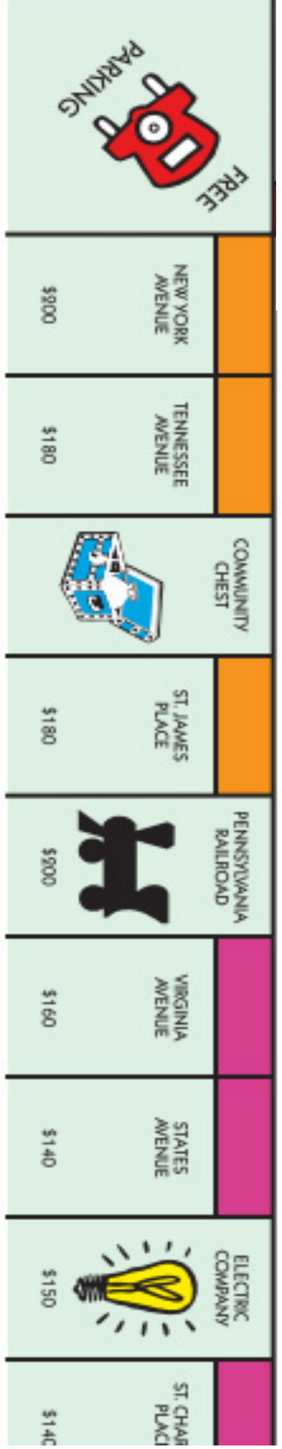
Geometric Distribution Examples

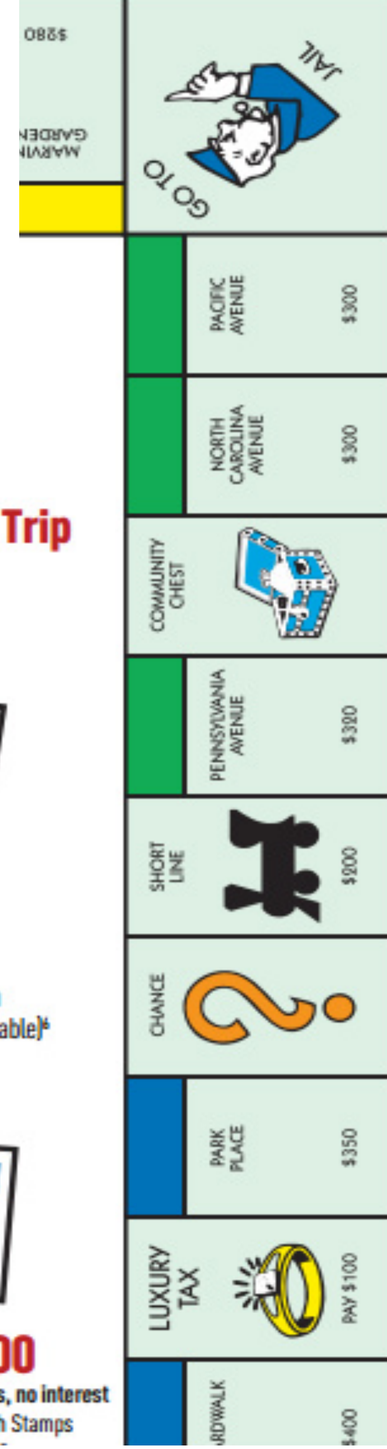
MONOPOLY®

COLLECT & WIN!

Complete Winning Combinations to win awesome prizes!

GAME BOARD





COLLECT & WIN!

DELTA VACATIONS

WIN a Dream Trip

to 1 of 250+ Destinations (20 available)[1]

Collect All 3 Stamps

Stamps: 512, 513 & 514

COLLECT & WIN!

Beaches

Make Your Getaway

Beaches® Resorts Luxury Included® Caribbean Vacation (75 available)[2]

Collect All 3 Stamps

Stamps: 515, 516 & 517

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Cessna

Cessna Private Jet Trip

(2 available)[3]

Collect All 3 Stamps

Stamps: 518, 519 & 520

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Mobile Wallet

$2,500

and a smartphone with Mobile Wallet (238 available)[4]

Collect All 3 Stamps

Stamps: 509, 510 & 511

COLLECT & WIN!

$10,000

Cash Prize to Help Get Your Bills Paid (4 available)[5]

Collect Both Stamps

Stamps: 527 & 528

COLLECT & WIN!

$5,000

Target Shopping Experience with Early Access on Black Friday (10 available)[6]

Collect All 4 Stamps

Stamps: 523, 524, 525 & 526

COLLECT & WIN!

$5,000

Cash Prize (5 available)[7]

Collect All 3 Stamps

COLLECT & WIN!

Fuel for a Year

(4 available)[8]

Collect All 3 Stamps

COLLECT & WIN!

$50

Cash Prize (1,000 available)[9]

Collect Both Stamps

COLLECT & WIN!

$1,000,000

Payable $50,000/yr for 20 yrs, no interest (1 available)[10] Collect Both Stamps

Startup Statistics

P(startup success) = 20%, independent of previous attempts

Expected # startups till first success

$X \sim G_{0.2}$ $E(X) = \frac{1}{.2} = 5$

Home-Grown Entrepreneur

One of first three startups succeeds

Dad will fund up to three startups P(success)?

$$P(X \leq 3) = F(3) = 1 - (0.8)^3 \approx 0.49$$

Cry Uncle

Even wealthier uncle funds next three startups (4,5,6)

P(success with uncle if dad's help did not suffice)?

$$P(X \in \{4,5,6\}|X > 3) = P(4|X > 3) + P(5|X > 3) + P(6|X > 3)$$
$$= P(1) + P(2) + P(3) = P(X \leq 3) \approx 49\%$$

P(success with uncle)? 1,2,3 failed but one of 4, 5, 6 succeeded

$$P(3 < X \leq 6) = P(X > 3 \cap X \leq 6) = P(X > 3) \cdot P(x \leq 6|x > 3)$$
$$= (0.8)^3 \cdot 0.49 \approx 25\%$$

$P(X_1, X_2, X_3 \text{ failed}) = q^3$

$$P(3 < X \leq 6) = F(6) - F(3) = (1 - 0.8^6) - (1 - 0.8^3)$$
$$= 0.8^3 - 0.8^6 \approx 25\%$$

Foreign-Born Entrepreneur

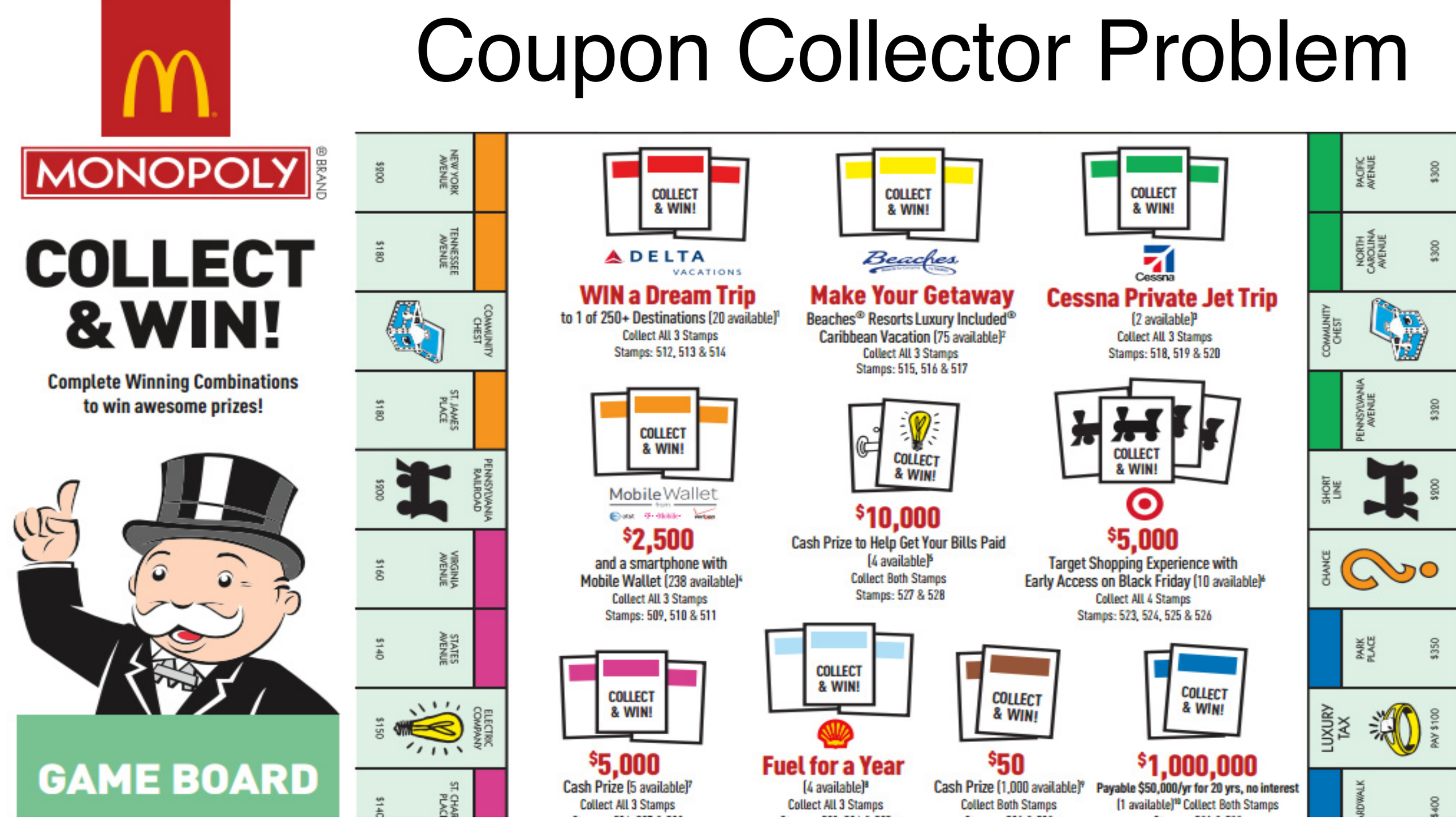
X - time to first success p=0.2

r^X - fraction of company you keep r=0.5

$$E(r^X) = \sum_{k=1}^{\infty} r^k P(X=k) = \sum_{k=1}^{\infty} pq^{k-1}r^k = pr\sum_{i=0}^{\infty}(qr)^i$$

$$= \frac{pr}{1-qr} = \frac{0.2\cdot 0.5}{1-0.8\cdot 0.5} = \frac{0.1}{1-0.4} = \frac{0.1}{0.6} \approx 16.67\%$$

Coupon Collector Problem



Pre GROUPON

n coupons

Each item contains one coupon selected uniformly

Collect all coupons, get a prize

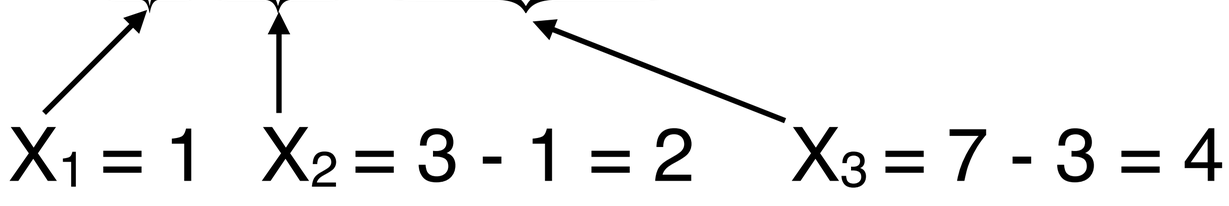
How many items need to buy to collect all?

Expectation

X - # items to collect all coupons

n = 3 Items 1 2 3 4 5 6 7 X = 7 EX?

Coupon 2 2 3 2 3 3 1



$X_1 = 1$ $X_2 = 3 - 1 = 2$ $X_3 = 7 - 3 = 4$

X_i - # items to get i^{th} coupon after getting i - 1 coupons

$X = X_1 + X_2 + X_3$

$7 = 1 + 2 + 4$

$X_1 = 1$

$X_2 \sim G_{2/3}$

$X_3 \sim G_{1/3}$

$\perp\!\!\!\perp$

General n

$$X_i \sim G(\tfrac{n-(i-1)}{n}) = G(\tfrac{n-i+1}{n})$$

$$EX_i = \tfrac{n}{n-i+1}$$

$$X = \sum_{i=1}^{n} X_i$$

$$EX = \sum_{i=1}^{n} EX_i = \sum_{i=1}^{n} \tfrac{n}{n-i+1} = \tfrac{n}{n} + \tfrac{n}{n-1} + \tfrac{n}{n-2} + \ldots + \tfrac{n}{1}$$

$$= n(\tfrac{1}{1} + \tfrac{1}{2} + \ldots + \tfrac{1}{n}) = nH_n \approx n \ln n + 0.577n$$

Harmonic Sum $H_n = \tfrac{1}{1} + \tfrac{1}{2} + \ldots + \tfrac{1}{n} \to \ln n + 0.577\ldots$

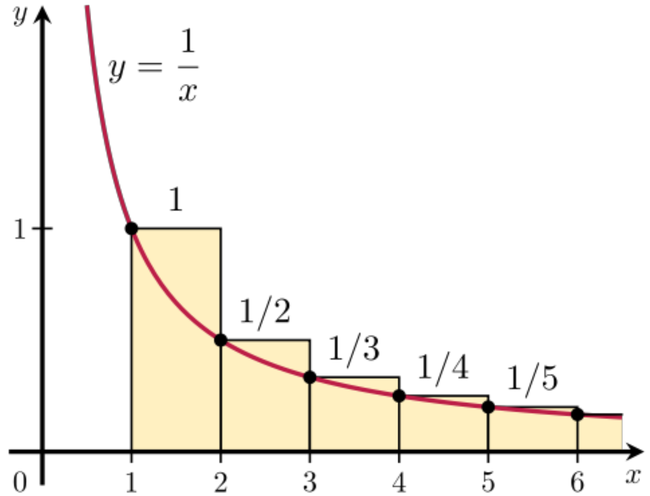
Harmonic Sum

$$H_n > \int_1^{n+1} \frac{1}{x} dx$$

$$= \ln x \Big|_1^{n+1} = \ln(n+1)$$

$$H_n \leq 1 + \int_1^n \frac{1}{x} dx = 1 + \ln x \,\Big|_1^n = 1 + \ln n$$

$$H_n \to \ln n + 0.577\ldots$$



Variance

$X \sim G(P)$ $\qquad V(X) = \frac{1-p}{p^2} \leq \frac{1}{p^2}$

$$V(X) = V\left(\sum_{i=1}^{n} X_i\right)$$

$$\overset{\perp\!\!\!\perp}{=} \sum_{i=1}^{n} V(X_i)$$

$$\leq \sum_{i=1}^{n} \frac{1}{(\frac{n-i+1}{n})^2}$$

$$= n^2 \left(\frac{1}{n^2} + \frac{1}{(n-1)^2} + \cdots + \frac{1}{1^2}\right)$$

$$\leq \frac{\pi^2}{6} n^2 \qquad \sigma \leq \frac{\pi}{\sqrt{6}} n$$

Summary

Geometric-distribution examples

Coupon collector problem

Discrete distribution families

Bernoulli, Binomial, Poisson, Geometric