

Geometric Distribution Examples



MONOPOLY

@ BRAND

COLLECT & WIN!

Complete Winning Combinations to win awesome prizes!



GAME BOARD

FREE PARKING	
NEW YORK AVENUE \$200	
TENNESSEE AVENUE \$180	
COMMUNITY CHEST	
ST. JAMES PLACE \$180	
PENNSYLVANIA RAILROAD \$200	
VIRGINIA AVENUE \$160	
STATES AVENUE \$140	
ELECTRIC COMPANY \$150	
ST. CHARLES PLACE \$140	

COLLECT & WIN!

DELTA VACATIONS

WIN a Dream Trip
to 1 of 250+ Destinations (20 available)¹
Collect All 3 Stamps
Stamps: 512, 513 & 514

COLLECT & WIN!

Beaches

Make Your Getaway
Beaches® Resorts Luxury Included®
Caribbean Vacation (75 available)²
Collect All 3 Stamps
Stamps: 515, 516 & 517

COLLECT & WIN!

Cessna

Cessna Private Jet Trip
(2 available)³
Collect All 3 Stamps
Stamps: 518, 519 & 520

COLLECT & WIN!

Mobile Wallet
from AT&T, T-Mobile, Verizon

\$2,500
and a smartphone with
Mobile Wallet (238 available)⁴
Collect All 3 Stamps
Stamps: 509, 510 & 511

COLLECT & WIN!

\$10,000
Cash Prize to Help Get Your Bills Paid
(4 available)⁵
Collect Both Stamps
Stamps: 527 & 528

COLLECT & WIN!

Target

\$5,000
Target Shopping Experience with
Early Access on Black Friday (10 available)⁶
Collect All 4 Stamps
Stamps: 523, 524, 525 & 526

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\$5,000
Cash Prize (5 available)⁷
Collect All 3 Stamps

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Shell

Fuel for a Year
(4 available)⁸
Collect All 3 Stamps

COLLECT & WIN!

\$50
Cash Prize (1,000 available)⁹
Collect Both Stamps

COLLECT & WIN!

\$1,000,000
Payable \$50,000/yr for 20 yrs, no interest
(1 available)¹⁰ Collect Both Stamps

MARVIN THE MONSTER GO TO JAIL	
PACIFIC AVENUE \$300	
NORTH CAROLINA AVENUE \$300	
COMMUNITY CHEST	
PENNSYLVANIA AVENUE \$320	
SHORT LINE \$200	
CHANCE	
PARK PLACE \$350	
LUXURY TAX PAY \$100	
BOARDWALK \$400	

Startup Statistics

$P(\text{startup success}) = 20\%$, independent of previous attempts

Expected # startups till first success

$$X \sim G_{0.2} \qquad E(X) = \frac{1}{.2} = 5$$

Home-Grown Entrepreneur

One of first three
startups succeeds

Dad will fund up to three startups $P(\text{success})?$

$$P(X \leq 3) = F(3) = 1 - (0.8)^3 \approx 0.49$$

Cry Uncle

Even wealthier uncle funds next three startups (4,5,6)

P(success with uncle if dad's help did not suffice)?

$$\begin{aligned}P(X \in \{4, 5, 6\} | X > 3) &= P(4 | X > 3) + P(5 | X > 3) + P(6 | X > 3) \\&= P(1) + P(2) + P(3) = P(X \leq 3) \approx 49\%\end{aligned}$$

P(success with uncle)?  1,2,3 failed but one of 4, 5, 6 succeeded

$$\begin{aligned}P(3 < X \leq 6) &= P(X > 3 \cap X \leq 6) = P(X > 3) \cdot P(x \leq 6 | x > 3) \\&= (0.8)^3 \cdot 0.49 \approx 25\%\end{aligned}$$

$P(X_1, X_2, X_3 \text{ failed}) = q^3$

$$\begin{aligned}P(3 < X \leq 6) &= F(6) - F(3) = (1 - 0.8^6) - (1 - 0.8^3) \\&= 0.8^3 - 0.8^6 \approx 25\%\end{aligned}$$

Foreign-Born Entrepreneur

X - time to first success $p=0.2$

r^X - fraction of company you keep $r=0.5$

$$\begin{aligned} E(r^X) &= \sum_{k=1}^{\infty} r^k P(X = k) = \sum_{k=1}^{\infty} pq^{k-1} r^k = pr \sum_{i=0}^{\infty} (qr)^i \\ &= \frac{pr}{1-qr} = \frac{0.2 \cdot 0.5}{1-0.8 \cdot 0.5} = \frac{0.1}{1-0.4} = \frac{0.1}{0.6} \approx 16.67\% \end{aligned}$$

Coupon Collector Problem



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	CHANCE	
	PARK PLACE	\$350
	LUXURY TAX	PAY \$100
	JAILWALK	\$400

Pre **GROUPON**

n coupons

Each item contains one coupon selected uniformly

Collect all coupons, get a prize



How many items need to buy to collect all?

Expectation

X - # items to collect all coupons

$n = 3$ Items 1 2 3 4 5 6 7 $X = 7$ EX?

Coupon 2 2 3 2 3 3 1

$X_1 = 1$ $X_2 = 3 - 1 = 2$ $X_3 = 7 - 3 = 4$

X_i - # items to get i^{th} coupon after getting $i - 1$ coupons

$$\left. \begin{array}{l} X = X_1 + X_2 + X_3 \\ 7 = 1 + 2 + 4 \end{array} \right\} \perp \begin{array}{l} X_1 = 1 \\ X_2 \sim G_{2/3} \\ X_3 \sim G_{1/3} \end{array}$$

General n

$$X_i \sim G\left(\frac{n-(i-1)}{n}\right) = G\left(\frac{n-i+1}{n}\right)$$

$$EX_i = \frac{n}{n-i+1}$$

$$X = \sum_{i=1}^n X_i$$

$$EX = \sum_{i=1}^n EX_i = \sum_{i=1}^n \frac{n}{n-i+1} = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \dots + \frac{n}{1}$$

$$= n\left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n}\right) = nH_n \approx n \ln n + 0.577n$$

Harmonic Sum $H_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} \rightarrow \ln n + 0.577\dots$

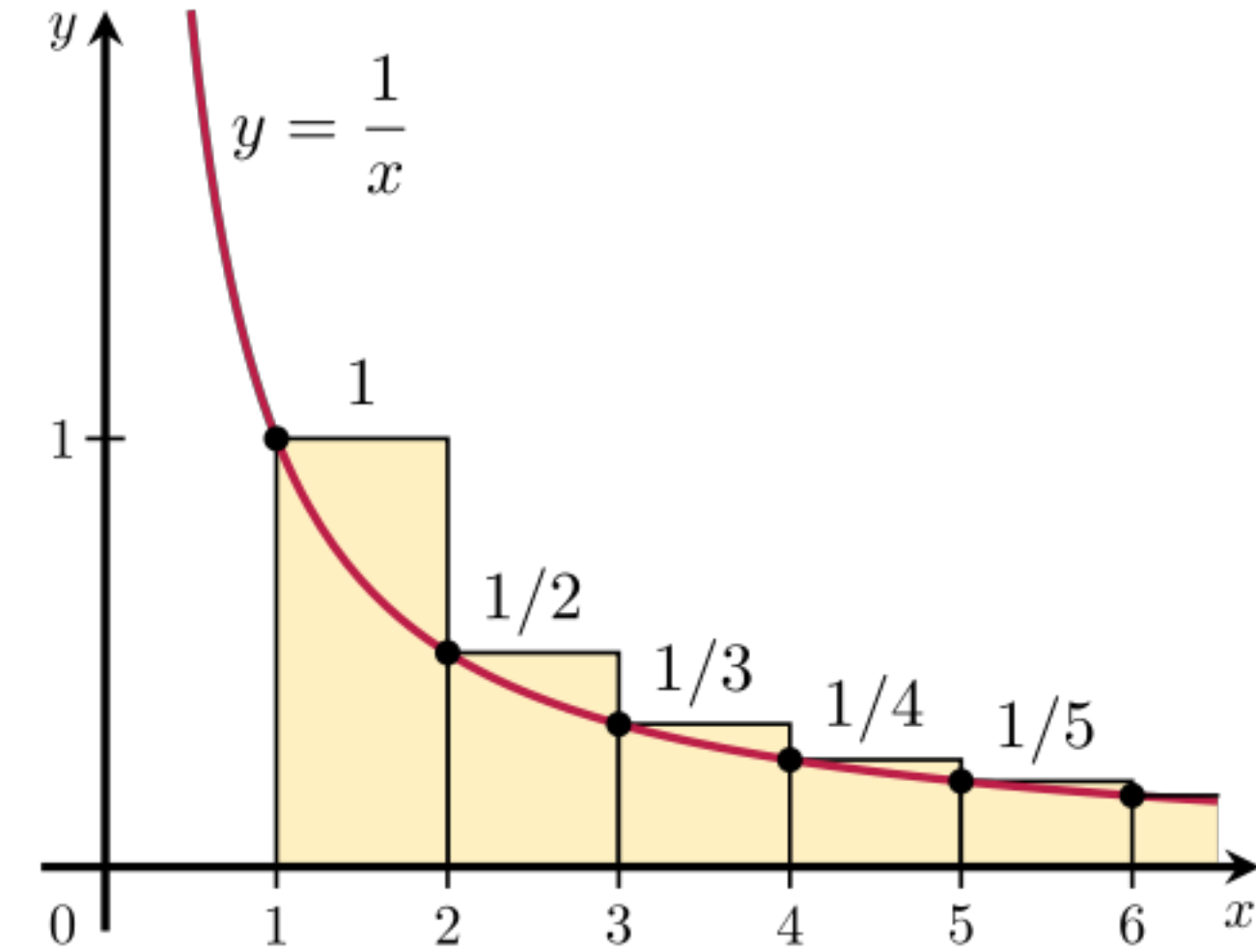
Harmonic Sum

$$H_n > \int_1^{n+1} \frac{1}{x} dx$$

$$= \ln x \Big|_1^{n+1} = \ln(n+1)$$

$$H_n \leq 1 + \int_1^n \frac{1}{x} dx = 1 + \ln x \Big|_1^n = 1 + \ln n$$

$$H_n \rightarrow \ln n + 0.577 \dots$$



Variance

$$X \sim G(P)$$

$$V(X) = \frac{1-p}{p^2} \leq \frac{1}{p^2}$$

$$V(X) = V\left(\sum_{i=1}^n X_i\right)$$

$$\stackrel{\textcircled{\parallel}}{=} \sum_{i=1}^n V(X_i)$$

$$\leq \sum_{i=1}^n \frac{1}{\left(\frac{n-i+1}{n}\right)^2}$$

$$= n^2 \left(\frac{1}{n^2} + \frac{1}{(n-1)^2} + \dots + \frac{1}{1^2} \right)$$

$$\leq \frac{\pi^2}{6} n^2$$

$$\sigma \leq \frac{\pi}{\sqrt{6}} n$$

Summary

Geometric-distribution examples

Coupon collector problem

Discrete distribution families

Bernoulli, Binomial, Poisson, Geometric

