Poisson distribution

Definition

Applications

Motivation

Derivation

μ, σ

Example

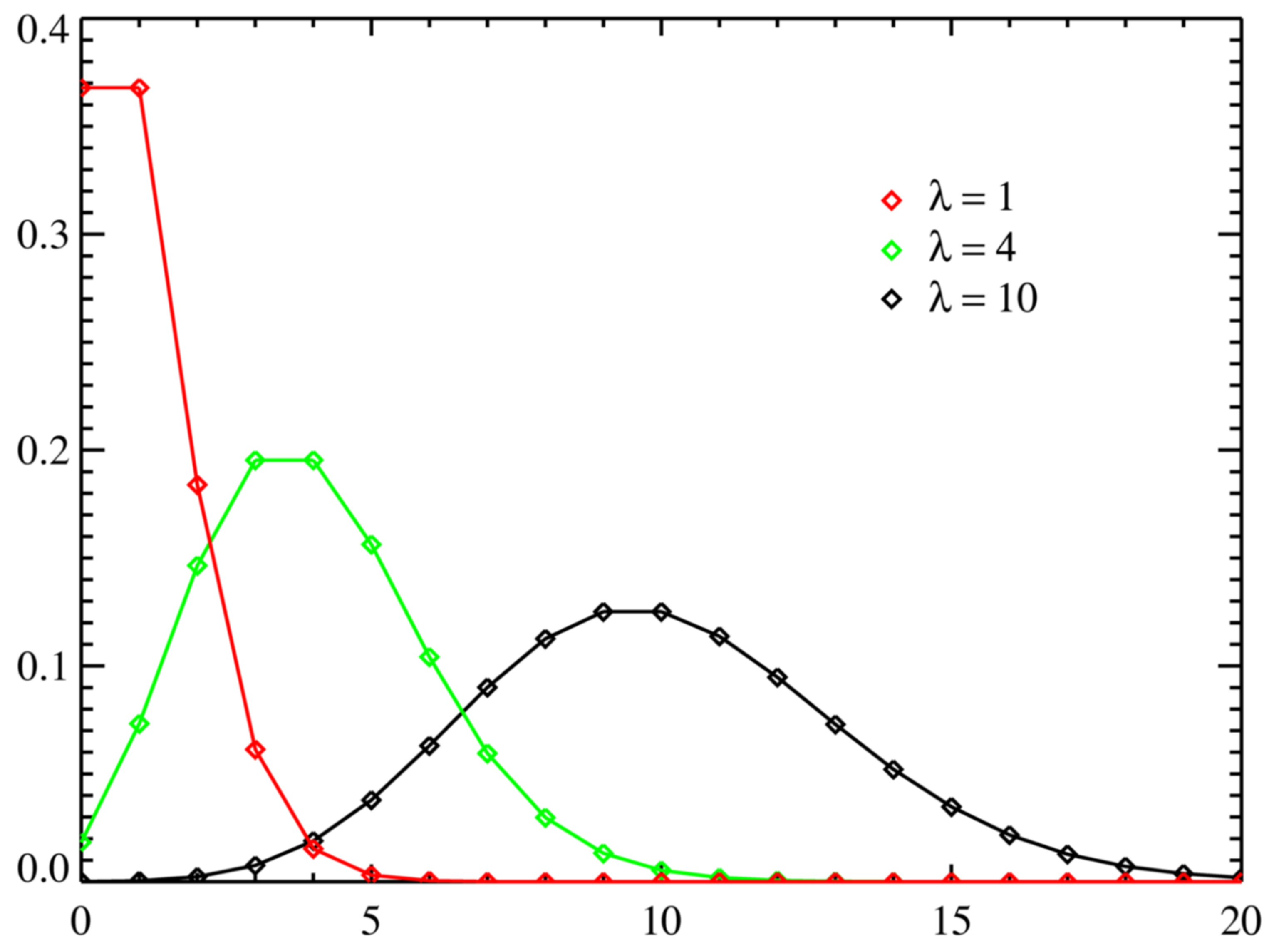
The Poisson Distribution

Parameter $\lambda \geq 0$ (some > 0)

Support $\mathbb{N}$

PMF

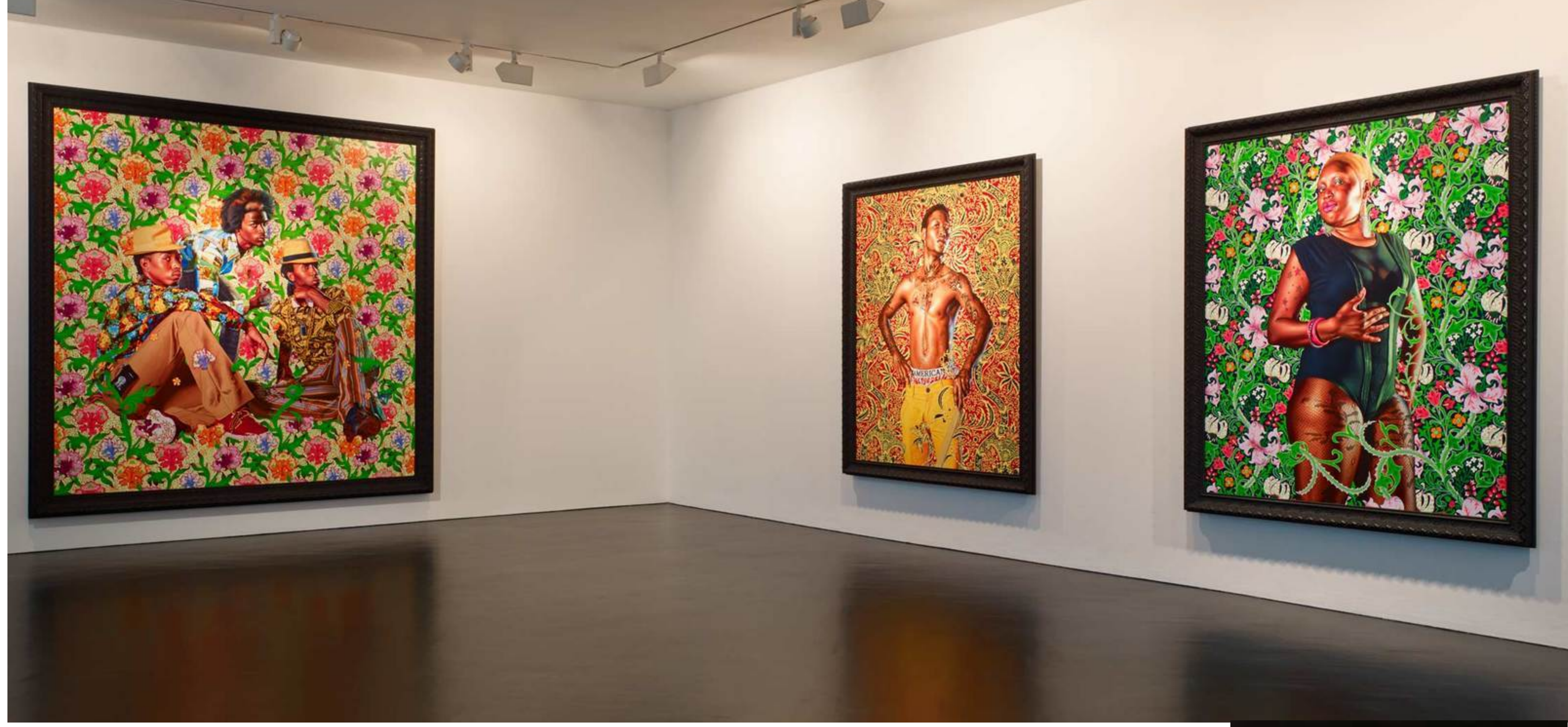
$$P_\lambda(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$



More in notebook

Significance Approximates $B_{p,n}$ for large n and small p so that np = λ is moderate

We are Poisson



P_λ approximates $B_{p,n}$ for small p, large n

Numerous applications

$450M

\# People clicking ad

Responses to spam

Rare-disease infections

Daily 911 calls

Daily store customers

Gallery purchasing customers

Flight no shows

Typos in a page

Small k

	λ	$P_\lambda(k)$	0	1	2	3
			k			
$\frac{\lambda^k}{e^\lambda}$	General	$e^{-\lambda}\frac{\lambda^k}{k!}$	$\frac{1}{e^\lambda}$	$\frac{\lambda}{e^\lambda}$	$\frac{\lambda^2}{2e^\lambda}$	$\frac{\lambda^3}{6e^\lambda}$
$\frac{1}{e}$	1	$\frac{1}{ek!}$	$\frac{1}{e}$	$\frac{1}{e}$	$\frac{1}{2e}$	$\frac{1}{6e}$
$\frac{2^k}{e^2}$	2	$\frac{2^k}{e^2k!}$	$\frac{1}{e^2}$	$\frac{2}{e^2}$	$\frac{2}{e^2}$	$\frac{4}{3e^2}$
0^k	0	$\frac{0^k}{k!}$	1	0	0	0

Binomial Approximation

P_λ approximates $B_{p,n}$ for $\lambda = pn$, when $n \gg 1 \gg p$

$$B_{p,n}(k) = \binom{n}{k} p^k q^{n-k} \qquad q = 1 - p$$

$$p = \frac{\lambda}{n}$$

$$= \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n^{\underline{k}}}{k!} \cdot \frac{\lambda^k}{n^k} \cdot \frac{(1 - \frac{\lambda}{n})^n}{(1 - \frac{\lambda}{n})^k}$$

Fix k and λ, let n ↗ and p ↘

Derive Poisson

Limit of Binomial

$$B_{p,n}(k) = \frac{\lambda^k}{k!} \cdot \cancelto{1}{\frac{n^k}{n^k}} \cdot \frac{\cancelto{e^{-\lambda}}{(1-\frac{\lambda}{n})^n}}{\cancelto{1}{(1-\frac{\lambda}{n})^k}} \to e^{-\lambda}\,\frac{\lambda^k}{k!}$$ ✅

$$\lambda = p \cdot n$$

λ and k fixed, n $\to \infty$

① $\frac{n^k}{n^k} = \cancelto{1}{\frac{n}{n}} \cdot \cancelto{1}{\frac{(n-1)}{n}} \cdot \ldots \cdot \cancelto{1}{\frac{(n-k+1)}{n}} \to 1$ fixed # (k) terms, each → 1

② $(1-\frac{\lambda}{n})^k \to 1$ fixed # (k) terms, each → 1

③ $(1-\frac{\lambda}{n})^n = ((1-\frac{\lambda}{n})^{\frac{n}{\lambda}})^\lambda \to (e^{-1})^\lambda = e^{-\lambda}$

increasing # terms, each → 1 $(1-\frac{1}{m})^m \to e^{-1}$

ΣWill It Add?

$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

k≥0

≥ 0 ✅

Taylor expansion

$$e^\lambda = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$$

$$\sum_{k=0}^{\infty} P_\lambda(k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

YES IT ADDS!

Mean and Variance

P_λ approximates $B_{p,n}$ for $\lambda = np$ when $n \gg 1 \gg p$

	μ	V	
$B_{p,n}$	np	npq	
P_λ	λ	λ	$\longleftarrow$ Expect

Calculate next

Observation

$$\frac{d}{d\lambda}\lambda^k = k\lambda^{k-1} = \frac{k}{\lambda}\lambda^k$$

$$\frac{d^2}{d\lambda^2}\lambda^k = k^{\underline{2}}\lambda^{k-2} = \frac{k^{\underline{2}}}{\lambda^2}\lambda^k$$

$$\frac{d^r}{d\lambda^r}\lambda^k = k^{\underline{r}}\lambda^{k-r} = \frac{k^{\underline{r}}}{\lambda^r}\lambda^k$$

$$k^{\underline{r}}\lambda^k = \lambda^r\frac{d^r}{d\lambda^r}\lambda^k$$

Falling Moments

$X \sim P_\lambda$

$$k^{\underline{r}} \lambda^k = \lambda^r \frac{d^r}{d\lambda^r} \lambda^k$$

$$E(X^{\underline{r}}) = \sum_{k=0}^{\infty} k^{\underline{r}} P_\lambda(k) = \sum_k k^{\underline{r}} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_k k^{\underline{r}} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_k \frac{\lambda^r}{k!} \frac{d^r}{d\lambda^r} \lambda^k$$

$$= e^{-\lambda} \lambda^r \frac{d^r}{d\lambda^r} \sum_k \frac{\lambda^k}{k!} = e^{-\lambda} \lambda^r \frac{d^r}{d\lambda^r} e^\lambda$$

$$= e^{-\lambda} \lambda^r e^\lambda = \lambda^r$$

$EX = EX^{\underline{1}} = \lambda$ $\qquad$ $EX(X-1) = EX^{\underline{2}} = \lambda^2$

Mean and Variance

$$EX = EX^{\underline{1}} = \lambda$$ ✅

$$EX(X-1) = EX^{\underline{2}} = \lambda^2$$

$$E(X^2) = E(X(X-1) + X) = E(X(X-1)) + E(X) = \lambda^2 + \lambda$$

$$V(X) = E(X^2) - (EX)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$ ✅

$\sigma = \sqrt{\lambda}$ Small relative to the mean

Approximation Example

Factory produces 200 items, each defective with probability 1%

P(3 defective)?

Binomial (precise) $B_{0.01,200}(3) = \binom{200}{3}(0.01)^3(0.99)^{197} \approx 0.181$

Poisson (approximation) $\lambda = 200 \cdot 0.01 = 2$ $P_2(3) = e^{-2}\frac{2^3}{3!} \approx 0.18$

P(some defective)?

$B_{0.01,200}(0) = \binom{200}{0}(0.99)^{200} \approx 0.134$

$P_2(0) = e^{-2}\frac{2^0}{0!} = e^{-2} \approx 0.135$

$B_{0.01,200}(\geq 1) = 1 - 0.134 \approx 0.866$

$P_2(\geq 1) = 1 - 0.135 \approx 0.865$

$$P_\lambda(k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$\lambda \geq 0$

$k \geq 0$

Approximates $B_{p,n}$ for $\lambda = np$, when $n \gg 1 \gg p$

# of ad clicks, rare diseases, production defects

$\mu = \lambda$

$V = \lambda$

$\sigma = \sqrt{\lambda}$

Geometric Distribution

Poisson

distribution