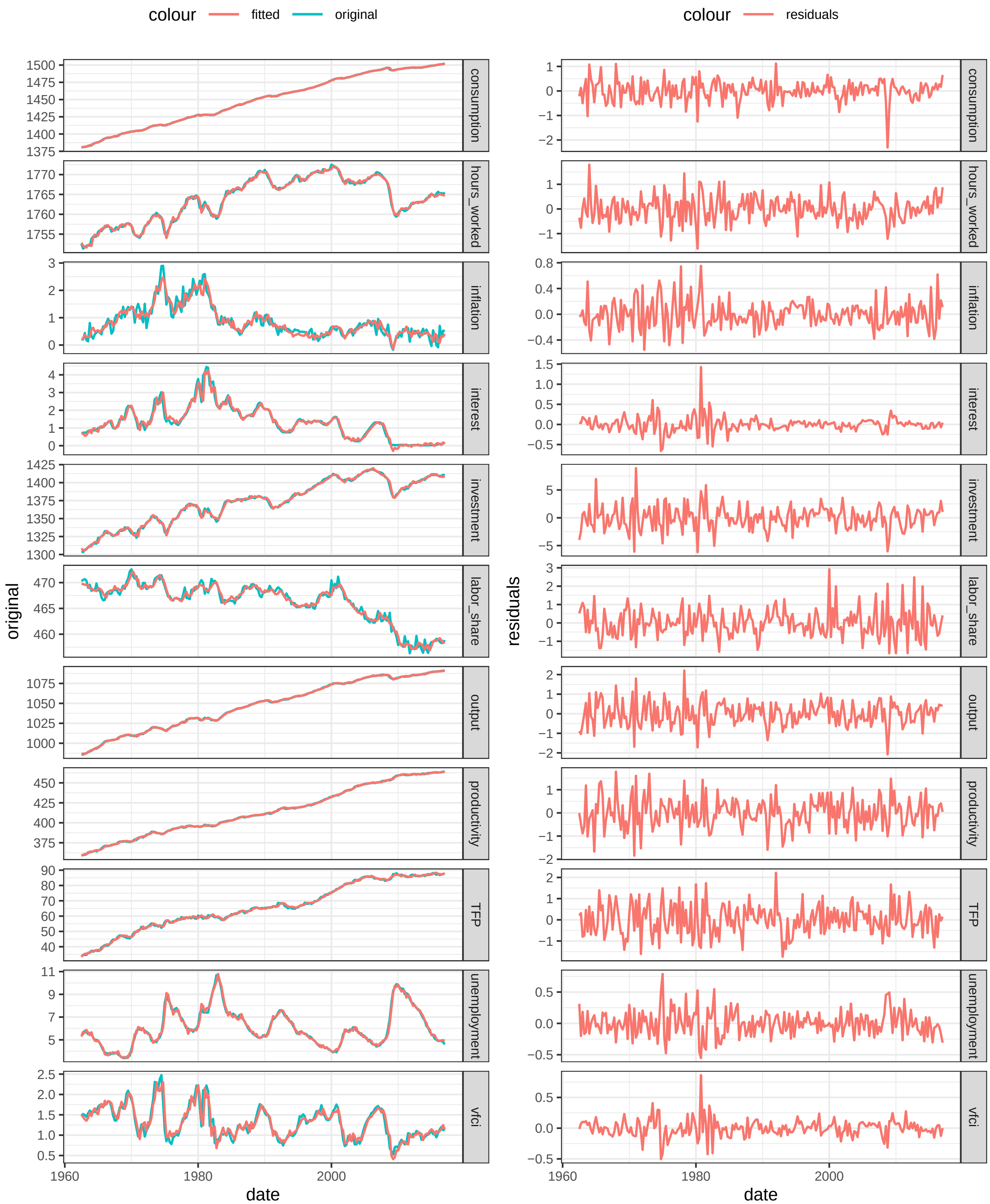
(2) the predicted volatility:

$$\widehat{\text{Vol}}_t \tag{7}$$

**Note:** There are $k$ mean-vol relationships, one for each variable in the VAR.

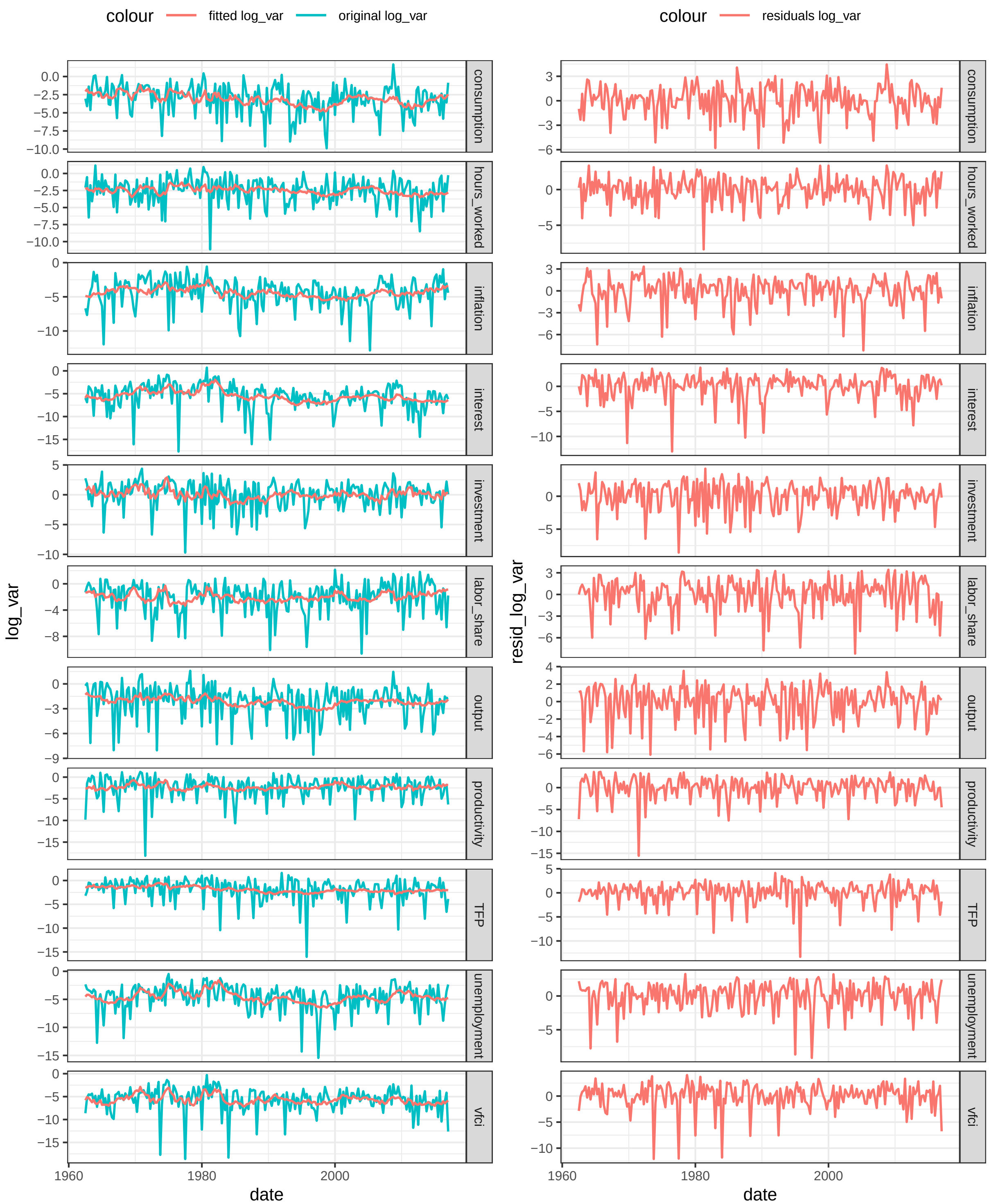
# Fitting the VAR

(1) Fit the VAR to the raw (original) data, find the fitted values and the residuals.



## Fitting the heteroskedastic variance regression

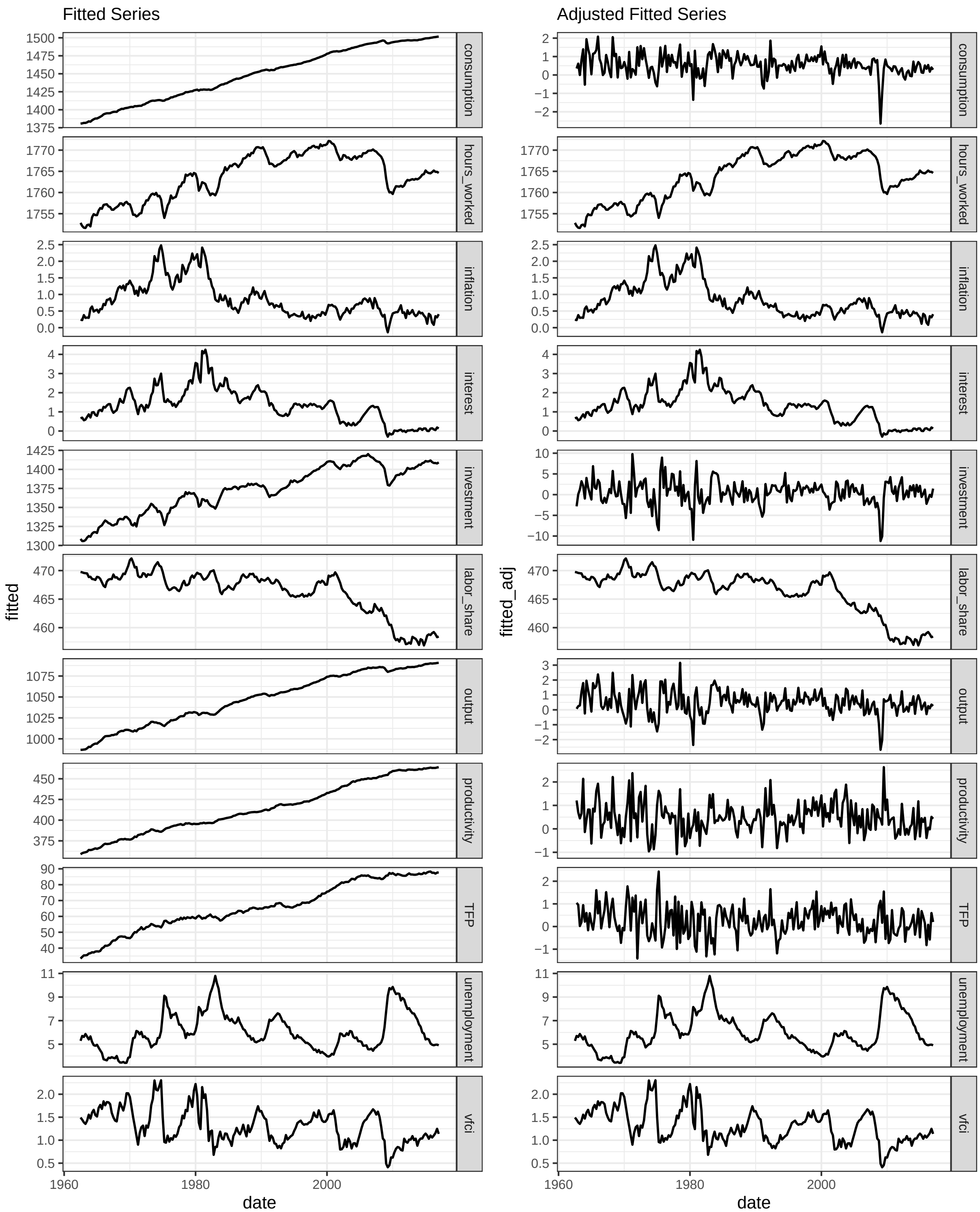
(2) Construct the log-variance for each residual. Fit a regression on the same variables as the VAR, find the fitted value.



## Nonstationary Series

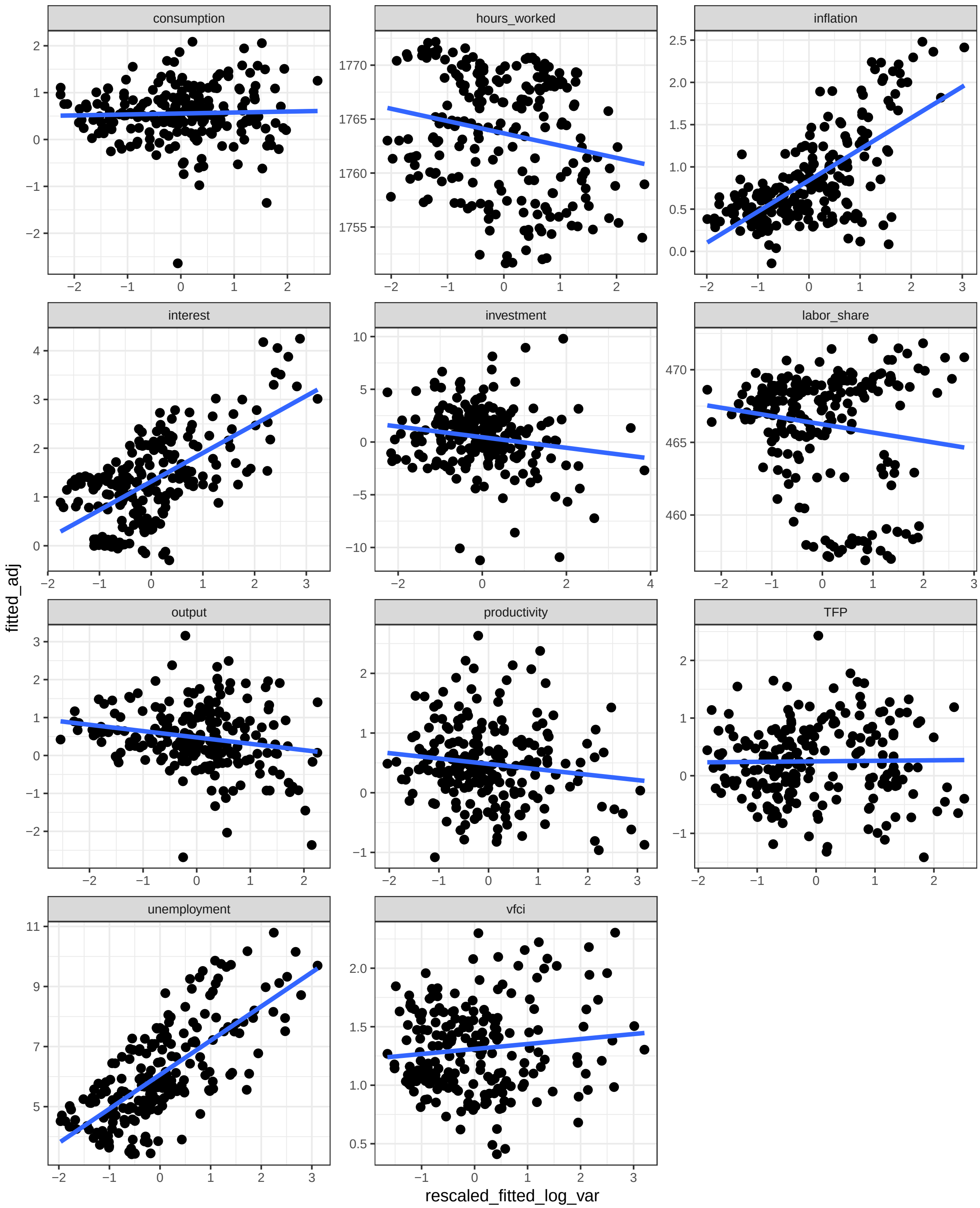
(3) Some of the values in $\widehat{y}_t$ are non-stationary (they have a trend), so we take differences before presenting them in the mean-vol and mean-var figures.

The differences variables are: output, consumption, investment, productivity, and TFP



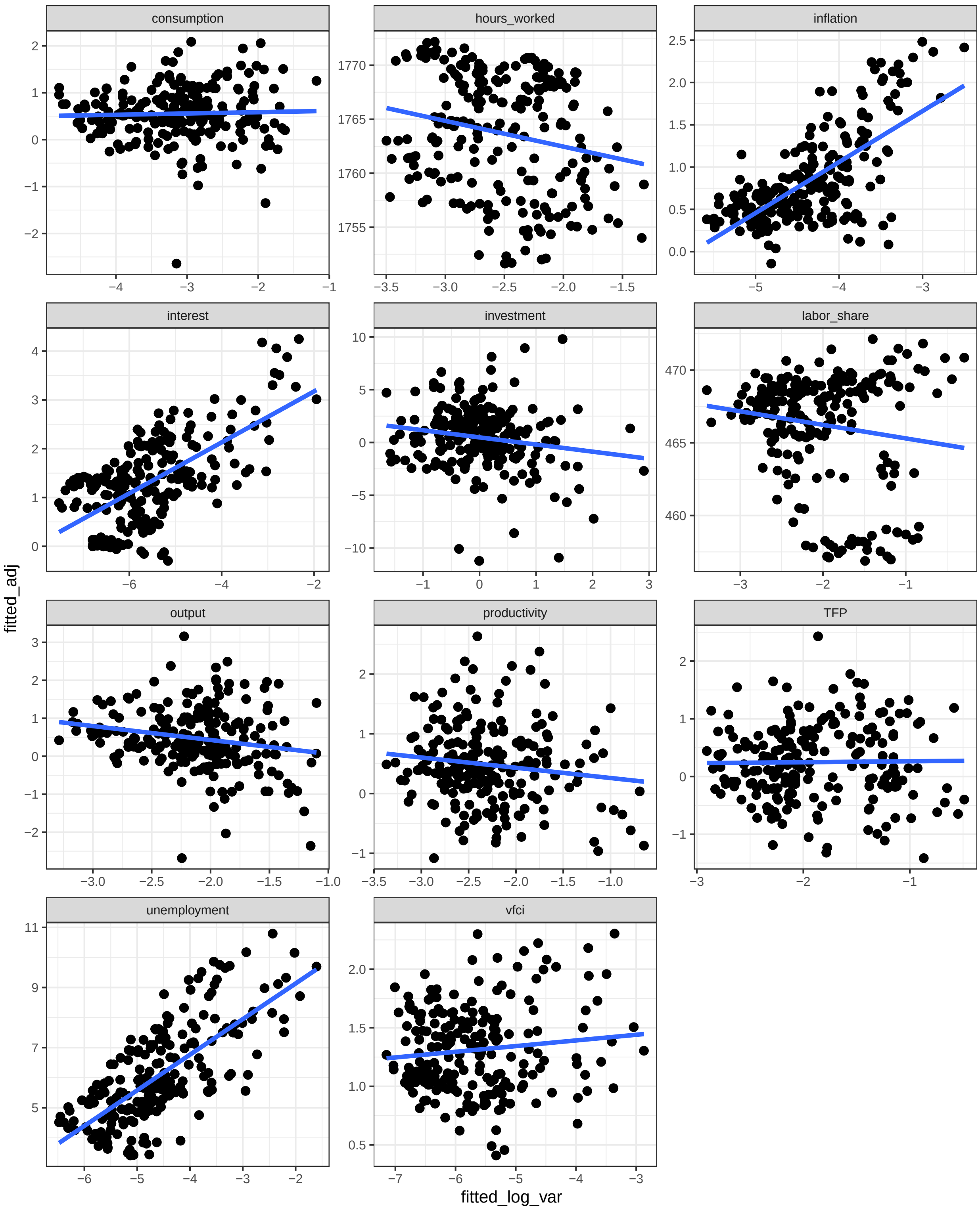
## Mean-Vol Relationship

(4) Chart the relationship between the fitted means and the fitted volatilities.



## Mean-Var Relationship

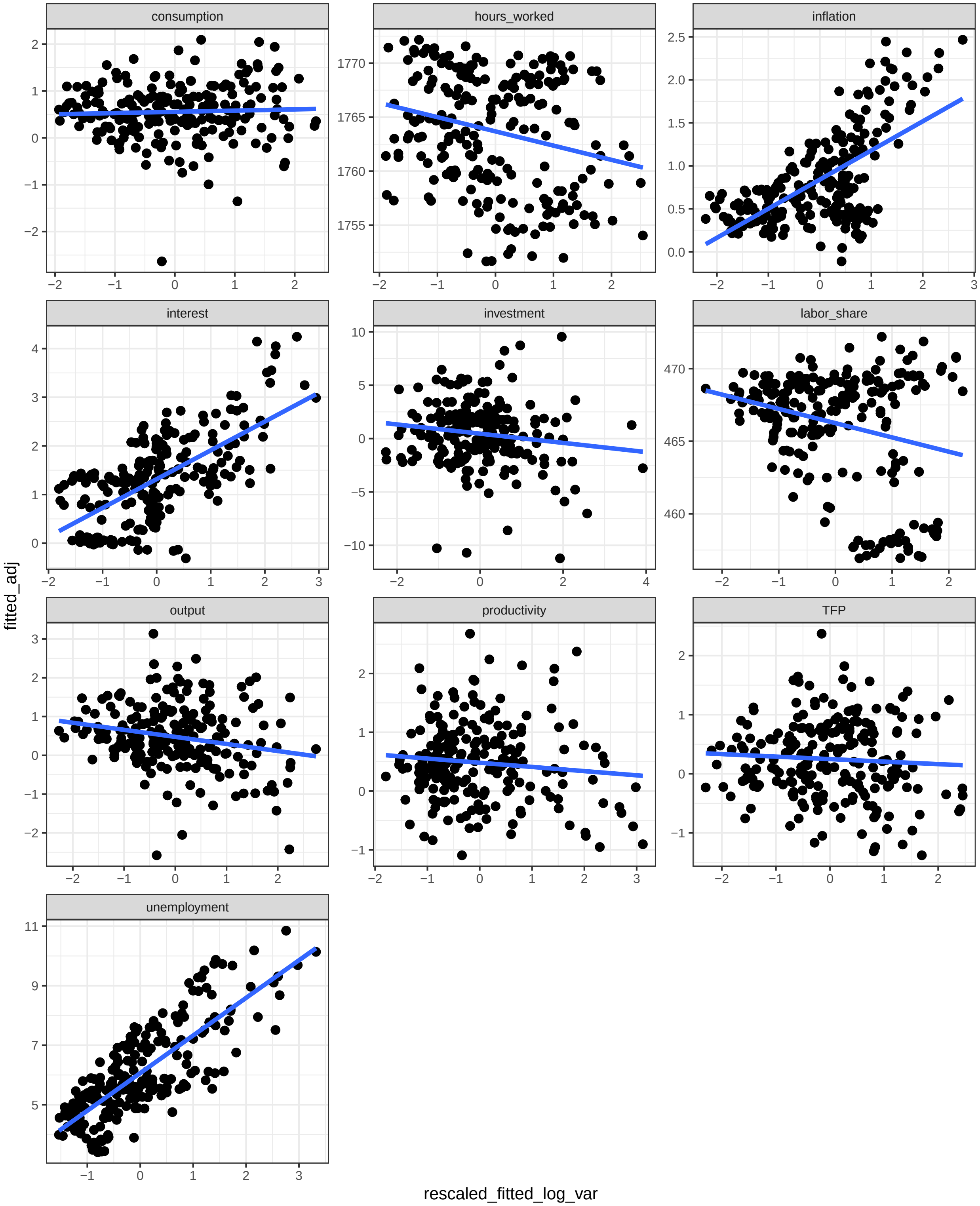
(5) Chart the relationship between the fitted means and the fitted log-variances.



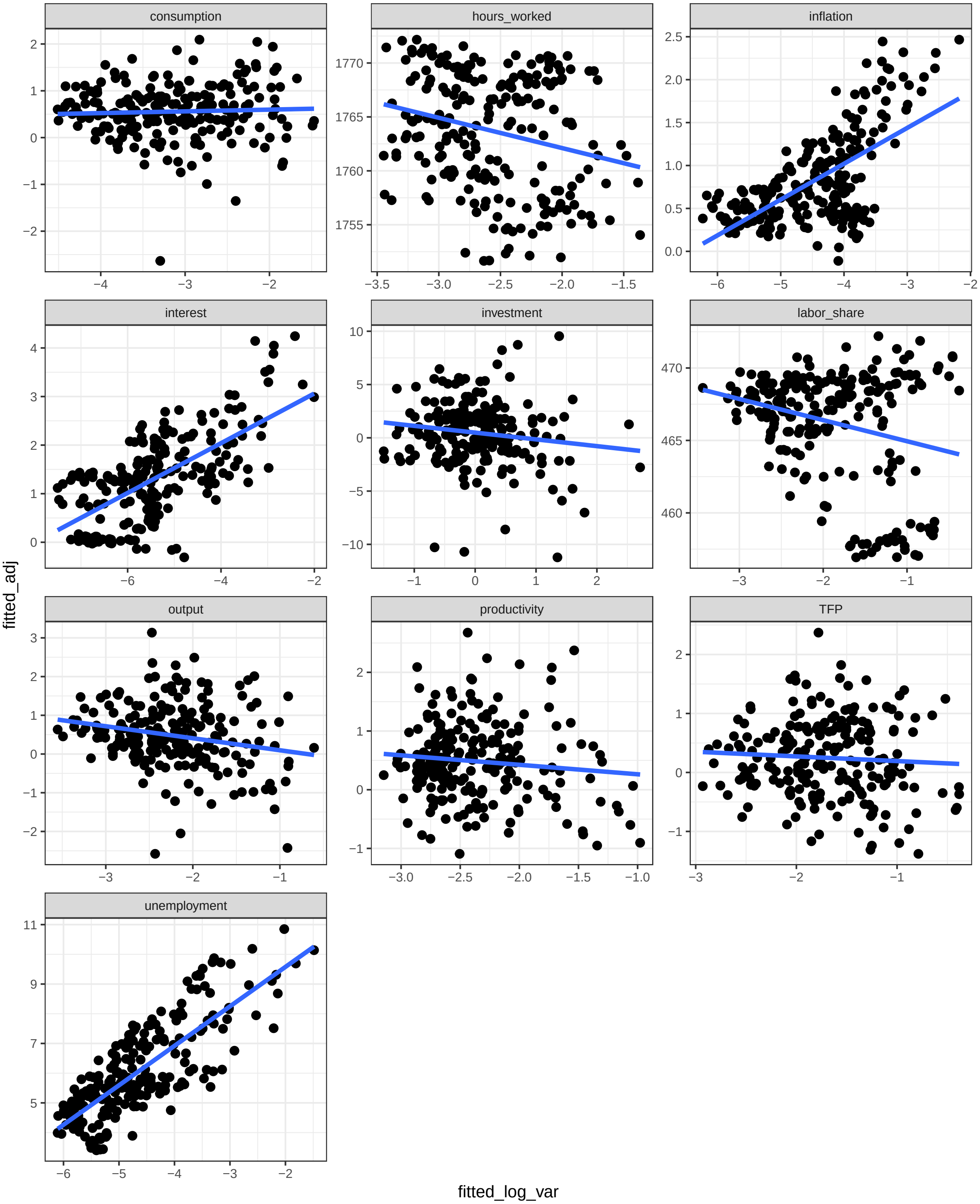
# Using 10-variable VAR

We can repeat the above analysis with just 10 variables in the VAR (dropping vfci).

## Mean-Vol Relationship - 10 variable VAR

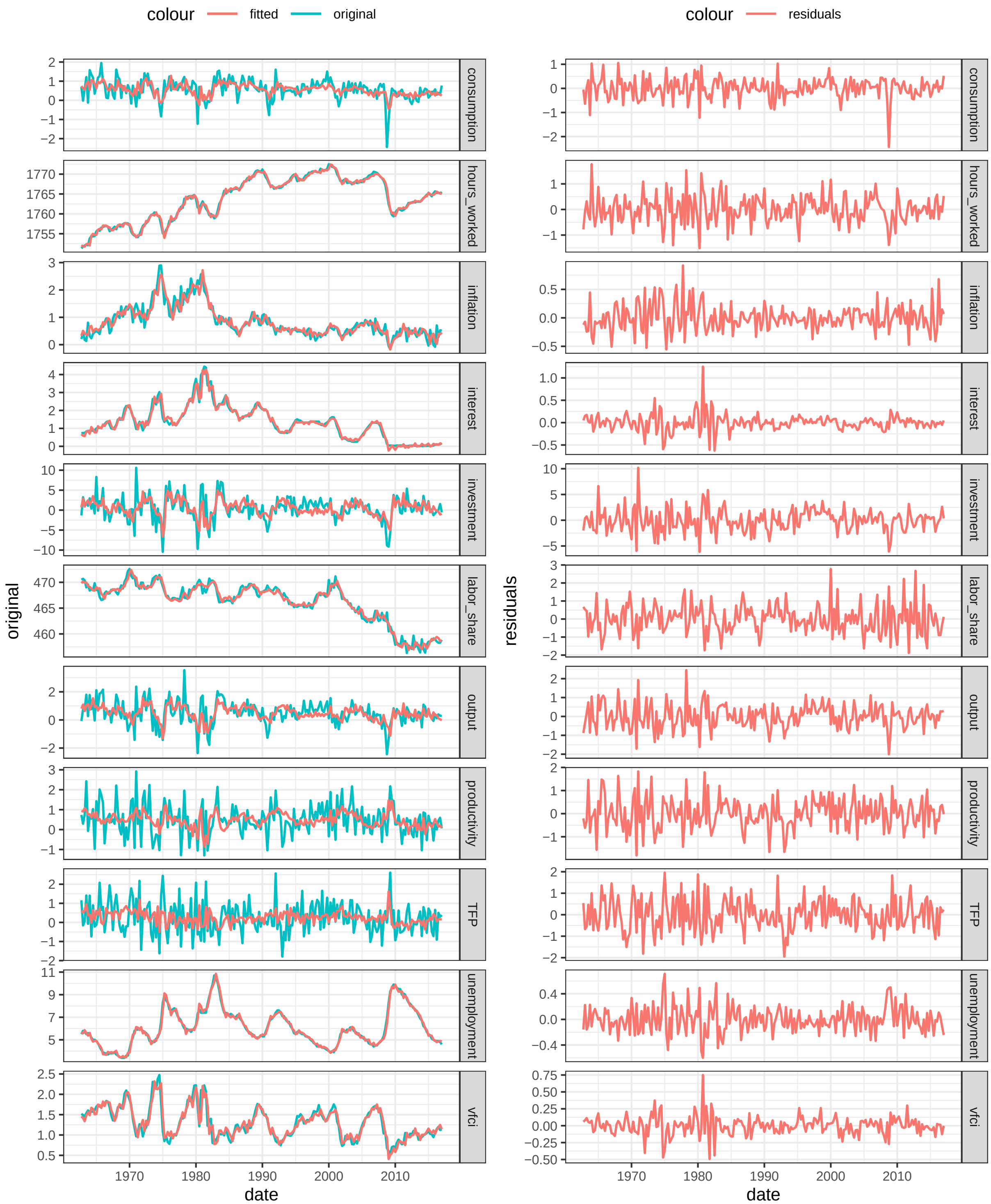


## Mean-Var Relationship - 10 variable VAR

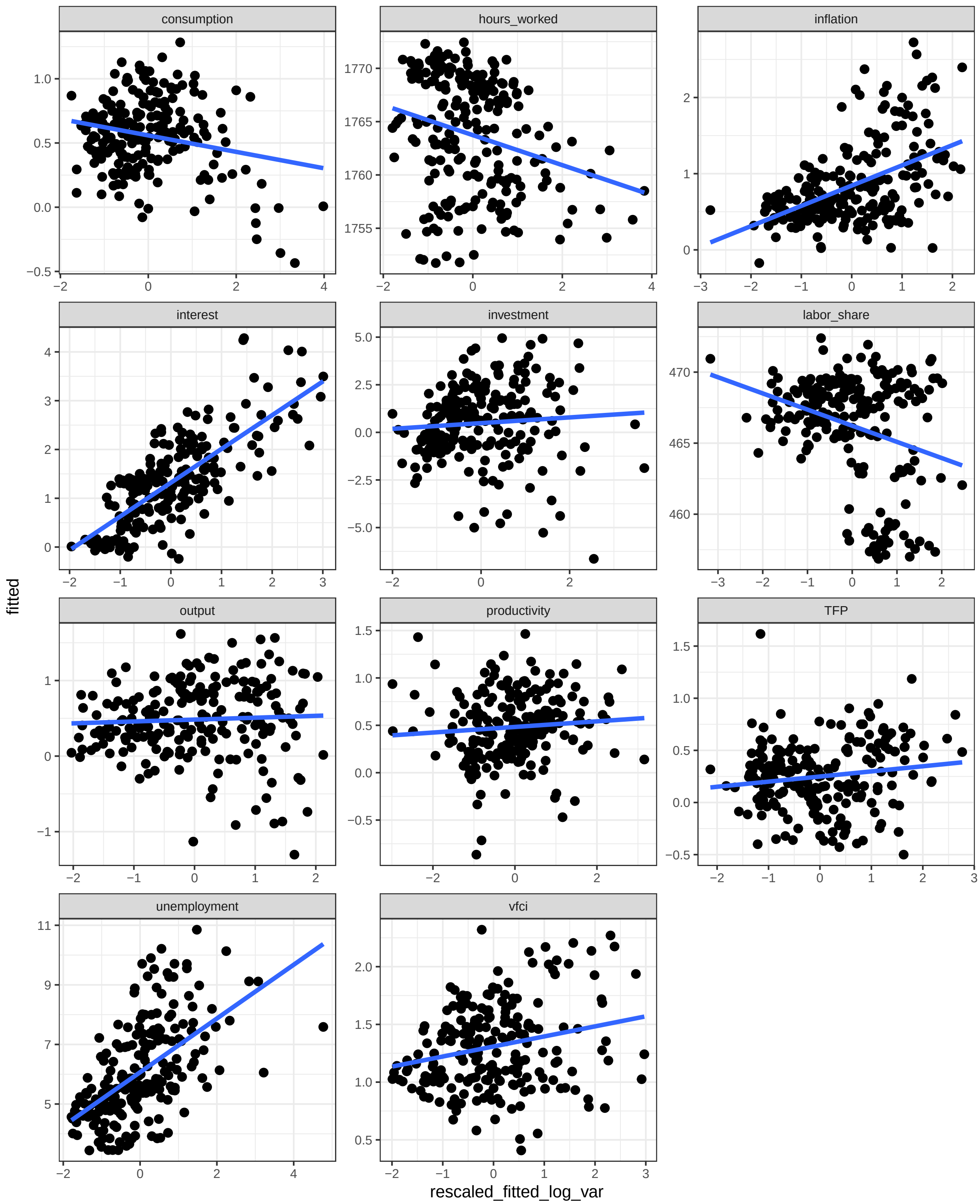


# Estimating the VAR in differences

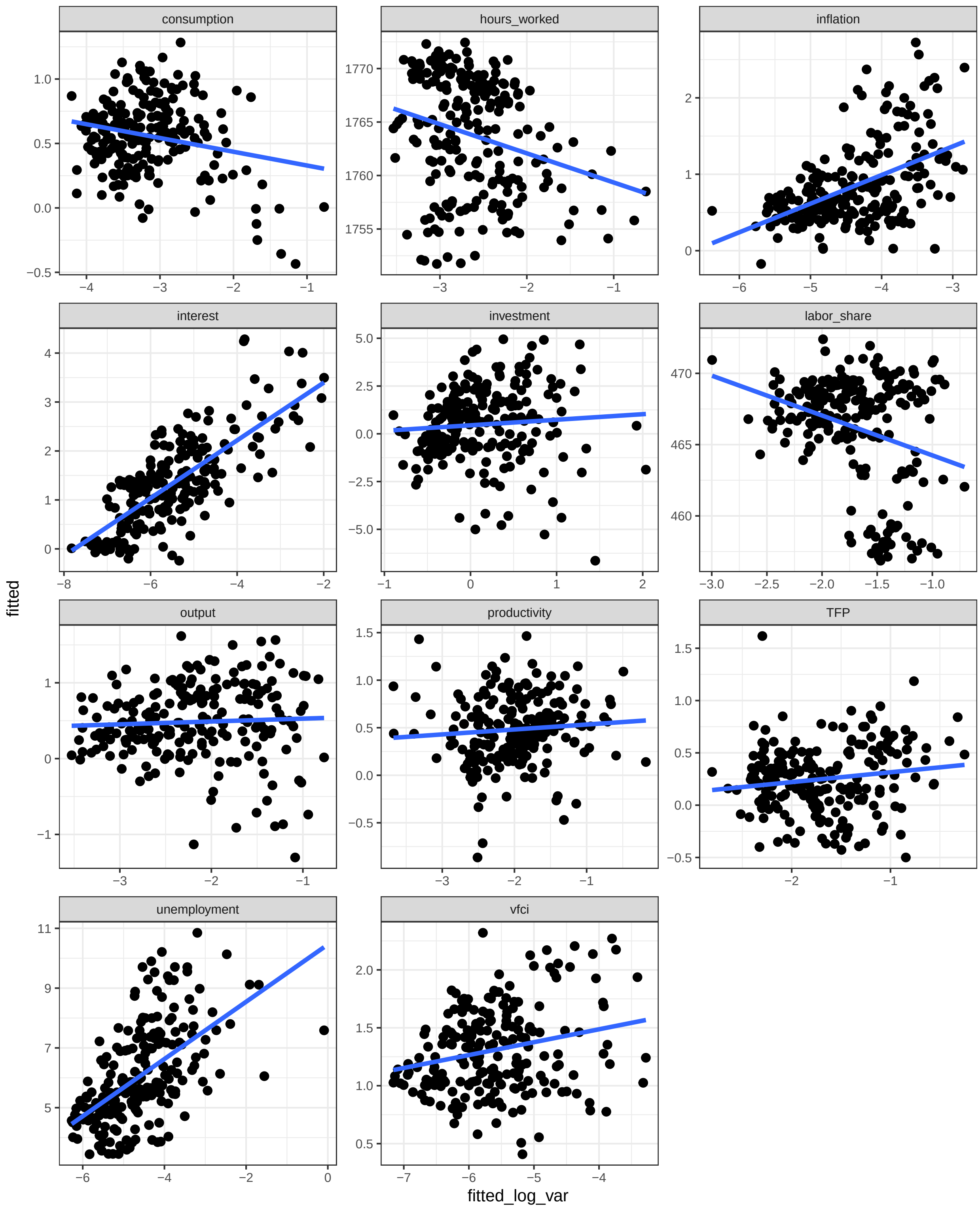
We can estimate the VAR in differences rather than taking differences of the non-stationary variables.



## Mean-Vol Relationship - VAR estimated in differences



## Mean-Var Relationship - VAR estimated in differences



## Business Cycle Shock Mean-Vol

This section attempts to construct the same mean-vol relationship for just the identified business cycle shock.

This requires a change in the definition of the "mean" values.

We would expect (and we find) a null result here, as the business cycle shock should be homoskedastic.

### Max Share ID

Using the max share identification method, we identify one structural shock, $w_t^{\text{BC}}$, which drives the business cycle.

$$u_t = PQ^* w_t \tag{8}$$

where $P$ is the choleskey of $\Sigma_u$ and $Q^*$ is the identified rotation matrix that returns $w_t^{\text{BC}}$ in the first column.

Then we can write,

$$y_t = A_1 y_{t-1} + ... + A_p y_{t-p} + PQ^* \begin{bmatrix} w_t^{\text{BC}} \\ w_t^2 \\ w_t^3 \\ ... \\ w_t^k \end{bmatrix} \tag{9}$$

### Business Cycle Contribution

Using the identified business cycle shock, we can construct the time series for $y_t^{\text{BC}}$ which are the contributions to each variable of the VAR driven by the identified shock.

$$\widehat{y^{\text{BC}}}_t = \begin{bmatrix} \widehat{\text{output}}_t^{\text{BC}} \\ \text{unemployment}_t^{\text{BC}} \\ \text{inflation}_t^{\text{BC}} \\ ... \\ \text{vfci}_t^{\text{BC}} \end{bmatrix} \tag{10}$$

### Business Cycle Volatility

The log-variance for the business cycle can be defined as:

$$\text{Var}_t^{\text{BC}} = log(\left[\widehat{w}_t^{\text{BC}}\right]^2) \tag{11}$$

This can also be modeled using the variables of the VAR:

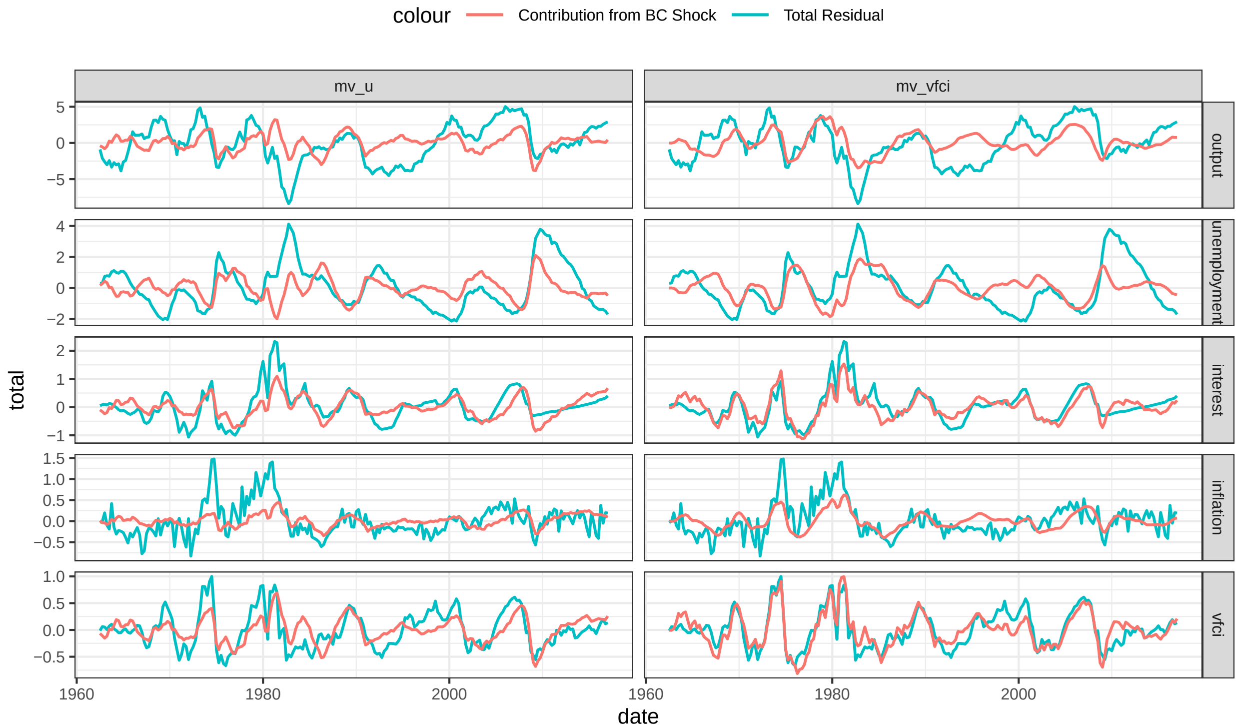
$$\text{Var}_t^{\text{BC}} = \alpha_1 y_{t-1} + ... + \alpha_p y_{t-p} + \epsilon_t \tag{12}$$

where $\alpha_i$ is a $1xk$ vector of coefficients.

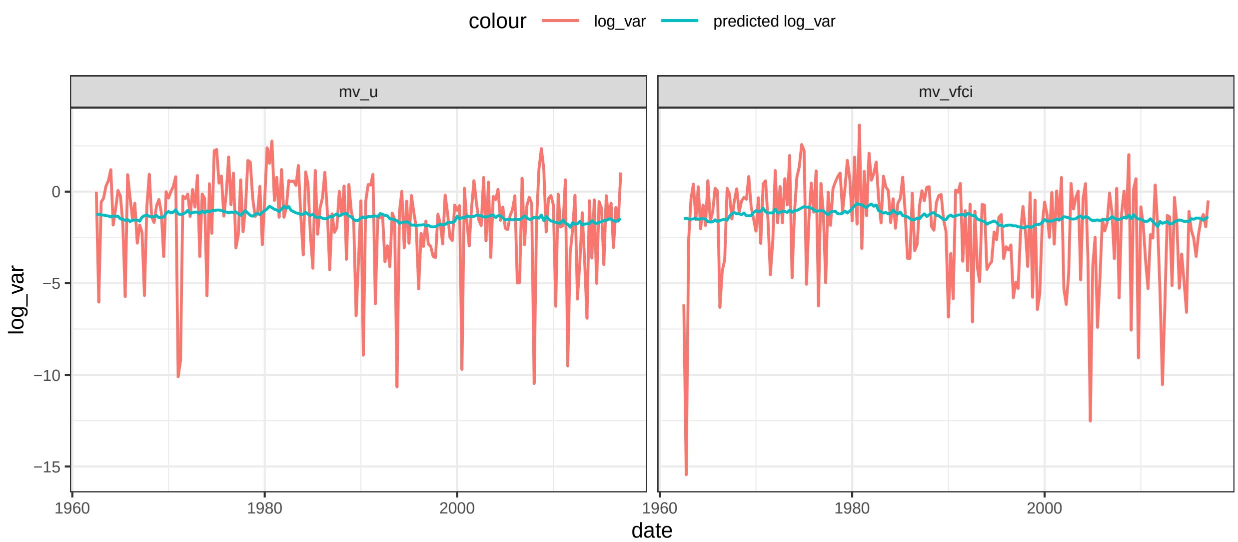
We can then estimate $\widehat{\text{Var}_t}^{\text{BC}}$ as a time series.

This can be done for each model that identifies a structural shock. In our case, we can compare the model targetting unemployment and the one targetting vfci.

Here are the contributions of the Business Shock, $\widehat{y_t^{\mathrm{BC}}}$, to a subset of the variables, compared to the total unexplained varaition in those series.




And here is the log-variance of the business cycle shock, $\mathrm{Var}_t^{\mathrm{BC}}$, and the predicted value, $\widehat{\mathrm{Var}_t^{\mathrm{BC}}}$:



The correlation between the two sets of predicted values are not particularly high. This is just showing a small subsample of the variables in the VAR.