Piecewise Notes

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1 Introduction

These notes serve as a single source of knowledge in which those reading should ideally be able to garner a deeper understanding of the thoughts and ideas about piecewise objects I have had and continue to have.

To consider something piecewise is to enumerate something under differing conditions. For example, one might consider the function $|x|$ piecewise, namely when $x \geq 0$ or $x \leq 0$.

An object which is piecewise is an object which contains a set of pieces, which themselves contain values and conditions under which those values are taken for the overall object. Continuing with the example of $|x|$, we have the piece value x for when $x \geq 0$ (which forms a piece), and $-x$ for when $x \leq 0$.

We shall develop ideas such as above more explicitly throughout these notes. These ideas often intersect with other areas of maths, which one might be familiar with - but if not, don't panic; such ideas are not strictly foundational to the presented notes.

Furthermore, these notes focus on construction (not to be confused with constructivism); rather than evaluating existing concepts or formulas, we focus on deriving existing or new tools, ideas and formulas. For you, reader, this must be a process you should become familiar with, and rather than just reading these notes, attempt to follow along by hand, and construct your own objects using the ideas presented here. We also stress that with ideas that intersect with more mainstream mathematics, that existing concepts be used to evaluate the validity of the constructions.

Finally, if you have trouble understanding some of the ideas presented here, you should consider taking a look at the blog posts on https://piecewise.org or contacting myself at ally@piecewise.org. The things you'll see here are the culmination of many hundreds of hours scribbling, doodling and refining thoughts over the course of several years. More importantly than the fact I am myself still learning the fundamentals of mathematics at a higher level, is that you understand that trial and lots of error forms the majority of this work.

Thank you in advance for reading.

2 Notation

2.1 Piecewise functions

A piecewise *function* is a function defined over several pieces.

Let us consider some function $f : \mathbb{R} \to \mathbb{R}$ and some intervals $D_1, D_2, \ldots, D_n \subseteq \mathbb{R}$ such that $D_1 \cap D_2 \cap \cdots \cap D_n = \varnothing$. Suppose we have some functions $f_1 : D_1 \to \mathbb{R}$ and so on. We describe this function using the following notation:

$$f(x) = \begin{cases} f_1(x) & x \in D_1 \\ f_2(x) & x \in D_2 \\ \vdots & \vdots \\ f_n(x) & x \in D_n \end{cases}$$

This same function could instead be written as the following:

$$\begin{gathered} f : D_1 \to \mathbb{R}, f(x) = f_1(x), \\ f : D_2 \to \mathbb{R}, f(x) = f_2(x), \\ \vdots \\ f : D_n \to \mathbb{R}, f(x) = f_n(x) \end{gathered}$$

This essentially describes the same function over different domains. Alternatively, one might think about it as different functions under the same label, f.

Reading this function more verbosely, we say that if x is in the domain D_1 and so on, then we might 'choose' to have $f(x) = f_1(x)$ as per our definition.

Example 2.1 (Absolute value function)**.** Let us define the absolute value function, $|x|$ as the following:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$