Basic Notions

Vector Space

Set $\mathcal{D}$ with operations $+$ and $\alpha\cdot$, $\alpha \in \mathbb{R}$. Axioms:

1. $x + (y + z) = (x + y) + z$
2. $x + y = y + x$
3. $\exists 0 \in \mathcal{D} : 0 + x = x$
4. $\forall x \exists (-x) : x + (-x) = 0$
5. $\alpha(x + y) = \alpha x + \alpha y$
6. $(\alpha + \beta)x = \alpha x + \beta x$
7. $\alpha(\beta x) = (\alpha\beta)x$
8. $1x = x$

Eucledian (Inner Product) Space

Vector space $\mathcal{D}$ with inner product $\langle x, y \rangle \in \mathbb{R}$, $x, y \in \mathcal{D}$.

1. $\langle x, y \rangle = \langle y, x \rangle$
2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
4. $\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \iff x = 0$

$\|x\| = \sqrt{\langle x, x \rangle}$ - norm of x.
Orthonormal basis (ONB) $e_1, \ldots, e_n \in \mathcal{D}$:
$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$
Coordinates in the ONB $e_1, \ldots, e_n \in \mathcal{D}$:

$$x_1, \ldots, x_n \in \mathbb{R} : \quad x = \sum_{i=1}^{n} x_i e_i$$

$$x_i = \langle x, e_i \rangle, \quad \bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ - column of coordinates.}$$

Linear Transformation

$A : \mathcal{D} \to \mathcal{R}$ - Linear transformation (mapping, map) if
$\forall x, y \in \mathcal{D}$ and $\forall \alpha \in \mathbb{R}$

1. $A(x+y) = Ax + Ay$

2. $A(\alpha x) = \alpha A x$

If $A : \mathcal{D} \to \mathcal{D}$ - linear **operator**.
Matrix of a linear map $A : \mathcal{D} \to \mathcal{R}$ in ONBs

$$\varphi_1, \ldots, \varphi_m \in \mathcal{D}, \quad e_1, \ldots, e_n \in \mathcal{R}$$

$$\bar{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \qquad a_{ij} = \langle e_i, A\varphi_j \rangle$$

$$y = Ax \iff \bar{y} = \bar{A}\bar{x}$$

Adjoint linear map. If $A : \mathcal{D} \to \mathcal{R}$ then $A^* : \mathcal{R} \to \mathcal{D}$ - adjoint to A if

$$\forall x \in \mathcal{D} \;\; \forall y \in \mathcal{R} \;\; \langle Ax, y \rangle = \langle x, A^* y \rangle$$

$$\overline{A^*} = \bar{A}^T$$

$$(AB)^* = B^* A^*$$

$S : \mathcal{R} \to \mathcal{R}$ is **self-adjoint operator** if $S^* = S$.
Its matrix is symmetric: $\bar{S}^T = \bar{S}$.
Let S - self-adjoint operator in $\mathcal{R}$. If $\forall x \in \mathcal{R} : x \neq 0$
$\langle Sx, x \rangle \geq 0$ $-$ **nonnegative definite**, $S \geq 0$.
$\langle Sx, x \rangle > 0$ $-$ **positive definite**, $S > 0$.

$$S \geq 0 \implies s_{ii} \geq 0, \qquad S > 0 \implies s_{ii} > 0.$$

$$\forall A : \mathcal{D} \to \mathcal{R} \quad AA^* \geq 0, \quad A^* A \geq 0.$$

Let $S, T : \mathcal{R} \to \mathcal{R}$ - self-adjoint. Define:

$$S \geq T \quad \text{if} \quad S - T \geq 0, \qquad S > T \quad \text{if} \quad S - T > 0.$$

Trace of $S : \mathcal{R} \to \mathcal{R}$

$$\operatorname{tr} S = \sum_{i=1}^{n} s_{ii} = \sum_{i=1}^{n} \langle e_i, S e_i \rangle .$$

- Linear: $\operatorname{tr}(\alpha S + \beta T) = \alpha \operatorname{tr} S + \beta \operatorname{tr} T$.

- Monotone: $S \geq T \implies \operatorname{tr} S \geq \operatorname{tr} T$.

Invertible operator $S : \mathcal{R} \to \mathcal{R}$ if

$$\exists S^{-1} : \quad S^{-1} S = I = S S^{-1}.$$

$$S \text{ is invertible iff} \quad Sx = 0 \implies x = 0.$$

$$S > 0 \implies S - \text{is invertible.}$$

$$(S^*)^{-1} = (S^{-1})^* \qquad (ST)^{-1} = T^{-1} S^{-1}.$$

Random Vector

ν - random vector in $\mathcal{R}$ if its coordinates in (any) basis are random variables.

$$\nu_i = \langle \nu, e_i \rangle, \quad \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} \text{- column of coordinates.}$$

Mathematical Expectation of ν, $\mathsf{E}\nu$:

$$\mathsf{E} \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix} = \begin{bmatrix} \mathsf{E}\nu_1 \\ \vdots \\ \mathsf{E}\nu_n \end{bmatrix}$$

Linearity: ν, μ - random vectors in $\mathcal{R}$, $\alpha, b \in \mathbb{R}$,

$$\mathsf{E}(\alpha\nu + \beta\mu) = \alpha\mathsf{E}\nu + \beta\mathsf{E}\mu.$$

$$x \in \mathcal{R} \qquad \mathsf{E} \langle \nu, x \rangle = \langle \mathsf{E}\nu, x \rangle .$$

$$A : \mathcal{R} \to \mathcal{D} \qquad \mathsf{E}Ax = A\mathsf{E}x.$$

Independence: ν, μ - rand.vec. in $\mathcal{R}$, α - rand.var.

$$\nu, \alpha \text{ independent} \implies \mathsf{E}(\alpha\nu) = \mathsf{E}\alpha\, \mathsf{E}\nu.$$

$$\nu, \mu \text{ independent} \implies \mathsf{E} \langle \nu, \mu \rangle = \langle \mathsf{E}\nu, \mathsf{E}\mu \rangle .$$

Variance operator

of a random vector $\nu \in \mathcal{R}$. $S = \mathrm{Var}(\nu) : \mathcal{R} \to \mathcal{R}$

$$\forall x \in \mathcal{R} \quad Sx = \mathsf{E} \langle \nu - \mathsf{E}\nu, x \rangle (\nu - \mathsf{E}\nu).$$

$$s_{ij} = \mathrm{Cov}(\nu_i, \nu_j).$$

$$\bar{S} = \begin{bmatrix} \mathrm{Var}(\nu_1) & \mathrm{Cov}(\nu_1, \nu_2) & \cdots & \mathrm{Cov}(\nu_1, \nu_n) \\ \mathrm{Cov}(\nu_2, \nu_1) & \mathrm{Var}(\nu_2) & \cdots & \mathrm{Cov}(\nu_2, \nu_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(\nu_n, \nu_1) & \mathrm{Cov}(\nu_n, \nu_2) & \cdots & \mathrm{Var}(\nu_n) \end{bmatrix}$$

- variance-covariance matrix of $\begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix}$.

- $S \geq 0 \quad (\implies s_{ii} \geq 0 \implies \mathrm{tr}\, S \geq 0)$.
- If $B : \mathcal{R} \to \mathcal{D} \qquad \mathrm{Var}(B\nu) = BSB^*$.
- If $\mathsf{E}\nu = 0 \qquad \mathsf{E}\, \|\nu\|^2 = \mathrm{tr}\, S$.