Spring 2020 - Knowledge Discovery in Data at Scale Technologies - Homework #3

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Homework description

Linear Regression

Write a program which illustrates simple linear regression (or a more general variant of linear regression) and implements accumulation of canonical information.

a) For some fixed parameters a and b (or, in a more general case, $a1, \dots, am$) generate a sequence of *observations* (xi, yi):

$$y_i = f(x_i) + \varepsilon_i,$$

where

$$f(x) = a + bx$$

or

$$f(x) = a_1 + a_2 x + a_3 x^2 + \dots + a_m x^{(m-1)}$$

ε_i are i.i.d. with zero mean and $E\varepsilon i = \sigma$. Values x_i can be generated randomly with some mean and variance.

b) Accumulate canonical information, i.e.,at each step, when a new observation (x_i, y_i) is produced, update canonical information

c) Illustrate the real function $f(x)$ and its estimate $\widehat{f(x)}$.

d) Illustrate $Var(\widehat{f(x)})$, assuming that σ^2 is known.

e) Illustrate $\widehat{Var(\widehat{f(x)})}$, assuming that σ^2 is NOT known.

In your report present the source code and a few (around 3) nice graphs showing estimations for *"small"*, *"intermediate"*, and *"large"* number of observations.

God damn! How to solve it?

Relax, don't panic!

Function used in the demo - polynomial:

$$y_i = 1 + 1 \cdot x_i - 1 \cdot x_i^2 + 0.2 \cdot x_i^3 + \varepsilon_i$$

Data:

$$(x_i, y_i), i = 1, \dots, n$$

Canonical information:

$$(T, v, V, n)$$

Elementary information:

$$(T_i, v_i, V_i, n_i)$$

Update:

$$(T, v, V, n) + (T_i, v_i, V_i, n_i) = (T + T_i, v + v_i, V + V_i, n + n_i)$$

Estimate $f(x)$:

$$(T, v, V, n) * x =>$$

$$\widehat{f(x)} = F_x T^{-1} v$$

$$Var(\widehat{f(x)}) = \sigma^2 F_x T^{-1} F_x^T$$

$$\widehat{Var(\widehat{f(x)})} = \frac{V - v^T T^{-1} v}{n - m} * F_x T^{-1} F_x^T$$

Now, as you see it becomes simple! Just define the functions and plot three pictures?!

Ok, but I don't got it! What is the T,v,V,and n?

Hmm... Let's look at the HW paper?!

$$n_i = 1,$$

$$V_i = y_i^2,$$

$$v = F_{x_i}^T \cdot y_i = \begin{pmatrix} f_1(x_i)y_i \\ \vdots \\ f_4(x_i)y_i \end{pmatrix}$$

$$T_i = F_{x_i}^T * F_{x_i} = \begin{pmatrix} f_1(x_i)^2 & f_1(x_i)f_2(x_i) & \cdots & f_1(x_i)f_4(x_i) \\ \vdots & \vdots & \ddots & \vdots \\ f_4(x_i)f_1(x_i) & f_4(x_i)f_2(x_i) & \cdots & f_4(x_i)^2 \end{pmatrix}$$

Oh... it is simple now and I know how to solve the task?! I need just implement four functions that calculates these variables n, v, V, T

In [1]:
```
import numpy as np
import numpy.random as rnd

import matplotlib.pyplot as plt

%matplotlib inline
```

In [2]:
```
WINDOW_START = 0
WINDOW_END   = 4
WINDOW_STEP  = 0.1

FIGURE_WIDTH = 20
FIGURE_HEIGHT= 10

M_SIZE = 4
A = np.array([1, 1, -1, 0.2])
```

So, let's define the canonical information. I thinks the tuple4 is the best solution, isn't it?

Canonical information in python is a tuple4 (n, v, V, T)

In [3]:
```
def get_initial_cannonical_info(x, y):
    n = 1
    V = y * y
    Fxi = np.array([x ** k for k in range(M_SIZE)])
    v = Fxi * y
    T = np.matmul(np.expand_dims(Fxi, 1), np.expand_dims(Fxi, 1).T)

    return (n, V, v, T)
```

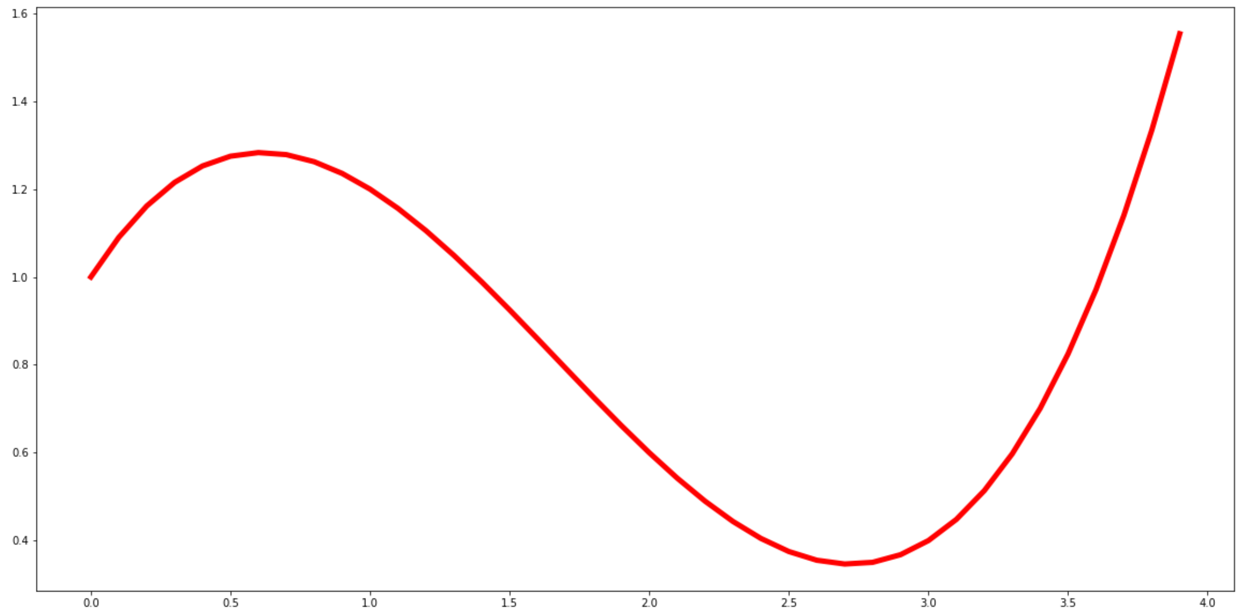
In [4]:
```
def update_canonical_info(prev, x, y):
    n_prev, V_prev, v_prev, T_prev = prev
    n_curr, V_curr, v_curr, T_curr = get_initial_cannonical_info(x, y)

    n_new = n_prev + n_curr
    V_new = V_prev + V_curr
    v_new = v_prev + v_curr
    T_new = T_prev + T_curr

    return (n_new, V_new, v_new, T_new)
```

Let's try to draw a right function (red)

In [5]:
```
def f(x):
    x_p = np.array([x ** k for k in range(M_SIZE)])
    return np.dot(A, x_p)
```

In [6]:
```
def draw_right_function(plot):
    x = np.arange(WINDOW_START, WINDOW_END, WINDOW_STEP)

    y = [f(xi) for xi in x]

    plot.figure(figsize=(FIGURE_WIDTH, FIGURE_HEIGHT))
    plot.plot(x, y, 'r', label='f_real', linewidth=5)
```

In [7]:
```
draw_right_function(plt)
```



Let's try to define estimate function

In [8]:
```
def estimate_f(cannonical_info, x_i):
    # f(x)^=FxT−1v
    n_prev, V_prev, v_prev, T_prev = cannonical_info
    Fx = np.expand_dims(np.array([x_i ** k for k in range(M_SIZE)]), 0)
    T_1 = np.linalg.inv(T_prev)
    v = np.expand_dims(v_prev, 1)

    ## Please, don't ask me "What there happening, because I don't know. I
    return np.squeeze(np.matmul(np.matmul(Fx, T_1), v)).item()
```

Good. Next is the variance of the estimate function

In [9]:
```
def variance_of_estimate_f(cannonical_info, x_i, eps):
    # Var(f(x)^)=σ2FxT−1FTx

    n_prev, V_prev, v_prev, T_prev = cannonical_info
    Fx = np.expand_dims(np.array([x_i ** k for k in range(M_SIZE)]), 0)
    T_1 = np.linalg.inv(T_prev)
    ## Please, don't ask me "What there happening, because I don't know. I
    return np.squeeze((eps * np.matmul(np.matmul(Fx, T_1), Fx.T))).item()
```

The last is monstrous function - Estimation of variance of the estimate functions... God damn!

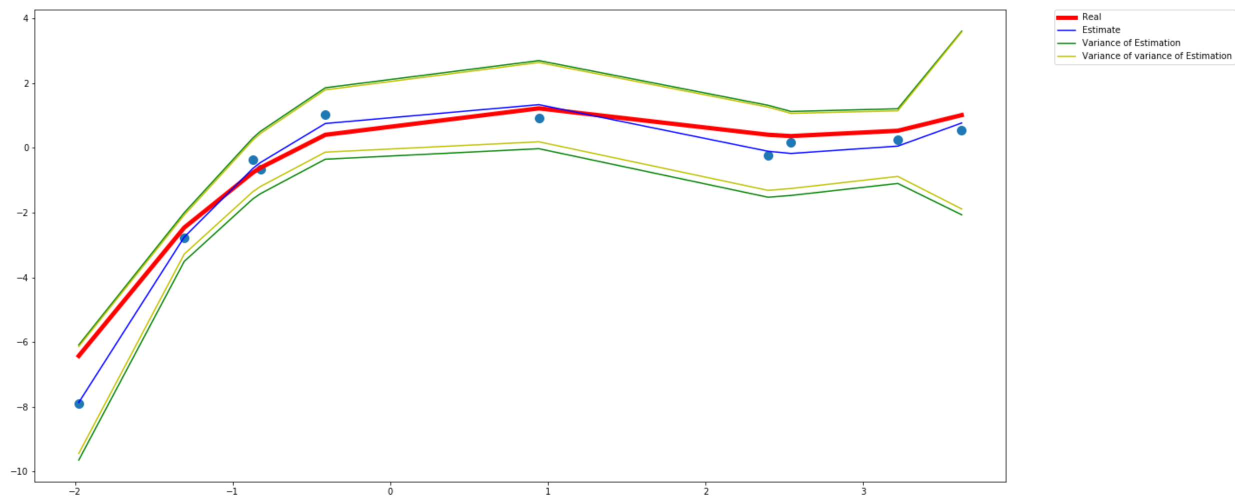
$$\widehat{Var(\widehat{f(x)})} = \frac{V - v^T T^{-1} v}{n - m} * F_x T^{-1} F_x^T$$

In [221]:
```
#Var(f(x)^)^=(V−vTT−1v/(n−m))∗FxT−1FTx
def my_monstrous_function(cannonical_info, x_i):
    n, V, v, T = cannonical_info

    Fx = np.expand_dims(np.array([x_i ** k for k in range(M_SIZE)]), 0)
    T_1 = np.linalg.inv(T)
    numerator = V - np.matmul(np.matmul(v.T, T_1), v)
    denominator = n - M_SIZE

    multiplier = np.matmul(np.matmul(Fx, T_1), Fx.T)
    result = (numerator / denominator) * multiplier

    return result.item()
```

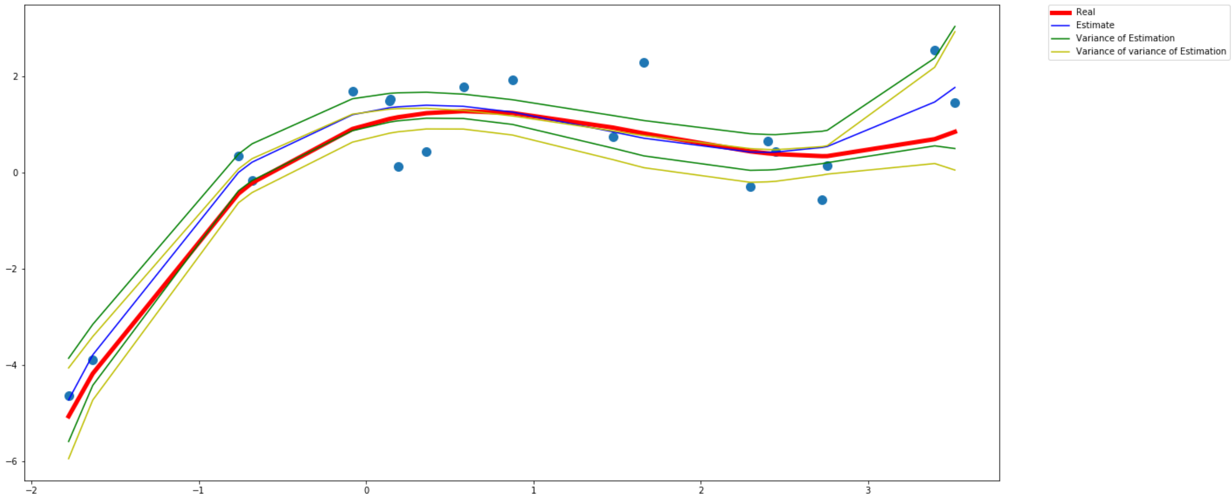
OK, Let's move on - make a sample with n = 10 and plot all the results

In [256]:
```
def draw(plot, cannonical_info, observations_x, y, eps):
    right_y = [f(x) for x in observations_x]
    estimated_y = np.array([estimate_f(cannonical_info, x) for x in observa

    var_y = np.array([variance_of_estimate_f(cannonical_info, x, eps) for i
    var_y_plus = estimated_y + var_y
    var_y_minus = estimated_y - var_y

    var_var_of_est_y = np.array([my_monstrous_function(cannonical_info, x)
    var_var_of_est_y_plus = estimated_y + var_var_of_est_y
    var_var_of_est_y_minus = estimated_y - var_var_of_est_y


    #print(var_y)
    #print(var_var_of_est_y)

    #print(var_var_of_est_y_plus)
    #print(var_y_minus)

    #print(x)
    #print(var_var_of_est_y_plus)

    plot.figure(figsize=(FIGURE_WIDTH, FIGURE_HEIGHT))
    plot.plot(x, y, 'o', markersize=10)
    plot.plot(x, right_y, 'r', label = "Real", linewidth=5)
    plot.plot(x, estimated_y, 'b', label = "Estimate")
    plot.plot(x, var_y_plus, 'g', label = "Variance of Estimation")
    plot.plot(x, var_y_minus, 'g')

    plot.plot(x, var_var_of_est_y_plus, 'y', label = "Variance of variance
    plot.plot(x, var_var_of_est_y_minus, 'y')

    plot.legend(bbox_to_anchor=(1.05, 1), loc='upper left', borderaxespad=0
    plot.show()
```

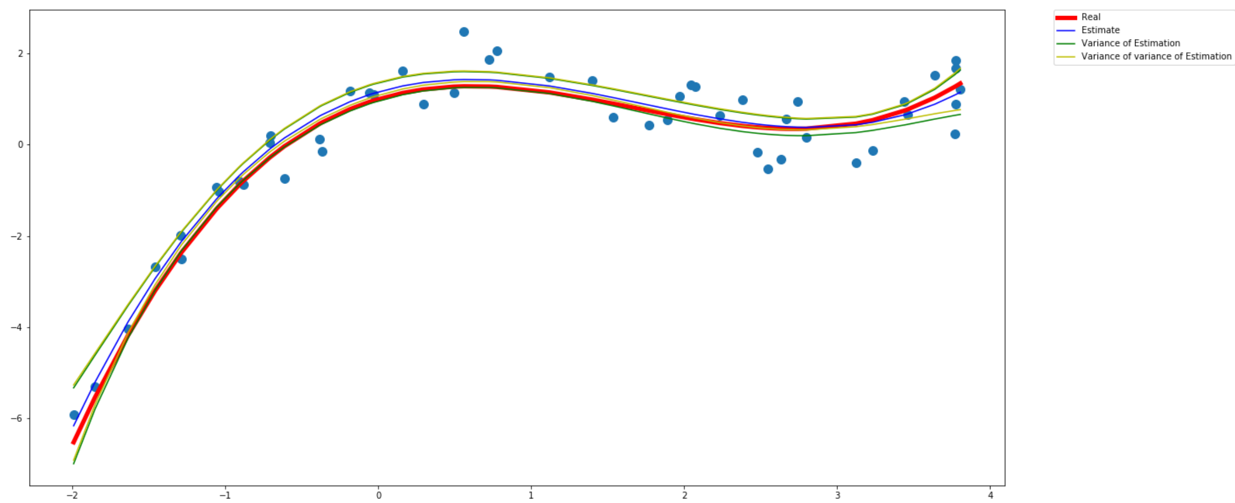
In [251]:
```
SAMPLE = 10

x = np.array([rnd.uniform(-2, 4) for i in range(SAMPLE)])
MEAN_EPS = 0
VAR_EPS = 0.6
eps = rnd.normal(MEAN_EPS, VAR_EPS, SAMPLE)

sigma_s = np.var(x)

x.sort()
y = f(x) + eps

cannonical_info = get_initial_cannonical_info(x[0], y[0])
for xi, yi in zip(x, y):
    cannonical_info = update_canonical_info(cannonical_info, xi, yi)

draw(plt, cannonical_info, x, y, sigma_s)
```



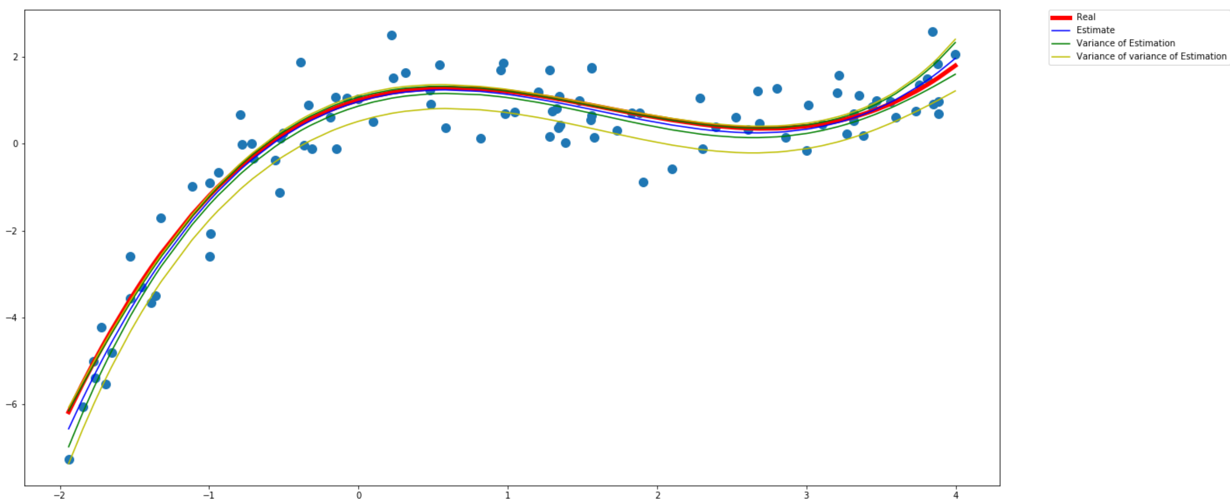
In [252]:
```
SAMPLE = 20

x = np.array([rnd.uniform(-2, 4) for i in range(SAMPLE)])
MEAN_EPS = 0
VAR_EPS = 0.6
eps = rnd.normal(MEAN_EPS, VAR_EPS, SAMPLE)

sigma_s = np.var(x)

x.sort()
y = f(x) + eps

cannonical_info = get_initial_cannonical_info(x[0], y[0])
for xi, yi in zip(x, y):
    cannonical_info = update_canonical_info(cannonical_info, xi, yi)

draw(plt, cannonical_info, x, y, sigma_s)
```



In [253]:
```
SAMPLE = 50

x = np.array([rnd.uniform(-2, 4) for i in range(SAMPLE)])
MEAN_EPS = 0
VAR_EPS = 0.6
eps = rnd.normal(MEAN_EPS, VAR_EPS, SAMPLE)

sigma_s = np.var(x)

x.sort()
y = f(x) + eps

cannonical_info = get_initial_cannonical_info(x[0], y[0])
for xi, yi in zip(x, y):
    cannonical_info = update_canonical_info(cannonical_info, xi, yi)

draw(plt, cannonical_info, x, y, sigma_s)
```



In [254]:
```
SAMPLE = 100

x = np.array([rnd.uniform(-2, 4) for i in range(SAMPLE)])
MEAN_EPS = 0
VAR_EPS = 0.6
eps = rnd.normal(MEAN_EPS, VAR_EPS, SAMPLE)

sigma_s = np.var(x)

x.sort()
y = f(x) + eps

cannonical_info = get_initial_cannonical_info(x[0], y[0])
for xi, yi in zip(x, y):
    cannonical_info = update_canonical_info(cannonical_info, xi, yi)

draw(plt, cannonical_info, x, y, sigma_s)
```



In [255]:
```
SAMPLE = 1000

x = np.array([rnd.uniform(-2, 4) for i in range(SAMPLE)])
MEAN_EPS = 0
VAR_EPS = 0.6
eps = rnd.normal(MEAN_EPS, VAR_EPS, SAMPLE)

sigma_s = np.var(x)

x.sort()
y = f(x) + eps

cannonical_info = get_initial_cannonical_info(x[0], y[0])
for xi, yi in zip(x, y):
    cannonical_info = update_canonical_info(cannonical_info, xi, yi)

draw(plt, cannonical_info, x, y, sigma_s)
```