PERTH MODERN SCHOOL
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Independent Public School

Course Specialist Year 11

Student name: ___________________________ Teacher name: _________________

Date: 18 Sep 2020

Task type: **Response**

Time allowed for this task: **_____45______ mins**

Number of questions: **_____6______**

Materials required: **Calculator-Free**

Standard items: Pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: Drawing instruments, templates

Marks available: **__45____ marks**

Task weighting: **__16__%**

Formula sheet provided: **Yes**

Note: All part questions worth more than 2 marks require working to obtain full marks.

Question 1 **(2.2.1, 2.2.2, 2.2.3)** **(6 marks)**

If $A = \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix}$, O is the 2×2 zero matrix and I is the 2×2 identity matrix, find

a) Matrix B given that $A - B = I$ (1 mark)

$$B = A - I = \begin{bmatrix} 1 & 0 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix} \checkmark$$

b) Matrix C given that $2A + C = \text{O}$ (1 mark)

$$C = -2A$$
$$= \begin{bmatrix} -2 & 0 \\ 4 & 6 \end{bmatrix} \checkmark$$

c) Matrix D given that $D = B - AD$, where B is from part a) (4 marks)

$$D + AD = B$$
$$(I + A)D = B \checkmark$$
$$D = (I + A)^{-1} B \checkmark$$
$$= \begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix}$$
$$= -\frac{1}{4} \begin{bmatrix} -2 & 0 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix} \checkmark$$
$$= -\frac{1}{4} \begin{bmatrix} 0 & 0 \\ -4 & -8 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

either ✓

or Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$a = 0$ ✓

$b = 0$ ✓ R/W

$c = 1$ ✓

$d = 2$ ✓

Question 2 (2.2.11) (8 marks)

If $A = \begin{bmatrix} 7 & 5 \\ -4 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 10 \\ -8 & -14 \end{bmatrix}$

a) Determine AB. (2 marks)

$$\begin{bmatrix} 42-40 & 70-70 \\ -24+24 & -40+42 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

b) Express A^{-1} in terms of B. (3 marks)

$$AB = 2I$$
$$A^{-1}AB = 2A^{-1}I$$
$$B = 2A^{-1}$$
$$A^{-1} = \frac{B}{2}$$

c) Solve the system $\begin{cases} 7x + 5y = 1 \\ -4x - 3y = 1 \end{cases}$, clearly showing your use of A^{-1}. (3 marks)

Award full mark if student used $4x+3y=1$

$$\begin{bmatrix} 7 & 5 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ -4 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \\ -4 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -11 \end{bmatrix}$$

$x = 8, \quad y = -11$

$$\begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

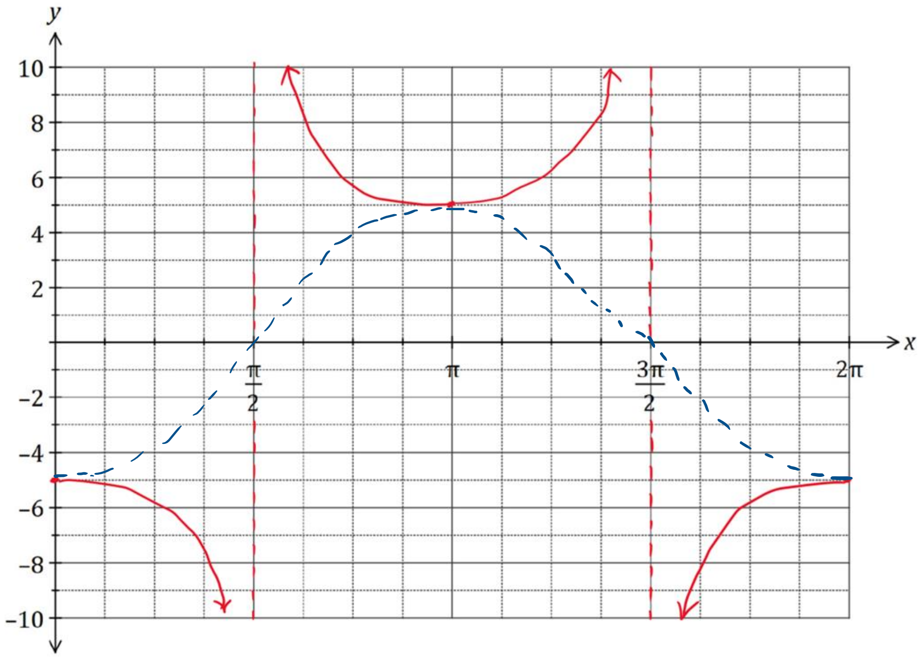
$$= \frac{1}{21-20}\begin{bmatrix} 3 & -5 \\ -4 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$\therefore x = -2, \quad y = 3$

Question 3 **(2.1.4, 2.1.7)** **(7 marks)**

a) On the axes below, sketch the graph of $y = 5\,sec(x - \pi), 0 \le x \le 2\pi$. (3 marks)



✓ Asymptotes

✓ Maximum & Minimum turning points

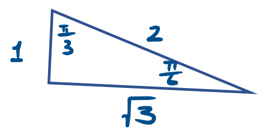
✓ Shape with smooth curves

b) Find the general solution for $\sqrt{3}\cos(x) - sin(x) = 1$. (4 marks)

$$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\cos x \cos\frac{\pi}{6} - \sin x \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2} \checkmark$$



$$\therefore \quad x + \frac{\pi}{6} = \pm\frac{\pi}{3} + 2k\pi, \ k \in \mathbb{Z} \checkmark$$

$$\therefore \quad x = \frac{\pi}{6} + 2k\pi \text{ or } -\frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$$

✓ ✓

or

$$\sin x - \sqrt{3}\cos x = -1$$

$$\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x = -\frac{1}{2}$$

$$\sin x \cos\frac{\pi}{3} - \cos x \sin\frac{\pi}{3} = -\frac{1}{2}$$

$$\therefore \quad \sin\left(x - \frac{\pi}{3}\right) = -\frac{1}{2} \checkmark$$

$$\therefore x - \frac{\pi}{3} = -\frac{\pi}{6} + 2k\pi \text{ or } -\frac{5\pi}{6} + 2k\pi, \ k \in \mathbb{Z} \checkmark$$

$$\therefore \quad x = \frac{\pi}{6} + 2k\pi \text{ or } -\frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$$

✓ ✓

Question 4 **(2.2.3, 2.1.3)** **(6 marks)**

Let $A = \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}$ and $B = \begin{bmatrix} cos(\beta) \\ sin(\beta) \end{bmatrix}$, such that $AB = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$.

Find two different sets of possible values for α and β given that $0 \leq \alpha < \frac{\pi}{2}$ and $0 \leq \beta < \frac{\pi}{2}$

$$AB = \begin{bmatrix} \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{bmatrix} \checkmark\checkmark = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\therefore \quad \cos(\alpha - \beta) = \frac{1}{2} \checkmark, \quad -\frac{\pi}{2} < \alpha - \beta < \frac{\pi}{2}$$

$$\sin(\alpha + \beta) = \frac{\sqrt{3}}{2} \checkmark, \quad 0 \leq \alpha + \beta < \pi$$

$$\therefore \begin{cases} \alpha - \beta = \frac{\pi}{3} \\ \alpha + \beta = \frac{\pi}{3} \end{cases} \text{ or } \begin{cases} \alpha - \beta = \frac{\pi}{3} \\ \alpha + \beta = \frac{2\pi}{3} \end{cases} \text{ or } \begin{cases} \alpha - \beta = -\frac{\pi}{3} \\ \alpha + \beta = \frac{\pi}{3} \end{cases} \text{ or } \begin{cases} \alpha - \beta = -\frac{\pi}{3} \\ \alpha + \beta = \frac{2\pi}{3} \end{cases}$$

$$\begin{cases} \alpha = \frac{\pi}{3} \\ \beta = 0 \end{cases} \checkmark \text{ or } \begin{cases} \alpha = \frac{\pi}{2} \\ \beta = \frac{\pi}{6} \end{cases} \text{ or } \begin{cases} \alpha = 0 \\ \beta = \frac{\pi}{3} \end{cases} \checkmark \text{ or } \begin{cases} \alpha = \frac{\pi}{6} \\ \beta = \frac{\pi}{2} \end{cases}$$

(exclude) (exclude)

* Award 1 mark if student gives $\alpha - \beta = \frac{\pi}{3}$ or $\alpha + \beta = \frac{\pi}{3}$ only

Question 5 **(2.1.3, 2.1.5)** **(6 marks)**

a) First show that

$$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$$

(2 marks)

$$\text{LHS} = \frac{\tan\theta - 1}{1 + \tan\theta} \quad ✓$$

$$= \frac{\frac{\sin\theta}{\cos\theta} - 1}{1 + \frac{\sin\theta}{\cos\theta}}$$

$$= \frac{\sin\theta - \cos\theta}{\cos\theta} \div \frac{\cos\theta + \sin\theta}{\cos\theta}$$

$$= \frac{\sin - \cos\theta}{\cos\theta} \times \frac{\cos\theta}{\cos\theta + \sin\theta}$$

either ✓ or $= \dfrac{\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}$ or $= \dfrac{\frac{\sin\theta}{\cos\theta} - 1}{1 + \frac{\sin\theta}{\cos\theta}} \times \dfrac{\cos\theta}{\cos\theta}$

$$= \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta}$$

b) Hence (or otherwise) prove the following identity:

$$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\sin(2\theta) \ - 1}{1 \ - 2\sin^2\theta}$$

(4 marks)

$$\text{LHS} = \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta} \times \frac{\cos\theta - \sin\theta}{\cos\theta - \sin\theta} \quad ✓$$

$$= \frac{\sin\theta\cos\theta - \sin^2\theta - \cos^2\theta + \sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta - (\sin^2\theta + \cos^2\theta)}{\cos 2\theta}$$

$$= \frac{\sin 2\theta - 1}{1 - 2\sin^2\theta}$$

✓ Uses double angle identity for sine

✓ Uses Pythagorean identity

✓ Uses double angle identities for cosine

Question 6 (2.2.4, 2.2.5, 2.2.6, 2.2.7, 2.2.8, 2.2.9, 2.2.10) (12 marks)

a) Determine the matrices that produce each of the transformations described below:

i. a rotation clockwise about the origin by $90°$ (1 mark)

$$\begin{bmatrix} \cos(-90°) & -\sin(-90°) \\ \sin(-90°) & \cos(-90°) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \checkmark$$

ii. a dilation parallel to the y-axis by a scale factor of 2 (1 mark)

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \checkmark$$

iii. a reflection in the line $y = x$ (1 mark)

$$\begin{bmatrix} \cos 90° & \sin 90° \\ \sin 90° & -\cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \checkmark$$

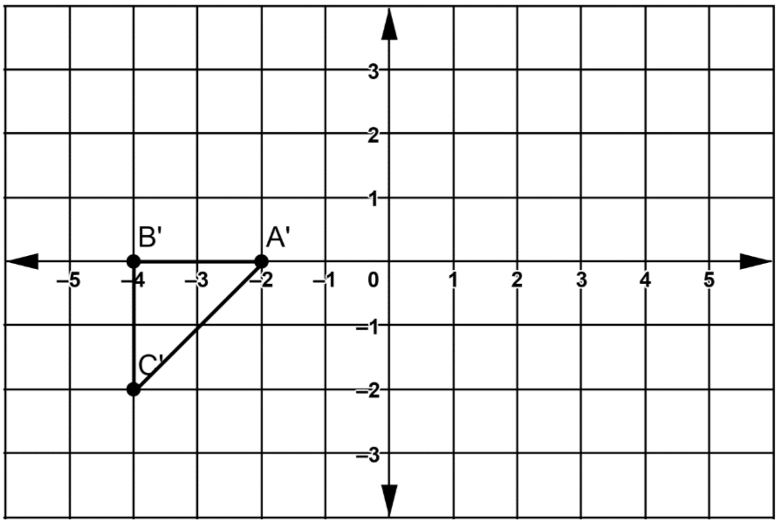
b) Show how to obtain the single transformation matrix T, given that T is the result of applying the transformations given in part a) in the order listed [i.e. a rotation clockwise about the origin by 90°, followed by a dilation parallel to the y-axis by a scale factor of 2, then a reflection in the line $y = x$]. (2 marks)

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \checkmark$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

c) ΔABC is first translated left by 1 unit and down by 2 units, then transformed by the transformation matrix T in part b). The final image $\Delta A'B'C'$ is shown below:



i. Determine the coordinates of points A, B and C. (5 marks)

$$\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & -4 & -4 \\ 0 & 0 & -2 \end{bmatrix} \checkmark$$

$$= -\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -2 & -4 & -4 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= -\frac{1}{2} \begin{bmatrix} -2 & -4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \checkmark$$

$\therefore$ A $(2, 2)$ ✓
B $(3, 2)$ ✓
C $(3, 0)$ ✓

Allow follow through here

ii. $\Delta A'B'C'$ is now transformed by a matrix

$$M = \begin{bmatrix} 2\sqrt{5} & \sqrt{5} \\ 3 & -1 \end{bmatrix}$$

Determine the exact area of the image of $\Delta A'B'C'$ under this transformation. (2 marks)

$$|\det(M)| = |-2\sqrt{5} - 3\sqrt{5}| = 5\sqrt{5} \checkmark$$

$$\therefore A = 5\sqrt{5} \times \left(\frac{1}{2} \times 2 \times 2\right)$$

$$= 10\sqrt{5} \text{ units}^2 \checkmark$$

Additional working space

Question number: ________