1. (a) $f(x) = \frac{1}{2}x^TAx + b^Tx$

$\nabla f(x) = Ax + b$

(b) $f(x) = g(h(x))$

$\nabla f(x) = \nabla g(h(x))$

$= g'(h(x))\nabla h(x)$

(c) $\nabla^2 f(x) = \nabla(Ax + b)$

$= A$

(d) $\nabla f(x) = \nabla g(a^Tx)$

$= g'(a^Tx)\nabla(a^Tx)$

$= g'(a^Tx)\cdot a$ ✓

$\nabla^2 f(x) = \nabla(g'(a^Tx)a)$

$= g''(a^Tx)a^2$ ✗

$$\frac{\partial^2 g(h(x))}{\partial x_i \partial x_j} = \frac{\partial^2 g(h(x))}{\partial(h(x))^2}\cdot\frac{\partial h(x)}{\partial x_i}\frac{\partial h(x)}{\partial x_j}$$

$$\frac{\partial^2 g(a^Tx)}{\partial x_i \partial x_j} = g''(a^Tx)\, a_i a_j$$

$$\nabla^2 f(x) = g''(a^Tx)\begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_n \\ a_2a_1 & \ddots & \vdots \\ \vdots & & \\ a_na_1 & \cdots & a_na_n \end{bmatrix} = g''(a^Tx)\, aa^T$$

2. (a) symmetric: $A^T = (zz^T)^T = zz^T = A$.

$x^TAx = x^Tzz^Tx = x^Tz(x^Tz)^T = (x^Tz)^2 \geq 0$ PSD

(b) $Ax = 0$, $zz^Tx = 0$. z is non-zero vector

so $z^Tx = 0$. $N(A) = \{x \in \mathbb{R}^n : z^Tx = 0\}$

$$zz^T = \begin{bmatrix} z_1z_1 & z_1z_2 & \cdots & z_1z_n \\ z_2z_1 & z_2z_2 & \cdots & z_2z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_nz_1 & z_2z_n & \cdots & z_nz_n \end{bmatrix}$$

row$j$ − row$1 \times \frac{z_j}{z_1} = 0$

$$\therefore zz^T = \begin{bmatrix} z_1z_1 & z_1z_2 & \cdots & z_1z_n \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$ rank$(A) = 1$.

Null$(A) = n - 1$.

(c) symmetric: $(BAB^T)^T = BA^TB^T$, A is PSD so $A^T = A$

$\therefore (BAB^T)^T = BAB^T$

A is PSD, so for all x $x^TAx \geq 0$.

$x^TBAB^Tx = (B^Tx)^TA(B^Tx) \geq 0$

$\therefore BAB^T$ is PSD

3. (a) $A = T\Lambda T^{-1}$

$AT = T\Lambda$

$$A[t^{(1)}\ t^{(2)} \ldots t^{(n)}] = [t^{(1)}\ t^{(2)} \ldots t^{(n)}]\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$[At^{(1)}\ At^{(2)} \ldots At^{(n)}] = [\lambda_1 t^{(1)}\ \lambda_2 t^{(2)} \ldots \lambda_n t^{(n)}]$

$At^{(i)} = \lambda_i t^{(i)}$

(b) $AU = U\Lambda U^TU$, $U^TU = I$.

$AU = U\Lambda$

$A[u^{(1)}\ u^{(2)} \ldots u^{(n)}] = [u^{(1)}\ u^{(2)} \ldots u^{(n)}]\,\mathrm{diag}(\lambda_1, \lambda_2 \ldots \lambda_n)$

$[Au^{(1)} \; Au^{(2)} \ldots Au^{(n)}] = [\lambda_1 u^{(1)} \; \lambda_2 u^{(2)} \ldots \lambda_n u^{(n)}]$

$\therefore Au^{(i)} = \lambda_i u^{(i)}$.

$u^{(i)}$ is eigen vector of A.

k) A is PSD, so $x^T A x \geq 0$ for all x

A is symmetric, according to spectral theorem, A is diagonalizable.

$A = U \Lambda U^T \quad \Lambda = \text{diag}(\lambda_1 \ldots \lambda_n)$

$x^T A x = x^T U \Lambda U^T x = (U^T x)^T \Lambda (U^T x)$

Let $U^T x = z$, $z \in \mathbb{R}^n$.

$$z^T \Lambda z = z^T \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} z = z^T \begin{bmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \\ \vdots \\ \lambda_n z_n \end{bmatrix} = \sum_{i=1}^{n} \lambda_i z_i^2 \geq 0$$

so for $\lambda_i(A) \geq 0$ for each i.