

Sequential Probability

General product rule

Chain rule

$$P(E \cap F) = P(E) \cdot P(F|E)$$

More events

Useful in applications

Birthday paradox

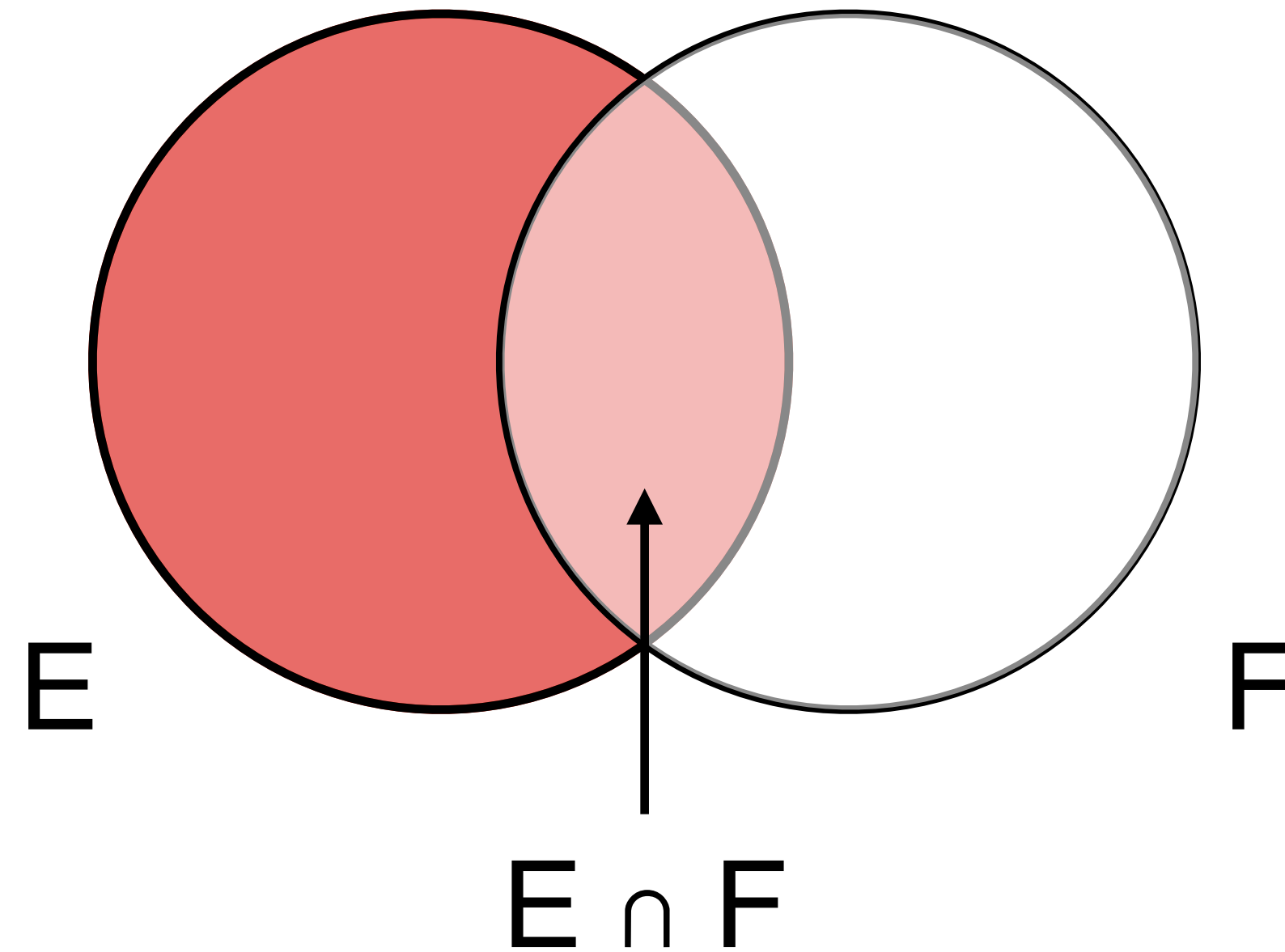
Of 23 people, $p > 0.5$ that two share a birthday

Chain Rule

Conditional probability

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)}$$

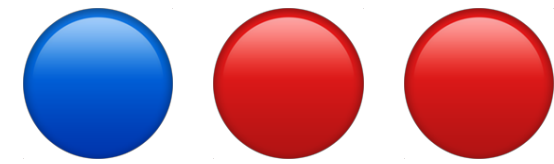
$$P(E \cap F) = P(E) \cdot P(F \mid E)$$



Helps calculate regular (not conditional) probabilities

Sequential Selection

1 blue, 2 red balls



Draw 2 balls without replacement

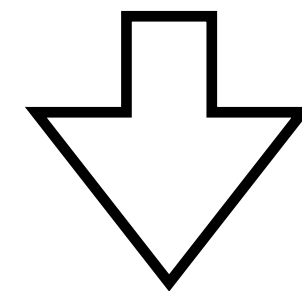
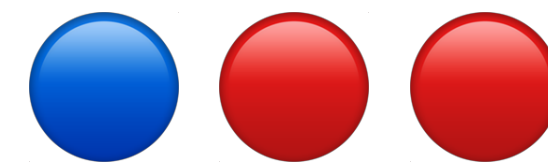
$P(\text{both red}) = ?$

R_i - i'th ball is red

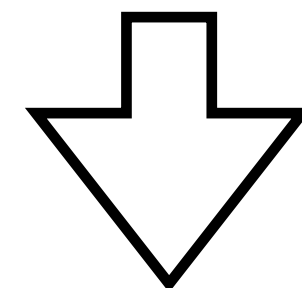
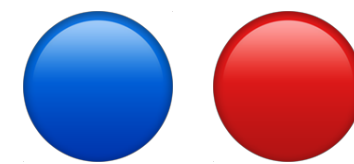
$$P(\text{both red}) = P(R_1, R_2)$$

$$= P(R_1) \cdot P(R_2 \mid R_1)$$

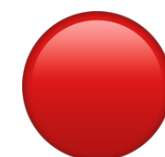
$$= \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$



$$P(R_1) = \frac{2}{3}$$



$$P(R_2 \mid R_1) = \frac{1}{2}$$



General Product Rule

For 3 events

$$P(E \cap F \cap G) = P((E \cap F) \cap G)$$

$$= P(E \cap F) \cdot P(G \mid E \cap F)$$

$$= P(E) \cdot P(F \mid E) \cdot P(G \mid E \cap F)$$

Similarly for more events

Odd Ball

$n-1$ red balls and one blue ball

Pick n balls without replacement

$P(\text{last ball is blue}) = ?$

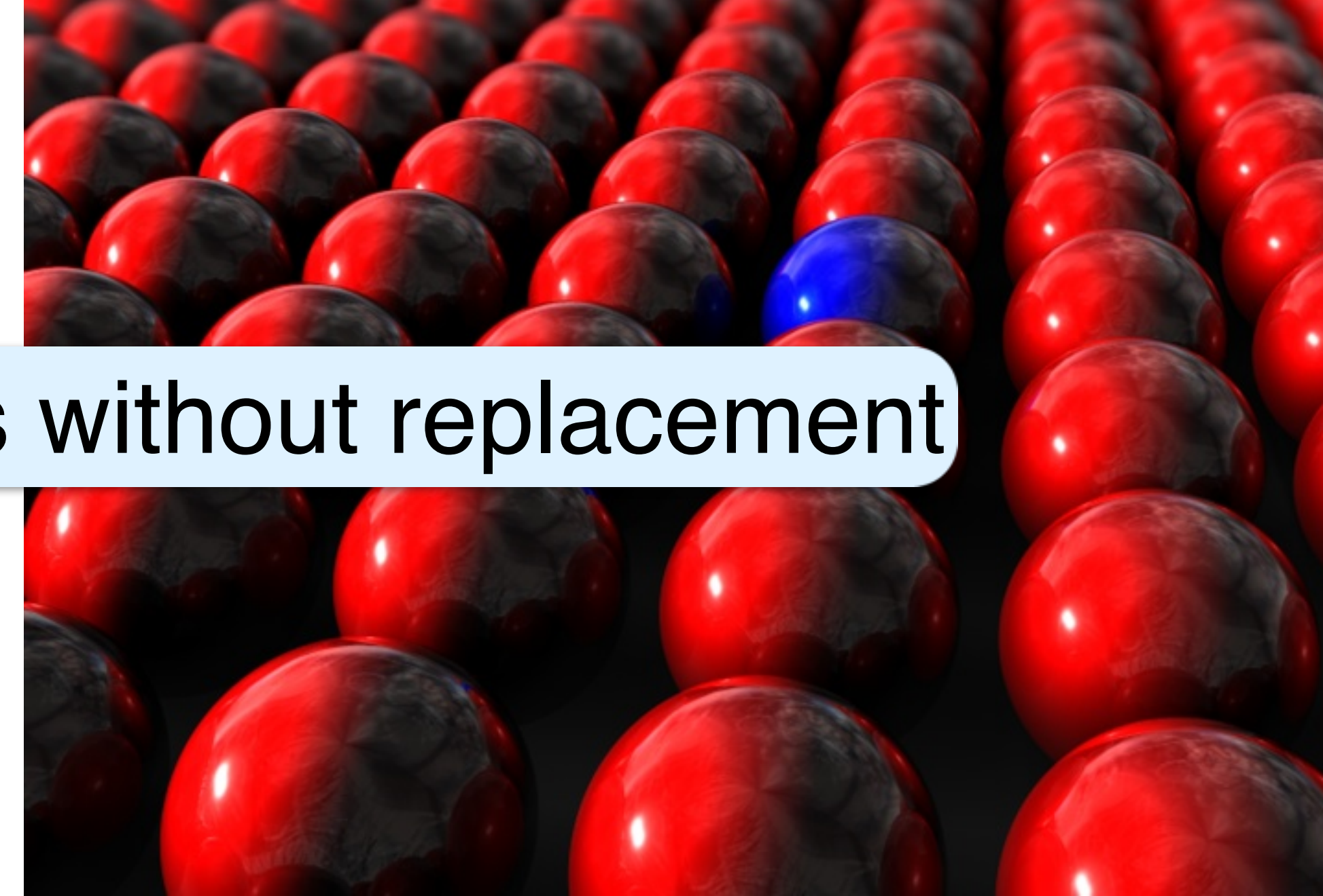
R_i - i th ball is red

$R^i - R_1, R_2, \dots, R_i$

$$P(\text{last ball blue}) = P(R_1)P(R_2|R_1)P(R_3|R^2)\dots P(R_{n-1}|R^{n-2})$$

$$= \frac{n-1}{n} \frac{n-2}{n-1} \frac{n-3}{n-2} \dots \frac{2}{3} \frac{1}{2} = \frac{1}{n}$$

Or.. Arrange in row, probability last ball is blue = $1/n$



The Birthday Paradox

How many people does it take so that two will likely share a birthday?

Assume that every year has 365 days

Everyone is equally likely to be born on any given day

Probabilistically

Choose n random integers, each $\in \{1, \dots, 365\}$, with replacement

$B(n)$ - probability that two (or more) are the same

For which n does $B(n)$ exceed, say, $1/2$?

Some first think it $n \approx 365$, but in fact much smaller

First Attempt

Consider the n people in order, say alphabetically

List their birthdays

2, 10, 365, 180, 10, ...

Selection with replacement

Set of all possible birthdays sequences

$\Omega = \{1, 2, \dots, 365\}^n$

$|\Omega| = 365^n$

Individual birthday uniform

→

Ω uniform

B_n {sequences with repetition}

$P(\text{repetition}) = |B_n| / |\Omega|$

Evaluating $|B_n|$ involved

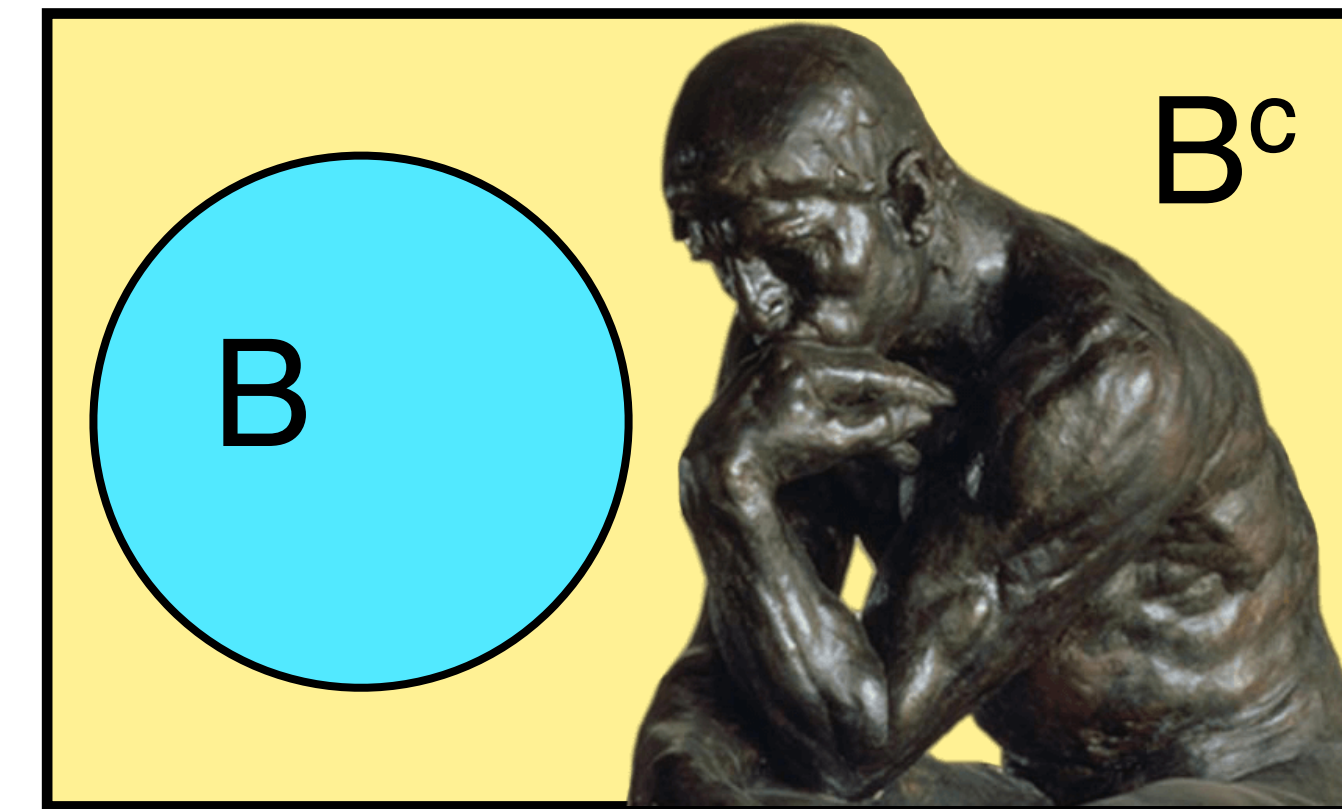
Complement

B_n n people have birthday repetition

B_n^c n people, no two share a birthday

Evaluate sequentially

Person i different b/day from all previous



Calculation

$$1 - x \leq e^{-x}$$

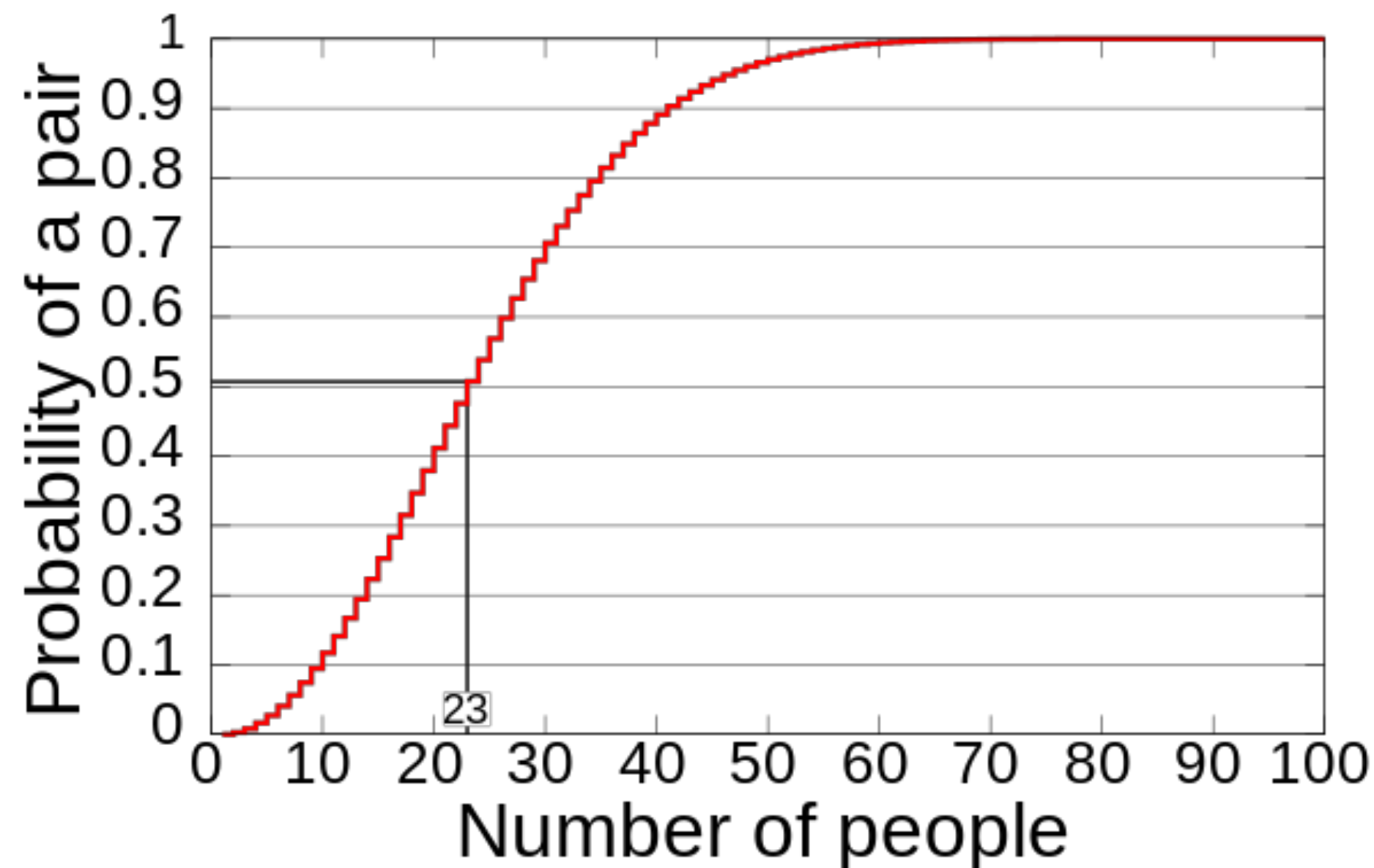
Among n people

P(no two people share a birthday)

When the probability is 0.5

$$-\frac{n^2}{2 \cdot 365} = \ln 0.5 = -\ln 2$$

$$n \approx \sqrt{-2 \cdot 365 \cdot \ln 0.5} = 22.494$$



$$= \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \cdot \frac{365 - n + 1}{365}$$

$$= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365} \right)$$

$$\leq \prod_{i=1}^n e^{-\frac{i}{365}}$$

$$= \exp \left(-\frac{1}{365} \cdot \sum_{i=1}^{n-1} i \right)$$

$$= \exp \left(-\frac{n(n-1)}{2 \cdot 365} \right)$$

$$\approx \exp \left(-\frac{n^2}{2 \cdot 365} \right) = 0.5$$

Sequential Probability

General product rule

Chain rule

$$P(E \cap F) = P(E) \cdot P(F|E)$$

More events

Useful in applications

Birthday paradox

Of 23 people, $p > 0.5$ that two share a birthday

Next: Total Probability