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Article in *Australian Surveyor* · June 1999

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Transforming Cartesian coordinates X, Y, Z to Geographical coordinates ϕ, λ, h

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[published in *The Australian Surveyor*, Vol. 44, No. 1, pp. 55-63, June 1999]

ABSTRACT

In November 1994, the Intergovernmental Committee on Surveying and Mapping (ICSM) adopted a new geodetic datum for Australia and recommended its progressive implementation nationally. This datum, designated *Geocentric Datum of Australia (GDA)* – compatible with the Global Positioning System (GPS), a US Department of Defense satellite positioning system – will eventually replace the existing Australian Geodetic Datum (AGD). Consequently, “old” AGD coordinates need to be converted to “new” GDA coordinates, which may require the transformation of Cartesian coordinates X, Y, Z to geographical coordinates ϕ, λ, h (latitude, longitude, height) related to an ellipsoid of revolution.

Given X, Y, Z , longitude is easily derived, but not so latitude, which requires more sophisticated evaluation – usually by iterative techniques, or complicated direct methods – the height following readily once latitude is obtained. Thus Cartesian-to-geographic transformations revolve around the determination of latitude; this paper reviews published techniques, some quite recent, which may be of use to practitioners.

INTRODUCTION

Conversion from geographic to Cartesian coordinates is a simple operation since closed formulae relate X, Y, Z to ϕ, λ, h . Inverse computations, on the other hand, are more complex, since no simple relationships link ϕ to X, Y, Z . Various techniques for computing ϕ are well documented in the literature, some more efficient than others when comparing computational time; indirect (or iterative) techniques are often preferred for their relative simplicity when compared with more complicated direct methods. This paper compares six techniques; two direct, *Paul* (1973) and *Ozone* (1985) and four indirect, *Bowring* (1976), *Borkowski* (1989), *Lin & Wang* (1995) and *Simple Iteration*. The last is described in several geodesy texts, for instance Bomford (1980, p. 679).

These are not the only techniques for computing ϕ given X, Y, Z ; Borkowski (1989) refers to fourteen other methods; three direct, five iterative, four using closed approximate formulae cast as series expansions and two others, as well as another direct method of his own (Borkowski 1987). Vincenty (1985) claims that the earliest solution to this problem was given by Doerrie in his book *Kubische und biquadratische Gleichungen*, Muenchen, 1948.

This review is not definitive, since only a small selection of the published techniques are chosen, but amongst them, *Bowring's* method and *Simple Iteration* have often been used for comparative analysis. For instance, Borkowski (1989) compares his iterative method against nine others, including three variants of *Simple Iteration*, concluding that whilst simple iterative procedures are the easiest to program, his solution is more appropriate – requiring fewer iterations to reach acceptable accuracy. Laskowski (1991) compares *Borkowski's* and *Bowring's* methods, demonstrating that *Bowring's* is faster whilst Lin & Wang (1995) find their method to be faster than *Bowring's*. Lin & Wang mention a review of methods by Rapp (1984) who concluded (at the time) that *Bowring's* procedure was the most efficient. Thus from the literature, it could be concluded that *Lin & Wang's* method is the most efficient Cartesian-to-geographic transformation. This is in fact confirmed by computational analysis of the six techniques, which are regarded as representative of the numerous published methods.

GEODETIC COORDINATE SYSTEMS IN USE IN AUSTRALIA

In 1966, under the direction of the National Mapping Council (NMC) all geodetic surveys in Australia were recomputed and adjusted on the then new AGD, an astronomically derived *topocentric* datum having a physical origin near the centroid of the geodetic network and fixing an ellipsoid of revolution, the *Australian National Spheroid* (ANS), with respect to the Earth's rotational axis. The national adjustment yielded an homogeneous set of geographical coordinates (latitudes and longitudes) for the geodetic network. At the same time, the NMC defined a system of rectangular grid coordinates (eastings and northings) known as the *Australian Map Grid* (AMG), based on a *Universal Transverse Mercator* (UTM) projection of AGD latitudes and longitudes.

After 1966 there were several readjustments of the national geodetic network, densified and strengthened by the inclusion of improved measurements, each readjustment referred to as a *Geodetic Model of Australia* (GMA). In 1984 the NMC, recognizing the eventual need for Australia to convert to a geocentric datum, adopted the latest readjustment at the time, GMA82, as an interim step in this process. This geographical coordinate set was defined as AGD84 with AMG84 grid coordinates, and to avoid confusion, earlier coordinate sets derived from the 1966 adjustment were defined as AGD66 and AMG66. Both AGD66 and AGD84 coordinates have a common datum (defined in 1966) excepting that AGD84 coordinates were derived from an adjustment which more correctly allowed for the separation between the *geoid* and the ANS over Australia (NMC 1986).

In 1988, the NMC was superseded by the ICSM, representing the mapping organizations of the States and Territories of the Commonwealth of Australia and New Zealand. The GDA was adopted by the ICSM in November 1994 in response to anticipated demand by major users of GPS technology such as the Australian Defence Force, the International Civil Aviation Organization, the International Hydrographic Organization and the International Association of Geodesy (Steed 1996, p. 24). The new datum is primarily based on the coordinates of eight geologically stable sites across Australia with permanent GPS tracking facilities known as the *Australian Fiducial Network* (AFN), supplemented by a network of seventy survey stations (covering Australia at approximately 500km intervals) which together form the *Australian National Network* (ANN).

Geocentric Cartesian coordinates of these stations were derived from an adjustment of precise GPS observations obtained from – (i) a two week global observation period in 1992 conducted by the *International GPS Geodynamics Service* at approximately two hundred sites around the world (including all the AFN sites) and (ii) ICSM campaigns in 1992, '93 and '94 linking all AFN and ANN sites. These coordinates are related to the *International Earth Rotation Service* (IERS) Terrestrial Reference Frame for 1992 (ITRF92) at epoch 1994.0 [The epoch 1994.0 (1st Jan. 1994) reflects the fact that monitoring stations used by IERS are moving with respect to each other due to earth crustal motion; the epoch date indicating the datum is ITRF92 adjusted for station motion in the intervening period]. The ICSM has defined GDA94 coordinates as latitudes and longitudes related to the ellipsoid of the *Geodetic Reference System 1980* (GRS80) [BG 1988, p. 348] and *Map Grid Australia 1994* (MGA94) grid coordinates as a UTM projection of those latitudes and longitudes.

The GRS80 was adopted by the International Union of Geodesy and Geophysics (IUGG) at its XVIITH General Assembly in Canberra, December 1979, as the best representation of the size and shape of the Earth. The GRS80 is based on the theory of a geocentric equipotential ellipsoid and is defined by four physical parameters of the Earth – (i) a the equatorial radius, (ii) GM the geocentric gravitational constant, (iii) J_2 the dynamical form factor, and (iv) ω the angular velocity – from which geometric constants, e^2 (square of the first eccentricity) and f (flattening) are derived. The *World Geodetic System 1984* (WGS84), the datum for GPS, is also based on GRS80, except that the dynamical form factor of the Earth is expressed in a modified form, causing very small differences between derived constants of the WGS84 and GRS80 ellipsoids. These differences can be regarded as negligible for all practical purposes and GDA94 is considered compatible with WGS84.

Geometric parameters of relevant ellipsoids are:

Ellipsoid	semi-major axis a	flattening f	semi-minor axis $b = a(1 - f)$
ANS	6378160 m	1/298.25	6356774.7192 m
GRS80	6378137 m	1/298.257222101	6356752.3141 m
WGS84	6378137 m	1/298.257223563	6356752.3142 m

THE GENERAL METHOD OF CONVERSION BETWEEN COORDINATE DATUMS

X, Y, Z Cartesian coordinates have an origin at the centre of the ellipsoid. The Z -axis is in the direction of the rotational axis of the ellipsoid of revolution, the X - Z plane is the Greenwich meridian plane (the origin of longitudes), the X - Y plane is the equatorial plane of the ellipsoid (the origin of latitudes), the X -axis is in the direction of the intersection of the Greenwich meridian plane and the equatorial plane and the Y -axis is advanced 90° east along the equator. In Australia, there are two reference ellipsoids to be considered, the ANS (for AGD66 and AGD84) and the GRS80 (for GDA94) and three derived sets of Cartesian coordinates X, Y, Z_{AGD66} , X, Y, Z_{AGD84} and X, Y, Z_{GDA94} , the last properly termed geocentric. It is known that the origin of the ANS is not coincident with the geocentre (the origin of the GRS80 ellipsoid) and also that the Cartesian axes of the ANS are slightly rotated with respect to the axes of GRS80 ellipsoid. A generally accepted method of conversion consists of the following steps (Featherstone 1994, pp. 8-10)

1. Compute Cartesian coordinates from geographical coordinates using

$$X = (\nu + h) \cos \phi \cos \lambda \quad (1.1)$$

$$Y = (\nu + h) \cos \phi \sin \lambda \quad (1.2)$$

$$Z = (\nu(1 - e^2) + h) \sin \phi \quad (1.3)$$

where ν (nu), the radius of curvature in the prime vertical plane, and e^2 are given by

$$\nu = \frac{a}{\sqrt{1 - e^2 \sin^2 \phi}} \quad (1.4)$$

$$e^2 = f(2 - f) \quad (1.5)$$

2. Convert Cartesian coordinates from AGD to GDA94 via a 7-parameter conformal transformation given in matrix notation as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{GDA94} = (1 + ds) \begin{bmatrix} 1 & R_Z & -R_Y \\ -R_Z & 1 & R_X \\ R_Y & -R_X & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{AGD} + \begin{bmatrix} X_T \\ Y_T \\ Z_T \end{bmatrix} \quad (1.6)$$

where ds is a scale change, R_X, R_Y, R_Z are small rotations about the X, Y, Z axes respectively and X_T, Y_T, Z_T are translations between the two ellipsoid origins.

3. Convert Cartesian coordinates to geographical coordinates using

$$\tan \lambda = \frac{Y}{X} \quad (1.7)$$

$$\tan \phi = \frac{Z + \nu e^2 \sin \phi}{p} \quad (1.8)$$

$$h = \frac{p}{\cos \phi} - \nu \quad (1.9)$$

where p is the perpendicular distance from the rotational axis

$$p = \sqrt{X^2 + Y^2} \quad (1.10)$$

Note 1: If AMG66 or AMG84 coordinates are to be transformed to MGA94 coordinates then they must be converted from grid to geographic prior to step 1 and from geographic to grid after step 3 using Redfearn's formulae (NMC 1986).

Note 2: In (1.1) to (1.3), the ellipsoidal height h , the height above the ellipsoid measured along the normal, is required. In Australia, all heights are related to the *Australian Height Datum* (AHD), a practical approximation of the geoid over Australia, and the change to GDA94 will not affect AHD heights. Cartesian coordinates for P on the Earth's surface must be calculated using h (obtained by adding the geoid-ellipsoid separation N to the AHD value). Geoid models, such as AUSGEOID93 (Steed & Holtznagel 1994) can be used to determine N . [AUSGEOID93 is a grid of computed N values, produced by the Australian Surveying and Land Information Group (AUSLIG), which can be interpolated to give values at desired locations].

Note 3: Sets of transformation parameters (ds , R_X , R_Y , R_Z and X_T , Y_T , Z_T) for AGD66 and AGD84 are available from AUSLIG.

The Cartesian-to-geographic conversion problem is embodied in (1.8) where functions of ϕ ($\tan \phi$, ν and $\sin \phi$) appear on both sides of the equation making the evaluation of ϕ difficult. The following techniques, direct and iterative, are solutions to this problem.

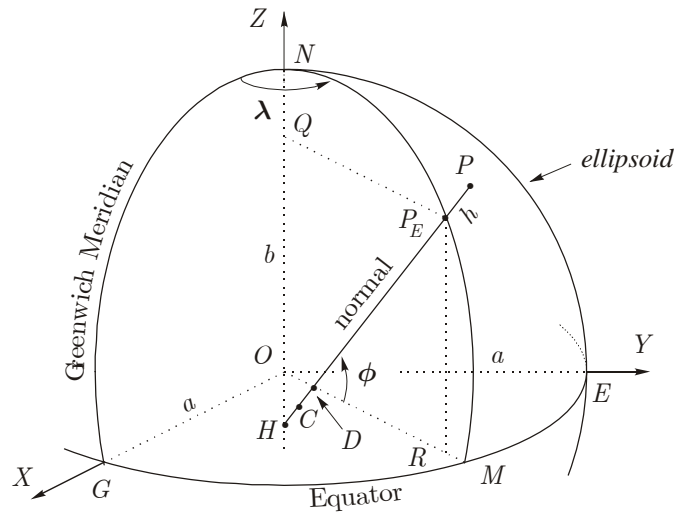


Figure 1 Section of ellipsoid

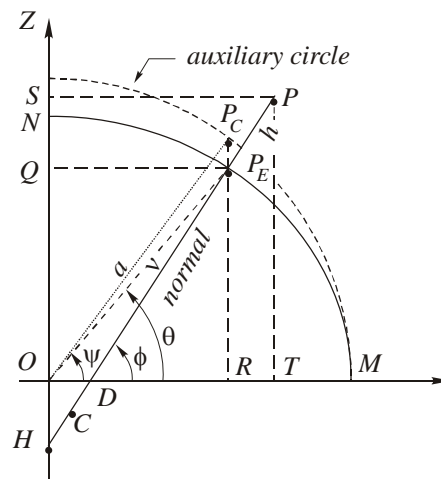


Figure 2 Meridian ellipse

ELLIPSOID RELATIONSHIPS

Referring to Figures 1 and 2, the following relationships will be of use in the derivations that follow.

- (i) $OE = OG = a$ and $ON = b$ are the semi-major and semi-minor axes of the ellipsoid
- (ii) PH is the ellipsoidal normal and $PP_E = h$. P_E is the projection of P on the ellipsoid and P_C is the projection of P_E on the auxiliary circle of the meridian ellipse
- (iii) $HP_E = \nu$ and $CP_E = \rho$ are the radii of curvature in the prime vertical and meridian planes of P respectively
- (iv) $OH = \nu e^2 \sin \phi$ and $DH = \nu e^2$
- (v) latitudes: $PDM = \phi$ (geodetic), $P_C OM = \psi$ (parametric), $P_E OM = \theta$ (geocentric)
- (vi) latitudes are related by: $\tan \psi = \frac{b}{a} \tan \phi$ and $\tan \theta = \frac{b}{a} \tan \psi = \frac{b^2}{a^2} \tan \phi$
- (vii) $PS = p = \sqrt{X^2 + Y^2}$ and $PT = Z$
- (viii) the ratio $P_C R / P_E R = a/b$ and $P_E Q = p_E = a \cos \psi$, $P_E R = Z_E = b \sin \psi$
- (ix) perpendicular distances from C (the centre of curvature of the meridian ellipse) to the Z - and OM -axes respectively are:

$$\left(\frac{a^2 - b^2}{a^2} \right) a \cos^3 \psi = e^2 a \cos^3 \psi \quad \text{and} \quad - \left(\frac{a^2 - b^2}{b^2} \right) b \sin^3 \psi = -e'^2 b \sin^3 \psi$$

The following relations between geometric constants of the ellipsoid are also of use.

$$f = \frac{a - b}{a} \quad \text{flattening} \quad (2.1)$$

$$b = a(1 - f) \quad \text{semi-minor axis} \quad (2.2)$$

$$e^2 = \frac{a^2 - b^2}{a^2} = f(2 - f) \quad \text{first eccentricity squared} \quad (2.3)$$

$$e'^2 = \frac{a^2 - b^2}{b^2} = \frac{e^2}{1 - e^2} \quad \text{second eccentricity squared} \quad (2.4)$$

SIMPLE ITERATION

This technique, described in various forms in geodesy texts, owes its popularity to its programming simplicity. The basis for the method is (1.8)

$$\tan \phi = \frac{Z + \nu e^2 \sin \phi}{p} \quad (1.8)$$

An approximate value ϕ_0 is used in the right hand side (RHS) of (1.8) to evaluate $\tan \phi_1$, (and hence ϕ_1) on the left hand side (LHS). This “new” value, ϕ_1 is then used in the RHS to give the next value, $\tan \phi_2$. This procedure is repeated until the difference between successive LHS values, $\tan \phi_i$, $\tan \phi_{i+1}$, reaches an acceptable limit; the iteration thus converges to a solution of $\tan \phi$.

The starting value ϕ_0 is obtained from $\tan \theta' = \frac{Z}{p}$, θ' being a good approximation of the geocentric latitude θ .

Hence, since $\tan \phi = \frac{a^2}{b^2} \tan \theta \approx \frac{a^2}{b^2} \tan \theta'$, using (2.4) gives

$$\tan \phi_0 \approx \frac{Z(1 + e'^2)}{p} \quad (3.1)$$

After evaluating ϕ the height h is calculated from (1.9).

PAUL'S METHOD

This method (Paul 1973) is direct in so far as $\tan \phi$ is obtained from a simple closed equation, but only after several intermediate variables have been evaluated. An outline of the development of the necessary equations is given below.

From the ellipsoid relationships

$$p \tan \phi - Z = \nu e^2 \sin \phi \quad (4.1)$$

After manipulations involving the equation for ν , a quartic equation in $\tan \phi$ is obtained

$$p^4 \tan^4 \phi - 2p^3 Z \tan^3 \phi + (Z^2 + \beta)p^2 \tan^2 \phi - \frac{2p^3 Z}{1-e^2} \tan \phi + \frac{p^2 Z^2}{1-e^2} = 0 \quad (4.2)$$

$$\text{with } \beta = \frac{p^2 - a^2 e^4}{1-e^2} \quad (4.3)$$

Now, in common with all direct methods, (4.2) is solved by successive substitutions which eventually yield a cubic equation in a subsidiary variable, for which standard solutions exist (Spiegel 1968, p. 32). Making the substitution

$$p \tan \phi = \zeta + \frac{Z}{2} \quad (4.4)$$

expressions for $p^2 \tan^2 \phi$, $p^3 \tan^3 \phi$ and $p^4 \tan^4 \phi$ are substituted into (4.2) giving

$$\zeta^4 + \left(\beta - \frac{Z^2}{2} \right) \zeta^2 - \alpha Z \zeta + Z^2 \left(\frac{\beta}{4} + \frac{Z^2}{16} \right) = 0 \quad (4.5)$$

$$\text{where } \alpha = \frac{p^2 + a^2 e^4}{1-e^2} \quad (4.6)$$

Assuming a solution of (4.5) is $\zeta = \sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3}$, t_1, t_2, t_3 being the roots of the cubic equation of the form $t^3 + a_1 t^2 + a_2 t + a_3 = 0$, expressions for ζ^2 and ζ^4 are substituted into (4.5), giving

$$\zeta^4 - 2(t_1 + t_2 + t_3)\zeta^2 - 8\sqrt{t_1 t_2 t_3} \zeta + \left\{ (t_1 + t_2 + t_3)^2 - 4(t_1 t_2 + t_1 t_3 + t_2 t_3) \right\} = 0 \quad (4.7)$$

Comparing coefficients of (4.5) and (4.7) gives expressions for $t_1 + t_2 + t_3$, $t_1 t_2 + t_1 t_3 + t_2 t_3$ and $t_1 t_2 t_3$ (the coefficients $-a_1$, a_2 and $-a_3$) respectively. Using the sums and products, and manipulating the expression for ζ gives a solution in terms of the single real root t_1

$$\zeta = \sqrt{t_1} + \sqrt{\frac{Z^2}{4} - \frac{\beta}{2} - t_1 + \frac{\alpha Z}{4\sqrt{t_1}}} \quad (4.8)$$

Substituting expressions for a_1 , a_2 and a_3 into the general cubic equation gives

$$t^3 + \left(\frac{\beta}{2} + \frac{Z^2}{4} \right) t^2 + \left(\frac{\beta^2}{16} - \frac{\beta Z^2}{8} \right) t - \frac{\alpha^2 Z^2}{64} = 0 \quad (4.9)$$

A further substitution

$$t = \left(\frac{\beta + Z^2}{6} \right) \rho + \frac{Z^2}{12} - \frac{\beta}{6} \quad (4.10)$$

enables (4.9) to be reduced to a standard form

$$4\rho^3 - 3\rho = q \quad (4.11)$$

with a single real root ρ_1 having the solution

$$\rho_1 = \frac{1}{2} \left\{ \left(q + \sqrt{q^2 - 1} \right)^{1/3} + \left(q - \sqrt{q^2 - 1} \right)^{-1/3} \right\} \quad (4.12)$$

$$\text{where } q = 1 + \frac{27Z^2(\alpha^2 - \beta^2)}{2(\beta + Z^2)^3} \quad (4.13)$$

Thus, having X, Y, Z (hence p) for a point related to an ellipsoid, $\tan \phi$ is obtained by computing the following variables in order: α from (4.6), β from (4.3), q from (4.13), ρ_1 from (4.12), t_1 from (4.10), ζ from (4.8) noting that all square-roots in (4.8) have the same sign as Z , and finally $\tan \phi$ from (4.4). [ρ_1 and t_1 are the real roots of (4.11) and (4.9) respectively]

In common with many direct solutions, the method fails when Z approaches zero, in which case q and ρ_1 both approach unity, t_1 becomes infinitely small and ζ is indeterminate. In such cases, Paul (1973, p. 136) gives an approximate equation for the calculation of $\tan \phi$.

OZONE'S METHOD

This method (Ozone 1985) is direct in so far as $\tan \phi$ is obtained from a simple closed equation, but only after several intermediate variables have been evaluated. An outline of the development of the necessary equations is given below.

From the ellipsoid relationships, where subscripts E refer to points on the ellipsoid

$$\tan \phi = \frac{Z - Z_E}{p - p_E} = \frac{a}{b} \tan \psi \quad (5.1)$$

which, with $p_E = a \cos \psi$ and $Z_E = b \sin \psi$ can be rearranged to give

$$\frac{ap}{\cos \psi} - \frac{bZ}{\sin \psi} = a^2 - b^2 \quad (5.2)$$

Making the substitution $\tan \frac{\psi}{2} = \frac{1}{u}$ and using trigonometric identities leads to

$$\cos \psi = \frac{u^2 - 1}{u^2 + 1}, \quad \sin \psi = \frac{2u}{u^2 + 1}, \quad \tan \psi = \frac{2u}{u^2 - 1} \quad (5.3)$$

and substituting expressions for $\cos \psi$ and $\sin \psi$ into (5.2) gives

$$\frac{ap(u^2 + 1)}{u^2 - 1} - \frac{bZ(u^2 + 1)}{2u} = a^2 - b^2$$

which can be simplified to give a quartic equation in u

$$u^4 - 4Mu^3 - 4Nu - 1 = 0 \quad (5.4)$$

$$\text{where } M = \frac{ap - (a^2 - b^2)}{2bZ} \quad (5.5)$$

$$N = \frac{ap + (a^2 - b^2)}{2bZ} \quad (5.6)$$

Introducing variables I, J, K enables (5.4) to be expressed in two forms

(i) the product of two quadratic equations

$$\left[u^2 - (2M - J)u + (I + J) \right] \left[u^2 - (2M + J)u + (I - J) \right] = 0 \quad (5.7)$$

(ii) the difference of two squares

$$(u^2 - 2Mu + I)^2 - (Ju + K)^2 = 0 \quad (5.8)$$

Both (5.7) and (5.8) give

$$u^4 - 4Mu^3 + (2I - J^2 + 4M^2)u^2 - (2KJ + 4MI)u + (I^2 - K^2) = 0 \quad (5.9)$$

Comparing coefficients of u and u^2 , and the constant terms of (5.4) and (5.9) gives

$$J^2 = 2I + 4M^2, \quad K^2 = I^2 + 1, \quad JK = 2(N - MI) \quad (5.10)$$

which can be rearranged to give a cubic equation in I

$$I^3 + VI - W = 0 \quad (5.11)$$

for which standard solutions exist for the real root of I (Spiegel 1968, p. 32)

$$I = \left(\sqrt{\left(\frac{V}{3}\right)^3 + \left(\frac{W}{2}\right)^2} + \frac{W}{2} \right)^{1/3} - \left(\sqrt{\left(\frac{V}{3}\right)^3 + \left(\frac{W}{2}\right)^2} - \frac{W}{2} \right)^{1/3} \quad (5.12)$$

$$\text{where } V = 4NM + 1 \quad (5.13)$$

$$W = 2(N^2 - M^2) \quad (5.14)$$

After solving for I , J and K follow from (5.10)

$$J = \sqrt{2I + 4M^2} \quad (5.15)$$

$$K = \frac{2(N - MI)}{J} \quad (5.16)$$

The parameter u can now be obtained from the real solution of (5.7), a quadratic equation in u for which the positive root is

$$u = \frac{2M + J + \sqrt{G}}{2} \quad (5.17)$$

where $G = (2M + J)^2 - 4(I - K)$ (5.18)

Now since $\tan \phi = \frac{a}{b} \tan \psi$ and from (5.3), where $\tan \psi = \frac{2u}{u^2 - 1}$ then

$$\tan \phi = \frac{2au}{b(u^2 - 1)} \quad (5.19)$$

Thus, having X, Y, Z (hence p) for a point related to an ellipsoid, $\tan \phi$ is obtained by computing the following variables in order: M and N from (5.5) and (5.6), noting that Z is taken as positive, V and W from (5.13) and (5.14), I from (5.12), J and K from (5.15) and (5.16), G from (5.18), then u from (5.17) and finally $\tan \phi$ from (5.19).

After calculating ϕ , its sign is determined by inspection of Z .

BOWRING'S METHOD

This method (Bowring 1976) is iterative, but owing to the nature of the equation used can be regarded as exact, since for all practical purposes, second or third iterations are not required (Bowring 1976, p.326).

From the ellipsoid relationships, perpendicular distances from C (the centre of curvature of the meridian ellipse) to the Z - and OM -axes are $e^2 a \cos^3 \psi$ and $-e'^2 b \sin^3 \psi$ respectively, thus an expression for $\tan \phi$ in terms of the parametric latitude ψ is

$$\tan \phi = \frac{Z + b e'^2 \sin^3 \psi}{p - a e^2 \cos^3 \psi} \quad (6.1)$$

This equation, having functions of ϕ on both sides of the equation, clearly requires an iterative solution for $\tan \phi$. An initial value ψ_0 is obtained from $\tan \theta' = \frac{Z}{p}$, θ' being a good approximation of the geocentric

latitude θ , and since $\tan \psi = \frac{a}{b} \tan \theta \approx \frac{a}{b} \tan \theta'$, then

$$\tan \psi_0 \approx \frac{aZ}{bp} \quad (6.2)$$

Bowring (1976) shows that for all Earth-bound points ($-5,000m \leq h \leq 10,000m$) the maximum error in ϕ , induced by using only a single iteration, is $0.000000030''$. Thus for all practical purposes, the evaluation of $\tan \phi$ by (6.1), with a first approximation of ψ from (6.2), can be regarded as exact.

BORKOWSKI'S METHOD

This is an indirect method using Newton's iterative technique to solve for the parametric latitude ψ (Borkowski 1989). An outline of the development of the necessary equations is given below.

From the ellipsoid relationships, the rectangular components of h in the meridian plane are given by

$$h \sin \phi = Z - b \sin \psi \quad (7.1)$$

$$h \cos \phi = p - a \cos \psi \quad (7.2)$$

Eliminating h by division and substituting for $\tan \phi$ gives

$$\frac{a}{b} \tan \psi = \frac{Z - b \sin \psi}{p - a \cos \psi} \quad (7.3)$$

which can be rearranged as

$$2ap \sin \psi - 2bZ \cos \psi - 2 \sin \psi \cos \psi (a^2 - b^2) = 0 \quad (7.4)$$

Replacing ap and bZ with $q \cos \Omega$ and $q \sin \Omega$ respectively, gives

$$2q(\cos \Omega \sin \psi - \sin \Omega \cos \psi) - 2 \sin \psi \cos \psi (a^2 - b^2) = 0 \quad (7.5)$$

where $\tan \Omega = \frac{bZ}{ap}$ (7.6)

$$q = \sqrt{(ap)^2 + (bZ)^2} \quad (7.7)$$

Simplifying (7.5) using trigonometric identities gives a function of ψ as

$$f(\psi) = 2 \sin(\psi - \Omega) - c \sin 2\psi = 0 \quad (7.8)$$

where $c = \frac{a^2 - b^2}{q} = \frac{a^2 - b^2}{\sqrt{(ap)^2 + (bZ)^2}}$ (7.9)

and ψ can be computed by Newton's iterative technique

$$\psi_{n+1} = \psi_n - \frac{f(\psi_n)}{f'(\psi_n)} \quad (7.10)$$

where the function and its derivative are

$$f(\psi_n) = 2 \sin(\psi_n - \Omega) - c \sin 2\psi_n \quad (7.11)$$

$$f'(\psi_n) = 2 \{ \cos(\psi_n - \Omega) - c \cos 2\psi_n \} \quad (7.12)$$

An initial approximation ψ_0 can be obtained from

$$\tan \psi_0 \approx \frac{aZ}{bp} \quad (7.13)$$

After solving for ψ , $\tan \phi$ is obtained from

$$\tan \phi = \frac{a}{b} \tan \psi \quad (7.14)$$

Thus, having X, Y, Z (hence p) for a point related to an ellipsoid, ψ is obtained by iteration after first calculating constants Ω , q and c from (7.6), (7.7) and (7.9) respectively, then using (7.10), with (7.11) and (7.12) calculated using successively improved values of ψ .

The standard method of computing h once ϕ is known is via (1.9); which unfortunately fails at the poles (division by zero). Borkowski (1989, p. 51, eq. 6) gives an alternative, obtained by multiplying (7.1) and (7.2) by $\sin \phi$ and $\cos \phi$ respectively and adding the resultants to give

$$h = (Z - b \sin \psi) \sin \phi + (p - a \cos \psi) \cos \phi \quad (7.15)$$

This formula, whilst more robust, has more trigonometric functions to evaluate, and thus is demanding of computer resources; (1.9) is more efficient in non-polar regions.

LIN AND WANG'S METHOD

This elegant method (Lin & Wang 1995) uses Newton's iterative technique to evaluate a scalar multiplier m of the unit vector \mathbf{n} , normal to the ellipsoid. Once obtained, simple relationships between Cartesian coordinates of P and its projection on the ellipsoid P_E are used to evaluate X_E, Y_E, Z_E giving the geocentric latitude θ directly and thus ϕ . An outline of the development of the necessary equations is given below.

The Cartesian equation of the ellipsoid of revolution is

$$\frac{X_E^2}{a^2} + \frac{Y_E^2}{a^2} + \frac{Z_E^2}{b^2} = 1 \quad (8.1)$$

which, when partially differentiated, gives the normal unit vector \mathbf{n}

$$\mathbf{n} = \frac{2X_E}{a^2} \mathbf{i} + \frac{2Y_E}{a^2} \mathbf{j} + \frac{2Z_E}{b^2} \mathbf{k} \quad (8.2)$$

The vector equation of h is

$$\mathbf{h} = (X - X_E) \mathbf{i} + (Y - Y_E) \mathbf{j} + (Z - Z_E) \mathbf{k} \quad (8.3)$$

h can also be given by

$$\mathbf{h} = m \mathbf{n}$$

where m is a scalar multiplier of the normal unit vector, hence \mathbf{h} is also given by

$$\mathbf{h} = \frac{2mX_E}{a^2} \mathbf{i} + \frac{2mY_E}{a^2} \mathbf{j} + \frac{2mZ_E}{b^2} \mathbf{k} \quad (8.4)$$

Equating the vector components of (8.3) and (8.4) and rearranging, gives

$$\frac{X_E}{a} = \frac{X}{a + \frac{2m}{a}}, \quad \frac{Y_E}{a} = \frac{Y}{a + \frac{2m}{a}}, \quad \frac{Z_E}{b} = \frac{Z}{b + \frac{2m}{b}} \quad (8.5)$$

which, when squared and substituted into (8.1), and simplified, gives a function of m as

$$f(m) = \frac{p^2}{\left(a + \frac{2m}{a}\right)^2} + \frac{Z^2}{\left(b + \frac{2m}{b}\right)^2} - 1 = 0 \quad (8.6)$$

and m can be obtained by Newton's iterative technique

$$m_{n+1} = m_n - \frac{f(m_n)}{f'(m_n)} \quad (8.7)$$

where the function and its derivative are

$$f(m_n) = \frac{p^2}{\left(a + \frac{2m_n}{a}\right)^2} + \frac{Z^2}{\left(b + \frac{2m_n}{b}\right)^2} - 1 \quad (8.8)$$

$$f'(m_n) = -4 \left\{ \frac{p^2}{a \left(a + \frac{2m_n}{a}\right)^3} + \frac{Z^2}{b \left(b + \frac{2m_n}{b}\right)^3} \right\} \quad (8.9)$$

An initial approximation m_0 can be obtained from (Lin & Wang 1995, p. 301)

$$m_0 = \frac{ab(a^2Z^2 + b^2p^2)^{3/2} - a^2b^2(a^2Z^2 + b^2p^2)}{2(a^4Z^2 + b^4p^2)} \quad (8.10)$$

After calculating m , p_E and Z_E are obtained from (8.5) as

$$p_E = \frac{p}{1 + \frac{2m}{a^2}} \quad (8.11)$$

$$Z_E = \frac{Z}{1 + \frac{2m}{b^2}} \quad (8.12)$$

and the latitude ϕ and height h obtained from

$$\tan \phi = \frac{a^2 Z_E}{b^2 p_E} \quad (8.13)$$

$$h = \pm \sqrt{(p - p_E)^2 + (Z - Z_E)^2} \quad (8.14)$$

Note: h is negative if $(p + |Z|)$ is less than $(p_E + |Z_E|)$

Thus, having X, Y, Z (hence p) for a point related to an ellipsoid, m is obtained by iteration after first calculating a starting value m_0 from (8.10) then using (8.7), with (8.8) and (8.9) calculated using successively improved values of m . Having calculated m the coordinates of P_E on the ellipsoid are obtained and ϕ follows from (8.13). The attraction of this method, compared to the other iterative techniques (*Simple Iteration*, *Bowring's* and *Borkowski's*) is that no trigonometric functions are used in the calculation of $\tan \phi$ or h .

COMPUTER TESTING OF THE TRANSFORMATION METHODS

The preference for one transformation method over another depends on the answer to three questions relevant to computer evaluation – (i) is it easy to code as a computer algorithm? (ii) is it accurate? and (iii) is it fast? In answer to the first two questions; all the methods are simple to code as computer algorithms, requiring similar numbers of lines of source code and each yielding accurate results, providing that *Paul's* and *Ozone's* direct methods are not used when $Z = 0$ or very small, in which cases they either fail or give unreliable results. The answer to the last question, often the determining factor in the adoption of a particular method, depends on two factors, (a) the number square roots, cube roots and trigonometric functions requiring evaluation and (b) the

number of iterations required for the indirect methods. With regard to these factors, *Lin & Wang's* method requires only a single evaluation of a square root to compute $\tan \phi$ and, if points are restricted to an Earth-bound region ($-5,000\text{ m} \leq h \leq 10,000\text{ m}$), the analysis shows that *Simple Iteration* is the only indirect method actually requiring multiple iterations to reach an acceptable accuracy for ϕ , the other indirect methods (*Bowring's*, *Borkowski's* and *Lin & Wang's*) giving, for all practical purposes, “exact” solutions for $\tan \phi$ without iteration.

In order to evaluate accuracy and relative speeds, each method was coded as a computer algorithm (using Borland's C++ version 5.0A compiler on a PC with a 133MHz Pentium processor) and tested over a grid of points at 0.1° intervals between latitudes 5°S and 50°S and longitudes 110°E and 160°E on the GRS80 ellipsoid, each assigned an ellipsoidal height $h = 10,000\text{ m}$ (225,450 points in total, covering the Australian continent, Tasmania and offshore islands). For each algorithm, the following procedure was adopted

- compute X, Y, Z coordinates for each point (ϕ, λ, h) using (1.1), (1.2) and (1.3) [these Cartesian and geographic values regarded as “exact”] then, using the X, Y, Z coordinates
- compute the longitude λ_C using (1.7)
- compute the latitude ϕ_C of each point using the relevant equations for each method
- compute the height h_C using (1.9) for each method, excepting *Lin* and *Wang's*, where (8.14) was used
- compare ϕ_C and h_C with the exact values to determine the magnitude of the “geographical errors”
- compute X, Y, Z again, using (1.1), (1.2) and (1.3) with computed values ϕ_C, λ_C, h_C and compare with exact values to determine the magnitude of the “Cartesian errors”.

Note: *Simple Iteration* was the only indirect method where iteration was used, the others, *Bowring's*, *Borkowski's* and *Lin & Wang's*, were used without iteration.

For each algorithm, the time required to perform the Cartesian-to-geographic transformations for all points was recorded (in processor “clock-ticks”) and converted to relative speed units, with *Bowring's* method given a value of 50. Results of the comparative tests are given in Table 1.

Conversion Method	Maximum Errors (magnitudes)					Relative Speed
	$\phi(\text{sec})$	$h(\text{m})$	$X(\text{m})$	$Y(\text{m})$	$Z(\text{m})$	
<i>Lin & Wang</i>	6.87e-11	2.53e-09	9.31e-10	9.31e-10	1.86e-09	42
<i>Bowring</i>	2.88e-08	1.01e-06	9.31e-10	9.31e-10	1.32e-06	50
<i>Paul</i>	3.90e-07	1.14e-06	9.31e-10	9.31e-10	1.21e-05	52
<i>Ozone</i>	6.87e-11	2.42e-09	9.31e-10	9.31e-10	2.33e-09	56
<i>Simple Iteration</i>	1.35e-05	6.12e-05	9.31e-10	9.31e-10	4.19e-04	62
<i>Borkowski</i>	2.88e-08	1.01e-06	9.31e-10	9.31e-10	1.32e-06	63

Table 1
Cartesian-to-geographic conversion algorithms ranked from
fastest (*Lin & Wang*) to slowest (*Borkowski*).

CONCLUSION AND RECOMMENDATION

This paper has presented the mathematical background to six Cartesian-to-geographic transformation techniques, two direct and four indirect. In theory, the indirect methods require iteration for an acceptable solution of ϕ , but in practice, by restricting the use of the techniques to Earth-bound regions ($-5,000\text{ m} \leq h \leq 10,000\text{ m}$), only the *Simple Iteration* technique actually requires iteration, the other indirect methods giving, for all practical purposes, “exact” solutions for $\tan \phi$ without iteration. All the methods were coded as computer algorithms, tested (on a region covering Australia) and found to give acceptable results for the computation of ϕ and h . *Lin & Wang's* method is appreciably faster than the other five methods and is recommended for use in Australia.

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