4) Exponential family distribution - probability density:

$$p(y;\eta) = b(y) \cdot \exp(\eta^T \cdot T(y) - a(\eta))$$

assume:

$\eta = \theta^T \cdot x$ is a scalar

$T(y) = y$

then: $p(y;\eta) = b(y) \cdot \exp(\eta \cdot y - a(\eta))$

a) Derive an expression for the mean of the distribution

Show that $\mathbb{E}[Y|X;\theta]$ can be represented as the gradient of the log partition function a with respect to η

$$\frac{\partial}{\partial\eta}\int p(y;\eta)\,dy = \int \frac{\partial}{\partial\eta} p(y;\eta)\,dy$$

$$= \int (y - a'(\eta)) \cdot b(y) \cdot \exp(\eta \cdot y - a(\eta)) \cdot dy$$

$$= \mathbb{E}_y[y|\eta] - a'(\eta)$$

$$\therefore a'(\eta) = \mathbb{E}[y|\eta]$$

b) Derive an expression for the variance of the distribution

Show that $Var(Y|X;\theta)$ can be expressed as the derivative of the mean w.r.t. η

$$\frac{\partial}{\partial^2\eta}\int p(y;\eta)\,dy = \frac{\partial}{\partial\eta}\underbrace{\int \frac{\partial}{\partial\eta} p(y;\eta)\,dy}$$

$$= \frac{\partial}{\partial\eta}\int (y - a'(\eta)) \cdot b(y) \cdot \exp(\eta \cdot y - a(\eta))\,dy$$

$$= \int (y^2 - a'(\eta)) \exp(\eta y - a(\eta))\,dy = a''(\eta)$$

$$= \mathbb{E}[y^2;\eta] - \mathbb{E}[y;\eta]^2 - a''(\eta)$$

$$= Var(y;\eta) - a''(\eta) = 0$$

$$\therefore Var(y;\eta) = a''(\eta)$$