Piecewise Notes

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Preface

These notes serve as a single source of knowledge in which those reading should ideally be able to garner a deeper understanding of the thoughts and ideas about piecewise objects I have had and continue to have.

If you have trouble understanding some of the ideas presented here, you should consider taking a look at the blog posts on `https://piecewise.org` or contacting myself at ally@piecewise.org. The things you'll see here are the culmination of many hundreds of hours scribbling, doodling and refining thoughts over the course of several years. More importantly than the fact I am myself still learning the fundamentals of mathematics at a higher level, is that you understand that trial and lots of error forms the majority of this work.

With this being said, there is the distinct possibility that these notes are the culmination of wasted effort and crackpotery :-). It's entirely possible people have stumbled on these same or similar ideas, simply notated differently, as is prevalent in all undergraduate students learning new (old) maths, including myself.

Thank you in advance for reading.

1 Introduction

To consider something piecewise is to enumerate something under differing conditions. For example, one might consider the function $|x|$ piecewise, namely $|x| = x$ when $x \geq 0$ or $|x| = -x$ when $x \leq 0$.

An object which is piecewise is something which contains a set of pieces, which themselves contain values and conditions, under which those values are taken for the overall object. Continuing with the example of $|x|$, we have the piece value x for when $x \geq 0$, the condition, (which forms a piece), and $-x$ for when $x \leq 0$.

We shall develop ideas such as above more explicitly throughout these notes. These ideas often intersect with other areas of maths, which one might be familiar with — but if not, don't panic; such ideas are not strictly foundational to the presented notes.

Furthermore, these notes focus on construction; rather than evaluating existing concepts or formulas, we focus on deriving existing or new tools, ideas and formulas. For you, reader, this must be a process you should become familiar with, and rather than just reading these notes, attempt to follow along by hand, and construct your own objects using the ideas presented here. We also stress that with ideas that intersect with more mainstream mathematics, that existing concepts be used to evaluate the validity of the constructions. Moreover, we attempt to make these ideas and tools systematic when applied to piecewise functions, so as to develop algorithms and consistently draw parallels to existing methods (for example, we may derive a formula which can then be verified by induction).

As to a preview of what's to come: we develop ideas and formulations for elementary function based formulations of typically piecewise-denoted functions, do some probably piecewise algebra, interpolate points and other functions, construct squares, shapes, re-develop the batman equation but continuously, paint some graphs, think about what happens when we mix in complex numbers, and more.

2 Notation

2.1 Single variable piecewise functions

A piecewise *function* is a function defined over several pieces.

Let us consider some function $f : \mathbb{R} \to \mathbb{R}$ and some intervals $D_1, D_2, \ldots, D_n \subseteq \mathbb{R}$ such that $D_i \cap D_j = \varnothing$ for all $i \neq j$, where $i, j = 1, 2, \ldots, n$. That is, we have some number real intervals which do not overlap. Suppose we have some functions $f_1 : D_1 \to \mathbb{R}$ and so on. We describe this function using the following notation:

$$f(x) = \begin{cases} f_1(x) & x \in D_1 \\ f_2(x) & x \in D_2 \\ \vdots & \vdots \\ f_n(x) & x \in D_n \end{cases}$$

This same function could instead be written as the following:

$$\begin{gathered} f : D_1 \to \mathbb{R}, f(x) = f_1(x), \\ f : D_2 \to \mathbb{R}, f(x) = f_2(x), \\ \vdots \\ f : D_n \to \mathbb{R}, f(x) = f_n(x) \end{gathered}$$

This essentially describes the same function over different domains. Alternatively, one might think about it as different functions under the same label, f.

Reading this function more verbosely, we say that if x is in the domain D_1 and so on, then we might 'choose' to have $f(x) = f_1(x)$ as per our definition.

Example 2.1.1. Let us define the absolute value function, $|x|$, as the following:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

To evaluate, for example, $|-3|$, we use the definition:

$$|-3| = \begin{cases} -3 & -3 \geq 0 \\ 3 & -3 \leq 0 \end{cases}$$

Since the condition in the second piece (or case) evaluates to true, we use the value in that piece. That is, since $-3 \leq 0$ is true, $|-3| = 3$.

2.2 Piecewise functions

Piecewise functions needn't be restricted to a single variable. More generally, piecewise functions needn't be restricted to any number of conditions or pieces. The perfect example, as we'll see later, is the floor function, which has an infinite number of pieces. Similarly, the bivariate function describing a square when it intersects a certain plane has 4 pieces.

In general, we can notate such a more general piecewise function like the following:

$$f(x) = \begin{cases} f_1(x) & C_{1,1} \quad C_{1,2} \quad \dots \quad C_{1,m} \\ f_2(x) & C_{2,1} \quad C_{2,2} \quad \dots \quad C_{2,m} \\ \vdots & \vdots \quad\quad \vdots \quad\quad \ddots \quad \vdots \\ f_n(x) & C_{n,1} \quad C_{n,2} \quad \dots \quad C_{n,m} \end{cases}$$

Where $C_{p,q}$ is some condition which evaluates to true or false for some inputs (e.g. a predicate). We interpret the string of conditions in each piece, such as $\{C_{1,1} \quad C_{1,2} \quad \dots \quad C_{1,m}\}$, as equivalent to the condition $C_{1,1} \wedge C_{1,2} \wedge \cdots \wedge C_{1,m}$.

Such conditions could be anything, and per the section above, we could have $C_{p,q}$ denote the predicate $x \in D_{p,q}$ for some set $D_{p,q}$.

Importantly, we note in general this piecewise object is written in a shortened form, without commas or English terms such as 'and' or 'or', or their respective logical operators. This is a conventional decision in these notes given the number of piecewise objects written. For example, the following is identical to the above:

$$f(x) = \begin{cases} f_1(x), & \text{if } C_{1,1} \wedge C_{1,2} \wedge \cdots \wedge C_{1,m} \\ f_2(x), & \text{if } C_{2,1} \wedge C_{2,2} \wedge \cdots \wedge C_{2,m} \\ \vdots & \vdots \\ f_n(x), & \text{if } C_{n,1} \wedge C_{n,2} \wedge \cdots \wedge C_{n,m} \end{cases}$$

And of course we might package each piece's conditions together, effectively making one condition per piece. Though, in practice, this doesn't help very much at all (this is because, as we will later see, considering each condition individually yields results which affect the way we interpolate functions and other manipulation of piecewise objects).

2.3 Generalised piecewise object

Finally, we might consider a set-like generalised form of a piecewise object. Let us denote the following:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

Where ϕ describes the piecewise object, and for each $i \in I$ (the iterator), $\phi = \varphi_i$ for when C_i is true.

Example 2.3.1. Suppose we have a set $I = \{1, 2, \dots, n\}$ and a piecewise function f such that $f = f_i$ when C_i is true.

We can express this function as:

$$f = \begin{cases} f_1 & C_1 \\ f_2 & C_2 \\ \vdots & \vdots \\ f_n & C_n \end{cases}$$

Alternatively, we might express it using the generalised notation:

$$f = \{f_i, \quad C_i \mid i \in \{1, 2, \dots, n\}\}$$

Example 2.3.2. We will see the floor function later on where it will be described more deeply, but we can introduce it like so using our notation:

$$\lfloor x \rfloor = \{n, \quad x \in [n, n+1) \mid n \in \mathbb{Z}\}$$

To evaluate this function at $x = 3.5$ consider that for all $n \in \mathbb{Z}$ there exists only one such interval $N = [n, n+1)$ such that $3.5 \in N$. This interval is $[3, 4)$, i.e. $n = 3$; the corresponding piece value is 3 and so $\lfloor 3.5 \rfloor = 3$.

If a single iterator (e.g. $i \in I$) isn't sufficient to describe the piecewise object we desire, we might consider using several; $i \in I$, $j \in J$ for example). Alternatively, we might consider joining two piecewise objects together using $\cup$:

$$\varphi = \varphi_1 \cup \varphi_2 = \{\varphi_{1,i}, \quad A_i \mid i \in I\} \cup \{\varphi_{2,j}, \quad B_j \mid j \in J\}$$

This can be interpreted similarly to before, though we now have that $\varphi = \varphi_{1,i}$ if A_i is true for some i, or $\varphi = \varphi_{2,j}$ is B_j is true for some j.

Notably, this notation can be used to express individual pieces explicitly in a piecewise object rather than collectively.

Example 2.3.3. Recall the absolute value function; see Example 2.1.1.

We can then express $|x|$ as:

$$|x| = \{mx, \quad mx \geq 0 \mid m \in \{-1, 1\}\}$$

2.3.1. Using our notation, we can also explicitly define unions:

Suppose $I = U \cup V$, then the following piecewise object

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

is equivalent to:

$$\phi = \{\varphi_u, \quad C_u \mid u \in U\} \cup \{\varphi_v, \quad C_v \mid v \in V\}$$

(There is no intersection equivalent.)

3 Algebra on Piecewise Objects

3.1 Piece association

Operations on piecewise objects are fairly straightforward and behave as any other object in the context you're working in (whether this be with the real numbers, complex numbers, certain algebras, spaces, and so on).

What this section aims to do is not to teach you directly how to perform exactly the operations you want to on each piecewise object, but instead provide a basis for which you can base your ideas: no piece of a piecewise object is independent of another. Equivalently, each piece is dependent on each other piece. There is an intuitive reason for this: you are evaluating an object not conditionally, but in full generality; each piece exists, in some sense, 'simultaneously'.

It is here we might recognise that this idea could fairly easily lead into combinatorics; an area of maths which deals heavily with the enumeration and construction of such objects.

3.2 Equality property

3.2.1. Consider the following piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

Then if for all $i, j \in I$ we have that $\varphi_i = \varphi_j$ (that is, all piece values are equal for when ϕ is defined), then $\phi = \varphi_i$ for any $i \in I$.

For example, consider the function $f : (-\infty, 0] \to \mathbb{R}$:

$$f(x) = \begin{cases} 0 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We have that $f(x) = 0$. This is because everywhere in $\mathrm{dom}(f)$, or both when $x \geq 0$ and $x < 0$, we have that $f(x) = 0$.

3.3 Piece and condition equivalence

Arguably one of the most fundamental property of piecewise objects is the relationship between each piece's conditions and its respective value. That is, the values for which a piecewise object takes is dependent on their respective conditions, and so can be treated explicitly as such.

3.3.1. Consider the following piecewise function:

$$f(x) = \{f_i(x), \quad C_i \mid i \in I\}$$

For all $i \in I$, let us define a substitution $x \sim y_i$, where $\sim$ represents equality under the condition C_i. Then we might consider writing $f(x)$ as:

$$f(x) = \{f_i(y_i), \quad C_i \mid i \in I\}$$

And $f(x)$ may still remain a non-constant function, but in this way we've given it another representation. For some C_i we might just have $x \sim x$ (and is effectively no substitution at all). This substitution can also go the other way, i.e. $y_i \sim x$.

3.3.2. Consider the following piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

For all $i \in I$, suppose that $C_i \leftrightarrow D_i$ in the context of ϕ (i.e. if a function, within its domain, etc.). Then we can rewrite this object as:

$$\phi = \{\varphi_i, \quad D_i \mid i \in I\}$$

That is, we've substituted the set of our conditions for another set of conditions (and again, we may not have substituted all of them). For example, we know that $x = 5$ when $(x-5)^2 \leq 0$, and so these two conditions are interchangeable.

This idea becomes far more important when working using other functions, such as max, min and $|x|$ (which will be covered later), although it can still be used in other contexts. In essence, for certain classes of piecewise functions, we can build off existing functions to construct not only transformations, but completely new functions and representations.

Example 3.3.1. Let us consider the function $f : (-6, 6) \to \mathbb{R}$:

$$f(x) = \begin{cases} x & x > 5 \\ 5 & x = 5 \\ x & x < 5 \end{cases}$$

Consider the piece for which $x = 5$: note that the piece value is equal to 5. Since $x = 5$ we can replace 5 with x to get:

$$f(x) = \begin{cases} x & x > 5 \\ x & x = 5 \\ x & x < 5 \end{cases}$$

And by the equality property, we know that this function is equivalent to $f(x) = x$.

Finally, depending on the context of the problem, function or object, we can rewrite conditions in some way to make it more straightforward for us to understand. This becomes relevant in the context of function domains.

Example 3.3.2. Let us consider the function from Example 3.3.1.

The following two representations are equivalent:

$$f(x) = \begin{cases} x & x > 5 \\ 5 & x = 5 \\ x & x < 5 \end{cases}$$

$$f(x) = \begin{cases} x & x \in (5, 6) \\ 5 & x = 5 \\ x & x \in (-6, 5) \end{cases}$$

The reason for this is that for all $x \in (-6, 5)$ we have that $x < 5$, despite the contrary not being

Example 3.3.2 (continued)**.**
always true. The reason the contrary needn't always be true for all $\mathbb{R}$ is because our function f is defined on the interval $(-6, 6)$. This same argument can also be applied to $x \in (5, 6) \implies x > 5$.

A strong note on functions which are not well-defined: Piecewise objects can easily define not well-defined functions. In these cases, care must be taken when working with pieces in general, values or conditions.

3.3.3. A piecewise function is only well-defined if all of its piece values denote well-defined functions, and in places where two conditions in separate pieces are true, their respective values must be equal.

Formally, given the piecewise object $\phi = \{\varphi_i, \quad C_i \mid i \in I\}$ if there exists $i \neq j \in I$ such that $C_i \wedge C_j$ is true, then ϕ is well-defined iff $\varphi_i = \varphi_j$ and each φ_k for $k \in I$ is well-defined.

For example, the following is not well-defined:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

This is because the latter piece is 0 for all real numbers that aren't integers, and the first piece is 1 for all rational numbers; these include non-integers. There are a few ways we could change this function to be well-defined:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

This is known as the Dirichlet function; it's the indicator function of the rationals (which we'll cover later on). Alternatively:

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

Which would simply make the function 0 for all $\mathbb{R}$.

3.4 (De)nesting pieces; logical 'and'

The third of several special operations on piecewise objects involves the nesting and denesting of pieces; essentially, we decouple the conditions in each piece from one another in order to group and subsequently simplify the piece values.

Example 3.4.1. Consider the following piecewise object:

$$\phi = \begin{cases} \varphi_1 & C_1 \\ \varphi_2 & C_2 \\ \vdots & \vdots \\ \varphi_n & C_n \end{cases}$$

Suppose that we have

$$\varphi_1 = \begin{cases} \mu_1 & D_1 \\ \mu_2 & D_2 \end{cases}$$

Example 3.4.1 (continued)**.**
Then we can rewrite ϕ as:

$$\phi = \begin{cases} \begin{cases} \mu_1 & D_1 \\ \mu_2 & D_2 \end{cases} & C_1 \\ \varphi_2 & C_2 \\ \vdots & \vdots \\ \varphi_n & C_n \end{cases}$$

Suppose that $D_1 \wedge C_1$ is true: then it stands to reason that $\phi = \mu_1$. Likewise, if $D_2 \wedge C_1$ then $\phi = \mu_2$. The behaviour is as normal for each other piece. This also means ϕ can be represented as:

$$\phi = \begin{cases} \mu_1 & C_1 & D_1 \\ \mu_2 & C_1 & D_2 \\ \varphi_2 & C_2 \\ \vdots & \vdots \\ \varphi_n & C_n \end{cases}$$

And so we might go back and forth between these two forms to represent the same object, keeping in mind each column of the piecewise object, other than the first, represents the conditions under which that piece value is taken (the logical 'and').

In general, if we have common, and multiple, conditions for pieces, we should be able to nest, or denest, piecewise objects. This often helps simplify piecewise problems in multiple variables, or single variables with intervals, etc.

3.4.1. Given the following piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

If we have that $C_i \leftrightarrow A_i \wedge B_i$ for some conditions A_i, B_i, then

$$\phi = \{\varphi_i, \quad A_i \wedge B_i \mid i \in I\}$$

This is a semi-obvious fact, but becomes useful when coupled with the grouping/nesting of piecewise objects via their piece's conditions. This was also touched on in Chapter 1.

3.4.2. We generalise Example 3.4.1 to piecewise objects. Given the following piecewise object(s):

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}, \quad \varphi_i = \{\alpha_{i,j}, \quad D_{i,j} \mid j \in J_i\}$$

The following are equivalent:

$$\begin{aligned} \phi &= \{\{\alpha_{i,j}, \quad D_{i,j} \mid j \in J_i\}, \quad C_i \mid i \in I\} \\ \phi &= \{\alpha_{i,j}, \quad C_i \wedge D_{i,j} \mid i \in I, \ j \in J_i\} \end{aligned}$$

As you may have noticed, j is dependent on i in either form of this piecewise object. This can't be avoided notationally without limiting how we might generalise such cases, but in most circumstances this piecewise object can be simplified using only one iterator.

3.4.2 (continued)**.**
Combined with the later-covered function property of piecewise objects, we can add, multiply (and so on), piecewise functions together.

3.4.3. As with 3.4.2, we have a similar property on conditions rather than piece values.

Given the following piecewise objects:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}, \quad C_i = \{C_{i,j}, \quad D_{i,j} \mid j \in J_i\}$$

The following are equivalent:

$$\phi = \{\varphi_i, \quad \{C_{i,j}, \quad D_{i,j} \mid j \in J_i\} \mid i \in I\}$$
$$\phi = \{\varphi_i, \quad C_{i,j} \wedge D_{i,j} \mid i \in I, \ j \in J_i\}$$

This gives us the ability to nest piecewise functions (i.e. piecewise functions as parameters to piecewise functions).

3.5 Combining, splitting piece conditions; logical 'or'

Finally, we'll look at explicit logical 'or' in piece conditions at a basic level and in full generality. This is just as important as the previous section, particularly for grouping.

3.5.1. Given the following piecewise object (e.g. $A_i \vee B_i \leftrightarrow C_i$):

$$\phi = \{\varphi_i, \quad A_i \vee B_i \mid i \in I\}$$

For some conditions A_i, B_i, we can rewrite this as the following:

$$\phi = \{\varphi_i, \quad A_i \mid i \in I\} \cup \{\varphi_i, \quad B_i \mid i \in I\}$$

And vice versa.

We are effectively using the logical 'or' condition in each piece to split up the pieces such that each piece only has one condition in the piecewise object representation. For example,

$$f(x) = \begin{cases} 5 & x \geq 5 \vee x \leq -5 \\ 0 & -5 < x < 5 \end{cases}$$

Is equivalent to:

$$f(x) = \begin{cases} 5 & x \geq 5 \\ 5 & x \leq -5 \\ 0 & -5 < x < 5 \end{cases}$$

Noting that the last piece cannot be split as it is a logical 'and'; that is, $x < 5 \wedge x > -5$.

3.5.2. Finally, there is no rule that says pieces are unique in a piecewise object (although generally redundant, duplicate pieces can be useful for grouping once more).

That is, the following, for some set $J \subseteq I$

3.5.2 (continued)**.**

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

is equivalent to:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\} \cup \{\varphi_j, \quad C_j \mid j \in J\}$$

The reason for this comes down to our well-definition argument as before (equal values).

3.6 Functions on piecewise objects

Arguably the most important property of piecewise objects is the following:

3.6.1. Given the following piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

We have that (when applicable):

$$f(\phi) = \{f(\varphi_i), \quad C_i \mid i \in I\}$$

A quick intuition is as follows: For all $i \in I$ suppose that at least one C_i is true. Then we have that $\phi = \varphi_i \implies f(\phi) = f(\varphi_i)$. Then, since $C_i \to f(\phi) = f(\varphi_i)$, we have by definition the above.

3.6.2. Let $\phi = \{\varphi_i, \quad C_i \mid i \in I\}$ and let a be some element in the appropriate domain. We have that:

1. $\phi + a = \{\varphi_i + a, \quad C_i \mid i \in I\}$ (if $\varphi_i + a$ is defined for $i \in I$)
2. $a \cdot \phi = \{a \cdot \varphi_i, \quad C_i \mid i \in I\}$ (if $a \cdot \varphi_i$ is defined for $i \in I$)

And so on, so forth, where applicable.

Example 3.6.1. Let us define the following function:

$$f(x) = \begin{cases} 3x + 2 & x \geq 1 \\ 5 & x \leq 1 \end{cases}$$

We should establish that this function is, in fact, continuous (although this is implied by conditions $x \geq 1$ and $x \leq 1$). For $x \geq 1$ we have that $f(x) = 3x + 2$ and so $x = 1 \implies f(1) = 5$. Likewise for $x \leq 1$ we have $x = 1 \implies f(1) = 5$.

We want to rewrite this function as a linear combination (and transformation) of the following function:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Note that $f(x)$ has conditions $x \geq 1$ and $x \leq 1$, which differs from the conditions in $|x|$. Therefore, we might rewrite these conditions, namely that $x \geq 1 \implies x - 1 \geq 0$ and $x \leq 1 \implies x - 1 \leq 0$. From here, let us define $y = x - 1 \implies x = y + 1$ and substitute into $f(x)$:

Example 3.6.1 (continued)**.**

$$f(y+1) = \begin{cases} 3(y+1)+2 & y \geq 0 \\ 5 & y \leq 0 \end{cases}$$

From here, we should focus on the piece values themselves: Since they differ from the form we want, we should manipulate them until we achieve something that looks like $|y|$. Namely:

Let us expand the first piece of $f(y+1)$:

$$f(y+1) = \begin{cases} 3y+5 & y \geq 0 \\ 5 & y \leq 0 \end{cases}$$

Noting that we have $+5$ as a constant term in each piece, let us 'extract' that using our function property:

$$f(y+1) = \begin{cases} 3y & y \geq 0 \\ 0 & y \leq 0 \end{cases} + 5$$

We're not sure what to do here - we want y in the first piece and $-y$ in the latter, so let us 'extract' the coefficient of y again by the same property:

$$f(y+1) = 3\begin{cases} y & y \geq 0 \\ 0 & y \leq 0 \end{cases} + 5$$

Nearly there. Notice that y is the same as $2y - y$ – that is, multiply by 2, then subtract a y. If we apply this process to the latter piece as well (as is important) we notice that $2 \cdot 0 - y = -y$, which is what we want. Therefore:

$$\begin{aligned} f(y+1) &= 3\begin{cases} \frac{1}{2}2y & y \geq 0 \\ 0 & y \leq 0 \end{cases} + 5 \\ &= \frac{3}{2}\begin{cases} y+y & y \geq 0 \\ -y+y & y \leq 0 \end{cases} + 5 \\ &= \frac{3}{2}\left(\begin{cases} y & y \geq 0 \\ -y & y \leq 0 \end{cases} + y\right) + 5 \end{aligned}$$

Notice now that we have $f(y+1) = \frac{3}{2}\left(|y| + y\right) + 5$. Substituting $y = x - 1$ back in, we have:

$$f(x) = \frac{3}{2}\left(|x-1| + x - 1\right) + 5$$

This is not the easiest method of rewriting such piecewise functions, nor is it even remotely scalable. But I would advise you to come up with your own problem with 2 linear functions such as the above (e.g. $4x + 10$ for $x \leq -2$ and 2 for $x \geq -2$) and perform the same process, thinking about why this works, and what sort of linear functions work in this way. Also, consider attempting an example with non-linear functions (and see what the result is). Such examples will help you garner an intuition for manipulating multiple pieces at once.

Example 3.6.2. Consider the following truth table:

Example 3.6.2 (continued).

x	y	$x \wedge y$
1	1	1
1	0	0
0	1	0
0	0	0

As it turns out, we can represent $x \wedge y$ as a piecewise function. That is, $x \wedge y : \{0,1\} \times \{0,1\} \to \{0,1\}$ such that:

$$x \wedge y = \begin{cases} 1 & x = 1 \quad y = 1 \\ 0 & x = 1 \quad y = 0 \\ 0 & x = 0 \quad y = 1 \\ 0 & x = 0 \quad y = 0 \end{cases}$$

Furthermore, we can perform some fancy algebra magic by nesting pieces (logical 'and'); we create one piece for when $x = 1$ and one for $x = 0$:

$$x \wedge y = \begin{cases} \begin{cases} 1 & y = 1 \\ 0 & y = 0 \end{cases} & x = 1 \\ \begin{cases} 0 & y = 1 \\ 0 & y = 0 \end{cases} & x = 0 \end{cases}$$

By observation, we might substitute each piece inside the first nested piecewise function with y. In our latter nested piecewise function, we simply have 0 for all possible values of y. Therefore:

$$x \wedge y = \begin{cases} \begin{cases} y & y = 1 \\ y & y = 0 \end{cases} & x = 1 \\ 0 & x = 0 \end{cases}$$

And performing the same argument as before on our final nested piecewise function, we have:

$$x \wedge y = \begin{cases} y & x = 1 \\ 0 & x = 0 \end{cases}$$

Luckily for us, this is fairly straightforward to simplify: 'extract' y, and then substitute the values with x. This gives:

$$x \wedge y = y \cdot \begin{cases} x & x = 1 \\ x & x = 0 \end{cases}$$

And since we have piece values x for all possible values of x, we get:

$$x \wedge y = xy$$

Example 3.6.3. Let us define the following functions:

$$f(x) = \begin{cases} x & x \geq 1 \\ 1 & x \leq 1 \end{cases}$$

$$g(x) = \begin{cases} x^2 & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Example 3.6.3 (continued)**.**
We wish to evaluate $(f \circ g)(x)$. To evaluate $(f \circ g)(x)$, or $f(g(x))$, we evaluate each piece in terms of $g(x)$ (rather than x):

$$(f \circ g)(x) = \begin{cases} g(x) & g(x) \geq 1 \\ 1 & g(x) \leq 1 \end{cases}$$

Using the definition of $g(x)$ we can evaluate the piece conditions:

$$(f \circ g)(x) = \begin{cases} g(x) & \begin{cases} x^2 & x \geq 0 \\ -x & x \leq 0 \end{cases} \geq 1 \\ 1 & \begin{cases} x^2 & x \geq 0 \\ -x & x \leq 0 \end{cases} \leq 1 \end{cases}$$

Using our 'and' property, we denest the piecewise objects inside the conditions and separate the pieces accordingly:

$$(f \circ g)(x) = \begin{cases} g(x) & x^2 \geq 1 & x \geq 0 \\ g(x) & -x \geq 1 & x \leq 0 \\ 1 & x^2 \leq 1 & x \geq 0 \\ 1 & -x \leq 1 & x \leq 0 \end{cases}$$

Evaluating $g(x)$ in each piece (per each piece's conditions) we have that:

$$(f \circ g)(x) = \begin{cases} x^2 & x^2 \geq 1 & x \geq 0 \\ -x & -x \geq 1 & x \leq 0 \\ 1 & x^2 \leq 1 & x \geq 0 \\ 1 & -x \leq 1 & x \leq 0 \end{cases}$$

We can now simplify the conditions of each piece as follows:

1. $x^2 \geq 1 \land x \geq 0 \implies x \geq 1$,
2. $-x \geq 1 \land x \leq 0 \implies x \leq -1$,
3. $x^2 \leq 1 \land x \geq 0 \implies 0 \leq x \leq 1$,
4. $-x \leq 1 \land x \leq 0 \implies -1 \leq x \leq 0$

Using these simplifications, we have:

$$(f \circ g)(x) = \begin{cases} x^2 & x \geq 1 \\ -x & x \leq -1 \\ 1 & 0 \leq x \leq 1 \\ 1 & -1 \leq x \leq 0 \end{cases}$$

Furthermore, since the values of the third and fourth pieces are identical, we can use our 'or' property to combine them:

$$(f \circ g)(x) = \begin{cases} x^2 & x \geq 1 \\ -x & x \leq -1 \\ 1 & -1 \leq x \leq 1 \end{cases}$$

4 Common Piecewise Functions

So far we've really only used, rather than having properly introduced, the function $|x|$. In this section we'll define a set of piecewise functions that crop up fairly frequently, in many areas of maths, and how we might apply them to the algebra-related problems we've encountered already.

4.1 Absolute value function

The absolute value function $|x|$ is a real function defined as follows:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Alternatively, you might see $|x|$ notated using $\sqrt{x^2}$ for real x and principal square root. This is something which will be introduced and explained in the next section, although $\sqrt{x^2}$ is only one 'representation' of $|x|$.

4.2 The maximum function

The max of two variables is a function defined as:

$$\max\{a, b\} = \begin{cases} a & a \geq b \\ b & a \leq b \end{cases}$$

That is, it is a function whose purpose is two return the larger of two numbers. It can also be written as $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$.

Proof. We can derive this representation of $\max\{a, b\}$ in much the same way as Example 3.6.1:

$$\begin{aligned}
\max\{a, b\} &= \begin{cases} a & a \geq b \\ b & a \leq b \end{cases} \\
&= \begin{cases} a - b & a - b \geq 0 \\ 0 & a - b \leq 0 \end{cases} + b \\
&= \frac{1}{2}\left(\begin{cases} 2(a - b) & a - b \geq 0 \\ 0 & a - b \leq 0 \end{cases}\right) + b \\
&= \frac{1}{2}\left(a - b + \begin{cases} a - b & a - b \geq 0 \\ b - a & a - b \leq 0 \end{cases}\right) + b \\
&= \frac{1}{2}(a - b + |a - b|) + b \\
&= \frac{1}{2}(a + b + |a - b|)
\end{aligned}$$

□

Topping this off, we have that $|x| = \max\{x, -x\}$.

We might also define an extension of the max function that takes in a finite set; let $\max\{a\} = a$. Then, for a set S, we have that:

$$\max(S) = \{x, \quad x \geq \max(S \setminus \{x\}) \mid x \in S\}$$

This helps us to derive formulas for $\max\{a, b, c\}$ and so on. Really, in full generality (and slightly out of scope), the max function can be defined as $\sup(S)$ if and only if $\sup(S) \in S$ (otherwise it does not exist at all).

4.2.1. The union property of the max function is as follows:

$$\max(U \cup V) = \max\{\max(U), \max(V)\}$$

Proof. By definition, the right hand side can be written as:

$$\max\{\max(U), \max(V)\} = \begin{cases} \max(U) & \max(U) \geq \max(V) \\ \max(V) & \max(V) \geq \max(U) \end{cases}$$

We let $x = \max(U \cup V)$. Therefore, $x \geq \max(U)$ and $x \geq \max(V)$, and so we have two cases:

1. $x \in U \implies x = \max(U)$. Since this means $x = \max(U) \geq \max(V)$, we have that $\max\{\max(U), \max(V)\} = x$.
2. $x \in V \implies x = \max(V)$. Since this means $x = \max(V) \geq \max(U)$, we have that $\max\{\max(U), \max(V)\} = x$.

We therefore have that $x = \max\{\max(U), \max(V)\}$. □

This property means we can explicitly write the following, for example:

$$\max\{a, b, c\} = \max\{\max\{a, b\}, c\}$$

(and hence write it in terms of the absolute value function)

4.2.2. It is worth noting that $f(\max(S)) \neq \max(f(S))$, where $f(x)$ is a function and $f(S)$ is $f(x)$ applied elementwise over the set S. However:

1. $\max(S) + a = \max(S + a)$ where $S + a$ is elementwise addition by a over S.
2. $c \cdot \max(S) = \max(c \cdot S)$ where $c \geq 0$ and $c \cdot S$ is elementwise multiplication by c over S.
3. $c \cdot \max(S) = \min(c \cdot S)$ where $c \leq 0$ and $c \cdot S$ is elementwise multiplication by c over S.

Each of these properties can be proven inductively, and such a proof is left to the reader.

4.3 The minimum function

The min of two variables is a function defined as:

$$\min\{a, b\} = \begin{cases} b & a \geq b \\ a & a \leq b \end{cases}$$

That is, it is a function whose purpose is two return the smaller of two numbers. It can also be written as $\max\{a, b\} = \frac{1}{2}(a + b - |a - b|)$. Such a proof of this is identical to the one given in Proof 4.2.1, and you would be encouraged, as the reader, to attempt it explicitly as a personal exercise.

We might also define an (identically motivated) extension of the min function that takes in a finite set; let $\min\{a\} = a$. Then, for a set S, we have that:

$$\min(S) = \{x, \quad x \leq \min(S \setminus \{x\}) \mid x \in S\}$$

This helps us to derive formulas for $\min\{a, b, c\}$ and so on. Really, in full generality (and slightly out of scope), the min function can be defined as $\inf(S)$ if and only if $\inf(S) \in S$ (otherwise it does not exist at all).

4.3.1. The union property of the min function is as follows:

$$\min(U \cup V) = \min\{\min(U), \min(V)\}$$

Proof. By definition, the right hand side can be written as:

$$\min\{\max(U), \max(V)\} = \begin{cases} \min(V) & \min(U) \geq \min(V) \\ \min(U) & \min(V) \geq \min(U) \end{cases}$$

We let $x = \min(U \cup V)$. Therefore, $x \leq \min(U)$ and $x \leq \min(V)$, and so we have two cases:

1. $x \in U \implies x = \min(U)$. Since this means $x = \min(U) \leq \min(V)$, we have that $\min\{\min(U), \min(V)\} = x$.
2. $x \in V \implies x = \min(V)$. Since this means $x = \min(V) \leq \min(U)$, we have that $\max\{\min(U), \min(V)\} = x$.

We therefore have that $x = \min\{\min(U), \min(V)\}$. $\square$

This property means we can explicitly write the following, for example:

$$\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$$

(and hence write it in terms of the absolute value function)

4.3.2. Just as with the max function, it is worth noting that $f(\min(S)) \neq \min(f(S))$, where $f(x)$ is a function and $f(S)$ is $f(x)$ applied elementwise over the set S. However:

1. $\min(S) + a = \min(S + a)$ where $S + a$ is elementwise addition by a over S.
2. $c \cdot \min(S) = \min(c \cdot S)$ where $c \geq 0$ and $c \cdot S$ is elementwise multiplication by c over S.
3. $c \cdot \min(S) = \max(c \cdot S)$ where $c \leq 0$ and $c \cdot S$ is elementwise multiplication by c over S.

Each of these properties can be proven inductively, and such a proof is left to the reader.

4.4 Mixed maximum and minimum

It is here we note some identities of max and min but also extend explicitly our ability to manipulate piecewise functions which would otherwise elude us, per Example 3.6.1.

Example 4.4.1. We wish to represent the following function in terms of functions such as max and min:

$$f(x) = \begin{cases} \sin(x) & x \geq \pi \\ \pi - x & x \leq \pi \end{cases}$$

Notice that $x \geq \pi \iff \max\{x, \pi\} = x$ and $x \leq \pi \iff \min\{x, \pi\} = x$, and so we can provide substitutions for our conditions as so:

$$f(x) = \begin{cases} \sin(x) & \max\{x, \pi\} = x \\ \pi - x & \min\{x, \pi\} = x \end{cases}$$

Example 4.4.1 (continued).
Furthermore, we have now have substitutions for x in each piece; in each piece value, we perform this substitution for x:

$$f(x) = \begin{cases} \sin(\max\{x, \pi\}) & \max\{x, \pi\} = x \\ \pi - \min\{x, \pi\} & \min\{x, \pi\} = x \end{cases}$$

You may be wondering what the point of this is — well, now that we've performed these substitutions that are only true under each piece, we can 'subtract' each piece out (that is, add and substract using our function property):

$$f(x) = \begin{cases} 0 - (\pi - \min\{x, \pi\}) & \max\{x, \pi\} = x \\ 0 - \sin(\max\{x, \pi\}) & \min\{x, \pi\} = x \end{cases}$$
$$+ \sin(\max\{x, \pi\}) + \pi - \min\{x, \pi\}$$

Now, we might evaluate each piece; notice that $\min\{x, \pi\} = \pi$ for $x \geq \pi$ and also that $\max\{x, \pi\} = \pi$ for $x \leq \pi$. Therefore we have:

$$f(x) = \begin{cases} 0 & \max\{x, \pi\} = x \\ 0 & \min\{x, \pi\} = x \end{cases}$$
$$+ \sin(\max\{x, \pi\}) + \pi - \min\{x, \pi\}$$

Finally, simplifying, we're left with:

$$f(x) = \sin(\max\{x, \pi\}) - \min\{x, \pi\} + \pi$$

4.4.1. We have the following basic identities to work with using max and min:

1. $\max\{a, b\} + \min\{a, b\} = a + b$; this result can be proven by definition, or using the max and min formulations in terms of the absolute value function.
2. $\max\{a, b\} - \min\{a, b\} = |a - b|$; this result can be similarly proven as above.
3. $|x| = \max\{x, -x\} = -\min\{x, -x\}$.

4.4.1 Clamping function

The clamping function is a function which restricts a number between an upper and lower bound, as per its definition:

$$\ell_a^b(x) = \begin{cases} b & x \geq b \\ x & a \leq x \leq b \\ a & x \leq a \end{cases}$$

We use this symbol to represent the clamping function as later on it will be given more usage, so it will be useful to have a quick and easy tool (also, this is LaTeX; my hboxes aren't infinite).

4.4.2. We first provide some properties of the clamping function:

1. $\ell_a^\infty(x) = \lim_{t\to\infty} \ell_a^t(x) = \max\{x, a\}$
2. $\ell_{-\infty}^b(x) = \lim_{t\to\infty} \ell_{-t}^b = \min\{x, b\}$
3. $-\ell_a^b(x) = \ell_{-b}^{-a}(-x)$

4.4.2 (continued)**.**

4. $c \cdot \ell_a^b(x) = \ell_{ac}^{bc}(cx)$, for $c \geq 0$
5. $\ell_a^b(x) + k = \ell_{a+k}^{b+k}(x+k)$

4.4.3. The clamping function can be written as any of the following:

1. $\ell_a^b(x) = \min\{\max\{x, a\}, b\}$
2. $\ell_a^b(x) = \max\{\min\{x, b\}, a\}$
3. $\ell_a^b(x) = \frac{1}{2}\left(a + b + |x - a| - |x - b|\right)$

We shall give proofs of the first and last of these formulations:

Proof. We begin by using the definition of $\ell_a^b(x)$:

$$\ell_a^b(x) = \begin{cases} b & x \geq b \\ x & a \leq x \leq b \\ a & x \leq a \end{cases}$$

We nest a piece under the conditions $x \geq a$ and $x \leq a$ in order to simplify the conditions we're working with:

$$\begin{aligned} \ell_a^b(x) &= \begin{cases} \begin{cases} b & x \geq b \\ x & x \leq b \end{cases} & x \geq a \\ a & x \leq a \end{cases} \\ &= \begin{cases} \min\{x, b\} & x \geq a \\ a & x \leq a \end{cases} \end{aligned}$$

Now using the definition of $\max\{x, a\}$ we substitute $x \geq a$ with $\max\{x, a\} = x$ and likewise with $x \leq a$, to give:

$$\ell_a^b(x) = \begin{cases} \min\{x, b\} & \max\{x, a\} = x \\ a & \max\{x, a\} = a \end{cases}$$

We then use the first piece's condition to substitute the value of x with $\max\{x, a\}$,

$$\ell_a^b(x) = \begin{cases} \min\{\max\{x, a\}, b\} & \max\{x, a\} = x \\ a & \max\{x, a\} = a \end{cases}$$

Substracting out $\min\{\max\{x, a\}, b\}$ gives us

$$\ell_a^b(x) = \begin{cases} 0 & x \geq a \\ a - \min\{\max\{x, a\}, b\} & x \leq a \end{cases} + \min\{\max\{x, a\}, b\}$$

And evaluating the second piece (since $x \leq a \leq b$) gives us 0, leaving us with

$$\ell_a^b(x) = \min\{\max\{x, a\}, b\}$$

□

4.4.3 (continued).

Proof. We now set out to prove the last representation of $\ell_a^b(x)$.

Using our previous derivation, we apply the addition property of min to give us:

$$\min\{\max\{x,a\},b\} = \min\{0, b - \max\{x,a\}\} + \max\{x,a\}$$

Which simplifies to:

$$\begin{aligned}\ell_a^b(x) &= \min\{0, b + \min\{-x,-a\}\} + \max\{x,a\}\\ &= \min\{0, \min\{b-x, b-a\}\} + \max\{x,a\}\\ &= \min\{0, b-x, b-a\} + \max\{x,a\}\end{aligned}$$

Since $b - a \geq 0$, we're left with:

$$\ell_a^b(x) = \min\{b-x, 0\} + \max\{x,a\}$$

Using the absolute value representations of max and min and simplifying, we get that

$$\ell_a^b(x) = \frac{1}{2}\left(a + b + |x-a| - |x-b|\right)$$

□

Example 4.4.2. We want to consider the following function, rewriting it in terms of max, min:

$$f(x) = \begin{cases} x & x > 1 \\ 1 & -1 \leq x \leq 1 \\ -x & x < -1 \end{cases}$$

This is not in the form we want in order to manipulate it, so we consider that this function is, in fact, continuous. It stands to reason, therefore, that $x > 1 \iff x \geq 1$ and $x < -1 \iff x \leq -1$. Therefore:

$$f(x) = \begin{cases} x & x \geq 1 \\ 1 & -1 \leq x \leq 1 \\ -x & x \leq -1 \end{cases}$$

Now, we note that $x \geq 1$ is equivalent to $\ell_1^\infty(x) = x$, $-1 \leq x \leq 1$ is equivalent to $\ell_{-1}^1(x) = x$ and $x \leq -1$ is equivalent to $\ell_{-\infty}^{-1}(x) = x$.

With our equivalences, we can now make the respective substitutions in both the conditions and piece values, as with previous examples (although we needn't really keep the conditions in this form):

$$f(x) = \begin{cases} \ell_1^\infty(x) & \ell_1^\infty(x) = x \\ 1 & \ell_{-1}^1(x) = x \\ -\ell_{-\infty}^{-1}(x) & \ell_{-\infty}^{-1}(x) = x \end{cases}$$

Example 4.4.2 (continued).
Now we 'extract' our piece values by addition:

$$f(x) = \left(\begin{cases} -(1 - \ell_{-\infty}^{-1}(x)) & x \geq 1 \\ -(\ell_1^{\infty}(x) - \ell_{-\infty}^{-1}(x)) & -1 \leq x \leq 1 \\ -(\ell_1^{\infty}(x) + 1) & x \leq -1 \end{cases}\right) + \left(\ell_1^{\infty}(x) + 1 - \ell_{-\infty}^{-1}(x)\right)$$

Evaluting each piece value using the definition of the clamping function gives us:

$$f(x) = \left(\begin{cases} -2 & x \geq 1 \\ -2 & -1 \leq x \leq 1 \\ -2 & x \leq -1 \end{cases}\right) + \left(\ell_1^{\infty}(x) + 1 - \ell_{-\infty}^{-1}(x)\right)$$

Finally simplifying, we have:

$$f(x) = \ell_1^{\infty}(x) - \ell_{-\infty}^{-1}(x) - 1$$

Notice that, in this example, we don't have any $\ell_{-1}^{1}(x)$ because, in fact, 1 is a constant function. Rewriting our clamping functions in terms of max and min per the limit identities gives us:

$$f(x) = \max\{x, 1\} - \min\{x, -1\} - 1$$

The clamping function is useful for formulating single expressions for continuous piecewise functions as we've seen above. If the function is not continuous, we may not actually have much luck in using it. In fact, we'll derive the general 'Gluing Formula' soon within these notes.

4.5 The sign function

The sign function is an interesting function in that it doesn't have a single explicit definition, but instead satisfies the following relation:

$$x \cdot \operatorname{sgn}(x) = |x|$$

This can be true for some domain $D \subseteq \mathbb{R}$, such as $\mathbb{R} \setminus \{0\}$. For the purposes of these notes, however, we shall define $\operatorname{sgn}(x)$ as the following:

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

This definition does indeed satisfy the definition of $|x|$, noting that we can write $|x|$ in the following explicit form[1]:

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

It is useful to note that the sign function is discontinuous; we can express discontinuous piecewise functions in terms of the sign function, although this is not common (we instead opt for step functions there, although step functions are just transformations of the sign function).

4.5.1. Given an odd function $f : A \to \mathbb{R}$ where $A = -A$, we have that $f(x)\operatorname{sgn}(x) = f(|x|)$ and $f(|x|)\operatorname{sgn}(x) = f(x)$.

[1]It is a good exercise to reconcile this form of $|x|$ with the previous forms displayed in these notes, using piecewise function properties (and when and why each form may be useful to us).

4.5.1 (continued).

Proof. We begin with the left hand side of the first property and perform the usual steps:

$$\begin{aligned}
f(x)\mathrm{sgn}(x) &= f(x)\begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases} \\
&= \begin{cases} -f(x) & x<0 \\ 0 & x=0 \\ f(x) & x>0 \end{cases} \\
&= \begin{cases} f(-x) & x<0 \\ 0 & x=0 \\ f(x) & x>0 \end{cases} \\
&= f\left(\begin{cases} -x & x<0 \\ 0 & x=0 \\ x & x>0 \end{cases}\right) \\
&= f(|x|)
\end{aligned}$$

□

The third and fourth steps of this proof follows from $f(x)$ being odd; that is, $f(0) = 0$ and $f(-x) = -f(x)$.

Proof. The second property is similar to the first:

$$\begin{aligned}
f(|x|)\mathrm{sgn}(x) &= \begin{cases} f(-x) & x<0 \\ f(0) & x=0 \\ f(x) & x>0 \end{cases} \cdot \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases} \\
&= \begin{cases} -f(-x) & x<0 \\ 0 & x=0 \\ f(x) & x>0 \end{cases} \\
&= \begin{cases} f(x) & x<0 \\ f(x) & x=0 \\ f(x) & x>0 \end{cases} \\
&= f(x)
\end{aligned}$$

□

4.6 Step functions

Step functions are useful functions which, usually in calculus at higher levels, allow us to provide a form in which to express a piecewise function regardless of continuity or not. We look at the Heaviside step function, which, like the sign function, doesn't have a single explicit formulation, but instead satisfies:

$$x \cdot \mathrm{H}(x) = \frac{1}{2}(x + |x|)$$

The right hand side of this relation is known as the ramp function and can be expressed as $\max\{x, 0\}$.

For the purposes of these notes we might define the Heaviside step function as the following:

$$\mathrm{H}(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

4.6.1. We can express $\mathrm{H}(x)$ in terms of $\operatorname{sgn}(x)$;

$$\mathrm{H}(x) = \frac{1}{2}(1 + \operatorname{sgn}(x))$$

This can be shown using the same techniques found in Example 3.6.1.

Example 4.6.1. We wish to express the following function (similar to that found in 4.4.2) in terms of step functions:

$$f(x) = \begin{cases} x & x > 1 \\ 1 & -1 < x < 1 \\ -x & x < -1 \end{cases}$$

In order to simplify this down, let us nest piecewise functions on the conditions $x > -1$ and $x < -1$:

$$f(x) = \begin{cases} \begin{cases} x & x > 1 \\ 1 & x < 1 \end{cases} & x > -1 \\ -x & x < -1 \end{cases}$$

We now wish to write the inner piecewise function in terms of the step function: to do so, we 'subtract' our second piece, 1 out and then factor out $x - 1$, as follows:

$$f(x) = \begin{cases} (x-1)\begin{cases} 1 & x > 1 \\ 0 & x < 1 \end{cases} + 1 & x > -1 \\ -x & x < -1 \end{cases}$$

Therefore, the inner piecewise function is equivalent to $\mathrm{H}(x - 1)$:

$$f(x) = \begin{cases} (x-1)\mathrm{H}(x-1) + 1 & x > -1 \\ -x & x < -1 \end{cases}$$

Instead of repeating this process with the outer piecewise function, we should instead subtract out the first piece, since we have a step function inside that:

$$f(x) = (x-1)\mathrm{H}(x-1) + 1 + \begin{cases} 0 & x > -1 \\ -x - (x-1)\mathrm{H}(x-1) - 1 & x < -1 \end{cases}$$

And from there, evaluate the second piece (wherein $\mathrm{H}(x - 1) = 0$ since $x < -1$):

$$f(x) = (x-1)\mathrm{H}(x-1) + 1 + \begin{cases} 0 & x > -1 \\ -x - 1 & x < -1 \end{cases}$$

We can factor out $(-x - 1)$ again and rewrite $x < -1$ as $-x - 1 > 0$:

$$f(x) = (x-1)\mathrm{H}(x-1) + 1 + (-x-1)\begin{cases} 0 & -x - 1 < 0 \\ 1 & -x - 1 > 0 \end{cases}$$

Example 4.6.0 (continued).
We therefore have that:

$$f(x) = (x-1)\mathrm{H}(x-1) + (-x-1)\mathrm{H}(-x-1) + 1$$

The astute may have noticed that, in fact, this function can be rewritten directly as a result of the definition of the step function:

$$f(x) = \max\{x-1, 0\} + \max\{-x-1, 0\} + 1$$

We have, as a result, that $f(x)$ is continuous (which we already knew from the previous example), as the max function is continuous. Note however that unlike the previous example, even though we can write this function in this form, that $f(-1)$ and $f(1)$ are not defined.

Explicitly, we write $f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}$.

This process is generalisable as with our continuous gluing function (i.e. through repeating nesting). I will, however, leave this process up to the reader.

4.7 The floor and ceiling functions

We've already introduced the floor function once previously, however we shall do so again here for the sake of completeness:

$$\lfloor x \rfloor = \{n, \quad x \in [n, n+1) \mid n \in \mathbb{Z}\}$$

We can alternatively write the floor function using the max function of an infinite set:

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$$

That is, the floor function is the greatest integer n such that n is less than or equal to x.

The ceiling function is similarly defined piecewise;

$$\lceil x \rceil = \{n, \quad x \in (n-1, n] \mid n \in \mathbb{Z}\}$$

or alternatively using the min function:

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$$

That is, the ceiling function is the smallest integer n such that n is larger than or equal to x.

4.8 The modulo operation/function

The modulo operation is an operation used when computing the remainder in division. As such it has several ambiguities regarding the sign of the divisor and remainder. With that being said, for the purposes of these notes we shall define it per the following, for $y > 0$:

$$x \bmod y = \{x - ny, \quad x \in [ny, (n+1)y) \mid n \in \mathbb{Z}\}$$

Which gives the positive remainder for positive divisors.

4.8.1. The modulo operation as defined in these notes can be written as:

$$x \bmod y = x - y\left\lfloor \frac{x}{y} \right\rfloor$$

Proof. We begin by using the definition of $x \bmod y$, and subsequently applying the function property

4.8.1 (continued).
to our piece values to write them in terms of n:

$$x \bmod y = x - y\,\{n, \quad x \in [ny, (n+1)y) \mid n \in \mathbb{Z}\}$$

Notice the piecewise function is close to the definition of the floor function. We rewrite our conditions accordingly:

$$x \bmod y = x - y\left\{n, \quad \frac{x}{y} \in [n, n+1) \;\middle|\; n \in \mathbb{Z}\right\}$$

The piecewise object now matches the definition for $\left\lfloor \frac{x}{y} \right\rfloor$ and therefore we have:

$$x \bmod y = x - y\left\lfloor \frac{x}{y} \right\rfloor$$

(If we like, we can also use this definition for modulo a negative number.) □

4.9 Characteristic functions and Iverson brackets

Briefly, a characteristic (or indicator) function for a set S is a function defined such that:

$$\mathbf{1}_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

4.9.1. The characteristic function for the integers is given by:

$$\mathbf{1}_{\mathbb{Z}}(x) = 1 - (\lceil x \rceil - \lfloor x \rfloor)$$

Proof. We use our piecewise notation for the ceiling and floor functions for convenience (where for the ceiling function we've just reindexed $n \to n+1$ in order to match the conditions):

$$\lceil x \rceil - \lfloor x \rfloor = \{n+1, \quad x \in (n, n+1] \mid n \in \mathbb{Z}\} - \{n, \quad x \in [n, n+1) \mid n \in \mathbb{Z}\}$$

We can separate out the $x = n$ and $x = n+1$ pieces in each:

$$\begin{aligned} \lceil x \rceil - \lfloor x \rfloor = &\{n+1, \quad x \in (n, n+1) \mid n \in \mathbb{Z}\} \cup \\ &\{n+1, \quad x = n+1 \mid n \in \mathbb{Z}\} \\ &- \{n, \quad x \in (n, n+1) \mid n \in \mathbb{Z}\} \cup \{n, \quad x = n \mid n \in \mathbb{Z}\} \end{aligned}$$

Reindexing the $x = n+1$ pieces with $n+1 \to n$, and then combining all pieces for $x \in (n, n+1)$, we have that:

$$\begin{aligned} \lceil x \rceil - \lfloor x \rfloor = & \qquad \{1, \quad x \in (n, n+1) \mid n \in \mathbb{Z}\} \cup \{n, \quad x = n \mid n \in \mathbb{Z}\} \\ & \qquad - \{n, \quad x = n \mid n \in \mathbb{Z}\} \\ = & \qquad \{1, \quad x \in (n, n+1) \mid n \in \mathbb{Z}\} \cup \{0, \quad x = n \mid n \in \mathbb{Z}\} \end{aligned}$$

That is, for all integers n, if $x = n$ (i.e. x is an integer) we have $\lceil x \rceil - \lfloor x \rfloor = 0$. Otherwise, for all

4.9.1 (continued)**.**
non-integer x, we have $\lceil x \rceil - \lfloor x \rfloor = 1$. We therefore have that:

$$\begin{aligned}
\lceil x \rceil - \lfloor x \rfloor &= \begin{cases} 1 & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} \\
&= -\begin{cases} -1 & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} \\
&= 1 - \begin{cases} 0 & x \notin \mathbb{Z} \\ 1 & x \in \mathbb{Z} \end{cases} \\
&= 1 - \mathbf{1}_{\mathbb{Z}}(x)
\end{aligned}$$

□

Iverson bracket notation is almost a generalisation to these characteristic or indicator functions, except instead of using a set and variable, they are provided with any sort of predicate, condition, etc.:

$$[S] = \begin{cases} 1 & S \\ 0 & \neg S \end{cases}$$

Essentially all piecewise functions we work with in these notes (to a point) are able to be written in terms of Iverson brackets. In turn, certain Iverson brackets can be written in terms of elementary functions or functions we've already seen.

4.9.2. We can write several of the previous functions in this section in terms of Iverson brackets:

1. $\mathbf{1}_S(x) = [x \in S]$ (noting that the complement of $x \in S$ is $x \in \mathbb{R} \setminus S$, for example)
2. $\mathrm{H}(x) = [x > 0]$
3. $\mathrm{sgn}(x) = [x > 0] - [x < 0]$
4. $|x| = x[x > 0] - x[x < 0]$
5. $\max\{a, b\} = a[a \geq b] + b[a < b]$
6. $\min\{a, b\} = b[a \geq b] + a[a < b]$

Proofs of these equivalences are left specifically for the reader to practice.

4.9.3. Even more generally, let us denote the piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

Such that $C_i \wedge C_j \leftrightarrow \perp$ for all $i \neq j$ and $i, j \in I$. Then ϕ can be expressed using Iverson bracket notation:

$$\phi = \sum_{i \in I} \varphi_i [C_i]$$

Proof. Let us begin by splitting up the piecewise object ϕ into separate piecewise objects such that all but 1 piece is equal to 0, in each:

$$\{\varphi_i, \quad C_i \mid i \in I\} = \sum_{i \in I} \{\varphi_m, \quad C_m \mid m = i\} \cup \{0, \quad C_n \mid n \in I \setminus \{i\}\}$$

4.9.3 (continued)**.**
Since each C_n is distinct, then $C_n \leftrightarrow \neg C_i$ for $n \neq i$. Therefore:

$$\{\varphi_i, \quad C_i \mid i \in I\} = \sum_{i \in I} \{\varphi_m, \quad C_m \mid m = i\} \cup \{0, \quad \neg C_i \mid n \in I \setminus \{i\}\}$$

Since each piecewise object has two pieces only, we can rewrite this as:

$$\{\varphi_i, \quad C_i \mid i \in I\} = \sum_{i \in I} \begin{cases} \varphi_i & C_i \\ 0 & \neg C_i \end{cases}$$

Then, factoring φ_i and writing in terms of Iverson brackets, we have our result. $\square$

5 Further Piecewise Algebra

Until now, we've been focusing on ways we can use existing functions to motivate forms for our own constructions. It turns out there are far simpler methods of deriving what we have before using an interpolation-style approach with piecewise notation (which do in fact yield two important interpolation polynomials; Newton polynomials and Lagrange polynomials).

5.1 Anonymous piecewise objects

The name anonymous piecewise object is more or less misleading, but we use these objects as a basis for extending the definition of a piecewise object where not defined. Namely, given a piecewise object, we can formulate an expression which defines the behaviour of the piecewise object overall.

When we describe an anonymous piecewise object, we're actually describing a (very infinite) set of possible objects which we can use to represent an existing piecewise object. We simply denote this by $\star$ inside a piecewise object for either conditions, piece values or both (though when in a condition, it's effectively equivalent to the condition 'otherwise').

Example 5.1.1. We wish to denote a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(1) = 1$ using an anonymous piecewise function. We do so by writing the following:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ \star & \star \end{cases}$$

This represents a set of functions f such that the above is satisfied. Also, by observation, a function $f(x) = x$ would satisfy this.

We can define, explicitly, $f(x) \in S$ where $S = \{f \in \mathcal{F} \mid f(0) = 0 \wedge f(1) = 1\}$ where $\mathcal{F}$ is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.

Any time we start manipulating such objects as will be discussed in this section, we reduce the number of functions that satisfy the piecewise definition provided. Generally, anonymous piecewise objects will be denoted (if the respective piecewise object is ϕ) $\phi^\star$. This will be explored further. However, for the purposes of most of the working here, we *minimally* interpolate.

5.2 Polynomial extraction

We should distinguish initially between the sort of 'extractions' we can perform on piecewise objects. The first of which is polynomial extraction, as we rely on the roots of a polynomial. In our case, these roots are formed by the pieces of our piecewise objects.

The premise of this extraction is as follows: We can reduce the number of pieces of a piecewise object by vanishing any number of those pieces, and 'extracting'. This extraction is done by means of multiplication and addition on the respective piece conditions.

Example 5.2.1. Let us use the logical NOT function; that is, $f : \{0, 1\} \to \{0, 1\}$:

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x = 1 \end{cases}$$

Recall that previously we would have manipulated the piece values to match the conditions so as to simplify this piecewise function. Instead, we're going to 'extract' the root corresponding to $x = 1$;

Example 5.2.1 (continued).
$x-1$ (since that piece is 0):

$$f(x) = (x-1)\begin{cases} \frac{1}{x-1} & x = 0 \\ 0 & x = 1 \end{cases}$$

Since the entire expression is 0 when $x = 1$ outside of the piecewise object, we can remove the piece when $x = 1$ inside the piecewise object (adding in $\star$ to denote all other pieces), thereby leaving us with:

$$f(x) = (x-1)\begin{cases} \frac{1}{x-1} & x = 0 \\ \star & \star \end{cases}$$

Evaluating the piece value for when $x = 0$ (using a substitution for x), we now have:

$$f(x) = (x-1)\begin{cases} -1 & x = 0 \\ \star & \star \end{cases}$$

Since there's only one piece we can use left, we reduce:

$$f(x) = (x-1)(-1)$$

And so $f(x) = 1 - x$. Or rather, $1 - x \in S$ where S is the set of all functions from $\{0, 1\} \to \{0, 1\}$ (though for simplicity we shall just say the former going forward).

As a disclaimer, this is only one possible function that can satisfy the above conditions. The reason we don't have all possible relevant functions as the result is because when we reduced our pieces (not when we extracted the root, because they are equivalent forms), we eliminated the $x = 1$ piece explicitly. Likewise, when we reduced the final piecewise object down to a constant, we eliminated any remaining cases.

Example 5.2.2. We revisit the absolute value formula for max once again. Recall the piecewise definition of this function per 4.2.

We wish to derive the absolute value formula for max:

$$\max\{a, b\} = \begin{cases} a & a \geq b \\ b & a \leq b \end{cases}$$

Recall that $a \geq b \iff |a-b| = a-b$ and $a \leq b \iff |a-b| = b-a$. We perform this substitution on these conditions, leaving us with:

$$\max\{a, b\} = \begin{cases} a & |a-b| = a-b \\ b & |a-b| = b-a \end{cases}$$

Subtracting b out and 'extracting' the second piece, we get:

$$\max\{a, b\} = b + (|a-b| + a - b)\begin{cases} \frac{a-b}{|a-b|+a-b} & |a-b| = a-b \\ \star & \star \end{cases}$$

Example 5.2.2 (continued)**.**
Evaluating the remaining piece value, we have:

$$\max\{a,b\} = b + (|a-b| + a - b)\begin{cases} \frac{a-b}{2(a-b)} & |a-b| = a-b \\ \star & \star \end{cases}$$

Simplifying:

$$\begin{aligned} \max\{a,b\} &= b + \frac{1}{2}(|a-b| + a - b) \\ &= \frac{1}{2}(a + b + |a-b|) \end{aligned}$$

Example 5.2.3. Let us revisit the Heaviside step function:

$$\mathrm{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

To rewrite this function in terms of $\mathrm{sgn}(x)$, recall that $x > 0 \iff \mathrm{sgn}(x) = 1$ and likewise $x < 0 \iff \mathrm{sgn}(x) = -1$:

$$\mathrm{H}(x) = \begin{cases} 1 & \mathrm{sgn}(x) = 1 \\ 0 & \mathrm{sgn}(x) = -1 \end{cases}$$

Extracting the second piece, we have:

$$\mathrm{H}(x) = (\mathrm{sgn}(x) + 1)\begin{cases} \frac{1}{\mathrm{sgn}(x)+1} & \mathrm{sgn}(x) = 1 \\ \star & \star \end{cases}$$

Evaluating the remaining piece and simplifying gives:

$$\mathrm{H}(x) = \frac{\mathrm{sgn}(x) + 1}{2}$$

An interesting byproduct of this is that should $\mathrm{H}(0)$ be defined, this interpolation would give $\mathrm{H}(0) = \frac{1}{2}$.

Furthermore if $|x|$ were used instead of $\mathrm{sgn}(x)$ we would have that:

$$\mathrm{H}(x) = \frac{|x| + x}{2x}$$

While we can continue to extract pieces progressively and do the appropriate manipulation to simplify, there are 2 slightly different processes we can perform which yield Newton and Lagrange polynomials.

5.2.1 Newton polynomials

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x_m) = y_m$ for $m \in \{0, 1, 2, \dots, n\}$. Then f can be given by a Newton polynomial:

$$f(x) = \sum_{m=0}^{n} [y_0, \dots, y_m] \prod_{l=0}^{m-1} (x - x_l)$$

Where $[y_0, \dots, y_m]$ denotes divided differences. That is, $[y_0] = y_0$, $[y_0, y_1] = \frac{y_1 - y_0}{x_1 - x_0}$, $[y_0, y_1, y_2] = \frac{[y_1, y_2] - [y_0, y_1]}{x_2 - x_0}$, and so on.

5.2.1. A Newton polynomial is formed by nesting pieces inside a piecewise object and recursively interpolating. Setting up the interpolation problem, we have:

$$f(x) = \begin{cases} y_0 & x = x_0 \\ y_1 & x = x_1 \\ \vdots & \vdots \\ y_n & x = x_n \\ \star & \star \end{cases}$$

Let us denote $f(x) = f_n(x)$ and then nest pieces $x = x_0, x_2, \ldots, x_{n-1}$:

$$f(x) = \begin{cases} \begin{cases} y_0 & x = x_0 \\ y_1 & x = x_1 \\ \vdots & \vdots \\ y_{n-1} & x = x_{n-1} \\ \star & \star \end{cases} & (x - x_0)(x - x_1)\ldots(x - x_{n-1}) = 0 \\ y_n & x = x_n \\ \star & \star \end{cases}$$

How we achieved the condition on the nested piecewise function will be discussed later. Let us denote the nested piecewise function $f_{n-1}(x)$ such that:

$$f_n(x) = \begin{cases} f_{n-1}(x) & (x - x_0)(x - x_1)\ldots(x - x_{n-1}) = 0 \\ y_n & x = x_n \\ \star & \star \end{cases}$$

Now, we have a recurrence relation in $f_n(x)$, given that $f_0(x) = y_0$ (since we a constant function satisfies the definition in this case). Extracting the first piece, containing the recurrence, we have:

$$f_n(x) = f_{n-1}(x) + \left(\prod_{m=0}^{n-1} (x - x_m)\right) \begin{cases} \frac{y_n - f_{n-1}(x)}{\prod_{m=0}^{n-1} (x - x_m)} & x = x_n \\ \star & \star \end{cases}$$

Evaluating the last piece value and simplifying, we get:

$$\begin{aligned} f_n(x) &= f_{n-1}(x) + \frac{y_n - f_{n-1}(x_n)}{\prod_{m=0}^{n-1} (x_n - x_m)} \prod_{m=0}^{n-1} (x - x_m) \\ f_0(x) &= y_0 \end{aligned}$$

Example 5.2.4. We wish to find a Newton polynomial f corresponding to $f(0) = 0$, $f(1) = 1$ and $f(2) = 4$.

We begin by writing this problem piecewise:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ 4 & x = 2 \\ \star & \star \end{cases}$$

Example 5.2.4 (continued)**.**
We now nest:

$$f(x) = \begin{cases} 0 & x = 0 \\ \begin{cases} 1 & x = 1 \\ 4 & x = 2 \\ \star & \star \end{cases} & (x-1)(x-2) = 0 \\ \star & \star \end{cases}$$

We needn't do this again for our nested piecewise object, but we should instead focus on finding a Newton polynomial form of it:

$$\begin{cases} 1 & x = 1 \\ 4 & x = 2 \\ \star & \star \end{cases} = 1 + (x-1)\begin{cases} \frac{3}{1} & x = 2 \\ \star & \star \end{cases} = 1 + 3(x-1)$$

Substituting this into $f(x)$ we have:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 + 3(x-1) & (x-1)(x-2) = 0 \\ \star & \star \end{cases}$$

Performing an extraction on the second piece once again we get:

$$f(x) = 1 + 3(x-1) + (x-1)(x-2)\begin{cases} \frac{-(1+3(x-1))}{(x-1)(x-2)} & x = 0 \\ \star & \star \end{cases}$$

We now evaluate the remaining piece, $x = 0$ gives 1, and simplify:

$$f(x) = 1 + 3(x-1) + (x-1)(x-2)$$

And this gives us a solution for our initial interpolation problem. Also notice that if $f(x)$ is simplified, it gives x^2.

5.2.2 Lagrange polynomials

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x_m) = y_m$ for $m \in \{0, 1, 2, \ldots, n\}$. Then f can be given by a Lagrange polynomial:

$$f(x) = \sum_{m=0}^{n} y_m \prod_{\substack{l=0 \\ l \neq m}}^{n} \frac{x - x_l}{x_m - x_l}$$

This form might seem familiar; recall 4.9.3. The form of this sum looks similar to the Iverson form of a piecewise object, except that each polynomial term in the sum approximates the Iverson bracket $[x = x_m]$ for some m. This is not, in fact, coincidence (and will be covered soon).

5.2.2. Lagrange polynomials can be formed by separating each piece into its own piecewise object with roots at every other piece, and then interpolating. Setting up our interpolation problem, we

5.2.2 (continued).
have:

$$f(x) = \begin{cases} y_0 & x = x_0 \\ y_1 & x = x_1 \\ \vdots & \vdots \\ y_n & x = x_n \end{cases}$$

We can now separate our pieces, similarly to 4.9.3:

$$f(x) = \begin{cases} y_0 & x = x_0 \\ 0 & x = x_1 \\ \vdots & \vdots \\ 0 & x = x_n \\ \star & \star \end{cases} + \begin{cases} 0 & x = x_0 \\ y_1 & x = x_1 \\ \vdots & \vdots \\ 0 & x = x_n \\ \star & \star \end{cases} + \cdots + \begin{cases} 0 & x = x_0 \\ 0 & x = x_1 \\ \vdots & \vdots \\ y_n & x = x_n \\ \star & \star \end{cases}$$

From here, we can interpolate each anonymous piecewise object independently, by extracting each zero piece and simplifying, giving us the Lagrange polynomial as needed.

5.2.3 General solutions

As we've seen so far, we've only been producing one solution of a polynomial kind for a set of values for which we wish to interpolate or create an expression for. In general, when we interpolate or perform these extractions, we produce an object in simplified form akin to the following:

$$\phi^\star = \mu + [\phi] \cdot r$$

Where μ is some solution to the problem, $[\phi]$ is an object which vanishes at the conditions given to us, called a decider (i.e. when those conditions are true), and r is some arbitrary object which generalises our single solution.

5.2.3. For a polynomial extraction-based anonymous piecewise object problem, we define $\phi^\star$ as the following:

$$\phi^\star = \begin{cases} \phi & \phi \text{ is defined} \\ \star & \phi \text{ is not defined} \end{cases}$$

There are an infinite number of ways to interpret this, however let us define μ such that μ is a 'solution' to $\phi^\star$ (that is, it satisfies the definition as above):

$$\mu = \begin{cases} \phi & \phi \text{ is defined} \\ \star & \phi \text{ is not defined} \end{cases}$$

Therefore, since $\mu = \phi$ for when ϕ is defined, we can write $\phi^\star$ as:

$$\phi^\star = \mu + \begin{cases} 0 & \phi \text{ is defined} \\ \star & \phi \text{ is not defined} \end{cases}$$

Let $[\phi]$ be a 'decider' object of this problem (that is, it vanishes where ϕ is defined). Then we're left with, for some arbitrary r:

$$\phi^\star = \mu + [\phi] \cdot r$$

5.2.4. The decider of a polynomial interpolation problem with points $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$ can be given by $\prod_{m=0}^{n}(x - x_m)$.

Proof. We denote the corresponding function F as per the following:

$$F(x) = \begin{cases} y_0 & x = x_0 \\ y_1 & x = x_1 \\ \vdots & \vdots \\ y_n & x = x_n \end{cases}$$

We can rewrite this function as the following:

$$F(x) = \begin{cases} y_0 & x = x_0 \\ y_1 & x = x_1 \\ \vdots & \vdots \\ y_n & x = x_n \\ \star & \star \end{cases} + \begin{cases} 0 & x = x_0 \\ 0 & x = x_1 \\ \vdots & \vdots \\ 0 & x = x_n \\ \star & \star \end{cases}$$

Extracting the pieces from the second piecewise function, we have:

$$F(x) = \begin{cases} y_0 & x = x_0 \\ y_1 & x = x_1 \\ \vdots & \vdots \\ y_n & x = x_n \\ \star & \star \end{cases} + \begin{cases} \star & \star \cdot \prod_{m=0}^{n}(x - x_m) \end{cases}$$

Noting that the first piecewise object can be considered a solution, f, to the original problem, and the latter piecewise object is arbitrary, e.g. r, we have:

$$F(x) = f(x) + r(x) \cdot \prod_{m=0}^{n}(x - x_m)$$

Per our definition, we have $\prod_{m=0}^{n}(x - x_m)$ as our decider. □

5.2.5. Let us define a polynomial interpolation problem with the points $\{(x_0, y_0), \ldots, (x_n, y_n)\}$.

Then we have that the function $F(x) = f(x) + r(x)\prod_{m=0}^{n}(x - x_m)$ describes all possible solutions to this problem, where $r(x)$ is an arbitrary polynomial.

Proof. While not relevant to the proof, note that the function $F(x) = f(x) + r(x)\prod_{m=0}^{n}(x - x_m)$ comes from our general solution form for $\phi^\star$, where the decider of this function is $\prod_{m=0}^{n}(x - x_m)$.

Let us denote the function $P(x) = F(x) - f(x)$

Since $P(x_m) = 0$ by definition for $m \in \{0, 1, \ldots, n\}$, we have that each $(x - x_m)$ is a root of $P(x)$ by the factor theorem. Therefore, the following is a polynomial:

$$Q(x) = \frac{P(x)}{\prod_{m=0}^{n}(x - x_m)} \implies P(x) = Q(x)\prod_{m=0}^{n}(x - x_m)$$

5.2.5 (continued)**.**
Using the definition of $P(x)$ we therefore have that:

$$F(x) = f(x) + Q(x) \prod_{m=0}^{n} (x - x_m)$$

Noting that $Q(x)$ is still some arbitrary polynomial as given. □

5.2.6. A polynomial of degree n can be minimally interpolated using $n + 1$ distinct points, and such polynomials are unique.

Proof. Convince yourself that a Lagrange or Newton polynomial as derived above with $n + 1$ points produces a polynomial of degree n (without a general solution). Since we've derived these formulas piecewise, it is straightforward to do so: count the number of extractions, corresponding to the number of roots.

In any case, suppose that two polynomials $p(x)$ and $q(x)$ minimally interpolate $n + 1$ points (of degree n). Let us define a polynomial $r(x) = p(x) - q(x)$. For each point being interpolated we have $r(x) = 0$; namely, $r(x)$ has $n + 1$ distinct zeroes. However, since $\deg r(x) \leq n$, the fundamental theorem of algebra states that $r(x)$ can have at most n zeroes unless $r(x) = 0$. It therefore follows that $p(x) = q(x)$. □

5.3 Non-trivial piece conditions

As per our regular piecewise properties, we can perform substitutions on piece conditions of a piecewise object. This is no different for anonymous piecewise objects, or even just piecewise objects when performing extractions as we have so far. With this being said, there are caveats.

As we've observed with Examples 5.2.2 and 5.2.3, particularly with $|x|$, overlapping conditions produce discontinuities and ambiguities. Sometimes, like in the former example, the resultant discontinuities are removable, but most times (such as in the latter example), they are not. Ultimately, operations on piecewise objects are only 'nice' when these piecewise objects are well defined; it is important, therefore, to keep this in mind when performing piece condition substitution.

5.3.1. Given some piecewise object ϕ as defined below:

$$\phi = \{\phi_i, \quad C_i \mid i \in I\}$$

If there exists a substitution $C_i \to x_i = y_i$, then not only can we write ϕ as the following:

$$\phi = \{\phi_i, \quad x_i = y_i \mid i \in I\}$$

But we can also perform our extraction techniques on the piecewise object. For example, $x \geq y \iff |x - y| = x - y$. This has previously been demonstrated, but we now make explicit this as a general technique.

This also means that when we interpolate a set of points, we needn't produce a polynomial in the variable being given.

Example 5.3.1. We wish to produce a function, not strictly a polynomial, $f(x)$ such that $f(0) = 0$, $f(1) = 1$ and $f(\frac{\pi}{2}) = 1$.

Example 5.3.1 (continued).
We can do so, for example, per the following:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ 1 & x = \frac{\pi}{2} \\ \star & \star \end{cases}$$

Let us substitute the condition $x = \frac{\pi}{2} \implies \sin(x) = 1$, noting that $f(x)$ remains well-defined:

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \\ 1 & \sin(x) = 1 \\ \star & \star \end{cases}$$

From here, we'll subtract 1 from the piecewise object so that we can extract the latter two pieces:

$$f(x) = 1 + (x-1)(\sin(x)-1) \begin{cases} \frac{-1}{(x-1)(\sin(x)-1)} & x = 0 \\ \star & \star \end{cases}$$

This yields $f(x) = 1 - (x-1)(\sin(x)-1) = x - x\sin(x) + \sin(x)$.

5.4 Multiple conditions

Up until this point we've been dealing with piecewise objects whose pieces only have one condition (or objects whose conditions can't particularly be simplified by nesting); dealing with multiple conditions is far more difficult. With this being said, there are multiple strategies which we can employ to deal with various situations, including formulations for logical 'or' and logical 'and'.

We begin by dealing with the cases wherein each condition is an equality of some kind.

5.4.1. Suppose $x \in S$ where S is some nicely-defined set. If the following expression exists, we can describe x in a single equality using the following:

$$x \in S \leftrightarrow \prod_{s \in S} (x - s) = 0$$

This is sometimes known as the null factor law.

Proof. Let $x \in S$ denote the following piecewise object which is not well-defined:

$$x = \{s, \quad x = s \mid s \in S\}$$

That is, x is equal to all elements in the set it is itself an element of. Using this, we have that:

$$\{x - s, \quad x = s \mid s \in S\} = 0$$

Since $x = s$ in each piece, we have $x - s = 0$ and so simplifying each piece we get:

$$\{0, \quad x = s \mid s \in S\} = 0$$

5.4.1 (continued)**.**
Extracting each piece yields:

$$\prod_{s \in S} (x - s) = 0$$

□

5.4.2. The following is a logical 'or' simplification for equalities.

Suppose we have a condition $A \leftrightarrow \bigvee_{i \in I} A_i$ such that $A_i \leftrightarrow (x_i = y_i)$. Then we have that A can be given by the following:

$$A \leftrightarrow \prod_{i \in I} (x_i - y_i) = 0$$

Proof. For all $i \in I$ we have that $x_i = y_i \implies x_i - y_i = 0$. It therefore stands to reason that $0 \in \{x_i - y_i \mid i \in I\} \leftrightarrow A$.

By 5.4.1 we have that $A \leftrightarrow \prod_{i \in I} (x_i - y_i) = 0$. □

5.4.3. The following is a logical 'or' simplification for equalities.

Suppose we have a condition $A \leftrightarrow \bigwedge_{i \in I} A_i$ such that $A_i \leftrightarrow (x_i = y_i)$. Then we have that A can be given by the following:

$$A \leftrightarrow \sum_{i \in I} p_i(x_i - y_i) = 0$$

Where p_i is some function such that $p_i(x) = 0 \iff x = 0$, and $p_i(x) \geq 0$.

Proof. Let $\vec{v}$ be a vector with components $p_i(x_i - y_i)$ for $i \in I$. Consider that the following are equivalent statements:

$$\vec{v} = \mathbf{0}$$

$$\bigwedge_{i \in I} (p_i(x_i - y_i) = 0)$$

We can take the norm of the latter statement, giving us:

$$\sum_{i \in I} |p_i(x_i - y_i)| = 0$$

Since $p_i \geq 0$, we have that:

$$\sum_{i \in I} p_i(x_i - y_i) = 0$$

To prove that this is an iff relationship, let $j \in I$. Using the above statement, we have:

$$p_j(x_j - y_j) = -\sum_{\substack{i \in I \\ i \neq j}} p_i(x_i - y_i) \leq 0$$

But since $p_j(x_j - y_j) \geq 0$, we have that $p_j(x_j - y_j) = 0$ for all $j \in I$. □

5.5 Further extraction

Note, this is very much a section I wish to work on further and requires more knowledge in abstract algebra than I currently possess. As it stands I'm missing terminology, concepts and whatever else.

So far, we've been doing what I call in these notes 'polynomial extraction'. Really, this is a process of two major steps: addition and multiplication, and then their respective inverses. That is, for each piece, we've vanished the piece value by adding (and subtracting), and then by reducing by multiplying the corresponding root (and dividing in each other piece), given by the piece condition. As it turns out, this process is not restricted to these operations.

Let S be a set such that we have binary operations $f : S^2 \to S$, $g : S^2 \to S$, $f^{-1} : S^2 \to S$, $g^{-1} : S^2 \to S$ with the properties (for all $x, y \in S$):

$$\begin{aligned}
f(f^{-1}(x,y),y) &= x \\
\exists a \in S \ni f^{-1}(x,x) &= a \\
f(a,x) &= x && \textit{(by the first two properties)} \\
g(g^{-1}(x,y),y) &= x \\
\exists b \in S \ni g(x,b) &= a
\end{aligned}$$

Where a, b are special elements of S (and $\ni$ refers to 'such that').

Then, our polynomial extraction processes uses the operations: $f(x,y) = x + y$, $f^{-1}(x,y) = x - y$, $g(x,y) = x \cdot y$ and $g^{-1}(x,y) = \frac{x}{y}$.

There is, in fact, another: exponential extraction. That is we define $f(x,y) = x \cdot y$, $f^{-1}(x,y) = \frac{x}{y}$, $g(x,y) = x^y$ and $g^{-1}(x,y) = x^{\frac{1}{y}}$.

A few notes to be made here: firstly, we don't strictly have to define our operations in such a way (there are other ways to do so, such as changing the order of inverses and whatnot). Furthermore, our substitution property holds regardless of extraction method, where appropriate or applicable. In any case, we can actually derive these properties from an 'ideal' process such as with the polynomial or exponential extractions.

Example 5.5.1. Let us motivate these properties using an interpolation problem for points $(0, u)$ and $(1, v)$.

Polynomial interpolation.

$$\begin{aligned}
F(x) &= \begin{cases} u & x = 0 \\ v & x = 1 \\ \star & \star \end{cases} \\
&= \begin{cases} 0 & x = 0 \\ v - u & x = 1 + u \\ \star & \star \end{cases} \\
&= \begin{cases} \frac{v-u}{x} & x = 1 \cdot x + u \end{cases} \\
&= (v - u)x + u
\end{aligned}$$

Example 5.5.1 (continued).
Generalised interpolation. Let $g : S \to S$ be a function such that $G(0) = b$.

$$\begin{aligned}
F(x) &= \begin{cases} u & x = 0 \\ v & x = 1 \\ \star & \star \end{cases} \\
&= f\left(\begin{cases} f^{-1}(u, u) & x = 0 \\ f^{-1}(v, u) & x = 1 \\ \star & \star \end{cases}, u \right) \\
&= f\left(\begin{cases} a & G(x) = b \\ f^{-1}(v, u) & G(x) = G(1) \\ \star & \star \end{cases}, u \right) \\
&= f\left(g\left(\begin{cases} g^{-1}\big(f^{-1}(v, u), G(x)\big) & G(x) = G(1) \\ \star & \star \end{cases}, G(x) \right), u \right) \\
&= f\big(g\big(g^{-1}\big(f^{-1}(v, u), G(1)\big), G(x)\big), u\big)
\end{aligned}$$

Example 5.5.2. A common example of interpolation is the Lerp function. Namely, consider the points (x_0, y_0) and (x_1, y_1) which, which minimally interpolated, forms a linear line. Its resulting equation is:

$$f(x) = \begin{cases} y_0 & x = x_0 \\ y_1 & x = x_1 \\ \star & \star \end{cases} = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

Now, for $y_0, y_1 > 0$ we can do the same thing using our 'exponential' extractions:

$$\begin{aligned}
f(x) &= \begin{cases} y_0 & x = x_0 \\ y_1 & x = x_1 \\ \star & \star \end{cases} \\
&= y_0 \cdot \begin{cases} 1 & x = x_0 \\ \frac{y_1}{y_0} & x = x_1 \\ \star & \star \end{cases} \\
&= y_0 \cdot \left(\begin{cases} (\frac{y_1}{y_0})^{\frac{1}{x - x_0}} & x = x_1 \\ \star & \star \end{cases} \right)^{x - x_0} \\
&= y_0 \cdot \left((\frac{y_1}{y_0})^{\frac{1}{x_1 - x_0}} \right)^{x - x_0} \\
&= y_0 \cdot \left(\frac{y_1}{y_0} \right)^{\frac{x - x_0}{x_1 - x_0}}
\end{aligned}$$

Obviously this function is no longer strictly linear; with this being said, it is linear under a logarithm. That is:

$$\ln(f(x)) = \begin{cases} \ln(y_0) & x = x_0 \\ \ln(y_1) & x = x_1 \\ \star & \star \end{cases}$$

When we use polynomial extractions to interpolate this function. This is because of the properties of the logarithm:

Example 5.5.2 (continued).

$$\begin{aligned}\ln(ab) &= \ln(a) + \ln(b)\\ \ln\left(a^b\right) &= b\ln(a)\end{aligned}$$

Namely, we have a direct mapping between the operations involved in polynomial and exponential extractions (within the appropriate domains).

5.5.1. An anonymous piecewise object is given by the following:

$$\phi^\star = \begin{cases}\phi & \phi \text{ is defined}\\ \star & \phi \text{ is not defined}\end{cases}$$

Let μ be a solution to $\phi^\star$ given our operations. Then we can write $\phi^\star$ as:

$$\phi^\star = f(g(r, [\phi]), \mu)$$

Where r is some arbitrary object, $[\phi]$ is the decider object (i.e. $[\phi] = b$ when ϕ is defined) and μ is a solution to $\phi^\star$.

6 Forming Piecewise Functions

So far, we've discussed the properties of piecewise functions independently of anything else, except with a few examples as to how we might rewrite certain piecewise functions in terms of other piecewise functions. This is a topic that we'll get back to soon, as well. However, one of the interesting, but more incomplete, parts which constitute piecewise objects as a whole is how exactly they come about in mathematics.

Piecewise functions, particularly, have some interesting ways they come about.

6.1 Function composition

Piecewise functions can themselves be created from the composition of two (or so) functions. These piecewise functions are themselves elementary and continuous if the functions being composed are themselves elementary and continuous (in fact, this is why $|x|$ is an elementary function... more on that later).

6.1.1. Let us define sets $U, V, D \subseteq \mathbb{R}$ and function $F : D \to V$. We then define $f = F\big|_U$ (i.e. f is the restriction of F to U) such that there exists a function $g : V \to U$ satisfying:

$$(g \circ f)(x) = x, \quad x \in U$$
$$(f \circ g)(x) = x, \quad x \in V$$

That is, f is bijective. We now wish to define $(g \circ F)(x)$ (noting that $(F \circ g)(x) = x$ for $x \in V$); this will be, in effect, an extension of the left inverse of f over the reals.

We let $p : D \to U$, $p(x) = (g \circ F)(x)$. Noting that $(F \circ g)(x) = x$ we have that $(F \circ p)(x) = F(x)$. Therefore, if this equation can be solved for $p(x)$, we have that:

$$p(x) = \{p_i(x), \quad U_i \mid i \in I\}$$

Where $p_i(x)$ is a solution of the above equation on some domain, and U_i is the condition that defines that domain. To find U_i, we can reason that $p(x) \in U \implies p_i(x) \in U \iff U_i$. Therefore:

$$p(x) = \{p_i(x), \quad p_i(x) \in U \mid i \in I\}$$

Which is our piecewise function.

6.1.2. The absolute value function is a function which can be created by composing any (well-behaved) even function with range $[0, \infty)$ and its principal inverse.

Proof. Let $F : \mathbb{R} \to [0, \infty)$ be an even function. Then we let $f = F\big|_{[0,\infty)}$ be a function for which there exists $g : [0, \infty) \to [0, \infty)$ such that $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$ for $x \in [0, \infty)$.

Let $p(x) = (g \circ F)(x)$, then $F(p(x)) = F(x)$. For $x \geq 0$, we have that $f(p(x)) = f(x) \iff p(x) = x$, by bijectivity of f. For $x \leq 0$, by the evenness of F we have $F(p(x)) = F(-x)$ and by the same argument gives $p(x) = -x$. Therefore:

$$p(x) = (g \circ F)(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases} = |x|$$

□

Example 6.1.1. We've built bottom-up in terms of theory; now we want to put this to practice. Let us prove that $|x| = \sqrt[2n]{x^{2n}}$ for any positive integer n by deriving it ourselves.

Example 6.1.1 (continued)**.**
Suppose $f(x) = \sqrt[2n]{x^{2n}}$; then we have $f(x)^{2n} = x^{2n}$. By difference of perfect squares, we have that:

$$(f(x)^n - x^n)(f(x)^n + x^n) = 0$$

If n is even, then the second factor itself has no real factors, but also that we can repeat difference of squares on the first factor (until reduced to an odd power, m).

If m is odd, then by observation, $f(x) = x$ and $f(x) = -x$ are solutions to each respective factor (we shall disregard complex factors until a later time).

Therefore:

$$f(x) = \begin{cases} x & C_1 \\ -x & C_2 \end{cases}$$

Remember that the range of $f(x)$ is $[0, \infty)$; therefore, when $f(x) = x$ we have $x \in [0, \infty)$. Likewise when $f(x) = -x$ we have $-x \in [0, \infty) \implies x \in (-\infty, 0]$. Therefore:

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases} = |x|$$

And we're done.

It is important to realise that piecewise functions do not strictly have a single representation in terms of elementary functions. Therefore, it is important to consider that non-trivial operations should be done piecewise instead with existing fundamental ideas and theorems, rather than on the representation/expression. A common example of this is using $\sqrt{x^2} = |x|$ to find $\frac{\mathrm{d}}{\mathrm{d}x}|x|$.

6.2 Piecewise functions as results of limits

Perhaps an arguably more natural way that piecewise functions can appear is through the parameterisation of limits.

Example 6.2.1. The function $\tanh(x)$ approximates $\operatorname{sgn}(x)$.

Proof. What do we mean by 'approximates'? In terms of behaviour, tanh is a smooth, bounded, odd function with asymptotes per the following: For positive large x we have $\tanh(x) = 1$ and for negative large x we have $\tanh(x) = -1$.

We begin by noting $\tanh(x)$ is odd; therefore, $\tanh(x) = \tanh(|x|)\operatorname{sgn}(x)$ by 4.5.1.

We can then rewrite this using the definition of tanh:

$$\begin{aligned} \tanh(|x|) &= \frac{e^{|x|} - e^{-|x|}}{e^{|x|} + e^{-|x|}}\operatorname{sgn}(x) \\ &= \frac{e^{2|x|} - 1}{e^{2|x|} + 1}\operatorname{sgn}(x) \\ &= \left(1 - \frac{2}{e^{2|x|} + 1}\right)\operatorname{sgn}(x) \end{aligned}$$

For positive or negative large x, we have that $\left(1 - \frac{2}{e^{2|x|}+1}\right) \approx 1$. Therefore:

$$\tanh(x) \approx 1 \cdot \operatorname{sgn}(x)$$

$\square$

6.2.1. There are some examples of limits we can use to represent certain piecewise functions:

1. $\operatorname{sgn}(x) = \lim_{n\to\infty} \tanh(nx)$
2. $[x = 0] = \lim_{n\to\infty} \exp\left(-nx^2\right)$

6.2.2. The Dirichlet function is an example of a piecewise function that can be constructed using limit(s):

$$\mathbf{1}_{\mathbb{Q}}(x) = \lim_{m\to\infty} \lim_{n\to\infty} \cos(m!\pi x)^{2n}$$

It's quite an astonishing result.

[1]I'd love to fill this section up with more limits and more ways to generate piecewise functions. As it stands, this section is woefuly underfilled as I haven't particularly focused on it; these are not concepts that often pop up inside piecewise functions themselves. Rather, this section is about the approximation and representation of piecewise functions, of which I have frighteningly little knowledge.

Part II

Constructions

7 Gluing Functions

Until now, we've discussed piecewise concepts in the more abstract sense, with explicit examples permeating the paradigm behind them as previously discussed. From here, we'll begin introducing new concepts with a heavier focus on explicit constructions. This begins with our gluing functions.

7.1 Continuous gluing function

Let us define intervals $D_1, D_2, \ldots, D_n$ such that for $m \in \{1, 2, \ldots, n-1\}$ we have $\max(D_m) = \min(D_{m+1})$. Then we define all respective functions to be glued $f_m : D_m \to \mathbb{R}$. To 'glue' such functions together sequentially in a function $F(x)$ we write:

$$F(x) = \sum_{m=1}^{n} f_m\big(\ell_{a_m}^{b_m}(x)\big) - \sum_{m=2}^{n} f_m(a_m)$$

Where $a_m = \min(D_m)$ if it exists, or $-\infty$ otherwise. Likewise $b_m = \max(D_m)$ if it exists, or ∞ otherwise. This also uses the clamping function, as given in 4.4.1.

Proof. This derivation actually follows as an extension of Example 4.4.1. We begin by letting $F : \mathbb{R} \to \mathbb{R}$ be a function such that:

$$F(x) = \{f_m(x), \quad x \in D_m \mid m \in \{1, 2, \ldots, n\}\}$$

Let us define the following:

$$a_m = \begin{cases} \min(D_m) & \text{if } \min(D_m) \text{ exists} \\ -\infty & \text{otherwise} \end{cases}$$

$$b_m = \begin{cases} \max(D_m) & \text{if } \max(D_m) \text{ exists} \\ \infty & \text{otherwise} \end{cases}$$

Therefore $x \in D_m \iff \ell_{a_m}^{b_m}(x) = x$ (note, per the definition of the clamping function, this skips the nuances of having closed/open intervals, etc.). We can accordingly rewrite $F(x)$:

$$F(x) = \big\{f_m(x), \quad \ell_{a_m}^{b_m}(x) = x \;\big|\; m \in \{1, 2, \ldots, n\}\big\}$$

Substituting the respective conditions where $x = \ell_{a_m}^{b_m}(x)$, we have that:

$$F(x) = \big\{f_m\big(\ell_{a_m}^{b_m}(x)\big), \quad \ell_{a_m}^{b_m}(x) = x \;\big|\; m \in \{1, 2, \ldots, n\}\big\}$$

And then we can 'subtract' each piece value out:

$$F(x) = \sum_{l=1}^{n} f_l\big(\ell_{a_l}^{b_l}(x)\big) + \left\{ -\sum_{\substack{l=1 \\ l \neq m}}^{n} f_l\big(\ell_{a_l}^{b_l}(x)\big), \quad \ell_{a_m}^{b_m}(x) = x \;\middle|\; m \in \{1, 2, \ldots, n\} \right\}$$

Focusing on the sum in the piecewise function, we have:

$$\sum_{\substack{l=1 \\ l \neq m}}^{n} f_l\big(\ell_{a_l}^{b_l}(x)\big) = \sum_{l=1}^{m-1} f_l\big(\ell_{a_l}^{b_l}(x)\big) + \sum_{l=m+1}^{n} f_l\big(\ell_{a_l}^{b_l}(x)\big)$$

Since $x \in D_m$ we know that for the first partial sum, we have $f_l(b_l)$ for each term. Likewise in the second sum, we have $f_l(a_l)$ for each term. Noting that $f_{l+1}(a_{l+1}) = f_l(b_l)$ by our continuity property, we can rewrite the sums respectively:

$$\sum_{\substack{l=1 \\ l \neq m}}^{n} f_l\big(\ell_{a_l}^{b_l}(x)\big) = \sum_{l=1}^{m-1} f_{l+1}(a_{l+1}) + \sum_{l=m+1}^{n} f_l(a_l)$$

In the first sum, we write $l \to l-1$ and so we have, for each $m \in \{1, 2, \dots, n\}$:

$$\sum_{\substack{l=1 \\ l \neq m}}^{n} f_l\big(\ell_{a_l}^{b_l}(x)\big) = \sum_{l=2}^{n} f_l(a_l)$$

Therefore, $F(x)$ is given by:

$$F(x) = \sum_{l=1}^{n} f_l\big(\ell_{a_l}^{b_l}(x)\big) + \left\{ -\sum_{l=2}^{n} f_l(a_l), \quad \ell_{a_m}^{b_m}(x) = x \;\middle|\; m \in \{1, 2, \dots, n\} \right\}$$

Since each piece is identical, we can reduce our piecewise object accordingly:

$$F(x) = \sum_{l=1}^{n} f_l\big(\ell_{a_l}^{b_l}(x)\big) - \sum_{l=2}^{n} f_l(a_l)$$

$\square$

As a corollary, if all of $f_1, f_2, \dots, f_n$ are elementary functions, then $F(x)$ is itself an elementary function as $\ell_{a_m}^{b_m}(x)$ is, also.

The derivation of the gluing function above doesn't actually do the function, nor the piecewise techniques involved, much justice. With this being said, it was derived with full generality, and other such derivations may need to deal with nuances more directly on a case-by-case basis (such as when $\min(D_1)$ doesn't exist). Following this, I leave it as an exercise to the reader to derive this formula using, perhaps, induction, where each interval is finite (bounded above/below). This article Gluing Functions can be used for inspiration.

7.2 Generalised gluing function in one dimension

Recall that in 4.9.3 we explored the Iverson bracket form of a piecewise object when each condition was 'disjoint'. That is, the following are equivalent:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$
$$\phi = \sum_{i \in I} \varphi_i [C_i]$$

We can, in fact, reduce this form down for the purposes of our gluing function. Let us denote the sets $D_1, D_2, \dots, D_n$ such that for all $i, j \in \{1, 2, \dots, n\},\ i \neq j$, we have $D_i \cap D_j = \varnothing$ (that is, every set is disjoint from one another). Then let us define functions $f_m : D_m \to \mathbb{R}$ for $m \in \{1, 2, \dots, n\}$, and $F : \mathbb{R} \to \mathbb{R}$ such that:

$$F(x) = \sum_{m=1}^{n} f_m(x)[x \in D_m]$$

More explicitly, what we're summing is the following:

$$F(x) = \begin{cases} f_1(x) & x \in D_1 \\ 0 & x \notin D_1 \end{cases} + \begin{cases} f_2(x) & x \in D_2 \\ 0 & x \notin D_2 \end{cases} + \dots + \begin{cases} f_n(x) & x \in D_n \\ 0 & x \notin D_n \end{cases}$$

In fact, we can reach our continuous gluing function formula if what we have is continuous, per our usual techniques (although since our sets are disjoint, we resort to using sup and inf rather than max and min, respectively).

7.3 The triangle wave

Let us consider the function $f(x) = \arcsin(\sin(x))$ (a similar function is $g(x) = \arccos(\cos x)$ though we won't derive it here). This function, is, in fact, piecewise, and can be written as:

$$f(x) = \frac{\pi}{2} - \left| x - \frac{\pi}{2}\left(1 + 4\left\lfloor \frac{x}{2\pi} + \frac{1}{4} \right\rfloor\right) \right|$$

Proof. We use the techniques established in 6.1 to derive the above formula. Using the definition of $f(x)$ we have that:

$$\begin{aligned} f(x) = \arcsin(\sin(x)) &\implies \sin(f(x)) = \sin(x) \\ &\implies \sin(f(x)) - \sin(x) = 0 \end{aligned}$$

Recall the identity $\sin(a) - \sin(b) = 2\sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$, and so the above is equivalent to:

$$2\sin\left(\frac{f(x) - x}{2}\right)\cos\left(\frac{f(x) + x}{2}\right) = 0$$

The solutions to this equation are as follows[2]:

$$\sin\left(\frac{f(x) - x}{2}\right) = 0 \implies f(x) = 2m\pi + x \qquad m \in \mathbb{Z} \tag{1}$$

$$\cos\left(\frac{f(x) + x}{2}\right) = 0 \implies f(x) = (2n+1)\pi - x \qquad n \in \mathbb{Z} \tag{2}$$

From here, we use the fact that $f(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to establish respective domains for our solutions:

$$2m\pi + x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies x \in \left[-\frac{\pi}{2}(4m+1), -\frac{\pi}{2}(4m-1)\right] \tag{3}$$

$$(2n+1)\pi - x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \implies x \in \left[\frac{\pi}{2}(4n+1), \frac{\pi}{2}(4n+3)\right] \tag{4}$$

From here, we write $f(x)$ in piecewise form (noting that it is a continuous function), and also rewriting $m \to -n$:

$$\begin{aligned} f(x) = &\left\{-2n\pi + x, \quad x \in \left[\frac{\pi}{2}(4n-1), \frac{\pi}{2}(4n+1)\right] \;\middle|\; n \in \mathbb{Z}\right\} \\ &\cup \left\{(2n+1)\pi - x, \quad x \in \left[\frac{\pi}{2}(4n+1), \frac{\pi}{2}(4n+3)\right] \;\middle|\; n \in \mathbb{Z}\right\} \end{aligned}$$

From here, let us define $g_n : x \in \left[\frac{\pi}{2}(4n-1), \frac{\pi}{2}(4n+3)\right] \to \mathbb{R}$ such that $g_n(x) = f(x)$. Therefore, we can write $g_n(x)$ as the following:

$$\begin{aligned} g_n(x) &= \begin{cases} -2n\pi + x & x \leq \frac{\pi}{2}(4n+1) \\ (2n+1)\pi - x & x \geq \frac{\pi}{2}(4n+1) \end{cases} \\ &= -2n\pi + \ell_{-\infty}^{\frac{\pi}{2}(4n+1)}(x) + (2n+1)\pi - \ell_{\frac{\pi}{2}(4n+1)}^{\infty}(x) - \frac{\pi}{2} \\ &= \frac{\pi}{2} - \left(\max\left\{x, \frac{\pi}{2}(4n+1)\right\} - \min\left\{x, \frac{\pi}{2}(4n+1)\right\}\right) \\ &= \frac{\pi}{2} - \left| x - \frac{\pi}{2}(4n+1) \right| \end{aligned}$$

[2]Note, if you're not familiar with how these identities and solutions have come about, while not necessary, it is a fun exercise to recognise how they have. I found that abusing Euler's formula to derive the formulas is pretty fun.

We can now rewrite $f(x)$ using $g_n(x)$:

$$f(x) = \left\{ \frac{\pi}{2} - \left| x - \frac{\pi}{2}(4n+1) \right|, \quad x \in \left[\frac{\pi}{2}(4n-1), \frac{\pi}{2}(4n+3) \right] \;\middle|\; n \in \mathbb{Z} \right\}$$

Since we know this function is continuous, we can rewrite the conditions of $f(x)$ to match the floor function, and simplify from there:

$$\begin{aligned}
f(x) &= \left\{ \frac{\pi}{2} - \left| x - \frac{\pi}{2}(4n+1) \right|, \quad \frac{x}{2\pi} + \frac{1}{4} \in [n, n+1) \;\middle|\; n \in \mathbb{Z} \right\} \\
&= \frac{\pi}{2} - \left| \left\{ x - \frac{\pi}{2}(4n+1), \quad \frac{x}{2\pi} + \frac{1}{4} \in [n, n+1) \;\middle|\; n \in \mathbb{Z} \right\} \right| \\
&= \frac{\pi}{2} - \left| x - \frac{\pi}{2}\left(1 + 4 \left\{ n, \quad \frac{x}{2\pi} + \frac{1}{4} \in [n, n+1) \;\middle|\; n \in \mathbb{Z} \right\} \right) \right| \\
&= \frac{\pi}{2} - \left| x - \frac{\pi}{2}\left(1 + 4 \left\lfloor \frac{x}{2\pi} + \frac{1}{4} \right\rfloor \right) \right|
\end{aligned}$$

□

This function can also be represented using the gluing formula we derived earlier and after simplification ends up being:

$$\begin{aligned}
f(x) &= \lim_{n \to \infty} f_n(x), \\
f_n(x) &= n\pi + \frac{1}{2} \sum_{k=-n}^{n} \left(\pi - \left| \left| x - \frac{\pi}{2}(4k-1) \right| - \left| x - \frac{\pi}{2}(4k+3) \right| \right| \right)
\end{aligned}$$

7.4 Representing periodic functions

There an infinite number of ways to represent periodic functions on $\mathbb{R}$, perhaps the most famous being the Fourier series. However, we can derive several representations here.

7.4.1. Consider the function $f : [a, b]$ (or $f : [a, b)$, $f : (a, b]$, where appropriate). We define F such that f is repeated on $\mathbb{R}$; $b - a$-periodic. Then F can be given by:

$$F(x) = f\left(x - \left\lfloor \frac{x-a}{b-a} \right\rfloor (b-a) \right)$$

Proof. We shall derive this formula by letting $f_0(x) = f(x)$ for $x \in [a, b]$. Then we we want some function, $f_1 : [a + (b-a), b + (b-a)]$ such that:

$$f_1(x) = \begin{cases} f(a) & x = a + (b-a) \\ f(b) & x = b + (b-a) \\ \star & \star \end{cases}$$

In the first piece, we have that $a = x - (b-a)$. Likewise in the second piece, $b = x - (b-a)$. Therefore:

$$f_1(x) = \begin{cases} f(x - (b-a)) & x = a + (b-a) \\ f(x - (b-a)) & x = b + (b-a) \\ \star & \star \end{cases} = f(x - (b-a))$$

Continuing this process, forward and backward for $n \in \mathbb{Z}$, we have that:

$$f_n : [a + n(b-a), b + n(b-a)], \quad f_n(x) = f(x - n(b-a))$$

7.4.1 (continued)**.**
Then constructing F by definition we have that:

$$F(x) = \{f_n(x), \quad x \in [a + n(b-a), b + n(b-a)] \mid n \in \mathbb{Z}\}$$

Note that this condition will change based on the continuitiy of the function in general. Going forward, we instead write:

$$F(x) = \{f_n(x), \quad x \in [a + n(b-a), b + n(b-a)) \mid n \in \mathbb{Z}\}$$

We wish to find some transformation of x, x' such that (to match the floor function):

$$x' = \begin{cases} n & x = a + n(b-a) \\ n+1 & x = b + n(b-a) \end{cases}$$

We'll leave this interpolation problem to the reader. However, we get:

$$x' = \frac{x-a}{b-a}$$

This allows us to rewrite $F(x)$:

$$F(x) = \left\{ f_n(x), \quad \frac{x-a}{b-a} \in [n, n+1) \;\middle|\; n \in \mathbb{Z} \right\}$$

Applying our function property of piecewise objects, this is equivalent to:

$$F(x) = f_{\left\{n, \quad \frac{x-a}{b-a} \in [n,n+1) \;\mid\; n \in \mathbb{Z}\right\}}(x)$$

Finally, we're left with:

$$F(x) = f_{\left\lfloor \frac{x-a}{b-a} \right\rfloor}(x)$$

Using the definition of $f_n(x)$ we have that:

$$F(x) = f\left(x - \left\lfloor \frac{x-a}{b-a} \right\rfloor (b-a) \right)$$

□

As an alternative to the above, we can represent continuous periodic functions using our gluing formula.

7.4.2. We consider a function $f : [a, b]$ such that $f(a) = f(b) = c \in \mathbb{R}$. We define F such that the function f is repeated on $\mathbb{R}$; $b - a$ periodic. Then F can be given by a limit:

$$F(x) := \lim_{n \to \infty} F_N(x),$$
$$F_N(x) = -2Nc + \sum_{n=-N}^{N} f\big(\ell_a^b(x - n(b-a))\big)$$

Proof. Let us consider the same functions f_n in Proof 7.4.1, that is:

$$f_n : [a + n(b-a), b + n(b-a)], \quad f_n(x) = f(x - n(b-a))$$

7.4.2 (continued)**.**
Then using the fact that gluing all of f_n sequentially would be continuous, we do so for $n \in \{-N, 1-N, \ldots, N-1, N\}$, denoting this using $F_N(x)$:

$$F_N(x) = \sum_{n=-N}^{N} f\left(\ell_{a+n(b-a)}^{b+n(b-a)}(x) - n(b-a)\right) - \sum_{n=1-N}^{N} f(a + n(b-a) - n(b-a))$$

Simplifying:

$$F_N(x) = \sum_{n=-N}^{N} f\big(\ell_a^b(x - n(b-a))\big) - \sum_{n=1-N}^{N} c$$

Taking $N \to \infty$, we have our result. $\square$

8 Discontinuous Piecewise Functions as Limits

We've already explored some elementary functions which can approximate piecewise functions (continuous or not). Some functions, however aren't covered, and, moreover, there is no systematic way to derive such approximations except via substitution using existing functions. This is what we cover here, with some extra steps.

Let f be some discontinuous piecewise function. Then we shall define f using a limit. For all real x, let us define the following:

$$\lim_{\varepsilon \to 0} F(x, |\varepsilon|) = f(x)$$

That is, there exists some function F which is exactly f everywhere except around its discontinuities. In order to formulate such a function, we have the properties below. For some small $\varepsilon > 0$ we have the following:

1. If $\lim_{x \to a^+} f(x) = b$ then $f(a + \varepsilon) \approx b$.
2. If $\lim_{x \to a^-} f(x) = b$ then $f(a - \varepsilon) \approx b$.
3. If $\lim_{x \to a} f(x) = b$ then $f(a) \approx b$ (if $f(a)$ exists).

This should look familiar; we're given a set of individual points, which means can interpolate these points where appropriately. Furthermore, since we interpolate only in the region of the discontinuity, we can then apply our gluing formula to the rest of the domain, as derived in Section 7.1.

Example 8.0.1. Let us find a function F as per the above which approximates $\operatorname{sgn}(x)$. Recall that $\operatorname{sgn}(x)$ is defined as the following:

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

For some small $x \in [-\varepsilon, \varepsilon]$ for $\varepsilon > 0$, therefore, we have that:

$$F(x, \varepsilon) = \begin{cases} 1 & x = \varepsilon \\ 0 & x = 0 \\ -1 & x = -\varepsilon \\ \star & \star \end{cases}$$

If we view this as an interpolation problem, this gives $F(x, \varepsilon) = \frac{x}{\varepsilon}$. Combining this with the remainder of the domain, $\mathbb{R}$, we have that:

$$F(x, \varepsilon) = \begin{cases} 1 & x \geq \varepsilon \\ \frac{x}{\varepsilon} & -\varepsilon \leq x \leq \varepsilon \\ -1 & x \leq -\varepsilon \end{cases}$$

Applying our gluing formula, we get:

$$F(x, \varepsilon) = \left(1 + \frac{\ell_{-\varepsilon}^{\varepsilon}(x)}{\varepsilon} + (-1)\right) - (1 + (-1))$$

Fully simplifying using the properties of the clamping function given in 4.4.1, this leaves:

$$F(x, \varepsilon) = \ell_{-1}^{1}\left(\frac{x}{\varepsilon}\right)$$

Example 8.0.1 (continued)**.**
It therefore stands to reason that:

$$\lim_{\varepsilon\to 0} \ell^{1}_{-1}\left(\frac{x}{|\varepsilon|}\right) = \mathrm{sgn}(x)$$

Example 8.0.2. Recall that the floor function is defined as the following:

$$\lfloor x \rfloor = \{n, \quad x \in [n, n+1)] \mid n \in \mathbb{Z}\}$$

Using our patching method, we can also define $\lfloor x \rfloor$ using the following:

$$\lfloor x \rfloor = \lim_{N\to\infty}\left(N + \sum_{n=-N}^{N} \lim_{\varepsilon\to 0} \ell^{0}_{-1}\left(\frac{x-n}{|\varepsilon|}\right)\right)$$

Proof. The floor function is handled differently to other functions in that it has infinitely many discontinuities. So instead of trying to handle it for each integer section, let us consider, for all $n \in \mathbb{Z}$ the interval $x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]$:

$$\lfloor x \rfloor = \begin{cases} n-1 & x < n \\ n & x \geq n \end{cases}$$

We can therefore construct $F_n(x, \varepsilon)$ as follows:

$$F_n(x,\varepsilon) = \begin{cases} n-1 & x = n - \varepsilon \\ n & x = n \\ \star & \star \end{cases}$$

Which gives:

$$F_n(x,\varepsilon) = n + \frac{x-n}{\varepsilon}$$

Gluing, we have:

$$F_n(x,\varepsilon) = \ell^{n}_{n-1}\left(n + \frac{x-n}{\varepsilon}\right)$$

We therefore have that our 'total' function is given by:

$$\lfloor x \rfloor = \left\{\lim_{\varepsilon\to 0} \ell^{n}_{n-1}\left(n + \frac{x-n}{|\varepsilon|}\right), \quad x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right] \;\middle|\; n \in \mathbb{Z}\right\}$$

Now that this function is continuous at the endpoints of each interval, we can glue using partial sums:

$$f_N(x) = \sum_{n=-N}^{N} \lim_{\varepsilon\to 0} \ell^{n}_{n-1}\left(n + \frac{\ell^{n+\frac{1}{2}}_{n-\frac{1}{2}}(x) - n}{|\varepsilon|}\right) - \sum_{n=1-N}^{N}(n-1)$$

Note that the latter sum is equal to $-N$ (left as an exercise to the reader). Simplifying the inside of the former sum (and doing the same with the n term which sums to 0), however, we get:

$$f_N(x) = N + \sum_{n=-N}^{N} \lim_{\varepsilon\to 0} \ell^{0}_{-1}\left(\ell^{\frac{1}{2|\varepsilon|}}_{-\frac{1}{2|\varepsilon|}}\left(\frac{x-n}{|\varepsilon|}\right)\right)$$

Example 8.0.2 (continued).
Since $\varepsilon \to 0$ the inner clamping function is unbounded (i.e. $\ell_{-\infty}^{\infty}(x) = x$), so we have:

$$
\begin{aligned}
f_N(x) &= N + \sum_{n=-N}^{N} \lim_{\varepsilon \to 0} \ell_{-1}^{0}\left(\frac{x-n}{|\varepsilon|}\right) \\
\lfloor x \rfloor &= \lim_{N \to \infty} f_N(x)
\end{aligned}
$$

□

8.0.1. Using the floor function per Example 8.0.2, we can restate our proposition as the following: For all $|x| \leq N \in \mathbb{Z}^+$ we have that:

$$\lfloor x \rfloor = N + \sum_{n=-N}^{N} \lim_{\varepsilon \to 0} \ell_{-1}^{0}\left(\frac{x-n}{|\varepsilon|}\right)$$

Proof. For all $|x| \leq N$, we know that $\exists m \in \{-N, \ldots, N-1\}$ such that $m \leq x < m+1$. Then:

$$\lfloor x \rfloor = N + \sum_{n=-N}^{m} \lim_{\varepsilon \to 0} \ell_{-1}^{0}\left(\frac{x-n}{|\varepsilon|}\right) + \sum_{n=m+1}^{N} \lim_{\varepsilon \to 0} \ell_{-1}^{0}\left(\frac{x-n}{|\varepsilon|}\right)$$

Then for all $n \leq m$ we have $x \geq n \implies \frac{x-n}{|\varepsilon|} \geq 0$. Using the definition of the clamping function, therefore, we have that the summand of the first sum is 0 (after each limit is taken). Likewise for the second sum, we have $m \leq n$ which means $\frac{x-n}{|\varepsilon|} \leq 0$. Taking the limits, this means we have -1 for all terms, giving us:

$$\lfloor x \rfloor = N + \sum_{n=m+1}^{N} (-1)$$

This gives $\lfloor x \rfloor = N - (N-m) = m$. □

9 Further Operations on Piecewise Functions

9.1 Differentiation

The differentiability of piecewise functions has been covered frequently in calculus courses, as both practice for differentiation and its applications, as well as for general awareness of piecewise functions (although these are introduced usually in a pre-calculus setting). The nuances of differentiating a piecewise function exceed the number of pages I wish to put in these notes, so I suggest sticking to your typical calculus or analysis book for all of that.

With this being said, we can apply differentiation to interpolated functions (anonymous piecewise functions).

Example 9.1.1. Let us find a polynomial function $f(x)$ such that $f(0) = 0$, $f(1) = 1$ and $f'(0) = 0$, $f'(1) = 0$.

Recall the general solution of the APO using a polynomial extraction as per 5.2.3:

$$f^\star(x) = f(x) + [f(x)] \cdot r(x)$$

In our problem, we note the solution to the first derivative problem is $f(x) = x$; the general solution is therefore:

$$f^\star(x) = x + x(x-1)r(x)$$

Differentiating $f^\star(x)$ we have that:

$$\frac{\mathrm{d}f^\star}{\mathrm{d}x} = 1 + (2x-1)r(x) + x(x-1)r'(x)$$

Using the definition of $f'(x)$ and the above equation, we substitute $x = 0$ and $x = 1$:

$$\frac{\mathrm{d}f^\star}{\mathrm{d}x} = 1 - r(0) = 0 \implies r(0) = 1$$
$$\frac{\mathrm{d}f^\star}{\mathrm{d}x} = 1 + r(1) = 0 \implies r(1) = -1$$

Regard, now, that we have an interpolation problem in $r(x)$:

$$r(x) = \begin{cases} 1 & x = 0 \\ -1 & x = 1 \end{cases}$$

Note that one could have solved for $r(x)$ directly, making use of the interpolation problem of $f'(x)$, with $r'(x)$ terms (as the term vanishes), and then substituted in the appropriate values accordingly. Furthermore, one solution to this problem is $r(x) = 1 - 2x$ (the corresponding general solution is $r^\star(x) = 1 - 2x + x(x-1)s(x)$).

Using this, we have a solution given by:

$$f(x) = x + x(x-1)(1-2x)$$

Example 9.1.2. Let us find a polynomial function which corresponds to the following table (that is, for values of x we have the corresponding values of $f(x)$, and so forth):

x	$f(x)$	$f'(x)$	$f''(x)$
1	1	0	1
2	2	0	2

Example 9.1.2 (continued).
Interpolating $f(x)$, we have the following general solution:

$$f(x) = x + (x-1)(x-2)r_0(x)$$

We can now get the first and second derivative:

$$f'(x) = 1 + (2x-3)r_0(x) + (x-1)(x-2)r_0'(x)$$
$$f''(x) = 2r_0(x) + 2(2x-3)r_0'(x) + (x-1)(x-2)r_0''(x)$$

Using our first derivative, we obtain the following:

$$f'(1) = 1 - r_0(1) = 0 \implies r_0(1) = 1$$
$$f'(2) = 1 + r_0(2) = 0 \implies r_0(2) = -1$$

Subsequently, we have the general solution to $r_0(x)$ (note that we are now repeating the process for r_0 instead of f):

$$r_0(x) = 3 - 2x + (x-1)(x-2)r_1(x)$$

Using these values, we can now obtain $r_0'(x)$:

$$f''(1) = 2r_0(1) - 2r_0'(1) = 0 \implies r_0'(1) = \frac{1}{2}$$
$$f''(2) = 2r_0(2) + 2r_0'(2) = 0 \implies r_0'(2) = 2$$

Taking the derivative of $r_0(x)$ we have that:

$$r_0'(x) = -2 + (2x-3)r_1(x) + (x-1)(x-2)r_1'(x)$$

And repeating the process as before for the values of $r_1(x)$:

$$r_0'(1) = -2 - r_1(1) = \frac{1}{2} \implies r_1(1) = -\frac{5}{2}$$
$$r_0'(2) = -2 + r_1(2) = 2 \implies r_1(2) = 4$$

We therefore have a solution to r_1; $r_1(x) = -\frac{5}{2} + \frac{13}{2}(x-1)$. Therefore we have for r_0:

$$r_0(x) = 3 - 2x + (x-1)(x-2)\left(-\frac{5}{2} + \frac{13}{2}(x-1)\right)$$

And therefore $f(x)$:

$$f(x) = x + (x-1)(x-2)\left(3 - 2x + (x-1)(x-2)\left(-\frac{5}{2} + \frac{13}{2}(x-1)\right)\right)$$

As given by the examples above, the process of interpolating with respect to not only points, but derivatives at those points, is given by iterating through functions until we've satisfied all of our general solutions. This is the process which gives rise to the derivation of the Taylor polynomial (and series).

9.1.1. All polynomials which share a finite number of points and up to the n^{th} derivatives at those points can be written in the form:

$$\phi^\star(x) = \mu(x) + [\phi(x)]^{n+1} \cdot r(x)$$

9.1.1 (continued)**.**
Where $[\phi(x)]$ is the decider function, $\mu(x)$ is a solution to the original interpolation problem and $r(x)$ is some arbitrary polynomial.

Proof. We begin by considering all polynomials of the form $\phi^\star(x) = \mu(x) + [\phi(x)] \cdot r(x)$.

Differentiating n times, we get:

$$\phi^{\star(n)}(x) = \phi^{(n)}(x) + \sum_{m=0}^{n} \binom{n}{m} r(x)^{(n-m)} \cdot [\phi(x)]^{(m)}$$

By definition, we require that $\phi^{\star(n)}(x) = \phi^{(n)}(x)$ when each $[\phi(x)]^{(m)}$ vanish, for $m \in \{0, 1, \ldots, n-1\}$. That is, we want $\phi^{(n)}(x)$ to be a solution to $\phi^{\star(n)}(x)$ for all of our appropriate points (so we satisfy the n^{th} derivative requirement when each of the derivatives before the n^{th} derivative, of the decider, vanish; almost like induction).

Using the above, we're left with the problem:

$$[\phi(x)]^{(n)} \cdot r(x) = 0$$

The case where $[\phi(x)]^{(n)}$ vanishes yields a differential equation (or set of) which yields the trivial solution, so we'll focus on the other case: where $r(x)$ is a non-zero polynomial which vanishes.

Using the fact given before, where each decider vanishes, we know that $r(x)$ should at least vanish where the deciders vanish; that is, $r(x)$ can be given by, for some arbitrary polynomial $p(x)$:

$$r(x) = p(x) \prod_{m=0}^{n-1} [\phi(x)]^{(m)}$$

Furthermore, regard that for all $m \in \{0, 1, \ldots, n-1\}$ we have that $[\phi(x)]^{(m)}$ vanishes when $[\phi(x)]$ does. That is, for some arbitrary polynomials $t_m(x)$:

$$[\phi(x)]^{(m)} = [\phi(x)] \cdot t_m(x)$$

Therefore:

$$r(x) = p(x) \prod_{m=0}^{n-1} [\phi(x)] \cdot t_m(x) = [\phi(x)]^n \cdot p(x)$$

Using the definition of $r(x)$, therefore, we have that:

$$\phi^\star(x) = \mu(x) + [\phi(x)]^{n+1} \cdot p(x)$$

□

9.1.2. As a corollary to 9.1.1, suppose we have two analytic functions which share infinitely many derivatives at an arbitrary non-zero number of points. Then these functions must be identical everywhere (where these functions are analytic).

Proof. Let $n \in \mathbb{Z}^+$. We denote the functions described above as $\phi^\star(x)$, and respective 'solutions' as $\mu(x)$.

Since $\phi^\star(x)$ and $\mu(x)$ are analytic functions, we have that $\phi^\star(x) = \mu^\star(x) + \cancelto{0}{O(x^n)}$ and $\mu(x) =$

9.1.2 (continued)**.**
$M(x) + \cancelto{0}{O(x^n)}$ (as $n \to \infty$). We therefore have that:

$$\varphi^\star(x) = \lim_{n\to\infty} M(x) + [\phi(x)]^{n+1} \cdot r(x)$$

Then $r(x)$ is necessarily 0 since $\varphi^\star(x)$ is continuous. Therefore:

$$\varphi^\star(x) = M(x) \implies \phi^\star(x) = \phi(x)$$

□

9.2 Taylor Polynomials

We have now established two theorems which allow us to express the equality of polynomials, and analytic functions, through shared derivatives. We are therefore ready to establish the idea of a Taylor series: a function which encodes an infinite number of derivatives at a single point.

9.2.1. Let $\{a_n\}$ be a sequence such that for some analytic function f and $a \in \mathbb{R}$ we have that $f^{(n)}(a) = a_n$. Then $f(x)$ can be given by:

$$f(x) = \sum_{k=0}^{n} \frac{a_k}{k!}(x-a)^k + F(x)(x-a)^{n+1}$$

Proof. From the above, we have the following system of APOs:

$$\begin{aligned}
f(x) &= \begin{cases} a_0 & x = a \\ \star & \star \end{cases} = a_0 + (x-a)F_{0,0}(x) \\
f'(x) &= \begin{cases} a_1 & x = a \\ \star & \star \end{cases} = a_1 + (x-a)F_{0,1}(x) \\
&\vdots \\
f^{(n)}(x) &= \begin{cases} a_n & x = a \\ \star & \star \end{cases} = a_n + (x-a)F_{0,n}(x)
\end{aligned}$$

Explicitly, we write that for $m \geq 0$:

$$\begin{aligned}
f(x) &= a_0 + (x-a)F_{0,0}(x) \\
f^{(m)}(x) &= a_m + (x-a)F_{0,m}(x)
\end{aligned}$$

From the first equation, we can differentiate m times, and so it can be inductively shown that:

$$f^{(m)}(x) = mF_{0,0}^{(m-1)}(x) + F_{0,0}^{(m)}(x)$$

Letting $x = a$ and equating this with the second of our equations, we have that:

$$F_{0,0}^{(m-1)}(a) = \frac{a_m}{m}$$

9.2.1 (continued).
We now have a set of new functions (similar to our examples), and so repeating the same technique as before gives us:

$$\begin{aligned} F_{0,0}(x) &= a_1 + (x-a)F_{1,1}(x) \\ F_{0,0}^{(m-1)} &= \frac{a_m}{m} + (x-a)F_{1,m}(x) \end{aligned}$$

This is nearly identical to the first set of equations, with two noticable differences:

- $m \geq 1$, and so there is one less equation in the system than before; fewer derivatives.
- The leading term $\frac{a_m}{m}$ has differed from the original a_m and will be affected by the derivatives (this will, in fact, result in a factorial when this process is repeated).

From this setup, we set up an induction (to repeat this process) for which $0 \leq k < n$:

$$\begin{aligned} F_{k,k}(x) &= \frac{a_{k+1}}{(k+1)!} + (x-a)F_{k+1,k+1}(x) \\ F_{k,k}^{(m-k-1)}(x) &= \frac{a_m}{m(m-1)\dots(m-k)} + (x-a)F_{k+1,m}(x) \end{aligned}$$

Since we know this holds for $k = 0$, as this case satisfies the above, let us suppose for $0 \leq k < n$ we have the above hold for the $k+1$ case. From the first equation, we can inductively prove that:

$$F_{k,k}^{(m-k-1)}(x) = (m-k-1)F_{k+1,k+1}^{(m-k-2)}(x) + (x-a)F_{k+1,k+1}^{(m-k-1)}(x)$$

Equating with the second equation as before, we have that, for $x = a$:

$$F_{k+1,k+1}^{(m-k-2)}(a) = \frac{a_m}{m(m-1)\dots(m-k-1)}$$

This yields the following:

$$\begin{aligned} F_{k+1,k+1}^{(m-k-2)}(x) &= \frac{a_m}{m(m-1)\dots(m-k-1)} + (x-a)F_{k+2,m}(x) \\ F_{k+1,k+1}(x) &= \frac{a_{k+2}}{(k+2)!} + (x-a)F_{k+2,k+2}(x) \end{aligned}$$

Where the latter equation is derived from the former; $m = k+2$. We now have the following relevant formulas:

$$\begin{aligned} f(x) &= a_0 + (x-a)F_{0,0}(x) \\ f_{k,k}(x) &= \frac{a_{k+1}}{(k+1)!} + (x-a)F_{k+1,k+1}(x) \end{aligned}$$

Expanding recursively on $F_{0,0}(x)$ gives:

$$f(x) = a_0 + \frac{a_1}{1!}(x-a) + \frac{a_2}{2!}(x-a)^2 + \dots + \frac{a_n}{n!}(x-a)^n + F_{n,n}(x)(x-a)^{n+1}$$

□

For a minimal polynomial with these derivatives, one can let $F_{n,n}(x) = 0$.

If we extend 9.2.1 to an infinite number of derivatives, we get a Taylor series. However, we don't have a theorem here which states that knowing an infinite number of derivatives gives us, at least locally, an infinite number of points (equal to a function). This is the property of being real analytic (locally or globally; though actually this definition extends to power series in general). This relates to Taylor's theorem. Furthermore, given the way we've motivated and derived the formula, we nicely move into Borel's theorem too (both of which won't actually be discussed here, but are nice rabbit holes to go down).

9.3 Dual numbers (and extensions)

The dual numbers are a number system in which we have an element, ε, such that $\varepsilon^2 = 0$. They extend the real numbers in such a way we have a real component and a dual number component;

$$w = a + b\varepsilon$$

What makes this system very interesting is that we can extend analytic functions to the dual numbers:

$$f(w) = f(a + b\varepsilon) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(b\varepsilon)^n = f(a) + f'(a)b\varepsilon$$

That is, we take the Taylor series of f centred at a, and evaluate the function at $a + b\varepsilon$. The reason for this being so interesting is that we now have a direct mapping between dual numbers and a function's derivatives.

Circling back in on the idea of interpolation with derivatives, what we'll use is the following fact, in particular:

$$f(a) = a_0 \wedge f'(a) = a_1 \leftrightarrow f(0) = a_0 \wedge f(a + \varepsilon) = a_0 + a_1\varepsilon$$

This allows us to interpolate functions with values and first derivatives. We do so by taking in function values for f, generating a corresponding dual number-valued function F and then extracting our whole solution for f.

This above idea is actually closely related to multivariate interpolation: for some values of x, f', we want to find f. In some sense, we can think of f' as another argument or dimension, although we don't explicitly want to use it.

Example 9.3.1. We wish to find a polynomial function and an exponential function, f, g such that $f(0) = g(0) = 1$ and $f'(0) = g'(0) = 1$.

Note that $f(0) = 0 \wedge f'(0) = 1 \iff f(\varepsilon) = 1 + \varepsilon$. Then, beginning with the polynomial, we have:

$$\begin{aligned} F(w) &= \begin{cases} 1 & w = 0 \\ 1 + \varepsilon & w = \varepsilon \\ \star & \star \end{cases} \\ &= 1 + w \begin{cases} \frac{\varepsilon}{\varepsilon} & w = \varepsilon \\ \star & \star \end{cases} \\ &= 1 + w \end{aligned}$$

Substituting in the appropriate values, we see that is correct. Furthermore, if we take $F(a + b\varepsilon)$, we get the real part is given by $1 + a$. Therefore, $f(x) = 1 + x$.

Example 9.3.1 (continued)**.**
We now perform similar to find an exponential function:

$$\begin{aligned} G(w) &= \begin{cases} 1 & w = 0 \\ 1+\varepsilon & w = \varepsilon \\ \star & \star \end{cases} \\ &= \left(\begin{cases} (1+\varepsilon)^{\frac{1}{\varepsilon}} & w = \varepsilon \\ \star & \star \end{cases} \right)^{w} \end{aligned}$$

Note that $(1+\varepsilon)^{\frac{1}{\varepsilon}} := e$, since we might (abusively) write it as $\exp\bigl(\frac{1}{\varepsilon}\ln(1+\varepsilon)\bigr)$, and recursively evaluate. This gives us:

$$G(w) = e^{w}$$

Since $g(a) = \mathrm{Re}(G(a+b\varepsilon))$ we have $g(x) = e^{x}$, after taking the real part.

As we saw in this example, we actually divided by ε. We can't normally do this. But since we're simply trying to assign a value to various coefficients, it's totally fine, if not a little bit abusive of notation. Though, by argument, we should verify that our answer is correct by substituting in the appropriate values; interestingly, the non-indeterminate values we assign to problematic expressions like $\frac{\varepsilon}{\varepsilon}$ end up being correct for our uses. This becomes a significantly more important idea in the extension of the dual numbers.

On another note, we needn't perform the final substitution $a + b\varepsilon$; we may take the real part directly. This is to do with how ε^2 vanishes anyway, and the we disregard the imaginary part at the end regardless (though if we didn't, we'd observe the derivative and vanishing points).

9.3.1 Dual number extension

When we define $\varepsilon^2 = 0$ but $\varepsilon \neq 0$, we're actually defining a nilpotent element and an indeterminate. That is, ε doesn't have any specific value per se, but there exists a power n for which $\varepsilon^n = 0$. There exists a matrix representation, or several, rather, for ε:

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

And so, in some sense, we can give ε, and therefore dual numbers, a value. $w = a + b\varepsilon$ ends up being represented by:

$$w = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

This can be given by multiplying a scalar, a, by the 2×2 identity matrix. Hooray notation.

But now, a fairly straightforward question emerges: Can we get further derivatives using this idea? Can we embed or encode such a thing by extending the dual numbers? Should such a thing exist? Well, I can't answer the last question, but let us define the following:

9.3.1. For some $N \in \mathbb{N}$ and for all $n \in \{1, 2, \ldots, N\}$, we define some extension of the dual numbers using the elements:

$$\varepsilon_n^{n+1} = 0$$

Therefore, we can represent all elements in this number system using the following:

$$w = \sum_{n=1}^{N} \sum_{m=1}^{n-1} a_{n,m} \cdot \varepsilon_n^{m}$$

9.3.1 (continued).
This is because we haven't defined ε_n^n and so on to be 0.

9.3.2. In our N-extended dual number system, for all $i, j \in \{1, 2, \ldots, N\}$ and $i \neq j$, we have that:

$$\varepsilon_i \cdot \varepsilon_j = \varepsilon_j \cdot \varepsilon_i = 0$$

Proof. Suppose we have that $A = \varepsilon_i \varepsilon_j$ for some $j \in \{1, 2, \ldots, N\}$.
Then $\varepsilon_i^i \cdot A = \varepsilon_i^i \cdot \varepsilon_i \cdot \varepsilon_j = \varepsilon_i^{i+1} \cdot \varepsilon_j$.
The RHS is 0 by definition, and so we have that $\varepsilon_i^i \cdot A = 0$. Since ε_i^i is a non-zero element, we must have $A = 0$. This argument can be identically repeated for $\varepsilon_j \cdot \varepsilon_i$. $\square$

In other words, we don't care about commutativity under multiplication.

9.3.3. The degree of an N-extended dual number is given by the following:

$$\deg w = \min\{k \in \mathbb{Z}^+ \mid (w - \mathrm{Re}(w))^{k+1} = 0\}$$

Where $\mathrm{Re}(w)$ is the real part of w. In essence, we describe the degree of w to be the smallest such integer so that the non real part of w, raised to $k+1$, is 0. Therefore:

$$\deg \varepsilon_k = k$$

9.3.4. A 'simple' analytic extension of an analytic function to the N-extended dual numbers can be given by:

$$f(a + b\varepsilon_m) = \sum_{n=0}^{m} \frac{f^{(n)}(a)}{n!} (b\varepsilon_m)^n$$

These are not the only extensions which are relevant, but the extensions with all possible terms are just messy and so as-needed is perhaps better for those individual cases.

Proof. The analytic function f can be represented by its Taylor series centred around a:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)(a)}}{n!} (x - a)^n$$

Suppose we evaluate $f(x)$ at $x = a + b\varepsilon_m$, we get that:

$$f(a + b\varepsilon_m) = \sum_{n=0}^{\infty} \frac{f^{(n)(a)}}{n!} (b\varepsilon_m)^n$$

Since all terms beyond $n = m$ vanish by $\varepsilon_m^{m+1} = 0$, we rewrite the sum as

$$f(a + b\varepsilon_m) = \sum_{n=0}^{m} \frac{f^{(n)(a)}}{n!} (b\varepsilon_m)^n$$

$\square$

In order to encode $f(a), f'(a), \ldots, f^{(n)}(a)$ as a function, we require a definition for all of $f(a), f(a + b\varepsilon_1), \ldots, f(a + b\varepsilon_n)$. Intuitively, this is because the value of the function at each extended point provides several pieces of data (i.e. up to whatever derivative), and so we need to recurse down in order to retrieve each derivative. More or less.

Furthermore, we apply the same technique as in the regular dual numbers in order to interpolate our function. We choose to encode it as:

$$f(x) = \mathrm{Re}(F(w)), \quad w = x + x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n$$

This is for simplicity more than anything.

Example 9.3.2. We wish to find a polynomial f such that $f(0) = 1$, $f'(0) = 2$ and $f''(0) = 3$.

From $f(0) = 1$ and $f'(0) = 2$, we have that $F(\varepsilon_1) = 1 + 2\varepsilon_1$. Likewise, from $f(0) = 1$, $f'(0) = 2$ and $f''(0) = 3$, we have that $F(\varepsilon_2) = 1 + 2\varepsilon_2 + \frac{3}{2}\varepsilon_2^2$.

We construct our APO and simplify accordingly:

$$\begin{aligned}
F(w) &= \begin{cases} 1 & w = 0 \\ 1 + 2\varepsilon_1 & w = \varepsilon_1 \\ 1 + 2\varepsilon_2 + \frac{3}{2}\varepsilon_2^2 & w = \varepsilon_2 \\ \star & \star \end{cases} \\
&= 1 + w\begin{cases} \frac{2\varepsilon_1}{\varepsilon_1} & w = \varepsilon_1 \\ \frac{2\varepsilon_2 + \frac{3}{2}\varepsilon_2^2}{\varepsilon_2} & w = \varepsilon_2 \\ \star & \star \end{cases} \\
&= 1 + w\begin{cases} 2 & w = \varepsilon_1 \\ 2 + \frac{3}{2}\varepsilon_2 & w = \varepsilon_2 \\ \star & \star \end{cases} \\
&= 1 + w\left(2 + (w - \varepsilon_1)\begin{cases} \frac{\frac{3}{2}\varepsilon_2}{\varepsilon_2 - \varepsilon_1} & w = \varepsilon_2 \\ \star & \star \end{cases}\right)
\end{aligned}$$

We now have a problematic expression:

$$\frac{\varepsilon_2}{\varepsilon_2 - \varepsilon_1}$$

If we attempt to find some extended dual number for which this can be simplified, we won't find a solution. With this being said, if we multiply numerator and denominator by $\varepsilon_2 + \varepsilon_1$ we can 'set' the expression to be 1.

Finally, we have that:

$$F(w) = 1 + w\left(2 + \frac{3}{2}(w - \varepsilon_1)\right)$$

Regard that when we evaluate this function at each of our points $(0, \varepsilon_1, \varepsilon_2)$, we get the correct output. If we expand into 'real' and 'non-real' parts (keeping in mind that w is not strictly real; see the remark about taking the real part), we should have our solution.

$$F(w) = 1 + 2w + \frac{3}{2}w^2 - \frac{3}{2}\varepsilon_1 w \implies f(x) = 1 + 2x + \frac{3}{2}x^2$$

Furthermore, regard that the real part of this function corresponds exactly to the Taylor series at 0. This is because we've chosen not only a polynomial, but also to interpolate at a single point.

Example 9.3.3. As an exercise, let us verify that the minimal polynomial corresponding to $f(0) = 2$, $f'(0) = 3$ and $f(1) = 6$ is given by $f(x) = x^2 + 3x + 2$.

From $f(0) = 2$ and $f'(0) = 3$ we have $F(\varepsilon_1) = 2 + 3\varepsilon_1$. Therefore:

$$
\begin{aligned}
F(w) &= \begin{cases} 2 & w = 0 \\ 6 & w = 1 \\ 2 + 3\varepsilon_1 & w = \varepsilon_1 \\ \star & \star \end{cases} \\
&= 2 + w \begin{cases} 4 & w = 1 \\ 3\frac{\varepsilon_1}{\varepsilon_1} & w = \varepsilon_1 \\ \star & \star \end{cases} \\
&= 2 + w\left(4 + (w-1)\begin{cases} \frac{-1}{\varepsilon_1 - 1} & w = \varepsilon_1 \\ \star & \star \end{cases}\right) \\
&= 2 + w\left(4 + (w-1)\begin{cases} \varepsilon + 1 & w = \varepsilon_1 \\ \star & \star \end{cases}\right) \\
&= 2 + w(4 + (w-1)(\varepsilon_1 + 1))
\end{aligned}
$$

Expanding this out and separating real and non-real parts, we get:

$$F(w) = 2 + 4w + w(w-1) + w(w-1)\varepsilon_1 \implies f(x) = x^2 + 3x + 2$$

9.3.5. Division by an extended dual number can be given by the following, for non-zero real part:

$$\frac{1}{w} = \frac{1}{\operatorname{Re}(w)^{N+1}} \sum_{n=0}^{N} (-1)^n \operatorname{Re}(w)^{N-n} (w - \operatorname{Re}(w))^n$$

Where $N = \deg w$. This can be derived from the Taylor series of $\frac{1}{x}$. Furthermore, division by a nilpotent, extended dual number isn't well-defined, as previously mentioned.

However, let us note that multiplying both sides of this equation by w and $\operatorname{Re}(w)^{N+1}$ gives us the following:

$$\operatorname{Re}(w)^{N+1} = w \cdot \sum_{n=0}^{N} (-1)^n \operatorname{Re}(w)^{N-n} (w - \operatorname{Re}(w))^n$$

This, at least in form, resembles the norm of some complex number, namely that for some $z \in \mathbb{C}$ we have $z\overline{z} = \|z\|^2$; instead, we have that $w\overline{w} = \|w\|^{N+1}$. We can define $\overline{w}$, therefore, to be:

$$\overline{w} = \frac{\operatorname{Re}(w)^{N+1}}{w}$$

Even when N is odd, this still isn't really a *norm* since we fail $\|w\| = 0 \iff w = 0$, despite the fact that:

$$|\operatorname{Re}(w)| = \sqrt[N+1]{w\overline{w}}$$

9.3.5 (continued)**.**

So really, this is a seminorm. For even N then there's definitely no hope of it being a seminorm; $\sqrt[2n]{x^{2n}} = |x|$ for integer n; for odd powers we have $\sqrt[2n+1]{x^{2n+1}} = x$, which is not non-negative[3].

This phrase is problematic in English, see this English Stack Exchange thread; what we mean to say is that x is not restricted to being non-negative, as a property.

10 Introduction to Multivariate Interpolation

While we are on the topic of interpolation, let us briefly introduce the concept of multivariate interpolation; interpolation in multiple variables. For example, instead of taking values on the real number line and then mapping them to other real numbers, we may wish, instead, to take values on a plane and map them to real numbers, i.e. some 'height', thus producing a surface when graphed. Alternatively, we could instead associate numbers with vectors or coordinates (in fact, this is what a parametric equation is).

10.1 'And' multiple condition strategy

Recall that we discussed how we might work with multiple conditions in a piecewise object in 5.4. Suppose we have an interpolation problem with the following APO representation (which can be complex, also), for $f : \mathbb{R}^n \to \mathbb{R}$:

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} y_1 & x_1 = x_{1,1} & x_2 = x_{1,2} & \ldots & x_n = x_{1,n} \\ y_2 & x_1 = x_{2,1} & x_2 = x_{2,2} & \ldots & x_n = x_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_k & x_1 = x_{k,1} & x_2 = x_{k,2} & \ldots & x_n = x_{k,n} \\ \star & \star & \star & \ldots & \star \end{cases}$$

Then by the 'and' property, we can rewrite this as (for some functions $p_{u,v}$):

$$f(x_1, x_2, \ldots, x_n) = \begin{cases} y_1 & \sum_{m=1}^{n} p_{1,m}(x_m - x_{1,m}) = 0 \\ y_2 & \sum_{m=1}^{n} p_{2,m}(x_m - x_{2,m}) = 0 \\ \vdots & \vdots \\ y_k & \sum_{m=1}^{n} p_{k,m}(x_m - x_{k,m}) = 0 \\ \star & \star \end{cases}$$

And so we can return to our standard interpolation techniques. The pitfalls of this method are that we don't necessarily produce the simplest equations which satisfy all of our points, and subsequent calculations may be significantly more difficult than they need be.

Furthermore, it's worth noting that we're not restricted to the iff requirement we originally imposed on our 'and' condition (that is, p needn't have the properties described in the section). So, for example, we could have $p_{u,v}(x) = x$, at the risk of losing well-definition in our subsequent function (if another piece satisfies the same condition; we would otherwise end up dividing by 0 by attempting to perform standard interpolation).

Example 10.1.1. Suppose we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(1, 0) = 1$, $f(0, 1) = 1$, $f(0, 0) = 0$. Then we represent this problem as the following:

$$f(x, y) = \begin{cases} 1 & x = 1 & y = 0 \\ 1 & x = 0 & y = 1 \\ 0 & x = 0 & y = 0 \end{cases}$$

By our 'and' property, we can rewrite this as:

$$f(x, y) = \begin{cases} 1 & p_{1,1}(x - 1) + p_{1,2}(y) = 0 \\ 1 & p_{2,1}(x) + p_{2,2}(y - 1) = 0 \\ 0 & p_{3,1}(x) + p_{3,2}(y) = 0 \\ \star & \star \end{cases}$$

Example 10.1.1 (continued)**.**
Interestingly, we can let each of our p functions be $p_{u,v}(x) = x$ (interesting because this function remains well-defined) so that we get:

$$f(x,y) = \begin{cases} 1 & x - 1 + y = 0 \\ 1 & x + y - 1 = 0 \\ 0 & x + y = 0 \\ \star & \star \end{cases}$$

Extracting the final case gives:

$$f(x,y) = (x+y)\left(\begin{cases} 1 & x + y = 1 \\ 1 & x + y = 1 \\ \star & \star \end{cases}\right) = x + y$$

An oddly coincidental result! Perhaps, instead, we should let all of our p functions be x^2:

$$f(x,y) = \begin{cases} 1 & (x-1)^2 + y^2 = 0 \\ 1 & x^2 + (y-1)^2 = 0 \\ 0 & x^2 + y^2 = 0 \\ \star & \star \end{cases}$$

Extracting the final piece gives:

$$f(x,y) = (x^2+y^2)\begin{cases} \frac{1}{x^2+y^2} & (x-1)^2 + y^2 = 0 \\ \frac{1}{x^2+y^2} & x^2 + (y-1)^2 = 0 \\ \star & \star \end{cases}$$

Recall the first piece corresponds to $x = 1$ and $y = 0$, and the second corresponds to $x = 0$ and $y = 1$. This yields:

$$f(x,y) = (x^2+y^2)\left(\begin{cases} 1 & (x-1)^2 + y^2 = 0 \\ 1 & x^2 + (y-1)^2 = 0 \\ \star & \star \end{cases}\right) = x^2 + y^2$$

So both $x^2 + y^2$ and $x + y$ satisfy our interpolation problem, and these solutions are dependent on how we choose our functions.

10.2 Parametric strategy

Rather than attempt to solve an interpolation problem $f : \mathbb{R}^n \to \mathbb{R}$, we could attempt to instead solve $f : \mathbb{R} \to \mathbb{R}^{n+1}$ by means of parameterisation. That is, we take the original arguments $x_1, x_2, \dots, x_n$ with the output f and change to instead have some argument t and outputs $x_1, x_2, \dots, x_n, f$. That is, we would write $x_1 = x_1(t), x_2 = x_2(t), \dots, x_n = x_n(t), f(x_1, \dots, x_n) = f(t)$.

This, obviously, is not an ideal approach. Usually when we want to perform multivariate interpolation, we want a specific form for a reason. However, this technique demonstrates how we can also work with vectors or tuples in piecewise objects. We can represent the above problem as:

$$(x_1, x_2, \dots, x_n, f) = \begin{cases} (x_{1,1}, x_{1,2}, \dots, x_{1,n}, y_1) & t = t_1 \\ (x_{2,1}, x_{2,2}, \dots, x_{2,n}, y_2) & t = t_2 \\ \vdots & \vdots \\ (x_{k,1}, x_{k,2}, \dots, x_{k,n}, y_k) & t = t_k \\ \star & \star \end{cases}$$

If we then distribute the piecewise object across each element in the tuples, we can write:

$$(x_1, x_2, \dots, x_n, f) = \left(\begin{cases} x_{1,1} & t = t_1 \\ x_{2,1} & t = t_2 \\ \vdots & \vdots \\ x_{k,1} & t = t_k \\ \star & \star \end{cases}, \begin{cases} x_{1,2} & t = t_1 \\ x_{2,2} & t = t_2 \\ \vdots & \vdots \\ x_{k,2} & t = t_k \\ \star & \star \end{cases}, \dots, \begin{cases} x_{1,n} & t = t_1 \\ x_{2,n} & t = t_2 \\ \vdots & \vdots \\ x_{k,n} & t = t_k \\ \star & \star \end{cases} \right)$$

In general, this is one method for handling interpolations problems of the form $f : \mathbb{R} \to \mathbb{R}^n$.

10.3 Nesting strategy

The gist of this strategy is to interpolate in one direction, then in another, and so on, for each of your arguments (or dimensions). For linear problems, this corresponds to multilinear interpolation.

In terms of the piecewise notation, or APOs, this simply means recursively nesting pieces on our arguments, and then treating nested piecewise objects as their own interpolation problems. This is a decent method for finding reasonably simple functions which satisfy our points, but is subject to the order in which we nest piecewise objects. It is furthermore algebraically straightforward to perform.

To resolve this, we can permute all possible orders and make use of the general solution to each nested interpolation problem. This gives a set of solutions. We then use the fact that for all our interpolated points, the arithmetic mean of all solutions is equal to the solution itself. That is, for some function f and solutions $f_1, f_2, \dots, f_n$, we use:

$$f = \frac{1}{n}(f_1 + f_2 + \cdots + f_n)$$

There are alternate ways to formulate this, but they're irrelevant: the idea is that we superimpose the solutions in the way we interpolate. We could use the geometric mean for exponential extraction, and the arithmetic mean for polynomial extraction.

In fact, recall Example 3.6.2. This method more or less extends this approach (or, rather, that example was one in a subset of problems we can solve using this strategy now).

Example 10.3.1. Suppose we have a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0, 0) = 0$, $f(1, 0) = 1$, $f(0, 1) = 1$ and $f(1, 1) = 2$. Then we can represent this problem as the following:

$$f(x, y) = \begin{cases} 0 & x = 0 & y = 0 \\ 1 & x = 1 & y = 0 \\ 1 & x = 0 & y = 1 \\ 2 & x = 1 & y = 1 \\ \star & \star \end{cases}$$

Example 10.3.1 (continued)**.**
Nesting on the x conditions, we have:

$$f(x,y) = \begin{cases} \begin{cases} 0 & y=0 \\ 1 & y=1 \\ \star & \star \end{cases} & x=0 \\ \begin{cases} 1 & y=0 \\ 2 & y=1 \\ \star & \star \end{cases} & x=1 \\ \star & \star \end{cases}$$

Solving each nested APO problem individually, we have, for arbitrary $A(x,y)$ and $B(x,y)$:

$$f(x,y) = \begin{cases} y + y(y-1)A(x,y) & x=0 \\ y+1+y(y-1)B(x,y) & x=1 \\ \star & \star \end{cases}$$

Iterating further on this APO, we get:

$$f(x,y) = y + x + y(y-1)\begin{cases} A(x,y) & x=0 \\ B(x,y) & x=1 \\ \star & \star \end{cases}$$

And so our general solution (for this order) is:

$$f(x) = f_1(x,y) = y + x + A(x,y)y(y-1) + (B(x,y) - A(x,y))xy(y-1) + C(x,y)xy(x-1)(y-1)$$

Had we instead interpolated 'in the other direction', we would have:

$$f(x) = f_2(x,y) = y + x + D(x,y)x(x-1) + (E(x,y) - D(x,y))xy(x-1) + F(x,y)xy(x-1)(y-1)$$

We can more or less superimpose these general solutions using the fact that for our interpolated points:

$$f(x,y) = \frac{1}{2}(f_1(x,y) + f_2(x,y))$$

Since $f(x,y) = f_1(x,y) = f_2(x,y)$ at these points. Therefore, after simplifying all of the arbitrary functions, we get:

$$\begin{aligned} f(x,y) = x + y + a(x,y)y(y-1) + b(x,y)x(x-1) + \\ c(x,y)xy(y-1) + d(x,y)xy(x-1) + e(x,y)xy(x-1)(y-1) \end{aligned}$$

10.4 Complex number strategy

This strategy works for functions with two arguments only (but can potentially be extended using other hypercomplex number systems). In essence, suppose we have a function $f : \mathbb{R}^2 \to \mathbb{R}$, then we find a function F such that:

$$f(x,y) = \mathrm{Re}(F(x+iy))$$

And for all our points, i.e. for all $k \in \{1, 2, \ldots, n\}$, $(x_k, y_k) \to z_k$, we set $F(x_k + iy_k) = z_k$.

Example 10.4.1. Let us re-use an example from before, where we have $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(1,1) = 2$, $f(1,0) = 1$ and $f(0,1) = 1$. Then we have F such that:

$$F(z) = \begin{cases} 0 & z = 0 \\ 1 & z = 1 \\ 1 & z = i \\ \star & \star \end{cases}$$

Following this through as normal, we have:

$$F(z) = 1 - i(z-1)(z-i) = 2 - iz^2 + (i-1)z$$

Expanding using $z = x + iy$ and factoring on real and imaginary parts:

$$F(x+iy) = (2 + 2xy - x - y) + i(-x^2 + y^2 + x - y)$$

Notice that at all of our given points, the imaginary part vanishes. However, by our definition, we have:

$$f(x,y) = 2 + 2xy - x - y$$

10.5 Decomposition strategy

Recall from the univariate interpolation section with Lagrange polynomials, Section 5.2.2, we decomposed our APO into a sum of elements which looked a lot like Iverson brackets:

$$(x)_{x_a} = \begin{cases} 1 & x = x_a \\ 0 & x \neq x_a \\ \star & \star \end{cases}$$

Where $x \neq x_a$ is given by $x \in \{x_1, \ldots, x_n\} \setminus \{x_a\}$; or, alternatively,

$$(x)_{x_a} = \prod_{\substack{m=1 \\ m \neq a}}^{n} \frac{x - x_m}{x_a - x_m}$$

We can extend this idea to higher dimensions or arguments; i.e.,

$$(x_1, x_2, \ldots, x_n)_{(a_{1,j}, a_{2,j}, \ldots, a_{n,j})} = \begin{cases} 1 & (x_1, x_2, \ldots, x_n) = (a_{1,j}, a_{2,j}, \ldots, a_{n,j}) \\ 0 & (x_1, x_2, \ldots, x_n) \neq (a_{1,j}, a_{2,j}, \ldots, a_{n,j}) \\ \star & \star \end{cases}$$

But, of course, we haven't solved our problem. Instead, we realise that, by our 'or' property, we can add in pieces to the above in order to decompose (overall) our problem. This is done by the following:

$$\prod_{m=1}^{n} (x_m)_{x_{m,j}} \subseteq (x_1, x_2, \ldots, x_n)_{(a_{1,j}, a_{2,j}, \ldots, a_{n,j})}$$

And therefore we can use the left hand side as a substitution for the right hand side.

Example 10.5.1. Let us re-use the example $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0,0) = 0$, $f(1,0) = 1$,

Example 10.5.1 (continued).
$f(0,1) = 1$ and $f(1,1) = 2$. Then we have:

$$f(x,y) = \begin{cases} 0 & x=0 & y=0 \\ 1 & x=1 & y=0 \\ 1 & x=0 & y=1 \\ 2 & x=1 & y=1 \\ \star & \star \end{cases}$$

Then we can split this APO up into several, giving us:

$$f(x,y) = \begin{cases} 0 & x=0 & y=0 \\ 0 & x=1 & y=0 \\ 0 & x=0 & y=1 \\ 0 & x=1 & y=1 \\ \star & \star \end{cases} + \begin{cases} 0 & x=0 & y=0 \\ 1 & x=1 & y=0 \\ 0 & x=0 & y=1 \\ 0 & x=1 & y=1 \\ \star & \star \end{cases} + \begin{cases} 0 & x=0 & y=0 \\ 0 & x=1 & y=0 \\ 1 & x=0 & y=1 \\ 0 & x=1 & y=1 \\ \star & \star \end{cases} + \begin{cases} 0 & x=0 & y=0 \\ 0 & x=1 & y=0 \\ 0 & x=0 & y=1 \\ 2 & x=1 & y=1 \\ \star & \star \end{cases}$$

As above, this means we have:

$$f(x,y) = 0 \cdot (x,y)_{(0,0)} + 1 \cdot (x,y)_{(1,0)} + 1 \cdot (x,y)_{(0,1)} + 2 \cdot (x,y)_{(1,1)}$$

Which gives, by our decomposition substitution:

$$f(x,y) = (x)_1(y)_0 + (x)_0(y)_1 + 2(x)_1(y)_1$$

We can now evaluate $(x)_0$, $(x)_1$, $(y)_0$ and $(y)_1$:

$$\begin{aligned} (x)_0 &= \begin{cases} 1 & x=0 \\ 0 & x=1 \end{cases} && = 1-x \\ (x)_1 &= \begin{cases} 1 & x=1 \\ 0 & x=0 \end{cases} && = x \\ (y)_0 &= \begin{cases} 1 & y=0 \\ 0 & y=1 \end{cases} && = 1-y \\ (y)_1 &= \begin{cases} 1 & y=1 \\ 0 & y=0 \end{cases} && = y \end{aligned}$$

Substituting, this gives:

$$f(x,y) = x(1-y) + (1-x)y + 2xy = x+y$$

11 Expressions for Relations; Multivariable Functions

12 Complex Numbers and Piecewise Objects

Part III

Appendix