Piecewise Notes

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1 Introduction

These notes serve as a single source of knowledge in which those reading should ideally be able to garner a deeper understanding of the thoughts and ideas about piecewise objects I have had and continue to have.

To consider something piecewise is to enumerate something under differing conditions. For example, one might consider the function $|x|$ piecewise, namely $|x| = x$ when $x \geq 0$ or $|x| = -x$ when $x \leq 0$.

An object which is piecewise is something which contains a set of pieces, which themselves contain values and conditions, under which those values are taken for the overall object. Continuing with the example of $|x|$, we have the piece value x for when $x \geq 0$, the condition, (which forms a piece), and $-x$ for when $x \leq 0$.

We shall develop ideas such as above more explicitly throughout these notes. These ideas often intersect with other areas of maths, which one might be familiar with — but if not, don't panic; such ideas are not strictly foundational to the presented notes.

Furthermore, these notes focus on construction; rather than evaluating existing concepts or formulas, we focus on deriving existing or new tools, ideas and formulas. For you, reader, this must be a process you should become familiar with, and rather than just reading these notes, attempt to follow along by hand, and construct your own objects using the ideas presented here. We also stress that with ideas that intersect with more mainstream mathematics, that existing concepts be used to evaluate the validity of the constructions.

Finally, if you have trouble understanding some of the ideas presented here, you should consider taking a look at the blog posts on `https://piecewise.org` or contacting myself at ally@piecewise.org. The things you'll see here are the culmination of many hundreds of hours scribbling, doodling and refining thoughts over the course of several years. More importantly than the fact I am myself still learning the fundamentals of mathematics at a higher level, is that you understand that trial and lots of error forms the majority of this work.

Thank you in advance for reading.

2 Notation

2.1 Single variable piecewise functions

A piecewise *function* is a function defined over several pieces.

Let us consider some function $f : \mathbb{R} \to \mathbb{R}$ and some intervals $D_1, D_2, \ldots, D_n \subseteq \mathbb{R}$ such that $D_i \cap D_j = \varnothing$ for all $i \neq j$, where $i, j = 1, 2, \ldots, n$. That is, we have some number real intervals which do not overlap. Suppose we have some functions $f_1 : D_1 \to \mathbb{R}$ and so on. We describe this function using the following notation:

$$f(x) = \begin{cases} f_1(x) & x \in D_1 \\ f_2(x) & x \in D_2 \\ \vdots & \vdots \\ f_n(x) & x \in D_n \end{cases}$$

This same function could instead be written as the following:

$$\begin{gathered} f : D_1 \to \mathbb{R}, f(x) = f_1(x), \\ f : D_2 \to \mathbb{R}, f(x) = f_2(x), \\ \vdots \\ f : D_n \to \mathbb{R}, f(x) = f_n(x) \end{gathered}$$

This essentially describes the same function over different domains. Alternatively, one might think about it as different functions under the same label, f.

Reading this function more verbosely, we say that if x is in the domain D_1 and so on, then we might 'choose' to have $f(x) = f_1(x)$ as per our definition.

Example 2.1.1. Let us define the absolute value function, $|x|$, as the following:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

To evaluate, for example, $|-3|$, we use the definition:

$$|-3| = \begin{cases} -3 & -3 \geq 0 \\ 3 & -3 \leq 0 \end{cases}$$

Since the condition in the second piece (or case) evaluates to true, we use the value in that piece. That is, since $-3 \leq 0$ is true, $|-3| = 3$.

2.2 Piecewise functions

Piecewise functions needn't be restricted to a single variable. More generally, piecewise functions needn't be restricted to any number of conditions or pieces. The perfect example, as we'll see later, is the floor function, which has an infinite number of pieces. Similarly, the bivariate function describing a square when it intersects a certain plane has 4 pieces.

In general, we can notate such a more general piecewise function like the following:

$$f(x) = \begin{cases} f_1(x) & C_{1,1} \quad C_{1,2} \quad \dots \quad C_{1,m} \\ f_2(x) & C_{2,1} \quad C_{2,2} \quad \dots \quad C_{2,m} \\ \vdots & \vdots \qquad \vdots \qquad \ddots \quad \vdots \\ f_n(x) & C_{n,1} \quad C_{n,2} \quad \dots \quad C_{n,m} \end{cases}$$

Where $C_{p,q}$ is some condition which evaluates to true or false for some inputs (e.g. a predicate). We interpret the string of conditions in each piece, such as $\{C_{1,1} \quad C_{1,2} \quad \dots \quad C_{1,m}\}$, as equivalent to the condition $C_{1,1} \wedge C_{1,2} \wedge \dots \wedge C_{1,m}$.

Such conditions could be anything, and per the section above, we could have $C_{p,q}$ denote the predicate $x \in D_{p,q}$ for some set $D_{p,q}$.

Importantly, we note in general this piecewise object is written in a shortened form, without commas or English terms such as 'and' or 'or', or their respective logical operators. This is a conventional decision in these notes given the number of piecewise objects written. For example, the following is identical to the above:

$$f(x) = \begin{cases} f_1(x), & \text{if } C_{1,1} \wedge C_{1,2} \wedge \dots \wedge C_{1,m} \\ f_2(x), & \text{if } C_{2,1} \wedge C_{2,2} \wedge \dots \wedge C_{2,m} \\ \vdots & \vdots \\ f_n(x), & \text{if } C_{n,1} \wedge C_{n,2} \wedge \dots \wedge C_{n,m} \end{cases}$$

And of course we might package each piece's conditions together, effectively making one condition per piece. Though, in practice, this doesn't help very much at all (this is because, as we will later see, considering each condition individually yields results which affect the way we interpolate functions and other manipulation of piecewise objects).

2.3 Generalised piecewise object

Finally, we might consider a set-like generalised form of a piecewise object. Let us denote the following:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

Where ϕ describes the piecewise object, and for each $i \in I$ (the iterator), $\phi = \varphi_i$ for when C_i is true.

Example 2.3.1. Suppose we have a set $I = \{1, 2, \ldots, n\}$ and a piecewise function f such that $f = f_i$ when C_i is true.

We can express this function as:

$$f = \begin{cases} f_1 & C_1 \\ f_2 & C_2 \\ \vdots & \vdots \\ f_n & C_n \end{cases}$$

Alternatively, we might express it using the generalised notation:

$$f = \{f_i, \quad C_i \mid i \in \{1, 2, \ldots, n\}\}$$

Example 2.3.2. We will see the floor function later on where it will be described more deeply, but we can introduce it like so using our notation:

$$\lfloor x \rfloor = \{n, \quad x \in [n, n+1) \mid n \in \mathbb{Z}\}$$

To evaluate this function at $x = 3.5$ consider that for all $n \in \mathbb{Z}$ there exists only one such interval $N = [n, n+1)$ such that $3.5 \in N$. This interval is $[3, 4)$, i.e. $n = 3$; the corresponding piece value is 3 and so $\lfloor 3.5 \rfloor = 3$.

If a single iterator (e.g. $i \in I$) isn't sufficient to describe the piecewise object we desire, we might consider using several; $i \in I$, $j \in J$ for example). Alternatively, we might consider joining two piecewise objects together using $\cup$:

$$\varphi = \varphi_1 \cup \varphi_2 = \{\varphi_{1,i}, \quad A_i \mid i \in I\} \cup \{\varphi_{2,j}, \quad B_j \mid j \in J\}$$

This can be interpreted similarly to before, though we now have that $\varphi = \varphi_{1,i}$ if A_i is true for some i, or $\varphi = \varphi_{2,j}$ is B_j is true for some j.

Notably, this notation can be used to express individual pieces explicitly in a piecewise object rather than collectively. If such a piece isn't dependent on the iterator, we can ignore that part entirely (keeping in mind we might be able to rewrite it using an iterator later on).

Example 2.3.3. Recall the absolute value function; see Example 2.1.1.

We can then express $|x|$ as:

$$|x| = \{x, \quad x \geq 0\} \cup \{-x, \quad x \leq 0\}$$

2.3.1. Using our notation, we can also explicitly define unions:

Suppose $I = U \cup V$, then the following piecewise object

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

is equivalent to:

$$\phi = \{\varphi_u, \quad C_u \mid u \in U\} \cup \{\varphi_v, \quad C_v \mid v \in V\}$$

(There is no intersection equivalent.)

3 Algebra on Piecewise Objects

3.1 Piece association

Operations on piecewise objects are fairly straightforward and behave as any other object in the context you're working in (whether this be with the real numbers, complex numbers, certain algebras, spaces, and so on).

What this section aims to do is not to teach you directly how to perform exactly the operations you want to on each piecewise object, but instead provide a basis for which you can base your ideas: no piece of a piecewise object is independent of another. Equivalently, each piece is dependent on each other piece. There is an intuitive reason for this: you are evaluating an object not conditionally, but in full generality; each piece exists, in some sense, 'simultaneously'.

It is here we might recognise that this idea could fairly easily lead into combinatorics; an area of maths which deals heavily with the enumeration and construction of such objects.

3.2 Equality property

3.2.1. Consider the following piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

Then if for all $i, j \in I$ we have that $\varphi_i = \varphi_j$ (that is, all piece values are equal for when ϕ is defined), then $\phi = \varphi_i$ for any $i \in I$.

For example, consider the function $f : (-\infty, 0] \to \mathbb{R}$:

$$f(x) = \begin{cases} 0 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We have that $f(x) = 0$. This is because everywhere in $\text{dom}(f)$, or both when $x \geq 0$ and $x < 0$, we have that $f(x) = 0$.

3.3 Piece and condition equivalence

Arguably one of the most fundamental property of piecewise objects is the relationship between each piece's conditions and its respective value. That is, the values for which a piecewise object takes is dependent on their respective conditions, and so can be treated explicitly as such.

3.3.1. Consider the following piecewise function:

$$f(x) = \{f_i(x), \quad C_i \mid i \in I\}$$

For all $i \in I$, let us define a substitution $x \sim y_i$, where $\sim$ represents equality under the condition C_i. Then we might consider writing $f(x)$ as:

3.3.1 (continued).

$$f(x) = \{f_i(y_i), \quad C_i \mid i \in I\}$$

And $f(x)$ may still remain a non-constant function, but in this way we've given it another representation. For some C_i we might just have $x \sim x$ (and is effectively no substitution at all). This substitution can also go the other way, i.e. $y_i \sim x$.

3.3.2. Consider the following piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

For all $i \in I$, suppose that $C_i \leftrightarrow D_i$ in the context of ϕ (i.e. if a function, within its domain, etc.). Then we can rewrite this object as:

$$\phi = \{\varphi_i, \quad D_i \mid i \in I\}$$

That is, we've substituted the set of our conditions for another set of conditions (and again, we may not have substituted all of them). For example, we know that $x = 5$ when $(x-5)^2 \leq 0$, and so these two conditions are interchangeable.

This idea becomes far more important when working using other functions, such as max, min and $|x|$ (which will be covered later), although it can still be used in other contexts. In essence, for certain classes of piecewise functions, we can build off existing functions to construct not only transformations, but completely new functions and representations.

Example 3.3.1. As an example, let us consider the function $f : (-6, 6) \to \mathbb{R}$:

$$f(x) = \begin{cases} x & x > 5 \\ 5 & x = 5 \\ x & x < 5 \end{cases}$$

Consider the piece for which $x = 5$: note that the piece value is equal to 5. Since $x = 5$ we can replace 5 with x to get:

$$f(x) = \begin{cases} x & x > 5 \\ x & x = 5 \\ x & x < 5 \end{cases}$$

Example 3.3.1 (continued)**.**
And by the equality property, we know that this function is equivalent to $f(x) = x$.

Finally, depending on the context of the problem, function or object, we can rewrite conditions in some way to make it more straightforward for us to understand. This becomes relevant in the context of function domains.

Example 3.3.2. Let us consider the function from Example 3.3.1.

The following two representations are equivalent:

$$f(x) = \begin{cases} x & x > 5 \\ 5 & x = 5 \\ x & x < 5 \end{cases}$$

$$f(x) = \begin{cases} x & x \in (5, 6) \\ 5 & x = 5 \\ x & x \in (-6, 5) \end{cases}$$

The reason for this is that for all $x \in (-6, 5)$ we have that $x < 5$, despite the contrary not being always true. The reason the contrary needn't always be true for all $\mathbb{R}$ is because our function f is defined on the interval $(-6, 6)$. This same argument can also be applied to $x \in (5, 6) \implies x > 5$.

A strong note on functions which are not well-defined: Piecewise objects can easily define not well-defined functions. In these cases, care must be taken when working with pieces in general, values or conditions.

3.3.3. A piecewise function is only well-defined if all of its piece values denote well-defined functions, and in places where two conditions in separate pieces are true, their respective values must be equal.

Formally, given the piecewise object $\phi = \{\varphi_i, \quad C_i \mid i \in I\}$ if there exists $i \neq j \in I$ such that $C_i \wedge C_j$ is true, then ϕ is well-defined iff $\varphi_i = \varphi_j$ and each φ_k for $k \in I$ is well-defined.

For example, the following is not well-defined:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

This is because the latter piece is 0 for all real numbers that aren't integers, and the first piece is 1 for all rational numbers; these include non-integers. There are a few ways we could change this function to be well-defined:

3.3.3 (continued).

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

This is known as the Dirichlet function; it's the indicator function of the rationals (which we'll cover later on). Alternatively:

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

Which would simply make the function 0 for all $\mathbb{R}$.

3.4 (De)nesting pieces; logical 'and'

The third of several special operations on piecewise objects involves the nesting and denesting of pieces; essentially, we decouple the conditions in each piece from one another in order to group and subsequently simplify the piece values.

Example 3.4.1. Consider the following piecewise object:

$$\phi = \begin{cases} \varphi_1 & C_1 \\ \varphi_2 & C_2 \\ \vdots & \vdots \\ \varphi_n & C_n \end{cases}$$

Suppose that we have

$$\varphi_1 = \begin{cases} \mu_1 & D_1 \\ \mu_2 & D_2 \end{cases}$$

Then we can rewrite ϕ as:

$$\phi = \begin{cases} \begin{cases} \mu_1 & D_1 \\ \mu_2 & D_2 \end{cases} & C_1 \\ \varphi_2 & C_2 \\ \vdots & \vdots \\ \varphi_n & C_n \end{cases}$$

Suppose that $D_1 \wedge C_1$ is true: then it stands to reason that $\phi = \mu_1$. Likewise, if $D_2 \wedge C_1$ then $\phi = \mu_2$. The behaviour is as normal for each other piece. This also means ϕ can be represented as:

Example 3.4.1 (continued).

$$\phi = \begin{cases} \mu_1 & C_1 & D_1 \\ \mu_2 & C_1 & D_2 \\ \varphi_2 & C_2 \\ \vdots & \vdots \\ \varphi_n & C_n \end{cases}$$

And so we might go back and forth between these two forms to represent the same object, keeping in mind each column of the piecewise object, other than the first, represents the conditions under which that piece value is taken (the logical 'and').

In general, if we have common, and multiple, conditions for pieces, we should be able to nest, or denest, piecewise objects. This often helps simplify piecewise problems in multiple variables, or single variables with intervals, etc.

3.4.1. Given the following piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

If we have that $C_i \leftrightarrow A_i \wedge B_i$ for some conditions A_i, B_i, then

$$\phi = \{\varphi_i, \quad A_i \wedge B_i \mid i \in I\}$$

This is a semi-obvious fact, but becomes useful when coupled with the grouping/nesting of piecewise objects via their piece's conditions. This was also touched on in Chapter 1.

3.5 Combining, splitting piece conditions; logical 'or'

Finally, we'll look at explicit logical 'or' in piece conditions at a basic level and in full generality. This is just as important as the previous section, particularly for grouping.

3.5.1. Given the following piecewise object (e.g. $A_i \vee B_i \leftrightarrow C_i$):

$$\phi = \{\varphi_i, \quad A_i \vee B_i \mid i \in I\}$$

For some conditions A_i, B_i, we can rewrite this as the following:

$$\phi = \{\varphi_i, \quad A_i \mid i \in I\} \cup \{\varphi_i, \quad B_i \mid i \in I\}$$

And vice versa.

We are effectively using the logical 'or' condition in each piece to split up

3.5.1 (continued)**.**
the pieces such that each piece only has one condition in the piecewise object representation. For example,

$$f(x) = \begin{cases} 5 & x \geq 5 \vee x \leq -5 \\ 0 & -5 < x < 5 \end{cases}$$

Is equivalent to:

$$f(x) = \begin{cases} 5 & x \geq 5 \\ 5 & x \leq -5 \\ 0 & -5 < x < 5 \end{cases}$$

Noting that the last piece cannot be split as it is a logical 'and'; that is, $x < 5 \wedge x > -5$.

3.5.2. Finally, there is no rule that says pieces are unique in a piecewise object (although generally redundant, duplicate pieces can be useful for grouping once more).

That is, the following, for some set $J \subseteq I$

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

is equivalent to:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\} \cup \{\varphi_j, \quad C_j \mid j \in J\}$$

The reason for this comes down to our well-definition argument as before (equal values).

3.6 Functions on piecewise objects

Arguably the most important property of piecewise objects is the following:

3.6.1. Given the following piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

We have that (when applicable):

$$f(\phi) = \{f(\varphi_i), \quad C_i \mid i \in I\}$$

A quick intuition is as follows: For all $i \in I$ suppose that at least one

3.6.1 (continued)**.**
C_i is true. Then we have that $\phi = \varphi_i \implies f(\phi) = f(\varphi_i)$. Then, since $C_i \to f(\phi) = f(\varphi_i)$, we have by definition the above.

3.6.2. Let $\phi = \{\varphi_i, \quad C_i \mid i \in I\}$ and let a be some element in the appropriate domain. We have that:

1. $\phi + a = \{\varphi_i + a, \quad C_i \mid i \in I\}$ (if $\varphi_i + a$ is defined for $i \in I$)
2. $a \cdot \phi = \{a \cdot \varphi_i, \quad C_i \mid i \in I\}$ (if $a \cdot \varphi_i$ is defined for $i \in I$)

And so on, so forth, where applicable.

Example 3.6.1. Let us define the following function:

$$f(x) = \begin{cases} 3x + 2 & x \geq 1 \\ 5 & x \leq 1 \end{cases}$$

We should establish that this function is, in fact, continuous (although this is implied by conditions $x \geq 1$ and $x \leq 1$). For $x \geq 1$ we have that $f(x) = 3x + 2$ and so $x = 1 \implies f(1) = 5$. Likewise for $x \leq 1$ we have $x = 1 \implies f(1) = 5$.

We want to rewrite this function as a linear combination (and transformation) of the following function:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Note that $f(x)$ has conditions $x \geq 1$ and $x \leq 1$, which differs from the conditions in $|x|$. Therefore, we might rewrite these conditions, namely that $x \geq 1 \implies x - 1 \geq 0$ and $x \leq 1 \implies x - 1 \leq 0$. From here, let us define $y = x - 1 \implies x = y + 1$ and substitute into $f(x)$:

$$f(y + 1) = \begin{cases} 3(y + 1) + 2 & y \geq 0 \\ 5 & y \leq 0 \end{cases}$$

From here, we should focus on the piece values themselves: Since they differ from the form we want, we should manipulate them until we achieve something that looks like $|y|$. Namely:

Let us expand the first piece of $f(y + 1)$:

$$f(y + 1) = \begin{cases} 3y + 5 & y \geq 0 \\ 5 & y \leq 0 \end{cases}$$

Example 3.6.1 (continued)**.**
Noting that we have $+5$ as a constant term in each piece, let us 'extract' that using our function property:

$$f(y+1) = \begin{cases} 3y & y \geq 0 \\ 0 & y \leq 0 \end{cases} + 5$$

We're not sure what to do here - we want y in the first piece and $-y$ in the latter, so let us 'extract' the coefficient of y again by the same property:

$$f(y+1) = 3\begin{cases} y & y \geq 0 \\ 0 & y \leq 0 \end{cases} + 5$$

Nearly there. Notice that y is the same as $2y - y$ – that is, multiply by 2, then subtract a y. If we apply this process to the latter piece as well (as is important) we notice that $2 \cdot 0 - y = -y$, which is what we want. Therefore:

$$\begin{aligned} f(y+1) &= 3\begin{cases} \frac{1}{2}2y & y \geq 0 \\ 0 & y \leq 0 \end{cases} + 5 \\ &= \frac{3}{2}\begin{cases} y + y & y \geq 0 \\ -y + y & y \leq 0 \end{cases} + 5 \\ &= \frac{3}{2}\left(\begin{cases} y & y \geq 0 \\ -y & y \leq 0 \end{cases} + y\right) + 5 \end{aligned}$$

Notice now that we have $f(y+1) = \frac{3}{2}\left(|y| + y\right) + 5$. Substituting $y = x - 1$ back in, we have:

$$f(x) = \frac{3}{2}\left(|x-1| + x - 1\right) + 5$$

This is not the easiest method of rewriting such piecewise functions, nor is it even remotely scalable. But I would advise you to come up with your own problem with 2 linear functions such as the above (e.g. $4x + 10$ for $x \leq -2$ and 2 for $x \geq -2$) and perform the same process, thinking about why this works, and what sort of linear functions work in this way. Also, consider attempting an example with non-linear functions (and see what the result is). Such examples will help you garner an intuition for manipulating multiple pieces at once.

Example 3.6.2. Consider the following truth table:

Example 3.6.2 (continued).

x	y	$x \wedge y$
1	1	1
1	0	0
0	1	0
0	0	0

As it turns out, we can represent $x \wedge y$ as a piecewise function. That is, $x \wedge y : \{0,1\} \times \{0,1\} \to \{0,1\}$ such that:

$$x \wedge y = \begin{cases} 1 & x = 1 \quad y = 1 \\ 0 & x = 1 \quad y = 0 \\ 0 & x = 0 \quad y = 1 \\ 0 & x = 0 \quad y = 0 \end{cases}$$

Furthermore, we can perform some fancy algebra magic by nesting pieces (logical AND); we create one piece for when $x = 1$ and one for $x = 0$:

$$x \wedge y = \begin{cases} \begin{cases} 1 & y = 1 \\ 0 & y = 0 \end{cases} & x = 1 \\ \begin{cases} 0 & y = 1 \\ 0 & y = 0 \end{cases} & x = 0 \end{cases}$$

By observation, we might substitute each piece inside the first nested piecewise function with y. In our latter nested piecewise function, we simply have 0 for all possible values of y. Therefore:

$$x \wedge y = \begin{cases} \begin{cases} y & y = 1 \\ y & y = 0 \end{cases} & x = 1 \\ 0 & x = 0 \end{cases}$$

And performing the same argument as before on our final nested piecewise function, we have:

$$x \wedge y = \begin{cases} y & x = 1 \\ 0 & x = 0 \end{cases}$$

Luckily for us, this is fairly straightforward to simplify: 'extract' y, and then substitute the values with x. This gives:

$$x \wedge y = y \cdot \begin{cases} x & x = 1 \\ x & x = 0 \end{cases}$$

And since we have piece values x for all possible values of x, we get:

$$x \wedge y = xy$$

4 Common Piecewise Functions

So far we've really only used, rather than having properly introduced, the function $|x|$. In this section we'll define a set of piecewise functions that crop up fairly frequently, in many areas of maths, and how we might apply them to the algebra-related problems we've encountered already.

4.1 Absolute value function

The absolute value function $|x|$ is a real function defined as follows:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Alternatively, you might see $|x|$ notated using $\sqrt{x^2}$ for real x and principal square root. This is something which will be introduced and explained in the next section, although $\sqrt{x^2}$ is only one 'representation' of $|x|$.

4.2 The maximum function

The max of two variables is a function defined as:

$$\max\{a, b\} = \begin{cases} a & a \geq b \\ b & a \leq b \end{cases}$$

That is, it is a function whose purpose is two return the larger of two numbers. It can also be written as $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$.

Proof. We can derive this representation of $\max\{a, b\}$ in much the same way as Example 3.6.1:

$$\begin{aligned} \max\{a, b\} &= \begin{cases} a & a \geq b \\ b & a \leq b \end{cases} \\ &= \begin{cases} a - b & a - b \geq 0 \\ 0 & a - b \leq 0 \end{cases} + b \\ &= \frac{1}{2}\left(\begin{cases} 2(a - b) & a - b \geq 0 \\ 0 & a - b \leq 0 \end{cases}\right) + b \\ &= \frac{1}{2}\left(a - b + \begin{cases} a - b & a - b \geq 0 \\ b - a & a - b \leq 0 \end{cases}\right) + b \\ &= \frac{1}{2}(a - b + |a - b|) + b \\ &= \frac{1}{2}(a + b + |a - b|) \end{aligned}$$

□

Topping this off, we have that $|x| = \max\{x, -x\}$.

We might also define an extension of the max function that takes in a finite set; let $\max\{a\} = a$. Then, for a set S, we have that:

$$\max(S) = \{x, \quad x \geq \max(S \setminus \{x\}) \mid x \in S\}$$

This helps us to derive formulas for $\max\{a, b, c\}$ and so on. Really, in full generality (and slightly out of scope), the max function can be defined as $\sup(S)$ if and only if $\sup(S) \in S$ (otherwise it does not exist at all).

4.2.1. The union property of the max function is as follows:

$$\max(U \cup V) = \max\{\max(U), \max(V)\}$$

Proof. By definition, the right hand side can be written as:

$$\max\{\max(U), \max(V)\} = \begin{cases} \max(U) & \max(U) \geq \max(V) \\ \max(V) & \max(V) \geq \max(U) \end{cases}$$

We let $x = \max(U \cup V)$. Therefore, $x \geq \max(U)$ and $x \geq \max(V)$, and so we have two cases:

1. $x \in U \implies x = \max(U)$. Since this means $x = \max(U) \geq \max(V)$, we have that $\max\{\max(U), \max(V)\} = x$.
2. $x \in V \implies x = \max(V)$. Since this means $x = \max(V) \geq \max(U)$, we have that $\max\{\max(U), \max(V)\} = x$.

We therefore have that $x = \max\{\max(U), \max(V)\}$. $\square$

This property means we can explicitly write the following, for example:

$$\max\{a, b, c\} = \max\{\max\{a, b\}, c\}$$

(and hence write it in terms of the absolute value function)

4.2.2. It is worth noting that $f(\max(S)) \neq \max(f(S))$, where $f(x)$ is a function and $f(S)$ is $f(x)$ applied elementwise over the set S. However:

1. $\max(S) + a = \max(S + a)$ where $S + a$ is elementwise addition by a over S.
2. $c \cdot \max(S) = \max(c \cdot S)$ where $c \geq 0$ and $c \cdot S$ is elementwise multiplication by c over S.
3. $c \cdot \max(S) = \min(c \cdot S)$ where $c \leq 0$ and $c \cdot S$ is elementwise multiplication by c over S.

Each of these properties can be proven inductively, and such a proof is left to the reader.

4.3 The minimum function

The min of two variables is a function defined as:

$$\min\{a, b\} = \begin{cases} b & a \geq b \\ a & a \leq b \end{cases}$$

That is, it is a function whose purpose is two return the smaller of two numbers. It can also be written as $\max\{a, b\} = \frac{1}{2}(a + b - |a - b|)$. Such a proof of this is identical to the one given in Proof 4.2.1, and you would be encouraged, as the reader, to attempt it explicitly as a personal exercise.

We might also define an (identically motivated) extension of the min function that takes in a finite set; let $\min\{a\} = a$. Then, for a set S, we have that:

$$\min(S) = \{x, \quad x \leq \min(S \setminus \{x\}) \mid x \in S\}$$

This helps us to derive formulas for $\min\{a, b, c\}$ and so on. Really, in full generality (and slightly out of scope), the min function can be defined as $\inf(S)$ if and only if $\inf(S) \in S$ (otherwise it does not exist at all).

4.3.1. The union property of the min function is as follows:

$$\min(U \cup V) = \min\{\min(U), \min(V)\}$$

Proof. By definition, the right hand side can be written as:

$$\min\{\max(U), \max(V)\} = \begin{cases} \min(V) & \min(U) \geq \min(V) \\ \min(U) & \min(V) \geq \min(U) \end{cases}$$

We let $x = \min(U \cup V)$. Therefore, $x \leq \min(U)$ and $x \leq \min(V)$, and so we have two cases:

1. $x \in U \implies x = \min(U)$. Since this means $x = \min(U) \leq \min(V)$, we have that $\min\{\min(U), \min(V)\} = x$.
2. $x \in V \implies x = \min(V)$. Since this means $x = \min(V) \leq \min(U)$, we have that $\max\{\min(U), \min(V)\} = x$.

We therefore have that $x = \min\{\min(U), \min(V)\}$. □

This property means we can explicitly write the following, for example:

$$\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$$

(and hence write it in terms of the absolute value function)

4.3.2. Just as with the max function, it is worth noting that $f(\min(S)) \neq \min(f(S))$, where $f(x)$ is a function and $f(S)$ is $f(x)$ applied elementwise over the set S. However:

1. $\min(S) + a = \min(S + a)$ where $S + a$ is elementwise addition by a over S.
2. $c \cdot \min(S) = \min(c \cdot S)$ where $c \geq 0$ and $c \cdot S$ is elementwise multiplication by c over S.
3. $c \cdot \min(S) = \max(c \cdot S)$ where $c \leq 0$ and $c \cdot S$ is elementwise multiplication by c over S.

Each of these properties can be proven inductively, and such a proof is left to the reader.

4.4 Mixed maximum and minimum

It is here we note some identities of max and min but also extend explicitly our ability to manipulate piecewise functions which would otherwise elude us, per Example 3.6.1.

Example 4.4.1. We wish to represent the following function in terms of functions such as max and min:

$$f(x) = \begin{cases} \sin(x) & x \geq \pi \\ \pi - x & x \leq \pi \end{cases}$$

Notice that $x \geq \pi \iff \max\{x, \pi\} = x$ and $x \leq \pi \iff \min\{x, \pi\} = x$, and so we can provide substitutions for our conditions as so:

$$f(x) = \begin{cases} \sin(x) & \max\{x, \pi\} = x \\ \pi - x & \min\{x, \pi\} = x \end{cases}$$

Furthermore, we have now have substitutions for x in each piece; in each piece value, we perform this substitution for x:

$$f(x) = \begin{cases} \sin(\max\{x, \pi\}) & \max\{x, \pi\} = x \\ \pi - \min\{x, \pi\} & \min\{x, \pi\} = x \end{cases}$$

You may be wondering what the point of this is — well, now that we've performed these substitutions that are only true under each piece, we can 'subtract' each piece out (that is, add and substract using our function property):

$$f(x) = \begin{cases} 0 - (\pi - \min\{x, \pi\}) & \max\{x, \pi\} = x \\ 0 - \sin(\max\{x, \pi\}) & \min\{x, \pi\} = x \end{cases}$$
$$+ \sin(\max\{x, \pi\}) + \pi - \min\{x, \pi\}$$

Example 4.4.1 (continued)**.**
Now, we might evaluate each piece; notice that $\min\{x, \pi\} = \pi$ for $x \geq \pi$ and also that $\max\{x, \pi\} = \pi$ for $x \leq \pi$. Therefore we have:

$$f(x) = \begin{cases} 0 & \max\{x, \pi\} = x \\ 0 & \min\{x, \pi\} = x \end{cases}$$
$$+ \sin(\max\{x, \pi\}) + \pi - \min\{x, \pi\}$$

Finally, simplifying, we're left with:

$$f(x) = \sin(\max\{x, \pi\}) - \min\{x, \pi\} + \pi$$

4.4.1. We have the following basic identities to work with using max and min:

1. $\max\{a, b\} + \min\{a, b\} = a + b$; this result can be proven by definition, or using the max and min formulations in terms of the absolute value function.
2. $\max\{a, b\} - \min\{a, b\} = |a - b|$; this result can be similarly proven as above.
3. $|x| = \max\{x, -x\} = -\min\{x, -x\}$.

4.4.1 Clamping function

The clamping function is a function which restricts a number between an upper and lower bound, as per its definition:

$$\ell_a^b(x) = \begin{cases} b & x \geq b \\ x & a \leq x \leq b \\ a & x \leq a \end{cases}$$

We use this symbol to represent the clamping function as later on it will be given more usage, so it will be useful to have a quick and easy tool (also, this is LaTeX; my hboxes aren't infinite).

4.4.2. We first provide some properties of the clamping function:

1. $\ell_a^\infty(x) = \lim_{t\to\infty} \ell_a^t(x) = \min(x, a)$
2. $\ell_{-\infty}^b(x) = \lim_{t\to\infty} \ell_{-t}^b = \max(x, b)$
3. $-\ell_a^b(x) = \ell_{-b}^{-a}(-x)$
4. $c \cdot \ell_a^b(x) = \ell_{ac}^{bc}(cx)$, for $c \geq 0$
5. $\ell_a^b(x) + k = \ell_{a+k}^{b+k}(x + k)$

4.4.3. The clamping function can be written as any of the following:

1. $\ell_a^b(x) = \min\{\max\{x, a\}, b\}$
2. $\ell_a^b(x) = \max\{\min\{x, b\}, a\}$
3. $\ell_a^b(x) = \frac{1}{2}\left(a + b + |x - a| - |x - b|\right)$

We shall give proofs of the first and last of these formulations:

Proof. We begin by using the definition of $\ell_a^b(x)$:

$$\ell_a^b(x) = \begin{cases} b & x \geq b \\ x & a \leq x \leq b \\ a & x \leq a \end{cases}$$

We nest a piece under the conditions $x \geq a$ and $x \leq a$ in order to simplify the conditions we're working with:

$$\begin{aligned} \ell_a^b(x) &= \begin{cases} \begin{cases} b & x \geq b \\ x & x \leq b \end{cases} & x \geq a \\ a & x \leq a \end{cases} \\ &= \begin{cases} \min\{x, b\} & x \geq a \\ a & x \leq a \end{cases} \end{aligned}$$

Now using the definition of $\max\{x, a\}$ we substitute $x \geq a$ with $\max\{x, a\} = x$ and likewise with $x \leq a$, to give:

$$\ell_a^b(x) = \begin{cases} \min\{x, b\} & \max\{x, a\} = x \\ a & \max\{x, a\} = a \end{cases}$$

We then use the first piece's condition to substitute the value of x with $\max\{x, a\}$,

$$\ell_a^b(x) = \begin{cases} \min\{\max\{x, a\}, b\} & \max\{x, a\} = x \\ a & \max\{x, a\} = a \end{cases}$$

Substracting out $\min\{\max\{x, a\}, b\}$ gives us

$$\ell_a^b(x) = \begin{cases} 0 & x \geq a \\ a - \min\{\max\{x, a\}, b\} & x \leq a \end{cases} + \min\{\max\{x, a\}, b\}$$

And evaluating the second piece (since $x \leq a \leq b$) gives us 0, leaving us with

$$\ell_a^b(x) = \min\{\max\{x, a\}, b\}$$

$\square$

4.4.3 (continued)**.**
Proof. We now set out to prove the last representation of $\ell_a^b(x)$.

Using our previous derivation, we apply the addition property of min to give us:

$$\min\{\max\{x,a\},b\} = \min\{0, b - \max\{x,a\}\} + \max\{x,a\}$$

Which simplifies to:

$$\begin{aligned}\ell_a^b(x) &= \min\{0, b + \min\{-x,-a\}\} + \max\{x,a\} \\ &= \min\{0, \min\{b-x, b-a\}\} + \max\{x,a\} \\ &= \min\{0, b-x, b-a\} + \max\{x,a\}\end{aligned}$$

Since $b - a \geq 0$, we're left with:

$$\ell_a^b(x) = \min\{b-x, 0\} + \max\{x,a\}$$

Using the absolute value representations of max and min and simplifying, we get that

$$\ell_a^b(x) = \frac{1}{2}\left(a + b + |x-a| - |x-b|\right)$$

□

Example 4.4.2. We want to consider the following function, rewriting it in terms of max, min:

$$f(x) = \begin{cases} x & x > 1 \\ 1 & -1 \leq x \leq 1 \\ -x & x < -1 \end{cases}$$

This is not in the form we want in order to manipulate it, so we consider that this function is, in fact, continuous. It stands to reason, therefore, that $x > 1 \iff x \geq 1$ and $x < -1 \iff x \leq -1$. Therefore:

$$f(x) = \begin{cases} x & x \geq 1 \\ 1 & -1 \leq x \leq 1 \\ -x & x \leq -1 \end{cases}$$

Now, we note that $x \geq 1$ is equivalent to $\ell_1^{\infty}(x) = x$, $-1 \leq x \leq 1$ is equivalent to $\ell_{-1}^{1}(x) = x$ and $x \leq -1$ is equivalent to $\ell_{-\infty}^{-1}(x) = x$.

With our equivalences, we can now make the respective substitutions in both the conditions and piece values, as with previous examples (although we needn't really keep the conditions in this form):

Example 4.4.2 (continued)**.**

$$f(x) = \begin{cases} \ell_1^\infty(x) & \ell_1^\infty(x) = x \\ 1 & \ell_{-1}^1(x) = x \\ -\ell_{-\infty}^{-1}(x) & \ell_{-\infty}^{-1}(x) = x \end{cases}$$

Now we 'extract' our piece values by addition:

$$f(x) = \left(\begin{cases} -(1 - \ell_{-\infty}^{-1}(x)) & x \geq 1 \\ -(\ell_1^\infty(x) - \ell_{-\infty}^{-1}(x)) & -1 \leq x \leq 1 \\ -(\ell_1^\infty(x) + 1) & x \leq -1 \end{cases} \right) + \left(\ell_1^\infty(x) + 1 - \ell_{-\infty}^{-1}(x)\right)$$

Evaluting each piece value using the definition of the clamping function gives us:

$$f(x) = \left(\begin{cases} -2 & x \geq 1 \\ -2 & -1 \leq x \leq 1 \\ -2 & x \leq -1 \end{cases} \right) + \left(\ell_1^\infty(x) + 1 - \ell_{-\infty}^{-1}(x)\right)$$

Finally simplifying, we have:

$$f(x) = \ell_1^\infty(x) - \ell_{-\infty}^{-1}(x) - 1$$

Notice that, in this example, we don't have any $\ell_{-1}^1(x)$ because, in fact, 1 is a constant function. Rewriting our clamping functions in terms of max and min per the limit identities gives us:

$$f(x) = \max\{x, 1\} - \min\{x, -1\} - 1$$

The clamping function is useful for formulating single expressions for continuous piecewise functions as we've seen above. If the function is not continuous, we may not actually have much luck in using it. In fact, we'll derive the general 'Gluing Formula' soon within these notes.

4.5 The sign function

The sign function is an interesting function in that it doesn't have a single explicit definition, but instead satisfies the following relation:

$$x \cdot \operatorname{sgn}(x) = |x|$$

This can be true for some domain $D \subseteq \mathbb{R}$, such as $\mathbb{R} \setminus \{0\}$. For the purposes of these notes, however, we shall define $\operatorname{sgn}(x)$ as the following:

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

This definition does indeed satisfy the definition of $|x|$, noting that we can write $|x|$ in the following explicit form[1]:

$$|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$$

It is useful to note that the sign function is discontinuous; we can express discontinuous piecewise functions in terms of the sign function, although this is not common (we instead opt for step functions there, although step functions are just transformations of the sign function).

4.6 Step functions

Step functions are useful functions which, usually in calculus at higher levels, allow us to provide a form in which to express a piecewise function regardless of continuity or not. We look at the Heaviside step function, which, like the sign function, doesn't have a single explicit formulation, but instead satisfies:

$$x \cdot \mathrm{H}(x) = \frac{1}{2}(x + |x|)$$

The right hand side of this relation is known as the ramp function and can be expressed as $\max\{x, 0\}$.

For the purposes of these notes we might define the Heaviside step function as the following:

$$\mathrm{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

4.6.1. We can express $\mathrm{H}(x)$ in terms of $\mathrm{sgn}(x)$;

$$\mathrm{H}(x) = \frac{1}{2}(1 + \mathrm{sgn}(x))$$

This can be shown using the same techniques found in Example 3.6.1.

Example 4.6.1. We wish to express the following function (similar to that found in 4.4.2) in terms of step functions:

$$f(x) = \begin{cases} x & x > 1 \\ 1 & -1 < x < 1 \\ -x & x < -1 \end{cases}$$

In order to simplify this down, let us nest piecewise functions on the

[1]It is a good exercise to reconcile this form of $|x|$ with the previous forms displayed in these notes, using piecewise function properties (and when and why each form may be useful to us).

Example 4.6.0 (continued).
conditions $x > -1$ and $x < -1$:

$$f(x) = \begin{cases} \begin{cases} x & x > 1 \\ 1 & x < 1 \end{cases} & x > -1 \\ -x & x < -1 \end{cases}$$

We now wish to write the inner piecewise function in terms of the step function: to do so, we 'subtract' our second piece, 1 out and then factor out $x - 1$, as follows:

$$f(x) = \begin{cases} (x-1)\begin{cases} 1 & x > 1 \\ 0 & x < 1 \end{cases} + 1 & x > -1 \\ -x & x < -1 \end{cases}$$

Therefore, the inner piecewise function is equivalent to $\mathrm{H}(x - 1)$:

$$f(x) = \begin{cases} (x-1)\mathrm{H}(x-1) + 1 & x > -1 \\ -x & x < -1 \end{cases}$$

Instead of repeating this process with the outer piecewise function, we should instead subtract out the first piece, since we have a step function inside that:

$$f(x) = (x-1)\mathrm{H}(x-1) + 1 + \begin{cases} 0 & x > -1 \\ -x - (x-1)\mathrm{H}(x-1) - 1 & x < -1 \end{cases}$$

And from there, evaluate the second piece (wherein $\mathrm{H}(x - 1) = 0$ since $x < -1$):

$$f(x) = (x-1)\mathrm{H}(x-1) + 1 + \begin{cases} 0 & x > -1 \\ -x - 1 & x < -1 \end{cases}$$

We can factor out $(-x - 1)$ again and rewrite $x < -1$ as $-x - 1 > 0$:

$$f(x) = (x-1)\mathrm{H}(x-1) + 1 + (-x-1)\begin{cases} 0 & -x - 1 < 0 \\ 1 & -x - 1 > 0 \end{cases}$$

We therefore have that:

$$f(x) = (x-1)\mathrm{H}(x-1) + (-x-1)\mathrm{H}(-x-1) + 1$$

Example 4.6.0 (continued)**.**
The astute may have noticed that, in fact, this function can be rewritten directly as a result of the definition of the step function:

$$f(x) = \max\{x - 1, 0\} + \max\{-x - 1, 0\} + 1$$

We have, as a result, that $f(x)$ is continuous (which we already knew from the previous example), as the max function is continuous. Note however that unlike the previous example, even though we can write this function in this form, that $f(-1)$ and $f(1)$ are not defined.

Explicitly, we write $f : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}$.

This process is generalisable as with our continuous gluing function (i.e. through repeating nesting). I will, however, leave this process up to the reader.

4.7 The floor and ceiling functions

We've already introduced the floor function once previously, however we shall do so again here for the sake of completeness:

$$\lfloor x \rfloor = \{n, \quad x \in [n, n+1) \mid n \in \mathbb{Z}\}$$

We can alternatively write the floor function using the max function of an infinite set:

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$$

That is, the floor function is the greatest integer n such that n is less than or equal to x.

The ceiling function is similarly defined piecewise;

$$\lceil x \rceil = \{n, \quad x \in (n-1, n] \mid n \in \mathbb{Z}\}$$

or alternatively using the min function:

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}$$

That is, the ceiling function is the smallest integer n such that n is larger than or equal to x.

4.8 Characteristic functions and Iverson brackets

Briefly, a characteristic (or indicator) function for a set S is a function defined such that:

$$\mathbf{1}_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

4.8.1. The characteristic function for the integers is given by:

$$\mathbf{1}_{\mathbb{Z}}(x) = 1 - (\lceil x \rceil - \lfloor x \rfloor)$$

Proof. We use our piecewise notation for the ceiling and floor functions for convenience (where for the ceiling function we've just reindexed $n \to n+1$ in order to match the conditions):

$$\lceil x \rceil - \lfloor x \rfloor = \{n+1, \quad x \in (n, n+1] \mid n \in \mathbb{Z}\} - \{n, \quad x \in [n, n+1) \mid n \in \mathbb{Z}\}$$

We can separate out the $x = n$ and $x = n+1$ pieces in each:

$$\begin{aligned}\lceil x \rceil - \lfloor x \rfloor =& \{n+1, \quad x \in (n, n+1) \mid n \in \mathbb{Z}\} \cup \\ & \{n+1, \quad x = n+1 \mid n \in \mathbb{Z}\} \\ & - \{n, \quad x \in (n, n+1) \mid n \in \mathbb{Z}\} \cup \{n, \quad x = n \mid n \in \mathbb{Z}\}\end{aligned}$$

Reindexing the $x = n+1$ pieces with $n+1 \to n$, and then combining all pieces for $x \in (n, n+1)$, we have that:

$$\begin{aligned}\lceil x \rceil - \lfloor x \rfloor =& \quad \{1, \quad x \in (n, n+1) \mid n \in \mathbb{Z}\} \cup \{n, \quad x = n \mid n \in \mathbb{Z}\} \\ & \qquad - \{n, \quad x = n \mid n \in \mathbb{Z}\} \\ =& \quad \{1, \quad x \in (n, n+1) \mid n \in \mathbb{Z}\} \cup \{0, \quad x = n \mid n \in \mathbb{Z}\}\end{aligned}$$

That is, for all integers n, if $x = n$ (i.e. x is an integer) we have $\lceil x \rceil - \lfloor x \rfloor = 0$. Otherwise, for all non-integer x, we have $\lceil x \rceil - \lfloor x \rfloor = 1$. We therefore have that:

$$\begin{aligned}\lceil x \rceil - \lfloor x \rfloor &= \begin{cases} 1 & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} \\ &= -\begin{cases} -1 & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases} \\ &= 1 - \begin{cases} 0 & x \notin \mathbb{Z} \\ 1 & x \in \mathbb{Z} \end{cases} \\ &= 1 - \mathbf{1}_{\mathbb{Z}}(x)\end{aligned}$$

□

Iverson bracket notation is almost a generalisation to these characteristic or indicator functions, except instead of using a set and variable, they are provided with any sort of predicate, condition, etc.:

$$[S] = \begin{cases} 1 & S \\ 0 & \neg S \end{cases}$$

Essentially all piecewise functions we work with in these notes (to a point) are able to be written in terms of Iverson brackets. In turn, certain Iverson brackets can be written in terms of elementary functions or functions we've already seen.

4.8.2. We can write several of the previous functions in this section in terms of Iverson brackets:

1. $\mathbf{1}_S(x) = [x \in S]$ (noting that the complement of $x \in S$ is $x \in \mathbb{R} \setminus S$, for example)
2. $\mathrm{H}(x) = [x > 0]$
3. $\operatorname{sgn}(x) = [x > 0] - [x < 0]$
4. $|x| = x[x > 0] - x[x < 0]$
5. $\max\{a, b\} = a[a \geq b] + b[a < b]$
6. $\min\{a, b\} = b[a \geq b] + a[a < b]$

Proofs of these equivalences are left specifically for the reader to practice.

4.8.3. Even more generally, let us denote the piecewise object:

$$\phi = \{\varphi_i, \quad C_i \mid i \in I\}$$

Such that $C_i \wedge C_j \leftrightarrow \perp$ for all $i \neq j$ and $i, j \in I$. Then ϕ can be expressed using Iverson bracket notation:

$$\phi = \sum_{i \in I} \varphi_i [C_i]$$

Proof. Let us begin by splitting up the piecewise object ϕ into separate piecewise objects such that all but 1 piece is equal to 0, in each:

$$\{\varphi_i, \quad C_i \mid i \in I\} = \sum_{i \in I} \{\varphi_m, \quad C_m \mid m = i\} \cup \{0, \quad C_n \mid n \in I \setminus \{i\}\}$$

Since each C_n is distinct, then $C_n \leftrightarrow \neg C_i$ for $n \neq i$. Therefore:

$$\{\varphi_i, \quad C_i \mid i \in I\} = \sum_{i \in I} \{\varphi_m, \quad C_m \mid m = i\} \cup \{0, \quad \neg C_i \mid n \in I \setminus \{i\}\}$$

Since each piecewise object has two cases only, we can rewrite this as:

$$\{\varphi_i, \quad C_i \mid i \in I\} = \sum_{i \in I} \begin{cases} \varphi_i & C_i \\ 0 & \neg C_i \end{cases}$$

Then, factoring φ_i and writing in terms of Iverson brackets, we have our result. $\square$

5 Forming Piecewise Functions