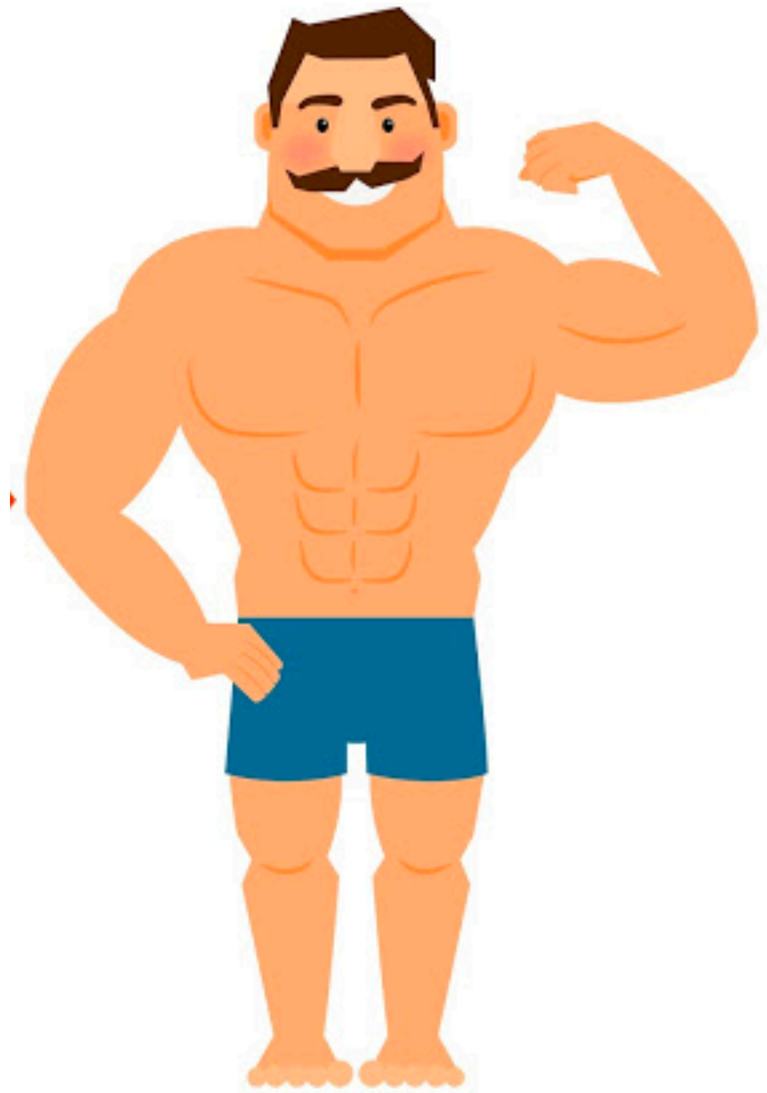
Two Variables

Why 2

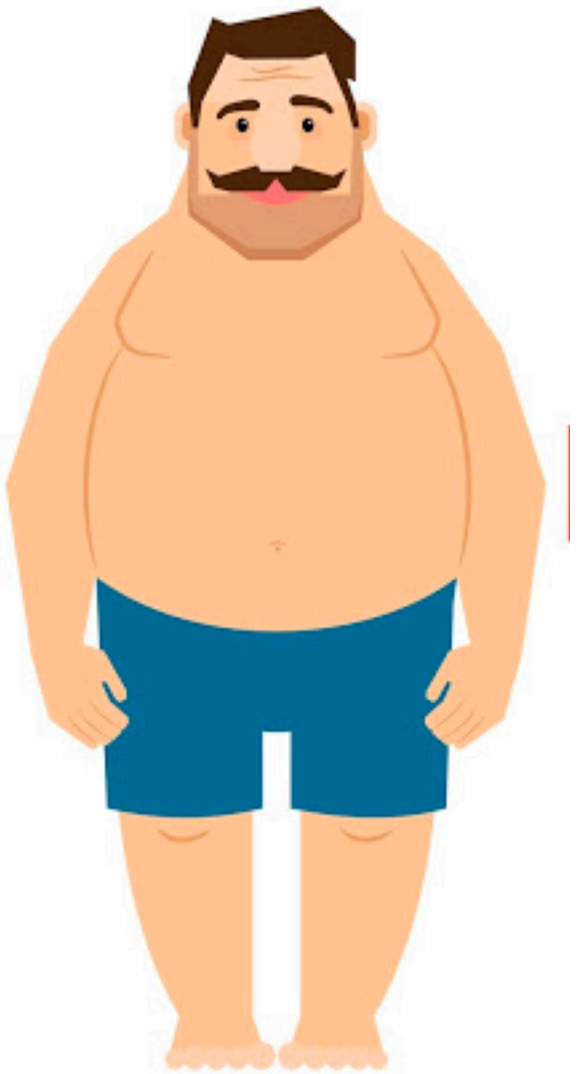
San Diego version

6 pack

6 are better than 1



LA version of SD version



Probabilistic Reasons

Outcomes often result from multiple factors

Rain temperature and humidity

Economy unemployment and inflation

Hiring experience and salary

Student # classes GPA

Human condition profession age cholesterol happiness salary location dinner plans …

Joint Distribution

Simple extension of one variable

# Variables	Variable Names	Sample Space	Probability	Abbreviation	properties
One	X	$\mathcal{X}$	p(X=x)	p(x)	$p(x) \geq 0$ $\sum_x p(x) = 1$
Two	X, Y	$\mathcal{X} \times \mathcal{Y}$	p((X,Y)=(x,y))	p(x,y)	$p(x,y) \geq 0$ $\sum_{x,y} p(x,y) = 1$

Specification

State probability of every possible (x,y) pair

Table

1-d

x	y	P(x,y)
0	0	.1
0	2	.2
1	2	.3
1	3	.4

Shows structure

2-d

y

x

	0	2	3
0	.1	.2	0
1	0	.3	.4

Structured distributions

More natural options

Coins

Story Two independent fair coins

Symbolic $U,V \sim B(1/2)$ $\perp\!\!\!\perp$

Explicit $P(u,v) \overset{\text{def}}{=} P(U{=}u, V{=}v) = ¼ \quad \forall \{u,v\} \in \{0,1\}$

Table 1-d 2-d

u	v	P(u,v)
0	0	¼
0	1	¼
1	0	¼
1	1	¼

v

u	0	1
0	¼	¼
1	¼	¼

Biased Coins

Independent B(p) and B(q) coins

1-d

u	v	P(u,v)
0	0	(1-p)(1-q)
0	1	(1-p)q
1	0	p(1-q)
1	1	pq

2-d

u \ v	0	1
0	¼	¼
1	¼	pq

Fair & Rigged

Two coins

Fair

B(1/2)

Rigged

B(0)

Randomly choose fair or rigged

Flip chosen coin

	0	1
fair	¼	¼
rigged	½	0

Event Probability

Joint distribution determines probabilities of all events

		0	1	y
x	0	0.1	0.2	
	1	0.3	0.4	

$$P(X \leq Y) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = 1)$$

$$= P(0, 0) + P(0, 1) + P(1, 1)$$

$$= 0.1 + 0.2 + 0.4$$

$$= 0.7$$

Quiz

Find a simple description of following events   And their probability

$X=Y$ $X\neq Y$

Show shape, ask for description

$X\leq Y$ $X<Y$

$X+Y=1$ $X\cdot Y=0$

$\min(X,Y)=1$ $\max(X,Y)=1$

$X=3$ $Y\neq 5$

Marginals

Special events Value of X, or Y

Marginal of X $P(x) \stackrel{\text{def}}{=} P_X(x) \stackrel{\text{def}}{=} P(X = x) = \sum_y p(x,y)$ Event that X=x

Rule of total probability

Marginal of Y $P(y) \stackrel{\text{def}}{=} P_Y(y) \stackrel{\text{def}}{=} P(Y = y) = \sum_x p(x,y)$ Event that Y=y

x \ y	0	1	
0	0.1	0.2	$\longleftarrow P(X = 0) = .3$
1	0.3	0.4	$\longleftarrow P(X = 1) = .7$

Write on "Margins"

$P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1)$ Total probability

$= P(0,0)+P(0,1) = .1 + .2 = .3$

Joint Matters

Very different joint distributions can have the same marginals

In all following, X, Y ~ B(.5), but very different joint distributions

y

0.5	0
0	0.5

x

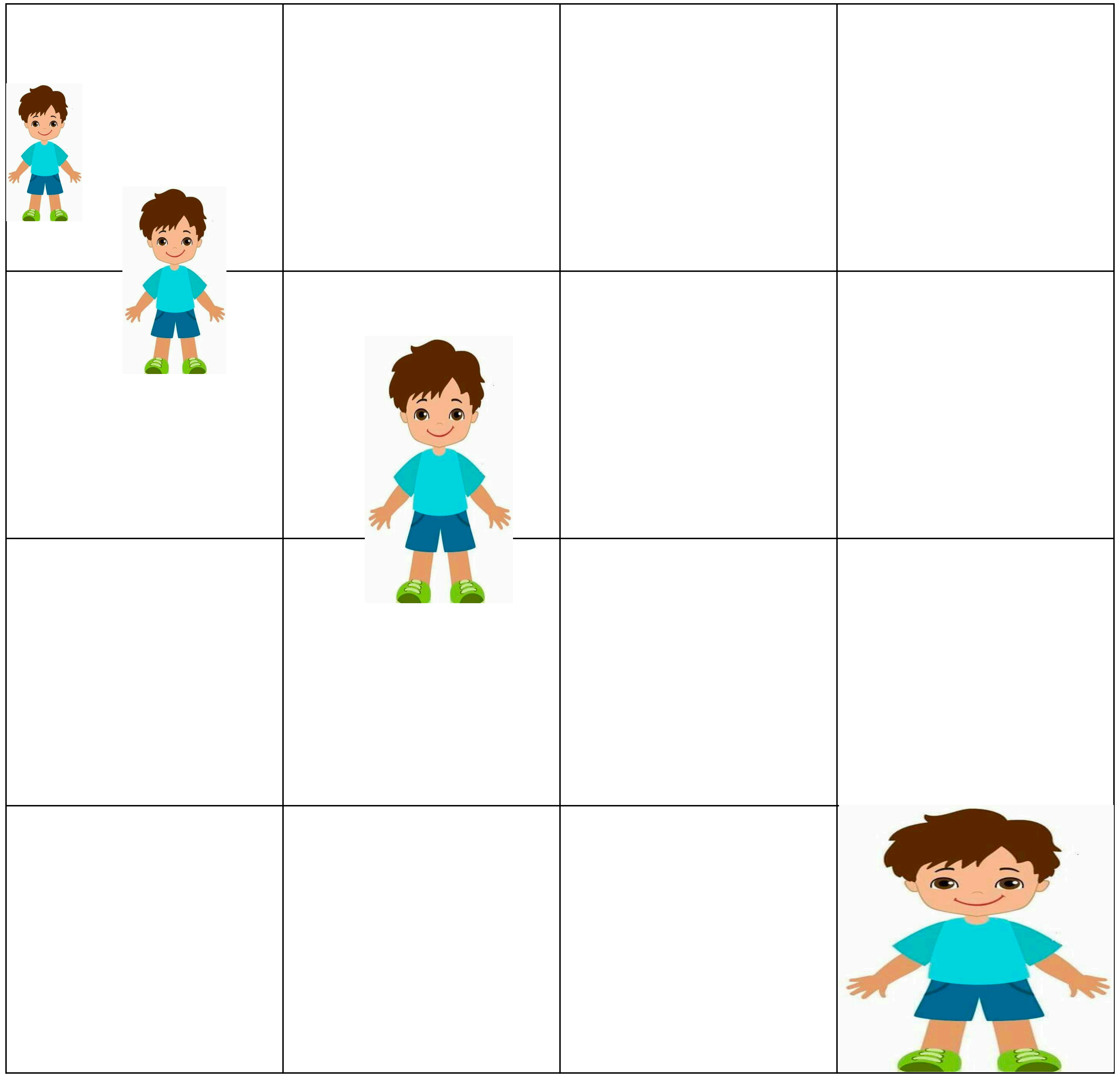
Y=X

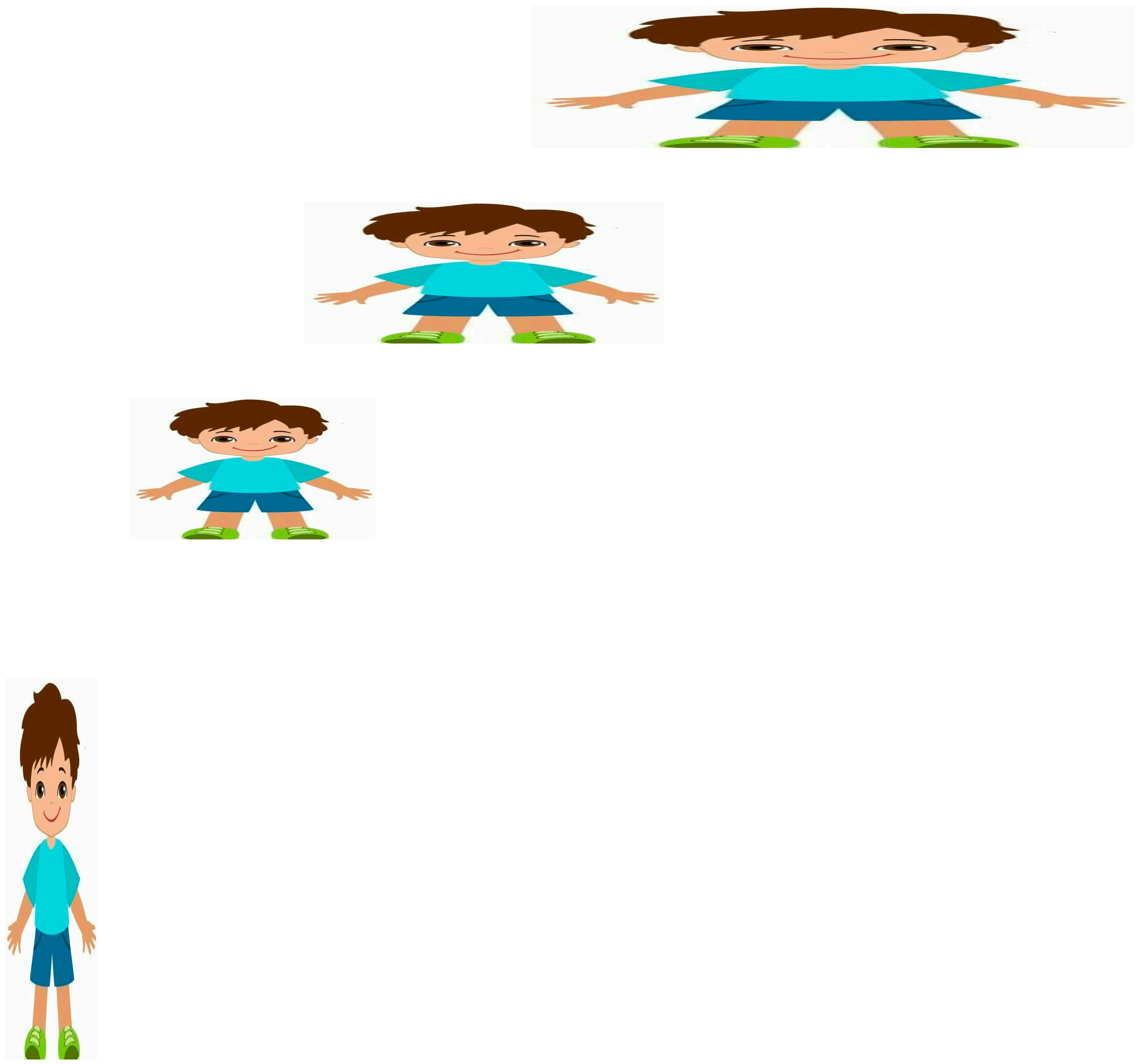
0	0.5
0.5	0

Y=1-X

0.25	0.25
0.25	0.25

0.4	0.1
0.1	0.4





Consider contents, not just marginals

Conditionals

$P(X=x \mid Y=y)$

$$P(x \mid y) = \frac{p(x,y)}{p(y)}$$

$$P(y \mid x) = \frac{p(x,y)}{p(x)}$$

x \ y	0	1	
0	0.1	0.2	← $P(X = 0) = 0.3$
1	0.3	0.4	

$P(Y = 0) = 0.4$

$$P(Y = 0 \mid X = 0) = \frac{P(X = 0 , Y = 0)}{P(X = 0)} = \frac{0.1}{0.3} = \frac{1}{3}$$

$$P(Y = 1 \mid X = 0) = \frac{P(X = 0 , Y = 1)}{P(X = 0)} = \frac{0.2}{0.3} = \frac{2}{3}$$

$$P(X = 0 \mid Y = 0) = \frac{P(X = 0 , Y = 0)}{P(Y = 0)} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$P(X = 1 \mid Y = 0) = 1 - P(X = 0 , Y = 0) = 1 - \frac{1}{4} = \frac{3}{4}$$

Independence

X, Y independent $X \perp\!\!\!\perp Y$

Intuitive

$\forall x,y$ $p(y \mid x) = p(y)$ Value of X does not affect distribution of Y

$p(x \mid y) = p(x)$ Value of Y does not affect distribution of X

$p(x,y) = p(x) \cdot p(y)$ ← more robust Formal

P(x) \ P(y)	0.2	0.8
0.6	0.12	0.48
0.4	0.08	0.32

$\perp\!\!\!\perp$

P(x) \ P(y)	0.4	0.6
0.3	0.1	0.2
0.7	0.3	0.4

$\not\perp\!\!\!\perp$

Independence Checks

Independent $\rightarrow$ rows proportional to each other

$\rightarrow$ columns proportional to each other

$X \sim B(½)$

$Y = X$

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{y} \\
 & & 0 & 1 \\
\hline
\multirow{2}{*}{x} & 0 & ½ & 0 \\
\hline
 & 1 & 0 & ½ \\
\hline
\end{array}$$

$\not\perp\!\!\!\perp$

$Y = 1 - X$

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{y} \\
 & & 0 & 1 \\
\hline
\multirow{2}{*}{x} & 0 & 0 & ½ \\
\hline
 & 1 & ½ & 0 \\
\hline
\end{array}$$

$\not\perp\!\!\!\perp$

$X \perp\!\!\!\perp Y$ For all x and all y Events $X = x \perp\!\!\!\perp Y = y$

Marginal Events

X-event Defined on X

Y-event Similar

If X $\perp\!\!\!\perp$ Y, then all X-events are $\perp\!\!\!\perp$ of all Y-events

Two Variables