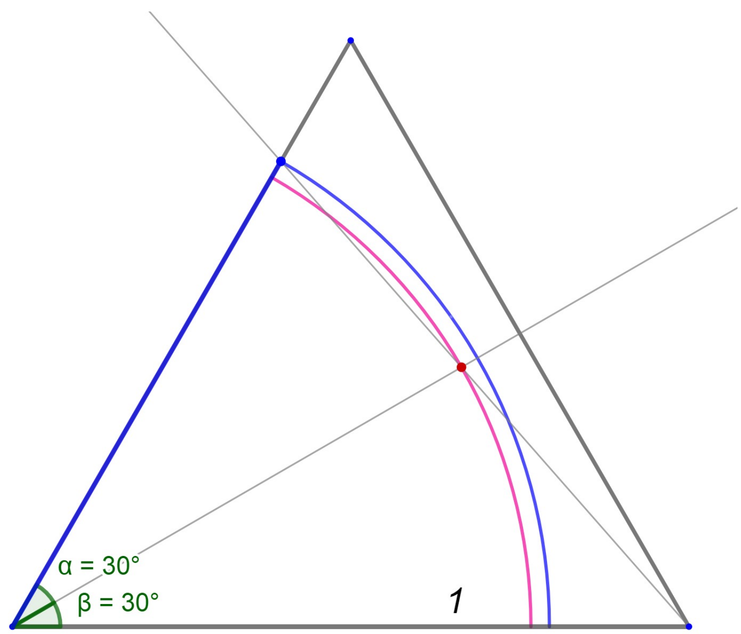
Doubling the cube

Same problem, different angle.



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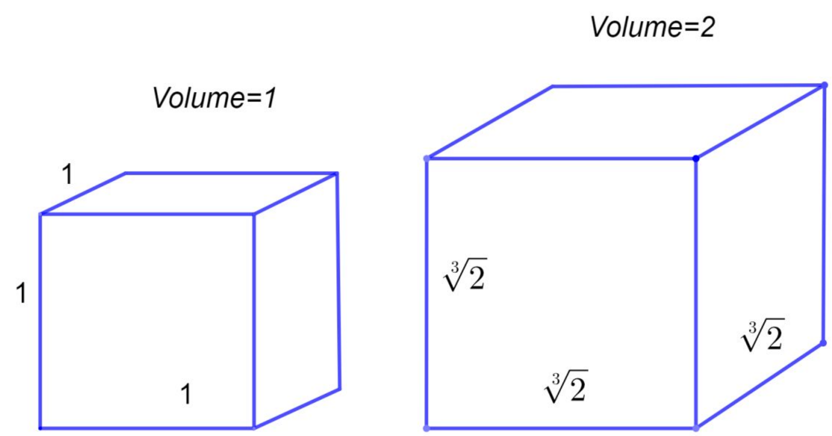
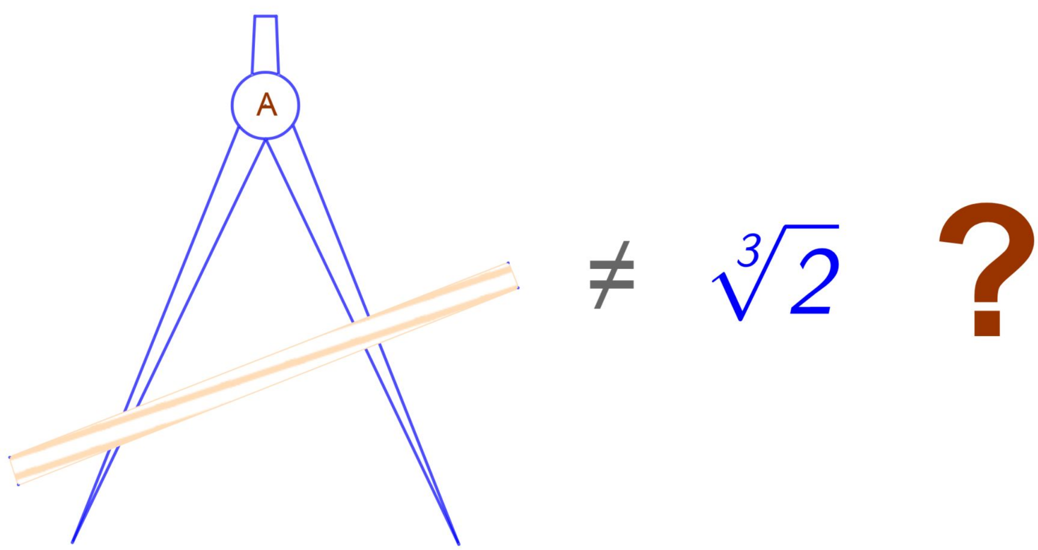
Abstract

Doubling the cube also known as the Delian problem is one of the three famous geometric problems of antiquity, unsolvable by compass and straightedge construction. Given the edge of a cube, the problem requires the construction of the edge of a second cube, whose volume is double that of the first.
The only tools allowed for the construction is the unmarked straightedge and compass.
In algebraic terms, doubling a unit cube requires the construction of a line segment of length x, where $x^3 = 2$; in other words $x = \sqrt[3]{2}$.
Pierre Wantzel proved in 1837 that the $\sqrt[3]{2}$ can not be constructed using a compass and straightedge.





Contents

Chapter 1

1.1 Construction

1. Let's construct base lines and circles with radius $r = 1$ as shown in fig. 1.1.

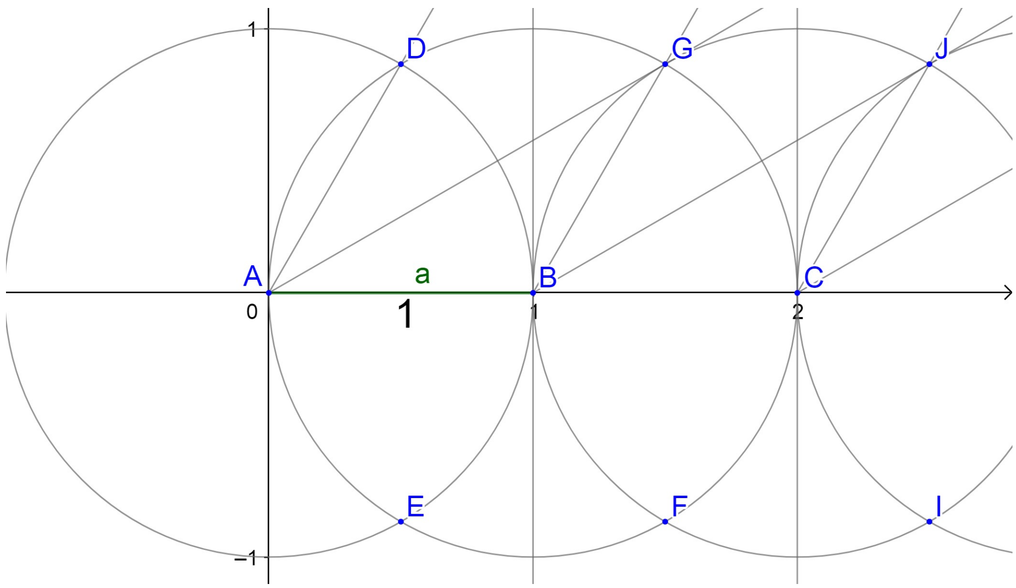


Figure 1.1: Basic circle structure

2. Then let's construct a right triangle $\triangle ANB$ (see fig. 1.2) where short leg $t = 0.75$.

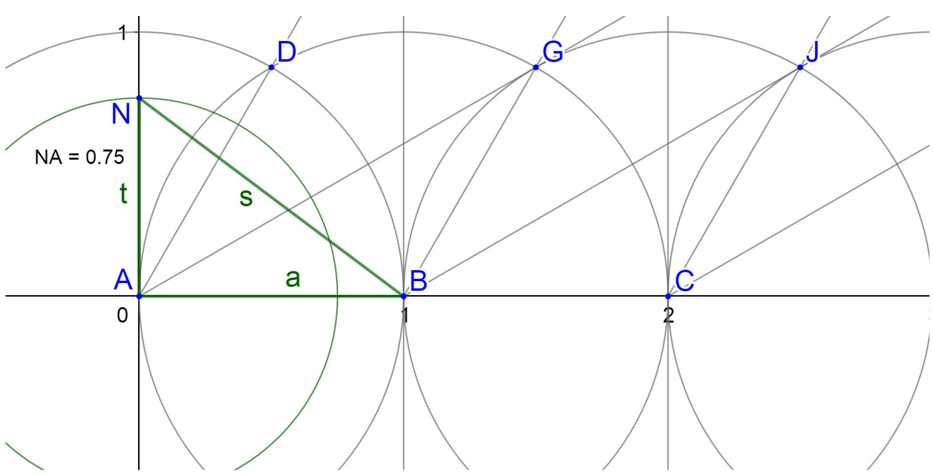


Figure 1.2: Right triangle

Then we calculate the hypotenuse using the Pythagorean theorem:

$$s = \overline{NB} = \sqrt{a^2 + t^2} = \sqrt{1^2 + 0.75^2} = 1.25 \tag{1.1}$$

3. Hence the ratio of the leg to the hypotenuse:

$$\frac{a}{s} = \cos \angle ABN = \frac{1}{1.25} = 0.8. \tag{1.2}$$

To build this relation, let's draw a line through the points N and B. We will get the point O and the point P. (see fig. 1.3)

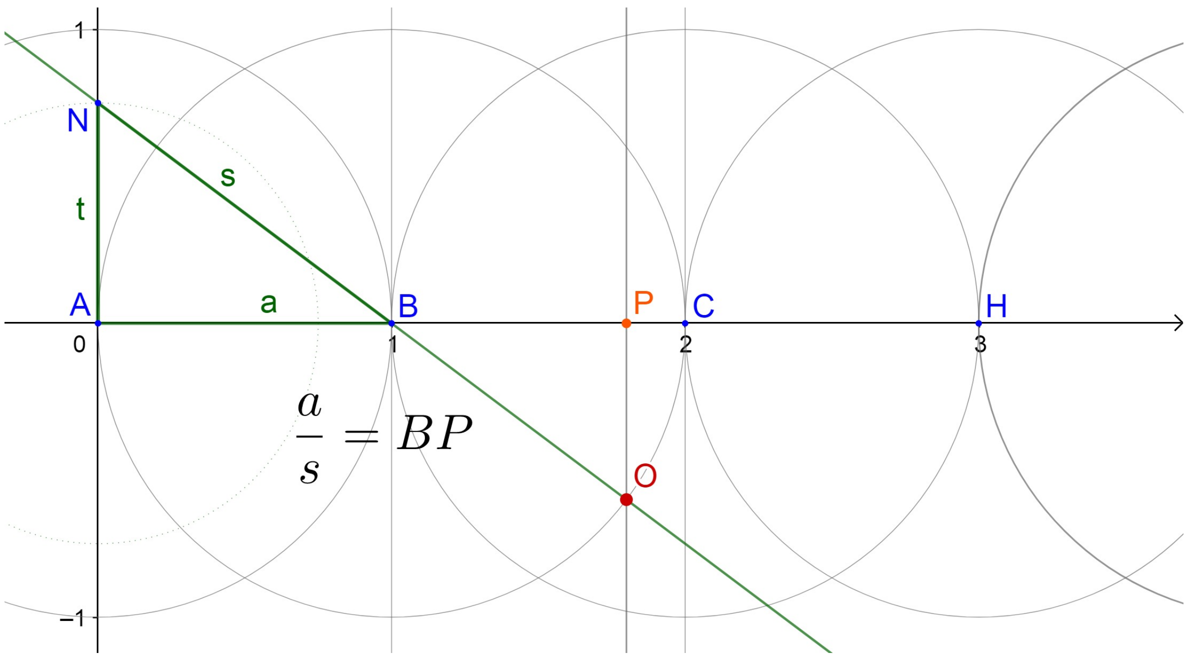


Figure 1.3

4. Now let's construct a circle with center B and radius $= BP$ (see fig. 1.4)

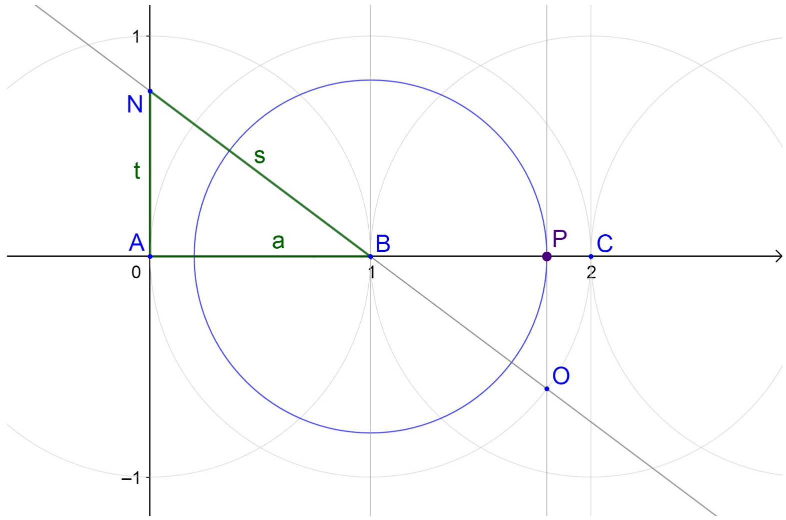


Figure 1.4

5. Then we construct a ray from the point $C(2, 0)$ through the point Q which is the intersection of the circle and the segment $\overline{BG}$. (see fig. 1.5).

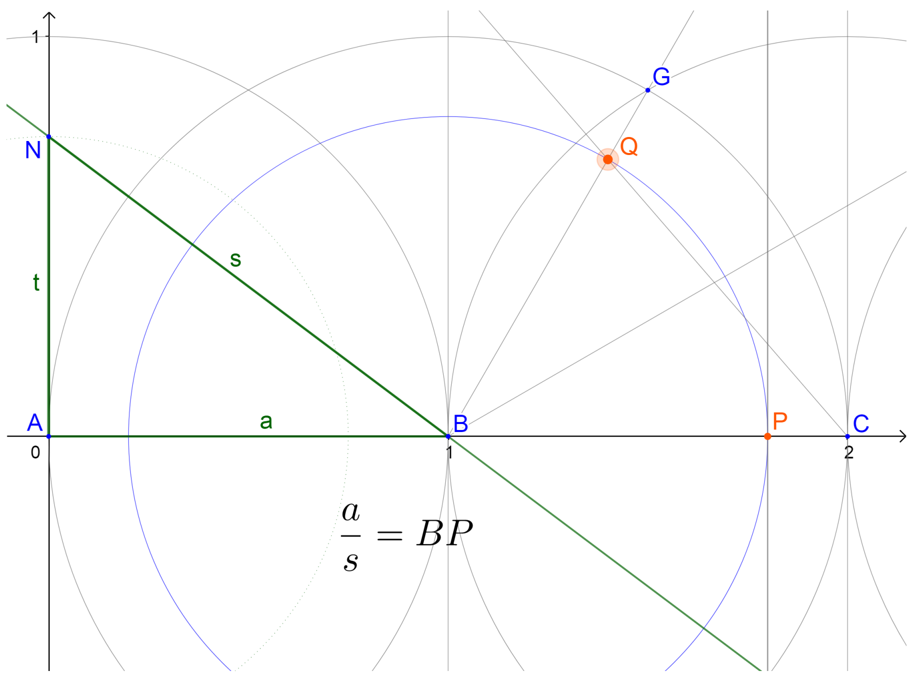


Figure 1.5

6. Then we get the point R as shown in fig. 1.6

   Segment BR is a bisector of a $\triangle\ BQC$ (see fig. 1.7)

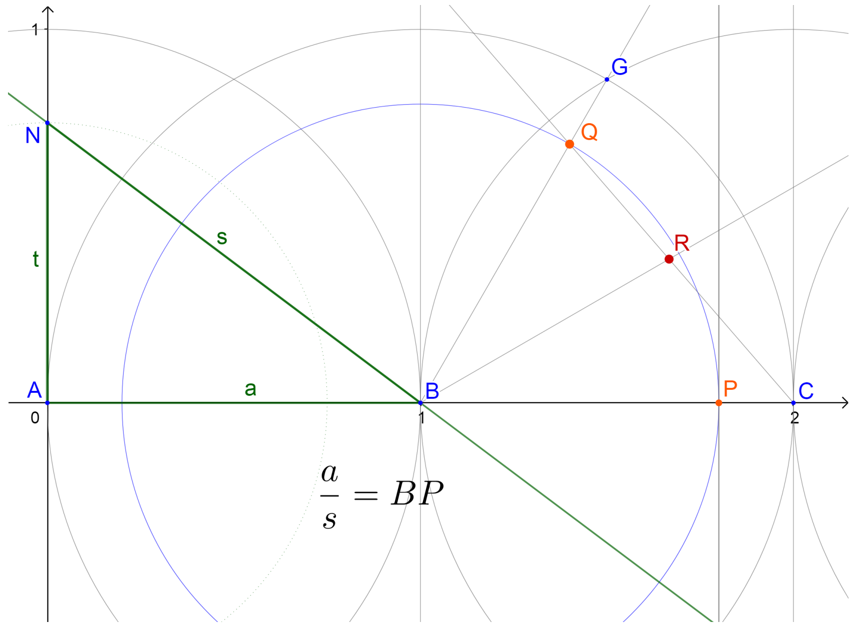


Figure 1.6

Now we have a $\triangle$ BQC where $BQ = BP = \dfrac{a}{s} = \cos\angle ABN$.

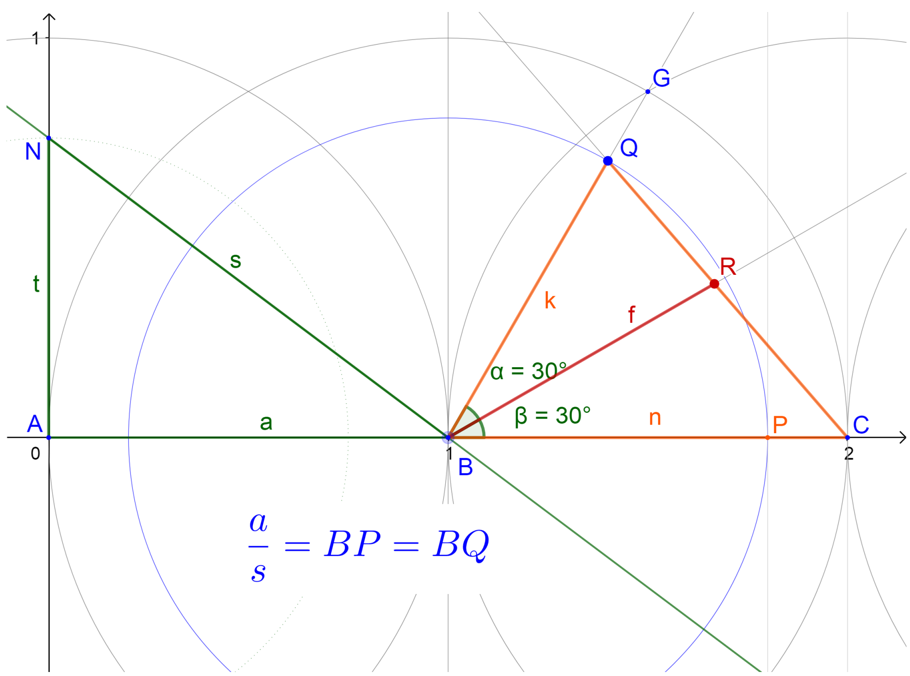


Figure 1.7: Triangle BQC

7. Let's calculate the bisector BR using the formula for the length of the bisector for an angle of 60 degrees:

$$f = \frac{2nk}{k+n} \times \cos 60^\circ = \frac{2nk}{k+n} \times \frac{\sqrt{3}}{2} \tag{1.3}$$

where: $k = BQ = 0.8$; $n = BC = 1$; hence:

$$f = \frac{2 \times 0.8}{1 + 0.8} \times \frac{\sqrt{3}}{2} \approx 0.769800358919501 \tag{1.4}$$

8. Let's construct a circle with radius $= f$ and center at the point $B(1,0)$ and get the point D (see Fig. 1.8)

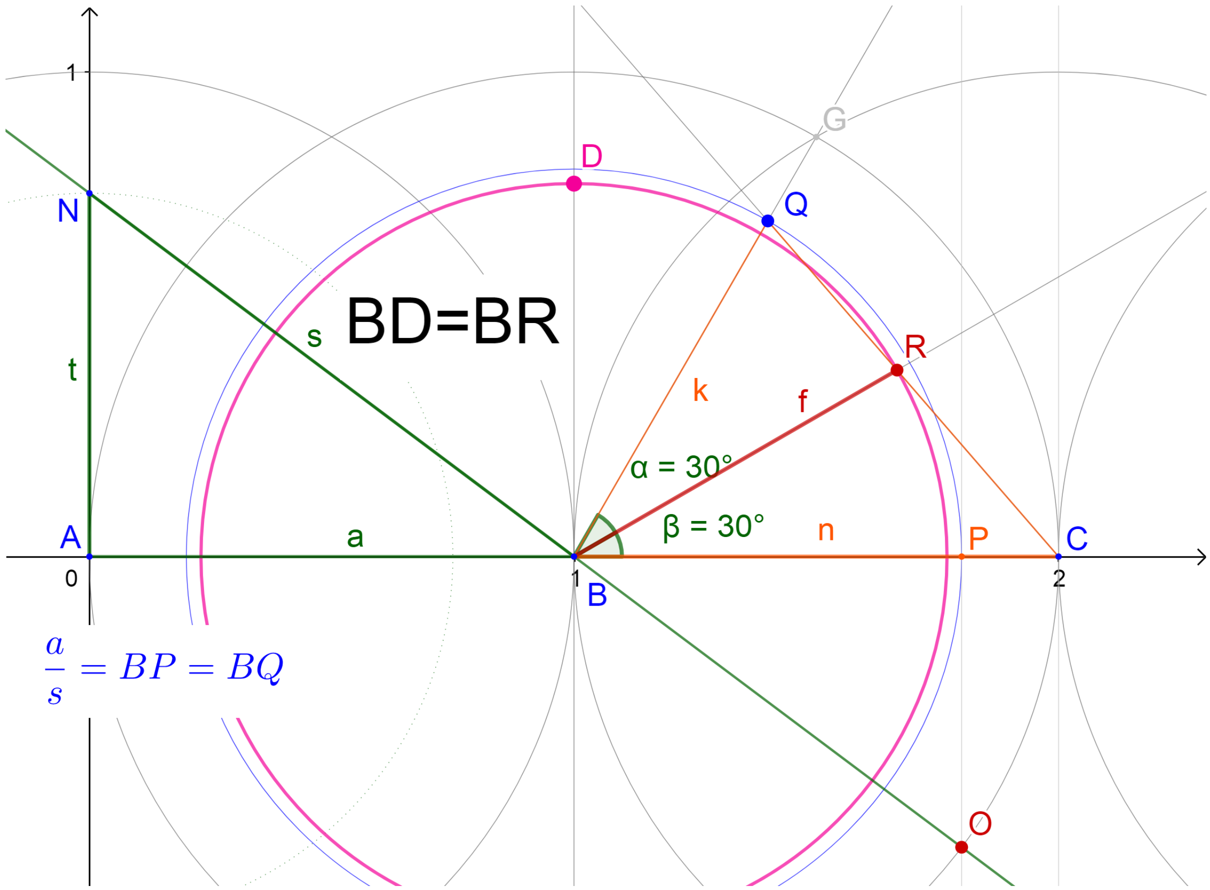


Figure 1.8: Circle where radius = bisector BR

9. Then again we build a right triangle $\triangle BDC$ as shown in the fig. 1.9.

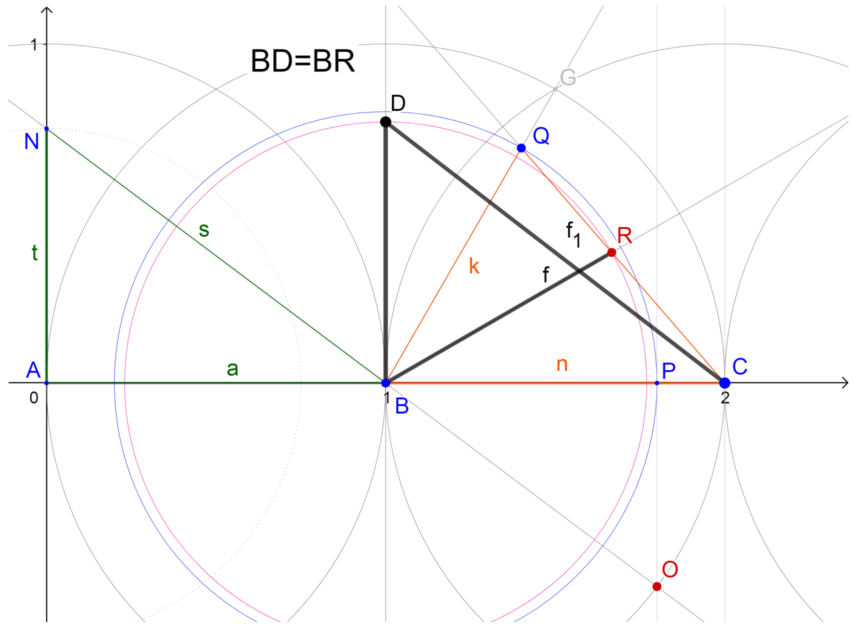


Figure 1.9

So we got a right triangle $\triangle BDC$. Accordingly, we then repeat the steps:

build the cosine value

we build a triangle with an angle of 60 degrees

find the bisector

build a right triangle

and repeat from start...

Repeating these constructions, through the n-th construction we get $\sqrt[3]{2}$ as the length of the hypotenuse of a right triangle. Thus, the length of the hypotenuse calculated from the given sequence tends to $\sqrt[3]{2}$.

1.2 Definition and proof

From the above constructions, we obtain a recurrent sequence:

$$(a_1 > 0, n \in \mathbb{N}) \; \lim_{n\to\infty} a_{n+1} = \frac{1}{\sqrt{1+\left(\frac{2a_n}{1+a_n}\frac{\sqrt{3}}{2}\right)^2}} = \frac{1}{\sqrt[3]{2}} \tag{1.5}$$

Which follows from the following:

$$\sqrt[3]{2} = \lim_{n\to\infty} \sqrt{1+b_n^2} \tag{1.6}$$

$$b_1 > 0, \;\; b_{n+1} = \frac{1}{\sqrt{1+b_n^2}}, \quad b_{n+2} = \frac{2\times b_{n+1}}{1+b_{n+1}} \times \frac{\sqrt{3}}{2}... \tag{1.7}$$

Where b_{n+1} is the cosine of the right triangle $\triangle CBD$ (see fig. 1.9),
then $b_{n+2} = \frac{2\times b_{n+1}}{1+b_{n+1}} \times \frac{\sqrt{3}}{2}$ is the bisector $\overline{BR}$ of the triangle $\triangle BQC$ of the angle $\angle CBQ$ 60-degree, where the shortest leg BQ is b_{n+1}

. Let's expand and simplify this:

$$a_{n+1} = \frac{1}{\sqrt{1+\left(\frac{2a_n}{1+a_n}\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \frac{1}{\sqrt{1+\frac{3a_n^2}{(a_n+1)^2}}}$$

$$= \frac{1}{\sqrt{\frac{3a_n^2+(a_n+1)^2}{(a_n+1)^2}}}$$

$$= \frac{1}{\sqrt{\frac{3a_n^2+(a_n^2+2a_n+1)}{(a_n+1)^2}}}$$

$$= \frac{1}{\sqrt{\frac{4a_n^2+2a_n+1}{(a_n+1)^2}}}$$

$$= \frac{1}{\frac{\sqrt{4a_n^2+2a_n+1}}{\sqrt{(a_n+1)^2}}}$$

$$= \frac{1}{\frac{\sqrt{4a_n^2+2a_n+1}}{a_n+1}}$$

$$= \frac{|a_n + 1|}{\sqrt{4a_n^2 + 2a_n + 1}}$$

Since $a_1 > 0$ we have that:

$$a_{n+1} = \frac{a_n + 1}{\sqrt{4a_n^2 + 2a_n + 1}}$$

Now let's evaluate the limit of the sequence defined by:

$$(0 < a_1 < 1, n \in \mathbb{N}), \lim_{n \to \infty} a_{n+1} = \frac{a_n + 1}{\sqrt{4a_n^2 + 2a_n + 1}} \tag{1.8}$$

First we need to prove that $\{a_n\}$ is bounded.
Let $a_1 = 1$, $a_{n+1} = a_{k+1}$, $k = n$ where $n \in \mathbb{N}$.
We will prove by induction that $a_{k+1} < 1$.

$$a_{k+1} = \frac{a_1 + 1}{\sqrt{4a_1^2 + 2a_1 + 1}} \Rightarrow \frac{1+1}{\sqrt{(4 \times 1^2) + (2 \times 1) + 1}}$$

$$= \frac{2}{\sqrt{7}} = \frac{2}{2.645751} \approx 0.755929$$

hence:

$$a_{k+1} \approx 0.755929 < 1$$

Now we will prove that $a_{k+2} < 1$

$$a_{k+2} = \frac{a_{k+1} + 1}{\sqrt{4a_{k+1}^2 + 2a_{k+1} + 1}} \Rightarrow \frac{0.755929 + 1}{\sqrt{(4 \times 0.755929^2) + (2 \times 0.755929) + 1}}$$

$$= \frac{1.755929}{\sqrt{4.797572}} = \frac{1.755929}{2.190336} \approx 0.801671$$

hence:

$$a_{k+2} \approx 0.801671 < 1$$

So it is easy to see that $\{a_k\} < 1$ when $a_1 \leq 1$.
Therefore the sequence (1.8) is bounded above by 1.
Also $\{a_n\}$ is bounded below since $a_1 > 0$.
Since:

$$f(x) = \frac{x + 1}{\sqrt{4x^2 + 2x + 1}} \tag{1.9}$$

is a continuous function, we have that:

$$L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{a_n + 1}{\sqrt{4a_n^2 + 2a_n + 1}} = \frac{L + 1}{\sqrt{4L^2 + 2L + 1}} \tag{1.10}$$

Therefore we need to solve the equation:

$$L = \frac{L+1}{\sqrt{4L^2+2L+1}} \tag{1.11}$$

We need to multiply both sides by $\sqrt{4L^2+2L+1}$ first:

$$L+1 = L\sqrt{4L^2+2L+1}$$

Isolate the radical to the left hand side:

$$L\sqrt{4L^2+2L+1} = L+1$$

Raise both sides to the power of two:

$$L^2(4L^2+2L+1) = (L+1)^2$$

Expand out terms:

$$4L^4+2L^3+L^2 = L^2+2L+1$$

Subtract L^2+2L+1 from both sides:

$$4L^4+2L^3-2L-1 = 0$$

The left hand side factors into a product with two terms:

$$(2L+1)(2L^3-1) = 0$$

Split into two equations:

$$2L+1 = 0$$

and

$$2L^3-1 = 0$$

Hence:

$$2L = -1$$

and

$$2L^3 = 1$$

Solve for L:

$$L = -\frac{1}{2}$$

and:

$$L^3 = \frac{1}{2}$$

since the $\{a_n\} > 0$, the only possible choice is:

$$L^3 = \frac{1}{2} \Rightarrow L = \frac{1}{\sqrt[3]{2}}$$

Let's check the solution:

$$\frac{L+1}{\sqrt{4L^2+2L+1}} \Rightarrow \frac{1+\dfrac{1}{\sqrt[3]{2}}}{\sqrt{4\left(\dfrac{1}{\sqrt[3]{2}}\right)^2+\dfrac{2}{\sqrt[3]{2}}+1}}$$

$$= \frac{2+2^{2/3}}{2+2\sqrt[3]{2}}$$

$$= \frac{2+\sqrt[3]{4}}{2+2\sqrt[3]{2}} = 0.793700525984$$

Hence:

$$L = 0.793700525984$$

While $\dfrac{1}{\sqrt[3]{2}} = 0.793700525984...$ rounded to 12 decimal places.

So this proves that $a_n \to \dfrac{1}{\sqrt[3]{2}}$ as $n \to \infty$ in the recurrence relation:

$$(a_1 > 0, n \in \mathbb{N})\ \lim_{n\to\infty} a_{n+1} = \frac{1}{\sqrt{1+\left(\dfrac{2a_n}{1+a_n}\dfrac{\sqrt{3}}{2}\right)^2}} = \frac{1}{\sqrt[3]{2}}$$

We have proved that the recurrent sequence 1.5 is equal to $\sqrt[3]{2}$.

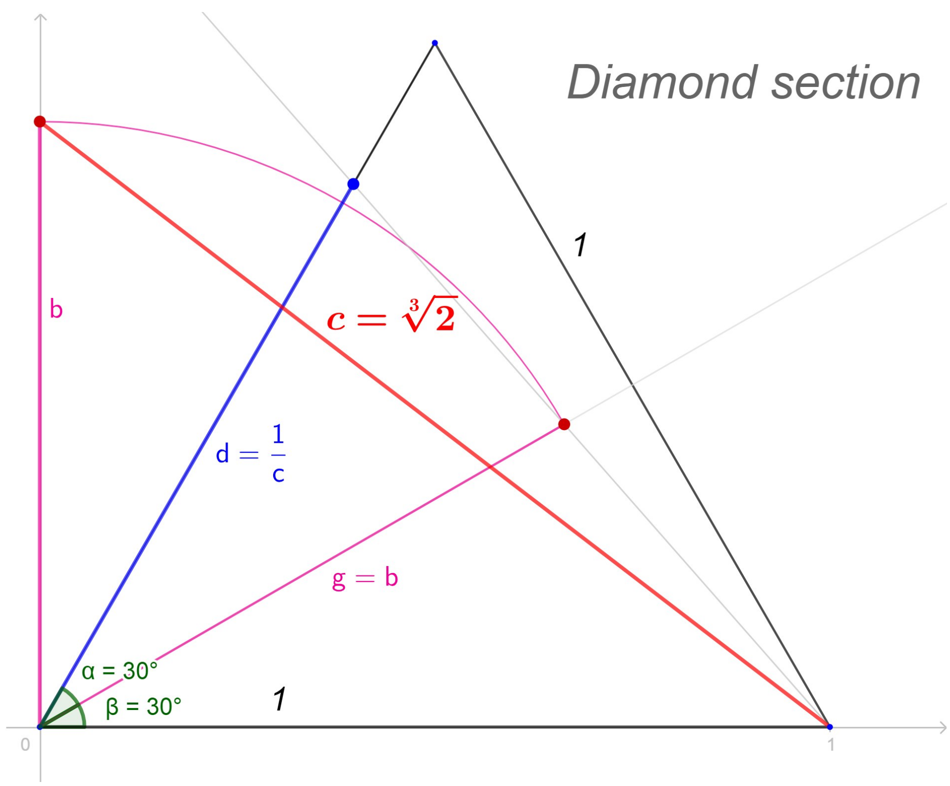
Notice that the physical world is limited in its minimum values by the Planck length:

$$l_p = 1,616255(18) \times 10^{-35}$$

which means that all values less than the Planck length have no physical meaning. In order to reach the maximum possible value limited by the Planck length we need to perform the above constructions 57 times.

Now we know for sure that it is possible to construct $\sqrt[3]{2}$ using only a compass and straightedge, and therefore the cube doubling problem is solved.

Most importantly, we discovered a new fundamental property of an equilateral triangle that is directly related to the cube root of two.



1.3 Appendix

The square root of the golden ratio

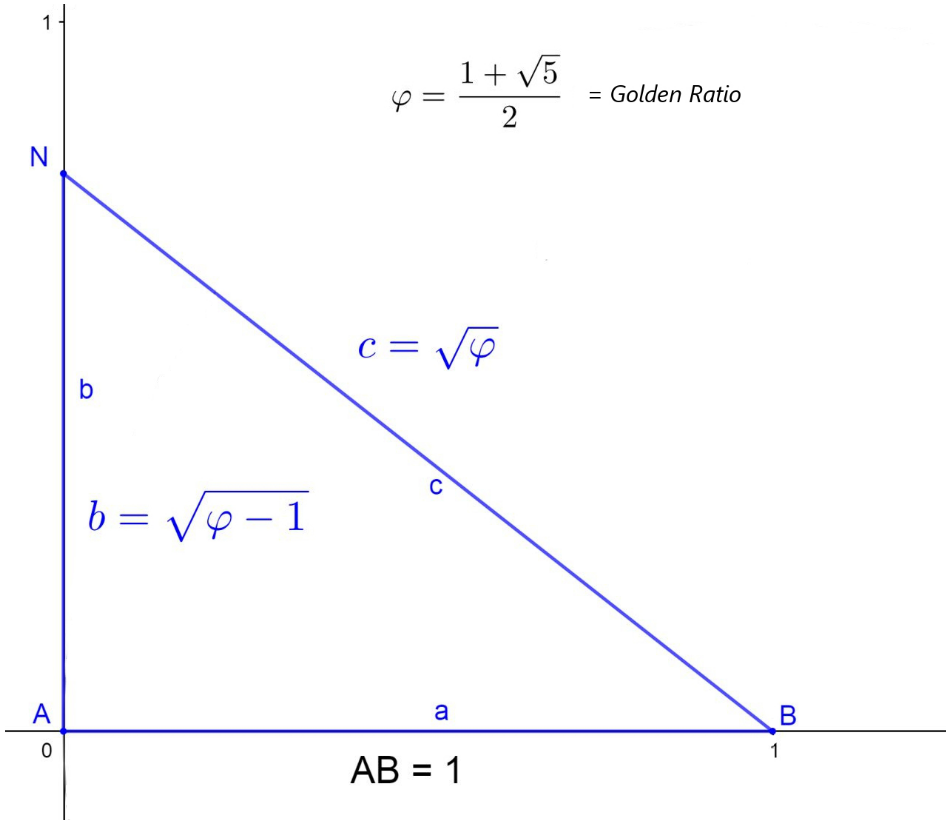


Figure 1.10: Right triangle

Let's take an **arbitrary** shortest leg b of the right triangle 1.10 with a long leg $a = 1$ and recursively do the following: find the hypotenuse using the formula c $=\sqrt{a^2+b^2}$, then get b_1 as the reciprocal of the hypotenuse c

$$\frac{1}{\sqrt{a^2+b^2}} = \frac{1}{c} = b_1; \tag{1.12}$$

Now let's take b_1 as a small leg of a right triangle, where $a = 1$, and repeat the above steps according to the formula:

$$\lim_{n\to\infty} \quad \frac{1}{\sqrt{a^2+b^2}} = b_1; \ \frac{1}{\sqrt{a^2+b_1^2}} = b_2; \ \frac{1}{\sqrt{a^2+b_2^2}} = b_3; \ \ldots \frac{1}{\sqrt{a^2+b_{n-1}^2}} = b_n \tag{1.13}$$

After the n-th number of iterations, we will get the following values:

$$a = 1; \quad b = \sqrt{\varphi - 1}; \quad c = \sqrt{\varphi}; \tag{1.14}$$

Thus, the value of a small leg b in the limit of the recursive sequence 1.13 tends to $\sqrt{\varphi - 1}$, and the hypotenuse c tends to $\sqrt{\varphi}$, where φ = Golden ratio = 1.6180339887... (see fig. 1.10)

1.4 Links

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github: `https://github.com/AlmasAskarbekov`

1.5 Thanks

`https://geogebra.org`

Keywords: Doubling the cube, the Delian problem, Geometric problems of Antiquity, geometry, ancient mission impossible of geometry

Bibliography

[1] https://en.wikipedia.org/wiki/Doubling_the_cube

[2] https://oeis.org/A002580