LAB REPORT: LAB 3

TNM079, MODELING AND ANIMATION

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Abstract

This lab investigates subdivision of cubic B-spline curves and triangular half-edge meshes. Subdivision enables smooth curves to for example guide animation trajectories, or increase detail on meshes. For the curves, new control points were found by leveraging on the conversion matrix between the bases of the old and the new control points. Mesh subdivision was computed with `Loop`'s subdivision scheme, which accounts for curvature and global surface layout. Subdivision of B-spline curves showed its robustness in comparison to an analytic reference curve in that it converges to the smooth version rather quickly. The `Loop subdivision` showcases its ability to remove bulges and improve detail levels, at the cost of shrinking meshes up to a limit.

This laboratory work aims for `grade 3`.

1 Background

B-spline curves are smooth, parametric curves constructed from connected spline segments. A set of control points placed around the curve (not necessarily on it) influence its shape - similar to how forces pull and bend a rubber band. The basis functions for a cubic B-spline curve determine how much the control points should influence different segments of the curve. The concept is illustrated in Figure 1 for a cubic B-spline curve.

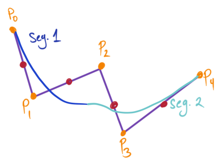


Figure 1: Sketch of a cubic B-spline curve (blue) and its segments, control points (orange), control polygon (purple), and new control points (red) added during subdivision.

If the basis functions $B_{i,d}$ (i is index for order and $d = 3$ is the degree) are defined recursively in terms of convolution of lower-degree functions, Equation 1 is obtained for a spline of degree d, where t is the global curve parameter, and s is the local curve parameter, unique to each spline segment.

$$B_d(t) = \int B_{d-1}B_0(t-s)ds \tag{1}$$

To refine a B-spline curve, new control points (red in Figure 1) were placed halfway between the old control points (orange in Figure 1). The coefficients c'_i controlling the influence of the new control points can be found form the basis of the old control points. If a B-spline curve $p(t)$ is described as the summation of a set of coefficients c_i multiplied with dilated and translated versions of the basis functions $B_{i,d}$, then the new coefficients c'_i are found as

$$\begin{cases} p(t) & = \sum_i c_i B_{i,3}(t) \\ c'_i & = \sum_j \mathbf{S}_{ij} c_j \end{cases} \tag{2}$$

, where **S** is a matrix containing the coefficients for translation and dilation to the new basis, j subdivisions away. Matrix **S** for halfway placement of new control points is shown in Equation 3, where the two top and bottom rows are the padding for the endpoints p_0 and p_4 in Figure 1. Important to remember is that matrix **S** is invalid for other cases as it is derived from translations and dilation on halfway placements and a number of five control points (hence five columns).

$$\mathbf{S} = \frac{1}{8}\begin{pmatrix} 8 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix} \tag{3}$$

Because the basis functions for the segments are only non-zero for certain ranges of parameter values, it is enough to consider only a neighborhood of points around each new control point c'_i. So rather than storing the entire matrix **S**, the more generic conditions in Equation 4 was enough to apply to each control point and its coefficient when computing the new basis.

$$\begin{cases} c'_i = \frac{1}{8}(c_{i-1} + 6c_i + c_{i+1}) \\ c'_{\frac{1}{2}} = \frac{1}{8}(4c_i + 4c_{i+1}) \\ c'_0 = c_0 \\ c'_{end} = c_{end} \end{cases} \tag{4}$$

Since new control points were added halfway between the old control points, Figure 1 shows that the total number of control points after each subdivision should be $2N - 1$, as a point is added after each original control point but the last one. This test was implemented to ensure subdivision correctness.

`Loop's subdivision algorithm` subdivide triangular meshes. The scheme splits each face into four by adding new vertices to the edges of the original face and moving its original vertices in such a way that the overall mesh shape remains intact. This high-level concept is illustrated in Figure 2.

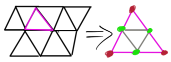
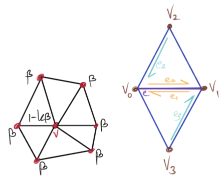


Figure 2: Sketch of Loop's algorithm creating four triangles (gray) out of every original triangle by adding new vertices (green) along the edges of the original triangles. Original vertices colored in red.

The new locations of the original red vertices are computed from weights β that depend on the number of incident edges k each vertex has. Because the new vertices should be distributed evenly around the current vertex v for mesh shape conservation, the same weight amount β is given to each vertex in the neighborhood, as is shown in Figure 3(a). Equation 5 accounts for how β in relation to the number of edges k that share the current vertex v was computed, and how to compute the weight w of the current vertex v.



(a) (b)

Figure 3: (a) shows weights β distribution for current vertex v and its neighborhood. (b) illustrates the logic of obtaining vertices (from half-edges) for the faces sharing an edge e on which a new vertex v should be placed.

$$\beta = \begin{cases} \frac{3}{8k}, & \text{if } k > 3 \\ \frac{3}{16}, & \text{if } k = 3 \\ w = 1 - k\beta \end{cases} \tag{5}$$

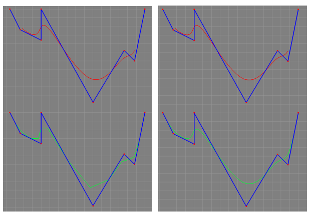
Each new vertex position along the edges of the face was computed according to Equation 6. Using the logic displayed in Figure 3(b), vertices (v_0, v_1, v_2, v_3) of the two faces sharing an edge e on which the new vertex v should be placed could be obtained, relying on the half-edge mesh structure (edge pairs connect adjacent faces and the back of each edge is connected to a vertex).

$$v = \frac{1}{8}(v_0 + v_1) + \frac{3}{8}(v_2 + v_3) \qquad (6)$$

2 Results

Figure 4 shows the cubic B-spline curve subdivided once and thrice, both compared to an analytical reference (red). Visibly, considering the endpoints in the transformation matrix **S** via padding, causes the curve to interpolate the end control points (which the analytical curve does not). This difference arises because the subdivision implementation includes boundary rules, that preserve the first and last control points.

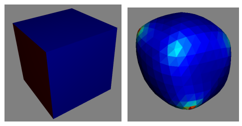
At around three subdivisions, the green B-spline curve approaches the reference curve, indicating that the iterative subdivision successfully converges to the same smooth limit shape as the analytical curve. This convergence demonstrates correctness and stability of the subdivision. This result highlights how uniform cubic B-spline subdivision is an effective approximation technique.



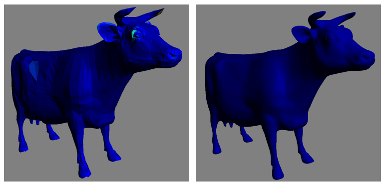
(a) (b)

Figure 4: A cubic B-spline curve, in green, subdivided once (a) and thrice (b). An analytical reference curve is illustrated in red.

The results of `Loop subdivision` on a cube and on a cow thrice are shown in Figure 5. Evidently, the cube turns into a sphere, with pointy discontinuities (highlighted in Figure 5(b)). This is because the cube's vertex normals are not perfectly straight, as they get contribution from a varying amount of faces.



(a) Original cube mesh. (b) Cube subdivided thrice.



(c) Original cow mesh. (d) Cow mesh subdivided thrice.

Figure 5: `Loop subdivision` applied to a cow mesh and a 2×2 cube. *"Jet"* color map applied on face normals.

The cow mesh is effectively smoothened, for example removing the bulge highlighted around the eye in Figure 5(c), and overall less color variation when coloring on surface normals.

Table 1: The area and volume measured for the cow and the cube in Figure 5 before and after each `Loop` subdivision.

	Start	**#1**	**#2**	**#3**
Cube Area	24.00	12.80	10.95	10.55
Cube Volume	8.00	3.96	3.30	3.16
Cow Area	1.090	1.057	1.050	1.049
Cow Volume	0.053	0.053	0.053	0.053

Table 1 details how area and volume change with subdivision. As anticipated, the area and volume of the cube shrink the more it is subdivided. Because each split of a face means relocating and adding vertices, sharp edges and/or corners get smoothened out, removing surface curvature and thereby volume and area it occupies.

The nearly constant changes of the cow exemplify that as a surface is gradually smoothened, it converges towards a limit. At this limit, the surface approaches continuous/smooth properties. Thus, it appears there is a point where further refinement does not significantly alter the global geometric characteristics. Such information can be valuable when mitigating computational- and rendering time, and visuals.