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Mathematics Specialist 3&4

TEST 4 – Resource Free

Systems of Equations, Differentiation and Integration

NAME: Solutions **DATE: Mon 1st August 2016**

Time: 50 min **Total: /52 mark**

1. Determine $\frac{dy}{dx}$ for each of the following: [2, 2, 3, 4 = 11 marks]

a) $y=\sqrt{\tan x}$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan x}}\times\frac{1}{\cos^2 x}$$

b) $y=\sin^3\left(\frac{\pi}{4}-x\right)$

$$\frac{dy}{dx}=-3\sin^2\left(\frac{\pi}{4}-x\right).\cos\left(\frac{\pi}{4}-x\right)$$

c) $(xy)^2+4\cos y=x$

$$2(xy)\left(y+x\frac{dy}{dx}\right)-4\sin y\frac{dy}{dx}=1$$

$$2xy^2+2x^2y\frac{dy}{dx}-4\sin y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1-2xy^2}{2x^2y-4\sin y}$$

d) $x=\cos(2t)$, $y=\sin(2t)$ (give answer in terms of x)

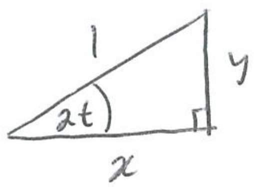
$$\frac{dx}{dt}=-2\sin(2t) \qquad \frac{dy}{dt}=2\cos(2t)$$

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$=\frac{2\cos 2t}{-2\sin 2t}$$

$$=\frac{-x}{y}$$

$$=\pm\frac{x}{\sqrt{1-x^2}}$$

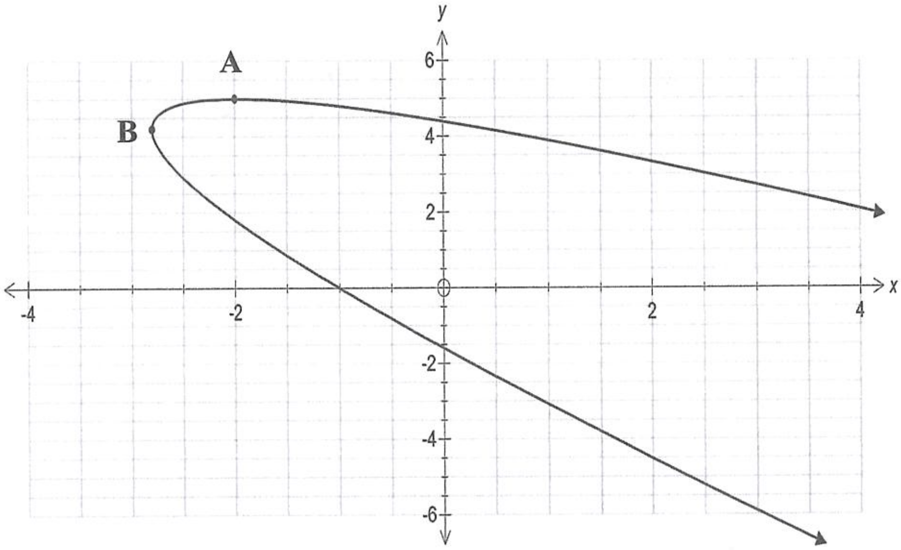


$$y^2=1-x^2$$

$$y=\pm\sqrt{1-x^2}$$

2. [3, 3 = 6 marks]

The diagram below has the parametric equations $x(t) = 5t^2 - 4t - 2$ and $y(t) = -5t^2 + 5$



a) Determine the exact coordinates of A, the point on the curve that is furthest above the horizontal axis.

$\frac{dx}{dt} = 10t - 4$ $\quad \frac{dy}{dt} = -10t$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{-10t}{10t-4}$ ✓

$\frac{dy}{dx} = 0$ $\quad 0 = \frac{-10t}{10t-4}$

$t = 0$ ✓

$\therefore x(0) = -2$

$y(0) = 5$ $\quad A(-2, 5)$ ✓

b) Determine the exact coordinates of B, the point on the curve that is furthest to the left of the vertical axis.

$\frac{dx}{dy} = \frac{10t-4}{-10t}$ ✓

$0 = 10t - 4$ $\quad \left(\frac{dx}{dy} = 0\right)$

$t = \frac{2}{5}$ ✓

$x\left(\frac{2}{5}\right) = 5\left(\frac{2}{5}\right)^2 - 4\left(\frac{2}{5}\right) - 2$

$= \frac{4}{5} - \frac{8}{5} - \frac{10}{5}$

$= -\frac{14}{5}$

$y\left(\frac{2}{5}\right) = -5\left(\frac{2}{5}\right)^2 + 5$

$= -\frac{4}{5} + \frac{25}{5}$

$= \frac{21}{5}$

$B(-2.8, 4.2)$ $\quad B\left(\frac{-14}{5}, \frac{21}{5}\right)$ ✓

3. Calculate the following integrals: [2, 2, 2, 2 = 8 marks]

a) $\int 2\sin(\cos x).\sin x \quad dx$

$= 2\cos(\cos x) + c$ ✓✓

Guess → check

If $y = 2\cos(\cos x) \Rightarrow y' = -2\sin(\cos x)(-\sin x) = 2\sin(\cos x)\sin x$

b) $\int \frac{4x}{1-x^2} dx$

$= -2\ln|1-x^2| + c$ ✓✓

c) $\int 1 + 2\sin^2 x \quad dx$

$= \int 1 + 2\left(\frac{1-\cos 2x}{2}\right) dx$ ✓

$= \int 2 - \cos 2x \, dx$

$= 2x - \frac{1}{2}\sin 2x + c$ ✓

d) $\int 2x^2 e^{x^2} + e^{x^2} dx$

$= xe^{x^2} + c$ ✓✓

product rule [Guess → check]

Let $y = xe^{x^2}$

then $y' = xe^{x^2}2x + e^{x^2}$

$= 2x^2 e^{x^2} + e^{x^2}$ checked

4. Determine the integral $\int 3^{x-1} dx$ using the substitution $u = 3^{x-1}$. [5 marks]

$\int 3^{x-1} dx$

$= \int u \frac{du}{u \ln 3}$ ✓

$= \int \frac{1}{\ln 3} du$ ✓ $= \frac{1}{\ln 3}\int du$

$= \frac{u}{\ln 3} + c = \frac{1}{\ln 3}(u) + c$

$= \frac{3^{x-1}}{\ln 3} + c$ ✓

$u = 3^{x-1}$

$\ln u = \ln 3^{x-1}$ ✓

$\ln u = (x-1)\ln 3$ diff wr

$\frac{1}{u}\frac{du}{dx} = \ln 3$

$\frac{du}{dx} = u \ln 3$ ✓

$dx = \frac{du}{u \ln 3}$

$\frac{x}{3} = \sin u$ $\therefore u = \sin^{-1}\left(\frac{x}{3}\right)$

5. Determine the integral $\int \frac{1}{\sqrt{9-x^2}}dx$ using an appropriate substitution. [6 marks]

$x = 3\sin u$ ✓

$\frac{dx}{du} = 3\cos u$

$dx = 3\cos u \, du$ ✓

$$\int \frac{1}{\sqrt{9-(3\sin u)^2}} \cdot 3\cos u \, du ✓$$

$$= \int \frac{1}{\sqrt{9-9\sin^2 u}} \cdot 3\cos u \, du$$

$$= \int \frac{1}{\sqrt{9\cos^2 u}} \cdot 3\cos u \, du ✓$$

$$= \int \frac{1}{3\cos u} \cdot 3\cos u \, du$$

$$= \int 1 \, du ✓$$

$$= u + C$$

$$= \sin^{-1}\left(\frac{x}{3}\right) + C ✓$$

$$\frac{1}{\sqrt{9(1-\sin^2\theta)}} = \frac{1}{\sqrt{9\cos^2\theta}} = \frac{1}{3\cos\theta}$$

6. Solve the following system of equations: [5 marks]

$2x + 3y - z = 15$ ①

$4x + 5y + 2z = 4$ ②

$2x - 4y - 3z = 13$ ③

$2\times① + ② \Rightarrow 8x + 11y = 34$ ④ ✓

$3\times① - ③ \Rightarrow 4x + 13y = 32$ ⑤ ✓

$④ - 2\times⑤ \Rightarrow -15y = -30$

$y = 2$ ✓

⑤ $4x + 26 = 32$

$4x = 6$

$x = 1.5$ ✓

① $2\times1.5 + 3\times2 - z = 15$

$z = 3 + 6 - 15$

$z = -6$ ✓

Therefore solution is $(1.5, 2, -6)$

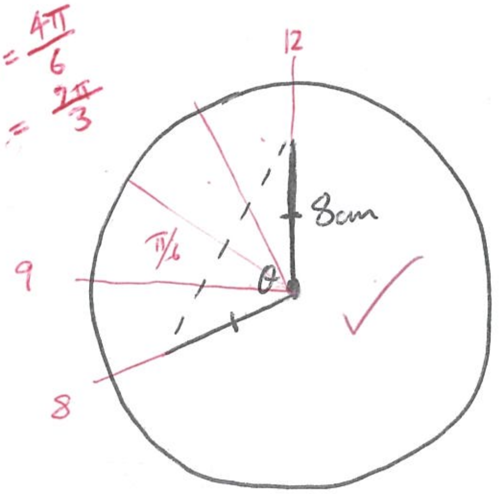
7. [6 marks]

Timex release a new clock with an identical minute and hour hand, each exactly 8 cm in length. An imaginary line is drawn joining the tips of each hand to form an isosceles triangle with centre angle θ. What is the rate of change of the area of the triangle at the instant the time is 8 o'clock?

At 8pm

$\theta = \frac{4\pi}{6}$

$= \frac{2\pi}{3}$



✓

$$\frac{d\theta}{dt} = \left(2\pi - \frac{2\pi}{12}\right) \text{ rad/hr}$$

$$= \frac{22\pi}{12}$$

$$= \frac{11\pi}{6} \checkmark$$

At 8 o'clock $\theta = \frac{2\pi}{3}$ ✓ (120°)

$$A = \tfrac{1}{2} ab \sin C$$

$$= \tfrac{1}{2} 8^2 \sin\theta$$

$$= \underline{32 \sin\theta} \checkmark \rightarrow \underline{\frac{dA}{d\theta} = 32\cos\theta}$$

$$\frac{dA}{dt} = 32\cos\theta \frac{d\theta}{dt} \checkmark$$

$$= 32 \cos\frac{2\pi}{3} \times \frac{11\pi}{6}$$

$$= 32\left(-\tfrac{1}{2}\right) \times \frac{11\pi}{6}$$

$$= -\frac{88\pi}{3} \text{ cm}^2/\text{hr} \checkmark$$

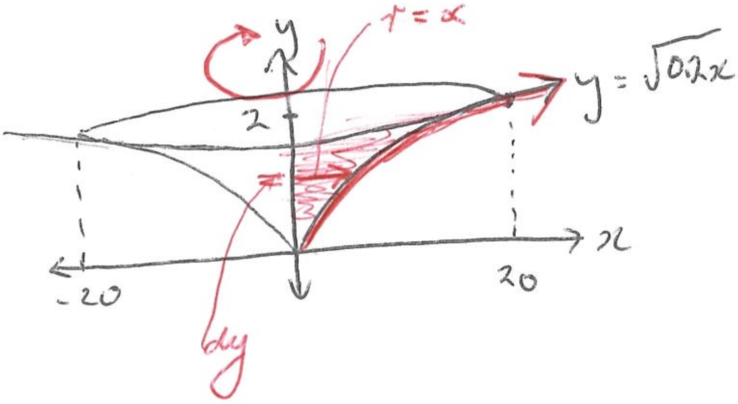
$$= -92.15 \text{ cm}^2/\text{hr}$$

$$= -1.54 \text{ cm}^2/\text{min} = 0.0256 \text{ cm}^2/\text{sec}$$

$$= \frac{176\pi}{21600}$$

8. [5 marks]

A pointed hat is modelled by rotating the line $y = \sqrt{0.2x}$ from $x = 0$ to $x = 20$ about the y-axis. If the measurements are in cm, find the volume of the hat.



when $x = 0$, $y = 0$

when $x = 20$, $y = 2$ ✓

$y = \sqrt{0.2 \times 20} = \sqrt{4} = 2$

$$V = \int_0^2 \pi x^2 \, dy$$

$$y = \sqrt{0.2x} = \sqrt{\frac{x}{5}}$$

$$y^2 = \frac{x}{5} = 5y^2 = x$$

$$\boxed{x^2 = 25y^4}$$ ✓

$$V = \pi \int_0^2 25 y^4 \, dy$$ ✓

$$= 25\pi \left[\frac{y^5}{5} \right]_0^2$$

$$= 25\pi \left(\frac{32}{5} \right)$$

$$= 160\pi$$

$\therefore$ Volume $= 160\pi$ ✓ cm^3 ✓

[−1 if no units]