Geometric distributions

Back to Basics

Independent B_p coin flips

$p(1) = p$

$p(0) = 1-p \stackrel{\text{def}}{=} q$

Two Derived Distributions

← Binomial	$B_{p,n}$	# 1's in flips
→ Geometric	G_p	# flips till first 1

Time to first success

Flips	X
101011	1
010111	2
001010	3
$\underbrace{0\ldots01}_{n}0$	n

n	$X_1,\ldots,X_n \sim B(p)$	p(n)
1	$X_1 = 1$	p
2	$X_1 = 0 \quad X_2 = 1$	qp
3	$X_1 = X_2 = 0 \quad X_3 = 1$	q^2p
n	$X_1 = \ldots = X_{n-1} = 0 \quad X_n = 1$	$q^{n-1}p$

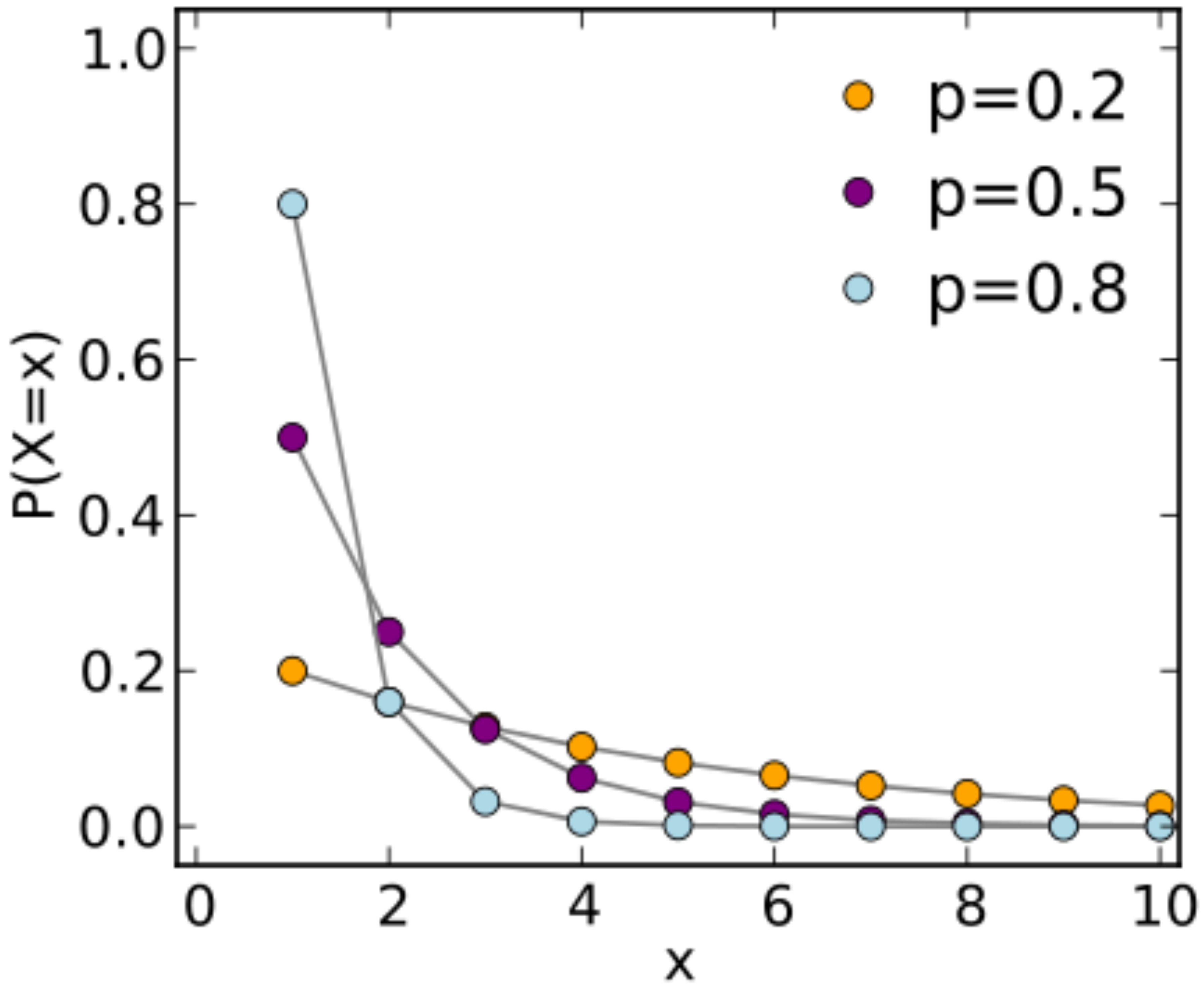
Geometric Distribution

G_p $0 < p \leq 1$

$p(n) = q^{n-1}p \stackrel{\text{def}}{=} g_p(n)$ $n \geq 1$

Range $p \neq 0$ n can be arbitrarily high

Notebook Plot different p



Who’s Geometric

30 years ago

Thief trying door keys

Trials to hit a target

Attempts till success

Till failure

Nowadays

ΣWill It ADD?

$$P(n) = pq^{n-1} \qquad n \geq 1 \qquad q = 1 - p$$

$$\begin{aligned}(1 + q + q^2 + \ldots)(1 - q) &= 1 + \cancel{q} + \cancel{q^2} + \ldots \\ &\qquad - \cancel{q} - \cancel{q^2} - \ldots\end{aligned}$$

$$\sum_{i=0}^{\infty} q^i = \frac{1}{1 - q}$$

YES IT ADDS!

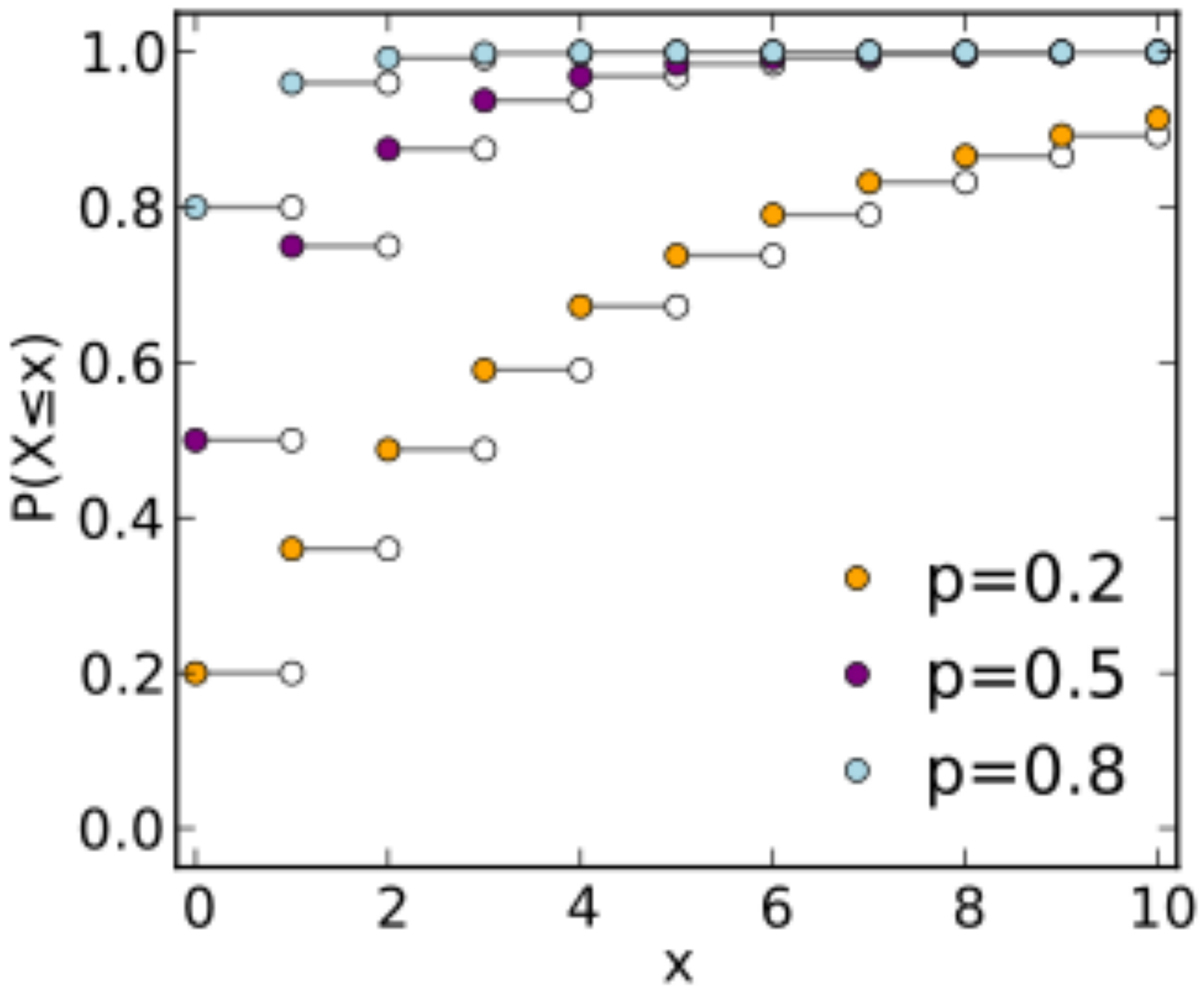
$$\sum_{n=1}^{\infty} p(1-p)^{n-1} = p\sum_{i=0}^{\infty}(1-p)^i = p \cdot \frac{1}{1-(1-p)} = \frac{p}{p} = 1$$

CDF

$n \in \mathbb{N}$ $X > n$ iff $X_1 = \ldots = X_n = 0$

$P(X > n) = P(X_1 = \ldots = X_n = 0) = q^n$

$$F(n) = P(X \leq n) = 1 - P(X > n) = 1 - q^n$$



Expectation via "Right" CDF

$x \in \mathbb{N} \qquad P_k = P(X = k)$

$$\begin{array}{cccc} P_1 & & & \\ P_2 & P_2 & & \\ P_3 & P_3 & P_3 & \\ P_4 & P_4 & P_4 & P_4 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

$$E(X) = \sum_{k=0}^{\infty} kP_k$$

$$= P_1 + 2P_2 + 3P_3 + \dots$$

$$= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \dots$$

Geometric distribution

$$EX = \sum_{k=1}^{\infty} P(X \geq k) = \sum_{i=0}^{\infty} P(X > i) = \sum_{i=0}^{\infty} q^i = \frac{1}{1-q} = \frac{1}{p}$$

Variance

$$EX(X-1) = \sum_{n=1}^{\infty} n(n-1)\cdot P(X=n) = p\sum_{n=2}^{\infty} n(n-1)q^{n-1}$$

$$= pq\sum_{n=2}^{\infty} \frac{d^2}{dq^2}q^n = pq\frac{d^2}{dq^2}\sum_{n=2}^{\infty} q^n$$

$$= pq\frac{d^2}{dq^2}\left(\frac{1}{1-q} - 1 - q\right)$$

$$= pq\frac{2}{(1-q)^3} = \frac{2q}{p^2}$$

$$\left(\frac{1}{1-q}\right)' = \frac{1}{(1-q)^2}$$

$$\left(\frac{1}{(1-q)^2}\right)' = \frac{2}{(1-q)^3}$$

$$EX^2 = EX(X-1) + EX = \frac{2q}{p^2} + \frac{1}{p} = \frac{2q+p}{p^2} = \frac{1+q}{p^2}$$

$$V(X) = EX^2 - (EX)^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2} \qquad \sigma = \frac{\sqrt{q}}{p}$$

Fair Coin

$$X \sim G_{\frac{1}{2}}$$

$$P(X = k) = g_{0.5}(k) = (\tfrac{1}{2})^{k-1} \cdot \tfrac{1}{2} = \tfrac{1}{2^k}$$

$$EX = \tfrac{1}{p} = 2$$

$$VX = \tfrac{q}{p^2} = 2$$

“Even” Odds

Normally, “even odds” means that two events have equal probabilities

Something a bit more “odd”

Probability that a Geometric random variable is even

$$P(1) = p \qquad P(2) = q \cdot p \; = q \cdot P(1)$$

Memoryless

A distribution over $\mathbb{P} = \{1,2,..\}$ is memoryless if for all n≥0, m>1

$$P(\ X=n+m \mid X>n\ ) = P(\ X=m\ )$$

$P(X=12 \mid X>10) = P(X=2)$

After observing or any number of samples, process behaves as at the start

Geometric → Memoryless

$$P(X = n + m | X > n) = \frac{P(X=n+m, X>n)}{P(X>n)}$$

$$= \frac{P(X=n+m)}{P(X>n)}$$

$$= \frac{p \cdot q^{n+m-1}}{q^n}$$

$$= p \cdot q^{m-1}$$

$$= P(X = m)$$

All geometric distributions are memoryless

Memoryless → Geometric

Any discrete memoryless distribution over $\mathbb{P}$ is geometric

Let $p \stackrel{\text{def}}{=} P(X = 1), \qquad q \stackrel{\text{def}}{=} 1 - p = P(X > 1)$

$$\forall\, n \geq 1, \quad P(X = n + 1) = P(X > 1 \;\wedge\; X = n + 1)$$

$$= P(X > 1) \cdot P(X = n + 1 | X > 1)$$

$$= q \cdot P(X = n)$$

Hence P(X=2)=qp, P(X=3)=q^2p, ….

P(X=n)=q^{n-1}p Geometric

r Successes

$$\text{Geometric } P(X = n) = P(\text{first success at n'th trial})$$

$$P(\text{r'th success at n'th trial}) = P(\text{r - 1 successes in n - 1 trials}) \cdot P(\text{n'th trial is success})$$

$$n \geq r$$

$$b_{n-1,p}(r-1) \qquad \text{p}$$

$$= \binom{n-1}{r-1} p^{r-1} q^{n-r} p$$

$$= \binom{n-1}{r-1} p^r q^{n-r}$$

$$r = 1 \rightarrow pq^{n-1} = g_p(n)$$

Negative binomial distribution

Geometric Distribution

$$P(n) = pq^{n-1} \qquad n \geq 1 \qquad q = 1 - p$$

Memoryless

$$E(X) = \frac{1}{p}$$

$$V(X) = \frac{q}{p^2} \qquad \sigma = \frac{\sqrt{q}}{p}$$

$$P(\text{r'th success at n'th trial}) = \binom{n-1}{r-1} p^r q^{n-r}$$

Next: Examples