Seminar 1

Problems

1. Find the number of binary operations on a set with n elements.
2. Describe explicitly all binary operations on a set with 2 elements and verify their properties.
3. Give definitions of S_n and A_n: the sets, the operations, and check the properties.
4. Explicitly describe elements of S_3, A_3, S_4, A_4.
5. Prove that
   (a) S_n is commutative if and only if $n \leqslant 2$.
   (b) A_n is commutative if and only if $n \leqslant 3$.
6. Check properties of the operations
$$\min\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \qquad \max\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$
   (a) Are they associative?
   (b) Do they have a neutral element?
   (c) If they do, describe the inverse elements.
   (d) Are they commutative?
7. Explicitly describe $(\mathbb{Z}_n, +)$ as a group. Here, the set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ is the set of remainders modulo n and the operation is addition modulo n.
8. Solve equation $ax = b$ in $\mathbb{Z}_n$ for some given $a, b \in \mathbb{Z}_n$.
9. Explicitly describe $(\mathbb{Z}_p^*, \cdot)$ in case of a prime number p. Prove that this is a group.
10. Find orders of all elements in the group $(\mathbb{Z}_7^*, \cdot)$.
11. Prove that $(\mathrm{GL}_n(\mathbb{R}), \cdot)$ is a group.
12. Explain why $(\mathrm{M}_n(\mathbb{R}), \cdot)$ is not a group.
13. Prove that $(\mathrm{M}_n(\mathbb{R}), +)$ is a group.
14. Write down the multiplication table for the group $\mathbb{Z}_3$.
15. Let G be a group and $g \in G$ be an element of order m. Find the order of g^k for any $k \in \mathbb{Z}$.
16. Find all subgroups in the group $\mathbb{Z}_4$.
17. Find all subgroups in the groups S_3 and A_4.
18. Give and examples of and element of order 13 in the group $\mathrm{GL}_2(\mathbb{R})$.

Homework

1. Let $\circ\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $(x, y) \mapsto x + y - \frac{\sqrt{\pi}}{2}$. Check if $(\mathbb{R}, \circ)$ is a group or not.
2. Find all subgroups in $(\mathbb{Z}_7^*, \cdot)$.
3. Solve the equation $\tau^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ in S_4.
4. For each element of $(\mathbb{Z}_{13}^*, \cdot)$, find its order and the inverse element.