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Cube root of 2.

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Abstract

In this paper I show the existence of the recurrence relations that allows us to construct a line segment equal to the cube root of 2, using a compass and straightedge in an infinite number of steps.

Keywords: cube root of 2, constructible number, euclidean number.

MSC: 97I30, 65Q10, 03DXX.

Introduction

It is well known that $\sqrt[3]{2}$ can not be constructed using compass and straightedge in a finite number of steps.[1]
However, there is at least two recurrence relations that allows us to construct this number in an infinite number of steps.
The recurrence relation (1) allows us to obtain a number that is reciprocal to $\sqrt[3]{2}$.

$$(a_1 > 0, n \in \mathbb{N}),\ a_{n+1} = \frac{1}{\sqrt{1+\left(\frac{2a_n}{1+a_n}\frac{\sqrt{3}}{2}\right)^2}} \tag{1}$$

This sequence follows from this:

$$\sqrt[3]{2} = \lim_{n\to\infty} \sqrt{1+b_n^2}$$

$$b_1 > 0,\ \ b_{n+1} = \frac{1}{\sqrt{1+b_n^2}},\quad b_{n+2} = \frac{2\times b_{n+1}}{1+b_{n+1}} \times \frac{\sqrt{3}}{2}$$

Where b_{n+1} is the cosine of the right triangle and $b_{n+2} = \frac{2\times b_{n+1}}{1+b_{n+1}} \times \frac{\sqrt{3}}{2}$ is the bisector of the triangle with angle of 60-degree whose shortest leg is equal to b_{n+1}

[1] Weisstein, 2018, *"Constructible Number."From MathWorld–A Wolfram Web Resource.*

Let's expand and simplify this:

$$a_{n+1} = \frac{1}{\sqrt{1 + \left(\frac{2a_n}{1+a_n}\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \frac{1}{\sqrt{1 + \frac{3a_n^2}{(a_n+1)^2}}}$$

$$= \frac{1}{\sqrt{\frac{3a_n^2 + (a_n+1)^2}{(a_n+1)^2}}}$$

$$= \frac{1}{\sqrt{\frac{3a_n^2 + (a_n^2 + 2a_n + 1)}{(a_n+1)^2}}}$$

$$= \frac{1}{\sqrt{\frac{4a_n^2 + 2a_n + 1}{(a_n+1)^2}}}$$

$$= \frac{1}{\frac{\sqrt{4a_n^2 + 2a_n + 1}}{\sqrt{(a_n+1)^2}}}$$

$$= \frac{1}{\frac{\sqrt{4a_n^2 + 2a_n + 1}}{a_n+1}}$$

$$= \frac{|a_n+1|}{\sqrt{4a_n^2 + 2a_n + 1}}$$

Since $a_1 > 0$ we have that:

$$a_{n+1} = \frac{a_n + 1}{\sqrt{4a_n^2 + 2a_n + 1}}$$

Now let's evaluate the limit of the sequence defined by:

$$(0 < a_1 < 1, n \in \mathbb{N}), \lim_{n \to \infty} a_{n+1} = \frac{a_n + 1}{\sqrt{4a_n^2 + 2a_n + 1}} \tag{2}$$

First we need to prove that $\{a_n\}$ is bounded.
Let $a_1 = 1$, $a_{n+1} = a_{k+1}$, $k = n$ where $n \in \mathbb{N}$.
We will prove by induction that $a_{k+1} < 1$.

$$a_{k+1} = \frac{a_1 + 1}{\sqrt{4a_1^2 + 2a_1 + 1}} \Rightarrow \frac{1+1}{\sqrt{(4 \times 1^2) + (2 \times 1) + 1}}$$

$$= \frac{2}{\sqrt{7}} = \frac{2}{2.645751} \approx 0.755929$$

hence:

$$a_{k+1} \approx 0.755929 < 1$$

Now we will prove that $a_{k+2} < 1$

$$a_{k+2} = \frac{a_{k+1} + 1}{\sqrt{4a_{k+1}^2 + 2a_{k+1} + 1}} \Rightarrow \frac{0.755929 + 1}{\sqrt{(4 \times 0.755929^2) + (2 \times 0.755929) + 1}}$$

$$= \frac{1.755929}{\sqrt{4.797572}} = \frac{1.755929}{2.190336} \approx 0.801671$$

hence:

$$a_{k+2} \approx 0.801671 < 1$$

So it is easy to see that $\{a_k\} < 1$ when $a_1 \le 1$.
Therefore the sequence (2) is bounded above by 1.
Also $\{a_n\}$ is bounded below since $a_1 > 0$.

Since:

$$f(x) = \frac{x+1}{\sqrt{4x^2+2x+1}}$$

is a continuous function, we have that:

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n+1}{\sqrt{4a_n^2+2a_n+1}} = \frac{L+1}{\sqrt{4L^2+2L+1}}$$

Therefore we need to solve the equation:

$$L = \frac{L+1}{\sqrt{4L^2+2L+1}}$$

We need to multiply both sides by $\sqrt{4L^2+2L+1}$ first:

$$L+1 = L\sqrt{4L^2+2L+1}$$

Isolate the radical to the left hand side:

$$L\sqrt{4L^2+2L+1} = L+1$$

Raise both sides to the power of two:

$$L^2(4L^2+2L+1) = (L+1)^2$$

Expand out terms:

$$4L^4+2L^3+L^2 = L^2+2L+1$$

Subtract L^2+2L+1 from both sides:

$$4L^4+2L^3-2L-1 = 0$$

The left hand side factors into a product with two terms:

$$(2L+1)(2L^3-1) = 0$$

Split into two equations:

$$2L+1 = 0$$

and

$$2L^3-1 = 0$$

Hence:

$$2L = -1$$

and

$$2L^3 = 1$$

Solve for L:

$$L = -\frac{1}{2}$$

and:

$$L^3 = \frac{1}{2}$$

since the $\{a_n\} > 0$, the only possible choice is:

$$L^3 = \frac{1}{2} \Rightarrow L = \frac{1}{\sqrt[3]{2}}$$

Let's check the solution:

$$\frac{L+1}{\sqrt{4L^2+2L+1}} \Rightarrow \frac{1+\frac{1}{\sqrt[3]{2}}}{\sqrt{4\left(\frac{1}{\sqrt[3]{2}}\right)^2+\frac{2}{\sqrt[3]{2}}+1}}$$

$$= \frac{2+2^{2/3}}{2+2\sqrt[3]{2}}$$

$$= \frac{2+\sqrt[3]{4}}{2+2\sqrt[3]{2}} \approx 0.793700525984$$

Hence:

$$L \approx 0.793700525984$$

While $\frac{1}{\sqrt[3]{2}} \approx 0.793700525984$ rounded to 12 decimal places.

So this proves that $a_n \to \frac{1}{\sqrt[3]{2}}$ as $n \to \infty$ in the recurrence relation:

$$(a_1 > 0, n \in \mathbb{N}) \lim_{n\to\infty} a_{n+1} = \frac{1}{\sqrt{1+\left(\frac{2a_n}{1+a_n}\frac{\sqrt{3}}{2}\right)^2}} = \frac{1}{\sqrt[3]{2}}$$

Recurrence relation 2

$$(a_1 > 0, n \in \mathbb{N}),\ \lim_{n\to\infty} a_{n+1} = \frac{1}{\sqrt{2a_n}} = \frac{1}{\sqrt[3]{2}}$$

Let $a_1 = a_k = 2$. It is easy to see that a_{k+1} is bounded above by 1 since:

$$\frac{1}{\sqrt{2\times 2}} = \frac{1}{2} = 0.5 < 1$$

Therefore $\{a_1 \le 2, a_{k+1}\} < 1$
It is clear that $\{a_{n+1}\}$ bounded below by 0 since $a_1 > 0$.
Now recall that since $f(x) = \dfrac{1}{\sqrt{2x}}$ is a continuous function we have that:

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{\sqrt{2a_n}} = \frac{1}{\sqrt{2\times \lim_{n\to\infty} a_n}} = \frac{1}{\sqrt{2L}}$$

Therefore we need to solve the equation:

$$L = \frac{1}{\sqrt{2L}}$$

hence:

$$L^2 = \frac{1}{2L}$$

Obviously, this equation has a solution with imaginary numbers, but we omit it and solve the part that gives a numbers belongs to $\mathbb{R}$, since $\{a_n\} \in \mathbb{R}$:

$$2\times L^2 = 2\times \frac{1}{2L}$$

$$2L^2 = \frac{1}{L} \Rightarrow 2L^2 \times L = 1$$

$$2L^3 = 1$$

Solve for L:

$$L^3 = \frac{1}{2}$$

Hence:

$$L = \frac{1}{\sqrt[3]{2}}$$

Now let's check the solution:

$$L = \frac{1}{\sqrt{2L}}$$

$$\frac{1}{\sqrt[3]{2}} = \frac{1}{\sqrt{2 \times \frac{1}{\sqrt[3]{2}}}}$$

$$= \frac{1}{\sqrt{\frac{2}{\sqrt[3]{2}}}} = \frac{1}{\sqrt{\frac{2}{\sqrt[3]{2}} \times \frac{2^{\frac{2}{3}}}{2^{\frac{2}{3}}}}}$$

$$= \frac{1}{\sqrt{\frac{2 \times 2^{\frac{2}{3}}}{2}}} = \frac{1}{\sqrt{2^{\frac{2}{3}}}} = \frac{1}{2^{\frac{2}{3 \times 2}}} = \frac{1}{2^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{2}}$$

This proves that $a_n \to \frac{1}{\sqrt[3]{2}}$ as $n \to \infty$ in the recurrence relation:

$$(a_1 > 0, n \in \mathbb{N}) \lim_{n \to \infty} a_{n+1} = \frac{1}{\sqrt{2a_n}} = \frac{1}{\sqrt[3]{2}}$$

References

Weisstein, E. W. (2018). *"Constructible Number."From MathWorld–A Wolfram Web Resource.* URL: `http://mathworld.wolfram.com/ConstructibleNumber.html` (visited on 05/19/2018) (cit. on p. 1).

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