Economic and mathematical modeling

Market microstructure

National Research University Higher School of Economics
Master's Program "Big Data Systems"

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Defining Market Microstructure

- Definition by Megginson, 1996
  Market microstructure is the study of how information is incorporated into security market prices through trading activities, and how market institutional arrangements impact security pricing efficiency.
- Definition by Cohen et al., 1986
  The microstructure literature examines the elements of the security trading process: The arrival and dissemination of information;...,the behaviour of specific types of market participants: nonprofessional investors, institutional investors, speculators, dealers, and specialists. Of concern are such issues as what the actions of profit motivated market makers might be, what the role of regulated specialists should be, how trading is affected by the market's organization, how such trading in turn affects prices, and how the system's performance affects investors.

Defining Market Microstructure 2

- Summarizing, Market Microstructure (MM) is the analysis of how the organization of markets affects:
  - the pricing process and
  - trade decisions of individuals

Issues addressed by MM:

- The reasons for trade: The private value of assets may be different from the cash equivalent that can be exchanged for the asset. This is normally captured by the MM models where informed traders operate.
- The protocol for trading: the MM models are normally very specific about the "rules of the game" for trading financial securities.
- Multiple prices: In the MM models there are many prices depending on whether you wish to buy or sell the asset, how much you wish to trade, etc.

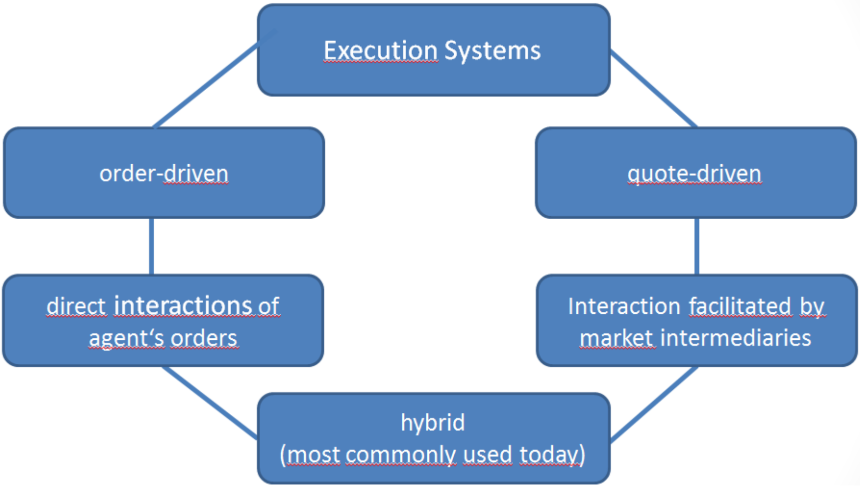
Liquidity and transparency

- Two special issues addressed by the MM theories
- Liquidity $\approx$ price elasticity – extent of price movement in response to quantity shocks
- Another aspect of liquidity is trading costs. In liquid markets we expect trading to be very cheap:
  - Direct trading costs (broker commission, etc)
  - Indirect trading costs associated with the difference in price depending on whether you want to buy or sell the asset.
- Transparency – how much information traders have when submitting an order

Market Microstructure vs Efficient Market Hypothesis

- MM often directly opposes Efficient Market Hypothesis because...
- If markets are efficient, then the details of how markets work are irrelevant since you always get the efficient price (less universally-known fee).

Trading Protocols



Order-driven markets

- Investors' buy and sell orders are matched directly, no intermediaries except for brokers
- No market makers, liquidity resulting from steady and constant flow of orders from market participants
- Brokers transmit orders but do not maintain own positions
- Rules for order matching are imposed, can be auction or crossing network markets

Quote-driven markets

- Market makers or dealers specify prices
- Monopoly of market makers, they are liquidity providers
- Market-makers act on own account, but can act as brokers
- Trading done at prices and volumes as quoted by market makers
- Low transparency, market-makers make price based on supply, demand and their risk of having large inventory
- Multiple dealers, geographically dispersed, electronically linked
- No consolidation of trading: no “floor”

Examples:

- NASDAQ
- London International Stock Exchange (SEAQ)
- OTC Bond Markets
- Foreign Exchange Markets

Hybrid markets

Hybrid markets mix aspects of the various structures:

- The most common hybrid markets are those with dealer-specialists
- These markets are order-driven auction markets in which the specialist must provide liquidity under some circumstances
- Most US stock exchanges and options exchanges have specialist systems

Market structures

Order-driven markets:

- Auction markets
  - Call markets
    - Oral auction (open outcry in floors)
    - Electronic auction
  - Continuous markets
- Crossing networks

Quote-driven markets:

- Screen-based markets – dealer markets
- Continuous auction markets

Hybrid markets

Auction markets

- Dominant type in leading markets
- Call auction characterized by simultaneous order submissions. That is, traders can trade in call markets only when the market is called
  - Pros for call markets:
    - Focus the attention of traders on the same security at the same time
    - Less volatile
  - Cons for call markets:
    - Information may need a lot of time to be incorporated into prices
  - used to open sessions in continuous markets (Bourse de Paris, NYSE,...). Also used for less active securities, bonds,....
- Continuous auction = possibility to submit orders at any time during the trading phase
  - Pros for continuous markets:
    - Traders can arrange their trades whenever they want
    - Information may be incorporated very fast into prices
  - Cons for continuous markets:
    - more volatile

Oral call auction

- Market participants physically present at trading floor
- Crying out of buy and sell offers directly
- Prices and executions open, public and transparent
- Price priority: highest bid and lowest ask preferred
- For given prices, those who bid first are preferred
- Equilibrium price determined either by auctioneer or by brokers

Electronic auction

- Electronic call auctions characterized by pre-determined time period, all submitted orders are traded at the same time at the same equilibrium price given limits
- Electronic continuous auction characterized by trading sequentially over time, with market participants observing order flow, mainly automated trading
- Electronic continuous auctions are most frequent for stocks and derivatives. Common form of order-driven markets, structured as open limit order book (OLOB)
- Precedence rules apply, price > time > others

Crossing networks

- No price priority, only time priority
- Call system that crosses prices several times per day
- Primary stock markets are indicative for prices that are determined
- Finding of prices through derivative pricing rules
- No equilibrium price, prices found through rules

Global market structures

Stock exchange	Structure	Call market, Market on close (MoC), Cross (Cr)			Pre-trade transparency					
		Mkt opening	Mkt closing	Intraday (trading halts)	Limit order book		Identities of liquidity providers		Hidden orders	Anonymity
					Members	Investors	Members	Investors		
Borsa Italiana (Bit)	AOD	yes	yes	yes	Full Book	5 Best B/A	no	no	yes	yes
Euro next	AOD	yes	yes	yes	Full Book	5 Best B/A	no	no	yes	yes
Frankfurt Stock Exchange (XETRA and Floor)	AOD/AQD	yes	yes	yes	Full Book	Best B/A	no	no	yes	yes
London Stock Exchange (SETs)	AOD	yes	yes	yes	Full Book	Full Book	no	no	yes	yes
London Stock Exchange (SEAQ-I)	SBQD	no	yes (Cr)	yes (2 Cr)	Best B/A	Best B/A	yes	no	no	no
NASDAQ (Integrated Single Book)	AOD	yes (Cr)	yes (Cr)	yes	Full Book	5 Best B/A	no	no	yes	yes
NYSE Hybrid Market (SuperDot)	AOD	yes	yes	yes	Full Book	Full Book	no	no	yes	yes
NYSE (Floor)	FBQD	yes (MoC)	yes (MoC)	yes	Full Book	Full Book	yes	no	yes	yes
Swiss Exchange (SWX)	AOD	yes	yes	yes	Full Book	Best B/A	no	no	yes	yes
Tokyo Stock Exchange (STP)	AOD	yes	yes	yes	5 Best B/A	5 Best B/A	no	no	no	yes
Toronto Stock Exchange (TOREX)	AOD/AQD	yes	no	yes	Full Book	Full Book	no	no	yes	yes

Note: AOD = Automated order-driven; AQD = Automated quote-driven; SBQD = Screen-based quote-driven; FBQD = Floor-based quote-driven; B/A = Bid/ask.

Order-driven markets: market orders

Market order – instruction to trade at the best price currently available in the market

- Executed with high probability, as long as demand and supply exist
- Buy at ask, sell at bid
- Price at execution is unknown

Example: the quote is $\$20$ bid, $\$24$ asked. Suppose that the best estimate of the true value of the security is $\$22$

- A market buy order would be executed at $\$24$ for a security worth $\$22$
- A market sell order would be executed at $\$20$ for a security worth $\$22$

Order-driven markets: limit orders

Limit order – instruction to trade at the best price available unless it is worse than the limit price specified by the trader

- For a limit buy order, the limit price specifies a maximum price
- For a limit sell order, the limit price specifies a minimum price

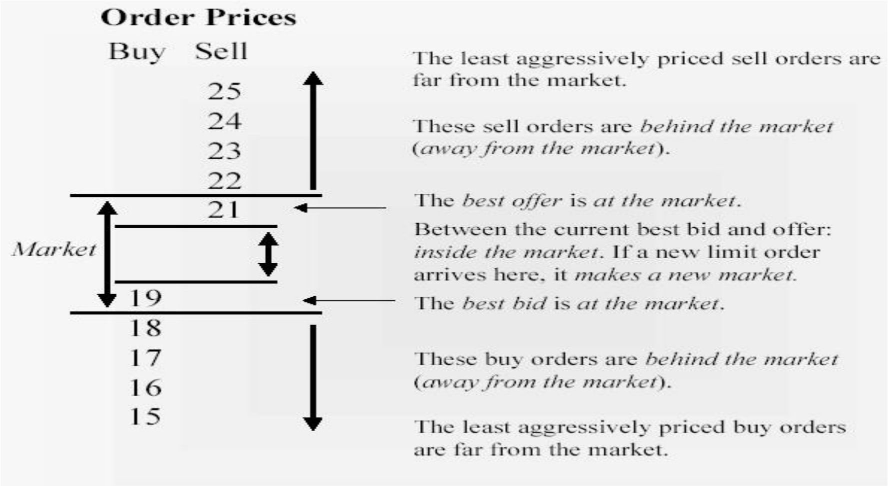
Example:

- One submits a limit buy order for 100 shares of Dell with a limit price of $\$20$. It means that he/she will buy those stocks at most for $\$20$
- One submits a limit sell order for 100 shares (round lot) of Dell with limit price of $\$24$. This means that he/she will sell those shares at least for $\$24$

Order-driven markets: limit orders 2

- If the limit order is executable (marketable), than the broker (or an exchange) will fill the order right away
- If the order is not executable, the order will be a standing offer to trade
  - waiting for incoming order to obtain a fill or ...
  - until cancellation
- Standing orders are placed in a file called limit order book

Limit order book



Limit order book: example

- Pure price-time precedence (the best shown in red):

Time	Trader	Buy/Sell	Size	Price
12:02	Sammy	Sell	100	$20.05
12:06	Steve	Sell	200	$20.06
12:15	Bern	Buy	500	$20.06
12:16	Susie	Sell	300	$20.08
12:20	Ben	Buy	200	Infinite
12:21	Bob	Buy	100	$20.08
12:24	Sandy	Sell	500	$20.12
12:25	Bev	Buy	500	$20.08
12:27	Bill	Buy	200	$20.05
12:27	Seth	Sell	200	$20.10

Limit order book: example 2

- The order book:

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	100	$20.05	200	Bill
Steve	200	$20.06	500	Bern
		$20.08	100	Bob
Susie	300	$20.08	500	Bev
Seth	200	$20.10		
Sandy	500	$20.12		
		Infinite	200	Ben

Limit order book: example 3

- Clearing the order book with a call at 12:30

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	~~100~~ 0	$20.05	200	Bill
Steve	~~200~~ ~~100~~ 0	$20.06	500	Bern
		$20.08	~~100~~	Bob
Susie	~~300~~ 0	$20.08	~~500~~ 200	Bev
Seth	200	$20.10		
Sandy	500	$20.12		
		Infinite	~~200~~ 0	Ben

Limit order book: example 4

- Trades in the example - call

Buyer	Seller	Quantity	Price?
Ben	Sammy	100	Infinity, $20.05
Ben	Steve	100	Infinity, $20.06
Bob	Steve	100	$20.08, $20.06
Bev	Susie	300	$20.08

Limit order book: example 5

- Example: the order book after the call

Sellers			Buyers	
Trader	Size	Price	Size	Trader
		$20.05	200	Bill
		$20.06	500	Bern
		$20.08	200	Bev
Seth	200	$20.10		
Sandy	500	$20.12		

Limit order book: clearing price

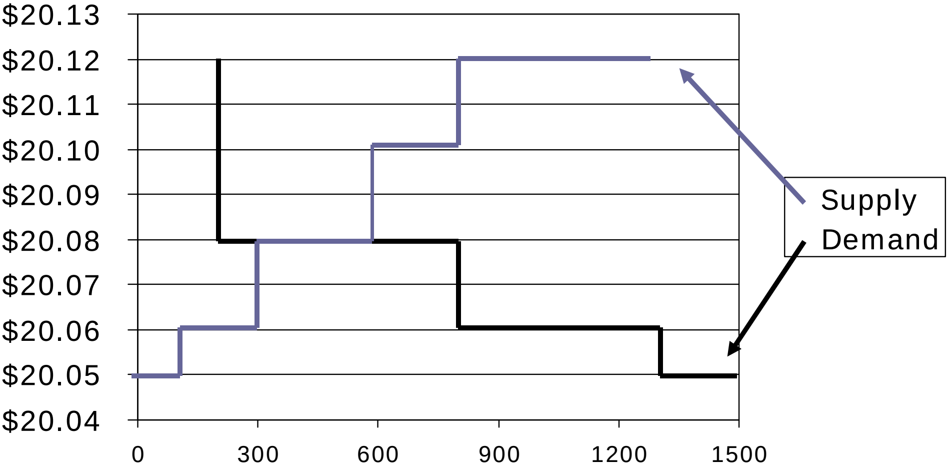
What should be the price/prices?

- Possibilities include:
  - Infinite
  - $20.05
  - $20.06
  - $20.08
- The price/prices depends on the trade pricing rules
- Single price auctions use the uniform pricing rule: everyone gets the same price
- Continuous two-sided auctions and a few call markets use the discriminatory pricing rule: trades occur at different prices
- Crossing networks use the derivative pricing rule: the price is determined by another market

Single price rule

- All trades take place at the same market clearing price
- The market clearing price is determined by the last feasible trade
- In the Example, the last feasible trade is between Bev and Susie, so the market clearing price is $\$20.08$
- The single price auction clears at the price where supply equals demand (see Slides 27,28)

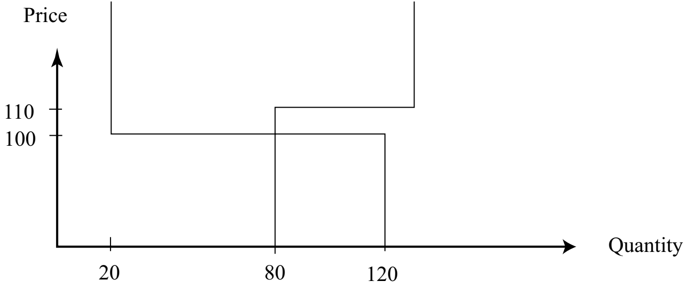
Example: supply-demand curve



Another example: left to students

Order	Buy quantity	Buy limit price	Sell quantity	Sell limit price
Limit buy	100	100		
Limit sell	–	–	50	110
Market buy	20	∞	–	–
Market sell	–	–	80	0

- Supply-demand curve:



Discriminatory Pricing Rule

- The buy and sell orders are separately sorted by their precedence
- The highest bid and the lowest offer are the best bid and offer, respectively
- When a new order arrives, the system tries to match this order with orders on the other side
  - If a trade is possible, e.g., the limit buy order is for a price at or above the best offer, the order is called a marketable order
  - If a trade is not possible, the order will be sorted into the book according to its precedence

Continuous trading at 12:02

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	100	$20.05		
		$20.06		
		$20.08		
		$20.08		
		$20.10		
		$20.12		
		Infinite		

Continuous trading at 12:06

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	100	\$20.05		
Steve	200	\$20.06		
		\$20.08		
		\$20.08		
		\$20.10		
		\$20.12		
		Infinite		

Continuous trading at 12:15

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	~~100~~ 0	$20.05		
Steve	~~200~~ 0	$20.06	~~500~~ 200	Bern
		$20.08		
		$20.08		
		$20.10		
		$20.12		
		Infinite		

Continuous trading at 12:16

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	~~100~~ 0	$20.05		
Steve	~~200~~ 0	$20.06	~~500~~ 200	Bern
Susie	300	$20.08		
		$20.08		
		$20.10		
		$20.12		
		Infinite		

Continuous trading at 12:20

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	~~100~~ 0	$20.05		
Steve	~~200~~ 0	$20.06	~~500~~ 200	Bern
Susie	~~300~~ 100	$20.08		
		$20.08		
		$20.10		
		$20.12		
		Infinite	~~200~~ 0	Ben

Continuous trading at 12:21

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	~~100~~ 0	$20.05		
Steve	~~200~~ 0	$20.06	~~500~~ 200	Bern
Susie	~~300~~ ~~100~~ 0	$20.08	~~100~~ 0	Bob
		$20.08		
		$20.10		
		$20.12		
		Infinite	200 0	Ben

Continuous trading at 12:24

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	~~100~~ 0	$20.05		
Steve	~~200~~ 0	$20.06	~~500~~ 200	Bern
Susie	~~300~~ ~~100~~ 0	$20.08	~~100~~ 0	Bob
		$20.08		
		$20.10		
Sandy	500	$20.12		
		Infinite	200 0	Ben

Continuous trading at 12:25

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	~~100~~ 0	$20.05		
Steve	~~200~~ 0	$20.06	~~500~~ 200	Bern
Susie	~~300~~ ~~100~~ 0	$20.08	~~100~~ 0	Bob
		$20.08	500	Bev
		$20.10		
Sandy	500	$20.12		
		Infinite	200 0	Ben

Continuous trading at 12:27

Sellers			Buyers	
Trader	Size	Price	Size	Trader
Sammy	~~100~~ 0	$20.05	200	Bill
Steve	~~200~~ 0	$20.06	~~500~~ 200	Bern
Susie	~~300~~ ~~100~~ 0	$20.08	~~100~~ 0	Bob
		$20.08	500	Bev
Seth	200	$20.10		
Sandy	500	$20.12		
		Infinite	~~200~~ 0	Ben

Summary on continuous trading

Buyer	Seller	Size	Price	Bid	Offer
					$20.05x100
					$20.06x100
Bern	Sammy	100	$20.05		
Bern	Steve	200	$20.06		
				$20.06x200	
				$20.06x200	$20.08x300
Ben	Susie	200	$20.08		
				$20.06x200	$20.08x100
Bob	Susie	100	$20.08		
				$20.06x200	
				$20.06x200	$20.12x500
				$20.08x500	$20.12x500
				$20.08x500	$20.10x200

Order-driven markets: stop orders

- Activated when the price of the stock reaches a predetermined limit (stop price).
- Buy only after price rises to the stop price
- Sell only after price falls to the stop price
- Stop orders are typically used to close down losing positions (stop loss orders).

Example: Suppose that the market for Dell is currently $\$20$ bid, $\$24$ offered

- Suppose that you place a stop loss order for 1000 shares of Dell at a stop price of $\$15$
- Suppose that after having placed that order, the market falls to: $\$13$ bid, $\$15$ offered. The bid price passed the stop price
- Your order is then executed at $\$13$ provided there is enough quantity at that price
- The stop price may not be the price at which you are executed as above

Order validity and expiration instructions

- Day orders (DAY)
- Good-till-cancel (GTC) orders
- Good-this-week (GTW) orders, good-this-month (GTM) orders
- Immediate-or-cancel (IOC) orders
- Fill-or-kill (FOK) orders, good-on-sight orders
- Market-on-open (MOO) orders, market-on-close (MOC) orders

Quantity instructions:

- All-or-none (AON) orders
- Minimum-or-none (MON) orders
- Minimum acceptable quantity instructions

Roll model (1984)

- The basic black dress of microstructure models:
  - appropriate in many different situations
  - easy to accessorize
  - offers an excellent pedagogical framework
- describes a dealer (quote-driven) market with fixed transaction costs

Roll model 2

- Random walk model of the efficient (log) price

$$m_t = m_{t-1} + u_t$$

  where u_t is the i.i.d. noise

$$u_t \sim N(0, \sigma_u^2)$$

- All trades are conducted through a dealer who posts the bid and ask prices
- Dealer incurs a constant cost c per trade
- The bid and the ask are

$$b_t = m_t - c, \quad a_t = m_t + c$$

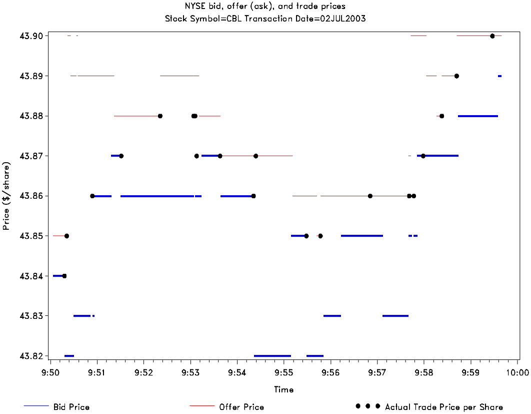
  hence, the bid-ask spread is $2c$
- The transaction price

$$p_t = m_t + cq_t$$

  where $q_t = 1$ (for buyer) and $q_t = -1$ (for seller)

Roll model 3

- Assumptions about q_t:
  - buys and sells are equally likely
  - q_t are serially independent
  - q_t and u_t are independent
- Actual bid/ask/trade data:



Roll model 4

As Figure in Slide 44 suggests,

- the spread between the bid and ask assumed to be a constant $2c$ in the Roll model is actually varying
- trades at the bid tend to cause a downward revision in the bid, and trades at the ask cause an upward revision in the ask. This raises a question about the assumed independence of q_t and u_t
- although it is not obvious in this particular sample, the q_t tend to be positively autocorrelated: buys tend to follow buys and sells tend to follow sells

Roll model 5

- The Roll model has two parameters, c and σ_u^2.
- Their estimates come from the variance and lag-one autocovariance of the price changes $\Delta p_t = p_t - p_{t-1}$:

$$\text{Var}(\Delta p_t) = E(\Delta p_t)^2 = E(p_t - p_{t-1})^2 = E(m_t + q_t c - m_{t-1} - q_{t-1} c)^2 =$$
$$E(u_t + (q_t - q_{t-1})c)^2 = E(u_t^2 + 2u_t(q_t - q_{t-1})c + (q_t - q_{t-1})^2 c^2) =$$
$$E(u_t^2) + c^2 E(q_t - q_{t-1})^2 = E(u_t^2) + c^2[E(q_t^2) + E(q_{t-1}^2)]$$

since u_t, q_t and q_{t-1} are uncorrelated

- $\text{Var}(u_t^2) = \sigma_t^2$ and

$$\text{Var}(q_t) = E(q_t^2) - [E(q_t)]^2 = \frac{1}{2} \times 1^2 + \frac{1}{2} \times (-1)^2 - 0^2 = 1$$

- So, $\text{Var}(\Delta p_t) = \sigma_u^2 + 2c^2 \equiv \gamma_0$

Roll model 6

- Autocovariance of price changes:

$$\text{Cov}(\Delta p_t, \Delta p_{t-1}) = E(\Delta p_t \Delta p_{t-1})^2 =$$
$$E[(m_{t-1} + q_{t-1}c - m_{t-2} - q_{t-2}c)(m_t + q_t c - m_{t-1} - q_{t-1}c)] =$$
$$E[(u_{t-1}+(q_{t-1}-q_{t-2})c)(u_t+(q_t-q_{t-1})c)] = -c^2[E(q_{t-1}^2)] = -c^2 \equiv \gamma_1$$

- Parameters $\gamma_0 = \text{Var}(\Delta p_t)$ and $\gamma_1 = -c^2$ can be estimated directly from market data
- For CBL on July 2, 2003, there were 821 NYSE trades. This implies $c = 0.005$ (\$/share) and a spread of $2c = 0.01$ (\$/share).
- The (time-weighted) average NYSE spread in the sample is 0.022 (\$/share), so the Roll underestimates the spread

Sequential trade models of asymmetric information

- When all agents are identical, they are said to be symmetric
  - This does not rule out private values or private information
  - It simply means that all individual-specific variables (e.g., the coefficient of risk aversion, a value signal) are identically distributed across all participants
- In an asymmetric information model, some subset of the agents has superior private information
- Two main sorts of asymmetric information models
  - In the sequential trade models, randomly-selected traders arrive at the market singly, sequentially, and independently, e.g., model of Glosten and Milgrom (1985).
  - The other class of models usually features a single informed agent who can trade multiple times (continuous trading), e.g., model of Kyle (1985).

Sequential trade models of asymmetric information 2

- The essential feature of both models is that a trade reveals something about agent's private information.
- A "buy" might result from a trader who has private positive information, but it won't originate from a trader who has private negative information.
  - Rational market makers will set their bid and ask quotes accordingly.
  - More extreme information asymmetries lead to wider quotes.
- The spread and trade-impact effects are the principal empirical implications of these models

Glosten-Milgrom model

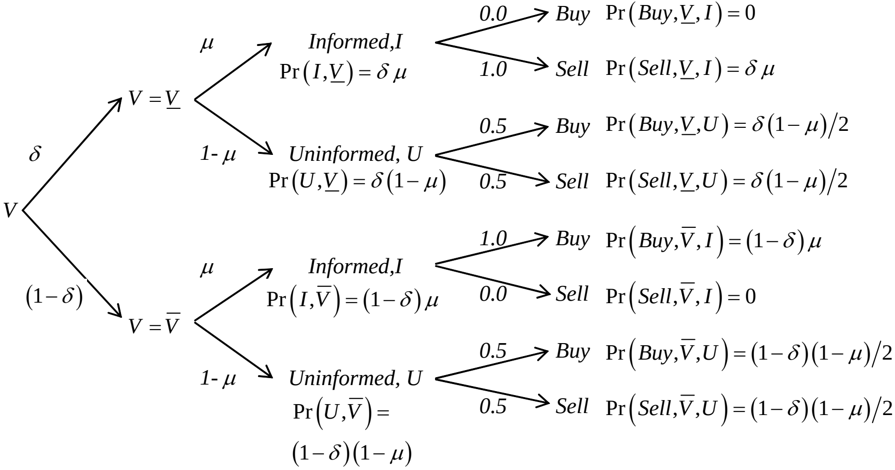
- The terminal security value is V , which at the end of the day takes just two values

$$V = \begin{cases} \bar{V}, & \Pr(V = \bar{V}) = 1 - \delta \\ \underline{V}, & \Pr(V = \underline{V}) = \delta \end{cases}$$

- The trading population consists of informed and uninformed traders.
  - The proportion of informed traders is μ
- The MM posts bid and ask quotes, B and A.
- A trader is drawn at random from the population
  - If the trader is informed, he/she buys if $V = \bar{V}$ and sells if $V = \underline{V}$
  - If the trader is uninformed, he buys or sells randomly and with equal probability (=1/2)
- The MM cannot determine whether the trader is informed or not.

Glosten-Milgrom model 2

The event tree for the first trade looks like this



Glosten-Milgrom model 3

- The unconditional buy and sell probabilities are:

$$\Pr(\text{Buy}) = \frac{1}{2}\left(-2\delta\mu + \mu + 1\right), \quad \Pr(\text{Sell}) = \left(\delta - \frac{1}{2}\right)\mu + \frac{1}{2}$$

- The unconditional expectation of terminal value is:

$$E(V) = \bar{V}\left(1 - \delta\right) + \underline{V}\delta$$

- The MMs compete with one another => MM's market is efficient => MM's expected profit is zero
- Suppose a customer buys (trades at the MM's ask price A). The MM's realized profit on the trade is $\pi = V - A$, or in expectation, conditional on the customer's purchase

$$E(\pi|\text{Buy}) = A - E(V|\text{Buy}) \tag{1}$$

- The LHS of (1) must be zero

Glosten-Milgrom model 4

- The MM's expected profit via conditional probabilities:

$$E(\pi|\text{Buy}) = A - [E(V|U,\text{Buy})\Pr(U|\text{Buy}) + E(V|I,\text{Buy})\Pr(I|\text{Buy})]$$

- The ask price is

$$A = [E(V|U,\text{Buy})\Pr(U|\text{Buy}) + E(V|I,\text{Buy})\Pr(I|\text{Buy})]$$

- Finalizing calculations...

$$E(V|U,\text{Buy}) = E(V) = \bar{V}(1-\delta) + \underline{V}\delta, \quad E(V|I,\text{Buy}) = \bar{V}$$

verify making use of the probability tree in Slide 51

- By Bayes' theorem

$$\Pr(U|\text{Buy}) = \frac{\Pr(\text{Buy}|U)\Pr(U)}{\Pr(\text{Buy})} = \frac{1-\mu}{-2\delta\mu+\mu+1},$$

$$\Pr(I|\text{Buy}) = \frac{\Pr(\text{Buy}|I)\Pr(I)}{\Pr(\text{Buy})} = \frac{2(\delta-1)\mu}{-2\delta\mu+\mu+1}.$$

Glosten-Milgrom model 5

- The ask price

$$A = \frac{\underline{V}\delta(\mu - 1) + \bar{V}(\delta - 1)(\mu + 1)}{(2\delta - 1)\mu - 1}$$

- The bid price

$$B = \frac{\underline{V}\delta(\mu + 1) + \bar{V}(\delta - 1)(\mu - 1)}{(2\delta - 1)\mu - 1}$$

- In the symmetric case $\delta = 1/2$ the bid-ask spread is

$$A - B = (\bar{V} - \underline{V})\mu$$

- The value of δ is revised once the dealer has got the order, e.g.:

$$\Pr(V = \underline{V}|\text{Sell}) = \frac{\Pr(\text{Sell}|V = \underline{V})\Pr(V = \underline{V})}{\Pr(\text{Sell})} = \frac{\delta(1 + \mu)}{1 - \mu(1 - 2\delta)} > \delta$$

Glosten-Milgrom model 6

- The revised probability of downward move in case of incoming buy order

$$\Pr(V = \underline{V}|\text{Buy}) = \frac{\Pr(\text{Buy}|V = \underline{V})\Pr(V = \underline{V})}{\Pr(\text{Buy})} = \frac{\delta(1-\mu)}{1+\mu(1-2\delta)} < \delta$$

- Sequential update of conditional probabilities of the price drop

$$\Pr(V = \underline{V}|\text{Sell}_k) \equiv \delta_k(\text{Sell}_k) = \frac{\delta_{k-1}(1+\mu)}{1-\mu(1-2\delta_{k-1})} \qquad (2)$$

$$\Pr(V = \underline{V}|\text{Buy}_k) \equiv \delta_k(\text{Buy}_k) = \frac{\delta_{k-1}(1-\mu)}{1+\mu(1-2\delta_{k-1})} \qquad (3)$$

- Getting the ask quotes sequentially

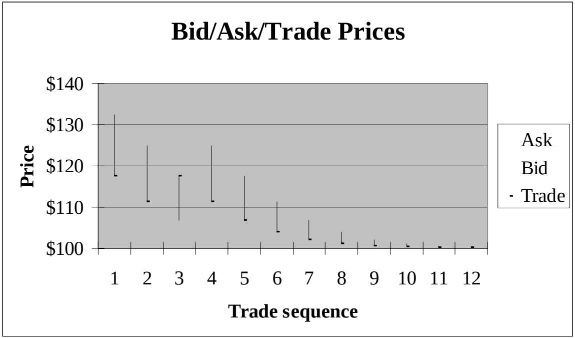
$$A_k = \underline{V}\Pr(V = \underline{V}|\text{Buy}_k) + \bar{V}\Pr(V = \bar{V}|\text{Buy}_k)$$

where probabilities come from (2)-(3). Similarly for B_k

Glosten-Milgrom model 7

Path of bid, ask and trade prices is for a given sequence of buys or sells within one trading day

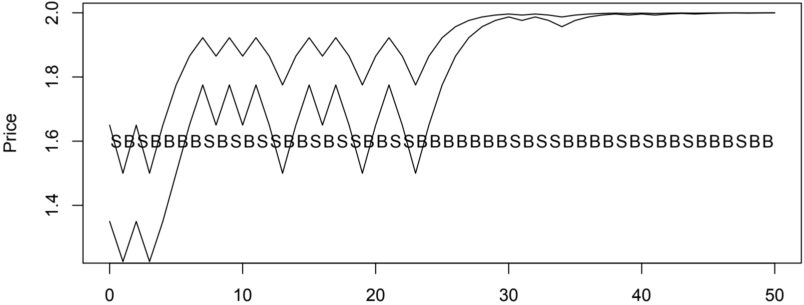
- Here is the case where the third trade is a buy, but all the others are sells



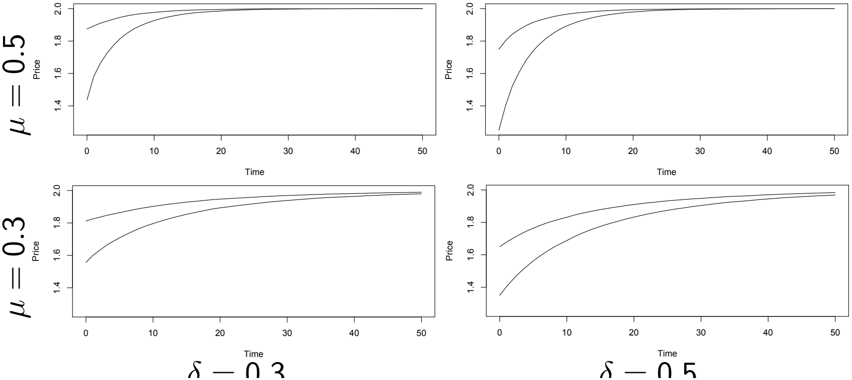
Glosten-Milgrom model 7

Another sample path

- Here $\bar{V} = 2$, $\underline{V} = 1$, $\mu = 0.3$, $\delta = 0.5$.



Bid and asked prices averaged over many sample paths



Glosten-Milgrom model 9

General features

- The trade price series is a martingale
  - the sequence of trade prices is a sequence of conditional expectations $E(V|\Phi_k)$. A sequence of expectations conditioned on expanding information sets is a martingale
- The order flow is not symmetric.
  - If $q_k = \pm 1$ for buy/sell, $E(q_k) \neq 0$
- The orders are serially correlated
  - one subset of the population (the informed traders) always trades in the same direction
- There is a price impact of trades
  - a buy increases B and A
  - The trade price impact is a particularly useful empirical implication of the model. It can be estimated from market data, and is a proxy for information asymmetries

Glosten-Milgrom model 9

General features

- The spread declines over time as the MM figures out V within the trading day

Kyle model

- A dynamic model of security prices where the MM is setting efficient prices (as in the Glosten- Milgrom)
- The informed trader chooses optimal trading strategy
  - The insiders can choose how much to trade
- The insiders wants to trade aggressively, e.g., buying a large quantity if his information is positive
- But the MM knows that if he sells into a large net customer "buy", he is likely to be on the wrong side of the trade
  - He protects himself by setting a price that is increasing in the net order flow
- This acts as a brake on the informed trader's desires: if he wishes to buy a lot, he'll have to pay a high price
- The solution to the model is a formal expression of this trade-off.

Kyle model 2

- The terminal security value

$$\nu \sim N(p_0, \Sigma_0)$$

- There is one informed trader who knows ν and places an order $x > 0$ (buying) or $x < 0$ (selling)
- Uninformed traders submit a net order flow independent of ν

$$u \sim N(0, \sigma_u^2)$$

- The MM observes only the total demand

$$y = x + u$$

and then sets a price $p(y)$
- The MM uses a linear price rule

$$p = \lambda y + \mu$$

- The informed trader's profit is

$$\pi = (\nu - p)x$$

Kyle model 3

- The MM is facing perfect competition, so the market price must be
$$p(y) = E(\nu|y)$$

Kyle model 4

The informed trader's perspective:

- The informed trader assumes that the MM uses a linear price rule

$$p = \lambda y + \mu$$

- The informed trader's profit is

$$\pi = (\nu - p)x \quad \text{or} \quad \pi = x(\nu - (u + x)\lambda - \mu)$$

- The expected profit is

$$E(\pi) = x(\nu - x\lambda - \mu)$$

- The informed trader maximizes expected profit by trading x:

$$x = \frac{\nu - \mu}{2\lambda}$$

- The second-order condition for the max is

$$-2\lambda < 0$$

Kyle model 5: market maker's perspective

- The MM assumes that the informed trader's demand is linear in ν

$$x = \alpha + \beta\nu$$

- Knowing the optimization process that the informed trader follows, the MM can solve for α and β:

$$\alpha + \beta\nu = \frac{\nu - \mu}{2\lambda} \quad \Rightarrow \quad \alpha = -\frac{\mu}{2\lambda}, \; \beta = \frac{1}{2\lambda}$$

- Recall, $p = E(\nu|y)$
- Probability rule: if $(X, Y)^T \sim N(\mu, \Sigma)$ then

$$E_{Y|X}(x) = \frac{\text{Cov}(X, Y)(x - E(X))}{\text{Var}(X)} + E(Y)$$

- So,

$$E_{\nu|y}(y) = p_0 + \frac{\beta(y - \alpha - \beta p_0)\Sigma_0}{\Sigma_0\beta^2 + \sigma_u^2} \tag{4}$$

Kyle model 6

- From $E_{\nu|y}(y) = p$ and (4) we get

$$p_0 + \frac{\beta(y - \alpha - \beta p_0)\Sigma_0}{\Sigma_0\beta^2 + \sigma_u^2} = y\lambda + \mu$$

- Hence,

$$\mu = -\frac{\alpha\beta\Sigma_0 - \sigma_u^2 p_0}{\Sigma_0\beta^2 + \sigma_u^2}, \quad \lambda = \frac{\beta\Sigma_0}{\Sigma_0\beta^2 + \sigma_u^2}, \quad \alpha = -\frac{\mu}{2\lambda}, \ \beta = \frac{1}{2\lambda} \qquad (5)$$

- Solving (5) for the parameters

$$\alpha = -\frac{\sigma_u p_0}{\sqrt{\Sigma_0}}, \quad \mu = p_0, \quad \lambda = \frac{\sqrt{\Sigma_0}}{2\sigma_u}, \quad \beta = \frac{\sigma_u}{\sqrt{\Sigma_0}}$$

Kyle model 7: : properties of the solution

- Liquidity λ and order flow β depend on uncertainty Σ_0 relative to the intensity of noise trading σ_u
- The informed trader's expected profit

$$E(\pi) = \frac{\sigma_u(\nu - p_0)^2}{2\sqrt{\Sigma_0}}$$

increases at large deviations of the value (known by the informed trader) from the expectation of the uninformed agents p_0

- Profit also increases with the variance of noise trading σ_u – a camouflage for the informed trader.
  - an agent trading on inside information will be able to make more money in a widely held and frequently traded stock
- Variance of the price. Given $(Y, X)^T \sim N(\mu, \Sigma)$,

$$\text{Var}_{Y|X} = \text{Var}(Y) - \frac{\text{Cov}^2(X, Y)}{\text{Var}(X)}$$

Kyle model 8: properties of the solution

- Thus,

$$\mathrm{Var}_{\nu|y} = \mathrm{Var}(\nu) - \frac{\mathrm{Cov}^2(\nu, y)}{\mathrm{Var}(y)} = \Sigma_0 - \frac{\beta^2 \Sigma_0^2}{\Sigma_0 \beta^2 + \sigma_u^2} = \frac{\Sigma_0}{2}$$

- Thus, half of the insider's information gets into the price

Multiperiod Kyle model

- $k = 1, ..., N$ auctions, time step between auctions Δt
- At the kth auction, noise traders submit an order flow $u_k \sim N(0, \sigma_u^2 \Delta t)$
- The informed trader submits an order flow Δx_k
- The informed traders profit is given recursively

$$\pi_k = (\nu - p_k)\Delta x_k + \pi_{k+1}, \quad k = 1, ..., N, \quad \pi_{N+1} = 0$$

Solution to multiperiod Kyle model

- The informed trader's demand in auction n is linear in the difference between the true value ν and the price on the preceding auction, p_{n-1}

$$\Delta x_n = \Delta t \beta_n (\nu - p_{n-1})$$

- The MM's price adjustment rule is linear in the total order flow

$$\Delta p_n = (\Delta u_n + \Delta x_n)\lambda_n$$

- Expected profits are quadratic

$$E\pi_n = \alpha_{n-1}(\nu - p_{n-1})^2 + \delta_n$$

Solution to multiperiod Kyle model 2

- The constants above solve the (backward) difference equation system subject to the terminal conditions $\alpha_N = \beta_N = 0$

$$\alpha_k = \frac{1}{4\lambda_{k+1}(1 - \alpha_{k+1}\lambda_{k+1})}, \quad \delta_k = \Delta t \alpha_{k+1}\lambda_{k+1}^2\sigma_u^2 + \delta_{k+1}$$

$$\beta_k = \frac{1 - 2\alpha_k\lambda_k}{2\Delta t \lambda_k(1 - \alpha_k\lambda_k)}, \quad \lambda_k = \frac{\beta_k\Sigma_k}{\sigma_u^2}$$

- Forward difference equation for the variance of ν conditional on all order flow and prices through auction k

$$\Sigma_k = (1 - \Delta t \beta_k\lambda_k)\Sigma_{k-1}$$

References

See Chapter 3 of [?] for more.