ABOUT THIS BOOK

Although their work is indispensable for an appreciation of much of science and philosophy, the great mathematicians are far less well-known than the great scientists and philosophers. Yet of them have had interesting lives in military affairs, statecraft, and other practical pursuits. On the personal side, mathematicians have been all sorts and conditions of men, poor and rich, honest and dishonest, open-handed and close-fisted, peaceable and quarrelsome, arrogant and humble – in fact, almost anything except stupid. Like some of the famous poets and musicians, most of the creative mathematicians have matured early; some accomplished lasting work before they were twenty. The following chapters present a fair sample of the lives of these extraordinarily gifted men. This volume includes: Zeno, Eudoxus, Archimedes, Descartes, Fermat, Pascal, Newton, Leibniz, the Bernoullis, Euler, Lagrange, Laplace, Monge, Fourier, Poncelet, Gauss, and Cauchy. Volume 2 continues the survey up to Poincaré and Cantor, who were famous figures in the early part of the twentieth century.

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MEN OF MATHEMATICS

E. T. BELL

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VOLUME ONE

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WITHOUT a mass of footnotes it would be impossible to cite authority for every statement of historical fact in the following pages. But little of the material consulted is available outside large university libraries, and most of it is in foreign languages. For the principal dates and leading facts in the life of a particular man I have consulted the obituary notices (of the moderns); these are found in the proceedings of the learned societies of which the man in question was a member. Other details of interest are given in the correspondence between mathematicians and in their collected works. In addition to the few specific sources cited presently, bibliographies and references in the following have been especially helpful.

(1) The numerous historical notes and papers abstracted in the *Jahrbuch über die Fortschritte der Mathematik* (section on history of mathematics).

(2) The same in *Bibliotheca Mathematica.*

Only three of the sources are sufficiently 'private' to need explicit citation. The life of Galois is based on the classic account by P. Dupuy in the *Annales scientifiques de l'École normale supérieure* (3^me^ série, tome 13, 1896), and the edited notes by Jules Tannéry. The correspondence between Weierstrass and Sonja Kowalewski was published by Mittag-Leffler in the *Acta Mathematica* (also partly in the *Comptes rendus du 2^me^ Congrès international des Mathématiciens*, Paris, 1902). Many of the details concerning Gauss are taken from the book by W. Sartorius von Waltershausen, *Gauss zum Gedächtniss*, Leipzig, 1856.

It would be rash to claim that every date or spelling of proper names in the book is correct. Dates are used chiefly with the purpose of orienting the reader as to a man's age when he made his most original inventions. As to spellings, I confess my helplessness in the face of such variants as Basle, Bâle, Basel for one

Swiss town, or Utzendorff, Uitzisdorf for another, each preferred by some admittedly reputable authority. When it comes to choosing between James and Johann, or between Wolfgang and Farkas, I take the easier way and identify the man otherwise.

As on a previous occasion (*The Search for Truth*), it gives me great pleasure to thank Doctor Edwin Hubble and his wife, Grace, for their invaluable assistance. While I alone am responsible for all statements in the book, nevertheless it was a great help to have scholarly criticism (even if I did not always profit by it) from two experts in fields in which I cannot claim to be expert, and I trust that their constructive criticisms have lightened my own deficiencies. Doctor Morgan Ward also has criticized certain of the chapters and has made many helpful suggestions on matters in which he is expert. Toby, as before, has contributed much; in acknowledgement for what she has given, I have dedicated the book to her (if she will have it) – it is as much hers as mine.

The drawings have been constructed accurately by Mr Eugene Edwards.

Last, I wish to thank the staffs of the various libraries which have generously helped with the loan of rare books and bibliographical material. In particular I should like to thank the librarians at Stanford University, the University of California, the University of Chicago, Harvard University, Brown University, Princeton University, Yale University, the John Crerar Library (Chicago), and the California Institute of Technology.

E. T. BELL

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MEN OF MATHEMATICS

VOLUME ONE

CHAPTER ONE

INTRODUCTION

THIS section is headed *Introduction* rather than *Preface* (which it really is) in the hope of decoying habitual preface-skippers into reading – for their own comfort – at least the following paragraphs down to the row of stars before going on to meet some of the great mathematicians. I should like to emphasize first that this book is not intended, in any sense, to be a history of mathematics, or any section of such a history.

The lives of mathematicians presented here are addressed to the general reader and to others who may wish to see what sort of human beings the men were who created *modern* mathematics. Our object is to lead up to some of the dominating ideas governing vast tracts of mathematics as it exists to-day and to do this through the lives of the men responsible for those ideas.

Two criteria have been applied in selecting names for inclusion: the importance for modern mathematics of a man's work; the human appeal of the man's life and character. Some qualify under both heads, for example Pascal, Abel, and Galois; others, like Gauss and Cayley, chiefly under the first, although both had interesting lives. When these criteria clash or overlap in the case of several claimants to remembrance for a particular advance, the second has been given precedence, as we are primarily interested here in mathematicians as human beings.

Of recent years there has been a tremendous surge of general interest in science, particularly physical science, and its bearing on our rapidly changing philosophical outlook on the universe. Numerous excellent accounts of current advances in science, written in as untechnical language as possible, have served to lessen the gap between the professional scientist and those who must make their livings at something other than science. In many of these expositions, especially those concerned with relativity and the modern quantum theory, names occur with which the general reader cannot be expected to be familiar – Gauss, Cayley, Riemann, and Hermite, for instance. With a knowledge of who these men were, their part in preparing for the explosive growth of physical science since 1900, and an appreciation of their rich personalities, the magnificent achievements of science fall into a truer perspective and take on a new significance.

The great mathematicians have played a part in the evolution of scientific and philosophic thought comparable to that of the philosophers and scientists themselves. To portray the leading features of that part through the lives of master mathematicians, presented against a background of some of the dominant problems of their times, is the purpose of the following chapters. The emphasis is wholly on modern mathematics, that is, on those great and simple guiding ideas of mathematical thought that are still of vital importance in living, creative science and mathematics.

It must not be imagined that the sole function of mathematics – 'the handmaiden of the sciences' – is to serve science. Mathematics has also been called 'the Queen of the Sciences.' If occasionally the Queen has seemed to beg from the sciences she has been a very proud sort of beggar, neither asking nor accepting favours from any of her more affluent sister sciences. What she gets she pays for. Mathematics has a light and wisdom of its own, above any possible application to science, and it will richly reward any intelligent human being to catch a glimpse of what mathematics means to itself. This is not the old doctrine of art for art's sake; it is art for humanity's sake. After all, the whole purpose of science is not technology – God knows we have gadgets enough already; science also explores depths of a universe that will never, by any stretch of the imagination, be visited by human beings or affect our material existence. So we shall attend also to some of the things which the great mathematicians have considered worthy of loving understanding for their intrinsic beauty.

Plato is said to have inscribed 'Let no man ignorant of geometry enter here' above the entrance to his Academy. No similar warning need be posted here, but a word of advice may save some over-conscientious reader unnecessary anguish. The gist of the story is in the lives and personalities of the creators of modern mathematics, not in the handful of formulas and diagrams scattered through the text. The basic ideas of modern mathematics, from which the whole vast and intricate complexity has been woven by thousands of workers, are simple, of boundless scope, and well within the understanding of any human being with normal intelligence. Lagrange (whom we shall meet later) believed that a mathematician has not thoroughly understood his own work till he has made it so clear that he can go out and explain it effectively to the first man he meets on the street.

This of course is an ideal and not always attainable. But it may be recalled that only a few years before Lagrange said this the Newtonian 'law' of gravitation was an incomprehensible mystery to even highly educated persons. Yesterday the Newtonian 'law' was a commonplace which every educated person accepted as simple and true; to-day Einstein's relativistic theory of gravitation is where Newton's 'law' was in the early decades of the eighteenth century; to-morrow or the day after Einstein's theory will seem as 'natural' as Newton's 'law' seemed yesterday. With the help of time Lagrange's ideal is not unattainable.

Another great French mathematician, conscious of his own difficulties no less than his readers', counselled the conscientious not to linger too long over anything hard but to 'Go on, and faith will come to you.' In brief, if occasionally a formula, a diagram, or a paragraph seems too technical, skip it. There is ample in what remains.

Students of mathematics are familiar with the phenomenon of 'slow development', or subconscious assimilation: the first time something new is studied the details seem too numerous and hopelessly confused, and no coherent impression of the whole is left on the mind. Then, on returning after a rest, it is found that everything has fallen into place with its proper emphasis – like the development of a photographic film. The majority of those who attack analytical geometry seriously for the first time experience something of the sort. The calculus on the other hand, with its aims clearly stated from the beginning, is usually grasped quickly. Even professional mathematicians often skim the work of others to gain a broad, comprehensive view of the whole before concentrating on the details of interest to them. Skipping is not a vice, as some of us were told by our puritan teachers, but a virtue of common sense.

As to the amount of mathematical knowledge necessary to understand *everything* that some will wisely skip, I believe it may be said honestly that a high school course in mathematics is sufficient. Matters far beyond such a course are frequently mentioned, but wherever they are, enough description has been given to enable anyone with high school mathematics to follow. For some of the most important ideas discussed in connexion with their originators – groups, space of many dimensions, non-Euclidean geometry, and symbolic logic, for example – *less* than a high school course is ample for an understanding of the basic concepts. All that is needed is interest and an undistracted head. Assimilation of some of these invigorating ideas of modern mathematical thought will be found as refreshing as a drink of cold water on a hot day and as inspiring as any art.

To facilitate the reading, important definitions have been repeated where necessary, and frequent references to earlier chapters have been included from time to time.

The chapters need not be read consecutively. In fact, those with a speculative or philosophical turn of mind may prefer to read the last chapter first. With a few trivial displacements to fit the social background the chapters follow the chronological order.

It would be impossible to describe *all* the work of even the least prolific of the men considered, nor would it be profitable in an account for the general reader to attempt to do so.

Moreover, much of the work of even the greater mathematicians of the past is now of only historical interest, having been included in more general points of view. Accordingly only some of the conspicuously new things each man did are described, and these have been selected for their originality and importance in modern thought.

Of the topics selected for description we may mention the following (among others) as likely to interest the general reader: the modern doctrine of the infinite (chapters 2, 29); the origin of mathematical probability (chapter 5); the concept and importance of a group (chapter 15); the meanings of invariance (chapter 21); non-Euclidean geometry (chapter 16 and part of 14); the origin of the mathematics of general relativity (last part of chapter 26); properties of the common whole numbers (chapter 4), and their modern generalization (chapter 25); the meaning and usefulness of so-called imaginary numbers – like $\sqrt{-1}$ (chapters 14, 19); symbolic reasoning (chapter 23). But anyone who wishes to get a glimpse of the power of the mathematical method, especially as applied to science, will be repaid by seeing what the calculus is about (chapters 2, 6).

Modern mathematics began with two great advances, analytical geometry and the calculus. The former took definite shape in 1637, the latter about 1666, although it did not become public property till a decade later. Though the idea behind it all is childishly simple, yet the method of analytical geometry is so powerful that very ordinary boys of seventeen can use it to prove results which would have baffled the greatest of the Greek geometers – Euclid, Archimedes, and Apollonius. The man, Descartes, who finally crystallized this great method, had a particularly full and interesting life.

In saying that Descartes was responsible for the creation of analytical geometry we do not mean to imply that the new method sprang full-armed from his mind alone. Many before him had made significant advances toward the new method, but it remained for Descartes to take the final step and actually to put out the method as a definitely workable engine of geometrical proof, discovery, and invention. But even Descartes must share the honour with Fermat.

Similar remarks apply to most of the other advances of modern mathematics. A new concept may be 'in the air' for generations until some one man – occasionally two or three together – sees clearly the essential detail that his predecessors missed, and the new thing comes into being. Relativity, for example, is sometimes said to have been the great invention reserved by time for the genius of Minkowski. The fact is, however, that Minkowski did not create the theory of relativity and that Einstein did. It seems rather meaningless to say that So-and-so might have done this or that if circumstances had been other than they were. Any one of us no doubt could jump over the moon if we and the physical universe were different from what we and it are, but the truth is that we do not make the jump.

In other instances, however, the credit for some great advance is not always justly placed, and the man who first used a new method more powerfully than its inventor sometimes gets more than his due. This seems to be the case, for instance, in the highly important matter of the calculus. Archimedes had the fundamental notion of limiting sums from which the integral calculus springs, and he had not only had the notion but showed that he could apply it. Archimedes also used the method of the differential calculus in one of his problems. As we approach Newton and Leibniz in the seventeenth century the history of the calculus becomes extremely involved. The new method was more than merely 'in the air' before Newton and Leibniz brought it down to earth; Fermat actually had it. He also invented the method of Cartesian geometry independently of Descartes. In spite of indubitable facts such as these we shall follow tradition and ascribe to each great leader what a majority vote says he should have, even at the risk of giving him a little more than his just due. Priority after all gradually loses its irritating importance as we recede in time from the men to whom it was a hotly contested cause of verbal battles while they and their partisans lived.

Those who have never known a professional mathematician may be rather surprised on meeting some, for mathematicians as a class are probably less familiar to the general reader than any other group of brain workers. The mathematician is a much rarer character in fiction than his cousin the scientist, and when he does appear in the pages of a novel or on the screen he is only too apt to be a slovenly dreamer totally devoid of common sense – comic relief. What sort of mortal is he in real life? Only by seeing in detail what manner of men some of the *great* mathematicians were and what kind of lives they lived, can we recognize the ludicrous untruth of the traditional portrait of a mathematician.

Strange as it may seem, not all of the great mathematicians have been professors in colleges or universities. Quite a few were soldiers by profession; others went into mathematics from theology, the law, and medicine, and one of the greatest was as crooked a diplomat as ever lied for the good of his country. A few have had no profession at all. Stranger yet, not all professors of mathematics have been mathematicians. But this should not surprise us when we think of the gulf between the average professor of poetry drawing a comfortable salary and the poet starving to death in his garret.

The lives that follow will at least suggest that a mathematician can be as human as anybody else – sometimes distressingly more so. In ordinary social contacts the majority have been normal. There have been eccentrics in mathematics, of course; but the percentage is no higher than in commerce or the professions. As a group the great mathematicians have been men of all-round ability, vigorous, alert, keenly interested in many things outside mathematics and, in a fight, men with their full share of backbone. As a rule mathematicians have been bad customers to persecute; they have usually been capable of returning what they received with compound interest. For the rest they were geniuses of tremendous accomplishment marked off from the majority of their gifted fellowmen only by an irresistible impulse to do mathematics. On occasion mathematicians have been (and some still are in France) extremely able administrators.

In their politics the great mathematicians have ranged over the whole spectrum from reactionary conservatism to radical liberalism. It is probably correct to say that as a class they have tended slightly to the left in their political opinions. Their religious beliefs have included everything from the narrowest orthodoxy – sometimes shading into the blackest bigotry – to complete scepticism. A few were dogmatic and positive in their assertions concerning things about which they knew nothing, but most have tended to echo the great Lagrange's 'I do not know'.

Another characteristic calls for mention here, as several writers and artists (some from Hollywood) have asked that it be treated – the sex life of great mathematicians. In particular these inquirers wish to know how many of the great mathematicians have been perverts – a somewhat indelicate question, possibly, but legitimate enough to merit a serious answer in these times of preoccupation with such topics. None. Some lived celibate lives, usually on account of economic disabilities, but the majority were happily married and brought up their children in a civilized, intelligent manner. The children, it may be noted in passing, were often gifted far above the average. A few of the great mathematicians of bygone centuries kept mistresses when such was the fashionable custom of their times. The only mathematician discussed here whose life might offer something of interest to a Freudian is Pascal.

Returning for a moment to the movie ideal of a mathematician, we note that sloppy clothes have not been the invariable attire of great mathematicians. All through the long history of mathematics about which we have fairly detailed knowledge, mathematicians have paid the same amount of attention to their personal appearance as any other equally numerous group of men. Some have been fops, others slovens; the majority, decently inconspicuous. If to-day some earnest individual affecting spectacular clothes, long hair, a black sombrero, or any other mark of exhibitionism, assures you that he is a mathematician, you may safely wager that he is a psychologist turned numerologist.

The psychological peculiarities of great mathematicians are another topic in which there is considerable interest. Poincaré will tell us something about the psychology of mathematical creation in a later chapter. But on the general question not much can be said till psychologists call a truce and agree among themselves as to what is what. On the whole the great mathematicians have lived richer, more virile lives than those that fall to the lot of the ordinary hard-working mortal. Nor has this richness been wholly on the side of intellectual adventuresomeness. Several of the greater mathematicians have had more than their share of physical danger and excitement, and some of them have been implacable haters – or, what is ultimately the same, expert controversialists. Many have known the lust of battle in their prime, reprehensibly enough, no doubt, but still humanly enough, and in knowing it they have experienced something no jellyfish has ever felt: 'Damn braces, Bless relaxes', as that devout Christian William Blake put it in his *Proverbs of Hell.*

This brings us to what at first sight (from the conduct of several of the men considered here) may seem like a significant trait of mathematicians – their hair-trigger quarrelsomeness. Following the lives of several of these men we get the impression that a great mathematician is more likely than not to think others are stealing his work, or disparaging it, or not doing him sufficient honour, and to start a row to recover imaginary rights. Men who should have been above such brawls seem to have gone out of their way to court battles over priority in discovery and to accuse their competitors of plagiarism. We shall see enough dishonesty to discount the superstition that the pursuit of truth necessarily makes a man truthful, but we shall not find indubitable evidence that mathematics makes a man bad-tempered and quarrelsome.

Another 'psychological' detail of a similar sort is more disturbing. Envy is carried up to a higher level. Narrow nationalism and international jealousies, even in impersonal pure mathematics, have marred the history of discovery and invention to such an extent that it is almost impossible in some important instances to get at the facts or to form a just estimate of the significance of a particular man's work for modern thought. Racial fanaticism – especially in recent times – has also complicated the task of anyone who may attempt to give an unbiased account of the lives and work of scientific men outside his own race or nation.

An impartial account of western mathematics, including the award to each man and to each nation of its just share in the intricate development, could be written only by a Chinese historian. He alone would have the patience and the detached cynicism necessary for disentangling the curiously perverted pattern to discover whatever truth may be concealed in our variegated occidental boasting.

Even in restricting our attention to the modern phase of mathematics we are faced with a problem of selection that must be solved somehow. Before the solution adopted here is indicated it will be of interest to estimate the amount of labour that would be required for a detailed history of mathematics on a scale similar to that of a political history for any important epoch, say that of the French Revolution or the American Civil War.

When we begin unravelling a particular thread in the history of mathematics we soon get a discouraged feeling that mathematics itself is like a vast necropolis to which constant additions are being made for the eternal preservation of the newly dead. The recent arrivals, like some of the few who were shelved for perpetual remembrance 5,000 years ago, must be so displayed that they shall seem to retain the full vigour of the manhood in which they died; in fact the illusion must be created that they have not yet ceased living. And the deception must be so natural that even the most sceptical archaeologist prowling through the mausoleums shall be moved to exclaim with living mathematicians themselves that mathematical truths are immortal, imperishable; the same yesterday, to-day, and forever; the very stuff of which eternal verities are fashioned and the one glimpse of changelessness behind all the recurrent cycles of birth, death, and decay our race has ever caught. Such may indeed be the fact; many, especially those of the older generation of mathematicians, hold it to be no less.

But the mere spectator of mathematical history is soon over whelmed by the appalling mass of mathematical inventions that still maintain their vitality and importance for modern work, as discoveries of the past in any other field of scientific endeavour do not, after centuries and tens of centuries.

A span of less than a hundred years covers everything of significance in the French Revolution or the American Civil War, and less than five hundred leaders in either played parts sufficiently memorable to merit recording. But the army of those who have made at least one definite contribution to mathematics as we know it soon becomes a mob as we look back over history; 6,000 or 8,000 names press forward for some word from us to preserve them from oblivion, and once the bolder leaders have been recognized it becomes largely a matter of arbitrary, illogical legislation to judge who of the clamouring multitude shall be permitted to survive and who be condemned to be forgotten.

This problem scarcely presents itself in describing the development of the physical sciences. They also reach far back into antiquity; yet for the most of them 350 years is a sufficient span to cover everything of importance to modern thought. But whoever attempts to do full, human justice to mathematics and mathematicians will have a wilderness of 6,000 years in which to exercise such talents as he may have, with that mob of 6,000 to 8,000 claimants before him for discrimination and attempted justice.

The problem becomes more desperate as we approach our own times. This is by no means due to our closer proximity to the men of the two centuries immediately preceding our own, but to the universally acknowledged fact (among professional mathematicians) that the nineteenth century, prolonged into the twentieth, was, and is, the greatest age of mathematics the world has ever known. Compared to what glorious Greece did in mathematics the nineteenth century is a bonfire beside a penny candle.

What threads shall we follow to guide us through this labyrinth of mathematical inventions? The main thread has already been indicated: that which leads from the half-forgotten past to some of those dominating concepts which now govern boundless empires of mathematics – but which may themselves be dethroned to-morrow to make room for yet vaster generalizations. Following this main thread we shall pass by the *developers* in favour of the *originators*.

Both inventors and perfectors are necessary to the progress of any science. Every explorer must have, in addition to his scouts, his followers to inform the world as to what he has discovered. But to the majority of human beings, whether justly or not is beside the point, the explorer who first shows the new way is the more arresting personality, even if he himself stumbles forward but half a step. We shall follow the originators in preference to the developers. Fortunately for historical justice the majority of the great originators in mathematics have also been peerless developers.

Even with this restriction the path from the past to the present may not always be clear to those who have not already followed it. So we may state here briefly what the main guiding clue through the whole history of mathematics is.

From the earliest times two opposing tendencies, sometimes helping one another, have governed the whole involved development of mathematics. Roughly these are the *discrete* and the *continuous*.

The discrete struggles to describe all nature and all mathematics atomistically, in terms of distinct, recognizable individual elements, like the bricks in a wall, or the numbers 1,2,3, ... The continuous seeks to apprehend natural phenomena – the course of a planet in its orbit, the flow of a current of electricity, the rise and fall of the tides, and a multitude of other appearances which delude us into believing that we know nature – in the mystical formula of Heraclitus: 'All things flow'. To-day (as will be seen in the concluding chapter), 'flow', or its equivalent, 'continuity', is so unclear as to be almost devoid of meaning. However, let this pass for the moment.

Intuitively we *feel* that we *know* what is meant by 'continuous motion' – as of a bird or a bullet through the air, or the fall of a raindrop. The motion is *smooth*; it *does not proceed by jerks*; it is *unbroken*. In *continuous* motion or, more generally, in the concept of continuity itself, the *individualized* numbers 1,2,3, ..., are *not* the appropriate mathematical image. *All* the points on a segment of a straight line, for instance, have no such clear-cut individualities as have the numbers of the sequence 1,2,3, . . ., where *the step from one member of the sequence to the next is the same* (namely $1 : 1 + 2 = 3, 1 + 3 = 4$, and so on); for *between* any two points on the line segment, no matter how close together the points may be, we can always *find*, or at least *imagine*, *another* point: *there is no 'shortest' step from one point to the 'next'*. In fact there is no *next* point at all.

The last – the conception of *continuity*, 'no nextness' – when developed in the manner of Newton, Leibniz, and their successors leads out into the boundless domain of *the calculus* and its innumerable applications to science and technology, and to all that is to-day called *mathematical analysis*. The other, the *discrete* pattern based on 1,2,3, . . . , is the domain of algebra, the theory of numbers, and symbolic logic. Geometry partakes of both the continuous and the discrete.

A major task of mathematics to-day is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.

It may be doing our predecessors an injustice to emphasize modern mathematical thought with but little reference to the pioneers who took the first and possibly the most difficult steps. But nearly everything useful that was done in mathematics before the seventeenth century has suffered one of two fates: either it has been so greatly simplified that it is now part of every regular school course, or it was long since absorbed as a detail in work of greater generality.

Things that now seem as simple as common sense – our way of writing numbers, for instance, with its 'place system' of value and the introduction of a symbol for zero, which put the essential finishing touch to the place system – cost incredible labour to invent. Even simpler things, containing the very essence of mathematical thought – *abstractness and generality*, must have cost centuries of struggle to devise; yet their originators have vanished leaving not a trace of their lives and personalities. For example, as Bertrand Russell observed, 'It must have taken many ages to discover that a brace of pheasants and a couple of days were both instances of the number two.' And it took some twenty-five centuries of *civilization* to evolve Russell's own logical definition of 'two' or of any cardinal number (reported in the concluding chapter of Volume Two).

Again, the conception of a point, which we (erroneously) think we fully understand when we begin school geometry, must have come very late in man's career as an artistic, cave-painting animal. Horace Lamb, an English mathematical physicist, would 'erect a monument to the unknown mathematical inventor of the mathematical point as the supreme type of that abstraction which has been a necessary condition of scientific work from the beginning.'

Who, by the way, *did* invent the mathematical point? In one sense Lamb's forgotten man; in another, Euclid with his definition 'a point is that which has no parts and which has no magnitude'; in yet a third sense Descartes with his invention of the 'co-ordinates of a point'; until finally in geometry as experts practise it to-day the mysterious 'point' has joined the forgotten man and all his gods in everlasting oblivion, to be replaced by something more usable – *a set of numbers written in a definite order.*

The last is a modern instance of the abstractness and precision toward which mathematics strives constantly, only to realize when abstractness and precision are attained that a higher degree of abstractness and a sharper precision are demanded for clear understanding. Our own conception of a 'point' will no doubt evolve into something yet more abstract. Indeed the 'numbers' in terms of which points are described to-day dissolved about the beginning of this century into the shimmering blue of pure logic, which in its turn seems about to vanish in something rarer and even less substantial.

It is not necessarily true then that a step-by-step following of our predecessors is the sure way to understand either their conception of mathematics or our own. Such a retracing of the path that has led up to our present outlook would undoubtedly be of great interest in itself. But it is quicker to glance back over the terrain from the hilltop on which we now stand. The false steps, the crooked trails, and the roads that led nowhere fade out in the distance, and only the broad highways are seen leading straight back to the past, where we lose them in the mists of uncertainty and conjecture. Neither space nor number, nor even time, have the same significance for us that they had for the men whose great figures appear dimly through the mist.

A Pythagorean of the sixth century before Christ could intone 'Bless us, divine Number, thou who generatest gods and men'; a Kantian of the nineteenth century could refer confidently to 'space' as a form of 'pure intuition'; a mathematical astronomer could announce a decade ago that the Great Architect of the Universe is a pure mathematician. The most remarkable thing about all of these profound utterances is that human beings no stupider than ourselves once thought they made sense.

To a modern mathematician such all embracing generalities mean less than nothing. Yet in parting with its claim to be the universal generator of gods and men mathematics has gained something more substantial, a faith in itself and in its ability to create human values.

Our point of view has changed – and is still changing. To Descartes' 'Give me space and motion and I will give you a world,' Einstein to-day might retort that altogether too much is being asked, and that the demand is in fact meaningless: without a 'world' – matter – there is neither 'space' nor 'motion'. And to quell the turbulent, muddled mysticism of Leibniz in the seventeenth century, over the mysterious $\sqrt{-1}$: 'The Divine Spirit found a sublime outlet in that wonder of analysis, the portent of the ideal, that mean between being and not-being, which we call the imaginary [square] root of negative unity', Hamilton in the 1840's constructed a number-couple which any intelligent child can understand and manipulate, and which does for mathematics and science all that the misnamed 'imaginary' ever did. The mystical 'not-being' of the seventeenth century Leibniz is seen to have a 'being' as simple as ABC.

Is this a loss? Or does a modern mathematician lose anything of value when he seeks through the postulational method to track down that elusive 'feeling' described by Heinrich Hertz, the discoverer of wireless waves: 'One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them'?

Any competent mathematician will understand Hertz's feeling, but he will also incline to the belief that whereas continents and wireless waves are discovered, dynamos and mathematics are invented and do what we make them do. We can still dream but we need not deliberately court nightmares. If it is true, as Charles Darwin asserted, that 'Mathematics seems to endow one with something like a new sense', that sense is the sublimated common sense which the physicist and engineer Lord Kelvin declared mathematics to be.

Is it not closer to our own habits of thought to agree temporarily with Galileo that 'Nature's great book is written in mathematical symbols' and let it go at that, than to assert with Plato that 'God ever geometrizes', or with Jacobi that 'God ever arithmetizes'? If we care to inspect the symbols in nature's great book through the critical eyes of modern science we soon perceive that we ourselves did the writing, and that we used the particular script we did because we invented it to fit our own understanding. Some day we may find a more expressive shorthand than mathematics for correlating our experiences of the physical universe – unless we accept the creed of the scientific mystics that everything *is* mathematics and is not merely *described* for our convenience in mathematical language. *If* 'Number rules the universe' as Pythagoras asserted, Number is merely our delegate to the throne, for we rule Number.

When a modern mathematician turns aside for a moment from his symbols to communicate to others the feeling that mathematics inspires in him, he does not echo Pythagoras and Jeans, but he may quote what Bertrand Russell said about a quarter of a century ago: 'Mathematics, rightly viewed, possesses not only truth but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.'

Another, familiar with what has happened to our conception of mathematical 'truth' in the years since Russell praised the beauty of mathematics, might refer to the 'iron endurance' which some acquire from their attempt to understand what mathematics means. And if some devotee is reproached for spending his life on what to many may seem the selfish pursuit of a beauty having no immediate reflection in the lives of his fellow men, he may repeat Poincaré's 'Mathematics for mathematics' sake. People have been shocked by this formula and yet it is as good as life for life's sake, if life is but misery.'

To form an estimate of what modern mathematics compared to ancient has accomplished, we may first look at the mere bulk of the work in the period after 1800 compared to that before 1800. The most extensive history of mathematics is that of Moritz Cantor, *Geschichte der Mathematik*, in three large closely printed volumes (a fourth, by collaborators, supplements the three). The four volumes total about 3,600 pages. Only the outline of the development is given by Cantor; there is no attempt to go into details concerning the contributions described, nor are technical terms explained so that an outsider could understand what the whole story is about, and biography is cut to the bone; the history is addressed to those who have some technical training. This history *ends with the year* 1799 – just before modern mathematics began to feel its freedom. What if the *outline* history of mathematics in the nineteenth century alone were attempted on a similar scale? It has been estimated that nineteen or twenty volumes the size of Cantor's would be required to tell the story, say about 17,000 pages. The nineteenth century, on this scale, contributed to mathematical knowledge about *five times as much* as was done in the whole of preceding history.

The beginningless period before 1800 breaks quite sharply into two. The break occurs about the year 1700, and is due mainly to Isaac Newton (1642–1727). Newton's greatest rival in mathematics was Leibniz (1646–1716). According to Leibniz, of all mathematics up to the time of Newton, the more important half is due to Newton. This estimate refers to the power of Newton's general methods rather than to the bulk of his work; the *Principia* is still rated as the most massive addition to scientific thought ever made by one man.

Continuing back into time beyond 1700 we find nothing comparable till we reach the Golden Age of Greece – a step of nearly 2,000 years. Farther back than 600 B.C. we quickly pass into the shadows, coming out into the light again for a moment in ancient Egypt. Finally we arrive at the first great age of mathematics, about 2000 B.C., in the Euphrates Valley.

The descendants of the Sumerians in Babylon appear to have been the first 'moderns' in mathematics; certainly their attack on algebraic equations is more in the spirit of the algebra we know than anything done by the Greeks in their Golden Age. More important than the technical algebra of these ancient Babylonians is their recognition – as shown by their work – of the necessity for *proof* in mathematics. Until recently it had been supposed that the Greeks were the first to recognize that proof is demanded for mathematical propositions. This was one of the most important steps ever taken by human beings. Unfortunately it was taken so long ago that it led nowhere in particular so far as our own civilization is concerned – unless the Greeks followed consciously, which they may well have done. They were not particularly generous to their predecessors.

Mathematics then has had four great ages: the Babylonian, the Greek, the Newtonian (to give the period around 1700 a name), and the recent, beginning about 1800 and continuing to the present day. Competent judges have called the last the Golden Age of Mathematics.

To-day mathematical invention (discovery, if you prefer) is going forward more vigorously than ever. The only thing, apparently, that can stop its progress is a general collapse of what we have been pleased to call civilization. If that comes, mathematics may go underground for centuries, as it did after the decline of Babylon; but if history repeats itself, as it is said to do, we may count on the spring bursting forth again, fresher and clearer than ever, long after we and all our stupidities shall have been forgotten.

CHAPTER TWO

MODERN MINDS IN ANCIENT BODIES

Zeno, Eudoxus, Archimedes

To appreciate our own Golden Age of mathematics we shall do well to have in mind a few of the great, simple guiding ideas of those whose genius prepared the way for us long ago, and we shall glance at the lives and works of three Greeks: Zeno (495–435 B.C.), Eudoxus (408–355), and Archimedes (287–212). Euclid will be noticed much later, where his best work comes into its own.

Zeno and Eudoxus are representative of two vigorous opposing schools of mathematical thought which flourish to-day, the critical-destructive and the critical-constructive. Both had minds as penetratingly critical as their successors in the nineteenth and twentieth centuries. This statement can of course be inverted: Kronecker (1823–91) and Brouwer (1881—), the modern critics of mathematical analysis – the theories of the infinite and the continuous – are as ancient as Zeno; the creators of the modern theories of continuity and the infinite, Weierstrass (1815–97), Dedekind (1831–1916), and Cantor (1845–1918), are intellectual contemporaries of Eudoxus.

Archimedes, the greatest intellect of antiquity, is modern to the core. He and Newton would have understood one another perfectly, and it is just possible that Archimedes, could he come to life long enough to take a post-graduate course in mathematics and physics, would understand Einstein, Bohr, Heisenberg, and Dirac better than they understand themselves. Of all the ancients Archimedes is the only one who habitually thought with the unfettered freedom that the greater mathematicians permit themselves to-day with all the hard-won gains of twenty-five centuries to smooth their way, for he alone of all the Greeks had sufficient stature and strength to stride clear over the obstacles thrown in the path of mathematical progress by frightened geometers who had listened to the philosophers.

Any list of the three 'greatest' mathematicians of all history would include the name of Archimedes. The other two usually associated with him are Newton (1642–1727) and Gauss (1777–1855). Some, considering the relative wealth – or poverty – of mathematics and physical science in the respective ages in which these giants lived, and estimating their achievements against the background of their times, would put Archimedes first. Had the Greek mathematicians and scientists followed Archimedes rather than Euclid, Plato, and Aristotle, they might easily have anticipated the age of modern mathematics, which began with Descartes (1596–1650) and Newton in the seventeenth century, and the age of modern physical science inaugurated by Galileo (1564–1642) in the same century, by 2,000 years.

Behind all three of these precursors of the modern age looms the half-mythical figure of Pythagoras (569?–500? B.C.), mystic, mathematician, investigator of nature to the best of his self-hobbled ability, 'one-tenth of him genius, nine-tenths sheer fudge'. His life has become a fable, rich with the incredible accretions of his prodigies; but only this much is of importance for the development of mathematics as distinguished from the bizarre number-mysticism in which he clothed his cosmic speculations: he travelled extensively in Egypt, learned much from the priests and believed more; visited Babylon and repeated his Egyptian experiences; founded a secret Brotherhood for high mathematical thinking and nonsensical physical, mental, moral, and ethical speculation at Croton in southern Italy; and, out of all this, made two of the greatest contributions to mathematics in its entire history. He died, according to one legend, in the flames of his own school fired by political and religious bigots who stirred up the masses to protest against the enlightenment which Pythagoras sought to bring them. *Sic transit gloria mundi.*

Before Pythagoras it had not been clearly realized that *proof* must proceed from *assumptions*. Pythagoras, according to persistent tradition, was the first European to insist that the *axioms*, the *postulates*, be set down first in developing geometry and that the entire development thereafter shall proceed by applications of close deductive reasoning to the axioms. Following current practice we shall use 'postulate', instead of 'axiom', hereafter, as 'axiom' has a pernicious historical association of 'self-evident, necessary truth' which 'postulate' does not have; a postulate is an arbitrary assumption laid down by the mathematician himself and not by God Almighty.

Pythagoras then imported *proof* into mathematics. This is his greatest achievement. Before him geometry had been largely a collection of rules of thumb empirically arrived at without any clear indication of the mutual connexions of the rules, and without the slightest suspicion that all were deducible from a comparatively small number of postulates. Proof is now so commonly taken for granted as the very spirit of mathematics that we find it difficult to imagine the primitive thing which must have preceded mathematical reasoning.

Pythagoras' second outstanding mathematical contribution brings us abreast of living problems. This was the discovery, which humiliated and devastated him, that the common whole numbers 1,2,3, ... are insufficient for the construction of mathematics even in the rudimentary form in which he knew it. Before this capital discovery he had preached like an inspired prophet that all nature, the entire universe in fact, physical, metaphysical, mental, moral, mathematical – *everything* – is built on the *discrete* pattern of the integers 1,2,3, ... and is interpretable in terms of these God-given bricks alone; God, he declared indeed, *is* 'number', and by that he meant common whole number. A sublime conception, no doubt, and beautifully simple, but as unworkable as its echo in Plato – 'God ever geometrizes', or in Jacobi – 'God ever arithmetizes', or in Jeans – 'The Great Architect of the Universe now begins to appear as a mathematician.' One obstinate mathematical discrepancy demolished Pythagoras' discrete philosophy, mathematics, and metaphysics. But, unlike some of his successors, he finally accepted defeat – after struggling unsuccessfully to suppress the discovery which abolished his creed.

This was what knocked his theory flat: it is impossible to find two whole numbers such that the square of one of them is equal to twice the square of the other. This can be proved by a simple argument* within the reach of anyone who has had a few weeks of algebra, or even by anyone who thoroughly understands elementary arithmetic. Actually Pythagoras found his stumbling-block in geometry: the ratio of the side of a square to one of its diagonals cannot be expressed as the ratio of any two whole numbers. This is equivalent to the statement above about squares of whole numbers. In another form we would say that the square root of 2 is *irrational*, that is, is not equal to any whole number or decimal fraction, or sum of the two, got by dividing one whole number by another. Thus even so simple a geometrical concept as that of the diagonal of a square defies the integers 1,2,3, ... and negates the earlier Pythagorean philosophy. We can easily construct the diagonal *geometrically, but we cannot measure it in any finite number of steps.* This impossibility sharply and clearly brought irrational numbers and the infinite (non-terminating) processes which they seem to imply to the attention of mathematicians. Thus the square root of two can be calculated to any required *finite* number of decimal places by the process taught in school or by more powerful methods, but the decimal never 'repeats' (as that for 1/7 does, for instance), nor does it ever terminate. In this discovery Pythagoras found the taproot of modern mathematical analysis.

Issues were raised by this simple problem which are not yet disposed of in a manner satisfactory to all mathematicians. These concern the mathematical concepts of the infinite (the unending, the uncountable), limits, and continuity, concepts which are at the root of modern analysis. Time after time the paradoxes and sophisms which crept into mathematics with these apparently indispensable concepts have been regarded as

* Let $a^2 = 2b^2$, where, without loss of generality, a, b are whole numbers without any common factor greater than 1 (such a factor could be cancelled from the assumed equation). If a is *odd*, we have an immediate contradiction, since $2b^2$ is *even*; if a is *even*, say $2c$, then $4c^2 = 2b^2$, or $2c^2 = b^2$, so b is *even*, and hence a, b have the common factor 2, again a contradiction.

finally eliminated, only to reappear a generation or two later, changed but yet the same. We shall come across them, livelier than ever, in the mathematics of our time. The following is an extremely simple, intuitively obvious picture of the situation.

1/3 2/3 1 1/3

1/4 1 1/2

Consider a straight line two inches long, and imagine it to have been traced by the 'continuous' 'motion' of a 'point'. The words in quotes are those which conceal the difficulties. Without analyzing them we easily persuade ourselves that we picture what they signify. Now label the left-hand end of the line 0 and the right-hand end 2. Half-way between 0 and 2 we naturally put 1; half-way between 0 and 1 we put $\frac{1}{2}$; half-way between 0 and $\frac{1}{2}$ we put $\frac{1}{4}$, and so on. Similarly, between 1 and 2 we mark the place $1\frac{1}{2}$, between $1\frac{1}{2}$ and 2, the place $1\frac{3}{4}$, and so on. Having done this we may proceed in the same way to mark $\frac{1}{3}$, $\frac{2}{3}$, $1\frac{1}{3}$, $1\frac{2}{3}$, and then split each of the resulting segments into smaller equal segments. Finally, 'in imagination', we can conceive of this process having been carried out for *all* the common fractions and common mixed numbers which are greater than 0 and less than 2; the conceptual division-points give us *all the rational numbers between* 0 *and* 2. There are an infinity of them. Do they completely 'cover' the line? No. To what point does the square root of 2 correspond? No point, because this square root is not obtainable by dividing *any* whole number by another. But the square root of 2 is obviously a 'number' of some sort*; its representative point lies somewhere between 1·41 and 1·42, and we can cage it down as closely as we please. To cover the line completely we are forced to imagine or to invent infinitely more 'numbers' than the rationals. That is, if we accept the line as being *continuous*, and *postulate* that to each point of it corresponds one, and only one, 'real number'. The same kind of imagining can be carried on to the entire plane, and farther, but this is sufficient for the moment.

Simple problems such as these soon lead to very serious

* The inherent viciousness of such an assumption is obvious.

difficulties. With regard to these difficulties the Greeks were divided, just as we are, into two irreconcilable factions; one stopped dead in its mathematical tracks and refused to go on to analysis – the integral calculus, at which we shall glance when we come to it; the other attempted to overcome the difficulties and succeeded in convincing itself that it had done so. Those who stopped committed but few mistakes and were comparatively sterile of truth no less than of error; those who went on discovered much of the highest interest to mathematics and rational thought in general, some of which may be open to destructive criticism, however, precisely as has happened in our own generation. From the earliest times we meet these two distinct and antagonistic types of mind: the justifiably cautious who hang back because the ground quakes under their feet, and the bolder pioneers who leap the chasm to find treasure and comparative safety on the other side. We shall look first at one of those who refused to leap. For penetrating subtlety of thought we shall not meet his equal till we reach the twentieth century and encounter Brouwer.

Zeno of Elea (495–435 B.C.) was a friend of the philosopher Parmenides, who, when he visited Athens with his patron, shocked the philosophers out of their complacency by inventing four innocent paradoxes which they could not dissipate in words. Zeno is said to have been a self-taught country boy. Without attempting to decide what was his purpose in inventing his paradoxes – authorities hold widely divergent opinions – we shall merely state them. With these before us it will be fairly obvious that Zeno would have objected to our 'infinitely continued' division of that two-inch line a moment ago. This will appear from the first two of his paradoxes, the *Dichotomy* and the *Achilles*. The last two, however, show that he would have objected with equal vehemence to the *opposite* hypothesis, namely that the line is *not* 'infinitely divisible' but is composed of a *discrete* set of points that can be counted off 1,2,3, . . . All four together constitute an iron wall beyond which progress appears to be impossible.

First, the *Dichotomy*. Motion is impossible, because whatever moves must reach the middle of its course *before* it reaches the end; but *before* it has reached the middle it must have reached the quarter-mark, and so on, *indefinitely*. Hence the motion can never even start.

Second, the *Achilles*. Achilles running to overtake a crawling tortoise ahead of him can never overtake it, because he must first reach the place from which the tortoise started; when Achilles reaches that place, the tortoise has departed and so is still ahead. Repeating the argument we easily see that the tortoise will always be ahead.

Now for the other side.

The *Arrow*. A moving arrow at any instant is either at rest or not at rest, that is, moving. If the instant is indivisible, the arrow cannot move, for if it did the instant would immediately be divided. But time is made up of instants. As the arrow cannot move in any one instant, it cannot move in any time. Hence it always remains at rest.

The *Stadium*. 'To prove that half the time may be equal to double the time. Consider three rows of bodies.

First Position					*Second Position*						
(A)	0	0	0	0	(A)		0	0	0	0	
(B)	0	0	0	0	(B)	0	0	0	0		
(C)	0	0	0	0	(C)			0	0	0	0

one of which (A) is at rest while the other two (B), (C) are moving with equal velocities in opposite directions. By the time they are all in the same part of the course (B) will have passed twice as many of the bodies in (C) as in (A). Therefore the time which it takes to pass (A) is twice as long as the time it takes to pass (C). But the time which (B) and (C) take to reach the position of (A) is the same. Therefore double the time is equal to half the time.' (Burnet's translation.) It is helpful to imagine (A) as a circular picket fence.

These, in non-mathematical language, are the sort of difficulties the early grapplers with continuity and infinity encountered. In books written twenty years or so ago it was said that 'the positive theory of infinity' created by Cantor, and the like for 'irrational' numbers, such as the square root of 2, invented by Eudoxus, Weierstrass, and Dedekind, had disposed of all these difficulties once and forever. Such a statement would not be accepted to-day by all schools of mathematical thought. So in dwelling upon Zeno we have in fact been discussing ourselves. Those who wish to see any more of him may consult Plato's *Parmenides*. We need remark only that Zeno finally lost his head for treason or something of the sort, and pass on to those who did not lose their heads over his arguments. Those who stayed behind with Zeno did comparatively little for the advancement of mathematics, although their successors have done much to shake its foundations.

Eudoxus (408–355 B.C.) of Cnidus inherited the mess which Zeno bequeathed the world and not much more. Like more than one man who has left his mark on mathematics, Eudoxus suffered from extreme poverty in his youth. Plato was in his prime while Eudoxus lived and Aristotle was about thirty when Eudoxus died. Both Plato and Aristotle, the leading philosophers of antiquity, were much concerned over the doubts which Zeno had injected into mathematical reasoning and which Eudoxus, in his theory of proportion – 'the crown of Greek mathematics' – was to allay till the last quarter of the nineteenth century.

As a young man Eudoxus moved to Athens from Tarentum, where he had studied with Archytas (428–347 B.C.), a first-rate mathematician, administrator, and soldier. Arriving in Athens, Eudoxus soon fell in with Plato. Being too poor to live near the Academy, Eudoxus trudged back and forth every day from the Piraeus where fish and olive oil were cheap and lodging was to be had for a smile in the right place.

Although he himself was not a mathematician in the technical sense, Plato has been called 'the maker of mathematicians', and it cannot be denied that he did irritate many infinitely better mathematicians than himself into creating some real mathematics. As we shall see, his total influence on the development of mathematics was probably baneful. But he did recognize what Eudoxus was and became his devoted friend until he began to exhibit something like jealousy towards his brilliant protégé. It is said that Plato and Eudoxus made a journey to Egypt together. If so, Eudoxus seems to have been less credulous than his predecessor Pythagoras; Plato, however, shows the effects of having swallowed vast quantities of the number-mysticism of the East. Finding himself unpopular in Athens, Eudoxus finally settled and taught at Cyzicus, where he spent his last years. He studied medicine and is said to have been a practising physician and legislator on top of his mathematics. As if all this were not enough to keep one man busy he undertook a serious study of astronomy, to which he made outstanding contributions. In his scientific outlook he was centuries ahead of his verbalizing, philosophizing contemporaries. Like Galileo and Newton he had a contempt for speculations about the physical universe which could not be checked by observation and experience. If by getting to the sun, he said, he could ascertain its shape, size, and nature, he would gladly share the fate of Phaëthon, but in the meantime he would not guess.

Some idea of what Eudoxus did can be seen from a very simple problem. To find the area of a rectangle we multiply the length by the breadth. Although this sounds intelligible it presents serious difficulties unless both sides are measurable by *rational* numbers. Passing these particular difficulties we see them in a more evident form in the next simplest type of problem, that of finding the length of a *curved* line, or the area of a *curved* surface, or the volume enclosed by *curved* surfaces.

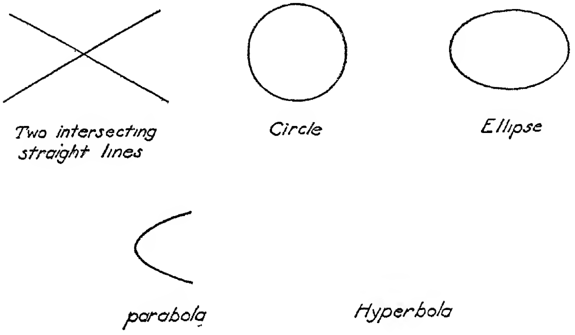
Any young genius wishing to test his mathematical powers may try to devise a method for doing these things. Provided he has never seen it done in school, how would he proceed to give a rigorous proof of the formula for the circumference of a circle of any given radius? Whoever does that entirely on his own initiative may justly claim to be a mathematician of the first rank. The moment we pass from figures bounded by *straight* lines or *flat* surfaces we run slap into all the problems of continuity, the riddles of the infinite and the mazes of irrational numbers. Eudoxus devised the first logically satisfactory method, which Euclid reproduced in Book V of his *Elements*, for handling such problems. In his *method of exhaustion*, applied to the computation of areas and volumes, Eudoxus showed that we need not assume the 'existence' of 'infinitely small quantities'. It is sufficient for the purposes of mathematics to be able to reach a magnitude *as small as we please* by the continued division of a given magnitude.

To finish with Eudoxus we shall state his epochal definition of equal ratios which enabled mathematicians to treat irrational numbers as rigorously as the rationals. This was, essentially, the starting-point of one modern theory of irrationals.

'The first of four magnitudes is said to have the *same ratio* to the second that the third has to the fourth when, any whatever equimultiples [the same multiples] of the *first* and *third* being taken, and any other equimultiples of the *second* and *fourth*, the multiple of the *first* is greater than, equal to, or less than the multiple of the *second*, according as the multiple of the *third* is greater than, equal to, or less than the multiple of the *fourth*.'

Of the Greeks not yet named whose work influenced mathematics after the year 1600 only Apollonius need be mentioned here. Apollonius (260?–200? B.C.) carried geometry in the manner of Euclid – the way it is still taught to hapless beginners – far beyond the state in which Euclid (330?–275? B.C.) left it. As a geometer of this type – a *synthetic*, 'pure' geometer – Apollonius is without a peer till Steiner in the nineteenth century.

If a cone standing on a circular base and extending indefinitely in both directions through its vertex is cut by a plane, the curve in which the plane intersects the surface of the cone is called a conic section. There are five possible kinds of conic sections: the ellipse; the hyperbola, consisting of two branches; the parabola, the path of a projectile in a vacuum; the circle; and a pair of intersecting straight lines. The ellipse, parabola, and hyperbola are 'mechanical curves' according to the Platonic formula; that is, these curves cannot be constructed by the use of straightedge and compass alone, although it is easy, with these implements, to construct any desired number of points lying on any one of these curves. The geometry of the conic sections, worked out to a high degree of perfection by Apollonius and his successors, proved to be of the highest importance in the celestial mechanics of the seventeenth and succeeding centuries. Indeed, had not the Greek geometers run ahead of Kepler it is unlikely that Newton could ever have come upon his law of universal gravitation, for which Kepler had prepared the way with his laboriously ingenious calculations on the orbits of the planets.



Among the later Greeks and the Arabs of the Middle Ages Archimedes seems to have inspired the same awe and reverence that Gauss did among his contemporaries and followers in the nineteenth century, and that Newton did in the seventeenth and eighteenth. Archimedes was the undisputed chieftain of them all, 'the old man', 'the wise one', 'the master', 'the *great* geometer.' To recall his dates, he lived in 287–212 B.C. Thanks to Plutarch more is known about his death than his life, and it is perhaps not unfair to suggest that the typical historical biographer Plutarch evidently thought the King of Mathematicians a less important personage historically than the Roman soldier Marcellus, into whose *Life* the account of Archimedes is slipped like a tissue-thin shaving of ham in a bull-choking sandwich. Yet Archimedes is to-day Marcellus' chief title to remembrance – and execration. In the death of Archimedes we shall see the first impact of a crassly practical civilization upon the greater thing which it destroyed – Rome, having half-demolished Carthage, swollen with victory and imperially purple with valour, falling upon Greece to shatter its fine fragility.

In body and mind Archimedes was an aristocrat. The son of the astronomer Pheidias, he was born at Syracuse, Sicily, and is said to have been related to Hieron II, tyrant (or king) of Syracuse. At any rate he was on intimate terms with Hieron and his son Gelon, both of whom had a high admiration for the king of mathematicians. His essentially aristocratic temperament expressed itself in his attitude to what would to-day be called applied science. Although he was one of the greatest mechanical geniuses of all time, if not the greatest when we consider how little he had to go on, the aristocratic Archimedes had a sincere contempt for his own practical inventions. From one point of view he was justified. Books could be written on what Archimedes did for applied mechanics; but great as this work was from our own mechanically biased point of view, it is completely overshadowed by his contributions to pure mathematics. We look first at the few known facts about him and the legend of his personality.

According to tradition Archimedes is a perfect museum specimen of the popular conception of what a great mathematician should be. Like Newton and Hamilton he left his meals untouched when he was deep in his mathematics. In the matter of inattention to dress he even surpasses Newton, for on making his famous discovery that a floating body loses in weight an amount equal to that of the liquid displaced, he leaped from the bath in which he had made the discovery by observing his own floating body, and dashed through the streets of Syracuse stark naked, shouting '*Eureka, eureka!*' (I have found it, I have found it!) What he had found was the first law of hydrostatics. According to the story a dishonest goldsmith had adulterated the gold of a crown for Hieron with silver and the tyrant, suspecting fraud, had asked Archimedes to put his mind on the problem. Any high school boy knows how it is solved by a simple experiment and some easy arithmetic on specific gravity; 'the principle of Archimedes' and its numerous practical applications are meat for youngsters and naval engineers to-day, but the man who first saw through them had more than common insight. It is not definitely known whether the goldsmith was guilty; for the sake of the story it is usually assumed that he was.

Another exclamation of Archimedes which has come down through the centuries is 'Give me a place to stand on and I will move the earth' (πᾶ βῶ καὶ κινῶ τὰν γᾶν, as he said it in Doric). He himself was strongly moved by his discovery of the laws of levers when he made his boast. The phrase would make a perfect motto for a modern scientific institute; it seems strange that it has not been appropriated. There is another version in better Greek but the meaning is the same.

In one of his eccentricities Archimedes resembled another great mathematician, Weierstrass. According to a sister of Weierstrass, he could not be trusted with a pencil when he was a young school teacher if there was a square foot of clear wall-paper or a clean cuff anywhere in sight. Archimedes beats this record. A sanded floor or dusted hard smooth earth was a common sort of 'blackboard' in his day. Archimedes made his own occasions. Sitting before the fire he would rake out the ashes and draw in them. After stepping from the bath he would anoint himself with olive oil, according to the custom of the time, and then, instead of putting on his clothes, proceed to lose himself in the diagrams which he traced with a finger-nail on his own oily skin.

Archimedes was a lonely sort of eagle. As a young man he had studied for a short time at Alexandria, Egypt, where he made two life-long friends, Conon, a gifted mathematician for whom Archimedes had a high regard both personal and intellectual, and Eratosthenes, also a good mathematician but quite a fop. These two, particularly Conon, seem to have been the only men of his contemporaries with whom Archimedes felt he could share his thoughts and be assured of understanding. Some of his finest work was communicated by letters to Conon. Later, when Conon died, Archimedes corresponded with Dositheus, a pupil of Conon.

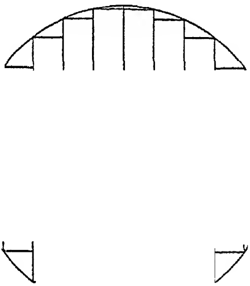
Leaving aside his great contributions to astronomy and mechanical invention we shall give a bare and inadequate summary of the principal additions which Archimedes made to pure and applied mathematics.

He invented general methods for finding the areas of curvilinear plane figures and volumes bounded by curved surfaces, and applied these methods to many special instances, including the circle, sphere, any segment of a parabola, the area enclosed between two radii and two successive whorls of a spiral, segments of spheres, and segments of surfaces generated by the revolution of rectangles (cylinders), triangles (cones), parabolas (paraboloids), hyperbolas (hyperboloids), and ellipses (spheroids) about their principal axes. He gave a method for calculating π (the ratio of the circumference of a circle to its diameter), and fixed π as lying between 3 1/7 and 3 10/71; he also gave methods for approximating to square roots which show that he anticipated the invention by the Hindus of what amount to periodic continued fractions. In arithmetic, far surpassing the incapacity of the unscientific Greek method of symbolizing numbers to write, or even to describe, large numbers, he invented a system of numeration capable of handling numbers as large as desired. In mechanics he laid down some of the fundamental postulates, discovered the laws of levers, and applied his mechanical principles (of levers) to calculate the areas and centres of gravity of several flat surfaces and solids of various shapes. He created the whole science of hydrostatics and applied it to find the positions of rest and of equilibrium of floating bodies of several kinds.

Archimedes composed not one masterpiece but many. How did he do it all? His severely economical, logical exposition gives no hint of the *method* by which he arrived at his wonderful results. But in 1906, J. L. Heiberg, the historian and scholar of Greek mathematics, made the dramatic discovery in Constantinople of a hitherto 'lost' treatise of Archimedes addressed to his friend Eratosthenes: *On Mechanical Theorems, Method.* In it Archimedes explains how by weighing, in imagination, a figure or solid whose area or volume was unknown against a known one, he was led to the knowledge of the fact he sought; the fact being known it was then comparatively easy (for him) to prove it mathematically. In short he used his mechanics to advance his mathematics. This is one of his titles to a modern mind: *he used anything and everything that suggested itself as a weapon to attack his problems.*

To a modern all is fair in war, love, and mathematics; to many of the ancients, mathematics was a stultified game to be played according to the prim rules imposed by the philosophically-minded Plato. According to Plato only a straight-edge and a pair of compasses were to be permitted as the implements of construction in geometry. No wonder the classical geometers hammered their heads for centuries against 'the three problems of antiquity': to trisect an angle; to construct a cube having double the volume of a given cube; to construct a square equal to a circle. *None of these problems is possible with only straight-edge and compass*, although it is hard to prove that the third is not, and the impossibility was finally proved only in 1882. All constructions effected with other implements were dubbed 'mechanical' and, as such, for some mystical reason known only to Plato and his geometrizing God, were considered shockingly vulgar and were rigidly taboo in respectable geometry. Not till Descartes, 1,985 years after the death of Plato, published his analytical geometry, did geometry escape from its Platonic straightjacket. Plato of course had been dead for sixty years or more before Archimedes was born, so he cannot be censured for not appreciating the lithe power and freedom of the methods of Archimedes. On the other hand, only praise is due to Archimedes for not appreciating the old-maidishness of Plato's rigidly corseted conception of what the muse of geometry should be.

The second claim of Archimedes to modernity is also based upon his methods. Anticipating Newton and Leibniz by more than 2,000 years he invented the integral calculus and in one of his problems anticipated their invention of the differential calculus. These two calculuses together constitute what is known as *the* calculus, which has been described as the most powerful instrument ever invented for the mathematical exploration of the physical universe. To take a simple example, suppose we wish to find the area of a circle. Among other ways of doing this we may slice the circle into any number of parallel strips of equal breadth, cut off the curved ends of the strips, so that the discarded bits shall total the least possible, by cuts perpendicular to the strips, and then add up the areas of all the resulting rectangles. This gives an approximation to the area sought. By increasing the number of strips indefinitely and taking the limit of the sum, we get the area of the circle. This (crudely described) process of taking the limit of the sum is called *integration*; the method of performing such summations is called the *integral calculus.* It was this calculus which Archimedes used in finding the area of a segment of a parabola and in other problems.



The problem in which he used the differential calculus was that of constructing a tangent at any given point of his spiral. If the angle which the tangent makes with any given line is known, the tangent can easily be drawn, for there is a simple construction for drawing a straight line through a given point parallel to a given straight line. The problem of finding the angle mentioned (for *any* curve, not merely for the spiral) is, in geometrical language, the main problem of the *differential* calculus. Archimedes solved this problem for his spiral. His spiral is the curve traced by a point moving with uniform speed along a straight line which revolves with uniform angular speed about a fixed point on the line. If anyone who has not studied the calculus imagines Archimedes' problem an easy one he may time himself doing it.

The life of Archimedes was as tranquil as a mathematician's should be if he is to accomplish all that is in him. All the action and tragedy of his life were crowded into its end. In 212 B.C. the second Punic war was roaring full blast. Rome and Carthage were going at one another hammer and tongs, and Syracuse, the city of Archimedes, lay temptingly near the path of the Roman fleet. Why not lay siege to it? They did.

Puffed up with conceit of himself ('relying on his own great fame', as Plutarch puts it), and trusting in the splendour of his 'preparedness' rather than in brains, the Roman leader, Marcellus, anticipated a speedy conquest. The pride of his confident heart was a primitive piece of artillery on a lofty harp-shaped platform supported by eight galleys lashed together. Beholding all this fame and miscellaneous shipping descending upon them the timider citizens would have handed Marcellus the keys of the city. Not so Hieron. He too was prepared for war, and in a fashion that the practical Marcellus would never have dreamed of.

It seems that Archimedes, despising applied mathematics himself, had nevertheless yielded in peace time to the importunities of Hieron, and had demonstrated to the tyrant's satisfaction that mathematics can, on occasion, become devastatingly practical. To convince his friends that mathematics is capable of more than abstract deductions, Archimedes had applied his laws of levers and pulleys to the manipulation of a fully loaded ship, which he himself launched single-handed. Remembering this feat when the war clouds began to gather ominously near, Hieron begged Archimedes to prepare a suitable welcome for Marcellus. Once more desisting from his researches to oblige his friend, Archimedes constituted himself a reception committee of one to trip the precipitate Romans. When they arrived his ingenious devilries stood grimly waiting to greet them.

The harp-shaped turtle affair on the eight quinqueremes lasted no longer than the fame of the conceited Marcellus. A succession of stone shots, each weighing over a quarter of a ton, hurled from the super-catapults of Archimedes, demolished the unwieldy contraption. Cranelike beaks and iron claws reached over the walls for the approaching ships, seized them, spun them round, and sank or shattered them against the jutting cliffs. The land forces, mowed down by the Archimedean artillery, fared no better. Camouflaging his rout in the official bulletins as a withdrawal to a previously prepared position in the rear, Marcellus backed off to confer with his staff. Unable to rally his mutinous troops for an assault on the terrible walls, the famous Roman leader retired.

At last evincing some slight signs of military common sense, Marcellus issued no further 'backs against the wall' orders of the day, abandoned all thoughts of a frontal attack, captured Megara in the rear, and finally sneaked up on Syracuse from behind. This time his luck was with him. The foolish Syracusans were in the middle of a bibulous religious celebration in honour of Artemis. War and religion have always made a bilious sort of cocktail; the celebrating Syracusans were very sick indeed. They woke up to find the massacre in full swing. Archimedes participated in the blood-letting.

His first intimation that the city had been taken by theft was the shadow of a Roman soldier falling across his diagram in the dust. According to one account the soldier had stepped on the diagram, angering Archimedes to exclaim sharply, 'Don't disturb my circles!' Another states that Archimedes refused to obey the soldier's order that he accompany him to Marcellus until he had worked out his problem. In any event the soldier flew into a passion, unsheathed his glorious sword, and dispatched the unarmed veteran geometer of seventy-five. Thus died Archimedes.

As Whitehead has observed, 'No Roman lost his life because he was absorbed in the contemplation of a mathematical diagram.'

CHAPTER THREE

GENTLEMAN, SOLDIER, AND MATHEMATICIAN

Descartes

'I DESIRE only tranquillity and repose.' These are the words of the man who was to deflect mathematics into new channels and change the course of scientific history. Too often in his active life René Descartes was driven to find the tranquillity he sought in military camps and to seek the repose he craved for meditation in solitary retreat from curious and exacting friends. Desiring only tranquillity and repose, he was born on 31 March, 1596 at La Haye, near Tours, France, into a Europe given over to war in the throes of religious and political reconstruction.

His times were not unlike our own. An old order was rapidly passing; the new was not yet established. The predatory barons, kings, and princelings of the Middle Ages had bred a swarm of rulers with the political ethics of highway robbers and, for the most part, the intellects of stable boys. What by common justice should have been thine was mine provided my arm was strong enough to take it away from thee. This may be an unflattering picture of that glorious period of European history known as the late Renaissance, but it accords fairly well with our own changing estimate, born of intimate experience, of what should be what in a civilized society.

On top of the wars for plunder in Descartes' day there was superimposed a rich deposit of religious bigotry and intolerance which incubated further wars and made the dispassionate pursuit of science a highly hazardous enterprise. To all this was added a comprehensive ignorance of the elementary rules of common cleanliness. From the point of view of sanitation the rich man's mansion was likely to be as filthy as the slums where the poor festered in dirt and ignorance, and the recurrent plagues which aided the epidemic wars in keeping the prolific population below the famine limit paid no attention to bank accounts. So much for the good old days.

On the immaterial, enduring side of the ledger the account is brighter. The age in which Descartes lived was indeed one of the great intellectual periods in the spotted history of civilization. To mention only a few of the outstanding men whose lives partly overlapped that of Descartes, we recall that Fermat and Pascal were his contemporaries in mathematics; Shakespeare died when Descartes was twenty; Descartes outlived Galileo by eight years, and Newton was eight when Descartes died; Descartes was twelve when Milton was born, and Harvey, the discoverer of the circulation of the blood, outlived Descartes by seven years, while Gilbert, who founded the science of electromagnetism, died when Descartes was seven.

René Descartes came from an old noble family. Although René's father was not wealthy his circumstances were a little better than easy, and his sons were destined for the careers of gentlemen – *noblesse oblige* – in the service of France. René was the third and last child of his father's first wife, Jeanne Brochard, who died a few days after René's birth. The father appears to have been a man of rare sense who did everything in his power to make up to his children for the loss of their mother. An excellent nurse took the mother's place, and the father, who married again, kept a constant, watchful, intelligent eye on his 'young philosopher' who always wanted to know the cause of everything under the sun and the reason for whatever his nurse told him about heaven. Descartes was not exactly a precocious child, but his frail health forced him to expend what vitality he had in intellectual curiosity.

Owing to René's delicate health his father let lessons slide. The boy however went ahead on his own initiative and his father wisely let him do as he liked. When Descartes was eight his father decided that formal education could not be put off longer. After much intelligent inquiry he chose the Jesuit college at La Flèche as the ideal school for his son. The rector, Father Charlet, took an instant liking to the pale, confiding little boy and made a special study of his case. Seeing that he must build up the boy's body if he was to educate his mind, and noticing that Descartes seemed to require much more rest than normal boys of his age, the rector told him to lie in bed as late as he pleased in the mornings and not to leave his room till he felt like joining his companions in the classroom. Thereafter, all through his life except for one unfortunate episode near its close, Descartes spent his mornings in bed when he wished to think. Looking back in middle age on his schooldays at La Flèche, he averred that those long, quiet mornings of silent meditation were the real source of his philosophy and mathematics.

His work went well and he became a proficient classicist. In line with the educational tradition of the time much attention was put on Latin, Greek, and rhetoric. But this was only a part of what Descartes got. His teachers were men of the world themselves and it was their job to train the boys under their charge to be 'gentlemen' – in the best sense of that degraded word – for their role in the world. When he left the school in August 1612, in his seventeenth year, Descartes had made a life-long friend in Father Charlet and was almost ready to hold his own in society. Charlet was only one of the many friends Descartes made at La Flèche; another, Mersenne (later Father) the famous amateur of science and mathematics, had been his older chum and was to become his scientific agent and protector-in-chief from bores.

Descartes' distinctive talent had made itself evident long before he left school. As early as the age of fourteen, lying meditating in bed, he had begun to suspect that the 'humanities' he was mastering were comparatively barren of human significance and certainly not the sort of learning to enable human beings to control their environment and direct their own destiny. The authoritative dogmas of philosophy, ethics, and morals offered for his blind acceptance began to take on the aspect of baseless superstitions. Persisting in his childhood habit of accepting nothing on mere authority, Descartes began unostentatiously questioning the alleged demonstrations and the casuistical logic by which the good Jesuits sought to gain the assent of his reasoning faculties. From this he rapidly passed to the fundamental doubt which was to inspire his life-work: how do we *know* anything? And further, perhaps more importantly, if we cannot say definitely that we know anything, how are we ever to find out those things which we may be capable of knowing?

On leaving school Descartes thought longer, harder, and more desperately than ever. As a first fruit of his meditations he apprehended the heretical truth that logic of itself – the great method of the schoolmen of the Middle Ages which still hung on tenaciously in humanistic education – is as barren as a mule for any creative human purpose. His second conclusion was closely allied to his first: compared to the demonstrations of mathematics – to which he took like a bird to the air as soon as he found his wings – those of philosophy, ethics, and morals are tawdry shams and frauds. How then, he asked, shall we ever find out anything? By the scientific method, although Descartes did not call it that: by *controlled experiment* and the application of rigid mathematical reasoning to the results of such experiment.

It may be asked what he got out of his rational scepticism. One fact, and only one: 'I exist.' As he put it, '*Cogito ergo sum*' (I think, therefore I am).

By the age of eighteen Descartes was thoroughly disgusted with the aridity of the studies into which he had put so much hard labour. He resolved to see the world and learn something of life as it is lived in flesh and blood and not in paper and printers' ink. Thanking God that he was well enough off to do as he pleased he proceeded to do it. By an understandable over-correction of his physically inhibited childhood and youth he now fell upon the pleasures appropriate to normal young men of his age and station and despoiled them with both hands. With several other young blades hungering for life in the raw he quit the depressing sobriety of the paternal estate and settled in Paris. Gambling being one of the accomplishments of a gentleman in that day, Descartes gambled with enthusiasm – and some success. Whatever he undertook he did with his whole soul.

This phase did not last long. Tiring of his bawdy companions, Descartes gave them the slip and took up his quarters in plain, comfortable lodgings in what is now the suburb of Saint-Germain where, for two years, he buried himself in incessant mathematical investigation. His gay deeds at last found him out, however, and his hare-brained friends descended whooping upon him. The studious young man looked up, recognized his friends, and saw that they were one and all intolerable bores. To get a little peace Descartes decided to go to war.

Thus began his first spell of soldiering. He went first to Breda, Holland, to learn his trade under the brilliant Prince Maurice of Orange. Being disappointed in his hopes for action under the Prince's colours, Descartes turned a disgusted back on the peaceful life of the camp, which threatened to become as exacting as the hurly-burly of Paris, and hastened to Germany. At this point of his career he first showed symptoms of an amiable weakness which he never outgrew. Like a small boy trailing a circus from village to village Descartes seized every favourable opportunity to view a gaudy spectacle. One was now about to come off at Frankfurt, where Ferdinand II was to be crowned. Descartes arrived in time to take in the whole rococo show. Considerably cheered up he again sought his profession and enlisted under the Elector of Bavaria, then waging war against Bohemia.

The army was lying inactive in its winter quarters near the little village of Neuburg on the banks of the Danube. There Descartes found in plenty what he had been seeking, tranquillity and repose. He was left to himself and he found himself.

The story of Descartes' 'conversion' – if it may be called that – is extremely curious. On St Martin's Eve, 10 November 1619, Descartes experienced three vivid dreams which, he says, changed the whole current of his life. His biographer (Baillet) records the fact that there had been considerable drinking in celebration of the saint's feast and suggests that Descartes had not fully recovered from the fumes of the wine when he retired. Descartes himself attributes his dreams to quite another source and states emphatically that he had touched no wine for three months before his elevating experience. There is no reason to doubt his word. The dreams are singularly coherent and quite unlike those (according to experts) inspired by a debauch, especially of stomach-filling wine. On the surface they are easily explicable as the subconscious resolution of a conflict between the dreamer's desire to lead an intellectual life and his realization of the futility of the life he was actually living. No doubt the Freudians have analysed these dreams, but it seems unlikely that any analysis in the classical Viennese manner could throw further light on the invention of analytical geometry, in which we are chiefly interested here. Nor do the several mystic or religious interpretations seem likely to be of much assistance in this respect.

In the first dream Descartes was blown by evil winds from the security of his church or college towards a third party which the wind was powerless to shake or budge; in the second he found himself observing a terrific storm with the unsuperstitious eyes of science, and he noted that the storm, once seen for what it was, could do him no harm; in the third he dreamed that he was reciting the poem of Ausonius which begins, '*Quod vitae sectabor iter?*' (What way of life shall I follow?)

There was much more. Out of it all Descartes says he was filled with 'enthusiasm' (probably intended in a mystic sense) and that there had been revealed to him, as in the second dream, the magic key which would unlock the treasure house of nature and put him in possession of the true foundation, at least, of all the sciences.

What was this marvellous key? Descartes himself does not seem to have told anyone explicitly, but it is usually believed to have been nothing less than the application of algebra to geometry, analytic geometry in short and, more generally, the exploration of natural phenomena by mathematics, of which mathematical physics to-day is the most highly developed example.

November 10th, 1619, then, is the official birthday of analytic geometry and therefore also of modern mathematics. Eighteen years were to pass before the method was published. In the meantime Descartes went on with his soldiering. On his behalf mathematics may thank Mars that no half-spent shot knocked his head off at the battle of Prague. A score or so of promising young mathematicians a few years short of three centuries later were less lucky, owing to the advance of that science which Descartes' dream inspired.

As never before the young soldier of twenty-two now realized that if he was ever to find truth he must first reject absolutely all ideas acquired from others and rely upon the patient questioning of his own mortal mind to show him the way. All the knowledge he had received from authority must be cast aside; the whole fabric of his inherited moral and intellectual ideas must be destroyed, to be refashioned more enduringly by the primitive, earthy strength of human reason alone. To placate his conscience he prayed the Holy Virgin to help him in his heretical project. Anticipating her assistance he vowed a pilgrimage to the shrine of Our Lady of Loreto and proceeded forthwith to subject the accepted truths of religion to a scorching, devastating criticism. However, he duly discharged his part of the contract when he found the opportunity.

In the meantime he continued his soldiering, and in the spring of 1620 enjoyed some very real fighting at the battle of Prague. With the rest of the victors Descartes entered the city chanting praises to God. Among the terrified refugees was the four-years-old Princess Elisabeth,* who was later to become Descartes' favourite disciple.

At last, in the spring of 1621, Descartes got his bellyful of war. With several other gay gentlemen soldiers he had accompanied the Austrians into Transylvania, seeking glory and finding it – on the other side. But if he was through with war for the moment he was not yet ripe for philosophy. The plague in Paris and the war against the Huguenots made France even less attractive than Austria. Northern Europe was both peaceful and clean; Descartes decided to pay it a visit. Things went well enough till Descartes dismissed all but one of his bodyguard before taking boat for east Frisia. Here was a Heaven-sent opportunity for the cut-throat crew. They decided to knock their prosperous passenger on the head, loot him, and pitch his

* Daughter of Frederick, Elector Palatine of the Rhine, and King of Bohemia, and a granddaughter of James I of England.

carcase to the fish. Unfortunately for their plans, Descartes understood their language. Whipping out his sword he compelled them to row him back to the shore, and once again analytical geometry escaped the accidents of battle, murder, and sudden death.

The following year passed quietly enough in visits to Holland and Rennes, where Descartes' father lived. At the end of the year he returned to Paris, where his reserved manner and somewhat mysterious appearance immediately got him accused of being a Rosicrucian. Ignoring the gossip, Descartes philosophized and played politics to get himself a commission in the army. He was not really disappointed when he failed, as he was left free to visit Rome where he enjoyed the most gorgeous spectacle he had yet witnessed, the ceremony celebrated every quarter of a century by the Catholic Church. This Italian interlude is of importance in Descartes' intellectual development for two reasons. His philosophy, so far as it fails to touch the common man, was permanently biased against that lowly individual by the fill which the bewildered philosopher got of unwashed humanity gathered from all corners of Europe to receive the papal benediction. Equally important was Descartes' failure to meet Galileo. Had the mathematician been philosopher enough to sit for a week or two at the feet of the father of modern science, his own speculations on the physical universe might have been less fantastic. All that Descartes got out of his Italian journey was a grudging jealousy of his incomparable contemporary.

Immediately after his holiday in Rome, Descartes enjoyed another bloody spree of soldiering with the Duke of Savoy, in which he so distinguished himself that he was offered a lieutenant-generalship. He had sense enough to decline. Returning to the Paris of Cardinal Richelieu and the swashing D'Artagnan – the latter near-fiction, the former less credible than a melodrama – Descartes settled down to three years of meditation. In spite of his lofty thoughts he was no grey-bearded savant in a dirty smock, but a dapper, well-dressed man of the world, clad in fashionable taffeta and sporting a sword as befitted his gentlemanly rank. To put the finishing touch to his elegance he crowned himself with a sweeping, broad-brimmed, ostrich-plumed hat. Thus equipped he was ready for the cut-throats infesting church, state, and street. Once when a drunken lout insulted Descartes' lady of the evening, the irate philosopher went after the rash fool quite in the stump-stirring fashion of D'Artagnan, and having flicked the sot's sword out of his hand, spared his life, not because he was a rotten swordsman, but because he was too filthy to be butchered before a beautiful lady.

Having mentioned one of Descartes' lady friends we may dispose of all but two of the rest here. Descartes liked women well enough to have a daughter by one. The child's early death affected him deeply. Possibly his reason for never marrying may have been, as he informed one expectant lady, that he preferred truth to beauty; but it seems more probable that he was too shrewd to mortgage his tranquillity and repose to some fat, rich, Dutch widow. Descartes was only moderately well off, but he knew when he had enough. For this he has been called cold and selfish. It seems juster to say that he knew where he was going and that he realized the importance of his goal. Temperate and abstemious in his habits he was not mean, never inflicting on his household the Spartan regimen he occasionally prescribed for himself. His servants adored him, and he interested himself in their welfare long after they had left his service. The boy who was with him at his death was inconsolable for days at the loss of his master. All this does not sound like selfishness.

Descartes also has been accused of atheism. Nothing could be farther from the truth. His religious beliefs were unaffectedly simple in spite of his rational scepticism. He compared his religion, indeed, to the nurse from whom he had received it, and declared that he found it as comforting to lean upon one as on the other. A rational mind is sometimes the queerest mixture of rationality and irrationality on earth.

Another trait affected all Descartes' actions till he gradually outgrew it under the rugged discipline of soldiering. The necessary coddling of his delicate childhood infected him with a deep tinge of hypochondria, and for years he was chilled by an oppressive dread of death. This, no doubt, is the origin of his biological researches. By middle age he could say sincerely that nature is the best physician and that the secret of keeping well is to lose the fear of death. He no longer fretted to discover means of prolonging existence.

His three years of peaceful meditation in Paris were the happiest of Descartes' life. Galileo's brilliant discoveries with his crudely constructed telescope had set half the natural philosophers of Europe pottering with lenses. Descartes amused himself in this way, but did nothing of striking novelty. His genius was essentially mathematical and abstract. One discovery which he made at this time, that of the principle of virtual velocities in mechanics, is still of scientific importance. This really was first-class work. Finding that few understood or appreciated it, he abandoned abstract matters and turned to what he considered the highest of all studies, that of man. But, as he dryly remarks, he soon discovered that the number of those who understand man is negligible in comparison with the number of those who think they understand geometry.

Up till now Descartes had published nothing. His rapidly mounting reputation again attracted a horde of fashionable dilettantes, and once more Descartes sought tranquillity and repose on the battlefield, this time with the King of France at the siege of La Rochelle. There he met that engaging old rascal Cardinal Richelieu, who was later to do him a good turn, and was impressed, not by the Cardinal's wiliness, but by his holiness. On the victorious conclusion of the war Descartes returned with a whole skin to Paris, this time to suffer his second conversion and abandon futilities forever.

He was now (1628) thirty-two, and only his miraculous luck had preserved his body from destruction and his mind from oblivion. A stray bullet at La Rochelle might easily have deprived Descartes of all claim to remembrance, and he realized at last that if he was ever to arrive it was high time that he be on his way. He was aroused from his sterile state of passive indifference by two Cardinals, De Bérulle and De Bagné, to the first of whom in particular the scientific world owes an everlasting debt of gratitude for having induced Descartes to publish.

The Catholic clergy of the time cultivated and passionately loved the sciences, in grateful contrast to the fanatical Protestants whose bigotry had extinguished the sciences in Germany. On becoming acquainted with De Bérulle and De Bagné, Descartes blossomed out like a rose under their genial encouragement. In particular, during soirées at De Bagné's, Descartes spoke freely of his new philosophy to a M. de Chandoux (who was later hanged for counterfeiting – not a result of Descartes' lessons in casuistry, let us hope). To illustrate the difficulty of distinguishing the true from the false Descartes undertook to produce twelve irrefutable arguments showing the falsity of any incontestable truth and, conversely, to do the like for the truth of any admitted falsehood. How then, the bewildered listeners asked, shall mere human beings distinguish truth from falsehood? Descartes confided that he had (what he considered) an infallible method, drawn from mathematics, for making the required distinction. He hoped and planned, he said, to show how his method could be applied to science and human welfare through the medium of mechanical invention.

De Bérulle was profoundly stirred by the vision of all the kingdoms of the earth with which Descartes had tempted him from the pinnacle of philosophic speculation. In no uncertain terms he told Descartes that it was his duty to God to share his discoveries with the world, and threatened him with hell-fire – or at least the loss of his chance of heaven – if he did not. Being a devout practising Catholic Descartes could not possibly resist such an appeal. He decided to publish. This was his second conversion, at the age of thirty-two. He straightway retired to Holland, where the colder climate suited him, to bring his decision to realization.

For the next twenty years he wandered about all over Holland, never settling for long in any one place, a silent recluse in obscure villages, country hotels and out-of-the-way corners of great cities, methodically carrying on a voluminous scientific and philosophical correspondence with the leading intellects of Europe, using as intermediary the trusted friend of his school days at La Flèche, Father Mersenne, who alone knew the secret at any time of Descartes' address. The parlour of the cloister of the Minims, not far from Paris, became the exchange (through Mersenne) for questions, mathematical problems, scientific and philosophical theories, objections, and replies.

During his long vagabondage in Holland Descartes occupied himself with a number of studies in addition to his philosophy and mathematics. Optics, chemistry, physics, anatomy, embryology, medicine, astronomical observations, and meteorology, including a study of the rainbow, all claimed their share of his restless activity. Any man to-day spreading his effort over so diversified a miscellany would write himself down a fiddling dilettante. But it was not so in Descartes' age; a man of talent might still hope to find something of interest in almost any science that took his fancy. Everything that came Descartes' way was grist to his mill. A brief visit to England acquainted him with the mystifying behaviour of the magnetic needle; forthwith magnetism had to be included in his comprehensive philosophy. The speculations of theology also called for his attention. All through his theorizing his mind was shadowed by the incubus of his early training. He would not have shaken it off if he could.

All of what Descartes had gathered and excogitated was to be incorporated into an imposing treatise, *Le Monde.* In 1634, Descartes being then thirty-eight, the treatise was undergoing its final revision. It was to have been a New Year's gift to Father Mersenne. All learned Paris was agog to see the masterpiece. Mersenne had been granted many previews of selected portions but as yet he had not seen the completed, dovetailed work. Without irreverence *Le Monde* may be described as what the author of the Book of Genesis might have written had he known as much science and philosophy as Descartes did. Descartes intended his account of God's creation of the universe to supply the lack which some readers had felt in the Bible story of the six days' creation, namely, an element of rationality. From the distance of 300 years there seems but little to choose between Genesis and Descartes, and it is somewhat difficult for us to realize that such a book as *Le Monde* could ever have caused a bishop or a pope to fly into a cold, murderous rage. As a matter of fact none did; Descartes saw to that.

Descartes was aware of the judgements of ecclesiastical justice. He also knew of the astronomical researches of Galileo and of that fearless man's championship of the Copernican system. In fact he was impatiently waiting to see Galileo's latest book before putting the final touches to his own. Instead of receiving the copy a friend had promised to send him, he got the stunning news that Galileo, in the seventieth year of his age, and in spite of the sincere friendship that the powerful Duke of Tuscany had for him, had been given up to the Inquisition and had been forced (22 June 1633) on his knees to abjure as a heresy the Copernican doctrine that the Earth moves round the Sun. What would have happened to Galileo had he refused to forswear his scientific knowledge Descartes could only conjecture, but the names of Bruno, Vanini, and Campanella recurred to his mind.

Descartes was crushed. In his own book he had expounded the Copernican system as a matter of course. On his own account he had been far more daring than Copernicus or Galileo had ever had occasion to be, because he was interested in the theology of science whereas they were not. He had proved to his own satisfaction the *necessity* of the cosmos as it exists, and he thought he had shown that if God had created any number of distinct universes they must all, under the action of 'natural law', sooner or later have fallen into line with *necessity* and have evolved into the universe as it actually is. Descartes, in short, professed with his scientific knowledge to know a great deal more about the nature and ways of God than either the author of Genesis or the theologians had ever dreamed of. If Galileo had been forced to get down on his knees for his mild and conservative heresy, what could Descartes expect?

To say that fear alone stopped Descartes from publishing *Le Monde* is to miss the more important part of the truth. He was not only afraid – as any sane man might well have been; he was deeply hurt. He was as convinced of the truth of the Copernican system as he was of his own existence. But he was also convinced of the infallibility of the Pope. Here now was the Pope making a silly ass of himself by contradicting Copernicus. This was his first thought. His casuistical schooling came to his aid. In some way, through the mystical incomprehensibilities of some superhuman synthesis, the Pope and Copernicus would yet both be proved right. From this as yet unrevealed Pisgah height Descartes confidently hoped and expected some day to look down in philosophic serenity on the apparent contradiction and see it vanish in a glory of reconciliation. It was simply impossible for him to give up either the Pope or Copernicus. So he suppressed his book and kept both his belief in the infallibility of the Pope and the truth of the Copernican system. As a sop to his subconscious self-respect he decided that *Le Monde* should be published after his death. By that time perhaps the Pope too would be dead and the contradiction would have resolved itself.

Descartes' determination not to publish extended to all his work. But in 1637, when Descartes was forty-one, his friends overcame his reluctance and induced him to permit the printing of his masterpiece, of which the title is translated as *A Discourse on the Method of rightly conducting the Reason and seeking Truth in the Sciences. Further, the Dioptric, Meteors, and Geometry, essays in this Method.* This work is known shortly as the *Method.* It was published on 8 June 1637. This is the day, then, on which analytical geometry was given to the world. Before describing wherein that geometry is superior to the synthetic geometry of the Greeks we shall finish with the life of its author.

After having given the reasons for Descartes' delay in publication it is only fair to tell now the other and brighter side of the story.

The Church which Descartes had feared but which had never actually opposed him now came most generously to his aid. Cardinal Richelieu gave Descartes the privilege of publishing either in France or abroad anything he cared to write. (In passing we may ask, however, by what right, divine, or other, did Cardinal Richelieu, or any other human being, dictate to a philosopher and man of science what he should or should not publish?) But in Utrecht, Holland, the Protestant theologians savagely condemned Descartes' work as atheistic and dangerous to that mystic entity known as 'The State'. The liberal Prince of Orange threw his great weight on Descartes' side and backed him to the limit.

Since the autumn of 1641 Descartes had been living at a quiet little village near The Hague in Holland, where the exiled Princess Elisabeth, now a young woman with a penchant for learning, rusticated with her mother. The Princess does indeed seem to have been a prodigy of learning. After mastering six languages and digesting much literature she had turned to mathematics and science, hoping to find more nourishing fare. One theory to account for this remarkable young woman's unusual appetite ascribes her hunger for knowledge to a disappointment in love. Neither mathematics nor science satisfied her. Then Descartes' book came her way and she knew that she had found what she needed to fill her aching void – Descartes. An interview was arranged with the somewhat reluctant philosopher.

It is very difficult to understand exactly what happened thereafter. Descartes was a gentleman with all the awe and reverence of a gentleman of those gallant, royalty-ridden times for even the least potent prince or princess. His letters are models of courtly discretion, but somehow they do not always ring quite true. One spiteful little remark, quoted in a moment, probably tells more of what he really thought of the Princess Elisabeth's intellectual capacity than do all the reams of subtle flattery he wrote to or about his eager pupil with one eye on his style and the other on publication after his death.

Elisabeth insisted upon Descartes giving her lessons. Officially he declared that 'of all my disciples she alone has understood my works completely.' There is no doubt that he was genuinely fond of her in a fatherly, cat-looking-at-a-king's-female-relative sort of way, but to believe that he meant what he said as a scientific statement of fact is to stretch credulity to the limit, unless, of course, he meant it as a wry comment on his own philosophy. Elisabeth may have understood too much, for it seems to be a fact that only a philosopher thoroughly understands his own philosophy, although any fool can think he does. Anyhow, he did not propose to her nor, so far as is known, did she propose to him.

Among other parts of his philosophy which he expounded to her was the method of analytical geometry. Now there is a certain problem in elementary geometry which can be quite simply solved by pure geometry, and which looks easy enough, but which is a perfect devil for analytical geometry to handle in the strict Cartesian form. This is to construct a circle which shall touch (be tangent to) any three circles given at random whose centres do not all lie on one straight line. There are eight solutions possible. The problem is a fine specimen of the sort that are *not* adapted to the crude brute force of elementary Cartesian geometry. *Elisabeth solved it by Descartes' methods*. It was rather cruel of him to let her do it. His comment on seeing her solution gives the whole show away to any mathematician. She was quite proud of her exploit, poor girl. Descartes said he would not undertake to carry out her solution and actually construct the required tangent circle in a month. If this does not convey his estimate of her mathematical aptitude it is impossible to put the matter plainer. It was an unkind thing to say, especially as she missed the point and he knew that she would.

When Elisabeth left Holland she corresponded with Descartes to almost the day of his death. His letters contain much that is fine and sincere, but we could wish that he had not been so dazzled by the aura of royalty.

In 1646 Descartes was living in happy seclusion at Egmond, Holland, meditating, gardening in a tiny plot, and carrying on a correspondence of incredible magnitude with the intellectuals of Europe. His greatest mathematical work lay behind him, but he still continued to think about mathematics, always with penetration and originality. One problem to which he gave some attention was Zeno's of Achilles and the tortoise. His solution of the paradox would not be universally accepted today but it was ingenious for its era. He was now fifty and world-famous, far more famous in fact than he would ever have cared to be. The repose and tranquillity he had longed for all his life still eluded him. He continued to do great work, but he was not to be left in peace to do all that was in him. Queen Christine of Sweden had heard of him.

This somewhat masculine young woman was then nineteen, already a capable ruler, reputedly a good classicist (of this, more later), a wiry athlete with the physical endurance of Satan himself, a ruthless huntress, an expert horsewoman who thought nothing of ten hours in the saddle without once getting off, and finally a tough morsel of femininity who was as hardened to cold as a Swedish lumberjack. With all this she combined a certain thick obtuseness toward the frailties of less thick-skinned beings. Her own meals were sparing; so were those of her courtiers. Like a hibernating frog she could sit for hours in an unheated library in the middle of a Swedish winter; her hangers-on begged her through their chattering teeth to throw all the windows wide open and let the merry snow in. Her cabinet, she noted without a qualm, always agreed with her. She knew everything there was to be known; her ministers and tutors told her so. As she got along on only five hours' sleep she kept her toadies hopping through the hoop nineteen hours a day. The very hour this holy terror saw Descartes' philosophy she decided she must annex the poor sleepy devil as her private instructor. All her studies so far had left her empty and hungering for more. Like the erudite Elisabeth she knew that only copious douches of philosophy from the philosopher himself could assuage her raging thirst for knowledge and wisdom.

But for that unfortunate streak of snobbery in his make-up Descartes might have resisted Queen Christine's blandishments till he was ninety and sans teeth, sans hair, sans philosophy, sans everything. Descartes held out till she sent Admiral Fleming in the spring of 1649 with a ship to fetch him. The whole outfit was generously placed at the reluctant philosopher's disposal. Descartes temporized till October. Then, with a last regretful look round his little garden, he locked up and left Egmond for ever.

His reception in Stockholm was boisterous, not to say royal. Descartes did not live at the Palace; that much was spared him. Importunately kind friends, however, the Chanutes, shattered his last remaining hope of reserving a little privacy. They insisted that he live with them. Chanute was a fellow-countryman, in fact the French ambassador. All might have gone well, for the Chanutes were really most considerate, had not the obtuse Christine got it into her immovable head that five o'clock in the morning was the proper hour for a busy, hard-boiled young woman like herself to study philosophy. Descartes would gladly have swapped all the headstrong queens in Christendom for a month's dreaming abed at La Flèche with the enlightened Charlet unobtrusively near to see that he did not get up too soon. However, he dutifully crawled out of bed at some ungodly hour in the dark, climbed into the carriage sent to collect him, and made his way across the bleakest, windiest square in Stockholm to the palace where Christine sat in the icy library impatiently waiting for her lesson in philosophy to begin promptly at five A.M.

The oldest inhabitants said Stockholm had never in their memory suffered so severe a winter. Christine appears to have lacked a normal human skin as well as nerves. She noticed nothing, but kept Descartes unflinchingly to his ghastly rendezvous. He tried to make up his rest by lying down in the afternoons. She soon broke him of that. A Royal Swedish Academy of Sciences was gestating in her prolific activity; Descartes was hauled out of bed to deliver her.

It soon became plain to the courtiers that Descartes and their Queen were discussing much more than philosophy in these interminable conferences. The weary philosopher presently realized that he had stepped with both feet into a populous and busy hornets' nest. They stung him whenever and wherever they could. Either the Queen was too thick to notice what was happening to her new favourite or she was clever enough to sting her courtiers through her philosopher. In any event, to silence the malicious whisperings of 'foreign influence', she resolved to make a Swede of Descartes. An estate was set aside for him by royal decree. Every desperate move he made to get out of the mess only bogged him deeper. By 1 January 1650, he was up to his neck with only a miracle of rudeness as his one dim hope of ever freeing himself. But with his inbred respect for royalty he could not bring himself to speak the magic words which would send him flying back to Holland, although he said plenty, with courtly politeness, in a letter to his devoted Elisabeth. He had chanced to interrupt one of the lessons in Greek.

To his amazement Descartes learned that the vaunted classicist Christine was struggling over grammatical puerilities which, he says, he had mastered by himself when he was a little boy. His opinion of her mentality thereafter appears to have been respectful but low. It was not raised by her insistence that he produce a ballet for the delectation of her guests at a court function when he resolutely refused to make a mountebank of himself by attempting at his age to master the stately capers of the Swedish lancers.

Presently Chanute fell desperately ill of inflammation of the lungs. Descartes nursed him. Chanute recovered; Descartes fell ill of the same disease. The Queen, alarmed, sent doctors. Descartes ordered them out of the room. He grew steadily worse. Unable in his debility to distinguish friend from pest he consented at last to being bled by the most persistent of the doctors, a personal friend, who all the time had been hovering about awaiting his chance. This almost finished him, but not quite.

His good friends the Chanutes, seeing that he was a very sick man, suggested that he might enjoy the last sacrament. He had expressed a desire to see his spiritual counsellor. Commending his soul to the mercy of God, Descartes faced his death calmly, saying the willing sacrifice of his life which he was making might possibly atone for his sins. La Flèche gripped him to the last. The counsellor asked him to signify whether he wished the final benediction. Descartes opened his eyes and closed them. He was given the benediction. Thus he died on 11 February 1650, aged 54, a sacrifice to the overweening vanity of a headstrong girl.

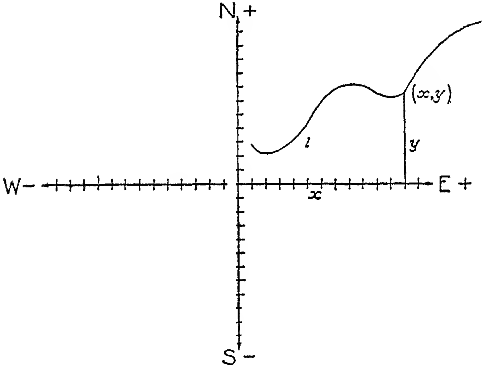
Christine lamented. Seventeen years later when she had long since given up her crown and her faith, the bones of Descartes were returned to France (all except those of the right hand, which were retained by the French Treasurer-General as a souvenir for his skill in engineering the transaction) and were re-entombed in Paris in what is now the Panthéon. There was to have been a public oration, but this was hastily forbidden by order of the crown, as the doctrines of Descartes were deemed to be still too hot for handling before the people. Commenting on the return of Descartes' remains to his native France, Jacobi remarks that 'It is often more convenient to possess the ashes of great men than to possess the men themselves during their lifetime.'

Shortly after his death Descartes' books were listed in the *Index* of that Church which, accepting Cardinal Richelieu's enlightened suggestion during the author's lifetime, had permitted their publication. 'Consistency, thou art a jewel!' But the faithful were not troubled by consistency, 'the bugbear of little minds' – and the ratbane of inconsistent bigots.

We are not concerned here with the monumental additions which Descartes made to philosophy. Nor can his brilliant part in the dawn of the experimental method detain us. These things fall far outside the field of pure mathematics in which, perhaps, his greatest work lies. It is given to but few men to renovate a whole department of human thought. Descartes was one of those few. Not to obscure the shining simplicity of his greatest contribution, we shall briefly describe it alone and leave aside the many beautiful things he did in algebra and particularly in algebraic notation and the theory of equations. This one thing is of the highest order of excellence, marked by the sensuous simplicity of the half dozen or so greatest contributions of all time to mathematics. Descartes remade geometry and made modern geometry possible.

The basic idea, like all the really great things in mathematics, is simple to the point of obviousness. Lay down any two intersecting lines on a plane. Without loss of generality we may assume that the lines are at right angles to one another. Imagine now a city laid out on the American plan, with avenues running north and south, streets east and west. The whole plan will be laid out with respect to *one* avenue and *one* street, called the *axes*, which intersect in what is called the *origin*, from which street-avenue numbers are read consecutively. Thus it is clear without a diagram where 1002 West 126 Street is, if we note that the *ten avenues* summarized in the number 1002 are stepped off to the *west*, that is, on the map, to the *left* of the origin. This is so familiar that we visualize the position of any particular address instantly. The avenue-number and street-number, with the necessary supplements of smaller numbers (as in the '2' in '1002' above) enable us to fix definitely and uniquely the position of any *point* whatever with respect to the *axes*, by giving the *pair* of numbers which measure its *east or west* and its *north or south* from the *axes;* this pair of numbers is called the *coordinates* of the point (with respect to the axes).

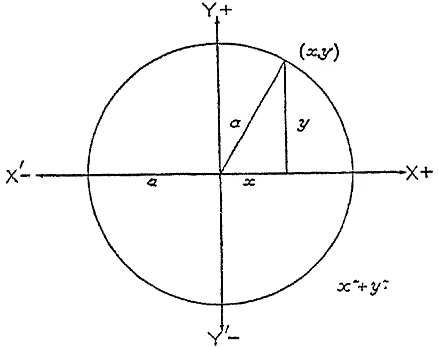
Now suppose a point to wander over the map. The *coordinates* (x, y) of *all* the points on the curve over which it wanders will be connected by an *equation* (this must be taken for granted by the reader who has never plotted a graph to fit data), which is



called *the equation of the curve.* Suppose now for simplicity that our curve is a circle. We have its equation. What can be done with it? Instead of this particular equation, we can write down the most general one of the same kind (for example, here, of the *second* degree, with no cross-product term, and with the coefficients of the highest powers of the coordinates equal), and then proceed to manipulate this equation algebraically. Finally we put back the results of all our algebraic manipulations into their equivalents in terms of coordinates of points on the diagram which, all this time, we have been deliberately forgetting. Algebra is easier to see through than a cobweb of lines in the

Greek manner of elementary geometry. What we have done has been to *use our algebra for the discovery and investigation of geometrical theorems concerning circles.*

For straight lines and circles this may not seem very exciting; we knew how to do it all before in another, a Greek way. Now comes the real power of the method. *We start with equations of any desired or suggested degree of complexity and interpret their algebraic and analytic properties geometrically.* Thus we have not only dropped geometry as our pilot; we have tied a sackful of bricks to his neck before pitching him overboard. *Henceforth algebra and analysis are to be our pilots to the uncharted seas of*



'space' and its 'geometry'. All that we have done can be extended, at one stride, to space of any number of dimensions; for the plane we need *two co-ordinates*, for ordinary 'solid' space *three*, for the geometry of mechanics and relativity, *four* co-ordinates, and finally, for 'space' as mathematicians like it, either n co-ordinates, or as many co-ordinates as there are of *all* the numbers 1, 2, 3, . . . , or as many as there are of *all* the points on a line. This is beating Achilles and the tortoise in their own race.

Descartes did not revise geometry; he created it.

It seems fitting that an eminent living mathematical fellow-countryman of Descartes should have the last word, so we shall quote Jacques Hadamard. He remarks first that the mere invention of co-ordinates was not Descartes' greatest merit, because that had already been done 'by the ancients' – a statement which is exact only if we read the unexpressed intention into the unaccomplished deed. Hell is paved with the half-baked ideas of 'the ancients' which they could never quite cook through with their own steam.

'It is quite another thing to recognize [as in the use of co-ordinates] a general method and to follow to the end the idea which it represents. It is exactly this merit, whose importance every real mathematician knows, that was pre-eminently Descartes' in geometry; it was thus that he was led to what ... is his truly great discovery in the matter; namely, the application of the method of co-ordinates not only to translate into equations curves already defined geometrically, but, looking at the question from an exactly opposite point of view, to the *a priori* definition of more and more complicated curves and, hence, more and more general. ...

'Directly, with Descartes himself, later, indirectly, in the return which the following century made in the opposite direction, it is the entire conception of the object of mathematical science that was revolutionized. Descartes indeed understood thoroughly the significance of what he had done, and he was right when he boasted that he had as far surpassed all geometry before him as Cicero's rhetoric surpasses the ABC.'

CHAPTER FOUR

THE PRINCE OF AMATEURS

Fermat

NOT all of our ducks can be swans; so after having exhibited Descartes as one of the leading mathematicians of all time, we shall have to justify the assertion, frequently made and seldom contradicted, that the greatest mathematician of the seventeenth century was Descartes' contemporary Fermat (1601?–65). This of course leaves Newton (1642–1727) out of consideration. But it can be argued that Fermat was *at least* Newton's equal *as a pure mathematician,* and anyhow nearly a third of Newton's life fell into the eighteenth century, whereas the whole of Fermat's was lived out in the seventeenth.

Newton appears to have regarded his mathematics principally as an instrument for scientific exploration and put his main effort on the latter. Fermat on the other hand was more strongly attracted to pure mathematics although he also did notable work in the applications of mathematics to science, particularly optics.

Mathematics had just entered its modern phase with Descartes' publication of analytical geometry in 1637, and was still for many years to be of such modest extent that a gifted man could reasonably hope to do good work in both the pure and applied divisions.

As a pure mathematician Newton reached his climax in the invention of the calculus, an invention also made independently by Leibniz. More will be said on this later; for the present it may be remarked that Fermat conceived and applied the leading idea of the differential calculus thirteen years before Newton was born and seventeen before Leibniz was born, although he did not, like Leibniz, reduce his method to a set of rules of thumb that even a dolt can apply to easy problems.

As for Descartes and Fermat, each of them, entirely independently of the other, invented analytical geometry. They corresponded on the subject, but this does not affect the preceding assertion. The major part of Descartes' effort went to miscellaneous scientific investigations, the elaboration of his philosophy, and his preposterous 'vortex theory' of the solar system – for long a serious rival, even in England, to the beautifully simple, unmetaphysical Newtonian theory of universal gravitation. Fermat seems never to have been tempted, as both Descartes and Pascal were, by the insidious seductiveness of philosophizing about God, man, and the universe as a whole; so, after having disposed of his part in the calculus and analytical geometry, and having lived a serene life of hard work all the while to earn his living, he still was free to devote his remaining energy to his favourite amusement – pure mathematics, and to accomplish his greatest work, the foundation of the theory of numbers, on which his undisputed and undivided claim to immortality rests.

It will be seen presently that Fermat shared with Pascal the creation of the mathematical theory of probability. If all these first-rank achievements are not enough to put him at the head of his contemporaries in pure mathematics we may ask who did more. Fermat was a born originator. He was also, in the strictest sense of the word, so far as his science and mathematics were concerned, an amateur. Without doubt he is one of the foremost amateurs in the history of science, if not the very first.

Fermat's life was quiet, laborious, and uneventful, but he got a tremendous lot out of it. The essential facts of his peaceful career are quickly told. The son of the leather-merchant Dominique Fermat, second consul of Beaumont, and Claire de Long, daughter of a family of parliamentary jurists, the mathematician Pierre Fermat was born at Beaumont-de-Lomagne, France, in August 1601 (the exact date is unknown; the baptismal day was 20 August). His earliest education was received at home in his native town; his later studies, in preparation for the magistracy, were continued at Toulouse. As Fermat lived temperately and quietly all his life, avoiding profitless disputes, and as he lacked a doting sister like Pascal's Gilberte to record his boyhood prodigies for posterity, singularly little appears to have survived of his career as a student. That it must have been brilliant will be evident from the achievements and accomplishments of his maturity; no man without a solid foundation of exact scholarship could have been the classicist and littérateur that Fermat became. His marvellous work in the theory of numbers and in mathematics generally cannot be traced to his schooling; for the fields in which he did his greatest work, not having been opened up while he was a student, could scarcely have been suggested by his studies.

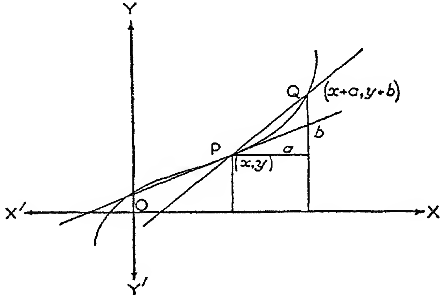
The only events worth noting in his material career are his installation at Toulouse, at the age of 30 (14 May 1631), as commissioner of requests; his marriage on 1 June of the same year to Louise de Long, his mother's cousin, who presented him with three sons, one of whom, Clément-Samuel, became his father's scientific executor, and two daughters, both of whom took the veil; his promotion in 1648 to a King's councillorship in the local parliament of Toulouse, a position which he filled with dignity, integrity, and great ability for seventeen years – his entire working life of thirty-four years was spent in the exacting service of the state; and finally, his death at Castres on 12 January 1665, in his sixty-fifth year, two days after he had finished conducting a case in the town of his death. 'Story?' he might have said: 'Bless you, sir! I have none.' And yet this tranquilly-living, honest, even-tempered, scrupulously just man has one of the finest stories in the history of mathematics.

His story is his work – his recreation, rather – done for the sheer love of it, and the best of it is so simple (to state, but not to carry through or imitate) that any schoolboy of normal intelligence can understand its nature and appreciate its beauty. The work of this prince of mathematical amateurs has had an irresistible appeal to amateurs of mathematics in all civilized countries during the past three centuries. This, the theory of numbers as it is called, is probably the one field of mathematics in which a talented amateur to-day may hope to turn up something of interest. We shall glance at his other contributions first after a passing mention of his 'singular erudition' in what many call the humanities. His knowledge of the chief European languages and literatures of Continental Europe was wide and accurate, and Greek and Latin philology are indebted to him for several important corrections. In the composition of Latin, French, and Spanish verses, one of the gentlemanly accomplishments of his day, he showed great skill and a fine taste. We shall understand his even, scholarly life if we picture him as an affable man, not touchy or huffy under criticism (as Newton in his later years was), without pride, but having a certain vanity which Descartes, his opposite in all respects, characterized by saying, 'Mr de Fermat is a Gascon; I am not.' The allusion to the Gascons may possibly refer to an amiable sort of braggadocio which some French writers (for example Rostand in *Cyrano de Bergerac*, Act II, Scene VII) ascribe to their men of Gascony. There may be some of this in Fermat's letters, but it is always rather naive and inoffensive, and nothing to what he might have justly thought of his work even if his head had been as big as a balloon. And as for Descartes it must be remembered that he was not exactly an impartial judge. We shall note in a moment how his own soldierly obstinacy caused him to come off a bad second-best in his protracted row with the 'Gascon' over the extremely important matter of tangents.

Considering the exacting nature of Fermat's official duties and the large amount of first-rate mathematics he did, some have been puzzled as to how he found time for it all. A French critic suggests a probable solution: Fermat's work as a King's councillor was an aid rather than a detriment to his intellectual activities. Unlike other public servants – in the army for instance – parliamentary councillors were expected to hold themselves aloof from their fellow townsmen and to abstain from unnecessary social activities lest they be corrupted by bribery or otherwise in the discharge of their office. Thus Fermat found plenty of leisure.

We now briefly state Fermat's part in the evolution of the calculus. As was remarked in the chapter on Archimedes, a geometrical equivalent of the fundamental problem of the *differential* calculus is to draw the straight line tangent to a given, unlooped, continuous arc of a curve at any given point. A sufficiently close description of what 'continuous' means here is 'smooth, without breaks or sudden jumps'; to give an exact, mathematical definition would require pages of definitions and subtle distinctions which, it is safe to say, would have puzzled and astonished the inventors of the calculus, including Newton and Leibniz. And it is also a fair guess that if all these subtleties which modern students demand had presented themselves to the originators, the calculus would never have got itself invented.

The creators of the calculus, including Fermat, relied on geometric and physical (mostly kinematical and dynamical) intuition to get them ahead: they *looked at* what passed in their imaginations for the *graph* of a 'continuous curve', pictured the process of drawing a straight line tangent to the curve at any point P on the curve by taking another point Q, also on the curve, drawing the straight line PQ joining P and Q, and then,



in imagination, letting the point Q slip along the arc of the curve from Q to P, till Q coincided with P, when the *chord PQ*, in the *limiting position* just described, became the *tangent PP* to the curve at the point P – the very thing they were looking for.

The next step was to translate all this into algebraical or analytical language. Knowing the co-ordinates x, y of the point P on the graph, and those, say $x + a$, $y + b$, of Q, before Q

started to slip along to coincidence with P, they inspected the graph and saw that the *slope* of the *chord* PQ was equal to b/a – obviously a measure of the 'steepness' of the chord with relation to the x-axis (the line along which x-distances are measured); this 'steepness' is precisely what is meant by slope. From this it was evident that the *required slope of the tangent at* P (after Q had slipped into coincidence with P) would be the *limiting value* of b/a as both b and a approached the value *zero* simultaneously; for $x + a$, $y + b$, the co-ordinates of Q, ultimately become x, y, the co-ordinates of P. This limiting value is the required slope. Having the slope and the point P they could now draw the tangent.

This is not exactly Fermat's process for drawing tangents, but his own process was, broadly, equivalent to what has been described.

Why should all this be worth the serious attention of any rational or practical man? It is a long story, only a hint of which need be given here; more will be said when we discuss Newton. One of the fundamental ideas in dynamics is that of the *velocity* (speed) of a moving particle. If we graph the number of units of distance passed over by the particle in a unit of time against the number of units of time, we get a line, straight or curved, which pictures at a glance the *motion* of the particle, and the *steepness* of this line at any given point of it will obviously give us the *velocity* of the particle at the instant corresponding to the point; the faster the particle is moving, the steeper the *slope* of the *tangent line*. This slope does in fact measure the velocity of the particle at any point of its path. The problem in *motion*, when translated into *geometry*, is exactly that of finding the slope of the tangent line at a given point of a curve. There are similar problems in connexion with *tangent planes* to surfaces (which also have important interpretations in mechanics and mathematical physics), and all are attacked by the differential calculus – whose fundamental problem we have attempted to describe as it presented itself to Fermat and his successors.

Another use of this calculus can be indicated from what has already been said. Suppose some quantity y is a 'function' of another, t, written $y = f(t)$, which means that when any definite number, say 10, is substituted for t, so that we get $f(10)$ – 'function f of 10' – we can calculate, from the *algebraical expression* of f, supposed given, the *corresponding* value of y, here $y = f(10)$. To be explicit, suppose $f(t)$ is that particular 'function' of t which is denoted in algebra by t^2, or $t \times t$. Then, when $t = 10$, we get $y = f(10)$, and hence *here* $y = 10^2, = 100$, for *this* value of t; when $t = \frac{1}{2}$, $y = \frac{1}{4}$, and so on, for *any* value of t.

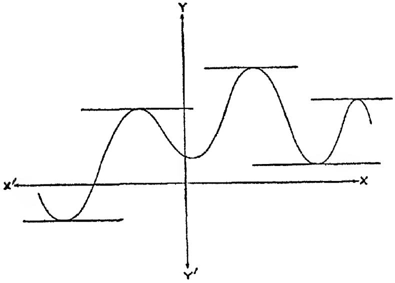
All this is familiar to anyone whose grammar-school education ended not more than thirty or forty years ago, but some may have forgotten what they did in arithmetic as children, just as others could not decline the Latin *mensa* to save their souls. But even the most forgetful will see that we could plot the graph of $y = f(t)$ for any particular form of f (when $f(t)$ is t^2 the graph is a parabola like an inverted arch). Imagine the

$y=t^2$

$-t$

$-y$

graph drawn. If it has on it *maxima* (highest) or *minima* (lowest) points – points higher or lower than those *in their immediate neighbourhoods* – we observe that the tangent at each of these *maxima* or *minima* is *parallel* to the t-axis. That is, the *slope* of the tangent at such an *extremum* (maximum *or* minimum) of the $f(t)$ we are plotting is *zero*. Thus if we were seeking the *extrema* of a given function $f(t)$ we should again have to solve our slope problem for the particular curve $y = f(t)$ and, having found the slope for the *general* point t, y, equate to zero the algebraical expression of this slope in order to find the values of t corresponding to the extrema. This is substantially what Fermat did in his method of maxima and minima invented in 1628–9, but not made semi-public till ten years later when Fermat sent an account of it through Mersenne to Descartes.



The scientific applications of this simple device – duly elaborated, of course, to take account of far more complicated problems than that just described – are numerous and far-reaching. In mechanics, for instance, as Lagrange discovered, there is a certain 'function' of the positions (co-ordinates) and velocities of the bodies concerned in a problem which, when made an extremum, furnishes us with the 'equations of motion' of the system considered, and these in turn enable us to determine the motion – to describe it completely – at any given instant. In physics there are many similar functions, each of which sums up most of an extensive branch of mathematical physics in the simple requirement that the function in question must be an extremum;* Hilbert in 1916 found one for general relativity. So Fermat was not fooling away his time when he amused himself in the leisure left from a laborious legal job by attacking the problem of maxima and minima. He himself

* This statement is sufficiently accurate for the present account. Actually, the values of the variables (co-ordinates and velocities) which make the function in question *stationary* (neither increasing nor decreasing, roughly) are those required. An *extremum* is stationary; but a *stationary* is not necessarily an extremum.

made one beautiful and astonishing application of his principles to optics. In passing it may be noted that this particular discovery has proved to be the germ of the newer quantum theory – in its mathematical aspect, that of 'wave mechanics' – elaborated since 1926. Fermat discovered what is usually called 'the principle of least time'. It would be more accurate to say 'extreme' (least *or* greatest) instead of 'least'.*

According to this principle, if a ray of light passes from a point A to another point B, being reflected and refracted ('refracted', that is, bent, as in passing from air to water, or through a jelly of variable density) in any manner during the passage, the path which it must take can be calculated – all its twistings and turnings due to refraction, and all its dodgings back and forth due to reflections – from the *single* requirement that the *time* taken to pass from A to B shall be an extremum (but see the preceding footnote).

From this principle Fermat deduced the familiar laws of reflection and refraction: the angle of incidence (in reflection) is equal to the angle of reflection; the sine of the angle of incidence (in refraction) is a *constant* number times the sine of the angle of refraction in passing from one medium to another.

The matter of analytical geometry has already been mentioned; Fermat was the first to apply it to space of three dimensions. Descartes contented himself with two dimensions. The extension, familiar to all students to-day, would not be self-evident to even a gifted man from Descartes' developments. It may be said that there is usually greater difficulty in finding a significant extension of a particular kind of geometry from space of two dimensions to three than there is in passing from three to four or five ... , or n. Fermat corrected Descartes in an essential point (that of the classification of curves by their degrees). It seems but natural that the somewhat touchy Descartes should have rowed with the imperturbable 'Gascon' Fermat. The soldier was frequently irritable and acid in his controversy over Fermat's method of tangents; the equable jurist was always unaffectedly courteous. As usually happens the man who kept his temper got the better of the argument.

* See footnote on page 67.

But Fermat deserved to win, not because he was a more skilful debater, but because he was right.

In passing, we should suppose that Newton would have heard of Fermat's use of the calculus and would have acknowledged the information. Until 1934 no evidence to this effect had been published, but in that year Professor L. T. More recorded in his biography of Newton a hitherto unnoticed letter in which Newton says explicitly that he got the hint of the method of the differential calculus from Fermat's method of drawing tangents.

We now turn to Fermat's greatest work, that which is intelligible to all, mathematicians and amateurs alike. This is the so-called 'theory of numbers', or 'the higher arithmetic', or finally, to use the unpedantic name which was good enough for Gauss, *arithmetic*.

The Greeks separated the miscellany which we lump together under the name 'arithmetic' in elementary textbooks into two distinct compartments, *logistica* and *arithmetica*, the first of which concerned the practical applications of reckoning to trade and daily life in general, and the second, arithmetic in the sense of Fermat and Gauss, who sought to discover the properties of numbers as such.

Arithmetic in its ultimate and probably most difficult problems investigates the mutual relationships of those common whole numbers 1, 2, 3, 4, 5, ... which we utter almost as soon as we learn to talk. In striving to elucidate these relationships, mathematicians have been driven to the invention of subtle and abstruse theories in algebra and analysis, whose forests of technicalities obscure the initial problems – those concerning 1, 2, 3, ... but whose real justification will be the solution of those problems. In the meantime the by-products of these apparently useless investigations amply repay those who undertake them by suggesting numerous powerful methods applicable to other fields of mathematics having direct contact with the physical universe. To give but one instance, the latest phase of algebra, that which is cultivated to-day by professional algebraists and which is throwing an entirely new light on the theory of algebraic equations, traces its origin directly to attempts to settle

Fermat's simple Last Theorem (which will be stated when the way has been prepared for it).

We begin with a famous statement Fermat made about prime numbers. A positive prime number, or briefly a *prime*, is any number greater than 1 which has as its divisors (without remainder) only 1 and the number itself; for example 2, 3, 5, 7, 13, 17 are primes, and so are 257, 65537. But 4294967297 is not a prime, because it has 641 as a divisor, nor is the number 18446744073709551617, because it is exactly divisible by 274177; both 641 and 274177 are primes. When we say in arithmetic that one number has as divisor another number, or is divisible by another, we mean *exactly divisible, without remainder*. Thus 14 is divisible by 7, 15 is not. The two large numbers were displayed above with malice aforethought for a reason that will be apparent in a moment. To recall another definition, the nth *power* of a given number, say N, is the result of multiplying together n N's, and is written N^n; thus $5^2 = 5 \times 5 = 25$; $8^4 = 8 \times 8 \times 8 \times 8 = 4096$. For uniformity N itself may be written as N^1. Again, such a pagoda as 2^{3^5} means that we are first to calculate 3^5 ($= 243$), and then 'raise' 2 to this power, 2^{243}; the resulting number has seventy-four digits.

The next point is of great importance in the life of Fermat, also in the history of mathematics. Consider the numbers 3, 5, 17, 257, 65537. They all belong to one 'sequence' of a specific kind, because they are all generated (from 1 and 2) by the same simple process, which will be seen from

$$3=2+1,\ 5=2^2+1,\ 17=2^4+1,\ 257=2^8+1,\ 65537=2^{16}+1;$$

and if we care to verify the calculation we easily see that the two large numbers displayed above are $2^{32} + 1$ and $2^{64} + 1$, also numbers of the sequence. We thus have seven numbers belonging to this sequence and *the first five of these numbers are primes, but the last two are not primes.*

Observing how the sequence is composed, we note the 'exponents' (the upper numbers indicating what powers of 2 are taken), namely 1, 2, 4, 8, 16, 32, 64, and we observe that these are 1 (which can be written 2^0, as in algebra, if we like, for uniformity), $2^1, 2^2, 2^3, 2^4, 2^5, 2^6$. Namely, our sequence is $2^{2^n}+1$, where n ranges over 0, 1, 2, 3, 4, 5, 6. We need not stop with $n = 6$; taking $n = 7, 8, 9, \ldots$, we may continue the sequence indefinitely, getting more and more enormous numbers.

Suppose we wish now to find out if a particular number of this sequence is a prime. Although there are many short cuts, and whole classes of trial divisors can be rejected by inspection, and although modern arithmetic limits the kinds of trial divisors that need be tested, our problem is of the same order of laboriousness as would be the dividing of the given number in succession by the primes 2, 3, 5, 7 ... which are less than the square root of the number. If none of these divides the number, the number is prime. Needless to say the labour involved in such a test, even using the known short cuts, would be prohibitive for even so small a value of n as 100. (The reader may assure himself of this by trying to settle the case $n = 8$.)

Fermat asserted that he was convinced that *all the numbers of the sequence are primes*. The displayed numbers (corresponding to $n = 5, 6$) contradict him, as we have seen. This is the point of historical interest which we wished to make: Fermat *guessed wrong, but he did not claim to have proved his guess*. Some years later he *did* make an obscure statement regarding what he had done, from which some critics infer that he had deceived himself. The importance of this fact will appear as we proceed.

As a psychological curiosity it may be mentioned that Zerah Colburn, the American lightning-calculating boy, when asked whether this sixth number of Fermat's (4294967297) was prime or not, replied after a short mental calculation that it was not, as it had the divisor 641. He was unable to explain the process by which he reached his correct conclusion. Colburn will occur again (in connexion with Hamilton).

Before leaving 'Fermat's numbers' $2^{2^n} + 1$ we shall glance ahead to the last decade of the eighteenth century where these mysterious numbers were partly responsible for one of the two or three most important events in all the long history of mathematics. For some time a young man in his eighteenth year had been hesitating – according to the tradition – whether to devote his superb talents to mathematics or to philology. He was equally gifted in both. What decided him was a beautiful discovery in connexion with a simple problem in elementary geometry familiar to every schoolboy.

A *regular* polygon of n sides has all its n sides equal and all its n angles equal. The ancient Greeks early found out how to construct regular polygons of 3, 4, 5, 6, 8, 10 and 15 sides by the use of straight-edge and compass alone, and it is an easy matter, with the same implements, to construct from a regular polygon having a given number of sides another regular polygon having twice that number of sides. The next step then would be to seek straight-edge and compass constructions for regular polygons of 7, 9, 11, 13, ... sides. Many sought, but failed to find, because such constructions are impossible, only they did not know it. After an interval of over 2200 years the young man hesitating between mathematics and philology took the next step – a long one – forward.

As has been indicated it is sufficient to consider only polygons having an *odd* number of sides. The young man proved that a straight-edge and compass construction of a regular polygon having an odd number of sides is possible when, and only when, that number is either a *prime* Fermat number (that is a prime of the form $2^{2^n} + 1$), or is made up by multiplying together *different* Fermat primes. Thus the construction is possible for 3, 5, or 15 sides as the Greeks knew, but not for 7, 9, 11 or 13 sides, and is also possible for 17 or 257 or 65537 or – for what the next prime in the Fermat sequence 3, 5, 17, 257, 65537, ... may be, *if there is one* – nobody yet (1936) knows – and the construction is also possible for 3×17, or $5 \times 257 \times 65537$ sides, and so on. It was this discovery, announced on 1 June 1796, but made on 30 March, which induced the young man to choose mathematics instead of philology as his life work. His name was Gauss.

As a discovery of another kind which Fermat made concerning numbers we state what is known as 'Fermat's Theorem' (*not* his 'Last Theorem'). If n is any whole number and p any prime, then $n^p - n$ is divisible by p. For example, taking $p = 3$, $n = 5$, we get $5^3 - 5$, or $125 - 5$, which is 120 and is 3×40; for $n = 2$, $p = 11$, we get $2^{11} - 2$, or $2048 - 2$, which is $2046 = 11 \times 186$.

It is difficult if not impossible to state why some theorems in arithmetic are considered 'important' while others, equally difficult to prove, are dubbed trivial. One criterion, although not necessarily conclusive, is that the theorem shall be of use in other fields of mathematics. Another is that it shall suggest researches in arithmetic or in mathematics generally, and a third that it shall be in some respect universal. Fermat's theorem just stated satisfies all of these somewhat arbitrary demands: it is of indispensable use in many departments of mathematics, including the theory of groups (see Chapter 15), which in turn is at the root of the theory of algebraic equations; it has suggested many investigations, of which the entire subject of primitive roots may be recalled to mathematical readers as an important instance; and finally it is universal in the sense that it states a property of *all* prime numbers – such general statements are extremely difficult to find and very few are known.

As usual, Fermat stated his theorem about $n^p - n$ without proof. The first proof was given by Leibniz in an undated manuscript, but he appears to have known a proof before 1683. The reader may like to test his own powers on trying to devise a proof. All that is necessary are the following facts, which can be proved but may be assumed for the purpose in hand: a given whole number can be built up in one way only – apart from rearrangements of factors – by multiplying together primes; if a prime divides the product (result of multiplying) of two whole numbers, it divides at least one of them. To illustrate: $24 = 2 \times 2 \times 2 \times 3$, and 24 cannot be built up by multiplication of primes in any essentially different way – we consider $2 \times 2 \times 2 \times 3$, $2 \times 2 \times 3 \times 2$, $2 \times 3 \times 2 \times 2$ and $3 \times 2 \times 2 \times 2$ as the same; 7 divides 42, and $42 = 2 \times 21 = 3 \times 14 = 6 \times 7$, in each of which 7 divides at least one of the numbers multiplied together to give 42; again, 98 is divisible by 7, and $98 = 7 \times 14$, in which case 7 divides both 7 and 14, and hence at least one of them. From these two facts the proof can be given in less than half a page. It is within the understanding of any normal fourteen-year-old, but it is safe to wager that out of a million human beings of normal intelligence of any or all ages, less than ten of those who had had no more mathematics than grammar-grade arithmetic would succeed in finding a proof within a reasonable time – say a year.

This seems to be an appropriate place to quote some famous remarks of Gauss concerning the favourite field of Fermat's interests and his own. The translation is that of the Irish arithmetician H. J. S. Smith (1826–83), from Gauss' introduction to the collected mathematical papers of Eisenstein published in 1847.

'The higher arithmetic presents us with an inexhaustible store of interesting truths – of truths, too, which are not isolated, but stand in a close internal connexion, and between which, as our knowledge increases, we are continually discovering new and sometimes wholly unexpected ties. A great part of its theories derives an additional charm from the peculiarity that important propositions, with the impress of simplicity upon them, are often easily discoverable by induction, and yet are of so profound a character that we cannot find their demonstration till after many vain attempts; and even then, when we do succeed, it is often by some tedious and artificial process, while the simpler methods may long remain concealed.'

One of these interesting truths which Gauss mentions is sometimes considered the most beautiful (but not the most important) thing about numbers that Fermat discovered: every prime number of the form $4n + 1$ is a sum of two squares, and is such a sum in only one way. It is easily proved that no number of the form $4n - 1$ is a sum of two squares. As all primes greater than 2 are readily seen to be of one or other of these forms, there is nothing to add. For an example, 37 when divided by 4 yields the remainder 1, so 37 must be the sum of two squares of whole numbers. By trial (there are better ways) we find indeed that $37 = 1 + 36, = 1^2 + 6^2$, and that there are no other squares x^2 and y^2 such that $37 = x^2 + y^2$. For the prime 101 we have $1^2 + 10^2$; for 41 we find $4^2 + 5^2$. On the other hand $19, = 4 \times 5 - 1$, is not a sum of two squares.

As in nearly all of his arithmetical work, Fermat left no proof of this theorem. It was first proved by the great Euler in 1749 after he had struggled, off and on, for *seven years* to find a proof. But Fermat does describe the ingenious method, which he invented, whereby he proved this and some others of his wonderful results. This is called 'infinite descent', and is infinitely more difficult to accomplish than Elijah's ascent to Heaven. His own account is both concise and clear, so we shall give a free translation from his letter of August 1659 to Carcavi.

'For a long time I was unable to apply my method to affirmative propositions, because the twist and the trick for getting at them is much more troublesome than that which I use for negative propositions. Thus, when I had to prove that *every prime number which exceeds a multiple of 4 by 1 is composed of two squares*, I found myself in a fine torment. But at last a meditation many times repeated gave me the light I lacked, and now affirmative propositions submit to my method, with the aid of certain new principles which necessarily must be adjoined to it. The course of my reasoning in affirmative propositions is such: if an arbitrarily chosen prime of the form $4n + 1$ is not a sum of two squares, [I prove that] there will be another of the same nature, less than the one chosen, and [therefore] next a third still less, and so on. Making an infinite descent in this way we finally arrive at the number 5, the least of all the numbers of this kind [$4n + 1$]. [By the proof mentioned and the preceding argument from it], it follows that 5 is not a sum of two squares. But it is. Therefore we must infer by a *reductio ad absurdum* that all numbers of the form $4n + 1$ are sums of two squares.'

All the difficulty in applying descent to a new problem lies in the first step, that of proving that *if* the assumed or conjectured proposition is *true* of any number of the kind concerned chosen at random, *then* it will be *true* of a *smaller* number of the *same kind*. There is no general method; applicable to all problems, for taking this step. Something rarer than grubby patience or the greatly overrated 'infinite capacity for taking pains' is needed to find a way through the wilderness. Those who imagine genius is nothing more than the ability to be a good bookkeeper may be recommended to exert their infinite patience on Fermat's Last Theorem. Before stating the theorem we give one more example of the deceptively simple problems Fermat attacked and solved. This will introduce the topic of *Diophantine analysis*, in which Fermat excelled.

Anyone playing with numbers might well pause over the curious fact that $27 = 25 + 2$. The point of interest here is that both 27 and 25 are exact powers, namely $27 = 3^3$ and $25 = 5^2$. Thus we observe that $y^3 = x^2 + 2$ has a solution in *whole numbers* x, y; the solution is $y = 3, x = 5$. As a sort of super-intelligence test the reader may now prove that $y = 3, x = 5$ are the *only* whole numbers which satisfy the equation. It is not easy. In fact it requires more innate intellectual capacity to dispose of this apparently childish thing than it does to grasp the theory of relativity.

The equation $y^3 = x^2 + 2$, *with the restriction that the solution* y, x *is to be in whole numbers*, is *indeterminate* (because there are more unknowns, namely two, x and y, than there are equations, namely one, connecting them) and *Diophantine*, after the Greek who was one of the first to insist upon *whole number* solutions of equations or, less stringently, on *rational* (fractional) solutions. There is no difficulty whatever in describing an infinity of solutions *without* the restriction to whole numbers: thus we may give x *any* value we please and then determine y by adding 2 to this x^2 and extracting the cube root of the result. But the *Diophantine* problem of finding *all* the *whole number* solutions is quite another matter. The solution $y = 3, x = 5$ is seen 'by inspection'; the difficulty of the problem is to prove that there are *no other* whole numbers y, x which will satisfy the equation. Fermat proved that there are none but, as usual, suppressed his proof, and it was not until many years after his death that a proof was found.

This time he was not guessing; the problem is hard; he asserted that he had a proof; a proof was later found. And so for all of his positive assertions with the one exception of the seemingly simple one which he made in his Last Theorem and which mathematicians, struggling for nearly 300 years, have been unable to prove; whenever Fermat asserted that he had *proved* anything, the statement, with the one exception noted, has subsequently been proved. Both his scrupulously honest character and his unrivalled penetration as an arithmetician substantiate the claim made for him by some, but not by all, that he knew what he was talking about when he asserted that he possessed a proof of his theorem.

It was Fermat's custom in reading Bachet's *Diophantus* to record the results of his meditations in brief marginal notes in his copy. The margin was not suited for the writing out of proofs. Thus, in commenting on the eighth problem of the Second Book of Diophantus' Arithmetic, which asks for the solution in rational numbers (fractions or whole numbers) of the equation $x^2 + y^2 = a^2$, Fermat comments as follows:

'On the contrary, it is impossible to separate a cube into two cubes, a fourth power into two fourth powers, or, generally, any power above the second into two powers of the same degree: I have discovered a truly marvellous demonstration [of this general theorem] which this margin is too narrow to contain' (Fermat, *Oeuvres*, III, p. 241). This is his famous Last Theorem, which he discovered about the year 1637.

To restate this in modern language: Diophantus' problem is to find whole numbers or fractions x, y, a such that $x^2 + y^2 = a^2$; Fermat asserts that *no* whole numbers or fractions exist such that $x^3 + y^3 = a^3$, or $x^4 + y^4 = a^4$, or, generally, such that $x^n + y^n = a^n$ if n is a whole number greater than 2.

Diophantus' problem has an infinity of solutions; specimens are $x = 3, y = 4, a = 5$; $x = 5, y = 12, a = 13$. Fermat himself gave a proof by his method of infinite descent for the impossibility of $x^4 + y^4 = a^4$. Since his day $x^n + y^n = a^n$ has been proved impossible in whole numbers (or fractions) for a great many numbers n (up to all primes* less than $n = 14{,}000$ if none of the numbers x, y, a is divisible by n), but this is not what is required. A proof disposing of *all* n's greater than 2 is demanded. Fermat said he possessed a 'marvellous' proof.

After all that has been said, is it likely that he had deceived himself? It may be left up to the reader. One great arithmeti

* The reader can easily see that it suffices to dispose of the case where n is an odd prime, since, in algebra, $u^{ab} = (u^a)^b$, where u, a, b are any numbers.

cian, Gauss, voted against Fermat. However, the fox who could not get at the grapes declared they were sour. Others have voted for him. Fermat was a mathematician of the first rank, a man of unimpeachable honesty, and an arithmetician without a superior in history*.

* In 1908 the late Professor Paul Wolfskehl (German) left 100,000 marks to be awarded to the first person giving a *complete* proof of Fermat's Last Theorem. The inflation after the World War reduced this prize to a fraction of a cent, which is what the mercenary will now get for a proof.

CHAPTER FIVE

'GREATNESS AND MISERY OF MAN'

Pascal

YOUNGER by twenty-seven years than his great contemporary Descartes, Blaise Pascal was born at Clermont, Auvergne, France, on 19 June 1623, and outlived Descartes by twelve years. His father, Étienne Pascal, president of the court of aids at Clermont, was a man of culture and had some claim to intellectual distinction in his own times; his mother, Antoinette Bégone, died when her son was four. Pascal had two beautiful and talented sisters, Gilberte, who became Madame Périer, and Jacqueline, both of whom, the latter especially, played important parts in his life.

Blaise Pascal is best known to the general reader for his two literary classics, the *Pensées* and the *Lettres écrites par Louis de Montalte à un provincial de ses amis* commonlyreferred to as the 'Provincial Letters', and it is customary to condense his mathematical career to a few paragraphs in the display of his religious prodigies. Here our point of view must necessarily be somewhat oblique, and we shall consider Pascal primarily as a highly gifted mathematician who let his masochistic proclivities for self-torturing and profitless speculations on the sectarian controversies of his day degrade him to what would now be called a religious neurotic.

On the mathematical side Pascal is perhaps the greatest might-have-been in history. He had the misfortune to precede Newton by only a few years and to be a contemporary of Descartes and Fermat, both more stable men than himself. His most novel work, the creation of the mathematical theory of probability, was shared with Fermat, who could easily have done it alone. In geometry, for which he is famous as a sort of infant prodigy, the creative idea was supplied by a man – Desargues – of much less celebrity.

In his outlook on experimental science Pascal had a far clearer vision than Descartes – from a modern point of view – of the scientific method. But he lacked Descartes' singleness of aim, and although he did some first-rate work, allowed himself to be deflected from what he might have done by his morbid passion for religious subtleties.

It is useless to speculate on what Pascal might have done. Let his life tell what he actually did. Then, if we choose, we can sum him up as a mathematician by saying that he did what was in him and that no man can do more. His life is a running commentary on two of the stories or similes in that New Testament which was his constant companion and unfailing comfort: the parable of the talents, and the remark about new wine bursting old bottles (or skins). If ever a wonderfully gifted man buried his talent, Pascal did; and if ever a medieval mind was cracked and burst asunder by its attempt to hold the new wine of seventeenth-century science, Pascal's was. His great gifts were bestowed upon the wrong person.

At the age of seven Pascal moved from Clermont with his father and sisters to Paris. About this time the father began teaching his son. Pascal was an extremely precocious child. Both he and his sisters appear to have had more than their share of nature's gifts. But poor Blaise inherited (or acquired) a wretched physique along with his brilliant mind, and Jacqueline, the more gifted of his sisters, seems to have been of the same stripe as her brother, for she too fell a victim to morbid religiosity.

At first everything went well enough. Pascal senior, astonished at the ease with which his son absorbed the stock classical education of the day, tried to hold the boy down to a reasonable pace to avoid injuring his health. Mathematics was taboo, on the theory that the young genius might overstrain himself by using his head. His father was an excellent drillmaster but a poor psychologist. His ban on mathematics naturally excited the boy's curiosity. One day when he was about twelve Pascal demanded to know what geometry was about. His father gave him a clear description. This set Pascal off like a hare after his true vocation. Contrary to his own opinion in later life he had been called by God, not to torment the Jesuits, but to be a great mathematician. But his hearing was defective at the time and he got his orders confused.

What happened when Pascal began the study of geometry has become one of the legends of mathematical precocity. In passing it may be remarked that infant prodigies in mathematics do not invariably blow up as they are sometimes said to do. Precocity in mathematics has often been the first flush of a glorious maturity, in spite of the persistent superstition to the contrary. In Pascal's case early mathematical genius was not extinguished as he grew up but stifled under other interests. The ability to do first-class mathematics persisted, as will be seen from the episode of the cycloid, late into his all too brief life, and if anything is to be blamed for his comparatively early mathematical demise it is probably his stomach. His first spectacular feat was to prove, entirely on his own initiative, and without a hint from any book, that the sum of the angles of a triangle is equal to two right angles. This encouraged him to go ahead at a terrific pace.

Realizing that he had begotten a mathematician, Pascal senior wept with joy and gave his son a copy of Euclid's *Elements*. This was quickly devoured, not as a task, but as play. The boy gave up his games to geometrize. In connexion with Pascal's rapid mastery of Euclid, sister Gilberte permits herself an over-appreciative fib. It is true that Pascal had found out and proved several of Euclid's propositions for himself before he ever saw the book. But what Gilberte romances about her brilliant young brother is less probable than a throw of a billion aces in succession with one die, for the reason that it is infinitely improbable. Gilberte declared that her brother had rediscovered for himself the first thirty-two propositions of Euclid, and that he had found them *in the same order* as that in which Euclid sets them forth. The thirty-second proposition is indeed the famous one about the sum of the angles of a triangle which Pascal rediscovered. Now, there may be only one way of doing a thing right, but it seems more likely that there are an infinite number of ways of doing it wrong. We know to-day that Euclid's allegedly rigorous demonstrations, even in the first four of his propositions, are no proofs at all. That Pascal faithfully duplicated all of Euclid's oversights on his own account is an easy story to tell but a hard one to believe. However, we can forgive Gilberte for bragging. Her brother was worth it. At the age of fourteen he was admitted to the weekly scientific discussions, conducted by Mersenne, out of which the French Academy of Sciences developed.

While young Pascal was fast making a geometer of himself, old Pascal was making a thorough nuisance of *him*self with the authorities on account of his honesty and general uprightness. In particular he disagreed with Cardinal Richelieu over a little matter of imposing taxes. The Cardinal was incensed; the Pascal family went into hiding till the storm blew over. It is said that the beautiful and talented Jacqueline rescued the family and restored her father to the light of the Cardinal's countenance by her brilliant acting, incognito, in a play presented for Richelieu's entertainment. On inquiring the name of the charming young artiste who had captivated his clerical fancy, and being told that she was the daughter of his minor enemy, Richelieu very handsomely forgave the whole family and planted the father in a political job at Rouen. From what is known of that wily old serpent, Cardinal Richelieu, this pleasing tale is probably a fish story. Anyhow, the Pascals once more found a job and security at Rouen. There young Pascal met the tragic dramatist Corneille, who was duly impressed with the boy's genius. At the time Pascal was all mathematician, so probably Corneille did not suspect that his young friend was to become one of the great creators of French prose.

All this time Pascal was studying incessantly. Before the age of sixteen (about 1639)* he had proved one of the most beautiful theorems in the whole range of geometry. Fortunately it can be

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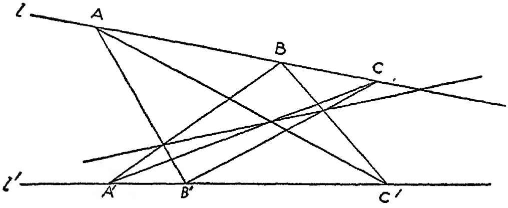
described in terms comprehensible to anyone. Sylvester, a mathematician of the nineteenth century whom we shall meet later, called Pascal's great theorem a sort of 'cat's cradle'. We state first a special form of the general theorem that can be constructed with the use of a ruler only.

Label two intersecting straight lines l and l'. On l take any three distinct points A, B, C, and on l' any three distinct points A', B', C'. Join up these points by straight lines, crisscross, as follows: A and B′, A′ and B, B and C′, B′ and C, C and A′, C′ and A. The two lines in each of these pairs intersect in a point. We thus get three points. The special case of Pascal's theorem which we are now describing states that these three points lie on one straight line.



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In Chapter 2 we stated what a conic section is. Imagine any conic section, for definiteness say an ellipse. On it mark any six points, A, B, C, D, E, F, and join them up, in this order, by straight lines. We thus have a six-sided figure inscribed in the conic section, in which AB and DE, BC and EF, CD and FA are pairs of opposite sides. The two lines in each of these three pairs intersect in a point; the three points of intersection lie on one straight line (see figure in Chapter 13, page 238). This is number of ways of doing it wrong. We know to-day that Euclid's allegedly rigorous demonstrations, even in the first four of his propositions, are no proofs at all. That Pascal faithfully duplicated all of Euclid's oversights on his own account is an easy story to tell but a hard one to believe. However, we can forgive Gilberte for bragging. Her brother was worth it. At the age of fourteen he was admitted to the weekly scientific discussions, conducted by Mersenne, out of which the French Academy of Sciences developed.

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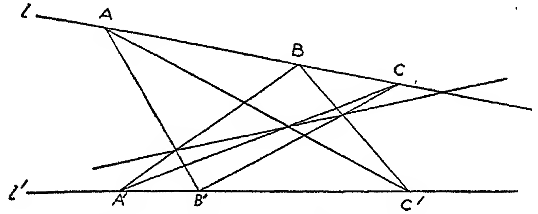

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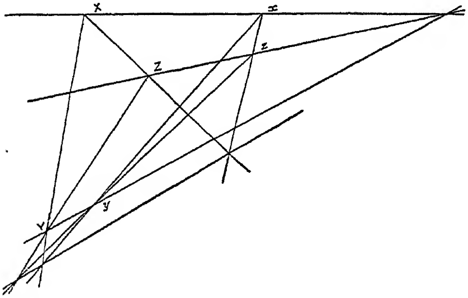
Pascal's theorem; the figure which it furnishes is what he called the 'mystic hexagram'. He probably first proved it true for a circle and then passed by projection to any conic section. Only a straight-edge and a pair of compasses are required if the reader wishes to see what the figure looks like for a circle.

There are several amazing things about this wonderful proposition, not the least of which is that it was discovered and proved by a boy of sixteen. Again, in his *Essai pour les Coniques* (Essay on Conics), written around his great theorem by this extraordinarily gifted boy, no fewer than 400 propositions on conic sections, including the work of Apollonius and others, were systematically deduced as corollaries, by letting pairs of the six points move into coincidence, so that a chord became a



tangent, and other devices. The full *Essai* itself was never published and is apparently lost irretrievably, but Leibniz saw and inspected a copy of it. Further, the *kind* of geometry which Pascal is doing here differs fundamentally from that of the Greeks; it is not *metrical*, but *descriptive*, or *projective*. Magnitudes of lines or angles cut no figure in either the statement or the proof of the theorem. This one theorem in itself suffices to abolish the stupid definition of mathematics, inherited from Aristotle and still sometimes reproduced in dictionaries, as the science of 'quantity'. There are no 'quantities' in Pascal's geometry.

To see what the *projectivity* of the theorem means, imagine a (circular) cone of light issuing from a point and pass a flat sheet of glass through the cone in varying positions. The boundary curve of the figure in which the sheet cuts the cone is a *conic section*. If Pascal's 'mystic hexagram' be drawn on the glass for any given position, and another flat sheet of glass be passed through the cone so that the shadow of the hexagram falls on it, *the shadow will be another 'mystic hexagram'* with its three points of intersection of opposite pairs of sides lying on one straight line, the shadow of the 'three-point-line' in the original hexagram. That is, Pascal's theorem is *invariant* (unchanged) *under conical projection*. The metrical properties of figures studied in common elementary geometry are *not* invariant under projection; for example, the shadow of a right angle is not a right angle for all positions of the second sheet. It is obvious that this kind of *projective*, or *descriptive* geometry, is one of the geometries naturally adapted to some of the problems of perspective. The *method* of projection was used by Pascal in proving his theorem, but had been applied previously by Desargues in deducing the result stated above concerning two triangles 'in perspective'. Pascal gave Desargues full credit for his great invention.

All this brilliance was purchased at a price. From the age of seventeen to the end of his life at thirty-nine, Pascal passed but few days without pain. Acute dyspepsia made his days a torment and chronic insomnia his nights half-waking nightmares. Yet he worked incessantly. At the age of eighteen he invented and made the first calculating machine in history – the ancestor of all the arithmetical machines that have displaced armies of clerks from their jobs in our own generation. We shall see farther on what became of this ingenious device. Five years later, in 1646, Pascal suffered his first 'conversion'. It did not take deeply, possibly because Pascal was only twenty-three and still absorbed in his mathematics. Up to this time the family had been decently enough devout; now they all seem to have gone mildly insane.

It is difficult for a modern to recreate the intense religious passions which inflamed the seventeenth century, disrupting families and hurling professedly Christian countries and sects at one another's throats. Among the would-be religious reformers of the age was Cornelius Jansen (1585–1638), a flamboyant Dutchman who became bishop of Ypres. A cardinal point of his dogma was the necessity for 'conversion' as a means to 'grace', somewhat in the manner of certain flourishing sects to-day. Salvation, however, at least to an unsympathetic eye, appears to have been the lesser of Jansen's ambitions. God, he was convinced, had especially elected him to blast the Jesuits in this life and toughen them for eternal damnation in the next. This was his call, his mission. His creed was neither Catholicism nor Protestantism, although it leaned rather toward the latter. Its moving spirit was, first, last and all the time, a rabid hatred of those who disputed its dogmatic bigotries. The Pascal family now (1646) ardently – but not too ardently at first – embraced this unlovely creed of Jansenism. Thus Pascal, at the early age of twenty-three, began to die off at the top. In the same year his whole digestive tract went bad and he suffered a temporary paralysis. But he was not yet dead intellectually.

His scientific greatness flared up again in 1648 in an entirely new direction. Carrying on the work of Torricelli (1608–47) on atmospheric pressure, Pascal surpassed him and demonstrated that he understood the scientific method which Galileo, the teacher of Torricelli, had shown the world. By experiments with the barometer, which he suggested, Pascal proved the familiar facts now known to every beginner in physics regarding the pressure of the atmosphere. Pascal's sister Gilberte had married a M. Périer. At Pascal's suggestion, Périer performed the experiment of carrying a barometer up the Puy de Dôme in Auvergne and noting the fall of the column of mercury as the atmospheric pressure decreased. Later Pascal, when he moved to Paris with his sister Jacqueline, repeated the experiment on his own account.

Shortly after Pascal and Jacqueline had returned to Paris they were joined by their father, now fully restored to favour as a state councillor. Presently the family received a somewhat formal visit from Descartes. He and Pascal talked over many things, including the barometer. There was little love lost between the two. For one thing, Descartes had openly refused to believe the famous *Essai pour les coniques* had been written by a boy of sixteen. For another, Descartes suspected Pascal of having filched the idea of the barometric experiments from himself, as he had discussed the possibilities in letters to Mersenne. Pascal, as has been mentioned, had been attending the weekly meetings at Father Mersenne's since he was fourteen. A third ground for dislike on both sides was furnished by their religious antipathies. Descartes, having received nothing but kindness all his life from the Jesuits, loved them; Pascal, following the devoted Jansen, hated a Jesuit worse than the devil is alleged to hate holy water. And finally, according to the candid Jacqueline, both her brother and Descartes were intensely jealous, each of the other. The visit was rather a frigid success.

The good Descartes however did give his young friend some excellent advice in a truly Christian spirit. He told Pascal to follow his own example and lie in bed every day till eleven. For poor Pascal's awful stomach he prescribed a diet of nothing but beef tea. But Pascal ignored the kindly meant advice, possibly because it came from Descartes. Among other things which Pascal totally lacked was a sense of humour.

Jacqueline now began to drag her genius of a brother down – or up; it all depends upon the point of view. In 1648, at the impressionable age of twenty-three, Jacqueline declared her intention of moving to Port Royal, near Paris, the main hang-out of the Jansenists in France, to become a nun. Her father sat down heavily on the project, and the devoted Jacqueline concentrated her thwarted efforts on her erring brother. She suspected he was not yet so thoroughly converted as he might have been, and apparently she was right. The family now returned to Clermont for two years.

During these two swift years Pascal seems to have become almost half human, in spite of sister Jacqueline's fluttering admonitions that he surrender himself utterly to the Lord. Even the recalcitrant stomach submitted to rational discipline for a few blessed months.

It is said by some and hotly denied by others that Pascal during this sane interlude and later for a few years discovered the predestined uses of wine and women. He did not sing. But these rumours of a basely human humanity may, after all, be nothing more than rumours. For after his death Pascal quickly passed into the Christian hagiocracy, and any attempts to get at the facts of his life as a human being were quietly but rigidly suppressed by rival factions, one of which strove to prove that he was a devout zealot, the other a sceptical atheist, but both of which declared that Pascal was a saint not of this earth.

During these adventurous years the morbidly holy Jacqueline continued to work on her frail brother. By a beautiful freak of irony Pascal was presently to be converted – for good, this time – and it was to be *his* lot to turn the tables on his too pious sister and drive *her* into the nunnery which now, perhaps, seemed less desirable. This, of course, is not the orthodox interpretation of what happened; but to anyone other than a blind partisan of one sect or the other – Christian or Atheist – it is a more rational account of the unhealthy relationship between Pascal and his unmarried sister than that which is sanctioned by tradition.

Any modern reader of the *Pensées* must be struck by a certain something or another which either completely escaped our more reticent ancestors or was ignored by them in their wiser charity. The letters, too, reveal a great deal which should have been decently buried. Pascal's ravings in the *Pensées* about 'lust' give him away completely, as do also the well-attested facts of his unnatural frenzies at the sight of his married sister Gilberte naturally caressing her children.

Modern psychologists, no less than the ancients with ordinary common sense, have frequently remarked the high correlation between sexual repression and morbid religious fervour. Pascal suffered from both, and his immortal *Pensées* are a brilliant if occasionally incoherent testimonial to his purely physiological eccentricities. If only the man could have been human enough to let himself go when his whole nature told him to cut loose, he might have lived out everything that was in him, instead of smothering the better half of it under a mass of meaningless mysticism and platitudinous observations on the misery and dignity of man.

Always shifting about restlessly the family returned to Paris in 1650. The next year the father died. Pascal seized the occasion to write Gilberte and her husband a lengthy sermon on death in general. This letter has been much admired. We need not reproduce any of it here; the reader who wishes to form his own opinion of it can easily locate it. Why this priggish effusion of pietistic and heartless moralizing on the death of a presumably beloved parent should ever have excited admiration instead of contempt for its author is, like the love of God which the letter in part dwells upon *ad nauseam*, a mystery that passeth all understanding. However, there is no arguing about tastes, and those who like the sort of thing that Pascal's much-quoted letter is, may be left to their undisturbed enjoyment of what is, after all, one of the masterpieces of self-conscious self-revelation in French literature.

A more practical result of Pascal senior's death was the opportunity which it offered Pascal, as administrator of the estate, of returning to normal intercourse with his fellow-men. Encouraged by her brother, sister Jacqueline now joined Port Royal, her father being no longer capable of objecting. Her sweet concern over her brother's soul was now spiced by a quite human quarrel over the division of the estate.

A letter of the preceding year (1650) reveals another facet of Pascal's reverent character, or possibly his envy of Descartes. Dazzled by the transcendent brilliance of the Swedish Christine, Pascal humbly begged to lay his calculating machine at the feet of 'the greatest princess in the world', who, he declares in liquid phrases dripping strained honey and melted butter, is as eminent intellectually as she is socially. What Christine did with the machine is not known. She did not invite Pascal to replace the Descartes whom she had done in.

At last, on 23 November 1654, Pascal was really converted. According to some accounts he had been living a fast life for three years. The best authorities seem to agree that there is not much in this tradition and that his life was not so fast after all. He had merely been doing his poor suffering best to live like a normal human being and to get something more than mathematics and piety out of life. On the day of his conversion he was driving a four-in-hand when the horses bolted. The leaders plunged over the parapet of the bridge at Neuilly, but the traces broke, and Pascal remained on the road.

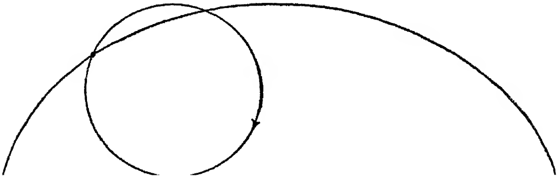
To a man of Pascal's mystical temperament this lucky escape from a violent death was a direct warning from Heaven to pull himself up sharply on the brink of the moral precipice over which he, the victim of his morbid self-analysis, imagined he was about to plunge. He took a small piece of parchment, inscribed on it some obscure sentiments of mystical devotion, and thenceforth wore it next to his heart as an amulet to protect him from temptation and remind him of the goodness of God which had snatched him, a miserable sinner, from the very mouth of hell. Only once thereafter did he fall from grace (in his own pitiable opinion), although all the rest of his life he was haunted by hallucinations of a precipice before his feet.

Jacqueline, now a postulant for the nunnery at Port Royal, came to her brother's aid. Partly on his own account, partly because of his sister's persuasive pleadings, Pascal turned his back on the world and took up his residence at Port Royal, to bury his talent thenceforth in contemplation on 'the greatness and misery of man.' This was in 1654, when Pascal was thirty-one. Before for ever quitting things of the flesh and the mind, however, he had completed his most important contribution to mathematics, the joint creation, with Fermat, of the mathematical theory of probability. Not to interrupt the story of his life we shall defer an account of this for the moment.

His life at Port Royal was at least sanitary if not exactly as sane as might have been wished, and the quiet, orderly routine benefited his precarious health considerably. It was while at Port Royal that he composed the famous *Provincial Letters*, which were inspired by Pascal's desire to aid in acquitting Arnauld, the leading light of the institution, of the charge of heresy. These famous letters (there were eighteen, the first of which was printed on 23 January 1656) are masterpieces of controversial skill, and are said to have dealt the Jesuits a blow from which their Society has never fully recovered. However, as a commonplace of objective observation which anyone with eyes in his head can verify for himself, the Society of Jesus still flourishes; so it may be reasonably doubted whether the *Provincial Letters* had in them the deadly potency ascribed to them by sympathetic critics.

In spite of his intense preoccupation with matters pertaining to his salvation and the misery of man, Pascal was still capable of doing excellent mathematics, although he regarded the pursuit of all science as a vanity to be eschewed for its derogatory effects on the soul. Nevertheless he did fall from grace once more, but only once. The occasion was the famous episode of the cycloid.

This beautifully proportioned curve (it is traced out by the motion of a fixed point on the circumference of a wheel rolling along a straight line on a flat pavement) seems to have turned up first in mathematical literature in 1501, when Charles



Bouvelles described it in connexion with the squaring of the circle. Galileo and his pupil Viviani studied it and solved the problem of constructing a tangent to the curve at any point (a problem which Fermat solved at once when it was proposed to him), and Galileo suggested its use as an arch for bridges. Since reinforced concrete has become common, cycloidal arches are frequently seen on highway viaducts. For mechanical reasons (unknown to Galileo) the cycloidal arch is superior to any other in construction. Among the famous men who investigated the cycloid was Sir Christopher Wren, the architect of St Paul's Cathedral, who determined the length of any arc of the curve and its centre of gravity, while Huygens, for mechanical reasons, introduced it into the construction of pendulum clocks. One of the most beautiful of all the discoveries of Huygens (1629–95) was made in connexion with the cycloid. He proved that it is the *tautochrone*, that is, the curve (when turned upside down like a bowl) down which beads placed *anywhere* on it will all slide to the lowest point under the influence of gravity *in the same time*. On account of its singular beauty, elegant properties, and the endless rows which it stirred up between quarrelsome mathematicians challenging one another to solve this or that problem in connexion with it, the cycloid has been called 'the Helen of Geometry', after the Graeco-Trojan lady whose mere face is said to have 'launched a thousand ships'.

Among the miseries which afflicted the wretched Pascal were persistent insomnia and bad teeth – in a day when such dentistry as was practised was done by the barber with a strong pair of forceps and brute force. Lying awake one night (1658) in the tortures of toothache, Pascal began to think furiously about the cycloid to take his mind off the excruciating pain. To his surprise he noticed presently that the pain had stopped. Interpreting this as a signal from Heaven that he was not sinning in thinking about the cycloid rather than his soul, Pascal let himself go. For eight days he gave himself up to the geometry of the cycloid and succeeded in solving many of the main problems in connexion with it. Some of the things he discovered were issued under the pseudonym of Amas Dettonville as challenges to the French and English mathematicians. In his treatment of his rivals in this matter Pascal was not always as scrupulous as he might have been. It was his last flicker of mathematical activity and his only contribution to science after his entry into Port Royal.

The same year (1658) he fell more seriously ill than he had yet been in all his tormented life. Racking and incessant headaches now deprived him of all but the most fragmentary snatches of sleep. He suffered for four years, living ever more ascetically. In June 1662 he gave up his own house to a poor family suffering from smallpox, as an act of self-denial, and went to live with his married sister. On 19 August 1662 his tortured existence came to an end in convulsions. He died at the age of thirty-nine.

The post-mortem revealed what had been expected regarding the stomach and vital organs; it also disclosed a serious lesion of the brain. Yet in spite of all this Pascal had done great work in mathematics and science and had left a name in literature that is still respected after nearly three centuries.

The beautiful things Pascal did in geometry, with the possible exception of the 'mystic hexagram', would all have been done by other men had he not done them. This holds in particular for the investigations on the cycloid. After the invention of the calculus all such things became incomparably easier than they had been before and in time passed into the textbooks as mere exercises for young students. But in the joint creation with Fermat of the mathematical theory of probabilities Pascal made a new world. It seems quite likely that Pascal will be remembered for his part in this great and ever-increasingly more important invention long after his fame as a writer has been forgotten. The *Pensées* and the *Provincial Letters*, apart from their literary excellences, appeal principally to a type of mind that is rapidly becoming extinct. The arguments for or against a particular point strike a modern mind as either trivial or unconvincing, and the very questions to which Pascal addressed himself with such fervent zeal now seem strangely ridiculous. If the problems which he discussed on the greatness and misery of man are indeed as profoundly important as enthusiasts have claimed, and not mere pseudo-problems mystically stated and incapable of solution, it seems unlikely that they will ever be solved by platitudinous moralizing. But in his theory of probabilities Pascal stated and solved a genuine problem, that of bringing the superficial lawlessness of pure chance under the domination of law, order, and regularity, and to-day this subtle theory appears to be at the very roots of human knowledge no less than at the foundation of physical science. Its ramifications are everywhere, from the quantum theory to epistemology.

The true founders of the mathematical theory of probability were Pascal and Fermat, who developed the fundamental principles of the subject in an intensely interesting correspondence during the year 1654. This correspondence is now readily available in the *Œuvres de Fermat* (edited by P. Tannéry and C. Henry, vol. 2, 1904). The letters show that Pascal and

Fermat participated equally in the creation of the theory. Their correct solutions of problems differ in details but not in fundamental principles. Because of the tedious enumeration of possible cases in a certain problem on 'points' Pascal tried to take a short cut and fell into error. Fermat pointed out the mistake, which Pascal acknowledged. The first letter of the series has been lost but the occasion of the correspondence is well attested.

The initial problem which started the whole vast theory was proposed to Pascal by the Chevalier de Méré, more or less of a professional gambler. The problem was that of 'points': each of two players (at dice, say) requires a certain number of points to win the game; if they quit the game before it is finished, how should the stakes be divided between them? The score (number of points) of each player is given at the time of quitting, and the problem amounts to determining the probability which each player has at a given stage of the game of winning the game. It is assumed that the players have equal chances of winning a single point. The solution demands nothing more than sound common sense; the *mathematics* of probability enters when we seek a method for enumerating possible cases without actually counting them off. For example, how many possible different hands each consisting of three deuces and three other cards, none a deuce, are there in a common pack of fifty-two? Or, in how many ways can a throw of three aces, five twos, and two sixes occur when ten dice are tossed? A third trifle of the same sort: how many different bracelets can be made by stringing ten pearls, seven rubies, six emeralds, and eight sapphires, if stones of one kind are considered as undistinguishable?

This detail of finding the number of ways in which a prescribed thing can be done or in which a completely specified event can happen, belongs to what is called *combinatorial analysis*. Its application to probability is obvious. Suppose, for example, we wish to know the probability of throwing two aces and one deuce in a single throw with three dice. If we know the *total* number of ways ($6 \times 6 \times 6$ or 216) in which the three dice can fall, and also the number of ways (say n, which the reader may find for himself) in which two aces and one deuce can fall, the required probability is $n/216$. (Here n is three, so the probability is 3/216.) Antoine Gombaud, Chevalier de Méré, who instigated all this, is described by Pascal as a man having a very good mind but no mathematics, while Leibniz, who seems to have disliked the gay Chevalier, dubs him a man of penetrating mind, a philosopher, and a gambler – quite an unusual combination.

In connexion with problems in combinatorial analysis and probability Pascal made extensive use of the arithmetical triangle.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

in which the numbers in any row after the first two are obtained from those in the preceding row by copying down the terminal 1's and adding together the successive pairs of numbers from left to right to give the new row; thus $5 = 1 + 4$, $10 = 4 + 6$, $10 = 6 + 4$, $5 = 4 + 1$. The numbers in the nth row, after the 1, are the number of different selections of one thing, two things, three things, ... that can be chosen from n distinct things. For example, 10 is the number of different pairs of things that can be selected from five distinct things. The numbers in the nth row are also the coefficients in the expansion of $(1 + x)^n$ by the binomial theorem, thus for $n = 4$, $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$. The triangle has numerous other interesting properties. Although it was known before the time of Pascal, it is usually named after him on account of the ingenious use he made of it in probabilities.

The theory which originated in a gamblers' dispute is now at the base of many enterprises which we consider more important than gambling, including all kinds of insurance, mathematical statistics and their application to biology and educational measurements, and much of modern theoretical physics. We no longer think of an electron being 'at' a given place at a given instant, but we do calculate its probability of being in a given region. A little reflection will show that even the simplest measurements we make (when we attempt to measure anything accurately) are statistical in character.

The humble origin of this extremely useful mathematical theory is typical of many: some apparently trivial problem, first solved perhaps out of idle curiosity, leads to profound generalizations which, as in the case of the new statistical theory of the atom in the quantum theory, may cause us to revise our whole conception of the physical universe or, as has happened with the application of statistical methods to intelligence tests and the investigation of heredity, may induce us to modify our traditional beliefs regarding the 'greatness and misery of man'. Neither Pascal nor Fermat of course foresaw what was to issue from their disreputable child. The whole fabric of mathematics is so closely interwoven that we cannot unravel and eliminate any particular thread which happens to offend our individual taste without danger of destroying the whole pattern.

Pascal however did make one application of probabilities (in the *Pensées*) which for his time was strictly practical. This was his famous 'wager'. The 'expectation' in a gamble is the value of the prize multiplied by the probability of winning the prize. According to Pascal the value of eternal happiness is infinite. He reasoned that even if the probability of winning eternal happiness by leading a religious life is very small indeed, nevertheless, since the expectation is infinite (*any* finite fraction of infinity is itself infinite) it will pay anyone to lead such a life. Anyhow, he took his own medicine. But just as if to show that he had not swallowed the bottle too, he jots down in another place in the *Pensées* this thoroughly sceptical query, 'Is probability probable?' 'It is annoying', as he says in another place, 'to dwell upon such trifles; but there is a time for trifling.' Pascal's difficulty was that he did not always see clearly when he was trifling, as in his wager against God, or when, as in the clearing up of the Chevalier de Méré's gambling difficulties for him, he was being profound.

CHAPTER SIX

ON THE SEASHORE

Newton

'I DO not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.'

Such was Isaac Newton's estimate of himself towards the close of his long life. Yet his successors capable of appreciating his work almost without exception have pointed to Newton as the supreme intellect that the human race has produced – 'he who in genius surpassed the human kind.'

Isaac Newton, born on Christmas Day ('old style' of dating), 1642, the year of Galileo's death, came of a family of small but independent farmers, living in the manor house of the hamlet of Woolsthorpe, about eight miles south of Grantham in the county of Lincoln, England. His father, also named Isaac, died at the age of thirty-seven before the birth of his son. Newton was a premature child. At birth he was so frail and puny that two women who had gone to a neighbour's to get 'a tonic' for the infant expected to find him dead on their return. His mother said he was so undersized at birth that a quart mug could easily have contained all there was of him.

Not enough of Newton's ancestry is known to interest students of heredity. His father was described by neighbours as 'a wild, extravagant, weak man'; his mother, Hannah Ayscough, was thrifty, industrious, and a capable manageress. After her husband's death Mrs Newton was recommended as a prospective wife to an old bachelor as 'an extraordinary good woman'. The cautious bachelor, the Reverend Barnabas Smith, of the neighbouring parish of North Witham, married the widow on this testimonial. Mrs Smith left her three-year-old son to the care of his grandmother. By her second marriage she had three children, none of whom exhibited any remarkable ability. From the property of his mother's second marriage and his father's estate Newton ultimately acquired an income of about £80 a year, which of course meant much more in the seventeenth century than it would now. Newton was not one of the great mathematicians who had to contend with poverty.

As a child Newton was not robust and was forced to shun the rough games of boys his own age. Instead of amusing himself in the usual way, Newton invented his own diversions, in which his genius first showed up. It is sometimes said that Newton was not precocious. This may be true so far as mathematics is concerned, but if it is so in other respects a new definition of precocity is required. The unsurpassed experimental genius which Newton was to exhibit as an explorer in the mysteries of light is certainly evident in the ingenuity of his boyish amusements. Kites with lanterns to scare the credulous villagers at night, perfectly constructed mechanical toys which he made entirely by himself and which worked – waterwheels, a mill that ground wheat into snowy flour, with a greedy mouse (who devoured most of the profits) as both miller and motive power, workboxes and toys for his many little girl friends, drawings, sundials, and a wooden clock (that went) for himself – such were some of the things with which this 'un-precocious' boy sought to divert the interests of his playmates into 'more philosophical' channels. In addition to these more noticeable evidences of talent far above the ordinary, Newton read extensively and jotted down all manner of mysterious recipes and out-of-the-way observations in his notebook. To rate such a boy as merely the normal, wholesome lad he appeared to his village friends is to miss the obvious.

The earliest part of Newton's education was received in the common village schools of his vicinity. A maternal uncle, the Reverend William Ayscough, seems to have been the first to recognize that Newton was something unusual. A Cambridge graduate himself, Ayscough finally persuaded Newton's mother to send her son to Cambridge instead of keeping him at home, as she had planned, to help her manage the farm on her return to Woolsthorpe after her husband's death when Newton was fifteen.

Before this, however, Newton had crossed his Rubicon on his own initiative. On his uncle's advice he had been sent to the Grantham Grammar School. While there, in the lowest form but one, he was tormented by the school bully who one day kicked Newton in the stomach, causing him much physical pain and mental anguish. Encouraged by one of the schoolmasters, Newton challenged the bully to a fair fight, thrashed him, and, as a final mark of humiliation, rubbed his enemy's cowardly nose on the wall of the church. Up to this young Newton had shown no great interest in his lessons. He now set out to prove his head as good as his fists and quickly rose to the distinction of top boy in the school. The Headmaster and Uncle Ayscough agreed that Newton was good enough for Cambridge, but the decisive die was thrown when Ayscough caught his nephew reading under a hedge when he was supposed to be helping a farmhand to do the marketing.

While at the Grantham Grammar School, and subsequently while preparing for Cambridge, Newton lodged with a Mr Clarke, the village apothecary. In the apothecary's attic Newton found a parcel of old books, which he devoured, and in the house generally, Clarke's stepdaughter, Miss Storey, with whom he fell in love and to whom he became engaged before leaving Woolsthorpe for Cambridge in June 1661, at the age of nineteen. But although Newton cherished a warm affection for his first and only sweetheart all her life, absence and growing absorption in his work thrust romance into the background, and Newton never married. Miss Storey became Mrs Vincent.

Before going on to Newton's student career at Trinity College we may take a short look at the England of his times and some of the scientific knowledge to which the young man fell heir. The bull-headed and bigoted Scottish Stuarts had undertaken to rule England according to the divine rights they claimed were vested in them, with the not uncommon result that mere human beings resented the assumption of celestial authority and rebelled against the sublime conceit, the stupidity, and the incompetence of their rulers. Newton grew up in an atmosphere of civil war – political and religious – in which Puritans and Royalists alike impartially looted whatever was needed to keep their ragged armies fighting. Charles I (born in 1600, beheaded in 1649) had done everything in his power to suppress Parliament; but in spite of his ruthless extortions and the villainously able backing of his own Star Chamber through its brilliant perversions of the law and common justice, he was no match for the dour Puritans under Oliver Cromwell, who in his turn was to back his butcheries and his roughshod march over Parliament by an appeal to the divine justice of his holy cause.

All this brutality and holy hypocrisy had a most salutary effect on young Newton's character: he grew up with a fierce hatred of tyranny, subterfuge, and oppression, and when King James later sought to meddle repressively in University affairs, the mathematician and natural philosopher did not need to learn that a resolute show of backbone and a united front on the part of those whose liberties are endangered is the most effective defence against a coalition of unscrupulous politicians; he knew it by observation and by instinct.

To Newton is attributed the saying 'If I have seen a little farther than others it is because I have stood on the shoulders of giants.' He had. Among the tallest of these giants were Descartes, Kepler, and Galileo. From Descartes, Newton inherited analytical geometry, which he found difficult at first; from Kepler three fundamental laws of planetary motion, discovered empirically after twenty-two years of inhuman calculation; while from Galileo he acquired the first two of the three laws of motion which were to be the cornerstone of his own dynamics. But bricks do not make a building; Newton was the architect of dynamics and celestial mechanics.

As Kepler's laws were to play the role of hero in Newton's development of his law of universal gravitation they may be stated here.

I. *The planets move round the Sun in ellipses; the Sun is at one focus of these ellipses.*

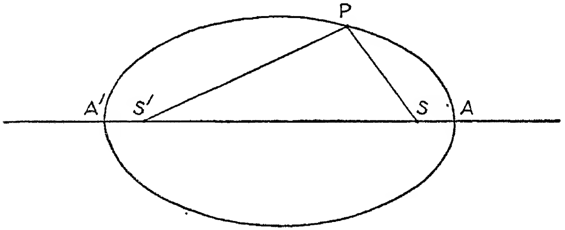
[If S, S′ are the foci, P any position of a planet in its orbit, SP + S′P is always equal to AA′, the major axis of the ellipse: fig. below.]

II. *The line joining the Sun and a planet sweeps out equal areas in equal times.*

III. *The square of the time for one complete revolution of each planet is proportional to the cube of its mean* [or average] *distance from the Sun.*

These laws can be proved in a page or two by means of the calculus applied to Newton's law of universal gravitation:

Any two particles of matter in the universe attract one another with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. Thus if m, M are the masses of the two particles and d the distance between them (all measured in appropriate units), the force of attraction between them is $\dfrac{k \times m \times M}{d^2}$, where k is some constant number (by suitably choosing the units of mass and distance k may be taken equal to 1, so that the attraction is simply $\dfrac{m \times M}{d^2}$).



For completeness we state Newton's three laws of motion.

I. *Every body will continue in its state of rest or of uniform* [unaccelerated] *motion in a straight line except in so far as it is compelled to change that state by impressed force.*

II. *Rate of change of momentum* ['mass times velocity', mass and velocity being measured in appropriate units] *is proportional to the impressed force and takes place in the line in which the force acts.*

III. *Action and reaction* [as in the collision on a frictionless table of perfectly elastic billiard balls] *are equal and opposite* [the momentum one ball loses is gained by the other].

The most important thing for mathematics in all this is the phrase opening the statement of the second law of motion, *rate of change*. What is a rate, and how shall it be measured? Momentum, as noted, is 'mass times velocity'. The masses which Newton discussed were assumed to remain constant during their motion – not like the electrons and other particles of current physics whose masses increase appreciably as their velocity approaches a measurable fraction of that of light. Thus, to investigate 'rate of change of momentum', it sufficed Newton to clarify *velocity*, which is rate of change of position. His solution of this problem – giving a workable mathematical method for investigating the velocity of any particle moving in any continuous manner, no matter how erratic – gave him the master key to the whole mystery of rates and their measurement, namely, the *differential* calculus.

A similar problem growing out of rates put the *integral* calculus into his hands. How shall the total distance passed over in a given time by a moving particle whose velocity is varying continuously from instant to instant be calculated? Answering this or similar problems, some phrased geometrically, Newton came upon the integral calculus. Finally, pondering the two types of problem together, Newton made a capital discovery: he saw that the differential calculus and the integral calculus are intimately and reciprocally related by what is to-day called 'the fundamental theorem of the calculus' – which will be described in the proper place.

In addition to what Newton inherited from his predecessors in science and mathematics he received from the spirit of his age two further gifts, a passion for theology and an unquenchable thirst for the mysteries of alchemy. To censure him for devoting his unsurpassed intellect to these things, which would now be considered unworthy of his serious effort, is to censure oneself. For in Newton's day alchemy *was* chemistry and it had *not* been shown that there was nothing much in it – except what was to come out of it, namely modern chemistry; and

Newton, as a man of inborn scientific spirit, undertook to find out *by experiment* exactly what the claims of the alchemists amounted to.

As for theology, Newton was an unquestioning believer in an all-wise Creator of the universe and in his own inability – like that of the boy on the seashore – to fathom the entire ocean of truth in all its depths. He therefore believed that there were not only many things in heaven beyond his philosophy but plenty on earth as well, and he made it his business to understand for himself what the majority of intelligent men of his time accepted without dispute (to them it was as natural as common sense) – the traditional account of creation.

He therefore put what he considered his really serious efforts into attempts to prove that the prophecies of Daniel and the poetry of the Apocalypse make sense, and into chronological researches whose object was to harmonize the dates of the Old Testament with those of history. In Newton's day theology was still queen of the sciences and she sometimes ruled her obstreperous subjects with a rod of brass and a head of cast iron. Newton however did permit his rational science to influence his beliefs to the extent of making him what would now be called a Unitarian.

In June 1661 Newton entered Trinity College, Cambridge, as a subsizar – a student who (in those days) earned his expenses by menial service. Civil war, the restoration of the monarchy in 1661, and uninspired toadying to the Crown on the part of the University had all brought Cambridge to one of the low-water marks in its history as an educational institution when Newton took up his residence. Nevertheless young Newton, lonely at first, quickly found himself and became absorbed in his work.

In mathematics Newton's teacher was Dr Isaac Barrow (1630–77), a theologian and mathematician of whom it has been said that brilliant and original as he undoubtedly was in mathematics, he had the misfortune to be the morning star heralding Newton's sun. Barrow gladly recognized that a greater than himself had arrived, and when (in 1669) the strategic moment came he resigned the Lucasian Professorship of Mathematics (of which he was the first holder) in favour of his incomparable pupil. Barrow's geometrical lectures dealt among other things with his own methods for finding areas and drawing tangents to curves – essentially the key problems of the integral and the differential calculus respectively, and there can be no doubt that these lectures inspired Newton to his own attack.

The record of Newton's undergraduate life is disappointingly meagre. He seems to have made no very great impression on his fellow-students, nor do his brief, perfunctory letters home tell anything of interest. The first two years were spent mastering elementary mathematics. If there is any reliable account of Newton's sudden maturity as a discoverer, none of his modern biographers seems to have located it. Beyond the fact that in the three years 1664–6 (age twenty-one to twenty-three) he laid the foundation of all his subsequent work in science and mathematics, and that incessant work and late hours brought on an illness, we know nothing definite. Newton's tendency to secretiveness about his discoveries has also played its part in deepening the mystery.

On the purely human side Newton was normal enough as an undergraduate to relax occasionally, and there is a record in his account book of several sessions at the tavern and two losses at cards. He took his B.A. degree in January 1664.

The Great Plague (bubonic plague) of 1664–5, with its milder recurrence the following year, gave Newton his great if forced opportunity. The University was closed, and for the better part of two years Newton retired to meditate at Woolsthorpe. Up till then he had done nothing remarkable – except make himself ill by too assiduous observation of a comet and lunar halos – or, if he had, it was a secret. In these two years he invented the method of fluxions (the calculus), discovered the law of universal gravitation, and proved experimentally that white light is composed of light of all the colours. All this before he was twenty-five.

A manuscript dated 20 May 1665 shows that Newton at the age of twenty-three had sufficiently developed the principles of the calculus to be able to find the tangent and curvature at any point of any continuous curve. He called his method 'fluxions' – from the idea of 'flowing' or variable quantities and their rates of 'flow' or 'growth'. His discovery of the binomial theorem, an essential step towards a fully developed calculus, preceded this.

The binomial theorem generalizes the simple results like

$(a + b)^2 = a^2 + 2ab + b^2$, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

and so on, which are found by direct calculation; namely,

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \times 2} a^{n-2}b^2 +$$

$$\frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^{n-3}b^3 +$$

where the dots indicate that the series is to be continued according to the same law as that indicated for the terms written; the next term is

$$\frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4} a^{n-4}b^4.$$

If n is one of the positive integers 1, 2, 3 . . . , the series automatically terminates after precisely $n + 1$ terms. This much is easily proved (as in the school algebras) by mathematical induction.

But if n is not a positive integer, the series does not terminate, and this method of proof is inapplicable. As a proof of the binomial theorem for fractional and negative values of n (also for more general values), with a statement of the necessary restrictions on a,b, came only in the nineteenth century, we need merely state here that in extending the theorem to these values of n Newton satisfied himself that the theorem was correct for such values of a,b as he had occasion to consider in his work.

If all modern refinements are similarly ignored in the manner of the seventeenth century it is easy to see how the calculus finally got itself invented. The underlying notions are those of *variable, function,* and *limit.* The last took long to clarify.

A letter, say s, which can take on several different values during the course of a mathematical investigation is called a *variable*; for example s is a variable if it denotes the height of a falling body above the earth.

The word *function* (or its Latin equivalent) seems to have been introduced into mathematics by Leibniz in 1694; the concept now dominates much of mathematics and is indispensable in science. Since Leibniz' time the concept has been made precise. If y and x are two variables so related that whenever a numerical value is assigned to x there is determined a numerical value of y, then y is called a (one-valued, or *uniform*) function of x, and this is symbolized by writing $y = f(x)$.

Instead of attempting to give a modern definition of a *limit* we shall content ourselves with one of the simplest examples of the sort which led the followers of Newton and Leibniz (the former especially) to the use of limits in discussing rates of change To the early developers of the calculus the notions of variables and limits were intuitive; to us they are extremely subtle concepts hedged about with thickets of semi-metaphysical mysteries concerning the nature of numbers, both rational and irrational.

Let y be a function of x, say $y = f(x)$. *The rate of change of y with respect to x*, or, as it is called, the *derivative of y with respect to x*, is defined as follows. To x is given any increment, say Δx (read, 'increment of x'), so that x becomes $x + \Delta x$, and $f(x)$, or y, becomes $f(x + \Delta x)$. The corresponding increment, Δy, of y is its *new* value *minus* its initial value; namely, $\Delta y = f(x + \Delta x) - f(x)$. As a crude approximation to the rate of change of y with respect to x we may take, by our intuitive notion of a rate as an 'average', the result of dividing the increment of y by the increment of x, that is, $\frac{\Delta y}{\Delta x}$.

But this obviously is too crude, as both x and y are varying and we cannot say that this average represents the rate for *any particular* value of x. Accordingly, we decrease the increment Δx *indefinitely*, till, 'in the limit', Δx approaches zero, and follow the 'average' $\frac{\Delta y}{\Delta x}$ all through the process: Δy similarly decreases indefinitely and ultimately approaches zero; but $\frac{\Delta y}{\Delta x}$ does not, thereby, present us with the meaningless symbol $\frac{0}{0}$, but with a definite *limiting value*, which is the required rate of change of y with respect to x.

To see how it works out, let $f(x)$ be the particular function x^2, so that $y = x^2$. Following the above outline we get first

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x}.$$

Nothing is yet said about limits. Simplifying the algebra we find

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

Having simplified the algebra as far as possible, we *now* let Δx approach zero and see that the limiting value of $\frac{\Delta y}{\Delta x}$ is $2x$. Quite generally, in the same way, if $y = x^n$, the limiting value of $\frac{\Delta y}{\Delta x}$ is nx^{n-1}, as may be proved with the aid of the binomial theorem.

Such an argument would not satisfy a student to-day, but something not much better was good enough for the inventors of the calculus and it will have to do for us here. If $y = f(x)$, the *limiting value* of $\frac{\Delta y}{\Delta x}$ (provided such a value exists) is called the *derivative of y with respect to x*, and is denoted by $\frac{dy}{dx}$. This symbolism is due (essentially) to Leibniz and is the one in common use to-day; Newton used another ($\dot{y}$) which is less convenient.

The simplest instances of rates in physics are velocity and acceleration, two of the fundamental notions of dynamics. Velocity is rate of change of *distance* (or 'position', or 'space') with respect to *time*; *acceleration* is rate of change of *velocity* with respect to *time*.

If s denotes the distance traversed in the time t by a moving particle (it being assumed that the distance is a function of the time), the velocity at the time t is $\frac{ds}{dt}$. Denoting this velocity by v, we have the corresponding acceleration, $\frac{dv}{dt}$.

This introduces the idea of a *rate of a rate*, or of a *second derivative*. For in accelerated motion the velocity is not constant but variable, and hence it has a rate of change: the acceleration is the rate of change of the rate of change of distance (both rates with respect to time); and to indicate this *second* rate, or 'rate of a rate', we write $\frac{d^2s}{dt^2}$ for the acceleration. This itself may have a rate of change with respect to the time; this *third* rate is written $\frac{d^3s}{dt^3}$. And so on for fourth, fifth, ... rates, namely for fourth, fifth, ... derivatives. The most important derivatives in the applications of the calculus to science are the first and second.

If now we look back at what was said concerning Newton's second law of motion and compare it with the like for acceleration, we see that 'forces' are proportional to the accelerations they produce. With this much we can 'set up' the *differential equation* for a problem which is by no means trivial – that of 'central forces': a particle is attracted toward a fixed point by a force whose direction always passes through the fixed point. Given that the force varies as some function of the distance s, say as $F(s)$, where s is the distance of the particle at the time t from the fixed point O,

O F(s) t

it is required to describe the motion of the particle. A little consideration will show that

$$\frac{}{dt^2} = - F(s),$$

the minus sign being taken because the attraction diminishes the velocity. This is the *differential equation* of the problem, so called because it involves a rate (the acceleration), and rates (or derivatives) are the object of investigation in the *differential* calculus.

Having translated the problem into a differential equation we are now required to solve this equation, that is, to find the relation between s and t, or, in mathematical language, to solve the differential equation by expressing s as a function of t. This is where the difficulties begin. It may be quite easy to translate a given physical situation into a set of differential equations which no mathematician can solve. In general every essentially new problem in physics leads to types of differential equations which demand the creation of new branches of mathematics for their solution. The particular equation above can however be solved quite simply in terms of elementary functions if $F(s) = \frac{1}{s^2}$ as in Newton's law of gravitational attraction. Instead of bothering with this particular equation, we shall consider a much simpler one which will suffice to bring out the point of importance:

$$\frac{dy}{dx} = x.$$

We are given that y is a function of x whose derivative is equal to x; it is required to express y as a function of x. More generally, consider in the same way

$$dx \quad = f(x).$$

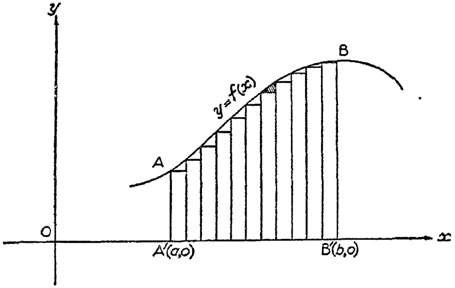
This asks, what is the function y (of x) whose derivative (rate of change) with respect to x is equal to $f(x)$? Provided we can find the function required (or provided such a function exists), we call it the *anti-derivative* of $f(x)$ and denote it by $\int f(x)dx$ – for a reason that will appear presently. For the moment we need note only that $\int f(x)dx$ symbolizes a function (if it exists) *whose derivative* is equal to $f(x)$.

By inspection we see that the first of the above equations has the solution $\frac{1}{2}x^2 + c$, where c is a constant (number not depending on the variable x); thus $\int x\,dx = \frac{1}{2}x^2 + c$.

Even this simple example may indicate that the problem of evaluating $\int f(x)dx$ for comparatively innocent-looking functions $f(x)$ may be beyond our powers. It does not follow that an 'answer' exists at all *in terms of known functions* when an $f(x)$ is chosen at random – the odds against such a chance are an infinity of the worst sort ('non-denumerable') to one. When a physical problem leads to one of these nightmares approximate methods are applied which give the result within the desired accuracy.

With the two basic notions, $\frac{dy}{dx}$ and $\int f(x)dx$, of the calculus we can now describe the *fundamental theorem of the calculus* connecting them. For simplicity we shall use a diagram, although this is not necessary and is undesirable in an exact account.

Consider a continuous, unlooped curve whose equation is $y = f(x)$ in Cartesian co-ordinates. It is required to find the area included between the curve, the x-axis and the two perpendiculars AA', BB' drawn to the x-axis from any two points A, B on the curve. The distances OA', OB' are a,b respectively – namely, the co-ordinates of A', B' are $(a,0)$, $(b,0)$. We proceed as Archimedes did, cutting the required area into parallel strips



of equal breadth, treating these strips as rectangles by disregarding the top triangular bits (one of which is shaded in the figure), adding the areas of all these rectangles, and finally evaluating *the limit of this sum* as the number of rectangles is increased indefinitely. This is all very well, but how are we to calculate the limit? The answer is surely one of the most astonishing things a mathematician ever discovered.

First, find $\int f(x)dx$. Say the result is $F(x)$. In this substitute a and b, getting $F(a)$ and $F(b)$. Then subtract the first from the second, $F(b) - F(a)$. *This is the required area.*

Notice the connexion between $y = f(x)$, the equation of the given curve; $\frac{dy}{dx}$, which (as seen in the chapter on Fermat) gives the *slope* of the tangent line to the curve at the point (x,y); and $\int f(x)dx$, or $F(x)$, which is the function whose *rate of change* with respect to x is equal to $f(x)$. We have just stated that the *area* required, which is a *limiting sum* of the kind described in connexion with Archimedes, is given by $F(b) - F(a)$. Thus we have connected *slopes*, or *derivatives*, with *limiting sums*, or, as they are called, *definite integrals*. The symbol $\int$ is an old-fashioned S, the first letter of the word *Summa*.

Summing all this up in symbols, we write for the area in question $\int_a^b f(x)dx$; a is the *lower limit* of the sum, b the *upper limit*; and

$$\int_a^b f(x)dx = F(b) - F(a),$$

in which $F(b)$, $F(a)$ are calculated by evaluating the '*indefinite integral*' $\int f(x)dx$, namely, by finding that function $F(x)$ such that its derivative with respect to x, $\frac{dF(x)}{dx}$ is equal to $f(x)$. This is the fundamental theorem of the calculus as it presented itself (in its geometrical form) to Newton and independently also to Leibniz. As a caution we repeat that numerous refinements demanded in a modern statement have been ignored.

Two simple but important matters may conclude this sketch of the leading notions of the calculus as they appeared to the pioneers. So far only functions of a single variable have been considered. But nature presents us with functions of several variables and even of an infinity of variables.

To take a very simple example, the volume, V, of a gas is a function of its temperature, T, and the pressure, P, on it; say $V = F\ (T,P)$ – the actual form of the function F need not be specified here. As T, P vary, V varies. But suppose *only one* of T, P varies while the other is held constant. We are then back essentially with a function of *one* variable, and the derivative of $F(T,P)$ can be calculated with respect to this variable. If T varies while P is held constant, the derivative of $F(T,P)$ with respect to T is called the *partial derivative* (with respect to T), and to show that the variable P is being held constant, a different symbol, ∂, is used for this partial derivative, $\frac{\partial F(T,P)}{\partial T}$. Similarly, if P varies while T is held constant, we get $\frac{\partial F(T,P)}{\partial P}$. Precisely as in the case of ordinary second, third, . . . derivatives, we have the like for partial derivatives; thus $\frac{\partial^2 F(T,P)}{\partial T^2}$ signifies the partial derivative of $\frac{\partial F(T,P)}{\partial T}$ with respect to T.

The great majority of the important equations of mathematical physics are *partial differential equations.* A famous example is Laplace's equation, or the 'equation of continuity', which appears in the theory of Newtonian gravitation, electricity and magnetism, fluid motion, and elsewhere:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

In fluid motion this is the mathematical expression of the fact that a 'perfect' fluid, in which there are no vortices, is indestructible. A derivation of this equation would be out of place here, but a statement of what it signifies may make it seem less mysterious. If there are no vortices in the fluid, the three component velocities parallel to the axes of x,y,z of any particle in the fluid are calculable as the partial derivatives

$$-\frac{\partial u}{\partial x}, \quad -\frac{\partial u}{\partial y}, \quad -\frac{\partial u}{\partial z}$$

of the *same* function u – which will be determined by the particular type of motion. Combining this fact with the obvious remark that if the fluid is incompressible and indestructible, as much fluid must flow out of any small volume in one second as flows into it; and noting that the amount of flow in one second across any small area is equal to the rate of flow multiplied by the area; we see (on combining these remarks and calculating the total inflow and total outflow) that Laplace's equation is more or less of a platitude.

The really astonishing thing about this and some other equations of mathematical physics is that a physical platitude, when subjected to mathematical reasoning, should furnish unforeseen information which is anything but platitudinous. The 'anticipations' of physical phenomena mentioned in later chapters arose from such commonplaces treated mathematically.

Two very real difficulties, however, arise in this type of problem. The first concerns the physicist, who must have a feeling for what complications can be lopped off his problem, without mutilating it beyond all recognition, so that he can state it mathematically at all. The second concerns the mathematician, and this brings us to a matter of great importance – the last we shall mention in this sketch of the calculus – that of what are called *boundary-value problems.*

Science does not fling an equation like Laplace's at a mathematician's head and ask him to find the *general* solution. What it wants is something (usually) much more difficult to obtain, a *particular* solution which will not only satisfy the equation but which *in addition will satisfy certain auxiliary conditions* depending on the particular problem to be solved.

The point may be simply illustrated by a problem in the conduction of heat. There is a *general* equation (Fourier's) for the 'motion' of heat in a conductor similar to Laplace's for fluid motion. Suppose it is required to find the final distribution of temperature in a cylindrical rod whose ends are kept at one constant temperature and whose curved surface is kept at another; 'final' here means that there is a 'steady state' – no further change in temperature – at all points of the rod. The solution must not only satisfy the *general* equation, it must also fit the *surface-temperatures*, or the *initial boundary conditions.*

The second is the harder part. For a cylindrical rod the problem is quite different from the corresponding problem for a bar of rectangular cross section. The theory of *boundary-value problems* deals with the fitting of solutions of differential equations to prescribed initial conditions. It is largely a creation of the past eighty years. In a sense mathematical physics is coextensive with the theory of boundary-value problems.

The second of Newton's great inspirations which came to him as a youth of twenty-three or four in 1666 at Woolsthorpe was his law of universal gravitation (already stated). In this connexion we shall not repeat the story of the falling apple. To vary the monotony of the classical account we shall give Gauss' version of the legend when we come to him.

Most authorities agree that Newton did make some rough calculations in 1666 (he was then twenty-four) to see whether his law of universal gravitation would account for Kepler's laws. Many years later (in 1684) when Halley asked him what law of attraction would account for the elliptical orbits of the planets Newton replied at once 'the inverse square'.

'How do you know?' Halley asked – he had been prompted by Sir Christopher Wren and others to put the question, as a great argument over the problem had been going on for some time in London.

'Why, I have calculated it,' Newton replied. On attempting to restore his calculation (which he had mislaid) Newton made a slip, and believed he was in error. But presently he found his mistake and verified his original conclusion.

Much has been made of Newton's twenty years' delay in the publication of the law of universal gravitation as an undeserved setback due to inaccurate data. Of three explanations a less romantic but more mathematical one than either of the others is to be preferred here.

Newton's delay was rooted in his inability to solve a certain problem in the integral calculus which was crucial for the whole theory of universal gravitation as expressed in the Newtonian law. Before he could account for the motion of both the apple and the Moon Newton had to find the total attraction of a solid homogeneous sphere on any mass particle outside the sphere. For *every* particle of the sphere attracts the mass particle outside the sphere with a force varying directly as the product of the masses of the two particles and inversely as the square of the distance between them: how are all these separate attractions, infinite in number, to be compounded or added into one resultant attraction?

This evidently is a problem in the integral calculus. To-day it is given in the textbooks as an example which young students dispose of in twenty minutes or less. Yet it held Newton up for twenty years. He finally solved it, of course: the attraction is the same as if the entire mass of the sphere were concentrated in *a single point* at its centre. The problem is thus reduced to finding the attraction between two mass particles at a given distance apart, and the immediate solution of this is as stated in Newton's law. If this is the correct explanation for the twenty years' delay, it may give us some idea of the enormous amount of labour which generations of mathematicians since Newton's day have expended on developing and simplifying the calculus to the point where very ordinary boys of sixteen can use it effectively.

Although our principal interest in Newton centres about his greatness as a mathematician we cannot leave him with his undeveloped masterpiece of 1666. To do so would be to give no idea of his magnitude, so we shall go on to a brief outline of his other activities without entering into detail (for lack of space) on any of them.

On his return to Cambridge Newton was elected a Fellow of Trinity in 1667 and in 1669, at the age of twenty-six, succeeded Barrow as Lucasian Professor of Mathematics. His first lectures were on optics. In these he expounded his own discoveries and sketched his corpuscular theory of light, according to which light consists in an emission of corpuscles and is not a wave phenomenon as Huygens and Hooke asserted. Although the two theories appear to be contradictory both are useful to-day in correlating the phenomena of light and are, in a purely mathematical sense, reconciled in the modern quantum theory. Thus it is not now correct to say, as it may have been a few years ago, that Newton was entirely wrong in his corpuscular theory.

The following year, 1668, Newton constructed a reflecting telescope with his own hands and used it to observe the satellites of Jupiter. His object doubtless was to see whether universal gravitation really was universal by observations on Jupiter's satellites. This year is also memorable in the history of the calculus. Mercator's calculation by means of infinite series of an area connected with a hyperbola was brought to

Newton's attention. The method was practically identical with Newton's own, which he had not published, but which he now wrote out, gave to Dr Barrow, and permitted to circulate among a few of the better mathematicians.

On his election to the Royal Society in 1672 Newton communicated his work on telescopes and his corpuscular theory of light. A commission of three, including the cantankerous Hooke, was appointed to report on the work on optics. Exceeding his authority as a referee Hooke seized the opportunity to propagandize for the undulatory theory and himself at Newton's expense. At first Newton was cool and scientific under criticism, but when the mathematician Lucas and the physician Linus, both of Liège, joined Hooke in adding suggestions and objections which quickly changed from the legitimate to the carping and the merely stupid, Newton gradually began to lose patience.

A reading of his correspondence in this first of his irritating controversies should convince anyone that Newton was not by nature secretive and jealous of his discoveries. The tone of his letters gradually changes from one of eager willingness to clear up the difficulties which others found, to one of bewilderment that scientific men should regard science as a battleground for personal quarrels. From bewilderment he quickly passes to cold anger and a hurt, somewhat childish resolution to play by himself in future. He simply could not suffer malicious fools gladly.

At last, in a letter of 18 November 1676 he says, 'I see I have made myself a slave to philosophy, but if I get free of Mr Lucas's business, I will resolutely bid adieu to it eternally, excepting what I do for my private satisfaction, or leave to come out after me; for I see a man must either resolve to put out nothing new, or become a slave to defend it.' Almost identical sentiments were expressed by Gauss in connexion with non-Euclidean geometry.

Newton's petulance under criticism and his exasperation at futile controversies broke out again after the publication of the *Principia*. Writing to Halley on 20 June 1688 he says, 'Philosophy [science] is such an impertinently litigious Lady, that a man had as good be engaged to lawsuits, as to have to do with her. I found it so formerly, and now I am no sooner come near her again, but she gives me warning.' Mathematics, dynamics, and celestial mechanics were in fact – we may as well admit it – secondary interests with Newton. His heart was in his alchemy, his researches in chronology, and his theological studies.

It was only because an inner compulsion drove him that he turned as a recreation to mathematics. As early as 1679, when he was thirty-seven (but when also he had his major discoveries and inventions securely locked up in his head or in his desk), he writes to the pestiferous Hooke: 'I had for some years past been endeavouring to bend myself from philosophy to other studies in so much that I have long grutched the time spent in that study unless it be perhaps at idle hours sometimes for diversion.' These 'diversions' occasionally cost him more incessant thought than his professed labours, as when he made himself seriously ill by thinking day and night about the motion of the Moon, the only problem, he says, that ever made his head ache.

Another side of Newton's touchiness showed up in the spring of 1673 when he wrote to Oldenburg resigning his membership in the Royal Society. This petulant action has been variously interpreted. Newton gave financial difficulties and his distance from London as his reasons. Oldenburg took the huffy mathematician at his word and told him that under the rules he could retain his membership without paying. This brought Newton to his senses and he withdrew his resignation, having recovered his temper in the meantime. Nevertheless Newton thought he was about to be hard pressed. However, his finances presently straightened out and he felt better. It may be noted here that Newton was no absent-minded dreamer when it came to a question of money. He was extremely shrewd and he died a rich man for his times. But if shrewd and thrifty he was also very liberal with his money and was always ready to help a friend in need as unobtrusively as possible. To young men he was particularly generous.

The years 1684–6 mark one of the great epochs in the history of all human thought. Skilfully coaxed by Halley, Newton at last consented to write up his astronomical and dynamical discoveries for publication. Probably no mortal has ever thought as hard and as continuously as Newton did in composing his *Philosophiae Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy). Never careful of his bodily health, Newton seems to have forgotten that he had a body which required food and sleep when he gave himself up to the composition of his masterpiece. Meals were ignored or forgotten, and on arising from a snatch of sleep he would sit on the edge of the bed half-clothed for hours, threading the mazes of his mathematics. In 1686 the *Principia* was presented to the Royal Society, and in 1687 was printed at Halley's expense.

A description of the contents of the *Principia* is out of the question here, but a small handful of the inexhaustible treasures it contains may be briefly exhibited. The spirit animating the whole work is Newton's dynamics, his law of universal gravitation, and the application of both to the solar system – 'the system of the world'. Although the calculus has vanished from the synthetic geometrical demonstrations, Newton states (in a letter) that he used it to *discover* his results and, having done so, proceeded to rework the proofs furnished by the calculus into geometrical shape so that his contemporaries might the more readily grasp the main theme – the dynamical harmony of the heavens.

First, Newton deduced Kepler's empirical laws from his own law of gravitation, and he showed how the mass of the Sun can be calculated, also how the mass of any planet having a satellite can be determined. Second, he initiated the extremely important theory of *perturbations*: the Moon, for example, is attracted not only by the Earth but by the Sun also; hence the orbit of the Moon will be perturbed by the pull of the Sun. In this manner Newton accounted for two ancient observations due to Hipparchus and Ptolemy. Our own generation has seen the now highly developed theory of perturbations applied to electronic orbits, particularly for the helium atom. In addition to these ancient observations, seven other irregularities of the Moon's motion observed by Tycho Brahe (1546–1601), Flamsteed (1646–1719), and others, were deduced from the law of gravitation.

So much for lunar perturbations. The like applies also to the planets. Newton began the theory of planetary perturbations, which in the nineteenth century was to lead to the discovery of the planet Neptune and, in the twentieth, to that of Pluto.

The 'lawless' comets – still warnings from an angered heaven to superstitious eyes – were brought under the universal law as harmless members of the Sun's family, with such precision that we now calculate and welcome their showy return (unless Jupiter or some other outsider perturbs them unduly), as we did in 1910 when Halley's beautiful comet returned promptly on schedule after an absence of seventy-four years.

He began the vast and still incomplete study of planetary evolution by calculating (from his dynamics and the universal law) the flattening of the earth at its poles due to diurnal rotation, and he proved that the shape of a planet determines the length of its day, so that if we knew accurately how flat Venus is at the poles, we could say how long it takes her to turn completely once round the axis joining her poles. He calculated the variation of weight with latitude. He proved that a hollow shell, bounded by concentric spherical surfaces, and homogeneous, exerts no force on a small body anywhere inside it. The last has important consequences in electrostatics – also in the realm of fiction, where it has been used as the motif for amusing fantasies.

The precession of the equinoxes was beautifully accounted for by the pull of the Moon and the Sun on the equatorial bulge of the Earth causing our planet to wobble like a top. The mysterious tides also fell naturally into the grand scheme – both the lunar and the solar tides were calculated, and from the observed heights of the spring and neap tides the mass of the Moon was deduced. The First Book laid down the principles of dynamics; the Second, the motion of bodies in resisting media, and fluid motion; the Third was the famous 'System of the World.'

Probably no other law of nature has so simply unified any such mass of natural phenomena as has Newton's law of universal gravitation in his *Principia.* It is to the credit of Newton's contemporaries that they recognized at least dimly the magnitude of what had been done, although but few of them could follow the reasoning by which the stupendous miracle of unification had been achieved, and made of the author of the *Principia* a demigod. Before many years had passed the Newtonian system was being taught at Cambridge (1699) and Oxford (1704). France slumbered on for half a century, still dizzy from the whirl of Descartes' angelic vortices. But presently mysticism gave way to reason and Newton found his greatest successor not in England but in France, where Laplace set himself the task of continuing and rounding out the *Principia*.

After the *Principia* the rest is anticlimax. Although the lunar theory continued to plague and 'divert' him, Newton was temporarily sick of 'philosophy' and welcomed the opportunity to turn to less celestial affairs. James II, obstinate Scot and bigoted Catholic that he was, had determined to force the University to grant a master's degree to a Benedictine over the protests of the academic authorities. Newton was one of the delegates who in 1687 went to London to present the University's case before the Court of High Commission presided over by that great and blackguardly lawyer the Lord High Chancellor George Jeffreys – 'infamous Jeffreys' as he is known in history. Having insulted the leader of the delegates in masterly fashion, Jeffreys dismissed the rest with the injunction to go and sin no more. Newton apparently held his peace. Nothing was to be gained by answering a man like Jeffreys in his own kennel. But when the others would have signed a disgraceful compromise it was Newton who put backbone into them and kept them from signing. He won the day; nothing of any value was lost – not even honour. 'An honest courage in these matters,' he wrote later, 'will secure all, having law on our sides.'

Cambridge evidently appreciated Newton's courage, for in January 1689 he was elected to represent the University at the Convention Parliament after James II had fled the country to make room for William of Orange and his Mary, and the faithful Jeffreys was burrowing into dunghills to escape the ready justice of the mob. Newton sat in Parliament till its dissolution in February 1690. To his credit he never made a speech in the place. But he was faithful to his office and not averse to politics; his diplomacy had much to do with keeping the turbulent University loyal to the decent King and Queen.

Newton's taste of 'real life' in London proved his scientific undoing. Influential and officious friends, including the philosopher John Locke (1632–1704) of *Human Understanding* fame, convinced Newton that he was not getting his share of the honours. The crowning imbecility of the Anglo-Saxon breed is its dumb belief in public office or an administrative position as the supreme honour for a man of intellect. The English finally (1699) made Newton Master of the Mint to reform and supervise the coinage of the Realm. For utter bathos this 'elevation' of the author of the *Principia* is surpassed only by the jubilation of Sir David Brewster in his life of Newton (1860) over the 'well-merited recognition' thus accorded Newton's genius by the English people. Of course if Newton really wanted anything of the sort there is nothing to be said; he had earned the right millions of times over to do anything he desired. But his busybody friends need not have egged him on.

It did not happen all at once. Charles Montagu, later Earl of Halifax, Fellow of Trinity College and a close friend of Newton, aided and abetted by the everlastingly busy and gossipy Samuel Pepys (1633–1703) of diary notoriety, stirred up by Locke and by Newton himself, began pulling wires to get Newton some recognition 'worthy' of him.

The negotiations evidently did not always run smoothly and Newton's somewhat suspicious temperament caused him to believe that some of his friends were playing fast and loose with him – as they probably were. The loss of sleep and the indifference to food which had enabled him to compose the *Principia* in eighteen months took their revenge. In the autumn of 1692 (when he was nearly fifty and should have been at his best) Newton fell seriously ill. Aversion to all food and an almost total inability to sleep, aggravated by a temporary persecution mania, brought on something dangerously close to a total mental collapse. A pathetic letter of 16 September 1693 to Locke, written after his recovery, shows how ill he had been.

SIR,

Being of opinion that you endeavoured to embroil me with women and by other means,* I was so much affected with it that when one told me you were sickly and would not live, I answered, 'twere better if you were dead. I desire you to forgive me for this uncharitableness. For I am now satisfied that what you have done is just, and I beg your pardon for having hard thoughts of you for it, and for representing that you struck at the root of morality, in a principle you laid down in your book of ideas, and designed to pursue in another book, and that I took you for a Hobbist. I beg your pardon also for saying or thinking that there was a design to sell me an office, or to embroil me.

I am your most humble
And unfortunate servant,
IS. NEWTON.

The news of Newton's illness spread to the Continent where, naturally, it was greatly exaggerated. His friends, including one who was to become his bitterest enemy, rejoiced at his recovery. Leibniz wrote to an acquaintance expressing his satisfaction that Newton was himself again. But in the very year of his recovery (1693) Newton heard for the first time that the calculus was becoming well known on the Continent and that it was commonly attributed to Leibniz.

The decade after the publication of the *Principia* was about equally divided between alchemy, theology, and worry, with more or less involuntary and headachy excursions into the lunar theory. Newton and Leibniz were still on cordial terms. Their respective 'friends', ignorant as Kaffirs of all mathematics and of the calculus in particular, had not yet decided to pit one against the other with charges of plagiarism in the invention of the calculus, and even grosser dishonesty, in the most shameful squabble over priority in the history of mathematics. Newton recognized Leibniz' merits, Leibniz recognized Newton's, and at this peaceful stage of their acquaintance neither for a mo-

* There had been gossip that Newton's favourite niece had used her charms to further Newton's advancement.

ment suspected that the other had stolen so much as a single idea of the calculus from the other.

Later, in 1712, when even the man in the street – the zealous patriot who knew nothing of the facts – realized vaguely that Newton had done something tremendous in mathematics (more, probably, as Leibniz said, than had been done in all history before him), the question as to who had invented the calculus became a matter of acute national jealousy, and all educated England rallied behind its somewhat bewildered champion, howling that his rival was a thief and a liar.

Newton at first was not to blame. Nor was Leibniz. But as the British sporting instinct presently began to assert itself, Newton acquiesced in the disgraceful attack and himself suggested or consented to shady schemes of downright dishonesty designed to win the international championship at any cost – even that of national honour. Leibniz and his backers did likewise. The upshot of it all was that the obstinate British practically rotted mathematically for a whole century after Newton's death, while the more progressive Swiss and French, following the lead of Leibniz, and developing his incomparably better way of merely *writing* the calculus, perfected the subject and made it the simple, easily applied implement of research that Newton's immediate successors should have had the honour of making it.

In 1696, at the age of fifty-four, Newton became Warden of the Mint. His job was to reform the coinage. Having done so, he was promoted in 1699 to the dignity of Master. The only satisfaction mathematicians can take in this degradation of the supreme intellect of ages is the refutation which it afforded of the silly superstition that mathematicians have no practical sense. Newton was one of the best Masters the Mint ever had. He took his job seriously.

In 1701–2 Newton again represented Cambridge University in Parliament, and in 1703 was elected President of the Royal Society, an honourable office to which he was re-elected time after time till his death in 1727. In 1705 he was knighted by good Queen Anne. Probably this honour was in recognition of his services as a money-changer rather than in acknowledgement of his pre-eminence in the temple of wisdom. This is all as it should be: if 'a riband to stick in his coat' is the reward of a turncoat politician, why should a man of intellect and integrity feel flattered if his name appears in the birthday list of honours awarded by the King? Caesar may be rendered the things that are his, ungrudgingly; but when a man of science, *as* a man of science, snaps up the droppings from the table of royalty he joins the mangy and starved dogs licking the sores of the beggars at the feast of Dives. It is to be hoped that Newton was knighted for his services to the money-changers and not for his science.

Was Newton's mathematical genius dead? Most emphatically no. He was still the equal of Archimedes. But the wiser old Greek, born aristocrat that he was – fortunately, cared nothing for the honours of a position which had always been his; to the very last minute of his long life he mathematicized as powerfully as he had in his youth. But for the accidents of preventable disease and poverty, mathematicians are a long-lived race intellectually; their creativeness outlives that of poets, artists, and even of scientists, by decades. Newton was still as virile of intellect as he had ever been. Had his officious friends but let him alone Newton might easily have created the calculus of variations, an instrument of physical and mathematical discovery second only to the calculus, instead of leaving it for the Bernoullis, Euler, and Lagrange to initiate. He had already given a hint of it in the *Principia* when he determined the shape of the surface of revolution which would cleave through a fluid with the least resistance. He had it in him to lay down the broad lines of the whole method. Like Pascal when he forsook this world for the mistier if more satisfying kingdom of heaven, Newton was still a mathematician when he turned his back on his Cambridge study and walked into a more impressive sanctum at the Mint.

In 1696 Johann Bernoulli and Leibniz between them concocted two devilish challenges to the mathematicians of Europe. The first is still of importance; the second is not in the same class. Suppose two points to be fixed at random in a vertical plane. What is the shape of the curve down which a particle must slide (without friction) under the influence of gravity so as to pass from the upper point to the lower in the *least time*? This is the problem of the *brachistochrone* (= 'shortest time'). After the problem had baffled the mathematicians of Europe for six months, it was proposed again, and Newton heard of it for the first time on 29 January 1696 when a friend communicated it to him. He had just come home, tired out, from a long day at the Mint. After dinner he solved the problem (and the second as well), and the following day communicated his solutions to the Royal Society anonymously. But for all his caution he could not conceal his identity – while at the Mint Newton resented the efforts of mathematicians and scientists to entice him into discussions of scientific interest. On seeing the solution Bernoulli at once exclaimed, 'Ah! I recognize the lion by his paw.' (This is not an exact translation of B.'s Latin.) They all knew Newton when they saw him, even if he did have a money-bag over his head and did not announce his name.

A second proof of Newton's vitality was to come in 1716 when he was seventy-four. Leibniz had rashly proposed what appeared to him a difficult problem as a challenge to the mathematicians of Europe and aimed at Newton in particular.* Newton received this at five o'clock one afternoon on returning exhausted from the blessed Mint. He solved it that evening. This time Leibniz somewhat optimistically thought he had trapped the Lion. In all the history of mathematics Newton has had no superior (and perhaps no equal) in the ability to concentrate all the forces of his intellect on a difficulty at an instant's notice.

The story of the honours that fall to a man's lot in his lifetime makes but trivial reading to his successors. Newton got all that were worth having to a living man. On the whole Newton had as fortunate a life as any great man has ever had. His bodily health was excellent up to his last years; he never wore glasses and he lost only one tooth in all his life. His hair whitened at thirty but remained thick and soft till his death.

* The problem was to find the orthogonal trajectories of any one-parameter family of curves (in modern language).

The record of his last days is more human and more touching. Even Newton could not escape suffering. His courage and endurance under almost constant pain during the last two or three years of his life add but another laurel to his crown as a human being. He bore the tortures of 'the stone' without flinching, though the sweat rolled from him, and always with a word of sympathy for those who waited on him. At last, and mercifully, he was seriously weakened by 'a persistent cough', and finally, after having been eased of pain for some days, died peacefully in his sleep between one and two o'clock on the morning of 20 March 1727, in his eighty-fifth year. He is buried in Westminster Abbey.

CHAPTER SEVEN

MASTER OF ALL TRADES

Leibniz

'JACK OF ALL TRADES, master of none' has its spectacular exceptions like any other folk proverb, and Gottfried Wilhelm Leibniz (1646–1716) is one of them.

Mathematics was but one of the many fields in which Leibniz showed conspicuous genius: law, religion, statecraft, history, literature, logic, metaphysics, and speculative philosophy all owe to him contributions, any one of which would have secured his fame and have preserved his memory. 'Universal genius' can be applied to Leibniz without hyperbole, as it cannot to Newton, his rival in mathematics and his infinite superior in natural philosophy.

Even in mathematics Leibniz' universality contrasts with Newton's undeviating direction to a unique end, that of applying mathematical reasoning to the phenomena of the physical universe: Newton imagined one thing of absolutely the first magnitude in mathematics; Leibniz, two. The first of these was the calculus, the second, combinatorial analysis. The calculus is the natural language of the *continuous*; combinatorial analysis does for the *discrete* (see Chapter 1) what the calculus does for the continuous. In combinatorial analysis we are confronted with an assemblage of distinct things, each with an individuality of its own, and we are asked, in the most general situation, to state what relations, if any, subsist between these completely heterogeneous individuals. Here we look, not at the smoothed-out resemblances of our mathematical population, but at whatever it may be that the individuals, *as individuals*, have in common – obviously not much. In fact it seems as if, in the end, all that we can say *combinatorially*, comes down to a matter of counting off the individuals in different ways, and comparing the results. That this apparently abstract and seemingly barren procedure should lead to anything of importance is in the nature of a miracle, but it is a fact. Leibniz was a pioneer in this field, and he was one of the first to perceive that the anatomy of logic – 'the laws of thought' – is a matter of combinatorial analysis. In our own day the entire subject is being arithmetized.

In Newton the mathematical spirit of his age took definite form and substance. It was inevitable after the work of Cavalieri (1598–1647), Fermat (1601–65), Wallis (1616–1703), Barrow (1630–77), and others that the calculus should presently get itself organized as an autonomous discipline. Like a crystal being dropped into a saturated solution at the critical instant, Newton solidified the suspended ideas of his time, and the calculus took definite shape. Any mind of the first rank might equally well have served as the crystal. Leibniz was the other first-rate mind of the age, and he too crystallized the calculus. But he was more than an agent for the expression of the spirit of his times, which Newton, in mathematics, was not. In his dream of a 'universal characteristic' Leibniz was well over two centuries ahead of his age, again only as concerns mathematics and logic. So far as historical research has yet shown, Leibniz was alone in his second great mathematical dream.

The union in one mind of the highest ability in the two broad, antithetical domains of mathematical thought, the analytical and the combinatorial, or the continuous and the discrete, was without precedent before Leibniz and without sequent after him. He is the one man in the history of mathematics who has had both qualities of thought in a superlative degree. His combinatorial side was reflected in the work of his German successors, largely in trivialities, and it was only in the twentieth century, when the work of Whitehead and Russell, following that of Boole in the nineteenth, partly realized the Leibnizian dream of a universal symbolic reasoning, that the supreme importance for all mathematical and scientic thought of the combinatorial side of mathematics became as significant as Leibniz had predicted that it must. To-day Leibniz' combinatorial method, as developed in symbolic logic and its extensions, is as important for the analysis that he and Newton started toward its present complexity as analysis itself is; for the symbolic method offers the only prospect in sight of clearing mathematical analysis of the paradoxes and antinomies that have infested its foundations since Zeno.

Combinatorial analysis has already been mentioned in connexion with the work of Fermat and Pascal in the mathematical theory of probability. This, however, is only a detail in the 'universal characteristic' which Leibniz had in mind and towards which (as will appear) he took a considerable first step. But the development and applications of the calculus offered an irresistible attraction to the mathematicians of the eighteenth century, and Leibniz' programme was not taken up seriously till the 1840's. Thereafter it was again ignored except by a few nonconformists to mathematical fashion until 1910, when the modern movement in symbolic reasoning originated in another *Principia*, that of Whitehead and Russell, *Principia Mathematica*.

Since 1910 the programme has become one of the major interests of modern mathematics. By a curious sort of 'eternal recurrence' the theory of probability, where combinatorial analysis in the narrow sense (as applied by Pascal, Fermat, and their successors) first appeared, has recently come under Leibniz' programme in the fundamental revision of the basic concepts of probability which experience, partly in the new quantum mechanics, has shown to be desirable; and to-day the theory of probability is on its way to becoming a province in the empire of symbolic logic – 'combinatoric' in the broad sense of Leibniz.

The part Leibniz played in the creation of the calculus was noted in the preceding chapter, also the disastrous controversy to which that part gave rise. For long after both Newton and Leibniz were dead and buried (Newton in Westminster Abbey, a relic to be reverenced by the whole English-speaking race; Leibniz, indifferently cast off by his own people, in an obscure grave where only the men with shovels and his own secretary heard the dirt thudding down on the coffin), Newton carried off all the honours – or dishonours, at least wherever English is spoken.

Leibniz did not himself elaborate his great project of reducing all exact reasoning to a symbolical technique. Nor, for that matter, has it been done yet. But he did imagine it all, and he did make a significant start. Servitude to the princelings of his day to earn worthless honours and more money than he needed, the universality of his mind, and exhausting controversies during his last years, all militated against the whole creation of a masterpiece such as Newton achieved in his *Principia*. In the bare summary of what Leibniz accomplished, his multifarious activities and his restless curiosity, we shall see the familiar tragedy of frustration which has prematurely withered more than one mathematical talent of the highest order – Newton, pursuing a popular esteem not worthy his spitting on, and Gauss seduced from his greater work by his necessity to gain the attention of men who were his intellectual inferiors. Only Archimedes of all the greatest mathematicians never wavered. He alone was born into the social class to which the others strove to elevate themselves; Newton crudely and directly; Gauss indirectly and no doubt subconsciously, by seeking the approbation of men of established reputation and recognized social standing, although he himself was the simplest of the simple. So there may after all be something to be said for aristocracy: its possession by birthright or other social discrimination is the one thing that will teach its fortunate possessor its worthlessness.

In the case of Leibniz the greed for money which he caught from his aristocratic employers contributed to his intellectual dalliance: he was forever disentangling the genealogies of the semi-royal bastards whose descendants paid his generous wages, and proving with his unexcelled knowledge of the law their legitimate claims to duchies into which their careless ancestors had neglected to fornicate them. But more disastrously than his itch for money his universal intellect, capable of anything and everything had he lived a thousand years instead of a meagre seventy, undid him. As Gauss blamed him for doing, Leibniz squandered his splendid talent for mathematics on a diversity of subjects in all of which no human being could hope to be supreme, whereas – according to Gauss – he had in him supremacy in mathematics. But why censure him? He was what he was, and willy-nilly he had to 'dree his weird'. The very diffusion of his genius made him capable of the dream which Archimedes, Newton, and Gauss missed – the 'universal characteristic'. Others may bring it to realization; Leibniz did his part in dreaming it to be possible.

Leibniz may be said to have lived not one life but several. As a diplomat, historian, philosopher, and mathematician he did enough in each field to fill one ordinary working life. Younger than Newton by about four years, he was born at Leipzig on 1 July 1646, and living only seventy years against Newton's eighty-five, died in Hanover on 14 November 1716. His father was a professor of moral philosophy and came of a good family which had served the government of Saxony for three generations. Thus young Leibniz' earliest years were passed in an atmosphere of scholarship heavily charged with politics.

At the age of six he lost his father, but not before he had acquired from him a passion for history. Although he attended a school in Leipzig, Leibniz was largely self-taught by incessant reading in his father's library. At eight he began the study of Latin and by twelve had mastered it sufficiently to compose creditable Latin verse. From Latin he passed on to Greek which he also learned largely by his own efforts.

At this stage his mental development parallels that of Descartes: classical studies no longer satisfied him and he turned to logic. From his attempts as a boy of less than fifteen to reform logic as presented by the classicists, the scholastics, and the Christian fathers, developed the first germs of his *Characteristica Universalis* or Universal Mathematics, which, as has been shown by Couturat, Russell, and others, is the clue to his metaphysics. The symbolic logic invented by Boole in 1847–54 (to be discussed in a later chapter) is only that part of the *Characteristica* which Leibniz called *calculus raticinator*. His own description of the universal characteristic will be quoted presently.

At the age of fifteen Leibniz entered the University of Leipzig as a student in law. The law, however, did not occupy all his time. In his first two years he read widely in philosophy and for the first time became aware of the new world which the modern, or 'natural' philosophers, Kepler, Galileo, and Descartes had discovered. Seeing that this newer philosophy could be understood only by one acquainted with mathematics, Leibniz passed the summer of 1663 at the University of Jena, where he attended the mathematical lectures of Erhard Weigel, a man of considerable local reputation but scarcely a mathematician.

On returning to Leipzig he concentrated on law. By 1666, at the age of twenty, he was thoroughly prepared for his doctor's degree in law. This is the year, we recall, in which Newton began the rustication at Woolsthorpe that gave him the calculus and his law of universal gravitation. The Leipzig faculty, bilious with jealousy, refused Leibniz his degree, officially on account of his youth, actually because he knew more about law than the whole dull lot of them.

Before this he had taken his bachelor's degree in 1663 at the age of seventeen with a brilliant essay foreshadowing one of the cardinal doctrines of his mature philosophy. We shall not take space to go into this, but it may be mentioned that one possible interpretation of Leibniz' essay is the doctrine of 'the organism as a whole', which one progressive school of biologists and another of psychologists has found attractive in our own time.

Disgusted at the pettiness of the Leipzig faculty Leibniz left his native town for good and proceeded to Nuremberg where, on 5 November 1666, at the affiliated University of Altdorf, he was not only granted his doctor's degree at once for his essay on a new method (the historical) of teaching law, but was begged to accept the University professorship of law. But, like Descartes refusing the offer of a lieutenant-generalship because he knew what he wanted out of life, Leibniz declined, saying he had very different ambitions. What these may have been he did not divulge. It seems unlikely that they could have been the higher pettifogging for princelets into which fate presently kicked him. Leibniz' tragedy was that he met the lawyers before the scientists.

His essay on the teaching of the law and its proposed recodification was composed on the journey from Leipzig to Nuremberg. This illustrates a lifelong characteristic of Leibniz, his ability to work anywhere, at any time, under any conditions. He read, wrote, and thought incessantly. Much of his mathematics, to say nothing of his other wonderings on everything this side of eternity and beyond, was written out in the jolting, draughty rattletraps that bumped him over the cow trails of seventeenth-century Europe as he sped hither and thither at his employers' erratic bidding. The harvest of all this ceaseless activity was a mass of papers, of all sizes and all qualities, as big as a young haystack, that has never been thoroughly sorted, much less published. To-day most of it lies baled in the royal Hanover library waiting the patient labours of an army of scholars to winnow the wheat from the straw.

It seems incredible that one head could have been responsible for all the thoughts, published and unpublished, that Leibniz committed to paper. As an item of interest to phrenologists and anatomists it has been stated (whether reliably or not I don't know) that Leibniz' skull was dug up, measured, and found to be markedly under the normal adult size. There may be something in this, as many of us have seen perfect idiots with noble brows bulging from heads as big as broth pots.

Newton's miraculous year 1666 was also the great year for Leibniz. In what he called a 'schoolboy's essay', *De arte combinatoria*, the young man of twenty aimed to create '*a general method in which all truths of the reason would be reduced to a kind of calculation. At the same time this would be a sort of universal language or script, but infinitely different from all those projected hitherto; for the symbols and even the words in it would direct the reason; and errors, except those of fact, would be mere mistakes in calculation. It would be very difficult to form or invent this language or characteristic, but very easy to understand it without any dictionaries.*' In a later description he confidently (and optimistically) estimates how long it would take to carry out his project: 'I think a few chosen men could turn the trick within five years.' Toward the end of his life Leibniz regretted that he had been too distracted by other things ever to work out his idea. If he were younger himself or had competent young assistants, he says, he could still do it – a common alibi for a talent squandered on snobbery, greed, and intrigue.

To anticipate slightly, it may be said that Leibniz' dream struck his mathematical and scientific contemporaries as a dream and nothing more, to be politely ignored as the fixed idea of an otherwise sane and universally gifted genius. In a letter of 8 September 1679 Leibniz (speaking of geometry in particular but of all reasoning in general) tells Huygens of a 'new characteristic, entirely different from Algebra, which will have great advantages for representing exactly and naturally to the mind, and without figures, everything that depends on the imagination.'

Such a direct, symbolic way of handling geometry was invented in the nineteenth century by Hermann Grassmann (whose work in algebra generalized that of Hamilton). Leibniz goes on to discuss the difficulties inherent in the project, and presently emphasizes what he considers its superiority over the Cartesian analytic geometry.

'But its principal utility consists in the consequences and reasonings which can be performed by the operations of characters [symbols], which could not be expressed by diagrams (or even by models) without too great elaboration, or without confusing them by an excessive number of points and lines, so that one would be obliged to make an infinity of useless trials: in contrast this method would lead surely and simply [to the desired end]. I believe mechanics could be handled by this method almost like geometry.'

Of the definite things that Leibniz did in that part of his universal characteristic which is now called symbolic logic, we may cite his formulation of the principal properties of logical addition and logical multiplication, negation, identity, the null class, and class inclusion. For an explanation of what some of these terms mean and the postulates of the algebra of logic we must refer ahead to the chapter on Boole. All this fell by the wayside. Had it been picked up by able men when Leibniz scattered it broadcast, instead of in the 1840's, the history of mathematics might now be quite a different story from what it is. Almost as well never as too soon.

Having dreamed his universal dream at the age of twenty, Leibniz presently turned to something more practical, and he became a sort of corporation lawyer and glorified commercial traveller for the Elector of Mainz. Taking one last spree in the world of dreams before plunging up to his chin into more or less filthy politics, Leibniz devoted some months to alchemy in the company of the Rosicrucians infesting Nuremberg.

It was his essay on a new method of teaching law that undid him. The essay came to the attention of the Elector's right-hand statesman, who urged Leibniz to have it printed so that a copy might be laid before the august Elector. This was done, and Leibniz, after a personal interview, was appointed to revise the code. Before long he was being entrusted with important commissions of all degrees of delicacy and shadiness. He became a diplomat of the first rank, always pleasant, always open and above-board, but never scrupulous, even when asleep. To his genius is due, at least partly, that unstable formula known as the 'balance of power'. And for sheer cynical brilliance, it would be hard to surpass, even to-day, Leibniz' great dream of a holy war for the conquest and civilization of Egypt. Napoleon was quite chagrined when he discovered that Leibniz had anticipated him in this sublime vision.

Up till 1672 Leibniz knew but little of what in his time was modern mathematics. He was then twenty-six when his real mathematical education began at the hands of Huygens, whom he met in Paris in the intervals between one diplomatic plot and another. Christian Huygens (1629–95), while primarily a physicist, some of whose best work went into horology and the undulatory theory of light, was an accomplished mathematician. Huygens presented Leibniz with a copy of his mathematical work on the pendulum. Fascinated by the power of the mathematical method in competent hands, Leibniz begged Huygens to give him lessons, which Huygens, seeing that Leibniz had a first-class mind, gladly did. Leibniz had already drawn up an impressive list of discoveries he had made by means of his own methods – phases of the universal characteristic. Among these was a calculating machine far superior to Pascal's, which handled only addition and subtraction; Leibniz' machine did also multiplication, division, and the extraction of roots. Under Huygens' expert guidance Leibniz quickly found himself. He was a born mathematician.

The lessons were interrupted from January to March 1673 during Leibniz' absence in London as an attaché for the Elector. While in London, Leibniz met the English mathematicians and showed them some of his work, only to learn that it was already known. His English friends told him of Mercator's quadrature of the hyperbola – one of the clues which Newton had followed to his invention of the calculus. This introduced Leibniz to the method of infinite series, which he carried on. One of his discoveries (sometimes ascribed to the Scotch mathematician James Gregory, 1638–75) may be noted: if π is the ratio of the circumference of a circle to its diameter,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots,$$

the series continuing in the same way indefinitely. This is not a practical way of calculating the numerical value of π (3.1415926 ...), but the simple connexion between π and *all* the odd numbers is striking.

During his stay in London Leibniz attended meetings of the Royal Society, where he exhibited his calculating machine. For this and his other work he was elected a foreign member of the Society before his return to Paris in March 1673. He and Newton subsequently (1700) became the first foreign members of the French Academy of Sciences.

Greatly pleased with what Leibniz had done while away, Huygens urged him to continue. Leibniz devoted every spare moment to his mathematics, and before leaving Paris for Hanover in 1676 to enter the service of the Duke of Brunswick-Lüneburg, had worked out some of the elementary formulas of the calculus and had discovered 'the fundamental theorem of the calculus' (see preceding chapter) – that is, if we accept his own date, 1675. This was not published till 11 July 1677, eleven years after Newton's unpublished discovery, which was not made public by Newton till after Leibniz' work had appeared. The controversy started in earnest, when Leibniz, diplomatically shrouding himself in editorial omniscience and anonymity, wrote a severely critical review of Newton's work in the *Acta Eruditorum*, which Leibniz himself had founded in 1682 and of which he was editor in chief. In the interval between 1677 and 1704 the Leibnizian calculus had been developed into an instrument of real power and easy applicability on the Continent, largely through the efforts of the Swiss Bernoullis, Jacob and his brother Johann, while in England, owing to Newton's reluctance to share his mathematical discoveries freely, the calculus was still a relatively untried curiosity.

One specimen of things that are now easy for beginners in the calculus, but which cost Leibniz (and possibly also Newton) much thought and many trials before the right way was found, may indicate how far mathematics has travelled since 1675. Instead of the infinitesimals of Leibniz we shall use the rates discussed in the preceding chapter. If u, v are functions of x, how shall the rate of change of uv with respect to x be expressed in terms of the respective rates of change of u and v with respect to x? In symbols, what is $\frac{d(uv)}{dx}$ in terms of $\frac{du}{dx}$ and $\frac{dv}{dx}$? Leibniz once thought it should be $\frac{du}{dx} \quad \frac{dv}{dx}$, which is nothing like the correct

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The Elector died in 1673 and Leibniz was more or less free during the last of his stay in Paris. On leaving Paris in 1676 to enter the service of the Duke John Frederick of Brunswick-Lüneburg, Leibniz proceeded to Hanover by way of London and Amsterdam. It was while in the latter city that he engineered one of the shadiest transactions in all his long career as a philosophic diplomat. The history of Leibniz' commerce with 'the God-intoxicated Jew' Benedict de Spinoza (1632–77) may be incomplete, but as the account now stands it seems that for once Leibniz was grossly unethical over a matter – of all things – of ethics. Leibniz seems to have believed in applying his ethics to practical ends. He carried off copious extracts from Spinoza's unpublished masterpiece *Ethica* (*Ordine Geometrica Demonstrata*) – a treatise on ethics developed in the manner of Euclid's geometry. When Spinoza died the following year Leibniz appears to have found it convenient to mislay his souvenirs of the Amsterdam visit. Scholars in this field seem to agree that Leibniz' own philosophy wherever it touches ethics was appropriated without acknowledgement from Spinoza.

It would be rash for anyone not an expert in ethics to doubt that Leibniz was guilty, or to suggest that his own thoughts on ethics were independent of Spinoza's. Nevertheless there are at least two similar instances in mathematics (elliptic functions, non-Euclidean geometry) where all the evidence at one time was sufficient to convict several men of dishonesty grosser than that attributed to Leibniz. When unsuspected diaries and correspondence were brought to light years after the death of all the accused it was seen that all were entirely innocent. It may pay occasionally to believe the best of human beings instead of the worst until all the evidence is in – which it can never be for a man who is tried after his death.

The remaining forty years of Leibniz' life were spent in the trivial service of the Brunswick family. In all he served three masters as librarian, historian, and general brains of the family. It was a matter of great importance to such a family to have an exact history of all its connexions with other families as highly favoured by heaven as itself. Leibniz was no mere cataloguer of books in his function as family librarian, but an expert genealogist and searcher of mildewed archives as well, whose function it was to confirm the claims of his employers to half the thrones of Europe or, failing confirmation, to manufacture evidence by judicious suppression. His historical researches took him all through Germany and thence to Austria and Italy in 1687–90.

During his stay in Italy Leibniz visited Rome and was urged by the Pope to accept the position of librarian at the Vatican. But as a prerequisite to the job was that Leibniz should become a Catholic he declined – for once scrupulous. Or was he? His reluctance to throw up one good post for another may have started him off on the next application of his 'universal characteristic', the most fantastically ambitious of all his universal dreams. Had he pulled this off he could have moved into the Vatican without leaving his face outside.

His grand project was no less than that of reuniting the Protestant and Catholic churches. It was then not so long since the first had split off from the second, so the project was not so insane as it now sounds. In his wild optimism Leibniz overlooked a law which is as fundamental for human nature as the second law of thermodynamics is for the physical universe – indeed it is of the same kind: all creeds tend to split into two, each of which in turn splits into two more, and so on, until after a certain finite number of generations (which can be easily calculated by logarithms) there are fewer human beings in any given region, no matter how large, than there are creeds, and further attenuations of the original dogma embodied in the first creed dilute it to a transparent gas too subtle to sustain faith in any human being, no matter how small.

A quite promising conference at Hanover in 1683 failed to effect a reconciliation as neither party could decide which was to be swallowed by the other, and both welcomed the bloody row of 1688 in England between Catholics and Protestants as a legitimate ground for adjourning the conference *sine die*.

Having learned nothing from this farce Leibniz immediately organized another. His attempt to unite merely the two Protestant sects of his day succeeded only in making a large number of excellent men more obstinate and sorer at one another than they were before. The Protestant Conference dissolved in mutual recriminations and curses.

It was about this time that Leibniz turned to philosophy as his major consolation. In an endeavour to assist Pascal's old Jansenist friend Arnauld, Leibniz composed a semi-casuistical treatise on metaphysics destined to be of use to Jansenists and others in need of something more subtle than the too subtle logic of the Jesuits. His philosophy occupied the remainder of Leibniz' life (when he was not engaged on the unending history of the Brunswick family for his employers), in all about a quarter of a century. That a mind like Leibniz' evolved a vast cloud of philosophy in twenty-five years need hardly be stated. Doubtless every reader has heard something of the ingenious theory of monads – miniature replicas of the universe out of which *everything* in the universe is composed, as a sort of one in all, all in one – by which Leibniz explained everything (except the monads) in this world and the next.

The power of Leibniz' method when applied to philosophy cannot be denied. As a specimen of the theorems *proved* by Leibniz in his philosophy, that concerning the existence of God may be mentioned. In his attempt to prove the fundamental theorem of optimism – 'everything is for the best in this best of all possible worlds' – Leibniz was less successful, and it was only in 1759, forty-three years after Leibniz had died neglected and forgotten, that a conclusive demonstration was published by Voltaire in his epoch-making treatise *Candide.* One further isolated result may be mentioned. Those familiar with general relativity will recall that 'empty space' – space totally devoid of matter – is no longer respectable. Leibniz rejected it as nonsensical.

The list of Leibniz' interests is still far from complete. Economics, philology, international law (in which he was a pioneer), the establishment of mining as a paying industry in certain parts of Germany, theology, the founding of academies, and the education of the young Electress Sophie of Brandenburg (a relative of Descartes' Elisabeth), all shared his attention, and in each of them he did something notable. Possibly his least successful ventures were in mechanics and physical science, where his occasional blunders show up glaringly against the calm, steady light of men like Galileo, Newton, and Huygens, or even Descartes.

Only one item in this list demands further attention here. On being called to Berlin in 1700 as tutor to the young Electress, Leibniz found time to organize the Berlin Academy of Sciences. He became its first president. The Academy was still one of the three or four leading learned bodies in the world till the Nazis 'purged' it. Similar ventures in Dresden, Vienna, and St Petersburg came to nothing during Leibniz' lifetime, but after his death the plans for the St Petersburg Academy of Sciences which he had drawn up for Peter the Great were carried out. The attempt to found a Viennese Academy was frustrated by the Jesuits when Leibniz visited Austria for the last time, in 1714. Their opposition was only to have been expected after what Leibniz had done for Arnauld. That they got the better of the master diplomat in an affair of petty academic politics shows how badly Leibniz had begun to slip at the age of sixty-eight. He was no longer himself, and indeed his last years were but a wasted shadow from his former glory.

Having served princes all his life he now received the usual wages of such service. Ill, fast ageing, and harassed by controversy, he was kicked out.

Leibniz returned to Brunswick in September 1714, to learn that his employer the Elector George Louis – 'the honest blockhead', as he is known in English history – having packed up his duds and his snuff, had left for London to become the first German King of England. Nothing would have pleased Leibniz better than to follow George to London, although his enemies at the Royal Society and elsewhere in England were now numerous and vicious enough owing to the controversy with Newton. But the boorish George, now socially a gentleman, had no further use for Leibniz' diplomacy, and curtly ordered the brains that had helped to lift him into civilized society to stick in the Hanover library and get on with their everlasting history of the illustrious Brunswick family.

When Leibniz died two years later (1716) the diplomatically doctored history was still incomplete. For all his hard labour Leibniz had been unable to bring the history down beyond the year 1005, and at that had covered less than three hundred years. The family was so very tangled in its marital adventures that even the universal Leibniz could not supply them all with unblemished scutcheons. The Brunswickers showed their appreciation of this immense labour by forgetting all about it till 1843, when it was published, but whether complete or expurgated will not be known until the rest of Leibniz' manuscripts have been sifted.

To-day, over 300 years after his death, Leibniz' reputation as a mathematician is higher than it was for many, many years after his secretary followed him to the grave, and it is still rising.

As a diplomat and statesman Leibniz was as good as the cream of the best of them in any time or any place, and far brainier than all of them together. There is but one profession in the world older than his, and until that is made respectable it would be premature to try any man for choosing diplomacy as his means of livelihood.

CHAPTER EIGHT

NATURE OR NURTURE?

The Bernoullis

SINCE the great depression began deflating western civilization eugenists, geneticists, psychologists, politicians, and dictators – for very different reasons – have taken a renewed interest in the still unsettled controversy of heredity versus environment. At one extreme the hundred-percenter proletarians hold that anyone can be a genius given the opportunity; while at the other, equally positive Tories assert that genius is inborn and will out even in a London slum. Between the two stretches a whole spectrum of belief. The average opinion holds that nature, not nurture, is the determining factor in the emergence of genius, but that without deliberate or accidental assistance genius perishes. The history of mathematics offers abundant material for a study of this interesting problem. Without taking sides – to do so at present would be premature – we may say that the evidence furnished by the life histories of mathematicians seems to favour the average opinion.

Probably the most striking case history is that of the Bernoulli family, which in three generations produced eight mathematicians, several of them outstanding, who in turn produced a swarm of descendants about half of whom were gifted above the average and nearly all of whom, down to the present day, have been superior human beings. No fewer than 120 of the descendants of the mathematical Bernoullis have been traced genealogically, and of this considerable posterity the majority achieved distinction – sometimes amounting to eminence – in the law, scholarship, science, literature, the learned professions, administration, and the arts. None were failures. The most significant thing about the majority of the mathematical members of this family in the second and third generations is that they did not deliberately choose mathematics as a profession but drifted into it in spite of themselves as a dipsomaniac returns to alcohol.

As the Bernoulli family played a leading part in developing the calculus and its applications in the seventeenth and eighteenth centuries, they must be given more than a passing mention in even the briefest account of the evolution of modern mathematics. The Bernoullis and Euler were in fact the leaders above all others who perfected the calculus to the point where quite ordinary men could use it for the discovery of results which the greatest of the Greeks could never have found. But the mere volume of the Bernoulli family's work is too vast for detailed description in an account like the present, so we shall treat them briefly together.

		Nicolaus Senior 1623-1708			
Jacob I 1654-1705	Nicolaus I 1662-1716	Johannes I 1667-1748			
	Nicolaus II 1687-1759	Nicolaus III 1695-1726	Daniel 1700-1782	Johannes II 1710-1790	
				Johannes III 1746-1807	Jacob II 1759-1789

The Bernoullis were one of many Protestant families who fled from Antwerp in 1583 to escape massacre by the Catholics (as on St Bartholomew's Eve) in the prolonged persecution of the Huguenots. The family sought refuge first in Frankfort, moving on presently to Switzerland, where they settled at Basle. The founder of the Bernoulli dynasty married into one of the oldest Basle families and became a great merchant. Nicolaus senior, who heads the genealogical table, was also a great merchant, as his grandfather and great-grandfather had been. All these men married daughters of merchants, and with one exception – the great-grandfather mentioned – accumulated large fortunes. The exception showed the first departure from the family tradition of trade by following the profession of medicine. Mathematical talent was probably latent for generations in this shrewd mercantile family, but its actual emergence was explosively sudden.

Referring now to the genealogical table we shall give a very brief summary of the chief scientific activities of the eight mathematicians descended from Nicolaus senior before continuing with the heredity.

Jacob I mastered the Leibnizian form of the calculus by himself. From 1687 to his death he was professor of mathematics at Basle. Jacob I was one of the first to develop the calculus significantly beyond the state in which Newton and Leibniz left it and to apply it to new problems of difficulty and importance. His contributions to analytical geometry, the theory of probability, and the calculus of variations were of the highest importance. As the last will recur frequently (in the work of Euler, Lagrange, and Hamilton), we may describe the nature of some of the problems attacked by Jacob I in this subject. We have already seen a specimen of the type of problem handled by the calculus of variations in Fermat's principle of least time.

The calculus of variations is of very ancient origin. According to one legend,* when Carthage was founded the city was granted as much land as a man could plough a furrow completely around in a day. What shape should the furrow be given that a man can plough a straight furrow of a certain length in a day? Mathematically stated, what is the figure which has the greatest area of all figures having perimeters of the same length? This is an *isoperimetrical* problem; the answer here is a circle. This seems obvious, but it is by no means easy to prove. (The elementary 'proofs' sometimes given in school geometries are rankly fallacious.) The mathematics of the problem comes down to making a certain integral a maximum subject to one restrictive condition. Jacob I solved this problem and generalized it.†

* Actually, here, I have combined *two* legends. Queen Dido was given a bull's hide to 'enclose' the greatest area. She cut it into one thong and enclosed a semicircle.

† Historical notes on this and other problems of the calculus of variations will be found in the book by G. A. Bliss, *Calculus of Variations*, Chicago, 1925. The Anglicized form of Jacob is James.

The discovery that the brachistochrone is a cycloid has been noted in previous chapters. This fact, that the cycloid is the curve of quickest descent, was discovered by the brothers Jacob I and Johannes I in 1697, and almost simultaneously by several others. But the cycloid is also the tautochrone. This struck Johannes I as something wonderful and admirable: 'With justice we may admire Huygens because he first discovered that a heavy particle falls on a cycloid in the same time always, no matter what the starting-point may be. But you will be petrified with astonishment when I say that exactly this same cycloid, the tautochrone of Huygens, is the brachistochrone we are seeking' (Bliss, *loc. cit.*, p. 54). Jacob also waxes enthusiastic. These again are instances of the sort of problem attacked by the calculus of variations. Lest they seem trivial, we repeat once more that a whole province of mathematical physics is frequently mapped into a simple *variational principle* – like Fermat's of least time in optics, or Hamilton's in dynamics.

After Jacob's death his great treatise on the theory of probability, the *Ars Conjectandi*, was published in 1713. This contains much that is still of the highest usefulness in the theory of probabilities and its applications to insurance, statistics, and the mathematical study of heredity.

Another research of Jacob's shows how far he had developed the differential and integral calculus: continuing the work of Leibniz, Jacob made a fairly exhaustive study of catenaries – the curves in which a uniform chain hangs suspended between two points, or in which loaded chains hang. This was no mere curiosity. To-day the mathematics developed by Jacob I in this connexion finds its use in applications to suspension bridges and high-voltage transmission lines. When Jacob I worked all this out it was new and difficult; to-day it is an exercise in the first course in the calculus or mechanics.

Jacob I and his brother Johannes I did not always get on well together. Johannes seems to have been the more quarrelsome of the two, and it is certain that he treated his brother with something pretty close to dishonesty in the matter of isoperimetrical problems. The Bernoullis took their mathematics in deadly earnest. Some of their letters about mathematics bristle with strong language that is usually reserved for horse thieves. For his part Johannes I not only attempted to steal his brother's ideas but threw his own son out of the house for having won a prize from the French Academy of Sciences for which Johannes himself had competed. After all, if rational human beings get excited about a game of cards, why should they not blow up over mathematics, which is infinitely more exciting?

Jacob I had a mystical strain which is of some significance in the study of the heredity of the Bernoullis. It cropped out once in an interesting way toward the end of his life. There is a certain spiral (the logarithmic or equiangular) which is reproduced in a similar spiral after each of many geometrical transformations. Jacob was fascinated by this recurrence of the spiral, several of whose properties he discovered, and directed that a spiral be engraved on his tombstone with the inscription *Eadem mutata resurgo* (Though changed I shall arise the same).

Jacob's motto was *Invito patre sidera verso* (Against my father's will I study the stars) – in ironic memory of his father's futile opposition to Jacob's devoting his talents to mathematics and astronomy. This detail favours the 'nature' view of genius over the 'nurture'. If his father had prevailed Jacob would have been a theologian.

Johannes I, brother of Jacob I, did not start as a mathematician but as a doctor of medicine. His dispute with the brother who had generously taught him mathematics has already been mentioned. Johannes was a man of violent likes and dislikes: Leibniz and Euler were his gods; Newton he positively hated and greatly under-estimated, as a bigoted champion of Leibniz was almost bound to do from envy or spite. The obstinate father attempted to cramp his younger son into the family business, but Johannes I, following the lead of his brother Jacob I, rebelled and went in for medicine and the humanities, unaware that he was fighting against his heredity. At the age of eighteen he took his M.A. degree. Before long he realized his mistake in choosing medicine and turned to mathematics. His first academic appointment was at Groningen in 1695 as professor of mathematics; on the death of Jacob I in 1705 Johannes I succeeded to the professorship at Basle.

Johannes I was even more prolific than his brother in mathematics and did much to spread the calculus in Europe. His range included physics, chemistry, and astronomy in addition to mathematics. On the applied side Johannes I contributed extensively to optics, wrote on the theory of the tides and the mathematical theory of ship sails, and enunciated the principle of virtual displacements in mechanics. Johannes I was a man of unusual physical and intellectual vigour, remaining active till within a few days of his death at the age of eighty.

Nicolaus I, the brother of Jacob I and Johannes I, was also gifted in mathematics. Like his brothers he made a false start. At the age of sixteen he took his doctor's degree in philosophy at the University of Basle, and at twenty earned the highest degree in law. He was first a professor of law at Bern before becoming one of the mathematical faculty at the Academy of St Petersburg. At the time of his death he was so highly thought of that the Empress Catherine gave him a public funeral at state expense.

Heredity came out curiously in the second generation. Johannes I tried to force his second son, Daniel, into business. But Daniel thought he preferred medicine and became a physician before landing, in spite of himself, in mathematics. At the age of eleven Daniel began taking lessons in mathematics from his elder brother Nicolaus III, only five years older than himself. Daniel and the great Euler were intimate friends and at times friendly rivals. Like Euler, Daniel Bernoulli has the distinction of having won the prize of the French Academy ten times (on a few occasions the prize was shared with other successful competitors). Some of Daniel's best work went into hydrodynamics, which he developed uniformly from the single principle that later came to be called the conservation of energy. All who work to-day in pure or applied fluid motion know the name of Daniel Bernoulli.

In 1725 (at the age of twenty-five) Daniel became professor of mathematics at St Petersburg, where the comparative barbarity of the life irked him so greatly that he returned at the first opportunity, eight years later, to Basle, where he became professor of anatomy and botany, and finally of physics. His mathematical work included the calculus, differential equations, probability, the theory of vibrating strings, an attempt at a kinetic theory of gases, and many other problems in applied mathematics. Daniel Bernoulli has been called the founder of mathematical physics.

From the standpoint of heredity it is interesting to note that Daniel had a marked vein of speculative philosophy in his nature – possibly a refined sublimation of the Huguenot religion of his ancestors. The like cropped out in numerous later descendants of the illustrious refugees from religious intolerance.

The third mathematician in the second generation, Johannes II, brother of Nicolaus III and Daniel, also made a false start and was pulled back into line by his heredity – or possibly by his brothers. Starting out in law he became professor of eloquence at Basle before succeeding his father in the chair of mathematics. His work was principally in physics and was sufficiently distinguished to capture the Paris prize on three occasions (once is usually enough to satisfy a good mathematician – provided he is good enough).

Johannes III, a son of Johannes II, repeated the family tradition of making a wrong start, and like his father began with law. At the age of thirteen he took his doctor's degree in philosophy. By nineteen Johannes III had found his true vocation and was appointed astronomer royal at Berlin. His interests embraced astronomy, geography, and mathematics.

Jacob II, another son of Johannes II, carried on the family blunder by starting in law, only to change over at twenty-one to experimental physics. He also turned to mathematics, becoming a member of the St Petersburg Academy in the section of mathematics and physics. His early death (at the age of thirty) by accidental drowning cut short a very promising career, and we do not know what Jacob II really had in him. He was married to a granddaughter of Euler.

The list of Bernoullis who showed mathematical talent is not yet exhausted, but the rest were less distinguished. It is sometimes asserted that the strain had worn thin. Quite the contrary seems to be the case. When mathematics was the most promising field for superior talent to cultivate, as it was immediately after the invention of the calculus, the gifted Bernoullis cultivated mathematics. But mathematics and science are only two of innumerable fields of human endeavour, and for gifted men to swarm into either when both are overcrowded with high ability indicates a lack of practical sense. The Bernoulli talent was not expended; it merely spent itself on things of equal – or perhaps greater – social importance than mathematics when that field began to resemble Epsom Downs on Derby Day.

Those interested in the vagaries of heredity will find plenty of material in the history of the Darwin and Galton families. The case of Francis Galton (a cousin of Charles Darwin) is particularly interesting, as the mathematical study of heredity was founded by him. To rail at the descendants of Charles Darwin because some of them have achieved eminence in mathematics or mathematical physics rather than in biology is slightly silly. The genius is still there, and one expression of it is not necessarily 'better' or 'higher' than another – unless we are the sort of bigots who insist that everything should be mathematics, or biology, or sociology, or bridge and golf. It may be that the abandonment of mathematics as the family trade by the Bernoullis was just one more instance of their genius.

Many legends and anecdotes have grown up round the famous Bernoullis, as is only natural in the case of a family as gifted and as violent in their language as the Bernoullis sometimes were. One of these ripe old chestnuts may be retailed again as it is one of the comparatively early authentic instances of a story which must be at least as old as ancient Egypt, and of which we daily see variants pinned on to all sorts of prominent characters from Einstein down. Once when travelling as a young man Daniel modestly introduced himself to an interesting stranger with whom he had been conversing: 'I am Daniel Bernoulli'. 'And I', said the other sarcastically, 'am Isaac Newton.' This delighted Daniel to the end of his days as the sincerest tribute he had ever received.

CHAPTER NINE

ANALYSIS INCARNATE

Euler

'EULER calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind' (as Arago said), is not an exaggeration of the unequalled mathematical facility of Léonard Euler (1707–83), the most prolific mathematician in history, and the man whom his contemporaries called 'analysis incarnate'. Euler wrote his great memoirs as easily as a fluent writer composes a letter to an intimate friend. Even total blindness during the last seventeen years of his life did not retard his unparalleled productivity; indeed, if anything, the loss of his eyesight sharpened Euler's perceptions in the inner world of his imagination.

The extent of Euler's work was not accurately known even in 1936, but it has been estimated that sixty to eighty large quarto volumes will be required for the publication of his collected works. In 1909 the Swiss Association for Natural Science undertook the collection and publication of Euler's scattered memoirs, with financial assistance from many individuals and mathematical societies throughout the world – rightly claiming that Euler belongs to the whole civilized world and not only to Switzerland. The careful estimates of the probable expense (about $80,000 in the money of 1909) were badly upset by the discovery in St Petersburg (Leningrad) of an unsuspected mass of Euler's manuscripts.

Euler's mathematical career opened in the year of Newton's death. A more propitious epoch for a genius like that of Euler's could not have been chosen. Analytical geometry (made public in 1637) had been in use ninety years, the calculus about fifty, and Newton's law of universal gravitation, the key to physical astronomy, had been before the mathematical public for forty years. In each of these fields a vast number of isolated problems had been solved, with here and there notable attempts at unification; but no systematic attack had yet been launched against the whole of mathematics, pure and applied, as it then existed. In particular the powerful analytical methods of Descartes, Newton, and Leibniz had not yet been exploited to the limit of what they were then capable of, especially in mechanics and geometry.

On a lower level algebra and trigonometry were then in shape for systematization and extension; the latter particularly was ready for essential completion. In Fermat's domain of Diophantine analysis and the properties of the common whole numbers no such 'temporary perfection' was possible (it is not even yet); but even here Euler proved himself the master. In fact one of the most remarkable features of Euler's universal genius was its equal strength in both of the main currents of mathematics, the continuous and the discrete.

As an algorist Euler has never been surpassed, and probably never even closely approached, unless perhaps by Jacobi. An algorist is a mathematician who devises 'algorithms' (or 'algorisms') for the solution of problems of special kinds. As a very simple example, we assume (or prove) that every positive real number has a real square root. How shall the root be calculated? There are many ways known; an algorist devises practical methods. Or again, in Diophantine analysis, also in the integral calculus, the solution of a problem may not be forthcoming until some ingenious (often simple) replacement of one or more of the variables by functions of other variables has been made; an algorist is a mathematician to whom such ingenious tricks come naturally. There is no uniform mode of procedure – algorists, like facile rhymesters, are born, not made.

It is fashionable to-day to despise the 'mere algorist'; yet, when a truly great one like the Hindu Ramanujan arrives unexpectedly out of nowhere, even expert analysts hail him as a gift from Heaven: his all but supernatural insight into apparently unrelated formulas reveals hidden trails leading from one territory to another, and the analysts have new tasks provided for them in clearing the trails. An algorist is a 'formalist' who loves beautiful formulas for their own sake.

Before going on to Euler's peaceful but interesting life we must mention two circumstances of his times which furthered his prodigious activity and helped to give it a direction.

In the eighteenth century the universities were not the principal centres of research in Europe. They might have become such sooner than they did but for the classical tradition and its understandable hostility to science. Mathematics was close enough to antiquity to be respectable, but physics, being more recent, was suspect. Further, a mathematician in a university of the time would have been expected to put much of his effort into elementary teaching; his research, if any, would have been an unprofitable luxury, precisely as in the average American institution of higher learning to-day. The Fellows of the British universities could do pretty well as they chose. Few, however, chose to do anything, and what they accomplished (or failed to accomplish) could not affect their bread and butter. Under such laxity or open hostility there was no good reason why the universities should have led in science, and they did not.

The lead was taken by the various royal academies supported by generous or far-sighted rulers. Mathematics owes an undischargeable debt to Frederick the Great of Prussia and Catherine the Great of Russia for their broadminded liberality. They made possible a full century of mathematical progress in one of the most active periods in scientific history. In Euler's case Berlin and St Petersburg furnished the sinews of mathematical creation. Both of these foci of creativity owed their inspiration to the restless ambition of Leibniz. The academies for which Leibniz had drawn up the plans gave Euler his chance to be the most prolific mathematician of all time; so, in a sense, Euler was Leibniz' grandson.

The Berlin Academy had been slowly dying of brainlessness for forty years when Euler, at the instigation of Frederick the Great, shocked it into life again; and the St Petersburg Academy, which Peter the Great did not live to organize in accordance with Leibniz' programme, was firmly founded by his successor.

These Academies were not like some of those to-day, whose chief function is to award membership in recognition of good work well done; they were research organizations which *paid* their leading members *to produce scientific research.* Moreover the salaries and perquisites were ample for a man to support himself and his family in decent comfort. Euler's household at one time consisted of no fewer than eighteen persons; yet he was given enough to support them all adequately. As a final touch of attractiveness to the life of an academician in the eighteenth century, his children, if worth anything at all, were assured of a fair start in the world.

This brings us to a second dominant influence on Euler's vast mathematical output. The rulers who paid the bills naturally wanted something in addition to abstract culture for their money. But it must be emphasized that when once the rulers had obtained a reasonable return on their investment, they did not insist that their employees spend the rest of their time on 'productive' labour; Euler, Lagrange, and the other academicians were free to do as they pleased. Nor was any noticeable pressure brought to bear to squeeze out the few immediately practical results which the state could use. Wiser in their generation than many a director of a research institute to-day, the rulers of the eighteenth century merely suggested occasionally what they needed at once, and let science take its course. They seem to have felt instinctively that so-called 'pure' research would throw off as by-products the instantly practical things they desired if given a hint of the right sort now and then.

To this general statement there is one important exception which neither proves nor disproves the rule. It so happened that in Euler's time the outstanding problem in mathematical research chanced also to coincide with what was probably the first practical problem of the age – control of the seas. That nation whose technique in navigation surpassed that of all its competitors would inevitably rule the waves. But navigation is ultimately an affair of accurately determining one's position at sea hundreds of miles from land, and of doing it so much better than one's competitors that they can be outsailed to the scene, unfavourable only for them, of a naval battle. Britannia, as everyone knows, rules the waves. That she does so is due in no small measure to the practical application which her navigators were able to make of purely mathematical investigations in celestial mechanics during the eighteenth century.

One such application concerned Euler directly – if we may anticipate slightly. The founder of modern navigation is of course Newton, although he himself never bothered his head about the subject and never (so far as seems to be known) planted his shoe on the deck of a ship. Position at sea is determined by observations on the heavenly bodies (sometimes including the satellites of Jupiter in really fancy navigation); and after Newton's universal law had suggested that with sufficient patience the positions of the planets and the phases of the Moon could be calculated for a century in advance if necessary, those who wished to govern the seas set their computers on the nautical almanac to grinding out tables of future positions.

In such a practical enterprise the Moon offers a particularly vicious problem, that of three bodies attracting one another according to the Newtonian law. This problem will recur many times as we proceed to the twentieth century; Euler was the first to evolve a *calculable* solution for the problem of the Moon ('the lunar theory'). The three bodies concerned are the Moon, the Earth, and the Sun. Although we shall defer what can be said here on this problem to later chapters, it may be remarked that the problem is one of the most difficult in the whole range of mathematics. Euler did not solve it, but his method of approximative calculation (superseded to-day by better methods) was sufficiently practical to enable an English computer to calculate the lunar tables for the British Admiralty. For this the computer received £5,000 (quite a sum for the time), and Euler was voted a bonus of £300 for the method.

Léonard (or Leonhard) Euler, a son of Paul Euler and his wife Marguerite Brucker, is probably the greatest man of science that Switzerland has produced. He was born at Basle on 15 April 1707, but moved the following year with his parents to the nearby village of Riechen, where his father became the Calvinist pastor. Paul Euler himself was an accomplished mathematician, having been a pupil of Jacob Bernoulli. The father intended Léonard to follow in his footsteps and succeed him in the village church, but fortunately made the mistake of teaching the boy mathematics.

Young Euler knew early what he wanted to do. Nevertheless he dutifully obeyed his father, and on entering the University of Basle studied theology and Hebrew. In mathematics he was sufficiently advanced to attract the attention of Johannes Bernoulli, who generously gave the young man one private lesson a week. Euler spent the rest of the week preparing for the next lesson so as to be able to meet his teacher with as few questions as possible. Soon his diligence and marked ability were noticed by Daniel and Nicolaus Bernoulli, who became Euler's fast friends.

Léonard was permitted to enjoy himself till he took his master's degree in 1724 at the age of seventeen, when his father insisted that he should abandon mathematics and give all his time to theology. But the father gave in when the Bernoullis told him that his son was destined to be a great mathematician and not the pastor of Riechen. Although the prophecy was fulfilled Euler's early religious training influenced him all his life and he never discarded a particle of his Calvinistic faith. Indeed as he grew older he swung round in a wide orbit toward the calling of his father, conducting family prayers for his whole household and usually finishing off with a sermon.

Euler's first independent work was done at the age of nineteen. It has been said that this first effort reveals both the strength and the weakness of much of Euler's subsequent work. The Paris Academy had proposed the masting of ships as a prize problem for the year 1727; Euler's memoir failed to win the prize but received an honourable mention. He was later to recoup this loss by winning the prize twelve times. The strength of the work was the analysis – the technical mathematics – it contained; its weakness the remoteness of the connexion, if any, with practicality. The last is not very surprising when we remember the traditional jokes about the non-existent Swiss navy. Euler might have seen a boat or two on the Swiss lakes, but he had not yet seen a ship. He has been criticized, sometimes justly, for letting his mathematics run away with his sense of reality. The physical universe was an occasion for mathematics to Euler, scarcely a thing of much interest in itself; and if the universe failed to fit his analysis it was the universe which was in error.

Knowing that he was a born mathematician, Euler applied for the professorship at Basle. Failing to get the position, he continued his studies, buoyed up by the hope of joining Daniel and Nicolaus Bernoulli at St Petersburg. They had generously offered to find a place for Euler in the Academy and kept him well posted.

At this stage of his career Euler seems to have been curiously indifferent as to what he should do, provided only it was something scientific. When the Bernoullis wrote of a prospective opening in the medical section of the St Petersburg Academy, Euler flung himself into physiology at Basle and attended the lectures on medicine. But even in this field he could not keep away from mathematics: the physiology of the ear suggested a mathematical investigation of sound, which in turn led out into another on the propagation of waves, and so on – this early work kept branching out like a tree gone mad in a nightmare all through Euler's career.

The Bernoullis were fast workers. Euler received his call to St Petersburg in 1727, officially as an associate of the medical section of the Academy. By a wise provision every imported member was obliged to take with him two pupils – actually apprentices to be trained. Poor Euler's joy was quickly dashed. The very day he set foot on Russian soil the liberal Catherine I died.

Catherine, Peter the Great's mistress before she became his wife, seems to have been a broadminded woman in more ways than one, and it was she who in her reign of only two years carried out Peter's wishes in establishing the Academy. On Catherine's death the power passed into the hands of an unusually brutal faction during the minority of the boy tsar (who perhaps fortunately for himself died before he could begin his reign). The new rulers of Russia looked upon the Academy as a dispensable luxury and for some anxious months contemplated suppressing it and sending all the foreign members home. Such was the state of affairs when Euler arrived in St Petersburg. Nothing was said in the confusion about the medical position to which he had been called, and he slipped into the mathematical section, after having almost accepted a naval lieutenancy in desperation.

Thereafter things went better and Euler settled down to work. For six years he kept his nose to the grindstone, not wholly because he was absorbed in his mathematics but partly because he dared not lead a normal social life on account of the treacherous spies everywhere.

In 1733 Daniel Bernoulli returned to free Switzerland, having had enough of holy Russia, and Euler, at the age of twenty-six, stepped into the leading mathematical position in the Academy. Feeling that he was to be stuck in St Petersburg for the rest of his life, Euler decided to marry, settle down, and make the best of things. The lady was Catharina, a daughter of the painter Gsell, whom Peter the Great had taken back to Russia with him. Political conditions became worse, and Euler longed more desperately than ever to escape. But with the rapid arrival of one child after another Euler felt more securely tied than before and took refuge in incessant work. Some biographers trace Euler's unmatched productivity to this first sojourn in Russia; common prudence forced him into an unbreakable habit of industry.

Euler was one of several great mathematicians who could work anywhere under any conditions. He was very fond of children (he had thirteen of his own, all but five of whom died very young), and would often compose his memoirs with a baby in his lap while the older children played all about him. The ease with which he wrote the most difficult mathematics is incredible.

Many legends of his constant outflow of ideas have survived. Some no doubt are exaggerations, but it is said that Euler would dash off a mathematical paper in the half hour or so before the first and second calls to dinner. As soon as a paper was finished it was laid on top of the growing stack awaiting the printer. When material to fill the transactions of the Academy was needed, the printer would gather up a sheaf from the top of the pile. Thus it happened that the dates of publication frequently ran counter to those of composition. The crazy effect was heightened by Euler's habit of returning many times to a subject in order to clarify or extend what he had already done, so that occasionally a sequence of papers on a given topic is seen in print through the wrong end of the telescope.

When the boy tsar died, Anna Ivanovna (niece of Peter) became Empress in 1730, and so far as the Academy was concerned, things brightened up considerably. But under the indirect rule of Anna's paramour, Ernest John de Biron, Russia suffered one of the bloodiest reigns of terror in its history, and Euler settled down to a spell of silent work that was to last ten years. Halfway through he suffered his first great misfortune. He had set himself to win the Paris prize for an astronomical problem for which some of the leading mathematicians had asked several months' time. (As a similar problem occurs in connexion with Gauss we shall not describe it here.) Euler solved it in three days. But the prolonged effort brought on an illness in which he lost the sight of his right eye.

It should be noted that the modern higher criticism which has been so effective in discrediting all the interesting anecdotes in the history of mathematics has shown that the astronomical problem was in no way responsible for the loss of Euler's eye. But how the scholarly critics (or anyone else) come to know so much about the so-called law of cause and effect is a mystery for David Hume's (a contemporary of Euler) ghost to resolve. With this caution we shall tell once more the famous story of Euler and the atheistic (or perhaps only pantheistic) French philosopher Denis Diderot (1713–84). This is slightly out of its chronological order, as it happened during Euler's second stay in Russia.

Invited by Catherine the Great to visit her Court, Diderot earned his keep by trying to convert the courtiers to atheism. Fed up, Catherine commissioned Euler to muzzle the windy philosopher. This was easy because all mathematics was Chinese to Diderot. De Morgan tells what happened (in his classic *Budget of Paradoxes*, 1872): 'Diderot was informed that a learned mathematician was in possession of an algebraical demonstration of the existence of God, and would give it before all the Court, if he desired to hear it. Diderot gladly consented. . . . Euler advanced toward Diderot, and said gravely, and in a tone of perfect conviction:

$$\text{`}\textit{Sir, } \frac{a + b^n}{n} = x, \textit{ hence God exists; reply!'}$$

It sounded like sense to Diderot. Humiliated by the unrestrained laughter which greeted his embarrassed silence, the poor man asked Catherine's permission to return at once to France. She graciously gave it.

Not content with this masterpiece, Euler in all seriousness painted his lily with solemn proofs, in deadly earnest, that God exists and that the soul is not a material substance. It is reported that both proofs passed into the treatises on theology of his day. These are probably the choicest flowers of the mathematically unpractical side of his genius.

Mathematics alone did not absorb all of Euler's energies during his stay in Russia. Wherever he was called upon to exercise his mathematical talents in ways not too far from pure mathematics he gave the government its full money's worth. Euler wrote the elementary mathematical textbooks for the Russian schools, supervised the government department of geography, helped to reform the weights and measures, and devised practical means for testing scales. These were but some of his activities. No matter how much extraneous work he did, Euler continued to pour out mathematics.

One of the most important works of this period was the treatise of 1736 on mechanics. Note that the date of publication lacks but a year of marking the centenary of Descartes' publication of analytical geometry. Euler's treatise did for mechanics what Descartes' had done for geometry – freed it from the shackles of synthetic demonstration and made it analytical. Newton's *Principia* might have been written by Archimedes; Euler's mechanics could not have been written by any Greek. For the first time the full power of the calculus was directed against mechanics and the modern era in that basic science began. Euler was to be surpassed in this direction by his friend

Lagrange, but the credit for having taken the decisive step is Euler's.

On the death of Anna in 1740 the Russian government became more liberal, but Euler had had enough and was glad to accept the invitation of Frederick the Great to join the Berlin Academy. The Dowager Queen took a great fancy to Euler and tried to draw him out. All she got was monosyllables.

'Why don't you want to speak to me?' she asked.

'Madame,' Euler replied, 'I come from a country where, if you speak, you are hanged.'

The next twenty-four years of his life were spent in Berlin, not altogether happily, as Frederick would have preferred a polished courtier instead of the simple Euler. Although Frederick felt it his duty to encourage mathematics he despised the subject, being no good at it himself. But he appreciated Euler's talents sufficiently to engage them in practical problems —the coinage, water conduits, navigation canals, and pension systems, among others.

Russia never let go of Euler completely and even while he was in Berlin paid part of his salary. In spite of his many dependents Euler was prosperous, owning a farm near Charlottenburg in addition to his house in Berlin. During the Russian invasion of the March of Brandenburg in 1760 Euler's farm was pillaged. The Russian general, declaring that he was 'not making war on the sciences', indemnified Euler for considerably more than the actual damage. When the Empress Elizabeth heard of Euler's loss she sent him a handsome sum in addition to the more than sufficient indemnity.

One cause of Euler's unpopularity at Frederick's court was his inability to keep out of arguments on philosophical questions about which he knew nothing. Voltaire, who spent much of his time toadying to Frederick, delighted with the other brilliant verbalists surrounding Frederick in tying the hapless Euler into metaphysical knots. Euler took it all good-naturedly and joined the others in roaring with laughter at his own ridiculous blunders. But Frederick gradually became irritated and cast about for a more sophisticated philosopher to head his Academy and entertain his Court.

D'Alembert (whom we shall meet later) was invited to Berlin to look over the situation. He and Euler had had a slight coolness over mathematics. But D'Alembert was not the man to let a personal difference cloud his judgement, and he told Frederick bluntly that it would be an outrage to put any other mathematician over Euler. This only made Frederick more stubborn and angrier than ever, and conditions became intolerable for Euler. His sons, he felt, would have no chance in Prussia. At the age of fifty-nine (in 1766) he pulled up his stakes once more and migrated back to St Petersburg at the cordial invitation of Catherine the Great.

Catherine received the mathematician as if he were royalty, setting aside a fully furnished house for Euler and his eighteen dependents, and donating one of her own cooks to run the kitchen.

It was at this time that Euler began to lose the sight of his remaining eye (by a cataract), and before long he was totally blind. The progress of his oncoming darkness is followed with alarm and sympathy in the correspondence of Lagrange, D'Alembert, and other leading mathematicians of the time. Euler himself watched the approach of blindness with equanimity. There can be no doubt that his deep religious faith helped him to face what was ahead of him. But he did not 'resign' himself to silence and darkness. He immediately set about repairing the irreparable. Before the last light faded he accustomed himself to writing his formulas with chalk on a large slate. Then, his sons (particularly Albert) acting as amanuenses, he would dictate the words explaining the formulas. Instead of diminishing, his mathematical productivity increased.

All his life Euler had been blessed with a phenomenal memory. He knew Virgil's *Aeneid* by heart, and although he had seldom looked at the book since he was a youth, could always tell the first and last lines on any page of his copy. His memory was both visual and aural. He also had a prodigious power for mental calculation, not only of the arithmetical kind but also of the more difficult type demanded in higher algebra and the calculus. All the leading formulas of the whole range of mathematics as it existed in his day were accurately stowed away in his memory.

As one instance of his prowess, Condorcet tells how two of Euler's students had summed a complicated convergent series (for a particular value of the variable) to seventeen terms, only to disagree by a unit in the fiftieth place of the result. To decide which was right Euler performed the whole calculation *mentally*; his answer was found to be correct. All this now came to his aid and he did not greatly miss the light. But even at that, one feat of his seventeen blind years almost passes belief. The lunar theory – the motion of the Moon, the only problem which had ever made Newton's head ache – received its first thorough workout at Euler's hands. All the complicated analysis was done entirely in his head.

Five years after Euler's return to St Petersburg another disaster overtook him. In the great fire of 1771 his house and all its furnishings were destroyed, and it was only by the heroism of his Swiss servant (Peter Grimm, or Grimmon) that Euler escaped with his life. At the risk of his own life Grimm carried his blind and ailing master through the flames to safety. The library was burned, but thanks to the energy of Count Orloff all of Euler's manuscripts were saved. The Empress Catherine promptly made good all the loss and soon Euler was back at work again.

In 1776 (when he was sixty-nine) Euler suffered a greater loss in the death of his wife. The following year he married again. The second wife, Salome Abigail Gsell, was a half-sister of the first. His greatest tragedy was the failure (through surgical carelessness, possibly) of an operation to restore the sight of his left eye – the only one for which there was any hope. The operation was 'successful' and Euler's joy passed all bounds. But presently infection set in, and after prolonged suffering which he described as hideous, he lapsed back into darkness.

In looking back over Euler's enormous output we may be inclined at the first glance to believe that any gifted man could have done a large part of it almost as easily as Euler. But an inspection of mathematics as it exists to-day soon disabuses us. For the present state of mathematics with its jungles of theories is relatively no more complicated, when we consider the power of the methods now at our disposal, than what Euler faced. Mathematics is ripe for a second Euler. In his day he systematized and unified vast tracts cluttered with partial results and isolated theorems, clearing the ground and binding up the valuable things by the easy power of his analytical machinery. Even to-day much of what is learned in a college course in mathematics is practically as Euler left it – the discussion of conic sections and quadrics in three-space from the unified point of view provided by the general equation of the second degree, for example, is Euler's. Again, the subject of annuities and all that grows out of it (insurance, old-age pensions, and so on) were put into the shape now familiar to students of the 'mathematical theory of investment' by Euler.

As Arago points out, one source of Euler's great and immediate success as a teacher through his writings was his total lack of false pride. If certain works of comparatively low intrinsic merit were demanded to clarify earlier and more impressive works, Euler did not hesitate to write them. He had no fear of lowering his reputation.

Even on the creative side Euler combined instruction with discovery. His great treatises of 1748, 1755, and 1768–70 on the calculus (*Introductio in analysin infinitorum*; *Institutiones calculi differentialis*; *Institutiones calculi integralis*) instantly became classic and continued for three-quarters of a century to inspire young men who were to become great mathematicians. But it was in his work on the calculus of variations (*Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, 1744) that Euler first revealed himself as a mathematician of the first rank. The importance of this subject has been noted in previous chapters.

Euler's great step forward when he made mechanics *analytical* has already been remarked; every student of rigid dynamics is familiar with Euler's analysis of rotations, to cite but one detail of this advance. Analytical mechanics is a branch of pure mathematics, so that Euler was not tempted here, as in some of his other flights toward the practical, to fly off on the first tangent he saw leading into the infinite blue of pure calculation. The severest criticism which Euler's contemporaries made of his work was his uncontrollable impulse to calculate merely for the sake of the beautiful analysis. He may occasionally have lacked a sufficient understanding of the physical situations he attempted to reduce to calculation without seeing what they were all about. Nevertheless, the fundamental equations of fluid motion, in use to-day in hydrodynamics, are Euler's. He could be practical enough when it was worth his trouble.

One peculiarity of Euler's analysis must be mentioned in passing, as it was largely responsible for one of the main currents of mathematics in the nineteenth century. This was his recognition that unless an infinite series is *convergent* it is unsafe to use. For example, by long division we find

$$\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots,$$

the series continuing indefinitely. In this put $x = \frac{1}{2}$. Then

$$\begin{aligned} -2 &= 2 + 2^2 + 2^3 + 2^4 + \dots, \\ &= 2 + 4 + 8 + 16 + \dots. \end{aligned}$$

The study of *convergence* (to be discussed in the chapter on Gauss) shows us how to avoid absurdities like this. (See also the chapter on Cauchy.) The curious thing is that, although Euler recognized the necessity for caution in dealing with *infinite* processes, he failed to observe it in much of his own work. His faith in analysis was so great that he would sometimes seek a preposterous 'explanation' to make a patent absurdity respectable.

But when all this is said, we must add that few have equalled or approached Euler in the mass of sound and novel work of the first importance which he put out. Those who love arithmetic – not a very 'important' subject, possibly – will vote Euler a palm in Diophantine analysis of the same size and freshness as those worn by Fermat and Diophantus himself. Euler was the first and possibly the greatest of the mathematical universalists.

Nor was he merely a narrow mathematician: in literature and all of the sciences, including the biological, he was at least well read. But even while he was enjoying his *Aeneid* Euler could not help seeing a problem for his mathematical genius to attack. The line 'The anchor drops, the rushing keel is stay'd' set him to working out the ship's motion under such circumstances. His omnivorous curiosity even swallowed astrology for a time, but he showed that he had not digested it by politely declining to cast the horoscope of Prince Ivan when ordered to do so in 1740, pointing out that horoscopes belonged in the province of the court astronomer. The poor astronomer had to do it.

One work of the Berlin period revealed Euler as a graceful (if somewhat too pious) writer, the celebrated *Letters to a German Princess*, composed to give lessons in mechanics, physical optics, astronomy, sound, etc., to Frederick's niece, the Princess of Anhalt-Dessau. The famous letters became immensely popular and circulated in book form in seven languages. Public interest in science is not the recent development we are sometimes inclined to imagine it is.

Euler remained virile and powerful of mind to the very second of his death, which occurred in his seventy-seventh year, on 18 September 1783. After having amused himself one afternoon calculating the laws of ascent of balloons – on his slate, as usual – he dined with Lexell and his family. 'Herschel's Planet' (Uranus) was a recent discovery; Euler outlines the calculation of its orbit. A little later he asked that his grandson be brought in. While playing with the child and drinking tea he suffered a stroke. The pipe dropped from his hand, and with the words 'I die', 'Euler ceased to live and calculate.' *

* The quotation is from Condorcet's *Éloge*.

CHAPTER TEN

A LOFTY PYRAMID

Lagrange

'Lagrange is the lofty pyramid of the mathematical sciences.' This was Napoleon Bonaparte's considered estimate of the greatest and most modest mathematician of the eighteenth century, Joseph-Louis Lagrange (1736–1813), whom he had made a Senator, a Count of the Empire, and a Grand Officer of the Legion of Honour. The King of Sardinia and Frederick the Great had also honoured Lagrange, but less lavishly than the imperial Napoleon.

Lagrange was of mixed French and Italian blood, the French predominating. His grandfather, a French cavalry captain, had entered the service of Charles Emmanuel II, King of Sardinia, and on settling at Turin had married into the illustrious Conti family. Lagrange's father, once Treasurer of War for Sardinia, married Marie-Thérèse Gros, the only daughter of a wealthy physician of Cambiano, by whom he had eleven children. Of this numerous brood only the youngest, Joseph-Louis, born on 25 January 1736, survived beyond infancy. The father was rich, both in his own right and his wife's. But he was also an incorrigible speculator, and by the time his son was ready to inherit the family fortune there was nothing worth inheriting. In later life Lagrange looked back on this disaster as the luckiest thing that had ever happened to him: 'If I had inherited a fortune I should probably not have cast my lot with mathematics.'

At school Lagrange's first interests were in the classics, and it was more or less of an accident that he developed a passion for mathematics. In line with his classical studies he early became acquainted with the geometrical works of Euclid and Archimedes. These do not seem to have impressed him greatly. Then an essay by Halley (Newton's friend) extolling the superiority of the calculus over the synthetic geometrical methods of the Greeks fell into young Lagrange's hands. He was captivated and converted. In an incredibly short time he had mastered entirely by himself what in his day was modern analysis. At the age of sixteen (according to Delambre there may be a slight inaccuracy here) Lagrange became professor of mathematics at the Royal Artillery School in Turin. Then began one of the most brilliant careers in the history of mathematics.

From the first Lagrange was an analyst, never a geometer. In him we see the first conspicuous example of that specialization which was to become almost a necessity in mathematical research. Lagrange's analytical preferences came out strongly in his masterpiece, the *Mécanique analytique* (Analytical Mechanics), which he had projected as a boy of nineteen at Turin, but which was published in Paris only in 1788 when Lagrange was fifty-two. 'No diagrams will be found in this work', he says in the preface. But with a half-humorous libation to the gods of geometry he remarks that the science of mechanics may be considered as the geometry of a space of four dimensions – three Cartesian co-ordinates with one time-coordinate sufficing to locate a moving particle in both space and time, a way of looking at mechanics that has become popular since 1915 when Einstein exploited it in his general relativity.

Lagrange's analytical attack on mechanics marks the first complete break with the Greek tradition. Newton, his contemporaries, and his immediate successors found diagrams helpful in their study of mechanical problems; Lagrange showed that greater flexibility and incomparably greater power are attained if general analytical methods are employed from the beginning.

At Turin the boyish professor lectured to students all older than himself. Presently he organized the more able into a research society from which the Turin Academy of Sciences developed. The first volume of the Academy's memoirs was published in 1759, when Lagrange was twenty-three. It is usually supposed that the modest and unobtrusive Lagrange was responsible for much of the fine mathematics in these early works published by others. One paper by Foncenex was so good that the King of Sardinia put the supposed author in charge of the Department of the Navy. Historians of mathematics have sometimes wondered why Foncenex never lived up to his first mathematical success.

Lagrange himself contributed a memoir on maxima and minima (the calculus of variations, described in Chapters 4, 8) in which he promises to treat the subject in a work from which he will deduce the whole of mechanics, of both solids and fluids. Thus at twenty-three – actually earlier – Lagrange had imagined his masterpiece, the *Mécanique analytique*, which does for general mechanics what Newton's law of universal gravitation did for celestial mechanics. Writing ten years later to the French mathematician D'Alembert (1717–83), Lagrange says he regards his early work, the calculus of variations, thought out when he was nineteen, as his masterpiece. It was by means of this calculus that Lagrange unified mechanics and, as Hamilton said, made of it 'a kind of scientific poem.'

When once understood the Lagrangian method is almost a platitude. As some have remarked, the Lagrangian equations dominating mechanics are the finest example in all science of the art of getting something out of nothing. But if we reflect a moment we see that any scientific principle which is general to the extent of uniting a whole vast universe of phenomena *must* be simple: only a principle of the utmost simplicity can dominate a multitude of diverse problems which on even a close inspection appear to be individual and distinct.

In the same volume of Turin memoirs Lagrange took another long step forward: he applied the differential calculus to the theory of probability. As if this were not enough for the young giant of twenty-three he advanced beyond Newton with a radical departure in the mathematical theory of sound, bringing that theory under the sway of the mechanics of systems of elastic particles (rather than of the mechanics of fluids), by considering the behaviour of all the air particles in one straight line under the action of a shock transmitted along the line from particle to particle. In the same general direction he also settled a vexed controversy that had been going on for years between the leading mathematicians over the correct mathematical formulation of the problem of a vibrating string – a problem of fundamental importance in the whole theory of vibrations. At twenty-three Lagrange was acknowledged the equal of the greatest mathematicians of the age – Euler and the Bernoullis.

Euler was always generously appreciative of the work of others. His treatment of his young rival Lagrange is one of the finest pieces of unselfishness in the history of science. When as a boy of nineteen Lagrange sent Euler some of his work the famous mathematician at once recognized its merit and encouraged the brilliant young beginner to continue. When four years later Lagrange communicated to Euler the true method for attacking the isoperimetrical problems (the calculus of variations, described in connexion with the Bernoullis), which had baffled Euler with his semi-geometrical methods for many years, Euler wrote to the young man saying that the new method had enabled him to overcome his difficulties. And instead of rushing into print with the long-sought solution, Euler held it back till Lagrange could publish his first, 'so as not to deprive you of any part of the glory which is your due.'

Private letters, however flattering, could not have helped Lagrange. Realizing this, Euler went out of his way when he published his work (after Lagrange's) to say how he had been held up by difficulties which, till Lagrange showed the way over them, were insuperable. Finally, to clinch the matter, Euler got Lagrange elected as a foreign member of the Berlin Academy (2 October 1759) at the unusually early age of twenty-three. This official recognition abroad was a great help to Lagrange at home. Euler and D'Alembert schemed to get Lagrange to Berlin. Partly for personal reasons they were eager to see their brilliant young friend installed as court mathematician at Berlin. After lengthy negotiations they succeeded, and the great Frederick, slightly outwitted in the whole transaction, was childishly (but justifiably) delighted.

Something must be said in passing about D'Alembert, Lagrange's devoted friend and generous admirer, if only for the grateful contrast one aspect of his character offers to that of the snobbish Laplace, whom we shall meet later.

Jean le Rond d'Alembert took his name from the little chapel of St-Jean-le-Rond hard by Notre-Dame in Paris. An illegitimate son of the Chevalier Destouches, D'Alembert had been abandoned by his mother on the steps of St-Jean-le-Rond. The parish authorities turned the foundling over to the wife of a poor glazier, who reared the child as if he were her own. The Chevalier was forced by law to pay for his bastard's education. D'Alembert's real mother knew where he was, and when the boy early gave signs of genius, sent for him, hoping to win him over.

'You are only my stepmother', the boy told her (a good pun in English, but not in French); 'the glazier's wife is my true mother.' And with that he abandoned his own flesh and blood as she had abandoned hers.

When he became famous and a great figure in French science D'Alembert repaid the glazier and his wife by seeing that they did not fall into want (they preferred to keep on living in their humble quarters), and he was always proud to claim them as his parents. Although we shall not have space to consider him apart from Lagrange, it must be mentioned that D'Alembert was the first to give a complete solution of the outstanding problem of the precession of the equinoxes. His most important purely mathematical work was in partial differential equations, particularly in connexion with vibrating strings.

D'Alembert encouraged his modest young correspondent to attack difficult and important problems. He also took it upon himself to make Lagrange take reasonable care of his health – his own was not good. Lagrange had in fact seriously impaired his digestion by quite unreasonable application between the ages of sixteen and twenty-six, and all his life thereafter he was forced to discipline himself severely, especially in the matter of overwork. In one of his letters D'Alembert lectures the young man for indulging in tea and coffee to keep awake; in another he lugubriously calls Lagrange's attention to a recent medical book on the diseases of scholars. To all of which Lagrange blithely replies that he is feeling fine and working like mad. But in the end he paid his tax.

In one respect Lagrange's career is a curious parallel to Newton's. By middle age prolonged concentration on problems of the first magnitude had dulled Lagrange's enthusiasm, and although his mind remained as powerful as ever, he came to regard mathematics with indifference. When only forty-five he wrote to D'Alembert, 'I begin to feel the pull of my inertia increasing little by little, and I cannot say that I shall still be doing mathematics ten years from now. It also seems to me that the mine is already too deep, and that unless new veins are discovered it will have to be abandoned.'

When he wrote this Lagrange was ill and melancholic. Nevertheless it expressed the truth so far as he was concerned. D'Alembert's last letter (September 1783), written a month before his death, reverses his early advice and counsels work as the only remedy for Lagrange's psychic ills: 'In God's name do not renounce work, for you the strongest of all distractions. Good-bye, perhaps for the last time. Keep some memory of the man who of all in the world cherishes and honours you the most.'

Happily for mathematics Lagrange's blackest depression, with its inescapable corollary that no human knowledge is worth striving for, was twenty glorious years in the future when D'Alembert and Euler were scheming to get Lagrange to Berlin. Among the great problems Lagrange attacked and solved before going to Berlin was that of the libration of the Moon. Why does the Moon always present the same face to the Earth – within certain slight irregularities that can be accounted for? It was required to deduce this fact from the Newtonian law of gravitation. The problem is an instance of the famous 'Problem of Three Bodies' – here the Earth, Sun, and Moon – mutually attracting one another according to the law of the inverse square of the distances between their centres of gravity. (More will be said on this problem when we come to Poincaré.)

For his solution of the problem of libration Lagrange was awarded the Grand Prize of the French Academy of Sciences in 1764 – he was then only twenty-eight.

Encouraged by this brilliant success the Academy proposed a yet more difficult problem, for which Lagrange again won the prize in 1766. In Lagrange's day only four satellites of Jupiter had been discovered. Jupiter's system (himself, the Sun, and his satellites) thus made a six-body problem. A *complete* mathematical solution is beyond our powers even to-day (1936) in a shape adapted to practical computation. But by using methods of approximation Lagrange made a notable advance in explaining the observed inequalities.

Such applications of the Newtonian theory were one of Lagrange's major interests all his active life. In 1772 he again captured the Paris prize for his memoir on the three-body problem, and in 1774 and 1778 he had similar successes with the motion of the Moon and cometary perturbations.

The earlier of these spectacular successes induced the King of Sardinia to pay Lagrange's expenses for a trip to Paris and London in 1766. Lagrange was then thirty. It had been planned that he was to accompany Caraccioli, the Sardinian minister to England, but on reaching Paris Lagrange fell dangerously ill – the result of an over-generous banquet of rich Italian dishes in his honour – and he was forced to remain in Paris. While there he met all the leading intellectuals, including the Abbé Mariè, who was later to prove an invaluable friend. The banquet cured Lagrange of his desire to live in Paris and he eagerly returned to Turin as soon as he was able to travel.

At last, on 6 November 1766, Lagrange was welcomed, at the age of thirty, to Berlin by Frederick, 'the greatest King in Europe', as he modestly styled himself, who would be honoured to have at his court 'the greatest mathematician'. The last, at least, was true. Lagrange became director of the physico-mathematical division of the Berlin Academy, and for twenty years crowded the transactions of the Academy with one great memoir after another. He was not required to lecture.

At first the young director found himself in a somewhat delicate position. Naturally enough the Germans rather resented foreigners being brought in over their heads and were inclined to treat Frederick's importations with a little less than cool civility. In fact they were frequently quite insulting. But in addition to being a mathematician of the first rank Lagrange was a considerate, gentle soul with the rare gift of knowing when to keep his mouth shut. In letters to trusted friends he could be outspoken enough, even about the Jesuits, whom he and D'Alembert seem to have disliked, and in his official reports to academies on the scientific work of others he could be quite blunt. But in his social contacts he minded his own business and avoided giving even justifiable offence. Until his colleagues got used to his presence he kept out of their way.

Lagrange's constitutional dislike of all disputes stood him in good stead at Berlin. Euler had blundered from one religious or philosophical controversy to another; Lagrange, if cornered and pressed, would always preface his replies with his sincere formula 'I do not know.' Yet when his own convictions were attacked he knew how to put up a spirited, reasoned defence.

On the whole Lagrange was inclined to sympathize with Frederick, who had sometimes been irritated by Euler's tilting at philosophical problems about which he knew nothing. 'Our friend Euler', he wrote to D'Alembert, 'is a great mathematician, but a bad enough philosopher.' And on another occasion, referring to Euler's effusion of pious moralizing in the celebrated *Letters to a German Princess*, he dubs the classic 'Euler's commentary on the Apocalypse' – incidentally a backhand allusion to the indiscretion which Newton permitted himself when he had lost his taste for natural philosophy. 'It is incredible', Lagrange said of Euler, 'that he could have been so flat and childish in metaphysics.' And for himself, 'I have a great aversion to disputes.' When he did philosophize in his letters it was with an unexpected touch of cynicism which is wholly absent from the works he published, as when he remarks, 'I have always observed that the pretensions of all people are in exact inverse ratio to their merits; this is one of the axioms of morals.' In religious matters Lagrange was, if anything at all, agnostic.

Frederick was delighted with his prize and spent many friendly hours with Lagrange, expounding the advantages of a regular life. The contrast Lagrange offered to Euler was particularly pleasing to Frederick. The King had been irritated by Euler's too obvious piety and lack of courtly sophistication. He had even gone so far as to call poor Euler a 'lumbering cyclops of a mathematician', because Euler at the time was blind in only one of his eyes. To D'Alembert the grateful

Frederick overflowed in both prose and verse. 'To your trouble and to your recommendation,' he wrote, 'I owe the replacement in my Academy of a mathematician blind in one eye by a mathematician with two eyes, which will be especially pleasing to the anatomical section.' In spite of sallies like this Frederick was not a bad sort.

Shortly after settling in Berlin Lagrange sent to Turin for one of his young lady relatives and married her. There are two accounts of how this happened. One says that Lagrange had lived in the same house with the girl and her parents and had taken an interest in her shopping. Having an economical streak in his cautious nature, Lagrange was scandalized by what he considered the girl's extravagance and bought her ribbons himself. From there on he was dragooned into marrying her.

The other version can be inferred from one of Lagrange's letters – certainly the strangest confession of indifference ever penned by a supposedly doting young husband. D'Alembert had joked to his friend: 'I understand that you have taken what we philosophers call the fatal plunge. . . . A great mathematician should know above all things how to calculate his happiness. I do not doubt then that after having performed this calculation you found the solution in marriage.'

Lagrange either took this in deadly earnest or set out to beat D'Alembert at his own game – and succeeded. D'Alembert had expressed surprise that Langrange had not mentioned his marriage in his letters.

'I don't know whether I calculated ill or well,' Lagrange replied, 'or rather, I don't believe I calculated at all; for I might have done as Leibniz did, who, compelled to reflect, could never make up his mind. I confess to you that I never had a taste for marriage. . . . but circumstances decided me to engage one of my young kinswomen to take care of me and all my affairs. If I neglected to inform you it was because the whole thing seemed to me so inconsequential in itself that it was not worth the trouble of informing you of it.'

The marriage was turning out happily for both when the wife declined in a lingering illness. Lagrange gave up his sleep to nurse her himself and was heartbroken when she died.

He consoled himself in his work. 'My occupations are reduced to cultivating mathematics, tranquilly and in silence.' He then tells D'Alembert the secret of the perfection of all his work which has been the despair of his hastier successors. 'As I am not pressed and work more for my pleasure than from duty, I am like the great lords who build: I make, unmake, and remake, until I am passably satisfied with my results, which happens only rarely.' And on another occasion, after complaining of illness brought on by overwork, he says it is impossible for him to rest: 'My bad habit of rewriting my memoirs several times till I am passably satisfied is impossible for me to break.'

Not all of Lagrange's main efforts during his twenty years at Berlin went into celestial mechanics and the polishing of his masterpiece. One digression – into Fermat's domain – is of particular interest as it may suggest the inherent difficulty of simple-looking things in arithmetic. We see even the great Lagrange puzzled over the unexpected effort his arithmetical researches cost him.

'I have been occupied these last few days', he wrote to D'Alembert on 15 August 1768, 'in diversifying my studies a little with certain problems of Arithmetic, and I assure you, I found many more difficulties than I had anticipated. Here is one, for example, at whose solution I arrived only with great trouble. Given any positive integer n which is not a square, to find a square integer, x^2, such that $nx^2 + 1$ shall be a square. This problem is of great importance in the theory of squares [to-day, *quadratic forms*, to be described in connexion with Gauss] which [squares] are the principal object in Diophantine analysis. Moreover I found on this occasion some very beautiful theorems of Arithmetic, which I will communicate to you another time if you wish.'

The problem Lagrange describes has a long history going back to Archimedes and the Hindus. Lagrange's classic memoir on making $nx^2 + 1$ a square is a landmark in the theory of numbers. He was also the first to prove some of Fermat's theorems and that of John Wilson (1741–93), who had stated that if p is any prime number, then if all the numbers 1, 2, ... up to $p - 1$ are multiplied together and 1 be added to the result, the sum is divisible by p. The like is not true if p is not prime. For example, if $p = 5$, $1 \times 2 \times 3 \times 4 + 1 = 25$. This can be proved by elementary reasoning and is another of those arithmetical super-intelligence tests.*

In his reply D'Alembert states his belief that Diophantine analysis may be useful in the integral calculus, but does not go into detail. Curiously enough, the prophecy was fulfilled in the 1870's by the Russian mathematician, G. Zolotarev.

Laplace also became interested in arithmetic for a while and told Lagrange that the existence of Fermat's unproved theorems, while one of the greatest glories of French mathematics, was also its most conspicuous blemish, and it was the duty of French mathematicians to remove the blemish. But he prophesied tremendous difficulties. The root of the trouble, in his opinion, is that *discrete* problems (those dealing ultimately with 1, 2, 3, ...) are not yet attackable by any general weapon such as the calculus provides for the continuous. D'Alembert also remarks of arithmetic that he found it 'more difficult than it seems at first.' These experiences of mathematicians like Lagrange and his friends may imply that arithmetic really is hard.

Another letter of Lagrange's (28 February 1769) records the conclusion of the matter. 'The problem I spoke of has occupied me much more than I anticipated at first; but finally I am happily finished and I believe I have left practically nothing to be desired in the subject of indeterminate equations of the second degree in two unknowns.' He was too optimistic here; Gauss had yet to be heard from – his father and mother had still seven years to go before meeting one another. Two years before the birth of Gauss (in 1777), Lagrange looked back over his work in a pessimistic mood: 'The arithmetical researches are those which have cost me most trouble and are perhaps the least valuable.'

* A ridiculous 'proof' by a Spanish gentleman is funny enough to be quoted. The customary abbreviation for $1 \times 2 \times \ldots \times n$ is $n!$ Now $p - 1 + 1 = p$, which is divisible by p. Put exclamation points throughout: $(p - 1)! + 1! = p!$. The right side is again divisible by p; hence $(p - 1)! + 1$ is divisible by p. Unfortunately this works equally well if p is not prime.

When he was feeling well Lagrange seldom lapsed into the error of estimating the 'importance' of his work. 'I have always regarded mathematics', he wrote to Laplace in 1777, 'as an object of amusement rather than of ambition, and I can assure you that I enjoy the works of others much more than my own, with which I am always dissatisfied. You will see by that, if you are exempt from jealousy by your own success, I am none the less so by my disposition.' This was in reply to a somewhat pompous declaration by Laplace that he worked at mathematics only to appease his own sublime curiosity and did not give a hang for the plaudits of 'the multitude' – which, in his case, was partly balderdash.

A letter of 15 September 1782 to Laplace is of great historical interest as it tells of the finishing of the *Mécanique analytique*: 'I have almost completed a Treatise on Analytical Mechanics, founded solely on the principle or formula in the first section of the accompanying memoir; but as I do not know when or where I can get it printed, I am not hurrying with the finishing touches.'

Legendre undertook the editing of the work for the press and Lagrange's old friend the Abbé Marie finally persuaded a Paris publisher to risk his reputation. This canny individual consented to proceed with the printing only when the Abbé agreed to purchase all stock remaining unsold after a certain date. The book did not appear until 1788, after Lagrange had left Berlin. A copy was delivered into his hands when he had grown so indifferent to all science and all mathematics that he did not even bother to open the book. For all he knew at the time the printer might have got it out in Chinese. He did not care.

One investigation of Lagrange's Berlin period is of the highest importance in the development of modern algebra, the memoir of 1767 *On the Solution of Numerical Equations* and the subsequent additions dealing with the general question of the algebraic solvability of equations. Possibly the greatest importance of Lagrange's researches in the theory and solution of equations is the inspiration they proved to be to the leading algebraists of the early nineteenth century. Time after time we shall see the men who finally disposed of a problem which had baffled algebraists for three centuries or more returning to Lagrange for ideas and inspiration. Lagrange himself did not resolve the central difficulty – that of stating necessary and sufficient conditions that a given equation shall be solvable algebraically, but the germ of the solution is to be found in his work.

As the problem is one of those major things in all algebra which can be simply described we may glance at it in passing; it will recur many times as a leading motive in the work of some of the great mathematicians of the nineteenth century — Cauchy, Abel, Galois, Hermite, and Kronecker, among others.

First it may be emphasized that there is no difficulty whatever in solving an algebraic equation with numerical coefficients. The labour may be excessive if the equation is of high degree, say

$$3x^{101} - 17.3x^{70} + x - 11 = 0,$$

but there are many straightforward methods known whereby a root of such a *numerical* equation can be found to any prescribed degree of accuracy. Some of these are part of the regular school course in algebra. But in Lagrange's day uniform methods for solving numerical equations to a preassigned degree of accuracy were not commonplace – if known at all. Lagrange provided such a method. Theoretically it did what was required, but it was not practical. No engineer faced with a numerical equation to-day would dream of using Lagrange's method.

The really significant problem arises when we seek an *algebraic* solution of an equation with *literal* coefficients, say $ax^2 + bx + c = 0$, or $ax^3 + bx^2 + cx + d = 0$, and so on for degrees higher than the third. What is required is a set of formulas expressing the *unknown* x in terms of the *given* a, b, c, . . . , such that if any one of these expressions for x be put in the left-hand side of the equation, that side shall reduce to zero. For an equation of degree n the unknown x has precisely n values. Thus for the above quadratic (second degree) equation,

$$\frac{1}{2a}(-b + \sqrt{b^2 - 4ac}), \frac{1}{2a}(-b - \sqrt{b^2 - 4ac})$$

are the two values which when substituted for x will reduce $ax^2 + bx + c$ to zero. *The required values of x in any case are to be expressed in terms of a, b, c, ... by means of only a* FINITE *number of additions, subtractions, multiplications, divisions, and extractions of roots.* This is the problem. Is it solvable? The answer to this was not given till about twenty years after Lagrange's death, but the clue is easily traced to his work.

As a first step towards a comprehensive theory Lagrange made an exhaustive study of all the solutions given by his predecessors for the general equations of the first four degrees, and succeeded in showing that all the dodges by which solutions had been obtained could be replaced by a uniform procedure. A detail in this general method contains the clue mentioned. Suppose we are given an algebraic expression involving letters $a, b, c, \ldots$: how many *different* expressions can be derived from the given one if the letters in it are interchanged in all possible ways? For example, from $ab + cd$ we get $ad + cb$ by interchanging b and d. This problem suggests another closely related one, also part of the clue Lagrange was seeking. What interchanges of letters will leave the given expression *invariant* (unaltered)? Thus $ab + cd$ becomes $ba + cd$ under the interchange of a and b, which is the same as $ab + cd$ since $ab = ba$. From these questions the *theory of finite groups* originated. This was found to be the key to the question of algebraic solvability. It will reappear when we consider Cauchy and Galois.

Another significant fact showed up in Lagrange's investigation. For degrees 2, 3, and 4 the general algebraic equation is solved by making the solution depend upon that of an equation of *lower degree* than the one under discussion. This works beautifully and uniformly for equations of degrees 2, 3, and 4, but when a precisely similar process is attempted on the general equation of degree 5,

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0,$$

the *resolvent equation,* instead of being of degree *less than* 5 turns out to be of degree 6. This has the effect of replacing the given equation by a harder one. *The method which works for* 2, 3, 4 *breaks down for* 5, and unless there is some way round the awkward 6 the road is blocked. As a matter of fact we shall see that there is no way of avoiding the difficulty. We might as well try to square the circle or trisect an angle by Euclidean methods.

After the death of Frederick the Great (17 August 1786) resentment against non-Prussians and indifference to science made Berlin an uncomfortable spot for Lagrange and his foreign associates in the Academy, and he sought his release. This was granted on condition that he continue to send memoirs to the proceedings of the Academy for a period of years, to which Lagrange agreed. He gladly accepted the invitation of Louis XVI to continue his mathematical work in Paris as a member of the French Academy. On his arrival in Paris in 1787 he was received with the greatest respect by the royal family and the Academy. Comfortable quarters were assigned him in the Louvre, where he lived till the Revolution, and he became a special favourite of Marie Antoinette – then less than six years from the guillotine. Marie was about nineteen years younger than Lagrange, but she seemed to understand him and did what she could to lighten his overwhelming depression.

At the age of fifty-one Lagrange felt that he was through. It was a clear case of nervous exhaustion from long-continued and excessive overwork. The Parisians found him gentle and agreeable in conversation, but he never took the lead. He spoke but little and appeared distrait and profoundly melancholy. At Lavoisier's gatherings of scientific men Lagrange would stand staring absently out of a window, his back to the guests who had come to do him honour, a picture of sad indifference. He said himself that his enthusiasm was extinct and that he had lost the taste for mathematics. If he were told that some mathematician was engaged on an important research he would say 'So much the better; I began it; I shall not have to finish it.' The *Mécanique analytique* lay unopened on his desk for two years.

Sick of everything smelling of mathematics Lagrange now turned to what he considered his real interests – as Newton had done after the *Principia*: metaphysics, the evolution of human thought, the history of religions, the general theory of languages, medicine, and botany. In this strange miscellany he surprised his friends with his extensive knowledge and the penetrating quality of his mind on matters alien to mathematics. Chemistry at the time was fast becoming a science – in distinction to the alchemy which preceded it, largely through the efforts of Lagrange's close friend Lavoisier (1743–94). In a sense which any student of elementary chemistry will appreciate Lagrange declared that Lavoisier had made chemistry 'as easy as algebra.'

As for mathematics, Lagrange considered that it was finished or at least passing into a period of decadence. Chemistry, physics, and science generally he foresaw as the future fields of greatest interest to first-class minds, and he even predicted that the chairs of mathematics in academies and universities would presently sink to the undistinguished level of those for Arabic. In a sense he was right. Had not Gauss, Abel, Galois, Cauchy, and others injected new ideas into mathematics the surge of the Newtonian impulse would have spent itself by 1850. Happily Lagrange lived long enough to see Gauss well started on his great career and to realize that his own forebodings had been unfounded. We may smile at Lagrange's pessimism to-day, thinking of the era before 1800 at its brightest as only the dawn of the modern mathematics in the first hour of whose morning we now stand, wondering what the noon will be like – if there is to be any; and we may learn from his example to avoid prophecy.

The Revolution broke Lagrange's apathy and galvanized him once more into a living interest in mathematics. As a convenient point of reference we may remember 14 July 1789, the day on which the Bastille fell.

When the French aristocrats and men of science at last realized what they were in for, they urged Lagrange to return to Berlin where a welcome awaited him. No objection would have been raised to his departure. But he refused to leave Paris, saying he would prefer to stay and see the 'experiment' through. Neither he nor his friends foresaw the Terror, and when it came Lagrange bitterly regretted having stayed until it was too late to escape. He had no fear for his own life. In the first place as a half-foreigner he was reasonably safe, and in the second he did not greatly value his life. But the revolting cruelties sickened him and all but destroyed what little faith he had left in human nature and common sense. '*Tu l'as voulu*' ('You wished it', or 'You *would* do it'), he would keep reminding himself as one atrocity after another shocked him into a realization of his error in staying to witness the inevitable horrors of a revolution.

The grandiose schemes of the revolutionists for the regeneration of mankind and the reform of human nature left him cold. When Lavoisier went to the guillotine – as he no doubt would have deserved had it been merely a question of social justice – Lagrange expressed his indignation at the stupidity of the execution: 'It took them only a moment to cause this head to fall, and a hundred years perhaps will not suffice to produce its like.' But the outraged and oppressed citizens had assured the tax-farmer Lavoisier that 'the people have no need of science' when the great chemist's contributions to science were urged as a common-sense reason that his head be left on his shoulders. They may have been right. Without the science of chemistry soap is impossible.

Although practically the whole of Lagrange's working life had been spent under the patronage of royalty his sympathies were not with the royalists. Nor were they with the revolutionists. He stood squarely and unequivocally on the middle ground of civilization which both sides had ruthlessly invaded. He could sympathize with the people who had been outraged beyond human endurance and wish them success in their struggle to gain decent living conditions. But his mind was too realistic to be impressed by any of the chimerical schemes put forth by the leaders of the people for the amelioration of human misery, and he refused to believe that the fabrication of such schemes was indubitable evidence of the greatness of the human mind as claimed by the enthusiastic guillotineers. 'If you wish to see the human mind truly great,' he said, 'enter Newton's study when he is decomposing white light or unveiling the system of the world.'

They treated him with remarkable tolerance. A special decree granted him his 'pension', and when the inflation by paper money reduced the pension to nothing, they appointed him on the committee of inventions to eke out his pay, and again on the committee for the mint. When the École Normale was established in 1795 (for an ephemeral first existence), Lagrange was appointed professor of mathematics. When the Normale closed and the great École Polytechnique was founded in 1797, Lagrange mapped out the course in mathematics and was the first professor. He had never taught before he was called upon to lecture to ill-prepared students. Adapting himself to his raw material, Lagrange led his pupils through arithmetic and algebra to analysis, seeming more like one of his pupils than their teacher. The greatest mathematician of the age became a great teacher of mathematics – preparing Napoleon's fierce young brood of military engineers for their part in the conquest of Europe. The sacred superstition that a man who knows anything is incapable of teaching was shattered. Advancing far beyond the elements Lagrange developed new mathematics before his pupil's eyes and presently they were taking part in the development themselves.

Two works thus developed were to exercise a great influence on the analysis of the first three decades of the nineteenth century. Lagrange's pupils found difficulty with the concepts of the infinitely small and the infinitely great permeating the traditional form of the calculus. To remove these difficulties Lagrange undertook the development of the calculus without the use of Leibniz' 'infinitesimals' and without Newton's peculiar conception of a limit. His own theory was published in two works, the *Theory of Analytic Functions* (1797), and the *Lessons on the Calculus of Functions* (1801). The importance of these works is not in their mathematics but in the impulse they gave Cauchy and others to construct a satisfactory calculus. Lagrange failed completely. But in saying this we must remember that even in our own day the difficulties with which Lagrange grappled unsuccessfully have not been completely overcome. His was a notable attempt and, for its epoch, satisfactory. If our own lasts as long as his did we shall have done well enough.

Lagrange's most important work during the period of the

Revolution was his leading part in perfecting the metric system of weights and measures. It was due to Lagrange's irony and common sense that 12 was not chosen as a base instead of 10. The 'advantages' of 12 are obvious and continue to the present day to be set forth in impressive treatises by earnest propagandists who escape the circle-squaring fraternity only by a hairbreadth. A base of 12 superimposed on the 10 of our number-system would be a hexagonal peg in a pentagonal hole. To bring home the absurdity of 12 even to the cranks, Lagrange proposed 11 as better yet – *any prime number* would have the advantage of giving all fractions in the system the same denominator. The disadvantages are numerous and obvious enough to anyone who understands short division. The committee saw the point and stuck to 10.

Laplace and Lavoisier were members of the committee as first constituted, but after three months they were 'purged' out of their seats with some others. Lagrange remained as president. 'I do not know why they kept me', he remarked, modestly unaware that his gift for silence had saved not only his seat but his head.

In spite of all his interesting work Lagrange was still lonely and inclined to despondency. He was rescued from this twilight between life and death at the age of fifty-six by a young girl nearly forty years his junior, the daughter of his friend the astronomer Lemonnier. She was touched by Lagrange's unhappiness and insisted on marrying him. Lagrange gave in, and contrary to all the laws of whatever it may be that governs the way of a man with a maid, the marriage turned out ideal. The young wife proved not only devoted but competent. She made it her life to draw her husband out and reawaken his desire to live. For his part Lagrange gladly made many concessions and accompanied his wife to balls where he would never have thought of going alone. Before long he could not bear to have her out of his sight for long, and during her brief absences – shopping – he was miserable.

Even in his new happiness Lagrange retained his curiously detached attitude to life and his perfect honesty about his own wishes. 'I had no children by my first marriage', he said; 'I don't know whether I shall have any by my second. I scarcely desire any.' Of all his successes the one he prized most highly, he said simply and sincerely, was having so tender and devoted a companion as his young wife.

Honours were showered on him by the French. The man who had been a favourite of Marie Antoinette now became an idol of the people who had put her to death. In 1796 when France annexed Piedmont, Talleyrand was ordered to wait in state on Lagrange's father, still living in Turin, to tell him that 'Your son, whom Piedmont is proud to have produced and France to possess, has done honour to all mankind by his genius.' When Napoleon turned to civil affairs between his campaigns he often talked with Lagrange on philosophical questions and the function of mathematics in a modern state, and conceived the highest respect for the gently-spoken man who always thought before he spoke and who was never dogmatic.

Beneath his calm reserve Lagrange concealed an ironic wit which flashed out unexpectedly on occasion. Sometimes it was so subtle that coarser men – Laplace for one – missed the point when it was directed at themselves. Once in defence of experiment and observation against mere woolgathering and vague theorizing Lagrange remarked, 'These astronomers are queer; they won't believe in a theory unless it agrees with their observations.' Noticing his rapt forgetfulness at a musicale, someone asked him why he liked music. 'I like it because it isolates me,' he replied. 'I hear the first three measures; at the fourth I distinguish nothing; I give myself up to my thoughts; nothing interrupts me; and it is thus that I solved more than one difficult problem.' Even his sincere reverence for Newton has a faint flavour of the same gentle irony. 'Newton', he declared, 'was assuredly the man of genius *par excellence*, but we must agree that he was also the luckiest: one finds only once the system of the world to be established.' And again: 'How lucky Newton was that in his time the system of the world still remained to be discovered!'

Lagrange's last scientific effort was the revision and extension of the *Mécanique analytique* for a second edition. All his old power returned to him although he was past seventy. Resuming his former habits he worked incessantly, only to discover that his body would no longer obey his mind. Presently he began to have fainting spells, especially on getting out of bed in the morning. One day his wife found him unconscious on the floor, his head badly cut by a fall against the edge of a table. Thereafter he moderated his pace but kept on working. His illness, which he knew to be grave, did not disturb his serenity; all his life Lagrange lived as a philosopher would like to live, indifferent to his fate.

Two days before Lagrange died Monge and other friends called, knowing that he was dying and that he wished to tell them something of his life. They found him temporarily better, except for lapses of memory which obliterated what he had wished to tell them.

'I was very ill yesterday, my friends,' he said. 'I felt I was going to die; my body grew weaker little by little; my intellectual and physical faculties were extinguished insensibly; I observed the well-graduated progression of the diminution of my strength, and I came to the end without sorrow, without regrets, and by a very gentle decline. Oh, death is not to be dreaded, and when it comes without pain, it is a last function which is not unpleasant.'

He believed that the seat of life is in all the organs, in the whole of the bodily machine, which, in his case, weakened equally in all its parts.

'In a few moments there will be no more functions anywhere, death will be everywhere; death is only the absolute repose of the body.

'I wish to die; yes, I wish to die, and I find a pleasure in it. But my wife did not wish it. In these moments I should have preferred a wife less good, less eager to revive my strength, who would have let me end gently. I have had my career; I have gained some celebrity in Mathematics. I never hated anyone, I have done nothing bad, and it would be well to end; but my wife did not wish it.'

He soon had his wish. A fainting spell from which he never awoke came on shortly after his friends had left. He died early on the morning of 10 April, 1813, in his seventy-sixth year.

CHAPTER ELEVEN

FROM PEASANT TO SNOB

Laplace

THE Marquis Pierre-Simon de Laplace (1749–1827) was not born a peasant nor did he die a snob. Yet to within small quantities of the second order his illustrious career is comprised within the limits indicated, and it is from this approximate point of view that he is of greatest interest as a specimen of humanity.

As a mathematical astronomer Laplace has justly been called the Newton of France; as a mathematician he may be regarded as the founder of the modern phase of the theory of probability. On the human side he is perhaps the most conspicuous refutation of the pedagogical superstition that noble pursuits necessarily ennoble a man's character. Yet in spite of all his amusing foibles – his greed for titles, his political suppleness, and his desire to shine in the constantly changing spotlight of public esteem – Laplace had elements of true greatness in his character. We may not believe all that he said about his unselfish devotion to truth for truth's sake, and we may smile at the care with which he rehearsed his sententious last words – 'What we know is not much; what we do not know is immense' – in an endeavour to telescope Newton's boy playing on the seashore into a neat epigram, but we cannot deny that Laplace in his generosity to unknown beginners was anything but a shifty and ungrateful politician. To give one young man a helping hand up Laplace once cheated himself.

Very little is known of Laplace's early years. His parents were peasants living in Beaumont-en-Auge, Department of Calvados, France, where Pierre-Simon was born on 23 March 1749. The obscurity surrounding Laplace's childbirth and youth is due to his own snobbishness: he was thoroughly ashamed of his humble parents and did everything in his power to conceal his peasant origin.

Laplace got his chance through the friendly interest of wealthy neighbours on the occasion, presumably, of his having shown remarkable talent in the village school. It is said that his first success was in theological disputations. If this is true it is an interesting prelude to the somewhat aggressive atheism of his maturity. He took to mathematics early. There was a military academy at Beaumont, which Laplace attended as an externe, and in which he is said to have taught mathematics for a time. One dubious legend states that the young man's prodigious memory attracted more attention than his mathematical ability and was responsible for the cordial recommendations from influential people which he carried with him to Paris when, at the age of eighteen, he wiped the mud of Beaumont off his boots for ever and set out to seek his fortune. His own estimate of his powers was high, but not too high. With justified self-confidence young Laplace invaded Paris to conquer the mathematical world.

Arriving in Paris, Laplace called on D'Alembert and sent in his recommendations. He was not received. D'Alembert was not interested in young men who came recommended only by prominent people. With remarkable insight for so young a man Laplace sensed what the trouble was. He returned to his lodgings and wrote D'Alembert a wonderful letter on the general principles of mechanics. This did the trick. In his reply inviting Laplace to call, D'Alembert wrote: 'Sir, you see that I paid little enough attention to your recommendations: you don't need any. You have introduced yourself better. That is enough for me; my support is your due.' A few days later, thanks to D'Alembert, Laplace was appointed professor of mathematics at the Military School of Paris.

Laplace now threw himself into his life work – the detailed application of the Newtonian law of gravitation to the entire solar system. If he had done nothing else he would have been greater than he was. The kind of man Laplace would have liked to be is described in a letter of 1777, when he was twenty-seven, to D'Alembert. The picture Laplace gives of himself is one of the strangest mixtures of fact and fancy a man ever perpetrated in the way of self-analysis.

'I have always cultivated mathematics by taste rather than from the desire for a vain reputation,' he declares. 'My greatest amusement is to study the march of the inventors, to see their genius at grips with the obstacles they have encountered and overcome. I then put myself in their place and ask myself how I should have gone about surmounting these same obstacles, and although this substitution in the great majority of instances has only been humiliating to my self-love, nevertheless the pleasure of rejoicing in their success has amply repaid me for this little humiliation. If I am fortunate enough to add something to their works, I attribute all the merit to their first efforts, well persuaded that in my position they would have gone much farther than I. . . .'

He may be granted the first sentence. But what about the rest of his smug little essay which might have been handed in by a priggish youngster of ten to his gullible Sunday-school teacher? Notice particularly the generous attribution of his own 'modest' successes to the preliminary work of his predecessors. Nothing could be farther from the truth than this frank avowal of indebtedness. To call a spade a spade, Laplace stole outrageously, right and left, wherever he could lay his hands on anything of his contemporaries and predecessors which he could use. From Lagrange, for example, he lifted the fundamental concept of the potential (to be described presently); from Legendre he took whatever he needed in the way of analysis; and finally, in his masterpiece, the *Mécanique céleste,* he deliberately omits references to the work of others incorporated in his own, with the intention of leaving posterity to infer that he alone created the mathematical theory of the heavens. Newton, of course, he cannot avoid mentioning repeatedly. Laplace need not have been so ungenerous. His own colossal contributions to the dynamics of the solar system easily overshadow the works of others whom he ignores.

The complications and difficulties of the problem Laplace attacked cannot be conveyed to anyone who has never seen anything similar attempted. In discussing Lagrange we mentioned the problem of three bodies. What Laplace undertook was similar, but on a grander scale. He had to work out from the Newtonian law the combined effects of the perturbations – cross-pulling and hauling – of all the members of the Sun's family of planets on one another and on the Sun. Would Saturn, in spite of an apparently steady decrease of his mean motion, wander off into space, or would he continue as a member of the Sun's family? Or would the acceleration of Jupiter and the Moon ultimately cause one to fall into the Sun and the other to smash down on the Earth? Were the effects of these perturbations cumulative and dissipative, or were they periodic and conservative? These and similar riddles were details of the grand problem: is the solar system stable or is it unstable? It is assumed that the Newtonian law of gravitation is indeed universal and the only one controlling the motions of the planets.

Laplace's first important step towards the general problem was taken in 1773, when he was twenty-four, in which he proved that the mean distances of the planets from the Sun are invariable to within certain slight periodic variations.

When Laplace attacked the problem of stability expert opinion was at best neutral. Newton himself believed that divine intervention might be necessary from time to time to put the solar system back in order and prevent it from destruction or dissolution. Others, like Euler, impressed by the difficulties of the lunar theory (motion of the Moon), rather doubted whether the motions of the planets and their satellites could be accounted for on the Newtonian hypothesis. The forces involved were too numerous, and their mutual interactions too complicated for any reasonably fair guess. Until Laplace *proved* the stability of the solar system one man's guess was as good as another's.

To dispose here of an objection which the reader doubtless has already raised, it may be stated that Laplace's solution of the problem of stability is good only for the highly idealized solar system which Newton and he imagined. Tidal friction (acting like a brake on diurnal rotation) among other things was ignored. Since the *Mécanique céleste* was published we have learned a great deal about the solar system and everything in it of which Laplace was ignorant. It is probably not too radical to say that the problem of stability for the actual solar system – as opposed to Laplace's ideal – is still open. However, the experts on celestial mechanics might disagree, and a competent opinion can be obtained only from them.

As a matter of temperament some find the Laplacian conception of an eternally stable solar system repeating the complicated cycle of its motions time after time for ever and ever as depressing as an endless nightmare. For these there is the recent comfort that the Sun will probably explode some day as a nova. Then stability will cease to trouble us, for we shall all quite suddenly become perfect gases.

For this brilliant start Laplace was rewarded with the first substantial honour of his career when he was barely twenty-four, associate membership in the Academy of Sciences. His subsequent scientific life is summarized by Fourier: 'Laplace gave to all his works a fixed direction from which he never deviated; the imperturbable constancy of his views was always the principal feature of his genius. He was already [when he began his attack on the solar system] at the extreme of mathematical analysis, knowing all that is most ingenious in this, and no one was more competent than he to extend its domain. He had solved a capital problem of astronomy [that communicated to the Academy in 1773], and he decided to devote all his talents to mathematical astronomy, which he was destined to perfect. He meditated profoundly on his great project and passed his whole life perfecting it with a perseverance unique in the history of science. The vastness of the subject flattered the just pride of his genius. He undertook to compose the *Almagest* of his age – the *Mécanique céleste*; and his immortal work carries him as far beyond that of Ptolemy as the analytical science [mathematical analysis] of the moderns surpasses the *Elements* of Euclid.'

This is no more than just. Whatever Laplace did in mathematics was designed as an aid to the solution of the grand problem. Laplace is the great example of the wisdom – for a man of genius – of directing all one's efforts to a single central objective worthy of the best that a man has in him. Occasionally Laplace was tempted to turn aside, but not for long. Once he was strongly attracted by the theory of numbers, but quickly abandoned it on realizing that its puzzles were likely to cost him more time than he could spare from the solar system. Even his epochal work in the theory of probabilities, although at first sight off the main road of his interests, was inspired by his need for it in mathematical astronomy. Once well into the theory he saw that it is indispensable in all exact science and felt justified in developing it to the limit of his powers.

The *Mécanique céleste*, which bound all Laplace's astronomical work into a reasoned whole, was published in parts over a period of twenty-six years. Two volumes appeared in 1799, dealing with the motions of the planets, their shapes (as rotating bodies), and the tides; two further volumes in 1802 and 1805 continued the investigation, which was finally completed in the fifth volume, 1823–25. The mathematical exposition is extremely concise and occasionally awkward. Laplace was interested in results, not in how he got them. To avoid condensing a complicated mathematical argument to a brief, intelligible form he frequently omits everything but the conclusion, with the optimistic remark '*Il est aisé à voir*' (It is easy to see). He himself would often be unable to restore the reasoning by which he had 'seen' these easy things without hours – sometimes days–of hard labour. Even gifted readers soon acquired the habit of groaning whenever the famous phrase appeared, knowing that as likely as not they were in for a week's blind work.

A more readable account of the main results of the *Mécanique céleste* appeared in 1796, the classic *Exposition du système du monde* (Exposition of the System of the World), which has been described as Laplace's masterpiece with all the mathematics left out. In this work, as in the long non-mathematical introduction (153 quarto pages) to the treatise on probabilities (third edition, 1820), Laplace revealed himself as almost as great a writer as he was a mathematician. Anyone wishing to glimpse the scope and fascination of the theory of probability, without being held up by technicalities intelligible only to mathematicians, could not do better than to read Laplace's introduction. Much has been done since Laplace wrote, especially in recent years and particularly in the foundations of the theory of probability, but his exposition is still classic and a perfect expression of at least one philosophy of the whole subject. The theory, it need scarcely be said, is not yet complete. Indeed it is beginning to seem as if it has not yet been begun – the next generation may have it all to do over again.

One interesting detail of Laplace's astronomical work may be mentioned in passing, the famous nebular hypothesis of the origin of the solar system. Apparently unaware that Kant had anticipated him, Laplace (only half seriously) proposed the hypothesis in a note. His mathematics was inadequate for a systematic attack, and it was not till Jeans in the present century resumed the discussion that it had any scientific meaning.

Lagrange and Laplace, the two leading French men of science of the eighteenth century, offer an interesting contrast, and one typical of a difference which was to become increasingly sharp with the expansion of mathematics: Laplace belongs to the tribe of mathematical physicists, Lagrange to that of pure mathematicians. Poisson, himself a mathematical physicist, seems to favour Laplace as the more desirable type:

'There is a profound difference between Lagrange and Laplace in all their work, whether in a study of numbers or the libration of the Moon. Lagrange often appeared to see in the questions he treated only mathematics, of which the questions were the occasion – hence the high value he put upon elegance and generality. Laplace saw in mathematics principally a tool, which he modified ingeniously to fit every special problem as it arose. One was a great mathematician; the other a great philosopher who sought to know nature by making higher mathematics serve it.'

Fourier (whom we shall consider later) was also struck by the radical difference between Lagrange and Laplace. Himself rather narrowly 'practical' in his mathematical outlook, Fourier was yet capable – at one time – of estimating Lagrange at his true worth:

'Lagrange was no less a philosopher than he was a great mathematician. By his whole life he proved, in the moderation of his desires, his immovable attachment to the general interests of humanity, by the noble simplicity of his manners and the elevation of his character, and finally by the accuracy and the depth of his scientific works.'

Coming from Fourier this statement is remarkable. It may smack of the bland rhetoric we are accustomed to expect in French funeral orations, yet it is true, at least to-day. Lagrange's great influence on modern mathematics is due to 'the depth and accuracy of his scientific works', qualities which are sometimes absent from Laplace's masterpieces.

To the majority of his contemporaries and immediate followers Laplace ranked higher than Lagrange. This was due partly to the magnitude of the problem Laplace attacked – the grandiose project of demonstrating that the solar system is a gigantic perpetual motion machine. A sublime project in itself, no doubt, but essentially illusory: not enough about the actual physical universe was known in Laplace's day – or even in our own – to give the problem any real significance, and it will probably be many years before mathematics is sufficiently advanced to handle the complicated mass of data we now have. Mathematical astronomers will doubtless continue to play with idealized models of 'the universe', or even of the infinitely less impressive solar system, and will continue to flood us with inspiring or depressing bulletins regarding the destiny of mankind; but in the end the by-products of their investigations – the perfection of the purely mathematical tools they have devised – will be their fairly permanent contribution to the advancement of science (as opposed to the propagation of guessing), precisely as has happened in the case of Laplace.

If the foregoing seems too strong, consider what has happened to the *Mécanique céleste*. Does anyone but an academic mathematician really believe to-day that Laplace's conclusions about the stability of the solar system are a reliable verdict on the infinitely complicated situation which Laplace replaced by an idealized dream? Possibly many do; but no worker in mathematical physics doubts the power and utility of the mathematical methods developed by Laplace to attack his ideal.

To take but one instance, the theory of the potential is more significant to-day than Laplace ever dreamed it would become.

Without the mathematics of this theory we should be halted almost at the beginning of our attempt to understand electromagnetism. Out of this theory grew one vigorous branch of the mathematics of boundary-value problems, to-day of greater significance for physical science than the whole Newtonian theory of gravitation. The concept of the potential was a mathematical inspiration of the first order – it made possible an attack on physical problems which otherwise would have been unapproachable.

The potential is merely the function u described in connexion with fluid motion and Laplace's equation in the chapter on Newton. The function u is there a 'velocity potential'; if it is a question of the force of Newtonian gravitational attraction, u is a 'gravitational potential'. The introduction of the potential into the theories of fluid motion, gravitation, electromagnetism, and elsewhere was one of the longest strides ever taken in mathematical physics. It had the effect of replacing partial differential equations in two or three unknowns by equations in one unknown.

In 1785, at the age of thirty-six, Laplace was promoted to full membership in the Academy. Important as this honour was in the career of a man of science, the year 1785 stands out as a landmark of yet greater significance in Laplace's career as a public character. For in that year Laplace had the unique distinction of examining a singular candidate of sixteen at the Military School. This youth was destined to upset Laplace's plans and deflect him from his avowed devotion to mathematics into the muddy waters of politics. The young man's name was Napoleon Bonaparte (1769–1821).

Laplace rode through the Revolution on horseback, as it were, and saw everything in comparative safety. But no man of his prominence and restless ambition could escape danger entirely. If De Pastoret knew what he was talking about in his eulogy, both Lagrange and Laplace escaped the guillotine only because they were requisitioned to calculate trajectories for the artillery and to help in directing the manufacture of saltpetre for gunpowder. Neither was forced to eat grass as some less necessary savants were driven to do, nor was either so careless as to betray himself, as their unfortunate friend Condorcet did, by ordering an aristocrat's omelet. Not knowing how many eggs go into a normal omelet Condorcet ordered a dozen. The good cook asked Condorcet his trade. 'Carpenter.' – 'Let me see your hands. You're no carpenter.' That was the end of Laplace's close friend Condorcet. They either poisoned him in prison or let him commit suicide.

After the Revolution Laplace went in heavily for politics, possibly in the hope of beating Newton's record. The French refer politely to Laplace's 'versality' as a politician. This is too modest. Laplace's alleged defects as a politician are his true greatness in the slippery game. He has been criticized for his inability to hold public office under successive regimes without changing his politics. It would seem that a man who is sharp enough to convince opposing parties that he is a loyal supporter of whichever one happens to be in power at the moment is a politician of no mean order. It was his patrons who played the game like amateurs, not Laplace. What would we think of a Republican Postmaster-General who gave all the fattest jobs to undeserving Democrats? Or the other way about? Laplace got a better job every time the government flopped. It cost him nothing to switch overnight from rabid republicanism to ardent royalism.

Napoleon shoved everything Laplace's way, including the portfolio of the Interior – about which more later. All the Napoleonic orders of any note adorned the versatile mathematician's chest – including the Grand Cross of the Legion of Honour and the Order of the Reunion, and he was made a Count of the Empire. Yet what did he do when Napoleon fell? Signed the decree which banished his benefactor.

After the restoration Laplace had no difficulty in transferring his loyalty to Louis XVIII, especially as he now sat in the Chamber of Peers as the Marquis de Laplace. Louis recognized his supporter's merits and in 1816 appointed Laplace president of the committee to reorganize the École Polytechnique.

Perhaps the most perfect expressions of Laplace's political genius are those found in his scientific writings. It takes real genius to doctor science according to fluctuating political opinion and get away with it. The first edition of the *Exposition* *du système du monde*, dedicated to the Council of Five Hundred, closes with these noble words: 'The greatest benefit of the astronomical sciences is to have dissipated errors born of ignorance of our true relations with nature, errors all the more fatal since the social order must rest solely on these relations. *Truth* and *justice* are its immutable bases. Far from us be the dangerous maxim that it may sometimes be useful to deceive or to enslave men the better to ensure their happiness! Fatal experiences have proved in all ages that these sacred laws are never infringed with impunity.' In 1824 this is suppressed and the Marquis de Laplace substitutes: 'Let us conserve with care and increase the store of this advanced knowledge, the delight of thinking beings. It has rendered important services to navigation and geography; but its greatest benefit is to have dissipated the fears produced by celestial phenomena and to have destroyed the errors born of ignorance of our true relations with nature, errors which will soon reappear if the torch of the sciences is extinguished.' In loftiness of sentiment there is but little to choose between these two sublime maxima.

This is enough on the debit side of the ledger. The last extract does indeed suggest one trait in which Laplace overtopped all courtiers – his moral courage where his true convictions were questioned. The story of Laplace's encounter with Napoleon over the *Mécanique céleste* shows the mathematician as he really was. Laplace had presented Napoleon with a copy of the work. Thinking to get a rise out of Laplace, Napoleon took him to task for an apparent oversight. 'You have written this huge book on the system of the world without once mentioning the author of the universe.' 'Sire', Laplace retorted, 'I had no need of that *hypothesis*.' When Napoleon repeated this to Lagrange, the latter remarked, 'Ah, but that is a fine hypothesis. *It explains so many things*.'

It took nerve to stand up to Napoleon and tell him the truth. Once at a session of the Institut when Napoleon was in one of his most insultingly bad tempers he caused poor old Lamarck to burst into tears with his deliberate brutality.

Also on the credit side was Laplace's sincere generosity to beginners. Biot tells how as a young man he read a paper before the Academy when Laplace was present, and was drawn aside afterward by Laplace who showed him the identical discovery in a yellowed old manuscript of his own, still unpublished. Cautioning Biot to secrecy, Laplace told him to go ahead and publish his work. This was but one of several such acts. Beginners in mathematical research were his stepchildren, Laplace liked to say, but he treated them as well as he did his own son.

As it is often quoted as an instance of the unpracticality of mathematicians we shall give Napoleon's famous estimate of Laplace, of which he is reported to have delivered himself while he was a prisoner at St Helena.

'A mathematician of the first rank, Laplace quickly revealed himself as only a mediocre administrator; from his first work we saw that we had been deceived. Laplace saw no question from its true point of view; he sought subtleties everywhere, had only doubtful ideas, and finally carried the spirit of the infinitely small into administration.'

This sarcastic testimonial was inspired by Laplace's short tenure – only six weeks – of the Ministry of the Interior. However, as Lucien Bonaparte needed a job at the moment and succeeded Laplace, Napoleon may have been rationalizing his well-known inclination to nepotism. Laplace's testimonial for Napoleon has not been preserved. It might have run somewhat as follows.

'A soldier of the first rank, Napoleon quickly revealed himself as only a mediocre politician; from his first exploits we saw that he was deceived. Napoleon saw all questions from the obvious point of view; he suspected treachery everywhere but where it was, had only a childlike faith in his supporters, and finally carried the spirit of infinite generosity into a den of thieves.'

Which, after all, was the more practical administrator? The man who could not hang on to his gains and who died a prisoner of his enemies, or the other who continued to gather wealth and honour to the day of his death?

Laplace spent his last days in comfortable retirement at his country estate at Arcueil, not far from Paris. After a short illness he died on 5 March 1827, in his seventy-eighth year. His last words have already been reported.

CHAPTER TWELVE

FRIENDS OF AN EMPEROR

Monge and Fourier

THE careers of Gaspard Monge (1746–1818) and Joseph Fourier (1768–1830) are curiously parallel and may be considered together. On the mathematical side each made one fundamental contribution: Monge invented descriptive geometry (not to be confused with the projective geometry of Desargues, Pascal, and others); Fourier started the current phase of mathematical physics with his classic investigations on the theory of heat-conduction.

Without Monge's geometry – originally invented for use in military engineering – the wholesale spawning of machinery in the nineteenth century would probably have been impossible. Descriptive geometry is the root of all the mechanical drawing and graphical methods that help to make mechanical engineering a fact.

The methods inaugurated by Fourier in his work on the conduction of heat are of a similar importance in boundary-value problems – a trunk nerve of mathematical physics.

Monge and Fourier between them are thus responsible for a considerable part of our own civilization, Monge on the practical and industrial side, Fourier on the purely scientific. But even on the practical side Fourier's methods are indispensable today; they are in fact a commonplace in all electrical and acoustical engineering (including wireless) beyond the rule of thumb and handbook stages.

A third man must be named with these mathematicians, although we shall not take space to tell his life: the chemist Count Claude-Louis Berthollet (1748–1822), a close friend of Monge, Laplace, Lavoisier, and Napoleon. With Lavoisier, Berthollet is regarded as one of the founders of modern chemistry. He and Monge became so thick that their admirers gave up trying to distinguish between them in their non-scientific labours and called them simply Monge-Berthollet.

Gaspard Monge, born on 10 May 1746, at Beaune, France, was a son of Jacques Monge, a peddler and knife grinder who had a tremendous respect for education and who sent his three sons through the local college. All the sons had successful careers; Gaspard was the genius of the family. At the college (run by a religious order) Gaspard regularly captured the first prize in everything and earned the unique distinction of having *puer aureus* inscribed after his name.

At the age of fourteen Monge's peculiar combination of talents showed up in the construction of a fire engine. 'How could you, without a guide or a model, carry through such an undertaking successfully?' he was asked by the astonished citizens. Monge's reply is a summary of the mathematical part of his career and of much of the rest. 'I had two infallible means of success: an invincible tenacity, and fingers which translated my thought with geometric fidelity.' He was in fact a born geometer and engineer with an unsurpassed gift for visualizing complicated space-relations.

At the age of sixteen he made a wonderful map of Beaune entirely on his own initiative, constructing his own surveying instruments for the purpose. This map got him his first great chance.

Impressed by his obvious genius, Monge's teachers recommended him for the professorship of physics at the college in Lyons run by their order. Monge was appointed at the age of sixteen. His affability, patience, and lack of all affectation, added to his sound knowledge, made him a great teacher. The order begged him to take their vows and cast his lot for life with them. Monge consulted his father. The astute knife grinder advised caution.

Some days later, on a visit home, Monge met an officer of engineers who had seen the famous map. The officer begged Jacques to send his son to the military school at Mézières. Perhaps fortunately for Monge's future career the officer omitted to state that on account of his humble birth Monge could never get a commission. Not knowing this, Monge eagerly accepted and proceeded to Mézières.

Monge quickly learned where he stood at Mézières. There were only twenty pupils at the school, of whom ten were graduated each year as lieutenants in engineering. The rest were destined for the 'practical' work – the dirty jobs. Monge did not complain. He rather enjoyed himself, as the routine work in surveying and drawing left him plenty of time for mathematics. An important part of the regular course was the theory of fortification, in which the problem was to design the works so that no part should be exposed to the direct fire of the enemy. The usual calculations demanded endless arithmetic. One day Monge handed in his solution of a problem of this sort. It was turned over a to a superior officer for inspection.

Sceptical that anyone could have solved the problem in the time, the officer declined to check the solution. 'Why should I give myself the trouble of subjecting a supposed solution to tedious verifications? The author has not even taken the time to group his figures. I can believe in a great facility in calculation, but not in miracles!' Monge persisted, saying he had not used arithmetic. His tenacity won; the solution was checked and found correct.

This was the beginning of descriptive geometry. Monge was at once given a minor teaching position to instruct the future military engineers in the new method. Problems which had been nightmares before – sometimes solved only by tearing down what had been built and beginning all over again – were now as simple as ABC. Monge was sworn not to divulge his method, and for fifteen years it was a jealously guarded military secret. Only in 1794 was he allowed to teach it publicly, at the École Normale in Paris, where Lagrange was among the auditors. Lagrange's reaction to descriptive geometry was like M. Jourdain's when he discovered that he had been talking prose all his life. 'Before hearing Monge,' Lagrange said after a lecture, 'I did not know that I knew descriptive geometry.'

The idea behind it all now seems as ridiculously simple to us as it did to Lagrange. Descriptive geometry is a method for representing solids and other figures in ordinary three-dimensional space on *one* plane. Imagine first two planes at right angles to one another, like two pages of a thin book opened at a 90 degree angle; one plane is horizontal, the other vertical. The figure to be represented is projected on to each of these planes by rays perpendicular to the plane. There are thus *two* projections of the figure; that on the horizontal plane is called a *plan* of the figure, that on the vertical plane an *elevation*. The vertical plane is now turned down ('rabbatted') till it and the horizontal plane lie in *one* plane (that of the horizontal plane) – as if the book were now opened out flat on a table.

The solid or other figure in space is now represented by two projections on one plane (that of the drawing board). A plane, for instance, is represented by its *traces* – the straight lines in which it cut the vertical and horizontal planes before the former was rabbatted; a solid, say a cube, is represented by the projections of its edges and vertices. Curved surfaces cut the vertical and horizontal planes in curves; these curves, or *traces* of the surface, represent the surface on the one plane.

When these and other equally simple remarks are developed we have a *descriptive* method which puts on one flat sheet of paper what we ordinarily visualize in space of three dimensions. A short training enables the draughtsman to read such representations as easily as others read good photographs – and to get a great deal more out of them. This was the simple invention that revolutionized military engineering and mechanical design. Like many of the first-rate things in applied mathematics its most conspicuous feature is its simplicity. There are many ways in which descriptive geometry can be developed or modified, but they all go back to Monge. The subject is now so thoroughly worked out that it is not of much interest to professional mathematicians.

To finish with Monge's contributions to mathematics before continuing with his life, we recall that his name is familiar to every student in the second course in the calculus to-day in connexion with the geometry of surfaces. Monge's great step forward was a systematic (and brilliant) application of the calculus to the investigation of the curvature of surfaces. In his general theory of curvature Monge prepared the way for Gauss, who in his turn was to inspire Riemann, who again was to develop the geometry known by his name in the theory of relativity.

It seems rather a pity that a born geometer like Monge should have lusted after the fleshpots of Egypt, but so he did. His work in differential equations, closely connected with that in geometry, also showed what he had in him. Years after he left Mézières, where these great things were dóne, Monge lectured on his discoveries to his colleagues at the École Polytechnique. Lagrange again was an auditor. 'My dear colleague', he told Monge after the lecture, 'you have just explained some very elegant things; I should have liked to have done them myself.' And on another occasion: 'With his application of analysis to geometry this devil of a man will make himself immortal!' He did; and it is interesting to note that although more urgent calls on his genius distracted him from mathematics, he never lost his talent. Like all the great mathematicians Monge was a mathematician to the last.

In 1768, at the age of twenty-two, Monge was promoted to the professorship of mathematics at Mézières, and three years later, on the death of the professor of physics, stepped into his place also. The double work did not bother him at all. Powerfully built and as strong of body as he was of mind, Monge was always capable of doing three or four men's work and frequently did.

His marriage had a touch of eighteenth-century romance. At a reception Monge heard some noble bounder slandering a young widow to get even with her for having rejected him. Shouldering his way through the cackling crowd, Monge demanded to know whether he had heard aright. 'What is it to you?' Monge demonstrated with a punch on the jaw. There was no duel. A few months later at another reception Monge was very much taken by a charming young woman. On being introduced he recognized her name – Madame Horbon – as that of the unknown lady he had tried to fight a duel for. She was the widow, only twenty, and somewhat reluctant to marry before her late husband's affairs were straightened out. 'Never mind all that,' Monge reassured her, 'I've solved lots of more difficult problems in my time.' Monge and she were married in 1777. She survived him and did what she could to perpetuate his memory – unaware that her husband had raised his own monument long before he ever met her. Monge's wife was the one human being who stuck to him through everything. Even Napoleon at the very last would have let him down on account of his age.

At about this time Monge began corresponding with D'Alembert and Condorcet. In 1780 these two had induced the Government to found an institute at the Louvre for the study of hydraulics. Monge was called to Paris to take charge, on the understanding that he spend half his time at Mézières. He was then thirty-four. Three years later he was relieved of his duties at Mézières and appointed examiner of candidates for commissions in the navy, a position which he held till the outbreak of the Revolution in 1789.

In looking back over the careers of all these mathematicians of the Revolutionary period we cannot help noticing how blind they and everyone else were to what now seems so obvious to us. Not one of them suspected that he was sitting on a mine and that the train was already sputtering. Possibly our successors in 2036 will be saying the same about us.

For the six years he held the naval job Monge proved himself an incorruptible public servant. Disgruntled aristocrats threatened him with dire penalties when he unmercifully disqualified their incompetent sons, but Monge never gave in. 'Get someone else to run the job if you don't like the way I am doing it.' As a consequence the navy was ready for business in 1789.

His birth and his experiences with snobs seeking unmerited favours made Monge a natural revolutionist. By first-hand experience he knew the corruption of the old order and the economic disabilities of the masses, and he believed that the time had come for a new deal. But like the majority of early liberals Monge did not know that a mob which has once tasted blood is not satisfied till no more is forthcoming. The early revolutionists had more faith in Monge than he had in himself. Against his better judgement they forced him into the Ministry of the Navy and the Colonies on 10 August 1792. He was the man for the position, but it was not healthy to be a public official in the Paris of 1792.

The mob was already out of hand; Monge was put on the Provisional Executive Council to attempt some measure of control. A son of the people himself, Monge felt that he understood them better than did some of his friends – Condorcet, for instance, who had wisely declined the naval job to save his head.

But there are people and people, all of whom together comprise 'the people'. By February 1793 Monge found himself suspect of being not quite radical enough, and on the 13th he resigned, only to be re-elected on the 18th to a job which stupid political interference, 'liberty, equality, and fraternity' among the sailors, and approaching bankruptcy of the state had made impossible. Any day during this difficult time Monge might have found himself on the scaffold. But he never truckled to ignorance and incompetence, telling his critics to their faces that he knew what was what while they knew nothing. His only anxiety was that dissension at home would lay France open to an attack which would nullify all the gains of the Revolution.

At last, on 10 April 1793, Monge was allowed to resign in order to undertake more urgent work. The anticipated attack was now plainly visible.

With the arsenals almost empty the Convention began raising an army of 900,000 men for defence. Only a tenth of the necessary munitions existed and there was no hope of importing the requisite materials – copper and tin for the manufacture of bronze cannon, saltpetre for gunpowder, and steel for firearms. 'Give us saltpetre from the earth and in three days we shall be loading our cannon,' Monge told the Convention. All very well, they retorted, but where were they to get the saltpetre? Monge and Berthollet showed them.

The entire nation was mobilized. Under Monge's direction bulletins were sent to every town, farmstead, and village in France telling the people what to do. Led by Berthollet the chemists invented new and better methods for refining the raw material and simplified the manufacture of gunpowder. The whole of France became a vast powder factory. The chemists also showed the people where to find tin and copper – in clock metal and church bells. Monge was the soul of it all. With his prodigious capacity for work he spent his days supervising the foundries and arsenals, and his nights writing bulletins for the direction of the workers, and throve on it. His bulletin on *The Art of Manufacturing Cannon* became the factory handbook.

Monge was not without enemies as the Revolution continued to fester. One day Monge's wife heard that Berthollet and her husband were to be denounced. Frantic with fear she ran to the Tuileries to learn the truth. She found Berthollet sitting quietly under the chestnut trees. Yes; he had heard the rumour, but believed nothing would happen for a week. 'Then', he added with his habitual composure, 'we shall certainly be arrested, tried, condemned, and executed.'

When Monge came home that evening his wife told him Berthollet's prediction. 'My word!' Monge exclaimed; 'I know nothing of all that. What I do know is that my cannon factories are going forward marvellously!'

Shortly after this Citizen Monge was denounced by the porter at his lodgings. This was too much, even for Monge. He prudently left Paris till the storm blew over.

The third stage of Monge's career opened in 1796 with a letter from Napoleon. The two had already met in 1792, but Monge was unaware of the fact. Monge at the time was fifty, Napoleon twenty-three years younger.

'Permit me', Napoleon wrote, 'to thank you for the cordial welcome that a young artillery officer, little in favour, received from the Minister of the Navy in 1792; he has preciously preserved its memory. You see this officer in the present general of the Army [of invasion] of Italy; he is happy to extend you a hand of recognition and friendship.'

Thus began the long intimacy between Monge and Napoleon. Commenting on this singular alliance, Arago* reports Napoleon's words 'Monge loved me as one loves a mistress.' On the other side Monge seems to have been the only man for whom Napoleon ever had an unselfish and abiding friendship.

* F. J. D. Arago, 1786–1853, astronomer, physicist, and scientific biographer.

Napoleon knew of course that Monge had helped to make his career possible; but that was not the root of his affection for the older man.

The 'recognition' mentioned in Napoleon's letter was the appointment of Monge and Berthollet by the Directory as commissioners sent to Italy to select the paintings, sculpture, and other works of art 'donated' by the Italians (after being bled white of money) as part of their contribution to the expenses of Napoleon's campaign. In picking over the loot Monge developed a keen appreciation of art and became quite a connoisseur.

The practical implications of the looting, however, disturbed him somewhat, and when enough to furnish the Louvre half a dozen times over had been lifted and shipped to Paris, Monge counselled moderation. It would not do, he said, in governing a people either for their own good or for that of the conquerors to beggar them completely. His advice was heeded, and the goose continued laying its golden eggs.

After the Italian adventure Monge joined Napoleon at his chateau near Udine. The two became great cronies, Napoleon revelling in Monge's conversation and inexhaustible fund of interesting information, and Monge basking in the commander-in-chief's genial humour. At public banquets Napoleon always ordered the band to strike up the *Marseillaise* – 'Monge is an enthusiast for it!' Indeed he was, shouting it at the top of his lungs before sitting down to meals.

Allons, enfants de la patrie,
Le jour de gloire est arrivé!

It will be our special privilege to see the day of glory arriving in the company of another great Napoleonic mathematician – Poncelet.

In December 1797 Monge made a second trip to Italy, this time as a member of the commission to investigate the 'great crime' of General Duphot's assassination. The General had been shot down in Rome while standing near Lucien Bonaparte. The commission (rudely anticipated by one of the martyred General's brothers-in-arms) somewhat lamely prescribed a republic modelled on the French for the obstreperous Italians. 'There must be an end of everything, even of the rights of conquest,' as one of the negotiators remarked when the matter of further extortions came up.

How right this canny diplomat was came out eight months later when the Italians scrapped their republic to the great embarrassment of Napoleon, then in Cairo, and to the greater embarrassment of Monge and Fourier who happened to be with him.

Monge was one of the dozen or so to whom Napoleon in 1798 confided his plan for the invasion, conquest, and civilization of Egypt. As Fourier enters naturally here we shall go back and pick him up.

Jean-Baptiste-Joseph Fourier, born on 21 March 1768, at Auxerre, France, was the son of a tailor. Orphaned at the age of eight, he was recommended to the Bishop of Auxerre by a charitable lady who had been captivated by the boy's good manners and serious deportment – little did she dream what he was to become. The Bishop got Fourier into the local military college run by the Benedictines, where the boy soon proved his genius. By the age of twelve he was writing magnificent sermons for the leading church dignitaries of Paris to palm off as their own. At thirteen he was a problem child, wayward, petulant, and full of the devil generally. Then, at his first encounter with mathematics, he changed as if by magic. He knew what had ailed him and cured himself. To provide light for his mathematical studies after he was supposed to be asleep he collected candle-ends in the kitchen and wherever he could find them in the college. His secret study was an inglenook behind a screen.

The good Benedictines prevailed upon the young genius to choose the priesthood as his profession, and he entered the abbey of Saint-Benoît to become a novice. But before Fourier could take his vows 1789 arrived. He had always wanted to be a soldier and had chosen the priesthood only because commissions were not given to sons of tailors. The Revolution set him free. His old friends at Auxerre were broad-minded enough to see that Fourier would never make a monk.

They took him back and made him professor of mathematics. This was the first step – a long one – toward his ambition. Fourier proved his versatility by teaching his colleagues' classes when they were ill, usually better than they did themselves, in everything from physics to the classics.

In December 1789 Fourier (then twenty-one) went to Paris to present his researches on the solution of numerical equations before the Academy. This work advanced beyond Lagrange, and is still of value, but as it is overshadowed by Fourier's methods in mathematical physics, we shall not discuss it further; it may be found in elementary texts on the theory of equations. The subject became one of his life-long interests.

On returning to Auxerre Fourier joined the people's party and used his natural eloquence, which had enabled him as a small boy to compose stirring sermons, to stir up the people to put an end to mere sermonizers (among others).

From the first Fourier was an enthusiast for the Revolution – till it got out of hand. During the Terror, ignoring the danger to himself, he protested against the needless brutality. If he were living to-day Fourier would probably belong to the intelligentsia, blissfully unaware that such are among the first to be swept into the gutter when the real revolution begins. He was all for the masses and the renaissance of science and culture which the intellectuals imagined they foresaw. Instead of the generous encouragement of the sciences which he had predicted, Fourier presently saw men of science riding in the tumbrils or fleeing the country, and science itself fighting for its life in a rapidly rising tide of barbarism.

It is to Napoleon's everlasting credit that he was one of the first to see with cold-blooded clarity that ignorance of itself can do nothing but destroy. His own remedy in the end may not have been much better, but he did recognize that such a thing as civilization might be possible. To check the mere bloodletting Napoleon ordered or encouraged the creation of schools. But there were no teachers. All the brains that might have been pressed into immediate service had long since fallen into the buckets. It became imperative to train a new teaching corps of 1,500, and for this purpose the École Normale was created in

1794. As a reward for his recruiting in Auxerre Fourier was called to the chair of mathematics.

With this appointment a new era in the teaching of French mathematics began. Remembering the deadly lectures of defunct professors, memorized and delivered verbatim the same year after dreary year, the Convention called in *creators* of mathematics to do the *teaching*, and forbade them to lecture from any notes at all. The lectures were to be delivered standing (not sitting half-asleep behind a desk), and were to be a free interchange of questions and explanations between the professor and his class. It was up to the lecturer to prevent a session from degenerating into a profitless debate.

The success of this scheme even surpassed expectations and led to one of the most brilliant periods in the history of French mathematics and science. Both at the short-lived Normale and the enduring Polytechnique Fourier demonstrated his genius for teaching. At the Polytechnique he enlivened his lectures on mathematics by out-of-the-way historical allusions (many of which he was the first to trace to their sources), and he skilfully tempered abstractions with interesting applications.

Fourier was still turning out engineers and mathematicians at the Polytechnique when Napoleon in 1798 decided to take him along as one of the Legion of Culture to civilize Egypt – 'to offer a succouring hand to unhappy peoples, to free them from the brutalizing yoke under which they have groaned for centuries, and finally to endow them without delay with all the benefits of European civilization.'

Incredible as it may seem, the quotation is not from Signor Mussolini in 1935 justifying an invasion of Ethiopia, but from Arago in 1833 setting forth the lofty and humane aims of Napoleon's assault on Egypt. It will be interesting to see how the unregenerate inhabitants of Egypt received 'all the benefits of European civilization' which MM. Monge, Berthollet, and Fourier strove to ram down their throats, and what those three musketeers of European culture themselves got out of their unselfish missionary work.

The French fleet of five hundred ships arrived at Malta on 9 June 1798, and three days later captured the place. As a first step toward civilizing the East, Monge started fifteen elementary schools and a higher school somewhat on the lines of the Polytechnique. A week later the fleet was on its way again, with Monge aboard Napoleon's flagship, *l'Orient.* Every morning Napoleon outlined a programme for discussion after dinner in the evening. Needless to say, Monge was the star of these soirées. Among the topics solemnly debated were the age of the earth, the possibility of the world coming to an end by fire or water, and 'Are the planets inhabited?' The last suggests that even at this comparatively early stage of his career Napoleon's ambitions outran Alexander's.

The fleet reached Alexandria on 1 July 1798. Monge was one of the first to leap ashore, and it was only by exercising his authority as Commander-in-Chief that Napoleon restrained the *Marseillaising* geometer from participating in the assault on the city. It would never do to have the Legion of Culture annihilated in the first skirmish before the work of civilization could begin; so Napoleon sent Monge and the rest of them up the Nile by boat to Cairo.

While Monge and company lolled like Cleopatra and her court under their sunshade, Napoleon marched resolutely along the bank, civilizing the uncultured (and poorly armed) inhabitants with shot and flame. Presently the intrepid General heard a devil of a cannonade from the direction of the river. Guessing the worst he abandoned the battle in which he was engaged at the moment and galloped to the rescue. The blessed boat was hard aground on a sand bar. There was Monge serving the cannon like a veteran. Napoleon arrived just in the nick of time to chase the attackers up the bank and give Monge his well-merited decoration for conspicuous bravery. So Monge after all had his way and got his sniff of powder. Napoleon was so overjoyed at having saved his friend that he did not regret the decisive victory Monge's rescue had cost him.

Following the victory of 20 July 1798, at the Battle of the Pyramids, the triumphant army whooped into Cairo. Everything went off like fireworks, precisely as that great idealist Napoleon had dreamed, but for one trifling fizzle. The obtuse Egyptians cared not a single curse for the cultural banquet which MM. Monge, Fourier, and Berthollet spread before them at the Egyptian Institute (founded, 27 August 1798, in parody of the *Institut de France*), but sat like mummies through the great chemist's scientific legerdemain, the enthusiastic Monge's concerts, and the historical disquisitions of the scholarly Fourier on the glories of their own mummified civilization. The sweating savants shed their sangfroid, damning their prospective enlightenees as tasteless cattle incapable of relishing the rich hash of French erudition offered for their spiritual nourishment, but to no avail. Once more the wily, 'unsophisticated' native made a complete ass of his determined uplifters by holding his peace and waiting for the plague of locusts to be blown away in the scavenging winds. To keep his self-respect till the breezes blew, the uncivilized Egyptian criticized the superior civilization of his conquerors in the one language they could understand. Three hundred of Napoleon's bravest had their hairy throats cut at one swipe in a street brawl. Monge himself saved his own windpipe and those of his beleaguered companions only by an exhibition of heroism for which any Boy Scout to-day in the English-speaking world might well receive a medal.

This ingratitude on the part of the unregenerate Egyptians cut Napoleon to the quick. His suspicion that it was his moral duty to desert his companions in arms was strengthened by disturbing news from Paris. During his absence things on the Continent had been going from purgatory to damnation; and now he must hurry back to preserve the honour of France and his own skin. Monge shared the General's confidence; the less beloved Fourier did not. Fourier, however, had the satisfaction of knowing that he was considerable enough in his commander's masterful eyes to be left in Cairo to educate Egypt or have his throat cut, when Napoleon, accompanied by the complaisant Monge, took secret passage for France without so much as an adieu to the troops who had suffered hell for him in the desert. Not being a Commander-in-Chief, Fourier was not entitled to take to his heels in the face of danger. He stayed, perforce. Only in 1801, when the French after Trafalgar finally acknowledged that the British, not they, were to regenerate the

Egyptians, did the devoted – but disillusioned – Fourier return to France.

The return trip of Monge and Napoleon was less amusing for both of them than the voyage out. Instead of speculating about the end of the world Napoleon spent much anxious thought on his own probable end should the British sailors bag him. The reward for desertion in the field, he recalled, was a strictly private interview with a firing squad. Would the British treat him as a deserter for having run away from his army? If he must die he would die theatrically.

'Monge,' he said one day, 'if we are attacked by the British, our ship must be blown up the instant they board us. I charge you to carry it out.'

The very next day a sail topped the horizon and all hands stood to their posts to repel the expected attack. But it turned out to be a French ship after all.

'Where's Monge?' somebody asked when all the excitement was over.

They found him in the powder magazine with a lighted lamp in his hand. If only that had been a British ship – . They always blow in fifteen minutes or fifteen years too late.

Berthollet and Monge arrived home looking like a pair of tramps. Neither had had a change of clothes since he left, and it was only with difficulty that Monge got by his wife's porter.

The friendship with Napoleon continued unmarred. Probably Monge was the only man in France who dared to stand up to Napoleon and tell him the truth in the days of his greatest arrogance. When Napoleon crowned himself Emperor the young men of the Polytechnique revolted. They were Monge's pride.

'Well, Monge,' Napoleon remarked one day, 'your pupils are nearly all in revolt against me; they have decidedly declared themselves my enemies.'

'Sire,' Monge replied, 'we have had trouble enough to make republicans out of them; give them time to become imperialists. Moreover, permit me to say, you have turned rather abruptly!'

Little spats like this meant nothing between old lovers. In 1804 Napoleon showed his appreciation of Monge's merits by creating him Count of Péluse (Pelusium). For his part Monge accepted the honour gratefully and lived up to the title with all the usual trappings of nobility, forgetting that he had once voted for the abolition of all titles.

And so it went, in an ever more dazzling blaze of splendour till the year 1812, which was to have ushered in the day of glory, but which brought instead the retreat from Moscow. Too old (he was sixty-six) to accompany Napoleon into Russia, Monge had stayed behind in France at his country estate, eagerly following the progress of the Grand Army through the official bulletins. When he read the fatal 'Bulletin 29' announcing the disaster to French arms, Monge suffered a stroke of apoplexy. On recovering he said, 'A little while ago I did not know something that I know now; I know how I shall die.'

Monge was to be spared for the final curtain; Fourier helped to lower it. On his return from Egypt Fourier was appointed (2 January 1802) prefect of the Department of Isère, with headquarters at Grenoble. The district was then in political turmoil; Fourier's first task was to restore order. He was met by a curious opposition which he subdued in a ludicrous fashion. While in Egypt Fourier had taken a leading part in administering the archaeological research of the Institute. The good citizens of Grenoble were much upset by the religious implications of some of the Institute's discoveries, particularly the great age assigned to the older monuments, which conflicted (they imagined) with the chronology of the Bible. They were quite satisfied however and took Fourier to their bosoms, when as the result of some further archaeological researches nearer home, he dug up a saint in his own family, the blessed Pierre Fourier, his great-uncle, whose memory was hallowed because he had founded a religious order. His respectability established, Fourier accomplished a vast amount of useful work, draining marshlands, stamping out malaria, and otherwise lifting his district out of the Middle Ages.

It was while at Grenoble that Fourier composed the immortal *Théorie analytique de la chaleur* (The Mathematical Theory of Heat), a landmark in mathematical physics. His first memoir on the conduction of heat was submitted in 1807. This was so promising that the Academy encouraged Fourier to continue by setting a contribution to the mathematical theory of heat as its problem for the Grand Prize in 1812. Fourier won the prize, but not without some criticism which he resented deeply but which was well taken.

Laplace, Lagrange, and Legendre were the referees. While admitting the novelty and importance of Fourier's work they pointed out that the mathematical treatment was faulty, leaving much to be desired in the way of rigour. Lagrange himself had discovered special cases of Fourier's main theorem but had been deterred from proceeding to the general result by the difficulties which he now pointed out. These subtle difficulties were of such a nature that their removal at the time would probably have been impossible. More than a century was to elapse before they were satisfactorily met.

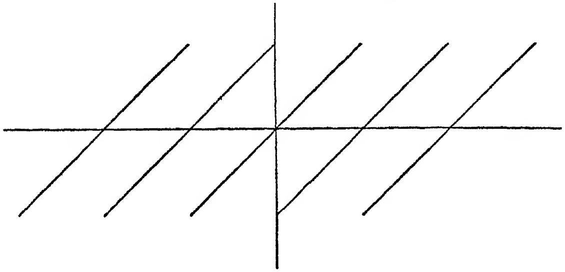
In passing it is interesting to observe that this dispute typifies a radical distinction between pure mathematicians and mathematical physicists. The only weapon at the disposal of pure mathematicians is sharp and rigid proof, and unless an alleged theorem can withstand the severest criticism of which its epoch is capable, pure mathematicians have but little use for it.

The applied mathematician and the mathematical physicist, on the other hand, are seldom so optimistic as to imagine that the infinite complexity of the physical universe can be described fully by any mathematical theory simple enough to be understood by human beings. Nor do they greatly regret that Airy's beautiful (or absurd) picture of the universe as a sort of interminable, self-solving system of differential equations has turned out to be an illusion born of mathematical bigotry and Newtonian determinism; they have something more real to appeal to at their own back door – the physical universe itself. They can *experiment* and check the deductions of their purposely imperfect mathematics against the verdict of experience – which, by the very nature of mathematics, is impossible for a pure mathematician. If their mathematical predictions are contradicted by experiment they do not, as a mathematician might, turn their backs on the physical evidence, but throw their mathematical tools away and look for a better kit.

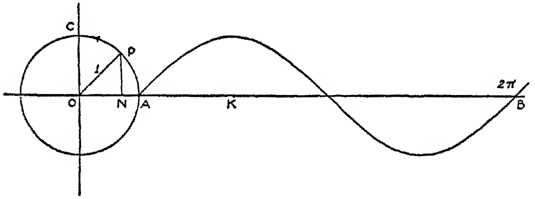
This indifference of scientists to mathematics for its own sake is as enraging to one type of *pure* mathematician as the omission of a doubtful iota subscript is to another type of pedant. The result is that but few *pure* mathematicians have ever made a significant contribution to science – apart, of course, from inventing many of the tools which scientists find useful (perhaps indispensable). And the curious part of it all is that the very purists who object to the boldly imaginative attack of the scientists are the loudest in their insistence that mathematics, contrary to a widely diffused belief, is not all an affair of grubbing, meticulous accuracy, but is as creatively imaginative, and sometimes as loose, as great poetry or music can be on occasion. Sometimes the physicists beat the mathematicians at their own game in this respect: ignoring the glaring lack of rigour in Fourier's classic on the analytical theory of heat, Lord Kelvin called it 'a great mathematical poem'.

As has already been stated Fourier's main advance was in the direction of boundary-value problems (described in the chapter on Newton) – the fitting of solutions of differential equations to prescribed initial conditions, probably the central problem of mathematical physics. Since Fourier applied this method to the mathematical theory of heat conduction a crowded century of splendidly gifted men has gone farther than he would ever have dreamed possible, but his step was decisive. One or two of the things he did are simple enough for description here.

In algebra we learn to plot the graphs of simple algebraic equations and soon notice that the curves we get, if continued sufficiently far, do not break off suddenly and end for good. What sort of an equation would result in a graph like that of the heavy line *segment* (finite length, terminated at both ends) repeated indefinitely as in the figure on p. 218? Such graphs, made up of disjointed fragments of straight or curved lines, recur repeatedly in physics, for example in the theories of heat, sound, and fluid motion. It can be proved that it is impossible to represent them by finite, closed, mathematical expressions; *an infinity* of terms occur in their equations. 'Fourier's Theorem' provides a means for representing and investigating such graphs mathematically: it expresses (within certain limitations) a given function continuous within a certain interval, or with only a finite number of discontinuities in the interval, and having in the interval only a finite number of turning-points, as an infinite sum of sines or cosines, or both. (This is only a rough description.)



Having mentioned sines and cosines we shall recall their most important property, *periodicity*. Let the radius of the circle in the figure be 1 unit in length. Through the centre O draw rectangular axes as in Cartesian geometry, and mark off AB equal to 2π units of length; thus AB is equal in length to the circumference of the circle (since the radius is 1). Let the point



P start from A and trace out the circle in the direction of the arrow. Drop PN perpendicular to OA. Then, for any position of P, the length of NP is called the *sine* of the angle AOP, and ON the *cosine*; NP and ON are to have their signs as in Cartesian geometry (NP is positive above OA, negative below; ON is positive to the right of OC, negative to the left).

For any position of P, the angle AOP will be that fraction of four right angles (360°) that the arc AP is of the whole circumference of the circle. So we may scale off these angles AOP by marking along AB the fractions of 2π which correspond to the arcs AP. Thus, when P is at C, $\frac{1}{4}$ the whole circumference has been traversed; hence, corresponding to the angle AOC we have the point K at $\frac{1}{4}$ of AB from A.

At each of these points on AB we erect a perpendicular equal in length to the sine of the corresponding angle, and above or below AB according as the sine is positive or negative.The ends of these perpendiculars not on AB lie on the continuous curve shown, the *sine curve*. When P returns to A and begins retracing the circle the curve is repeated beyond B, and so on indefinitely. If P revolves in the opposite direction, the curve is repeated to the left. After an interval of 2π the curve repeats: the sine of an angle (here AOP) is a *periodic function*, the *period* being 2π. The word 'sine' is abbreviated to 'sin'; and, if x is any angle, the equation

$$\sin (x + 2\pi) = \sin x$$

expresses the fact that $\sin x$ is a function of x having the period 2π.

It is easily seen that if the whole curve in the figure is shifted to the left a distance equal to AK, it now graphs the cosine of AOP. As before

$$\cos (x + 2\pi) = \cos x,$$

'cos' being the short for 'cosine'.

Inspection of the figure shows that $\sin 2x$ will go through its complete period 'twice as fast' as $\sin x$, and hence that the graph for a complete period will be one half as long as that for $\sin x$. Similarly $\sin 3x$ will require only $2\pi/3$ for its complete period, and so on. The same holds for $\cos x$, $\cos 2x$, $\cos 3x$, ...

Fourier's main mathematical result can now be described roughly. Within the restrictions already mentioned in connexion with 'broken' graphs, any function having a well-determined graph can be represented by an equation of the type

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

where the dots indicate that the two series are to continue indefinitely according to the rule shown, and the coefficients a_0, a_1, a_2, ... , b_1, b_2, b_3, ... are determinable when y, any given function of x, is known. In other words, any given function of x, say $f(x)$, can be expanded in a series of the type stated above, a *trigonometric* or *Fourier* series. To repeat, all this holds only within certain restrictions which, fortunately, are not of much importance in mathematical physics; the exceptions are more or less freak cases of little or no physical significance. Once more, Fourier's was the first great attack on boundary value problems. The specimens of such problems given in the chapter on Newton are solved by Fourier's method. In any given problem it is required to find the coefficients a_0, a_1, ..., b_0, b_1, ... in a form adapted to computation. Fourier's analysis provides this.

The concept of periodicity (*simple* periodicity) as described above is of obvious importance for natural phenomena; the tides, the phases of the Moon, the seasons, and a multitude of other familiar things are periodic in character. Sometimes a periodic phenomenon, such for example as the recurrence of sunspots, can be closely approximated by superposition of a certain number of graphs having simple periodicity. The study of such situations can then be simplified by analysing the individual periodic phenomena of which the original is the resultant.

The process is the same mathematically as the analysis of a musical sound into its fundamental and successive harmonics. As a first very crude approximation to the 'quality' of the sound only the fundamental is considered; the superposition of only a few harmonics usually suffices to produce a sound indistinguishable from the ideal (in which there is an infinity of harmonics). The like holds for phenomena attacked by 'harmonic' or 'Fourier' analysis. Attempts have even been made to detect long periods (the fundamentals) in the recurrence of earthquakes and annual rainfall. The notion of simple periodicity is as important in pure mathematics as it is in applied, and we shall see it being generalized to *multiple* periodicity (in connexion with elliptic functions and others), which in its turn reacts on applied mathematics.

Fully aware that he had done something of the first magnitude Fourier paid no attention to his critics. They were right, he wrong, but he had done enough in his own way to entitle him to independence.

When the work begun in 1807 was completed and collected in the treatise on heat-conduction in 1822, it was found that the obstinate Fourier had not changed a single word of his original presentations, thus exemplifying the second part of Francis Galton's advice to all authors: 'Never resent criticism, and never answer it.' Fourier's resentment was rationalized in attacks on pure mathematicians for minding their own proper business and not blundering about in mathematical physics.

All was going well with Fourier and France in general when Napoleon, having escaped from Elba, landed on the French coast on 1 March 1815. Veterans and all were just getting comfortably over their headache when the cause of it popped up again to give them a worse one. Fourier was at Grenoble at the time. Fearing that the populace would welcome Napoleon back for another spree, Fourier hastened to Lyons to tell the Bourbons what was about to happen. With their usual stupidity they refused to believe him. On his way back Fourier learned that Grenoble had capitulated. Fourier himself was taken prisoner and brought before Napoleon at Bourgoin. He was confronted by the same old commander he had known so well in Egypt and had learned to distrust with his head but not with his viscera. Napoleon was bending over a map, a pair of compasses in his hand. He looked up.

'Well, Monsieur Prefect! You too; you have declared war against me?'

'Sire,' Fourier stammered, 'my oaths made it a duty.'

'A duty, do you say? Don't you see that nobody in the country is of your opinion? And don't let yourself imagine that your plan of campaign frightens me much. I suffer only at seeing amongst my adversaries an *Egyptian*, a man who has eaten the bread of the bivouac with me, an old friend! How, moreover, Monsieur Fourier, have you been able to forget that I made you what you are?'

That Fourier, remembering Napoleon's callous abandonment of him in Egypt, could swallow such tripe and like it says a great deal for the goodness of his heart and the toughness of his stomach but precious little for the soundness of his head.

Some days later Napoleon asked the now loyal Fourier:

'What do you think of my plan?'

'Sire, I believe you will fail. You will meet a fanatic on your road, and everything will be over.'

'Bah! Nobody is for the Bourbons – not even a fanatic. As for that, you have read in the papers that they have put me outside the law. I myself will be more indulgent: I shall content myself with putting them outside the Tuileries!'

The leopard's spots and Napoleon's swellhead should be wedded in one proverb instead of pining apart in two.

The second restoration found Fourier in Paris pawning his effects to keep alive. But before he could starve to death old friends took pity on him and got him appointed director of the Bureau of Statistics for the Seine. The Academy tried to elect him to membership in 1816, but the Bourbon government ordered that no friend of their late kicker was to be honoured in any way. The Academy stuck to its guns and elected Fourier the following year. This action of the Bourbons against Fourier may seem petty, but beside what they did to poor old Monge it was princely. *Noblesse oblige!*

Fourier's last years evaporated in clouds of talk. As Permanent Secretary of the Academy he was always able to find listeners. To say that he bragged of his achievements under Napoleon is putting it altogether too mildly. He became an insufferable, shouting bore. And instead of continuing with his scientific work he entertained his audience with boastful accounts of what he was *going* to do. However, he had done far more than his share for the advancement of science, and if any human work merits immortality, Fourier's does. He did not need to boast or bluff.

Fourier's experiences in Egypt were responsible for a curious habit which may have hastened his death. Desert heat, he believed, was the ideal condition for health. In addition to swathing himself like a mummy he lived in rooms which his uncooked friends said were hotter than hell and the Sahara desert combined. He died of heart disease (some say an aneurism) on 16 May 1830, in the sixty-third year of his life. Fourier belongs to that select company of mathematicians whose work is so fundamental that their names have become adjectives in every civilized language.

Monge's decline was slower and more distressing. After the first restoration Napoleon felt embittered and vindictive toward the snobocracy of his own creation which, naturally, had let him down the moment his power waned. Once more in the saddle Napoleon was inclined to use the butt end of his crop on the skulls of the ungrateful. Monge, good old plebeian that he was, counselled mercy and common sense: Napoleon might some day find himself with his back to the wall (after an earthquake had cut off all means of flight), and be grateful for the support of the ingrates. Cooling off, Napoleon wisely tempered injustice with mercy. For this gracious dispensation Monge alone was responsible.

After Napoleon had run away from Waterloo, leaving his troops to get out of the mess as best they could, he returned to Paris. Fourier's devotion cooled then; Monge's boiled.

The school histories often tell of Napoleon's last dream – the conquest of America. The Mongian version differs and is on a much higher – in fact, incredibly high – plane. Hemmed in by enemies and appalled at the thought of enforced idleness for lack of further European conquest, Napoleon turned his eagle eye West, and in one flashing glance surveyed America from Alaska to Cape Horn. But, like the sick devil he was, Bonaparte longed to become a monk. The sciences alone could satisfy him, he declared; he would become a second and infinitely greater Alexander von Humboldt.

'I wish,' he confessed to Monge, 'in this new career to leave works, discoveries, worthy of me.'

What, precisely, are the works which could be worthy of a Napoleon? Continuing, the fallen eagle outlined his dream.

'I need a companion,' he admitted, 'to first put me abreast of the present state of the sciences. Then you [Monge] and I will traverse the whole continent, from Canada to Cape Horn; and in this immense journey we shall study all those prodigious phenomena of terrestrial physics on which the scientific world has not pronounced its verdict.' Paranoia?

'Sire,' Monge exclaimed – he was nearly sixty-seven – 'your collaborator is already found; I will go with you!'

His old self once more, Napoleon curtly dismissed the thought of the willing veteran hampering his lightning marches from Baffin Bay to Patagonia.

'You are too old, Monge. I need a younger man.'

Monge tottered off to find 'a younger man'. He approached the fiery Arago as the ideal travelling companion for his energetic master. But Arago, in spite of all his eloquent rhetoric on the gloriousness of glory, had learned his lesson. A general who could desert his troops as Napoleon had done at Waterloo, Arago pointed out, was no leader to follow anywhere, even in easy America.

Further negotiations were rudely halted by the British. By the middle of October Napoleon was exploring St Helena. The hoard of money which had been put aside for the conquest of America found its way into deeper pockets than those of the scientists, and no 'American Institute' rose on the banks of the Mississippi or the Amazon to match its fantastic twin overlooking the Nile.

Having enjoyed the bread of imperialism Monge now tasted the salt. His record as a revolutionist and favourite of the upstart Corsican made his head an extremely desirable object to the Bourbons, and Monge dodged from one slum to another in an endeavour to keep his head on his shoulders. For sheer human pettiness the treatment accorded Monge by the sanctified Bourbons would take a lot of beating. Small enough for anything they stripped the old man of his last honour – one with which the generosity of Napoleon had had nothing whatever to do. In 1816 they commanded that Monge be expelled from the Academy. The academicians, tame as rabbits now, obeyed.

The final touch of Bourbon pettiness graced the day of Monge's funeral. As he had foreseen he died after a prolonged stupor following a stroke. The young men at the Polytechnique, whom he had protected from Napoleon's domineering interference, were the pride of Monge's heart, and he was their idol. When Monge died on 28 July 1818, the Polytechnicians asked permission to attend the funeral. The King denied the request.

Well disciplined, the Polytechnicians observed the ban. Bı they were more resourceful or more courageous than the timi academicians. The King's order covered only the funeral. Tł following day they marched in a body to the cemetery and lai a wreath on the grave of their master and friend, Gaspaı Monge.

CHAPTER THIRTEEN

THE DAY OF GLORY

Poncelet

MORE than once during the World War when the French troops were hard pressed and reinforcements non-existent, the high command saved the day by routing some prima donna out of her boudoir, rushing her to the front, draping her from neck to heels in the tricolour, and ordering her to sing the *Marseillaise* to the exhausted men. Having sung her piece the lady rolled back to Paris in her limousine; the heartened troops advanced, and the following morning a cynically censored press once more unanimously assured a gullible public that 'the day of glory has arrived' – with unmentioned casualties.

In 1812 the day of glory was still on its way. Prima donnas did not accompany Napoleon Bonaparte's half-million troops on their triumphal march into Russia. The men did their own singing as the Russians retreated before the invincible Grand Army, and the endless plains rang to the stirring chant which had swept tyrants from their thrones and elevated Napoleon to their place.

All was going as gloriously as the most enthusiastic singer could have wished: six days before Napoleon crossed the Niemen his brilliant diplomatic strategy had indirectly exasperated President Madison into hurling the United States into a distracting war on England; the Russians were running harder than ever on their race back to Moscow, and the Grand Army was doing its valiant best to keep up with the reluctant enemy. At Borodino the Russians turned, fought, and retired. Napoleon continued without opposition – except from the erratic weather – to Moscow, whence he notified the Tsar of his willingness to consider an unconditional surrender of all the Russian forces. The competent inhabitants of Moscow, led by the

Governor, took matters into their own hands, fired their city, burned it to the ground, and smoked Napoleon and all his men out into the void. Chagrined but still master of the situation, Napoleon disregarded this broad hint – the second or third so far vouchsafed to his military obstinacy – that 'who killeth with the sword must perish by the sword', presently ordered his driver to give the horses the lash, and dashed back post-haste over the now frozen plains to prepare for his rendezvous with Blücher at Leipzig, leaving the Grand Army to walk home or freeze as it should see fit.

With the deserted French army was a young officer of engineers, Jean-Victor Poncelet (1 July 1788 – 23 December 1867) who, as a student at the École Polytechnique in Paris, later at the military academy at Metz, had been inspired by the new descriptive geometry of Monge (1746–1818) and the *Géométrie de position* (published in 1803) of the elder Carnot (Lazare-Nicolas-Marguerite Carnot, 13 May 1753 – 2 August 1823), whose revolutionary if somewhat reactionary programme was devised 'to free geometry from the hieroglyphics of analysis.'

In the preface to his classic *Applications d'analyse et de géométrie* (second edition 1862, of the work first published in 1822), Poncelet recounts his experiences in the disastrous retreat from Moscow. On 18 November 1812 the exhausted remnant of the French army under Marshal Ney was overwhelmed at Krasnoi. Among those left for dead on the frozen battlefield was young Poncelet. His uniform as an officer of engineers saved his life. A searching party, discovering that he still breathed, took him before the Russian staff for questioning.

As a prisoner of war the young officer was forced to march for nearly five months across the frozen plains in the tatters of his uniform, subsisting on a meagre ration of black bread. In a cold so intense that the mercury of the thermometer frequently froze, many of Poncelet's companions in misery died in their tracks, but his more rugged strength pulled him through, and in March 1813 he entered his prison at Saratov on the banks of the Volga. At first he was too exhausted to think. But when 'the splendid April sun' restored his vitality, he remembered that he had received a good mathematical education, and to soften the rigours of his exile he resolved to reproduce as much as he could of what he had learned. It was thus that he created projective geometry.

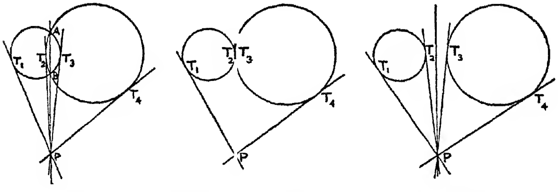
Without books and with only the scantiest writing materials at first, he retraced all that he had known of mathematics from arithmetic to higher geometry and the calculus. These first labours were enlivened by Poncelet's efforts to coach his fellow officers for the examinations they must take should they ever see France again. One legend states that at first Poncelet had only scraps of charcoal, salvaged from the meagre brazier which kept him from freezing to death for drawing his diagrams on the wall of his cell. He makes the interesting observation that practically all details and complicated developments of the mathematics he had been taught had evaporated, while the general, fundamental principles remained as clear as ever in his memory. The same was true of physics and mechanics.

In September 1814 Poncelet returned to France, carrying with him 'the material of seven manuscript notebooks written at Saratov in the prisons of Russia (1813–14), together with divers other writings, old and new', in which he, as a young man of twenty-four, had given projective geometry its strongest impulse since Desargues and Pascal initiated the subject in the seventeenth century. The first edition of his classic, as already mentioned, was published in 1822. It lacked the intimate 'apology for his life' which has been used above, but it started a tremendous nineteenth-century surge forward in projective geometry, modern synthetic geometry generally, and the geometric interpretation of the 'imaginary' numbers that present themselves in algebraic manipulations, giving to such 'imaginaries' geometrical interpretations as 'ideal' elements of space. It also proposed the powerful and (for a time) controversial 'doctrine of continuity', to be described presently, which greatly simplified the study of geometric configurations by unifying apparently unrelated properties of figures into uniform, self-contained complete wholes. Exceptions and awkward special cases appeared under Poncelet's broader point of view as merely different aspects of things already familiar. The classic treatise also made full use of the creative 'principle of duality' and introduced the method of 'reciprocation' devised by Poncelet himself. In short, a whole arsenal of new weapons was added to geometry by the young military engineer who had been left for dead on the field of Krasnoi, and who might indeed have died before morning had not his officer's uniform distinguished him as a likely candidate for questioning by the Russian staff.

For the next decade (1815–25) Poncelet's duties as a military engineer left him only odd moments for his real ambition – the exploitation of his new methods in geometry. Relief was not to come for many years. His high sense of duty and his fatal efficiency made Poncelet an easy prey for short-sighted superiors. Some of the tasks he was set could have been done only by a man of his calibre, for example the creation of the school of practical mechanics at Metz and the reform of mathematical education at the Polytechnique. But the reports on fortifications, his work on the Committee of Defence, and his presidency of the mechanical sections at the international expositions of London and Paris (1852–58), to mention only a few of his numerous routine jobs, could all have been done by lesser men. His high scientific merits, however, were not unappreciated. The Academy of Sciences elected him (1831) as successor to Laplace. For political reasons Poncelet declined the honour till three years later.

Poncelet's whole mature life was one long internal conflict between that half of him which was born to do lasting work and the other half which accepted all the odd or dirty jobs short-sighted politicians and obtuse militarists shoved in its way. Poncelet himself longed to escape, but a mistaken sense of duty, drilled into his very bones in Napoleon's armies, impelled him to serve the shadow and turn his back on the substance. That he did not suffer an early and permanent nervous breakdown is a remarkable testimonial to the ruggedness of his physique. And that he retained his creative abilities almost to his death at the age of seventy-nine is a shining proof of his unquenchable genius. When they could think of nothing better for this splendidly endowed man to do with his time they sent him traipsing about France to inspect cotton mills, silk mills, and linen mills. They did not need a Poncelet to do that sort of thing, and he knew it. He would have been the last man in France to object had his unique talents been indispensable in such affairs, for he was anything but the sort of intellectual prude who holds that science loses her perennial virginity every time she shakes hands with industry. But he was not the only man available for the work, as possibly Pasteur was in the equally important matters of the respective diseases of beer, silkworms, and human beings.

We now glance at one or two of the weapons either devised or remodelled by Poncelet for the conquest of projective geometry. First there is his 'principle of continuity', which refers to the permanence of geometrical properties as one figure shades, by projection or otherwise, into another. This no doubt is rather vague, but Poncelet's own statement of the principle was never very exact and, as a matter of fact, embroiled him in endless controversies with more conservative geometers whom he politely designated as old fossils – always in the dignified diction suitable to an officer and a gentleman, of course. With the caution that the principle is of great heuristic value but does not always of itself provide proofs of the theorems which it suggests, we may see something of its spirit from a few simple examples.



Imagine two intersecting circles. Say they intersect in the points A and B. Join A and B by a straight line. The figure presents ocular evidence of two *real* points A, B, and the common chord AB of the two circles. Now imagine the two circles pulled gradually apart. The common chord presently becomes a common tangent to the two circles at their point of contact. At any stage so far the following theorem (usually set as an exercise in school geometry) is true: if *any* point P be taken on the common chord, *four* tangent lines may be drawn from it to the two circles, and if the points in which these tangent lines touch the circles are T_1, T_2, T_3, T_4, then the segments PT_1, PT_2, PT_3, PT_4 are all equal in length. Conversely, if it is asked where do *all* the points P lie such that the four tangent-segments to the two circles shall all be equal, the answer is *on the common chord.* Stating all this briefly in the usual language, we say that the *locus* (which merely means *place*) of a point P which moves so that the lengths of the tangent-segments from it to two *intersecting* circles are equal, is the common chord of the two circles.* All this is familiar and straightforward; there is no element of mystery or incomprehensibility as some may say there is in the next where the 'principle of continuity' enters.

Pull the circles completely apart. Their two intersections (or in the last moment their one point of contact) are no longer visible on the paper and the 'common chord' is left suspended between the two circles, cutting neither visibly. But it is known that there is still a *locus* of equal tangent-segments, and it is easily proved that this locus is a straight line perpendicular to the line joining the centres of the two circles, just as the original locus (the common chord) was. Merely as a manner of speaking, if we object to 'imaginaries', we continue to *say* that the two circles intersect in two points in the infinite part of the plane, even when they have been pulled apart, and we *say* also that the new straight-line locus is still the common chord of the circles: the points of intersection are 'imaginary' or 'ideal', but the straight line joining them (the new 'common chord') is 'real' – we actually draw it on the paper.

If we write the equations of the circles and lines algebraically in the manner of Descartes, all that we do in the algebra of solving the equations for the intersections has its unique correlate in the enlarged geometry, whereas if we do not first expand our geometry – or at least increase its vocabulary, to take

* In what precedes the tangents are *real* (visible) if the point P lies *outside* the circles; if P is *inside*, the tangents are '*imaginary*'.

account of 'ideal' elements – much of the meaningful algebra is geometrically meaningless.

All this of course requires logical justification. Such justification has been given so far as is necessary, that is, up to the stage which includes the applications of the 'principle of continuity' useful in geometry.

A more important instance of the principle is furnished by parallel straight lines. Before describing this we may repeat the remark a venerable and distinguished judge relieved himself of recently when the matter was revealed to him. The judge had been under the weather; an amateur mathematician, thinking to cheer the old fellow up, told him something of the geometrical concept of infinity. They were strolling through the judge's garden at the time. On being informed that 'parallel lines meet at infinity', the judge stopped dead. 'Mr Blank,' he said with great emphasis, 'any man who says parallel lines meet at infinity, or anywhere else, simply hasn't got good sense.' To obviate an argument we may say as before that it is all a way of speaking to avoid irritating exceptions and separations into exasperating distinct cases. But once the language has been agreed upon, logical consistency demands that it be followed to the end without traversing the rules of logical grammar and syntax, and this is what is done.

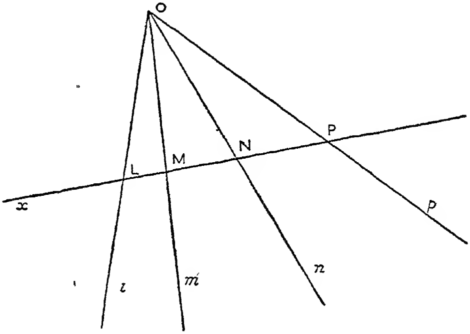
To see the reasonableness of the language, imagine a fixed straight line l and fixed point P not on l. Through P draw any straight line l' intersecting l in P', and imagine l' to rotate about P, so that P' recedes along l. When does P' stop receding? We say it stops when l, l' become parallel or, if we prefer, when the point of intersection P' is at infinity. For reasons already indicated this language is convenient and suggestive – not of a lunatic asylum, as the judge might think, but of interesting and sometimes highly practical things to do in geometry.

In a similar manner the visualizable, *finite* parts of lines, planes, and three-dimensional space (also of higher space) are enriched by the adjunction of 'ideal' points, lines, planes, or 'regions' *at infinity*. If the judge happens to see this he may enjoy the following shocking example of the behaviour of the infinite in geometry: *any two circles in a plane intersect in four points, two of which are imaginary and at infinity*. If the circles are concentric, they touch one another in two points lying on the line at infinity. Further, *all* circles in a plane go through *the same* two points at infinity – they are usually denoted by I and J, and are sometimes called Isaac and Jacob by irreverent students.

In the chapter on Pascal we described what is meant by projective properties in distinction to metrical properties in geometry. At this point we may glance back at Hadamard's remarks on Descartes' analytic geometry. Hadamard observed among other things that modern synthetic geometry repaid the debt of geometry in general to algebra by suggesting important researches in algebra and analysis. This modern synthetic geometry was the object of Poncelet's researches. Although all this may seem rather involved at the moment, we shall close the chain by taking a link from the 1840's, as the matter really is important, not only for the history of pure mathematics but for that of recent mathematical physics as well.

The link from the 1840's is the creation by Boole, Cayley, Sylvester and others, of the algebraic theory of invariance which (as will be explained in a later chapter) is of fundamental importance in current theoretical physics. The projective geometry of Poncelet and his school played a very important part in the development of the theory of invariance: the geometers had discovered a whole continent of properties of figures *invariant* under projection; the algebraists of the 1840's, notably Cayley, translated the geometrical *operations of projection* into analytical language, applied this translation to the *algebraic*, Cartesian mode of expressing geometric relationships, and were thus enabled to make phenomenally rapid progress in the elaboration of the theory of algebraic invariants. If Desargues, the daring pioneer of the seventeenth century, could have foreseen what his ingenious method of projection was to lead to, he might well have been astonished. He knew that he had done something good, but he probably had no conception of just how good it was to prove.

Isaac Newton was a young man of twenty when Desargues died. There is no evidence that Newton ever heard the name of Desargues. If he had, he also might have been astonished could he have foreseen that the humble link forged by his elderly contemporary was to form part of the strong chain which, in the twentieth century, was to pull his law of universal gravitation from its supposedly immortal pedestal. For without the mathematical machinery of the tensor calculus which developed naturally (as we shall see) from the algebraic work of Cayley and Sylvester, it is improbable that Einstein or anyone else could ever have budged the Newtonian theory of gravitation.



One of the useful ideas in projective geometry is that of *cross-ratio* or *anharmonic ratio*. Through a point O draw any four straight lines l, m, n, p. Across these four draw any straight line x, and label the points in which x cuts the others L, M, N, P

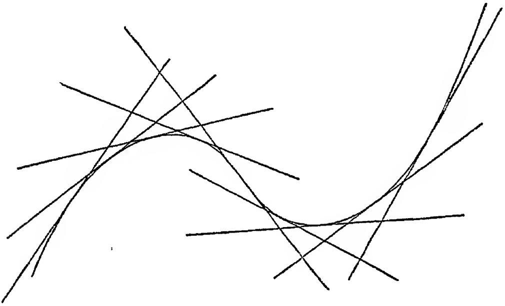
respectively. We thus have on x the line segments LM, MN, LP, PN. From these form the ratios $\frac{LM}{MN}$ and $\frac{LP}{PN}$. Finally we take the ratio of these two ratios, and get the *cross-ratio* $\frac{LM \times PN}{MN \times LP}$. The remarkable thing about this cross-ratio is that it has the same numerical magnitude for *all* positions of the line x.

Later we shall refer to Felix Klein's unification of Euclidean geometry and the common non-Euclidean geometries into one comprehensive geometry. This unification was made possible by Cayley's revision of the usual notions of *distance* and *angle* on which *metrical* geometry is founded. In this revision, cross-ratio played the leading part, and through it, by the introduction of 'ideal' elements of his own devising, Cayley was enabled to reduce *metrical* geometry to a species of *projective* geometry.

To close this inadequate description of the kind of weapons that Poncelet used we shall mention the extremely fruitful 'principle of duality'. For simplicity we consider only how the principle operates in plane geometry.

Note first that any continuous curve may be regarded in either of two ways: either as being generated by the motion of a point, or as being swept out by the turning motion of a straight line. To see the latter, imagine the tangent line drawn at each point of the curve. Thus *points* and *lines* are intimately and reciprocally associated with respect to the curve: *through* every *point* of the curve there is a *line* of the curve; *on* every line of the curve there is a point of the curve. Instead of 'through' in the preceding sentence, write 'on'. Then the two assertions separated by ';' after the ':' are identical except that the words 'point' and 'line' are interchanged.

As a matter of terminology we say that a line (straight or curved) is *on* a point if the line passes through the point, and we note that if a line is *on* a point, then the point is *on* the line, and conversely. To make this correspondence universal we 'adjoin' to the usual plane in which Euclidean geometry (common school geometry) is valid, a so-called *metric plane*, 'ideal elements' of the kind already described. The result of this adjunction is a *projective plane*: a projective plane consists of all the ordinary points and straight lines of a metric plane and, in addition, of a set of ideal points all of which are assumed to lie on one ideal line and such that one such ideal point lies on every ordinary line.*



In Euclidean language we would say that two parallel lines have the same direction; in projective phraseology this becomes 'two parallel lines have the same ideal point.' Again, in the old, if two or more lines have the same direction, they are parallel; in the new, if two or more lines have the same ideal point they are parallel. Every straight line in the projective plane is conceived of as having on it *one ideal point* ('at infinity'); *all* the ideal points are thought of as making up *one ideal line*, 'the line at infinity'.

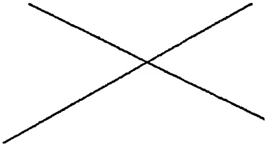
The purpose of these conceptions is to avoid the exceptional statements of Euclidean geometry necessitated by the postulated existence of parallels. This has already been commented on in connexion with Poncelet's formulation of the principle of continuity.

* This definition, and others of a similar character given presently, are taken from *Projective Geometry* (Chicago, 1930) by the late John Wesley Young. This little book is comprehensible to anyone who has had an ordinary school course in common geometry.

With these preliminaries the *principle of duality* in plane geometry can now be stated: All the propositions of plane projective geometry occur in dual pairs which are such that from either proposition of a particular pair another can be immediately inferred by interchanging the parts played by the words *point* and *line.*

In his projective geometry Poncelet exploited this principle to the limit. Opening almost any book on projective geometry at random we note pages of propositions printed in double columns, a device introduced by Poncelet. Corresponding propositions in the two columns are duals of one another; if either has been proved, a proof of the other is superfluous, as implied by the principle of duality. Thus geometry at one stroke is doubled in extent with no expenditure of extra labour. As a specimen of dual propositions we give the following pair. It

Two distinct points are on one, and only one, line.

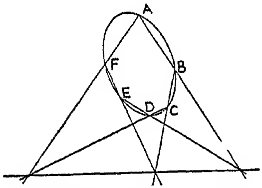


Two distinct lines are on one, and only one, point.

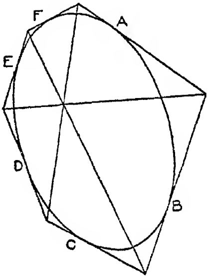
may be granted that this is not very exciting. The mountain has laboured and brought forth a mouse. Can it do any better?

The proposition in the left-hand column (page 238) is Pascal's concerning his *Hexagrammum Mysticum* which we have already seen; that on the right is Brianchon's theorem, which was *discovered* by means of the principle of duality. Brianchon (1785–1864) discovered his theorem while he was a student at the École Polytechnique; it was printed in the *Journal* of that school in 1806. The figures for the two propositions look nothing alike. This may indicate the power of the methods used by Poncelet.

Brianchon's discovery was the one which put the principle of duality on the map of geometry. Far more spectacular examples of the power of the principle will be found in any textbook on projective geometry, particularly in the extension of the principle to ordinary three-dimensional space. In this extension the parts played by the words *point* and *plane* are interchangeable; *straight line* stays as it was.



If A,B,C,D,E,F are any points on a conic section, the points of intersection of the pairs of lines AB and DE, BC and EF, CD and FA are on a straight line; and conversely.



If A,B,C,D,E,F are tangent straight lines on a conic section, the lines joining the pairs of intersections of A with B and D with E, B with C and E with F, C with D and F with A, meet in one point; and conversely.

The conspicuous beauty of projective geometry and the supple elegance of its demonstrations made it a favourite study with the geometers of the nineteenth century. Able men swarmed into the new goldfield and quickly stripped it of its more accessible treasures. To-day the majority of experts seem to agree that the subject is worked out so far as it is of interest to professional mathematicians. However, it is conceivable that there may yet be something in it as obvious as the principle of duality which has been overlooked. In any event it is an easy subject to acquire and one of fascinating delight to amateurs and even to professionals at some stage of their careers. Unlike some other fields of mathematics, projective geometry has been blessed with many excellent textbooks and treatises, some of them by master geometers, including Poncelet himself.

CHAPTER FOURTEEN

THE PRINCE OF MATHEMATICIANS

Gauss

ARCHIMEDES, NEWTON, AND GAUSS, these three, are in a class by themselves among the great mathematicians, and it is not for ordinary mortals to attempt to range them in order of merit. All three started tidal waves in both pure and applied mathematics: Archimedes esteemed his pure mathematics more highly than its applications; Newton appears to have found the chief justification for his mathematical inventions in the scientific uses to which he put them, while Gauss declared that it was all one to him whether he worked on the pure or the applied side. Nevertheless Gauss crowned the higher arithmetic, in his day the least practical of mathematical studies, the Queen of all.

The lineage of Gauss, Prince of Mathematicians, was anything but royal. The son of poor parents, he was born in a miserable cottage at Brunswick (Braunschweig), Germany, on 30 April 1777. His paternal grandfather was a poor peasant. In 1740 this grandfather settled in Brunswick, where he drudged out a meagre existence as a gardener. The second of his three sons, Gerhard Diederich, born in 1744, became the father of Gauss. Beyond that unique honour Gerhard's life of hard labour as a gardener, canal tender, and bricklayer was without distinction of any kind.

The picture we get of Gauss' father is that of an upright, scrupulously honest, uncouth man whose harshness to his sons sometimes bordered on brutality. His speech was rough and his hand heavy. Honesty and persistence gradually won him some measure of comfort, but his circumstances were never easy. It is not surprising that such a man did everything in his power to thwart his young son and prevent him from acquiring an education suited to his abilities. Had the father prevailed, the gifted boy would have followed one of the family trades, and it was only by a series of happy accidents that Gauss was saved from becoming a gardener or a bricklayer. As a child he was respectful and obedient, and although he never criticized his poor father in later life, he made it plain that he had never felt any real affection for him. Gerhard died in 1806. By that time the son he had done his best to discourage had accomplished immortal work.

On his mother's side Gauss was indeed fortunate. Dorothea Benz's father was a stonecutter who died at the age of thirty of tuberculosis, the result of insanitary working conditions in his trade, leaving two children, Dorothea and her younger brother Friederich.

Here the line of descent of Gauss' genius becomes evident. Condemned by economic disabilities to the trade of weaving, Friederich was a highly intelligent, genial man whose keen and restless mind foraged for itself in fields far from his livelihood. In his trade Friederich quickly made a reputation as a weaver of the finest damasks, an art which he mastered wholly by himself. Finding a kindred mind in his sister's child, the clever uncle Friederich sharpened his wits on those of the young genius and did what he could to rouse the boy's quick logic by his own quizzical observations and somewhat mocking philosophy of life.

Friederich knew what he was doing; Gauss at the time probably did not. But Gauss had a photographic memory which retained the impressions of his infancy and childhood unblurred to his dying day. Looking back as a grown man on what Friederich had done for him, and remembering the prolific mind which a premature death had robbed of its chance of fruition, Gauss lamented that 'a born genius was lost in him.'

Dorothea moved to Brunswick in 1769. At the age of thirty-four (in 1776) she married Gauss' father. The following year her son was born. His full baptismal name was Johann Friederich Carl Gauss. In later life he signed his masterpieces simply Carl Friederich Gauss. If a great genius was lost in Friederich Benz his name survives in that of his grateful nephew.

Gauss' mother was a forthright woman of strong character, sharp intellect, and humorous good sense. Her son was her pride from the day of his birth to her own death at the age of ninety-seven. When the 'wonder child' of two, whose astounding intelligence impressed all who watched his phenomenal development as something not of this earth, maintained and even surpassed the promise of his infancy as he grew to boyhood, Dorothea Gauss took her boy's part and defeated her obstinate husband in his campaign to keep his son as ignorant as himself.

Dorothea hoped and expected great things of her son. That she may sometimes have doubted whether her dreams were to be realized is shown by her hesitant questioning of those in a position to judge her son's abilities. Thus, when Gauss was nineteen, she asked his mathematical friend Wolfgang Bolyai whether Gauss would ever amount to anything. When Bolyai exclaimed 'The greatest mathematician in Europe!' she burst into tears.

The last twenty-two years of her life were spent in her son's house, and for the last four she was totally blind. Gauss himself cared little if anything for fame; his triumphs were his mother's life.* There was always the completest understanding between them, and Gauss repaid her courageous protection of his early years by giving her a serene old age. When she went blind he would allow no one but himself to wait on her, and he nursed her in her long last illness. She died on 19 April 1839.

Of the many accidents which might have robbed Archimedes and Newton of their mathematical peer, Gauss himself recalled one from his earliest childhood. A spring freshet had filled the canal which ran by the family cottage to overflowing. Playing

* The legend of Gauss' relations to his parents has still to be authenticated. Although, as will be seen later, the *mother* stood by her son, the *father* opposed him; and, as was customary *then* (usually, also, *now*), in a German household, the *father* had the last word. – I allude later to legends from living persons who had known members of the Gauss family, particularly in respect to Gauss' treatment of his sons. These allusions refer to first-hand evidence; but I do not vouch for them, as the people were very old.

near the water, Gauss was swept in and nearly drowned. But for the lucky chance that a labourer happened to be about his life would have ended then and there.

In all the history of mathematics there is nothing approaching the precocity of Gauss as a child. It is not known when Archimedes first gave evidence of genius. Newton's earliest manifestations of the highest mathematical talent may well have passed unnoticed. Although it seems incredible, Gauss showed his calibre before he was three years old.

One Saturday Gerhard Gauss was making out the weekly pay-roll for the labourers under his charge, unaware that his young son was following the proceedings with critical attention. Coming to the end of his long computations, Gerhard was startled to hear the little boy pipe up, 'Father, the reckoning is wrong, it should be. . . .' A check of the account showed that the figure named by Gauss was correct.

Before this the boy had teased the pronunciations of the letters of the alphabet out of his parents and their friends and had taught himself to read. Nobody had shown him anything about arithmetic, although presumably he had picked up the meanings of the digits 1, 2, . . . along with the alphabet. In later life he loved to joke that he knew how to reckon before he could talk. A prodigious power for involved mental calculations remained with him all his life.

Shortly after his seventh birthday Gauss entered his first school, a squalid relic of the Middle Ages run by a virile brute, one Büttner, whose idea of teaching the hundred or so boys in his charge was to thrash them into such a state of terrified stupidity that they forgot their own names. More of the good old days for which sentimental reactionaries long. It was in this hell-hole that Gauss found his fortune.

Nothing extraordinary happened during the first two years. Then, in his tenth year, Gauss was admitted to the class in arithmetic. As it was the beginning class none of the boys had ever heard of an arithmetical progression. It was easy then for the heroic Büttner to give out a long problem in addition whose answer he could find by a formula in a few seconds. The problem was of the following sort, $81297 + 81495 + 81693 + \ldots +$

100899, where the step from one number to the next is the same all along (here 198), and a given number of terms (here 100) are to be added.

It was the custom of the school for the boy who first got the answer to lay his slate on the table; the next laid his slate on top of the first, and so on. Büttner had barely finished stating the problem when Gauss flung his slate on the table: 'There it lies,' he said – '*Ligget se*' ' in his peasant dialect. Then, for the ensuing hour, while the other boys toiled, he sat with his hands folded, favoured now and then by a sarcastic glance from Büttner, who imagined the youngest pupil in the class was just another blockhead. At the end of the period Büttner looked over the slates. On Gauss' there appeared but a single number. To the end of his days Gauss loved to tell how the one number he had written was the correct answer and how all the others were wrong. Gauss had not been shown the trick for doing such problems rapidly. It is very ordinary once it is known, but for a boy of ten to find it instantaneously by himself is not so ordinary.

This opened the door through which Gauss passed on to immortality. Büttner was so astonished at what the boy of ten had done without instruction that he promptly redeemed himself and to at least one of his pupils became a humane teacher. Out of his own pocket he paid for the best textbook on arithmetic obtainable and presented it to Gauss. The boy flashed through the book. 'He is beyond me,' Büttner said; 'I can teach him nothing more.'

By himself Büttner could probably not have done much for the young genius. But by a lucky chance the schoolmaster had an assistant, Johann Martin Bartels (1769–1836), a young man with a passion for mathematics, whose duty it was to help the beginners in writing and cut their quill pens for them. Between the assistant of seventeen and the pupil of ten there sprang up a warm friendship which lasted out Bartels' life. They studied together, helping one another over difficulties and amplifying the proofs in their common textbook on algebra and the rudiments of analysis.

Out of this early work developed one of the dominating interests of Gauss' career. He quickly mastered the binomial theorem,

$$(1 + x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1 \times 2}x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}x^3 + \dots,$$

in which n is not necessarily a positive integer, but may be any number. If n is not a positive integer, the series on the right is *infinite* (non-terminating), and in order to state when this series is actually equal to $(1 + x)^n$, it is mandatory to investigate what restrictions must be imposed upon x and n in order that the infinite series shall *converge to a definite, finite limit.* Thus, if $x = -2$, and $n = -1$, we get the absurdity that $(1 - 2)^{-1}$, which is $(-1)^{-1}$ or $1/(-1)$, or finally -1, is equal to $1 + 2 + 2^2 + 2^3 + \dots$ and so on *ad infinitum*; that is, -1 is equal to the 'infinite number' $1 + 2 + 4 + 8 + \dots$, which is nonsense.

Before young Gauss asked himself whether infinite series *converge* and really do enable us to calculate the mathematical expressions (functions) they are used to represent, the older analysts had not seriously troubled themselves to explain the mysteries (and nonsense) arising from an uncritical use of infinite processes. Gauss' early encounter with the binomial theorem inspired him to some of his greatest work and he became the first of the 'rigorists'. A *proof* of the binomial theorem when n is not an integer greater than zero is even today beyond the range of an elementary textbook. Dissatisfied with what he and Bartels found in their book, Gauss made a proof. This initiated him to mathematical analysis. The very essence of analysis is the correct use of infinite processes.

The work thus well begun was to change the whole aspect of mathematics. Newton, Leibniz, Euler, Lagrange, Laplace – all great analysts for their times – had practically no conception of what is now acceptable as a proof involving infinite processes. The first to see clearly that a 'proof' which may lead to absurdities like 'minus 1 equals infinity' is no proof at all, was Gauss. Even if in *some* cases a formula gives consistent results, it has no place in mathematics until the precise conditions under which it will continue to yield consistency have been determined.

The rigor which Gauss imposed on analysis gradually overshadowed the whole of mathematics, both in his own habits and in those of his contemporaries – Abel, Cauchy – and his successors – Weierstrass, Dedekind, and mathematics after Gauss became a totally different thing from the mathematics of Newton, Euler, and Lagrange.

In the constructive sense Gauss was a revolutionist. Before his schooling was over the same critical spirit which left him dissatisfied with the binomial theorem had caused him to question the demonstrations of elementary geometry. At the age of twelve he was already looking askance at the foundations of Euclidean geometry; by sixteen he had caught his first glimpse of a geometry other than Euclid's. A year later he had begun a searching criticism of the proofs in the theory of numbers which had satisfied his predecessors and had set himself the extraordinarily difficult task of filling up the gaps and *completing* what had been only half done. Arithmetic, the field of his earliest triumphs, became his favourite study and the locus of his masterpiece. To his sure feeling for what constitutes proof Gauss added a prolific mathematical inventiveness that has never been surpassed. The combination was unbeatable.

Bartels did more for Gauss than to induct him into the mysteries of algebra. The young teacher was acquainted with some of the influential men of Brunswick. He now made it his business to interest these men in his find. They in turn, favourably impressed by the obvious genius of Gauss, brought him to the attention of Carl Wilhelm Ferdinand, Duke of Brunswick.

The Duke received Gauss for the first time in 1791. Gauss was then fourteen. The boy's modesty and awkward shyness won the heart of the generous Duke. Gauss left with the assurance that his education would be continued. The following year (February 1792) Gauss matriculated at the Collegium Carolinum in Brunswick. The Duke paid the bills and he continued to pay them till Gauss' education was finished.

Before entering the Caroline College at the age of fifteen, Gauss had made great headway in the classical languages by private study and help from older friends, thus precipitating a crisis in his career. To his crassly practical father the study of ancient languages was the height of folly. Dorothea Gauss put up a fight for her boy, won, and the Duke subsidized a two-years' course at the Gymnasium. There Gauss' lightning mastery of the classics astonished teachers and students alike.

Gauss himself was strongly attracted to philological studies, but fortunately for science he was presently to find a more compelling attraction in mathematics. On entering college Gauss was already master of the supple Latin in which many of his greatest works are written. It is an ever-to-be-regretted calamity that even the example of Gauss was powerless against the tides of bigoted nationalism which swept over Europe after the French Revolution and the downfall of Napoleon. Instead of the easy Latin which sufficed for Euler and Gauss, and which any student can master in a few weeks, scientific workers must now acquire a reading knowledge of two or three languages in addition to their own. Gauss resisted as long as he could, but even he had to submit when his astronomical friends in Germany pressed him to write some of his astronomical works in German.

Gauss studied at the Caroline College for three years, during which he mastered the more important works of Euler, Lagrange, and, above all, Newton's *Principia*. The highest praise one great man can get is from another in his own class. Gauss never lowered the estimate which as a boy of seventeen he had formed of Newton. Others – Euler, Laplace, Lagrange, Legendre – appear in the flowing Latin of Gauss with the complimentary *clarissimus*; Newton is *summus*.

While still at the college Gauss had begun those researches in the higher arithmetic which were to make him immortal. His prodigious powers of calculation now came into play. Going directly to the numbers themselves he experimented with them, discovering by induction recondite general theorems whose proofs were to cost even him an effort. In this way he rediscovered 'the gem of arithmetic', '*theorema aureum*', which Euler also had come upon inductively, which is known as the law of quadratic reciprocity, and which he was to be the first to prove. (Legendre's attempted proof slurs over a crux.)

The whole investigation originated in a simple question which many beginners in arithmetic ask themselves: How many digits are there in the period of a repeating decimal? To get some light on the problem Gauss calculated the decimal representations of all the fractions $1/n$ for $n = 1$ to 1000. He did not find the treasure he was seeking, but something infinitely greater – the law of quadratic reciprocity. As this is quite simply stated we shall describe it, introducing at the same time one of the revolutionary improvements in arithmetical nomenclature and notation which Gauss invented, that of *congruence*. All numbers in what follows are integers (common whole numbers).

If the *difference* ($a - b$ or $b - a$) of two numbers a, b is exactly divisible by the number m, we say that a, b are *congruent* with respect to the modulus m, or simply *congruent modulo* m, and we symbolize this by writing $a \equiv b \pmod{m}$. Thus $100 \equiv 2 \pmod{7}$, $35 \equiv 2 \pmod{11}$.

The advantage of this scheme is that it recalls the way we write algebraic equations, traps the somewhat elusive notion of arithmetical divisibility in a compact notation, and suggests that we try to carry over to arithmetic (which is much harder than algebra) some of the manipulations that lead to interesting results in algebra. For example we can 'add' equations, and we find that congruences also can be 'added', provided the modulus is the same in all, to give other congruences.

Let x denote an unknown number, r and m given numbers, of which r is not divisible by m. Is there a number x such that

$$x^2 \equiv r \pmod{m}?$$

If there is, r is called a *quadratic residue of* m, if not, a *quadratic non-residue of* m.

If r *is* a quadratic residue of m, then it must be possible to find at least one x whose square when divided by m leaves the remainder r; if r is a quadratic non-residue of m, then there is no x whose square when divided by m leaves the remainder r. These are immediate consequences of the preceding definitions.

To illustrate: is 13 a quadratic residue of 17? If so, it must be possible to solve the *congruence*

$$x^2 \equiv 13 \pmod{17}$$

Trying 1, 2, 3, ... , we find that $x = 8, 25, 42, 59, \ldots$ are solutions ($8^2 = 64 = 3 \times 17 + 13$; $25^2 = 625 = 36 \times 17 + 13$; etc.,) so that 13 *is* a quadratic residue of 17. But there is *no* solution of $x^2 \equiv 5 \pmod{17}$, so 5 is a quadratic non-residue of 17.

It is now natural to ask what are the quadratic residues and non-residues of a given m? Namely, given m in $x^2 \equiv r \pmod{m}$, what numbers r can appear and what numbers r cannot appear as x runs through all the numbers $1, 2, 3, \ldots$?

Without much difficulty it can be shown that it is sufficient to answer the question when both r and m are restricted to be primes. So we restate the problem: If p is a *given* prime, what primes q will make the congruence $x^2 \equiv q \pmod{p}$ solvable? This is asking altogether too much in the present state of arithmetic. However, the situation is not utterly hopeless.

There is a beautiful 'reciprocity' between the *pair* of congruences

$$x^2 \equiv q \pmod{p},\ x^2 \equiv p \pmod{q},$$

in which *both* of p, q are *primes*: *both* congruences are *solvable*, or *both* are *unsolvable, unless both of* p, q leave the remainder 3 when divided by 4, in which case *one* of the congruences *is* solvable and *the other* is *not*. This is the law of quadratic reciprocity.

It was not easy to prove. In fact it baffled Euler and Legendre. Gauss gave the first proof at the age of nineteen. As this reciprocity is of fundamental importance in the higher arithmetic and in many advanced parts of algebra, Gauss turned it over and over in his mind for many years, seeking to find its taproot, until in all he had given six distinct proofs, one of which depends upon the straightedge and compass construction of regular polygons.

A numerical illustration will illuminate the statement of the law. First, take $p = 5$, $q = 13$. Since both of 5, 13 leave the remainder 1 on division by 4, *both* of $x^2 \equiv 13 \pmod{5}$, $x^2 \equiv 5 \pmod{13}$ must be *solvable*, or *neither* is solvable. The latter is the case for this pair. For $p = 13$, $q = 17$, both of which leave the remainder 1 on division by 4, we get $x^2 \equiv 17 \pmod{13}$, $x^2 \equiv 13 \pmod{17}$, and *both*, or *neither* again must be solvable. The former is the case here: the first congruence has the solutions $x = 2, 15, 28, \ldots$; the second has the solutions $x = 8, 25, 42, \ldots$. There remains to be tested only the case when *both* of p, q leave the remainder 3 on division by 4. Take $p = 11$, $q = 19$. Then, according to the law, *precisely one* of $x^2 \equiv 19 \pmod{11}$, $x^2 \equiv 11 \pmod{19}$ must be solvable. The first congruence has no solution; the second has the solutions 7, 26, 45, ...

The mere discovery of such a law was a notable achievement. That it was first proved by a boy of nineteen will suggest to anyone who tries to prove it that Gauss was more than merely competent in mathematics.

When Gauss left the Caroline College in October 1795 at the age of eighteen to enter the University of Göttingen he was still undecided whether to follow mathematics or philology as his life work. He had already invented (when he was eighteen) the method of 'least squares', which to-day is indispensable in geodetic surveying, in the reduction of observations and indeed in all work where the 'most probable' value of anything that is measured is to be inferred from a large number of measurements. (The most probable value is furnished by making the sum of the squares of the 'residuals' – roughly, divergences from assumed exactness – a minimum.) Gauss shares this honour with Legendre who published the method independently in 1806. This work was the beginning of Gauss' interest in the theory of errors of observation. The Gaussian law of normal distribution of errors and its accompanying bell-shaped curve is familiar to-day to all who handle statistics, from high-minded intelligence testers to unscrupulous market manipulators.

March 30 1796 marks the turning point in Gauss' career. On that day, exactly a month before his twentieth year opened, Gauss definitely decided in favour of mathematics. The study of languages was to remain a lifelong hobby, but philology lost Gauss forever on that memorable day in March.

As has already been told in the chapter on Fermat the regular polygon of seventeen sides was the die whose lucky fall induced Gauss to cross his Rubicon. The same day Gauss began to keep his scientific diary (*Notizenjournal*). It is one of the most precious documents in the history of mathematics. The first entry records his great discovery.

The diary came into scientific circulation only in 1898, forty-three years after the death of Gauss, when the Royal Society of Göttingen borrowed it from a grandson of Gauss for critical study. It consists of nineteen small octavo pages and contains 146 extremely brief statements of discoveries or results of calculations, the last of which is dated 9 July 1814. A facsimile reproduction was published in 1917 in the tenth volume (part 1) of Gauss' collected works, together with an exhaustive analysis of its contents by several expert editors. Not all of Gauss' discoveries in the prolific period from 1796 to 1814 by any means are noted. But many of those that are jotted down suffice to establish Gauss' priority in fields – elliptic functions, for instance – where some of his contemporaries refused to believe he had preceded them. (Recall that Gauss was born in 1777.)

Things were buried for years or decades in this diary that would have made half a dozen great reputations had they been published promptly. Some were never made public during Gauss' lifetime, and he never claimed in anything he himself printed to have anticipated others when they caught up with him. But the record stands. He did anticipate some who doubted the word of his friends. These anticipations were not mere trivialities. Some of them became major fields of nineteenth-century mathematics.

A few of the entries indicate that the diary was a strictly private affair of its author's. Thus for 10 July 1796 there is the entry

$$\text{ΕΥΡΗΚΑ!} \quad \text{num} = \Delta + \Delta + \Delta.$$

Translated, this echoes Archimedes' exultant 'Eureka!' and states that every positive integer is the sum of three triangular numbers – such a number is one of the sequence 0,1,3,6,10,15, ... where each (after 0) is of the form $\frac{1}{2}n(n + 1)$, n being any positive integer. Another way of saying the same thing is that every number of the form $8n + 3$ is a sum of three odd squares: $3 = 1^2 + 1^2 + 1^2$; $11 = 1^2 + 1^2 + 3^2$; $19 = 1^2 + 3^2 + 3^2$, etc. It is not easy to prove this from scratch.

Less intelligible is the cryptic entry for 11 October 1796, 'Vicimus GEGAN.' What dragon had Gauss conquered this time? Or what giant had he overcome on 8 April 1799, when he boxes REV. GALEN up in a neat rectangle? Although the meaning of these is lost forever the remaining 144 are for the most part clear enough. One in particular is of the first importance, as we shall see when we come to Abel and Jacobi: the entry for 19 March 1797, shows that Gauss had already discovered the double periodicity of certain elliptic functions. He was then not quite twenty. Again, a later entry shows that Gauss had recognized the double periodicity in the general case. This discovery of itself, had he published it, would have made him famous. But he never published it.

Why did Gauss hold back the great things he discovered? This is easier to explain than his genius – if we accept his own simple statements, which will be reported presently. A more romantic version is the story told by W. W. R. Ball in his well-known history of mathematics. According to this, Gauss submitted his first masterpiece, the *Disquisitiones Arithmeticae*, to the French Academy of Sciences, only to have it rejected with a sneer. The undeserved humiliation hurt Gauss so deeply that he resolved thenceforth to publish only what anyone would admit was above criticism in both matter and form. There is nothing in this defamatory legend. It was disproved once for all in 1935, when the officers of the French Academy ascertained by an exhaustive search of the permanent records that the *Disquisitiones* was never even submitted to the Academy, much less rejected.

Speaking for himself Gauss said that he undertook his scientific works only in response to the deepest promptings of his nature, and it was a wholly secondary consideration to him whether they were ever published for the instruction of others. Another statement which Gauss once made to a friend explains both his diary and his slowness in publication. He declared that such an overwhelming horde of new ideas stormed his mind before he was twenty that he could hardly control them and had time to record but a small fraction. The diary contains only the final brief statements of the outcome of elaborate investigations, some of which occupied him for weeks. Contemplating as a youth the close, unbreakable chains of synthetic proofs in which Archimedes and Newton had tamed their inspirations, Gauss resolved to follow their great example and leave after him only finished works of art, severely perfect, to which nothing could be added and from which nothing could be taken away without disfiguring the whole. The work itself must stand forth, complete, simple, and convincing, with no trace remaining of the labour by which it had been achieved. A cathedral is not a cathedral, he said, till the last scaffolding is down and out of sight. Working with this ideal before him, Gauss preferred to polish one masterpiece several times rather than to publish the broad outlines of many as he might easily have done. His seal, a tree with but few fruits, bore the motto *Pauca sed matura* (Few, but ripe).

The fruits of this striving after perfection were indeed ripe but not always easily digestible. All traces of the steps by which the goal had been attained having been obliterated, it was not easy for the followers of Gauss to rediscover the road he had travelled. Consequently some of his works had to wait for highly gifted interpreters before mathematicians in general could understand them, see their significance for unsolved problems, and go ahead. His own contemporaries begged him to relax his frigid perfection so that mathematics might advance more rapidly, but Gauss never relaxed. Not till long after his death was it known how much of nineteenth-century mathematics Gauss had foreseen and anticipated before the year 1800. Had he divulged what he knew it is quite possible that mathematics would now be half a century or more ahead of where it is. Abel and Jacobi could have begun where Gauss left off, instead of expending much of their finest effort rediscovering things Gauss knew before they were born, and the creators of non-Euclidean geometry could have turned their genius to other things.

Of himself Gauss said that he was 'all mathematician'. This does him an injustice unless it is remembered that 'mathematician' in his day included also what would now be termed a mathematical physicist. Indeed his second motto*

* Shakespeare's *King Lear*, Act I, Scene ii, 1–2, with the essential change of 'laws' for 'law'.

Thou, nature, art my goddess; to thy laws
My services are bound ...,

truly sums up his life of devotion to mathematics and the physical sciences of his time. The 'all mathematician' aspect of him is to be understood only in the sense that he did not scatter his magnificent endowment broadcast over all fields where he might have reaped abundantly, as he blamed Leibniz for doing, but cultivated his greatest gift to perfection.

The three years (October 1795–September 1798) at the University of Göttingen were the most prolific in Gauss' life. Owing to the generosity of Duke Ferdinand the young man did not have to worry about finances. He lost himself in his work, making but few friends. One of these, Wolfgang Bolyai, 'the rarest spirit I ever knew', as Gauss described him, was to become a friend for life. The course of this friendship and its importance in the history of non-Euclidean geometry is too long to be told here; Wolfgang's son Johann was to retrace practically the same path that Gauss had followed to the creation of a non-Euclidean geometry, in entire ignorance that his father's old friend had anticipated him. The ideas which had overwhelmed Gauss since his seventeenth year were now caught – partly – and reduced to order. Since 1795 he had been meditating a great work on the theory of numbers. This now took definite shape, and by 1798 the *Disquisitiones Arithmeticae* (Arithmetical Researches) was practically completed.

To acquaint himself with what had already been done in the higher arithmetic and to make sure that he gave due credit to his predecessors, Gauss went to the University of Helmstedt, where there was a good mathematical library, in September 1798. There he found that his fame had preceded him. He was cordially welcomed by the librarian and the professor of mathematics, Johann Friedrich Pfaff (1765–1825), in whose house he roomed. Gauss and Pfaff became warm friends, although the Pfaff family saw but little of their guest. Pfaff evidently thought it his duty to see that his hard-working young friend took some exercise, for he and Gauss strolled together in the evenings, talking mathematics. As Gauss was not only modest but reticent about his own work, Pfaff probably did not learn as much as he might have. Gauss admired the professor tremendously (he was then the best-known mathematician in Germany), not only for his excellent mathematics, but for his simple, open character. All his life there was but one type of man for whom Gauss felt aversion and contempt, the pretender to knowledge who will not admit his mistakes when he knows he is wrong.

Gauss spent the autumn of 1798 (he was then twenty-one) in Brunswick, with occasional trips to Helmstedt, putting the finishing touches to the *Disquisitiones*. He had hoped for early publication, but the book was held up in the press owing to a Leipzig publisher's difficulties till September 1801. In gratitude for all that Ferdinand had done for him, Gauss dedicated his book to the Duke – '*Serenissimo Principi ac Domino Carolo Guilielmo Ferdinando*.'

If ever a generous patron deserved the homage of his protégé, Ferdinand deserved that of Gauss. When the young genius was worried ill about his future after leaving Göttingen – he tried unsuccessfully to get pupils – the Duke came to his rescue, paid for the printing of his doctoral dissertation (University of Helmstedt, 1799), and granted him a modest pension which would enable him to continue his scientific work unhampered by poverty. '*Your* kindness', Gauss says in his dedication, 'freed me from all other responsibilities and enabled me to assume this exclusively.'

Before describing the *Disquisitiones* we shall glance at the dissertation for which Gauss was awarded his doctor's degree *in absentia* by the University of Helmstedt in 1799: *Demonstratio nova theorematis omnem functionem algebraicam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse* (A New Proof that Every Rational Integral Function of One Variable Can Be Resolved into Real Factors of the First or Second Degree).

There is only one thing wrong with this landmark in algebra. The first two words in the title would imply that Gauss had merely added a *new* proof to others already known. He should have omitted 'nova'. His was the *first* proof. (This assertion will be qualified later.) Some before him had published what they supposed were proofs of this theorem – usually called the fundamental theorem of algebra – but none had attained a proof. With his uncompromising demand for logical and mathematical rigour Gauss insisted upon a *proof*, and gave the first. Another, equivalent, statement of the theorem says that every algebraic equation in one unknown has a root, an assertion which beginners often take for granted as being true without having the remotest conception of what it means.

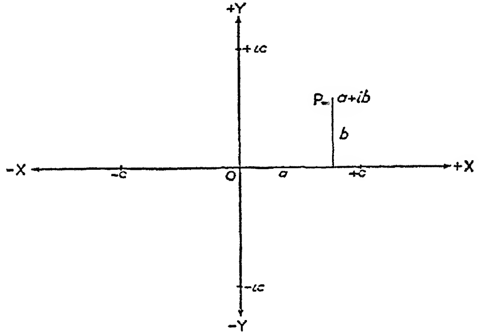
If a lunatic scribbles a jumble of mathematical symbols it does not follow that the writing means anything merely because to the inexpert eye it is indistinguishable from higher mathematics. It is just as doubtful whether the assertion that every algebraic equation has a root means anything until we say *what sort* of a root the equation has. Vaguely, we feel that a *number* will 'satisfy' the equation but that half a pound of butter will not.

Gauss made this feeling precise by proving that all the roots of any algebraic equation are 'numbers' of the form $a + bi$, where a,b are real numbers (the numbers that correspond to the distances, positive, zero, or negative, measured from a fixed point O on a given straight line, as on the x-axis in Descartes' geometry), and i is the square root of -1. The new sort of 'number' $a + bi$ is called *complex.*

Incidentally, Gauss was one of the first to give a coherent account of complex numbers and to interpret them as labelling the points of a plane, as is done to-day in elementary textbooks on algebra.

The Cartesian co-ordinates of P are (a,b); the point P is also labelled $a + bi$. Thus to every point of the plane corresponds precisely one complex number; the numbers corresponding to the points on XOX' are 'real', those on YOY' 'pure imaginary' (they are all of the type ic, where c is a real number).

The word 'imaginary' is the great algebraical calamity, but it is too well established for mathematicians to eradicate. It should never have been used. Books on elementary algebra give a simple interpretation of imaginary numbers in terms of rotations. Thus if we interpret the multiplication $i \times c$, where c is real, as a rotation about O of the segment Oc through one right angle, Oc is rotated on to OY; another multiplication by i, namely $i \times i \times c$, rotates Oc through another right angle, and hence the total effect is to rotate Oc through two right angles, so that $+ Oc$ becomes $- Oc$. As an operation, multiplication by $i \times i$ has the same effect as multiplication by -1; multiplication by i has the same effect as a rotation through a right



angle, and these interpretations (as we have just seen) are consistent. If we like we may now write $i \times i = -1$, in operations, or $i^2 = -1$; so that the operation of rotation through a right angle is symbolized by $\sqrt{-1}$.

All this of course proves nothing. It is not meant to prove anything. *There is nothing to be proved*; we *assign* to the symbols and operations of algebra *any meanings whatever* that will lead to consistency. Although the *interpretation* by means of rotations *proves* nothing, it may suggest that there is no occasion for anyone to muddle himself into a state of mystic wonderment over nothing about the grossly misnamed 'imaginaries'. For further details we must refer to almost any schoolbook on elementary algebra.

Gauss thought the theorem that every algebraic equation has a root in the sense just explained so important that he gave four distinct proofs, the last when he was seventy years old. To-day some would transfer the theorem from algebra (which restricts itself to processes that can be carried through in a finite number of steps) to analysis. Even Gauss *assumed* that the graph of a polynomial is a continuous curve and that if the polynomial is of odd degree the graph must cross the axis at least once. To any beginner in algebra this is obvious. But to-day it is *not obvious* without proof, and attempts to prove it again lead to the difficulties connected with continuity and the infinite. The roots of so simple an equation as $x^2 - 2 = 0$ cannot be computed exactly in any finite number of steps. More will be said about this when we come to Kronecker. We proceed now to the *Disquisitiones Arithmeticae*.

The *Disquisitiones* was the first of Gauss' masterpieces and by some considered his greatest. It was his farewell to pure mathematics as an exclusive interest. After its publication in 1801 (Gauss was then twenty-four), he broadened his activity to include astronomy, geodesy, and electromagnetism in both their mathematical and practical aspects. But arithmetic was his first love, and he regretted in later life that he had never found the time to write the second volume he had planned as a young man. The book is in seven 'sections'. There was to have been an eighth, but this was omitted to keep down the cost of printing.

The opening sentence of the preface describes the general scope of the book. 'The researches contained in this work appertain to that part of mathematics which is concerned with integral numbers, also fractions, surds [irrationals] being always excluded.'

The first three sections treat the theory of congruences and give in particular an exhaustive discussion of the binomial *congruence* $x^n \equiv A \pmod{p}$, where the given integers n, A are arbitrary and p is prime; the unknown integer is x. This beautiful *arithmetical* theory has many resemblances to the corresponding *algebraic* theory of the binomial *equation* $x^n = A$, but in its peculiarly arithmetical parts is incomparably richer and more difficult than the algebra which offers no analogies to the arithmetic.

In the fourth section Gauss develops the theory of quadratic residues. Here is found the first published *proof* of the law of quadratic reciprocity. The proof is by an amazing application of mathematical induction and is as tough a specimen of that ingenious logic as will be found anywhere.

With the fifth section the theory of *binary quadratic forms* from the arithmetical point of view enters, to be accompanied presently by a discussion of *ternary* quadratic forms which are found to be necessary for the completion of the binary theory. The law of quadratic reciprocity plays a fundamental part in these difficult enterprises. For the first forms named the general problem is to discuss the solution in integers x, y of the indeterminate equation.

$$ax^2 + 2bxy + cy^2 = m,$$

where a, b, c, m are any given integers; for the second, the integer solutions x, y, z of

$$ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 = m,$$

where a, b, c, d, e, f, m, are any given integers, are the subject of investigation. An easy-looking but hard question in this field is to impose necessary and sufficient restrictions upon a, c, f, m which will ensure the existence of a solution in integers x, y, z of the indeterminate equation

$$ax^2 + cy^2 + fz^2 = m.$$

The sixth section applies the preceding theory to various special cases, for example the integer solutions x, y of $mx^2 + ny^2 = A$, where m, n, A are any given integers.

In the seventh and last section, which many consider the crown of the work, Gauss applies the preceding developments, particularly the theory of binomial congruences, to a wonderful discussion of the algebraic equation $x^n = 1$, where n is any given integer, weaving together arithmetic, algebra, and geometry into one perfect pattern. The equation $x^n = 1$ is the *algebraic* formulation of the geometric problem to construct a regular polygon of n sides, or to divide the circumference of a circle into n equal parts (consult any secondary text book on algebra or trigonometry); the *arithmetical congruence* $x^m \equiv 1 \pmod{p}$, where m, p are given integers, and p is prime, is the thread which runs through the algebra and the geometry and gives the pattern its simple meaning. This flawless work of art is accessible to any student who has the usual algebra offered in school, but the *Disquisitiones* is not recommended for beginners (Gauss' concise presentation has been reworked by later writers into a more readily assimilable form).

Many parts of all this had been done otherwise before – by Fermat, Euler, Lagrange, Legendre and others; but Gauss treated the whole from his individual point of view, added much of his own, and deduced the isolated results of his predecessors from his general formulations and solutions of the relevant problems. For example, Fermat's beautiful result that every prime of the form $4n + 1$ is a sum of two squares, and is such a sum in only one way, which Fermat proved by his difficult method of 'infinite descent', falls out naturally from Gauss' general discussion of binary quadratic forms.

'The *Disquisitiones Arithmeticae* have passed into history,' Gauss said in his old age, and he was right. A new direction was given to the higher arithmetic with the publication of the *Disquisitiones*, and the theory of numbers, which in the seventeenth and eighteenth centuries had been a miscellaneous aggregation of disconnected special results, assumed coherence and rose to the dignity of a mathematical science on a par with algebra, analysis, and geometry.

The work itself has been called a 'book of seven seals'. It is hard reading, even for experts, but the treasures it contains (and partly conceals) in its concise, synthetic demonstrations are now available to all who wish to share them, largely the results of the labours of Gauss' friend and disciple Peter Gustav Lejeune Dirichlet (1805–59), who first broke the seven seals.

Competent judges recognized the masterpiece for what it was immediately. Legendre* at first may have been inclined to think that Gauss had done him but scant justice. But in the preface to the second edition of his own treatise on the theory of numbers (1808), which in large part was superseded by the *Disquisitiones*, he is enthusiastic. Lagrange also praised unstintedly. Writing to Gauss on 31 May 1804 he says 'Your

* Adrien-Marie Legendre (1752–1833). Considerations of space preclude an account of his life; much of his best work was absorbed or circumvented by younger mathematicians.

Disquisitiones have raised you at once to the rank of the first mathematicians, and I regard the last section as containing the most beautiful analytical discovery that has been made for a long time. ... Believe, sir, that no one applauds your success more sincerely than I.'

Hampered by the classic perfection of its style the *Disquisitiones* was somewhat slow of assimilation, and when finally gifted young men began studying the work deeply they were unable to purchase copies, owing to the bankruptcy of a bookseller. Even Eisenstein, Gauss' favourite disciple, never owned a copy. Dirichlet was more fortunate. His copy accompanied him on all his travels, and he slept with it under his pillow. Before going to bed he would struggle with some tough paragraph in the hope – frequently fulfilled – that he would wake up in the night to find that a re-reading made everything clear. To Dirichlet is due the marvellous theorem, mentioned in connexion with Fermat, that every arithmetical progression

$$a, a + b, a + 2b, a + 3b, a + 4b, \ldots,$$

in which a, b are integers with no common divisor greater than 1, contains an infinity of primes. This was proved by analysis, in itself a miracle, for the theorem concerns integers, whereas analysis deals with the *continuous*, the *non-integral*.

Dirichlet did much more in mathematics than his amplification of the *Disquisitiones*, but we shall not have space to discuss his life. Neither shall we have space (unfortunately) for Eisenstein, one of the brilliant young men of the early nineteenth century who died before their time and, what is incomprehensible to most mathematicians, as the man of whom Gauss is reported to have said, 'There have been but three epoch-making mathematicians, Archimedes, Newton, and Eisenstein'. If Gauss ever did say this (it is impossible to check) it deserves attention merely because he said it, and he was a man who did not speak hastily.

Before leaving this field of Gauss' activities we may ask why he never tackled Fermat's Last Theorem. He gives the answer himself. The Paris Academy in 1816 proposed the proof (or disproof) of the theorem as its prize problem for the period

1816–8. Writing from Bremen on 7 March 1816, Olbers tries to entice Gauss into competing: 'It seems right to me, dear Gauss, that you should get busy about this.'

But 'dear Gauss' resisted the tempter. Replying two weeks later he states his opinion of Fermat's Last Theorem. 'I am very much obliged for your news concerning the Paris prize. But I confess that Fermat's Theorem as an isolated proposition has very little interest for me, because I could easily lay down a multitude of such propositions, which one could neither prove nor dispose of.'

Gauss goes on to say that the question has induced him to recall some of his old ideas for a great extension of the higher arithmetic. This doubtless refers to the theory of algebraic numbers (described in later chapters) which Kummer, Dedekind, and Kronecker were to develop independently. But the theory Gauss has in mind is one of those things, he declares, where it is impossible to foresee what progress shall be made toward a distant goal that is only dimly seen through the darkness. For success in such a difficult search one's lucky star must be in the ascendant, and Gauss' circumstances are now such that, what with his numerous distracting occupations, he is unable to give himself up to such meditations, as he did 'in the fortunate years 1796–8 when I shaped the main points of the *Disquisitiones Arithmeticae*. Still I am convinced that if I am as lucky as I dare hope, and if I succeed in taking some of the principal steps in that theory, then Fermat's Theorem will appear as only one of the least interesting corollaries.'

Probably all mathematicians to-day regret that Gauss was deflected from his march through the darkness by 'a couple of clods of dirt which we call planets' – his own words – which shone out unexpectedly in the night sky and led him astray. Lesser mathematicians than Gauss – Laplace for instance – might have done all that Gauss did in computing the orbits of Ceres and Pallas, even if the problem was of a sort which Newton said belonged to the most difficult in mathematical astronomy. But the brilliant success of Gauss in these matters brought him instant recognition as the first mathematician in Europe and thereby won him a comfortable position where he could work in comparative peace; so perhaps those wretched lumps of dirt were after all his lucky stars.

The second great stage in Gauss' career began on the first day of the nineteenth century, also a red-letter day in the histories of philosophy and astronomy. Since 1781 when Sir William Herschel (1738–1822) discovered the planet Uranus, thus bringing the number of planets then known up to the philosophically satisfying seven, astronomers had been diligently searching the heavens for further members of the Sun's family, whose existence was to be expected, according to Bode's law, between the orbits of Mars and Jupiter. The search was fruitless till Giuseppe Piazzi (1746–1826) of Palermo, on the opening day of the nineteenth century, observed what he at first mistook for a small comet approaching the Sun, but which was presently recognized as a new planet – later named Ceres, the first of the swarm of minor planets known to-day.

By one of the most ironic verdicts ever delivered in the age-long litigation of fact versus speculation, the discovery of Ceres coincided with the publication by the famous philosopher Georg Wilhelm Friedrich Hegel (1770–1831) of a sarcastic attack on astronomers for presuming to search for an eighth planet. Would they but pay some attention to philosophy, Hegel asserted, they must see immediately that there can be precisely seven planets, no more, no less. Their search therefore was a stupid waste of time. Doubtless this slight lapse on Hegel's part has been satisfactorily explained by his disciples, but they have not yet talked away the hundreds of minor planets which mock his Jovian ban.

It will be of interest here to quote what Gauss thought of philosophers who busy themselves with scientific matters they have not understood. This holds in particular for philosophers who peck at the foundations of mathematics without having first sharpened their dull beaks on some hard mathematics. Conversely, it suggests why Bertrand A. W. Russell (1872–), Alfred North Whitehead (1861–1947) and David Hilbert (1862–1943) in our own times have made outstanding contributions to the philosophy of mathematics: these men are mathematicians.

Writing to his friend Schumacher on 1 November 1844, Gauss says 'You see the same sort of thing [mathematical incompetence] in the contemporary philosophers Schelling, Hegel, Nees von Essenbeck, and their followers; don't they make your hair stand on end with their definitions? Read in the history of ancient philosophy what the big men of that day – Plato and others (I except Aristotle) – gave in the way of explanations. But even with Kant himself it is often not much better; in my opinion his distinction between analytic and synthetic propositions is one of those things that either run out in a triviality or are false.' When he wrote this (1844) Gauss had long been in full possession of non-Euclidean geometry, itself a sufficient refutation of some of the things Kant said about 'space' and geometry, and he may have been unduly scornful.

It must not be inferred from this isolated example concerning purely mathematical technicalities that Gauss had no appreciation of philosophy. He had. All philosophical advances had a great charm for him, although he often disapproved of the means by which they had been attained. 'There are problems', he said once, 'to whose solution I would attach an infinitely greater importance than to those of mathematics, for example touching ethics, or our relation to God, or concerning our destiny and our future; but their solution lies wholly beyond us and completely outside the province of science.'

Ceres was a disaster for mathematics. To understand why she was taken with such devastating seriousness by Gauss we must remember that the colossal figure of Newton – dead for more than seventy years – still overshadowed mathematics in 1801. The 'great' mathematicians of the time were those who, like Laplace, toiled to complete the Newtonian edifice of celestial mechanics. Mathematics was still confused with mathematical physics – such as it was then – and mathematical astronomy. The vision of mathematics as an autonomous science which Archimedes saw in the third century before Christ had been lost sight of in the blaze of Newton's splendour, and it was not until the youthful Gauss again caught the vision that mathematics was acknowledged as a science whose first duty is to itself. But that insignificant clod of dirt, the minor planet Ceres, seduced his unparalleled intellect when he was twenty-four years of age, just as he was getting well into his stride in those untravelled wildernesses which were to become the empire of modern mathematics.

Ceres was not alone to blame. The magnificent gift for mental arithmetic whose empirical discoveries had given mathematics the *Disquisitiones Arithmeticae* also played a fatal part in the tragedy. His friends and his father, too, were impatient with the young Gauss for not finding some lucrative position now that the Duke had educated him and, having no conception of the nature of the work which made the young man a silent recluse, thought him deranged. Here now at the dawn of the new century the opportunity which Gauss had lacked was thrust at him.

A new planet had been discovered in a position which made it extraordinarily difficult of observation. To compute an orbit from the meagre data available was a task which might have exercised Laplace himself. Newton had declared that such problems are among the most difficult in mathematical astronomy. The mere arithmetic necessary to establish an orbit with accuracy sufficient to ensure that Ceres on her whirl round the sun should not be lost to telescopes might well deter an electrically-driven calculating machine even to-day; but to the young man whose inhuman memory enabled him to dispense with a table of logarithms when he was hard pressed or too lazy to reach for one, all this endless arithmetic – *logistica,* not *arithmetica* – was the sport of an infant.

Why not indulge his dear vice, calculate as he had never calculated before, produce the difficult orbit to the sincere delight and wonderment of the dictators of mathematical fashion and thus make it possible, a year hence, for patient astronomers to rediscover Ceres in the place where the Newtonian law of gravitation decreed that she *must* be found – *if* the law were indeed a law of nature? Why not do all this, turn his back on the insubstantial vision of Archimedes and forget his own unsurpassed discoveries which lay waiting for development in his diary? Why not, in short, be popular? The Duke's generosity, always ungrudged, had nevertheless wounded the young man's pride in its most secret place; honour, recognition, acceptance as a 'great' mathematician in the fashion of the time with its probable sequel of financial independence – all these were now within his easy reach. Gauss, the mathematical god of all time, stretched forth his hand and plucked the Dead Sea fruits of a cheap fame in his own young generation.

For nearly twenty years the sublime dreams whose fugitive glimpses the boyish Gauss had pictured with unrestrained joy in his diary lay cold and all but forgotten. Ceres was rediscovered, precisely where the marvellously ingenious and detailed calculations of the young Gauss had predicted she must be found. Pallas, Vesta, and Juno, insignificant sister planets of the diminutive Ceres, were quickly picked up by prying telescopes defying Hegel, and their orbits, too, were found to conform to the inspired calculations of Gauss. Computations which would have taken Euler three days to perform – one such is sometimes said to have blinded him – were now the simple exercises of a few laborious hours. Gauss had prescribed the *method*, the routine. The major part of his own time for nearly twenty years was devoted to astronomical calculations.

But even such deadening work as this could not sterilize the creative genius of a Gauss. In 1809 he published his second masterpiece, *Theoria motus corporum coelestium in sectionibus conicis solem ambientium* (Theory of the Motion of the Heavenly Bodies Revolving round the Sun in Conic Sections), in which an exhaustive discussion of the determination of planetary and cometary orbits from observational data, including the difficult analysis of perturbations, lays down the law which for many years is to dominate computational and practical astronomy. It was great work, but not as great as Gauss was easily capable of had he developed the hints lying neglected in his diary. No essentially new discovery was added to *mathematics* by the *Theoria motus*.

Recognition came with spectacular promptness after the rediscovery of Ceres. Laplace hailed the young mathematician at once as an equal and presently as a superior. Some time later when the Baron Alexander von Humboldt (1769–1859), the famous traveller and amateur of the sciences, asked Laplace who was the greatest mathematician in Germany, Laplace replied 'Pfaff'. 'But what about Gauss?' the astonished Von Humboldt asked, as he was backing Gauss for the position of director at the Göttingen observatory. 'Oh', said Laplace, 'Gauss is the greatest mathematician in the world.'

The decade following the Ceres episode was rich in both happiness and sorrow for Gauss. He was not without detractors even at that early stage of his career. Eminent men who had the ear of the polite public ridiculed the young man of twenty-four for wasting his time on so useless a pastime as the computation of a minor planet's orbit. Ceres might be the goddess of the fields, but it was obvious to the merry wits that no corn grown on the new planet would ever find its way into the Brunswick market of a Saturday afternoon. No doubt they were right, but they also ridiculed him in the same way thirty years later when he laid the foundations of the mathematical theory of electromagnetism and invented the electric telegraph. Gauss let them enjoy their jests. He never replied publicly, but in private expressed his regret that men of honour and priests of science could stultify themselves by being so petty. In the meantime he went on with his work, grateful for the honours the learned societies of Europe showered on him but not going out of his way to invite them.

The Duke of Brunswick increased the young man's pension and made it possible for him to marry (9 October 1805) at the age of twenty-eight. The lady was Johanne Osthof of Brunswick. Writing to his old university friend, Wolfgang Bolyai, three days after he became engaged, Gauss expresses his unbelievable happiness. 'Life stands still before me like an eternal spring with new and brilliant colours.'

Three children were born of this marriage: Joseph, Minna, and Louis, the first of whom is said to have inherited his father's gift for mental calculations. Johanne died on 11 October 1809, after the birth of Louis, leaving her young husband desolate. His eternal spring turned to winter. Although he married again the following year (4 August 1810) for the sake of his young children it was long before Gauss could speak without emotion of his first wife. By the second wife, Minna Waldeck, who had been a close friend of the first, he had two sons and a daughter.

According to gossip Gauss did not get on well with his sons, except possibly the gifted Joseph who never gave his father any trouble. Two are said to have run away from home and gone to the United States. As one of these sons is said to have left numerous descendants still living in America, it is impossible to say anything further here, except that one of the American sons became a prosperous merchant in St Louis in the days of the river boats; both first were farmers in Missouri. With his daughters Gauss was always happy. An exactly contrary legend (vouched for forty years ago by old people whose memories of the Gauss family might be considered trustworthy) to that about the sons asserts that Gauss was never anything but kind to his boys, some of whom were rather wild and caused their distracted father endless anxiety. One would think that the memory of his own father would have made Gauss sympathetic with his sons.

In 1808 Gauss lost his father. Two years previously he had suffered an even severer loss in the death of his benefactor under tragic circumstances.

The Duke Ferdinand was not only an enlightened patron of learning and a kindly ruler but a first-rate soldier as well who had won the warm praise of Frederick the Great for his bravery and military brilliance in the Seven Years' War (1756–63).

At the age of seventy Ferdinand was put in command of the Prussian forces in a desperate attempt to halt the French under Napoleon, after the Duke's mission to St Petersburg in an effort to enlist the aid of Russia for Germany had failed. The battle of Austerlitz (2 December 1805) was already history and Prussia found itself forsaken in the face of overwhelming odds. Ferdinand faced the French on their march toward the Saale at Auerstedt and Jena, was disastrously defeated and himself mortally wounded. He turned homeward.

Napoleon the Great here steps on the stage in person at his pot-bellied greatest. At the time of Ferdinand's defeat Napoleon was quartered at Halle. A deputation from Brunswick waited on the victorious Emperor of all the French to implore his generosity for the brave old man he had defeated. Would the mighty Emperor stretch a point of military etiquette and let his broken enemy die in peace by his own fireside? The Duke, they assured him, was no longer dangerous. He was dying.

It was the wrong time of the month and Napoleon was enjoying one of his womanish tantrums. He not only refused but did so with quite vulgar and unnecessary brutality. Revealing the true measure of himself as a man, Napoleon pointed his refusal with a superfluous vilification of his honourable opponent and a hysterical ridicule of the dying man's abilities as a soldier. There was nothing for the humiliated deputation to do but to try to save their gentle ruler from the disgrace of a death in prison. It does not seem surprising that these same Germans some nine years later fought like methodical devils at Waterloo and helped to topple the Emperor of the French into the ditch.

Gauss at the time was living in Brunswick. His house was on the main highway. One morning in late autumn he saw a hospital wagon hastening by. In it lay the dying Duke on his flight to Altona. With an emotion too deep for words Gauss saw the man who had been more than his own father to him hurried away to die in hiding like a hounded criminal. He said nothing then and but little afterwards, but his friends noticed that his reserve deepened and his always serious nature became more serious. Like Descartes in his earlier years Gauss had a horror of death, and all his life the passing of a close friend chilled him with a quiet, oppressive dread. Gauss was too vital to die or to witness death. The Duke died in his father's house in Altona on 10 November 1806.

His generous patron dead, it became necessary for Gauss to find some reliable livelihood to support his family. There was no difficulty about this as the young mathematician's fame had now spread to the farthest corners of Europe. St Petersburg had been angling for him as the logical successor of Euler who had never been worthily replaced after his death in 1783. In 1807 a definite and flattering offer was tendered Gauss. Alexander von Humboldt and other influential friends, reluctant to see Germany lose the greatest mathematician in the world, bestirred themselves, and Gauss was appointed director of the

Göttingen Observatory with the privilege – and duty, when necessary – of lecturing on mathematics to university students.

Gauss no doubt might have obtained a professorship of mathematics, but he preferred the observatory as it offered better prospects for uninterrupted research. Although it may be too strong to say that Gauss hated teaching, the instruction of ordinary students gave him no pleasure, and it was only when a real mathematician sought him out that Gauss, sitting at a table with his students, let himself go and disclosed the secrets of his methods in his perfectly prepared lessons. But such incentives were regrettably rare and for the most part the students who took up Gauss' priceless time had better have been doing something other than mathematics. Writing in 1810 to his intimate friend the astronomer and mathematician Friedrich Wilhelm Bessel (1784–1846), Gauss says 'This winter I am giving two courses of lectures to three students, of whom one is only moderately prepared, the other less than moderately and the third lacks both preparation and ability. Such are the burdens of a mathematical calling.'

The salary which Göttingen could afford to pay Gauss at the time – the French were then busy pillaging Germany in the interests of good government for the Germans by the French – was modest but sufficient for the simple needs of Gauss and his family. Luxury never attracted the Prince of Mathematicians whose life had been unaffectedly dedicated to science long before he was twenty. As his friend Sartorius von Waltershausen writes, 'As he was in his youth, so he remained through his old age to his dying day, the unaffectedly simple Gauss. A small study, a little work table with a green cover, a standing-desk painted white, a narrow sopha and, after his seventieth year, an arm chair, a shaded lamp, an unheated bedroom, plain food, a dressing gown and a velvet cap, these were so becomingly all his needs.'

If Gauss was simple and thrifty the French invaders of Germany in 1807 were simpler and thriftier. To govern Germany according to their ideas the victors of Auerstedt and Jena fined the losers more than the traffic would bear. As professor and astronomer at Göttingen Gauss was rated by the extortionists to be good for an involuntary contribution of 2,000 francs to the Napoleonic war chest. This exorbitant sum was quite beyond Gauss' ability to pay.

Presently Gauss got a letter from his astronomical friend Olbers enclosing the amount of the fine and expressing indignation that a scholar should be subjected to such petty extortion. Thanking his generous friend for his sympathy, Gauss declined the money and sent it back at once to the donor.

Not all the French were as thrifty as Napoleon. Shortly after returning Olbers' money Gauss received a friendly little note from Laplace telling him that the famous French mathematician had paid the 2,000-franc fine for the greatest mathematician in the world and had considered it an honour to be able to lift this unmerited burden from his friend's shoulders. As Laplace had paid the fine in Paris, Gauss was unable to return him the money. Nevertheless he declined to accept Laplace's help. An unexpected (and unsolicited) windfall was presently to enable him to repay Laplace with interest at the current market rate. Word must have got about that Gauss disdained charity. The next attempt to help him succeeded. An admirer in Frankfurt sent 1,000 guilders anonymously. As Gauss could not trace the sender he was forced to accept the gift.

The death of his friend Ferdinand, the wretched state of Germany under French looting, financial straits, and the loss of his first wife all did their part toward upsetting Gauss' health and making his life miserable in his early thirties. Nor did a constitutional predisposition to hypochondria, aggravated by incessant overwork, help matters. His unhappiness was never shared with his friends, to whom he is always the serene correspondent, but is confided – only once – to a private mathematical manuscript. After his appointment to the directorship at Göttingen in 1807 Gauss returned occasionally for three years to one of the great things noted in his diary. In a manuscript on elliptic functions purely scientific matters are suddenly interrupted by the finely pencilled words 'Death were dearer to me than such a life.' His work became his drug.

The years 1811–12 (Gauss was thirty-four in 1811) were brighter. With a wife again to care for his young children Gauss began to have some peace. Then, almost exactly a year after his second marriage, the great comet of 1811, first observed by Gauss deep in the evening twilight of 22 August, blazed up unannounced. Here was a worthy foe to test the weapons Gauss had invented to subjugate the minor planets.

His weapons proved adequate. While the superstitious peoples of Europe, following the blazing spectacle with awe-struck eyes as the comet unlimbered its flaming scimitar in its approach to the Sun, saw in the fiery blade a sharp warning from Heaven that the King of Kings was wroth with Napoleon and weary of the ruthless tyrant, Gauss had the satisfaction of seeing the comet follow the path he had quickly calculated for it to the last decimal. The following year the credulous also saw their own prediction verified in the burning of Moscow and the destruction of Napoleon's Grand Army on the icy plains of Russia.

This is one of those rare instances where the popular explanation fits the facts and leads to more important consequences than the scientific. Napoleon himself had a basely credulous mind – he relied on 'hunches', reconciled his wholesale slaughters with a childlike faith in a beneficent, inscrutable Providence, and believed himself a Man of Destiny. It is not impossible that the celestial spectacle of a harmless comet flaunting its gorgeous tail across the sky left its impress on the subconscious mind of a man like Napoleon and fuddled his judgement. The almost superstitious reverence of such a man for mathematics and mathematicians is no great credit to either, although it has been frequently cited as one of the main justifications for both.

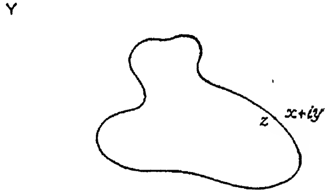
Beyond a rather crass appreciation of the value of mathematics in military affairs, where its utility is obvious even to a blind idiot, Napoleon had no conception of what mathematics as practised by masters like his contemporaries, Lagrange, Laplace, and Gauss, was all about. A quick student of trivial, elementary mathematics at school, Napoleon turned to other things too early to certify his promise and, mathematically, never grew up. Although it seems incredible that a man of Napoleon's demonstrated capacity could so grossly under-estimate the difficulties of matters beyond his comprehension as to patronize Laplace, it is a fact that he had the ludicrous audacity to assure the author of the *Mécanique céleste* that he would read the book the *first free month* he could find. Newton and Gauss might have been equal to the task; Napoleon no doubt could have turned the pages in his month without greatly tiring himself.

It is a satisfaction to record that Gauss was too proud to prostitute mathematics to Napoleon the Great by appealing to the Emperor's vanity and begging him in the name of his notorious respect for all things mathematical to remit the 2,000-franc fine, as some of Gauss' mistaken friends urged him to do. Napoleon would probably have been flattered to exercise his clemency. But Gauss could not forget Ferdinand's death, and he felt that both he and the mathematics he worshipped were better off without the condescension of a Napoleon.

No sharper contrast between the mathematician and the military genius can be found than that afforded by their respective attitudes to a broken enemy. We have seen how Napoleon treated Ferdinand. When Napoleon fell Gauss did not exult. Calmly and with a detached interest he read everything he could find about Napoleon's life and did his best to understand the workings of a mind like Napoleon's. The effort even gave him considerable amusement. Gauss had a keen sense of humour, and the blunt realism which he had inherited from his hard-working peasant ancestors also made it easy for him to smile at heroics.

The year 1811 might have been a landmark in mathematics comparable to 1801 – the year in which the *Disquisitiones Arithmeticae* appeared – had Gauss made public a discovery he confided to Bessel. Having thoroughly understood complex numbers and their geometrical representation as points on the plane of analytic geometry, Gauss proposed himself the problem of investigating what are to-day called *analytic functions* of such numbers.

The complex number $x + iy$, where i denotes $\sqrt{-1}$, represents the point (x, y). For brevity $x + iy$ will be denoted by the single letter z. As x, y independently take on real values in any prescribed continuous manner, the point z wanders about over the plane, obviously not at random but in a manner determined by that in which x, y assume their values. Any expression containing z, such as z^2, or $1/z$, etc., which takes on a *single* definite value when a value is assigned to z, is called a *uniform function* of z. We shall denote such a function by $f(z)$. Thus if $f(z)$ is the particular function z^2, so that here $f(z) = (x + iy)^2 = x^2 + 2ixy + i^2y^2, = x^2 - y^2 + 2ixy$ (because $i^2 = -1$), it is clear



that when any value is assigned to z, namely to $x + iy$, for example $x = 2$, $y = 3$, so that $z = 2 + 3i$, precisely one value of this $f(z)$ is thereby determined; here, for $z = 2 + 3i$ we get $z^2 = -5 + 12i$.

Not all uniform functions $f(z)$ are studied in the theory of functions of a complex variable; the *monogenic* functions are singled out for exhaustive discussion. The reason for this will be stated after we have described what 'monogenic' means.

Let z move to another position, say to z'. The function $f(z)$ takes on another value, $f(z')$, obtained by substituting z' for z. The *difference* $f(z') - f(z)$ of the new and old values of the function is now divided by the difference of the new and old values of the variable, thus $[f(z') - f(z)]/(z' - z)$, and, precisely as is done in calculating the slope of a graph to find the derivative of the function the graph represents, we here let z' approach z

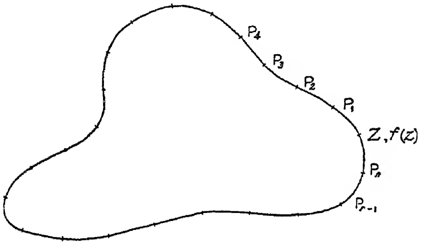
indefinitely, so that $f(z')$ approaches $f(z)$ simultaneously. But here a remarkable new phenomenon appears.

There is not here a unique way in which z' can move into coincidence with z, for z' may wander about all over the plane of complex numbers by any of an infinity of different paths before coming into coincidence with z. We should not expect the limiting value of $[f(z') - f(z)]/(z' - z)$ when z' coincides with z to be *the same* for *all* of these paths, and in general it is *not*. But if $f(z)$ is such that the limiting value just described *is* the same for *all* paths by which z' moves into coincidence with z, then $f(z)$ is said to be monogenic at z (or at the point representing z). *Uniformity* (previously described) and *monogenicity* are distinguishing features of *analytic* functions of a complex variable.

Some idea of the importance of analytic functions can be inferred from the fact that vast tracts of the theories of fluid motion (also of mathematical electricity and representation by maps which do not distort angles) are naturally handled by the theory of *analytic* functions of a complex variable. Suppose such a function $f(z)$ is separated into its 'real' part (that which does not contain the 'imaginary unit' i) and its 'imaginary' part, say $f(z) = U + iV$. For the special analytic function z^2 we have $U = x^2 - y^2$, $V = 2xy$. Imagine a film of fluid streaming over a plane. If the motion of the fluid is without vortices, a stream line of the motion is obtainable from *some* analytic function $f(z)$ by plotting the curve $U = a$, in which a is any real number, and likewise the equipotential lines are obtainable from $V = b$ (b any real number). Letting a, b range, we thus get a complete picture of the motion for as large an area as we wish. For a given situation, say that of a fluid streaming round an obstacle, the hard part of the problem is to find what analytic function to choose, and the whole matter has been gone at largely backwards: the simple analytic functions have been investigated and the physical problems which they fit have been sought. Curiously enough, many of these artificially prepared problems have proved of the greatest service in aerodynamics and other practical applications of the theory of fluid motion.

The theory of analytic functions of a complex variable was one of the greatest fields of mathematical triumphs in the nineteenth century. Gauss in his letter to Bessel states what amounts to the fundamental theorem in this vast theory, but he hid it away to be rediscovered by Cauchy and later Weierstrass. As this is a landmark in the history of mathematical analysis we shall briefly describe it, omitting all refinements that would be demanded in an exact formulation.

Imagine the complex variable z tracing out a closed curve of finite length without loops or kinks. We have an intuitive notion of what we mean by the 'length' of a piece of this curve.



Mark n points $P_1, P_2, \ldots, P_n$ on the curve so that each of the pieces $P_1P_2, P_2P_3, P_3P_4, \ldots, P_nP_1$ is not greater than some preassigned finite length l. On each of these pieces choose a point, not at either end of the piece; form the value of $f(z)$ for the value of z corresponding to the point, and multiply this value by the length of the piece in which the point lies. Do the like for *all* the pieces, and add the results. Finally take the limiting value of this sum as the number of pieces is indefinitely increased. This gives the '*line integral*' of $f(z)$ for the curve.

When will this line integral be zero? In order that the line integral shall be zero it is sufficient that $f(z)$ be *analytic* (uniform and monogenic) at every point z on the curve and inside the curve.

Such is the great theorem which Gauss communicated to Bessel in 1811 and which, with another theorem of a similar kind, in the hands of Cauchy who rediscovered it independently, was to yield many of the important results of analysis as corollaries.

Astronomy did not absorb the whole of Gauss' prodigious energies in his middle thirties. The year 1812, which saw Napoleon's Grand Army fighting a desperate rear-guard action across the frozen plains, witnessed the publication of another great work by Gauss, that on the *hypergeometric series*

$$1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)x^2}{c(c+1)1 \times 2} + \cdots,$$

the dots meaning that the series continues indefinitely according to the law indicated; the next term is

$$\frac{a(a+1)(a+2)b(b+1)(b+2) \qquad x^3}{c(c+1)(c+2) \qquad 1 \times 2 \times 3}.$$

This memoir is another landmark. As has already been noted Gauss was the first of the modern rigorists. In this work he determined the restrictions that must be imposed on the numbers a, b. c, x in order that the series shall converge (in the sense explained earlier in this chapter). The series itself was no mere textbook exercise that may be investigated to gain skill in analytical manipulations and then be forgotten. It includes as special cases – obtained by assigning specific values to one or more of a, b, c, x – many of the most important series in analysis, for example those by which logarithms, the trigonometric functions, and several of the functions that turn up repeatedly in Newtonian astronomy and mathematical physics are calculated and tabulated; the general binomial theorem also is a special case. By disposing of this series in its general form Gauss slew a multitude at one smash. From this work developed many applications to the differential equations of physics in the nineteenth century.

The choice of such an investigation for a serious effort is characteristic of Gauss. He never published trivialities. When he put out anything it was not only finished in itself but was also so crammed with ideas that his successors were enabled to apply what Gauss had invented to new problems. Although limitations of space forbid discussion of the many instances of this fundamental character of Gauss' contributions to pure mathematics, one cannot be passed over in even the briefest sketch: the work on the law of biquadratic reciprocity. The importance of this was that it gave a new and totally unforeseen direction to the higher arithmetic.

Having disposed of *quadratic* (second degree) reciprocity, it was natural for Gauss to consider the general question of binomial congruences of any degree. If m is a given integer not divisible by the prime p, and if n is a given positive integer, and if further an integer x can be found such that $x^n \equiv m \pmod{p}$, m is called an *n-ic residue* of p; when $n = 4$, m is a *biquadratic residue* of p.

The case of *quadratic* binomial congruences ($n = 2$) suggests but little to do when n exceeds 2. One of the matters Gauss was to have included in the discarded eighth section (or possibly, as he told Sophie Germain, in the projected but unachieved second volume) of the *Disquisitiones Arithmeticae* was a discussion of these higher congruences and a search for the corresponding laws of reciprocity, namely the interconnexions (as to solvability or non-solvability) of the pair $x^n \equiv p \pmod{q}$, $x^n \equiv q \pmod{p}$, where p, q are rational primes. In particular the cases $n = 3$, $n = 4$ were to have been investigated.

The memoir of 1825 breaks new ground with all the boldness of the great pioneers. After many false starts which led to intolerable complexity Gauss discovered the 'natural' way to the heart of his problem. The *rational* integers 1, 2, 3, ... are *not* those appropriate to the statement of the law of *biquadratic* reciprocity, as they are for *quadratic*; a totally new species of *integers* must be invented. These are called the *Gaussian complex integers* and are all those complex numbers of the form $a + bi$ in which a, b are *rational integers* and i denotes $\sqrt{-1}$.

To state the law of biquadratic reciprocity an exhaustive preliminary discussion of the laws of arithmetical divisibility for such *complex integers* is necessary. Gauss gave this, thereby inaugurating the theory of algebraic numbers – that which he probably had in mind when he gave his estimate of Fermat's Last Theorem. For *cubic* reciprocity ($n = 3$) he also found the right way in a similar manner. His work on this was found in his posthumous papers.

The significance of this great advance will become clearer when we follow the careers of Kummer and Dedekind. For the moment it is sufficient to say that Gauss' favourite disciple, Eisenstein, disposed of cubic reciprocity. He further discovered an astonishing connexion between the law of biquadratic reciprocity and certain parts of the theory of elliptic functions, in which Gauss had travelled far but had refrained from disclosing what he found.

Gaussian complex *integers* are of course a sub-class of *all* complex *numbers*, and it might be thought that the *algebraic* theory of *all* the numbers would yield the *arithmetical* theory of the included *integers* as a trivial detail. Such is by no means the case. Compared to the arithmetical theory the algebraic is childishly easy. Perhaps a reason why this should be so is suggested by the *rational numbers* (numbers of the form a/b where a, b are rational integers). We can *always* divide one rational number by another and get *another* rational number; a/b divided by c/d yields the rational number ad/bc. But a rational *integer* divided by another rational integer is not always another rational integer: 7 divided by 8 gives $\frac{7}{8}$. Hence if we must restrict ourselves to *integers*, the case of interest for the theory of numbers, we have tied our hands and hobbled our feet before we start. This is one of the reasons why the higher arithmetic is harder than algebra, higher or elementary.

Equally significant advances in geometry and the applications of mathematics to geodesy, the Newtonian theory of attraction, and electromagnetism were also to be made by Gauss. How was it possible for one man to accomplish this colossal mass of work of the highest order? With characteristic modesty Gauss declared that 'If others would but reflect on mathematical truths as deeply and as continuously as I have, they would make my discoveries.' Possibly. Gauss' explanation recalls Newton's. Asked how he had made discoveries in astronomy surpassing those of all his predecessors, Newton replied, 'By always thinking about them'. This may have been plain to Newton; it is not to ordinary mortals.

Part of the riddle of Gauss is answered by his *involuntary* preoccupation with mathematical ideas – which itself of course demands explanation. As a young man Gauss would be 'seized' by mathematics. Conversing with friends he would suddenly go silent, overwhelmed by thoughts beyond his control, and stand staring rigidly oblivious of his surroundings. Later he controlled his thoughts – or they lost their control over him – and he consciously directed all his energies to the solution of a difficulty till he succeeded. A problem once grasped was never released till he had conquered it, although several might be in the foreground of his attention simultaneously.

In one such instance (referring to the *Disquisitiones*, page 636) he relates how for four years scarcely a week passed that he did not spend some time trying to settle whether a certain sign should be plus or minus. The solution finally came of itself in a flash. But to imagine that it would have blazed out of itself like a new star without the 'wasted' hours is to miss the point entirely. Often after spending days or weeks fruitlessly over some research Gauss would find on resuming work after a sleepless night that the obscurity had vanished and the whole solution shone clear in his mind. The capacity for intense and prolonged concentration was part of his secret.

In this ability to forget himself in the world of his own thoughts Gauss resembles both Archimedes and Newton. In two further respects he also measures up to them, his gifts for precise observation and a scientific inventiveness which enabled him to devise the instruments necessary for his scientific researches. To Gauss geodesy owes the invention of the heliotrope, an ingenious device by which signals could be transmitted practically instantaneously by means of reflected light. For its time the heliotrope was a long step forward. The astronomical instruments he used also received notable improvements at Gauss' hands. For use in his fundamental researches in electromagnetism Gauss invented the bifilar magnetometer. And as a final example of his mechanical ingenuity it may be recalled that Gauss in 1833 invented the electric telegraph and that he and his fellow worker Wilhelm Weber (1804–91) used it as a matter of course in sending messages. The combination of mathematical genius with first-rate experimental ability is one of the rarest in all science.

Gauss himself cared but little for the possible practical uses of his inventions. Like Archimedes he preferred mathematics to all the kingdoms of the earth; others might gather the tangible fruits of his labours. But Weber, his collaborator in electromagnetic researches, saw clearly what the puny little telegraph of Göttingen meant for civilization. The railway, we recall, was just coming into its own in the early 1830's. 'When the globe is covered with a net of railroads and telegraph wires', Weber prophesied in 1835, 'this net will render services comparable to those of the nervous system in the human body, partly as a means of transport, partly as a means for the propagation of ideas and sensations with the speed of lightning.'

The admiration of Gauss for Newton has already been noted. Knowing the tremendous efforts some of his own masterpieces had cost him, Gauss had a true appreciation of the long preparation and incessant meditation that went into Newton's greatest work. The story of Newton and the falling apple roused Gauss' indignation. 'Silly!' he exclaimed. 'Believe the story if you like, but the truth of the matter is this. A stupid, officious man asked Newton how he discovered the law of gravitation. Seeing that he had to deal with a child in intellect, and wanting to get rid of the bore, Newton answered that an apple fell and hit him on the nose. The man went away fully satisfied and completely enlightened.'

The apple story has its echo in our own times. When teased as to what led him to his theory of the gravitational field Einstein replied that he asked a workman who had fallen off a building, to land unhurt on a pile of straw, whether he noticed the tug of the 'force' of gravity on the way down. On being told that no force had tugged, Einstein immediately saw that 'gravitation' in a sufficiently small region of space-time can be replaced by an acceleration of the observer's (the falling workman's) reference system. This story, if true, is also probably all rot. What gave Einstein his idea was the hard labour he expended for several years mastering the tensor calculus of two Italian mathematicians, Ricci and Levi-Civita, themselves disciples of Riemann and Christoffel, both of whom in their turn had been inspired by the geometrical work of Gauss.

Commenting on Archimedes, for whom he also had a boundless admiration, Gauss remarked that he could not understand how Archimedes failed to invent the decimal system of numeration or its equivalent (with some base other than 10). The thoroughly un-Greek work of Archimedes in devising a scheme for writing and dealing with numbers far beyond the capacity of the Greek symbolism had – according to Gauss – put the decimal notation with its all-important principle of place-value ($325 = 3 \times 10^2 + 2 \times 10 + 5$) in Archimedes' hands. This oversight Gauss regarded as the greatest calamity in the history of science. 'To what heights would science now be raised if Archimedes had made that discovery!' he exclaimed, thinking of his own masses of arithmetical and astronomical calculations which would have been impossible, even to him, without the decimal notation. Having a full appreciation of the significance for all science of improved methods of computation, Gauss slaved over his own calculations till pages of figures were reduced to a few lines which could be taken in almost at a glance. He himself did much of his calculating mentally; the improvements were intended for those less gifted than himself.

Unlike Newton in his later years, Gauss was never attracted by the rewards of public office, although his keen interest and sagacity in all matters pertaining to the sciences of statistics, insurance, and 'political arithmetic' would have made him a good minister of finance. Till his last illness he found complete satisfaction in his science and his simple recreations. Wide reading in the literatures of Europe and the classics of antiquity, a critical interest in world politics, and the mastery of foreign languages and new sciences (including botany and mineralogy) were his hobbies.

English literature especially attracted him, although its darker aspect as in Shakespeare's tragedies was too much for the great mathematician's acute sensitiveness to all forms of suffering, and he tried to pick his way through the happier masterpieces. The novels of Sir Walter Scott (who was a contemporary of Gauss) were read eagerly as they came out, but the unhappy ending of *Kenilworth* made Gauss wretched for days and he regretted having read the story. One slip of Sir Walter's tickled the mathematical astronomer into delighted laughter, 'the moon rises broad in the northwest', and he went about for days correcting all the copies he could find. Historical works in English, particularly Gibbon's *Decline and Fall of the Roman Empire* and Macaulay's *History of England*, gave him special pleasure.

For his meteoric young contemporary Lord Byron, Gauss had almost an aversion. Byron's posturing, his reiterated world-weariness, his affected misanthropy, and his romantic good looks had captivated the sentimental Germans even more completely than they did the stolid British who – at least the older males – thought Byron somewhat of a silly ass. Gauss saw through Byron's histrionics and disliked him. No man who guzzled good brandy and pretty women as assiduously as Byron did could be so very weary of the world as the naughty young poet with the flashing eye and the shaking hand pretended to be.

In the literature of his own country Gauss' tastes were somewhat unusual for an intellectual German. Jean Paul was his favourite German poet; Goethe and Schiller, whose lives partly overlapped his own, he did not esteem very highly. Goethe, he said, was unsatisfying. Being completely at variance with Schiller's philosophical tenets, Gauss disliked his poetry. He called *Resignation* a blasphemous, corrupt poem and wrote 'Mephistopheles!' on the margin of his copy.

The facility with which he mastered languages in his youth stayed with Gauss all his life. Languages were rather more to him than a hobby. To test the plasticity of his mind as he grew older he would deliberately acquire a new language. The exercise, he believed, helped to keep his mind young. At the age of sixty-two he began an intensive study of Russian without assistance from anyone. Within two years he was reading Russian prose and poetical works fluently, and carrying on his correspondence with scientific friends in St Petersburg wholly in Russian. In the opinion of Russians who visited him in Göttingen he also spoke the language perfectly. Russian literature he put on a par with English for the pleasure it gave him. He also tried Sanskrit but disliked it.

His third hobby, world politics, absorbed an hour or so of his time every day. Visiting the literary museum regularly, he kept abreast of events by reading all the newspapers to which the museum subscribed, from the London *Times* to the Göttingen local news.

In politics the intellectual aristocrat Gauss was conservative through and through, but in no sense reactionary. His times were turbulent, both in his own country and abroad. Mob rule and acts of political violence roused in him – as his friend Von Waltershausen reports – 'an indescribable horror'. The Paris revolt of 1848 filled him with dismay.

The son of poor parents himself, familiar from infancy with the intelligence and morality of 'the masses', Gauss remembered what he had observed, and his opinion of the intelligence, morality, and political acumen of 'the people' – taken in the mass, as demagogues find and take them – was extremely low. '*Mundus vult decipi*' he believed a true saying.

This disbelief in the innate morality, integrity, and intelligence of Rousseau's 'natural man' when massed into a mob or when deliberating in cabinets, parliaments, congresses, and senates, was no doubt partly inspired by Gauss' intimate knowledge, as a man of science, of what 'the natural man' did to the scientists of France in the early days of the French Revolution. It may be true, as the revolutionists declared, that 'the people have no need of science', but such a declaration to a man of Gauss' temperament was a challenge. Accepting the challenge, Gauss in his turn expressed his acid contempt for all 'leaders of the people' who lead the people into turmoil for their own profit. As he aged he saw peace and simple contentment as the only good things in any country. Should civil war break out in Germany, he said, he would as soon be dead. Foreign conquest in the grand Napoleonic manner he looked upon as an incomprehensible madness.

These conservative sentiments were not the nostalgia of a reactionary who bids the world defy the laws of celestial mechanics and stand still in the heavens of a dead and unchanging past. Gauss believed in reforms – when they were intelligent. And if brains are not to judge when reforms are intelligent and when they are not, what organ of the human body is ? Gauss had brains enough to see where the ambitions of some of the great statesmen of his own reforming generation were taking Europe. The spectacle did not inspire his confidence.

His more progressive friends ascribed Gauss' conservatism to the closeness with which he stuck to his work. This may have had something to do with it. For the last twenty-seven years of his life Gauss slept away from his observatory only once, when he attended a scientific meeting in Berlin to please Alexander von Humboldt who wished to show him off. But a man does not always have to be flying about all over the map to see what is going on. Brains and the ability to read newspapers (even when they lie) and government reports (especially when they lie) are sometimes better than any amount of sightseeing and hotel lobby gossip. Gauss stayed at home, read, disbelieved most of what he read, thought, and arrived at the truth.

Another source of Gauss' strength was his scientific serenity and his freedom from personal ambition. All his ambition was for the advancement of mathematics. When rivals doubted his assertion that he had anticipated them – not stated boastfully, but as a fact germane to the matter in hand – Gauss did not exhibit his diary to prove his priority but let his statement stand on its own merits.

Legendre was the most outspoken of these doubters. One experience made him Gauss' enemy for life. In the *Theoria motus* Gauss had referred to his early discovery of the method of least squares. Legendre published the method in 1806, before Gauss. With great indignation he wrote to Gauss practically accusing him of dishonesty and complaining that Gauss, so rich in discoveries, might have had the decency not to appropriate the method of least squares, which Legendre regarded as his own ewe lamb. Laplace entered the quarrel. Whether he believed the assurances of Gauss that Legendre had indeed been anticipated by ten years or more, he does not say, but he retains his usual suavity. Gauss apparently disdained to argue the matter further. But in a letter to a friend he indicates the evidence which might have ended the dispute then and there had Gauss not been 'too proud to fight'. 'I communicated the whole matter to Olbers in 1802', he says, and if Legendre had been inclinded to doubt this he could have asked Olbers, who had the manuscript.

The dispute was most unfortunate for the subsequent development of mathematics, as Legendre passed on his unjustified suspicions to Jacobi and so prevented that dazzling young developer of the theory of elliptic functions from coming to cordial terms with Gauss. The misunderstanding was all the more regrettable because Legendre was a man of the highest character and scrupulously fair himself. It was his fate to be surpassed by more imaginative mathematicians than himself in the fields where most of his long and laborious life was spent in toil which younger men – Gauss, Abel, and Jacobi – showed to have been superfluous. At every step Gauss strode far ahead of Legendre. Yet when Legendre accused him of unfair dealing Gauss felt that he himself had been left in the lurch. Writing to Schumacher (30 July 1806), he complains that 'It seems to be my fate to concur in nearly all my theoretical works with Legendre. So it is in the higher arithmetic, in the researches in transcendental functions connected with the rectification [the process for finding the length of an arc of a curve] of the ellipse, in the foundations of geometry and now again here [in the method of least squares, which] ... is also used in Legendre's work and indeed right gallantly carried through.'

With the detailed publication of Gauss' posthumous papers and much of his correspondence in recent years all these old disputes have been settled once for all in favour of Gauss. There remains another score on which he has been criticized, his lack of cordiality in welcoming the great work of others, particularly of younger men. When Cauchy began publishing his brilliant discoveries in the theory of functions of a complex variable, Gauss ignored them. No word of praise or encouragement came from the Prince of Mathematicians to the young Frenchman. Well, why should it have come? Gauss himself (as we have seen) had reached the heart of the matter years before Cauchy started. A memoir on the theory was to have been one of Gauss' masterpieces. Again, when Hamilton's work on quaternions (to be considered in a later chapter) came to his attention in

1852, three years before his death, Gauss said nothing. Why should he have said anything? The crux of the matter lay buried in his notes of more than thirty years before. He held his peace and made no claim for priority. As in his anticipations of the theory of functions of a complex variable, elliptic functions, and non-Euclidean geometry, Gauss was content to have done the work.

The gist of quaternions is the algebra which does for rotations in space of three dimensions what the algebra of complex numbers does for rotations in a plane. But in quaternions (Gauss called them mutations) one of the fundamental rules of algebra breaks down: it is no longer true that $a \times b = b \times a$, and it is impossible to make an algebra of rotations in three dimensions in which this rule *is* preserved. Hamilton, one of the great mathematical geniuses of the nineteenth century, records with Irish exuberance how he struggled for fifteen years to invent a consistent algebra to do what was required until a happy inspiration gave him the clue that $a \times b$ is not equal to $b \times a$ in the algebra he was seeking. Gauss does not state how long it took him to reach the goal; he merely records his success in a few pages of algebra that leave no mathematics to the imagination.

If Gauss was somewhat cool in his printed expressions of appreciation he was cordial enough in his correspondence and in his scientific relations with those who sought him out in a spirit of disinterested inquiry. One of his scientific friendships is of more than mathematical interest as it shows the liberality of Gauss' views regarding women scientific workers. His broad-mindedness in this respect would have been remarkable for any man of his generation; for a German it was almost without precedent.

The lady in question was Mademoiselle Sophie Germain (1776–1831) – just a year older than Gauss. She and Gauss never met, and she died (in Paris) before the University of Göttingen could confer the honorary doctor's degree which Gauss recommended to the faculty. By a curious coincidence we shall see the most celebrated woman mathematician of the nineteenth century, another Sophie, getting her degree from the same liberal University many years later after Berlin had refused her on account of her sex. Sophie appears to be a lucky name in mathematics for women – provided they affiliate with broadminded teachers. The leading woman mathematician of our own times, Emmy Noether (1882–1935) also came from Göttingen.*

Sophie Germain's scientific interests embraced acoustics, the mathematical theory of elasticity, and the higher arithmetic, in all of which she did notable work. One contribution in particular to the study of Fermat's Last Theorem led in 1908 to a considerable advance in this direction by the American mathematician Leonard Eugene Dickson (1874–).

Entranced by the *Disquisitiones Arithmeticae,* Sophie wrote to Gauss some of her own arithmetical observations. Fearing that Gauss might be prejudiced against a woman mathematician, she assumed a man's name. Gauss formed a high opinion of the talented correspondent whom he addressed in excellent French as 'Mr Leblanc'.

Leblanc dropped her – or his – disguise when she was forced to divulge her true name to Gauss on the occasion of her having done him a good turn with the French infesting Hanover. Writing on 30 April 1807, Gauss thanks his correspondent for her intervention on his behalf with the French General Pernety and deplores the war. Continuing, he pays her a high compliment and expresses something of his own love for the theory of numbers. As the latter is particularly of interest we shall quote from this letter which shows Gauss in one of his cordially human moods.

'But how describe to you my admiration and astonishment at seeing my esteemed correspondent Mr Leblanc metamorphose himself into this illustrious personage (Sophie Germain) who gives such a brilliant example of what I would find it difficult

* 'Came from' is right. When the sagacious Nazis expelled Fräulein Noether from Germany because she was a Jewess, Bryn Mawr College, Pennsylvania, took her in. She was the most creative abstract algebraist in the world. In less than a week of the new German enlightenment, Göttingen lost the liberality which Gauss cherished and which he strove all his life to maintain.

to believe. A taste for the abstract sciences in general and above all the mysteries of numbers is excessively rare: one is not astonished at it; the enchanting charms of this sublime science reveal themselves only to those who have the courage to go deeply into it. But when a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men to familiarize herself with these thorny researches, succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of them, then without doubt she must have the noblest courage, quite extraordinary talents and a superior genius. Indeed nothing could prove to me in so flattering and less equivocal manner that the attractions of this science, which has enriched my life with so many joys, are not chimerical, as the predilection with which *you* have honoured it.' He then goes on to discuss mathematics with her. A delightful touch is the date at the end of the letter: 'Bronsvic ce 30 Avril 1807 jour de ma naissance – Brunswick, this 30th of April, 1807, my birthday.'

That Gauss was not merely being polite to a young woman admirer is shown by a letter of 21 July 1807 to his friend Olbers. '... Lagrange is warmly interested in astronomy and the higher arithmetic; the two test-theorems (for what primes 2 is a cubic or a biquadratic residue), which I also communicated to him some time ago, he considers "among the most beautiful things and the most difficult to prove". But Sophie Germain has sent me the proofs of these; I have not yet been able to go through them, but I believe they are good; at least she had attacked the matter from the right side, only somewhat more diffusely than would be necessary. ...' The theorems to which Gauss refers are those stating for what odd primes p each of the congruences $x^3 \equiv 2 \pmod{p}$, $x^4 \equiv 2 \pmod{p}$ is solvable.

It would take a long book (possibly a longer one than would be required for Newton) to describe all the outstanding contributions of Gauss to mathematics, both pure and applied. Here we can only refer to some of the more important works that have not already been mentioned, and we shall select those which have added new techniques to mathematics or which rounded off outstanding problems. As a rough but convenient table of dates (from that adapted by the editors of Gauss' works) we summarize the principal fields of Gauss' interests after 1800 as follows: 1800–20, astronomy; 1820–30, geodesy, the theories of surfaces, and conformal mapping; 1830–40, mathematical physics, particularly electromagnetism, terrestrial magnetism, and the theory of attraction according to the Newtonian law; 1841–55, analysis situs, and the geometry associated with functions of a complex variable.

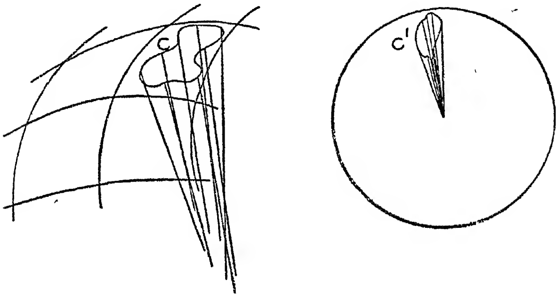
During the period 1821–48 Gauss was scientific adviser to the Hanoverian (Göttingen was then under the government of Hanover) and Danish governments in an extensive geodetic survey. Gauss threw himself into the work. His method of least squares and his skill in devising schemes for handling masses of numerical data had full scope but, more importantly, the problems arising in the precise survey of a portion of the earth's surface undoubtedly suggested deeper and more general problems connected with all curved surfaces. These researches were to beget the mathematics of relativity. The subject was not new: several of Gauss' predecessors, notably Euler, Lagrange, and Monge, had investigated geometry on certain types of curved surfaces, but it remained for Gauss to attack the problem in all its generality, and from his investigations the first great period of *differential geometry* developed.

Differential geometry may be roughly described as the study of properties of curves, surfaces, etc., in the immediate neighbourhood of a point, so that higher powers than the second of distances can be neglected. Inspired by this work, Riemann in 1854 produced his classic dissertation on the hypotheses which lie at the foundations of geometry, which, in its turn, began the second great period in differential geometry, that which is today of use in mathematical physics, particularly in the theory of general relativity.

Three of the problems which Gauss considered in his work on surfaces suggested general theories of mathematical and scientific importance: the measurement of *curvature*, the theory of *conformal representation* (or mapping), and the *applicability* of surfaces.

The unnecessarily mystical motion of a 'curved' space-time, which is a purely mathematical extension of familiar, visualizable curvature to a 'space' described by four co-ordinates instead of two, was a natural development of Gauss' work on curved surfaces. One of his definitions will illustrate the reasonableness of all. The problem is to devise some precise means for describing how the 'curvature' of a surface varies from point to point of the surface; the description must satisfy our intuitive feeling for what 'more curved' and 'less curved' signify.

The total curvature of any part of a surface bounded by an unlooped closed curve C is defined as follows. The *normal* to a



surface at a given point is that straight line passing through the point which is perpendicular to the plane which touches the surface at the given point. At each point of C there is a normal to the surface. Imagine all these normals drawn. Now, from the centre of a sphere (which may be anywhere with reference to the surface being considered), whose radius is equal to the unit length, imagine all the radii drawn which are parallel to the normals to C. These radii will cut out a curve, say C', on the sphere of unit radius. The *area* of that part of the spherical surface which is enclosed by C' is defined to be the *total curvature* of the part of the given surface which is enclosed by C. A little visualization will show that this definition accords with common notions as required.

Another fundamental idea exploited by Gauss in his study of surfaces was that of *parametric representation.*

It requires *two* co-ordinates to specify a particular point on a plane. Likewise on the surface of a sphere, or on a spheroid like the Earth: the co-ordinates in this case may be thought of as latitude and longitude. This illustrates what is meant by a *two-dimensional manifold.* Generally: if *precisely* n numbers are both necessary and sufficient to specify (individualize) each particular member of a class of things (points, sounds, colours, lines, etc.,) the class is said to be an *n-dimensional manifold.* In such specifications it is agreed that only certain characteristics of the members of the class shall be assigned numbers. Thus if we consider only the pitch of sounds, we have a one-dimensional manifold, because one number, the frequency of the vibration corresponding to the sound, suffices to determine the pitch; if we add loudness – measured on some convenient scale – sounds are now a two-dimensional manifold, and so on. If now we regard a *surface* as being made up of *points*, we see that it is a *two-dimensional manifold* (of points). Using the language of geometry we find it convenient to speak of *any* two-dimensional manifold as a 'surface', and to apply to the manifold the reasoning of geometry – in the hope of finding something interesting.

The foregoing considerations lead to the parametric representation of surfaces. In Descartes' geometry *one* equation between *three* co-ordinates represents a surface. Say the co-ordinates (Cartesian) are x, y, z. Instead of using a single equation connecting x,y,z to represent the surface, we now seek *three*:

$$x = f(u,v),\ y = g(u,v),\ z = h(u,v),$$

where $f(u,v)$, $g(u,v)$, $h(u,v)$ are such functions (expressions) of the new variables u,v that when these variables are eliminated (got rid of – 'put over the threshold' literally) there results between x,y,z the equation of the surface. The elimination is possible, because *two* of the equations can be used to solve for the *two* unknowns u,v; the results can then be substituted in the third. For example, if

$$x = u + v,\ y = u - v,\ z = vu,$$

we get $u = \frac{1}{2}(x + y)$, $v = \frac{1}{2}(x - y)$ from the first two, and hence $4z = x^2 - y^2$ from the third. Now as the variables u, v independently run through any prescribed set of numbers, the functions f,g,h will take on numerical values and x,y,z will move on the surface whose equations are the three written above. The variables u,v are called the *parameters* for the surface, and the three equations $x = f(u,v)$, $y = g(u,v)$, $z = h(u,v)$ their parametric equations. This method of representing surfaces has great advantages over the Cartesian when applied to the study of curvature and other properties of surfaces which vary rapidly from point to point.

Notice that the parametric representation is *intrinsic*; it refers to the surface itself for its co-ordinates, and not to an extrinsic, or extraneous, set of axes, not connected with the surface, as is the case in Descartes' method. Observe also that the *two* parameters u, v immediately show up the two-dimensionality of the surface. Latitude and longitude on the earth are instances of these intrinsic, 'natural' co-ordinates; it would be most awkward to have to do all our navigation with reference to three mutually perpendicular axes drawn through the centre of the Earth, as would be required for Cartesian sailing.

Another advantage of the method is its easy generalization to a space of any number of dimensions. It suffices to increase the number of parameters and proceed as before. When we come to Riemann we shall see how these simple ideas led naturally to a generalization of the metric geometry of Pythagoras and Euclid. The foundations of this generalization were laid down by Gauss, but their importance for mathematics and physical science was not fully appreciated till our own century.

Geodetic researches also suggested to Gauss the development of another powerful method in geometry, that of conformal mapping. Before a map can be drawn, say of Greenland, it is necessary to determine what is to be preserved. Are distances to be distorted, as they are on Mercator's projection, till Greenland assumes an exaggerated importance in comparison with North America? Or are distances to be preserved, so that one inch on the map, measured anywhere along the reference lines (say those for latitude and longitude) shall always correspond to the same distance measured on the surface of the earth? If so, one kind of mapping is demanded, and this kind will not preserve some other feature that we may wish to preserve; for example, if two roads on the earth intersect at a certain angle, the lines representing these roads on the map will intersect at a different angle. That kind of mapping which *preserves angles* is called conformal. In such mapping the theory of analytic functions of a complex variable, described earlier, is the most useful tool.

The whole subject of conformal mapping is of constant use in mathematical physics and its applications, for example in electrostatics, hydrodynamics and its offspring aerodynamics, in the last of which it plays a part in the theory of the airfoil.

Another field of geometry which Gauss cultivated with his usual thoroughness and success was that of the applicability of surfaces, in which it is required to determine what surfaces can be bent on to a given surface without stretching or tearing. Here again the methods Gauss invented were general and of wide utility.

To other departments of science Gauss contributed fundamental researches, for example in the mathematical theories of electromagnetism, including terrestrial magnetism, capillarity, the attraction of ellipsoids (the planets are ellipsoids of special kinds) where the law of attraction is the Newtonian, and dioptrics, especially concerning systems of lenses. The last gave him an opportunity to apply some of the purely abstract technique (continued fractions) he had developed as a young man to satisfy his curiosity in the theory of numbers.

Gauss not only mathematicized sublimely about all these things; he used his hands and his eyes, and was an extremely accurate observer. Many of the specific theorems he discovered, particularly in his researches on electromagnetism and the theory of attraction, have become part of the indispensable stock in trade of all who work seriously in physical science. For many years Gauss, aided by his friend Weber, sought a satisfying theory for all electromagnetic phenomena. Failing to find one that he considered satisfactory, he abandoned his attempt. Had he found Clerk Maxwell's (1831-79) equations of the electromagnetic field he might have been satisfied.

To conclude this long but still far from complete list of the great things that earned Gauss the undisputed title of Prince of Mathematicians we must allude to a subject on which he published nothing beyond a passing mention in his dissertation of 1799, but which he predicted would become one of the chief concerns of mathematics – *analysis situs*. A technical definition of what this means is impossible here (it requires the notion of a *continuous group*), but some hint of the type of problem with which the subject deals can be gathered from a simple instance. Any sort of a knot is tied in a string, and the ends of the string are then tied together. A 'simple' knot is easily distinguishable by eye from a 'complicated' one, but how are we to give an exact, *mathematical* specification of the difference between the two? And how are we to classify knots mathematically? Although he published nothing on this, Gauss had made a beginning, as was discovered in his posthumous papers. Another type of problem in this subject is to determine the least number of cuts on a given surface which will enable us to flatten the surface out on a plane. For a conical surface one cut suffices; for an anchor ring, two; for a sphere, no finite number of cuts suffices if no stretching is permitted.

These examples may suggest that the whole subject is trivial. But if it had been, Gauss would not have attached the extraordinary importance to it that he did. His prediction of its fundamental character has been fulfilled in our own generation. To-day a vigorous school (including many Americans – J. W. Alexander, S. Lefschetz, O. Veblen, among others) is finding that analysis situs, or the 'geometry of position' as it used sometimes to be called, has far-reaching ramifications in both geometry and analysis. What a pity it seems to us now that Gauss could not have stolen a year or two from Ceres to organize his thoughts on this vast theory which was to become the dream of his old age and a reality of our own young age.

His last years were full of honour, but he was not as happy as he had earned the right to be. As powerful of mind and as prolifically inventive as he had ever been, Gauss was not eager for rest when the first symptoms of his last illness appeared some months before his death.

A narrow escape from a violent death had made him more reserved than ever, and he could not bring himself to speak of the sudden passing of a friend. For the first time in more than twenty years he had left Göttingen on 16 June 1854 to see the railway under construction between his town and Cassel. Gauss had always taken a keen interest in the construction and operation of railroads; now he would see one being built. The horses bolted; he was thrown from his carriage, unhurt, but badly shocked. He recovered, and had the pleasure of witnessing the opening ceremonies when the railway reached Göttingen on 31 July 1854. It was his last day of comfort.

With the opening of the new year he began to suffer greatly from an enlarged heart and shortness of breath, and symptoms of dropsy appeared. Nevertheless he worked when he could, although his hand cramped and his beautifully clear writing broke at last. The last letter he wrote was to Sir David Brewster on the discovery of the electric telegraph.

Fully conscious almost to the end he died peacefully, after a severe struggle to live, early on the morning of 23 February 1855, in his seventy-eighth year. He lives everywhere in mathematics.

CHAPTER FIFTEEN

MATHEMATICS AND WINDMILLS

Cauchy

IN the first three decades of the nineteenth century mathematics quite suddenly became something noticeably different from what it had been in the heroic post-Newtonian age of the eighteenth. The change was in the direction of greater rigour in demonstration following an unprecedented generality and freedom of inventiveness. Something of the same sort is plainly visible again to-day, and he would be a rash prophet who would venture to forecast what mathematics will be like three-quarters of a century hence.

At the beginning of the nineteenth century only Gauss had any inkling of what was so soon to come, but his Newtonian reserve held him back from telling Lagrange, Laplace, and Legendre what he foresaw. Although the great French mathematicians lived well into the first third of the nineteenth century much of their work now appears to have been preparatory. Lagrange in the theory of equations prepared the way for Abel and Galois; Laplace, with his work on the differential equations of Newtonian astronomy – including the theory of gravitation – hinted at the phenomenal development of mathematical physics in the nineteenth century; while Legendre's researches in the integral calculus suggested to Abel and Jacobi one of the most fertile fields of investigation analysis has ever acquired. Lagrange's analytical mechanics is still modern; but even it was to receive magnificent additions at the hands of Hamilton and Jacobi and, later, Poincaré. Lagrange's work in the calculus of variations was also to remain classic and useful, but again the work of Weierstrass gave it a new direction under the rigorous, inventive spirit of the latter half of the nineteenth century, and this in its turn has been amplified and renovated in our own times (American and Italian mathematicians taking a leading part in the development).

Augustin-Louis Cauchy, the first of the great French mathematicians whose thought belongs definitely to the modern age, was born in Paris on 21 August 1789 – a little less than six weeks after the fall of the Bastille. A child of the Revolution, he paid his tax to liberty and equality by growing up with an undernourished body. It was only by the diplomacy and good sense of his father that Cauchy survived at all in the midst of semi-starvation. Having outlived the Terror, he graduated from the Polytechnique into the service of Napoleon. After the downfall of the Napoleonic order Cauchy got his full share of deprivations from revolutions and counter-revolutions, and in a measure his work was affected by the social unrest of his times. If revolutions and the like do affect a scientific man's work, Cauchy should be the prize laboratory specimen for proving the fact. He had an extraordinary fertility in mathematical inventiveness and a fecundity that has been surpassed only twice – by Euler and Cayley. His work, like his times, was revolutionary.

Modern mathematics is indebted to Cauchy for two of its major interests, each of which marks a sharp break with the mathematics of the eighteenth century. The first was the introduction of rigour into mathematical analysis. It is difficult to find an adequate simile for the magnitude of this advance; perhaps the following will do. Suppose that for centuries an entire people has been worshipping false gods and that suddenly their error is revealed to them. Before the introduction of rigour mathematical analysis was a whole pantheon of false gods. In this Cauchy was one of the great pioneers with Gauss and Abel. Gauss might have taken the lead long before Cauchy entered the field, but did not, and it was Cauchy's habit of rapid publication and his gift for effective teaching which really got rigour in mathematical analysis accepted.

The second thing of fundamental importance which Cauchy added to mathematics was on the opposite side – the combinatorial. Seizing on the heart of Lagrange's method in the theory of equations, Cauchy made it abstract and began the systematic creation of the theory of groups. The nature of this will be described later; for the moment we note only the modernity of Cauchy's outlook.

Without enquiring whether the thing he invented had any application or not, even to other branches of mathematics, Cauchy developed it on its own merits as an abstract system. His predecessors, with the exception of the universal Euler who was as willing to write a memoir on a puzzle in numbers as on hydraulics or the 'system of the world', had found their inspiration growing out of the applications of mathematics. This statement of course has numerous exceptions, notably in arithmetic; but before the time of Cauchy few if any sought profitable discoveries in the mere manipulations of algebra. Cauchy looked deeper, saw the *operations* and their *laws of combination* beneath the symmetries of algebraic formulas, isolated them, and was led to the theory of groups. To-day this elementary yet intricate theory is of fundamental importance in many fields of pure and applied mathematics, from the theory of algebraic equations to geometry and the theory of atomic structure. It is at the bottom of the geometry of crystals, to mention but one of its applications. Its later developments (on the analytical side) extend far into higher mechanics and the modern theory of differential equations.

Cauchy's life and character affect us like poor Don Quixote's we sometimes do not know whether to laugh or to cry, and compromise by swearing. His father, Louis-François, was a paragon of virtue and piety, both excellent things in their way, but easily overdone. Heaven only knows how Cauchy senior escaped the guillotine; for he was a parliamentary lawyer, a cultured gentleman, an accomplished classical and biblical scholar, a bigoted Catholic, and a lieutenant of police in Paris when the Bastille fell. Two years before the Revolution broke he had married Marie-Madeleine Desestre, an excellent, not very intelligent woman who, like himself, was also a bigoted Catholic.

Augustin was the eldest of six children (two sons, four daughters). From his parents Cauchy inherited and acquired all the estimable qualities which make their lives read like one of those charming love stories, insipid as stewed cucumbers, concocted for French schoolgirls under sixteen, in which the hero and heroine are as pure and sexless as God's holy angels. With parents such as his it was perhaps natural that Cauchy should have grown up to be the obstinate Quixote of French Catholicism in the 1830's and 1840's when the Church was on the defensive. He suffered for his religion, and for that he deserves respect (possibly even if he was the smug hypocrite his colleagues accused him of being), but he also richly deserved to suffer on more than one occasion. His everlasting preaching about the beauty of holiness put people's backs up and engendered an opposition to his pious schemes which they did not always deserve. Abel, himself the son of a minister and a decent enough Christian, expressed the general disgust which some of Cauchy's antics inspired when he wrote home, 'Cauchy is a bigoted Catholic – a strange thing for a man of science.' The emphasis of course is on 'bigoted', not on the word it qualifies. Two of the finest characters and greatest mathematicians we shall meet, Weierstrass and Hermite, were Catholics. They were devout but not bigoted.

Cauchy's childhood fell in the bloodiest period of the Revolution. The schools were closed. Having no need of science or culture at the moment, the Commune either left the cultured and men of science to starve or carted them off to the guillotine. To escape the obvious danger Cauchy senior moved his family to his country place in the village of Arcueil. There he sat out the Terror, half starved himself and feeding his wife and infant largely from what scanty fruits and vegetables he could raise. As a consequence Cauchy grew up delicate and under-developed physically. He was nearly twenty before he began to recover from this early malnutrition, and all his life had to watch his health.

This retreat, gradually becoming less strict, lasted nearly eleven years, during which Cauchy senior undertook the education of his children. He wrote his own textbooks, several of them in the fluent verse of which he was master. Verse, he believed, made grammar, history, and, above all, morals less repulsive to the juvenile mind. Young Cauchy thus acquired his own uncontrolled fluency in both French and Latin verse which he indulged all his life. His verse abounds in noble sentiments loftily expressed and admirably reflects the piety of his blameless life, but is otherwise undistinguished. A large share of the lessons was devoted to narrow religious instruction, in which the mother assisted ably.

Arcueil adjoined the imposing estates of the Marquis Laplace and Count Claude-Louis Berthollet (1748–1822), the distinguished and eccentric chemist who kept his head in the Terror because he knew all about gunpowder. The two were great friends. Their gardens were separated by a common wall with a gate to which each had a key. In spite of the fact that both the mathematician and the chemist were anything but pious, Cauchy senior scraped an acquaintance with his distinguished and well-fed neighbours.

Berthollet never went anywhere. Laplace was more sociable and presently began dropping in at his friend's cottage, where he was struck by the spectacle of young Cauchy, too feeble physically to be tearing round like a properly nourished boy, poring over his books and papers like a penitent monk and seeming to enjoy it. Before long Laplace discovered that the boy had a phenomenal mathematical talent and advised him to husband his strength. Within a few years Laplace was to be listening apprehensively to Cauchy's lectures on infinite series, fearing that the bold young man's discoveries in convergence might have destroyed the whole vast edifice of his own celestial mechanics. 'The system of the world' came within a hairsbreadth of going to smash that time; a slightly greater ellipticity of the Earth's almost circular orbit, and the infinite series on which Laplace had based his calculations would have diverged. Luckily his astronomical intuition had preserved him from disaster, as he discovered on rising with a sigh of infinite relief after a prolonged testing of the convergence of all his series by Cauchy's methods.

On 1 January 1800 Cauchy senior, who had kept discreetly in touch with Paris, was elected Secretary of the Senate. His office was in the Luxembourg Palace. Young Cauchy shared the office, using a corner as his study. Thus it came about that he frequently saw Lagrange – then Professor at the Polytechnique

– who dropped in frequently to discuss business with Secretary Cauchy. Lagrange soon became interested in the boy and, like Laplace, was struck by his mathematical talent. On one occasion when Laplace and several other notables were present, Lagrange pointed to young Cauchy in his corner and said, 'You see that little young man? Well! He will supplant all of us in so far as we are mathematicians.'

To Cauchy senior Lagrange gave some sound advice, believing that the delicate boy might burn himself out:'Don't let him touch a mathematical book till he is seventeen.' Lagrange meant higher mathematics. And on another occasion: 'If you don't hasten to give Augustin a solid literary education his tastes will carry him away; he will be a great mathematician but he won't know how to write his own language.' The father took this advice from the greatest mathematician of the age to heart and gave his son a sound literary education before turning him loose on advanced mathematics.

After his father had done all he could for him, Cauchy entered the Central School of the Panthéon, at about the age of thirteen. Napoleon had instituted several prizes in the school and a sort of grand sweepstakes prize for all the schools of France in the same class. From the first Cauchy was the star of the school, carrying off the first prizes in Greek, Latin composition, and Latin verse. On leaving the school in 1804 he won the sweepstakes and a special prize in humanities. The same year Cauchy received his first communion, a solemn and beautiful occasion in the life of any Catholic and trebly so to him.

For the next ten months he studied mathematics intensively with a good tutor, and in 1805 at the age of sixteen passed second into the Polytechnique. There his experiences were not altogether happy among the ribald young sceptics who hazed him unmercifully for making a public exhibition of his religious observances. But Cauchy kept his temper and even tried to convert some of his scorners.

From the Polytechnique Cauchy passed to the civil engineering school (*Ponts et Chaussées*) in 1807. Although only eighteen he easily beat young men of twenty who had been two years in the school, and was early marked for special service. On completing his training in March 1810, Cauchy was at once given an important commission. His ability and bold originality had singled him out as a man for whom red tape should be cut, even at the risk of lopping off some older man's head in the process. Whatever else may be said of Napoleon, he took ability where-ever he found it.

In March 1810, when Cauchy left Paris, 'light of baggage, but full of hope', for Cherbourg on his first commission, the battle of Waterloo (18 June 1815) was still over five years in the future, and Napoleon still confidently expected to take England by the neck and rub its nose in its own fragrant sod. Before an invasion could be launched an enormous fleet was necessary, and this had yet to be built. Harbours and fortifications to defend the shipyards from the seagoing British were the first detail to be disposed of in the glamorous dream. Cherbourg for many reasons was the logical point to begin all these grandiose operations which were to hasten 'the day of glory' the French had been yelling about ever since the fall of the Bastille. Hence the gifted young Cauchy's assignment to Cherbourg to become a great military engineer.

In his light baggage Cauchy carried only four books, the *Mécanique céleste* of Laplace, the *Traité des fonctions analytiques* of Lagrange, Thomas à Kempis' *Imitation of Christ*, and a copy of Virgil's works – an unusual assortment for an ambitious young military engineer. Lagrange's treatise was to be the very book which caused its author's prophecy that 'this young man will supplant all of us' to come true first, as it inspired Cauchy to seek some theory of functions free from the glaring defects of Lagrange's.

The third on the list was to occasion Cauchy some distress, for with it and his aggressive piety he rather got on the nerves of his practical associates who were anxious to get on with their job of killing. But Cauchy soon showed them by turning the other cheek that he had at least read the book. 'You'll soon get over all that', they assured him. To which Cauchy replied by sweetly asking them to point out what was wrong in his conduct and he would gladly correct it. What answer this drew has not survived.

Rumours that her darling boy was fast becoming an infidel or worse reached the ears of his anxious mother. In a letter long enough and full enough of pious sentiments to calm all the mothers who ever sent their sons to the front or anywhere near it Cauchy reassured her, and she was happy once more. The conclusion of the letter shows that the holy Cauchy was quite capable of holding his own against his tormentors, who had hinted he was slightly cracked.

'It is therefore ridiculous to suppose that religion can turn anybody's head, and if all the insane were sent to insane asylums, more philosophers than Christians would be found there.' Is this a slip on Cauchy's part, or did he really mean that no Christians are philosophers? He signs off with a flash from the other side of his head: 'But enough of this – it is more profitable for me to work at certain Memoirs on Mathematics'. Precisely; but every time he saw a windmill waving its gigantic arms against the sky he was off again full tilt.

Cauchy stayed approximately three years at Cherbourg. Outside of his heavy duties his time was well spent. In a letter of 3 July 1811, he describes his crowded life. 'I get up at four and am busy from morning to night. My ordinary work is augmented this month by the arrival of the Spanish prisoners. We had only eight days' warning, and during those eight days we had to build barracks and prepare camp beds for 1,200 men. . . . At last our prisoners are lodged and covered – since the last two days. They have camp beds, straw, food, and count themselves very fortunate. . . . Work doesn't tire me; on the contrary it strengthens me and I am in perfect health.'

On top of all this good work *pour la gloire de la belle France* Cauchy found time for research. As early as December 1810 he had begun 'to go over again all the branches of Mathematics, beginning with Arithmetic and finishing with Astronomy, clearing up obscurities, applying [my own methods] to the simplification of proofs and the discovery of new propositions.' And still on top of this the amazing young man found time to instruct others who begged for lessons so that they might rise in their profession, and he even assisted the mayor of Cherbourg by conducting school examinations. In this way he learned to teach. He still had time for hobbies.

The Moscow fiasco of 1812, war against Prussia and Austria, and the thorough drubbing he got at the battle of Leipzig in October 1813 all distracted Napoleon's attention from the dream of invading England, and the works at Cherbourg languished. Cauchy returned to Paris in 1813, worn out by overwork. He was then only twenty-four, but he had already attracted the attention of the leading mathematicians of France by his brilliant researches, particularly the memoir on polyhedra and that on symmetric functions. As the nature of both is easily understood, and each offers suggestions of the very first importance to the mathematics of to-day, we shall briefly describe them.

The first is of only minor interest in itself. What is significant regarding it to-day is the extraordinary acuteness of the criticism which Malus levelled at it. By a curious historical coincidence Malus was exactly 100 years ahead of his times in objecting to Cauchy's reasoning in the precise manner in which he did. The Academy had proposed as its prize problem 'To perfect in some essential point the theory of polyhedra', and Lagrange had suggested this as a promising research for young Cauchy to undertake. In February 1811 Cauchy submitted his first memoir on the theory of polyhedra. This answered negatively a question asked by Poinsot (1777–1859): is it possible that regular polyhedra other than those having 4, 6, 8, 12, or 20 faces exist? In the second part of this memoir Cauchy extended the formula of Euler, given in the school books on solid geometry, connecting the number of edges (E), faces (F), and vertices (V) of a polyhedron, $E + 2 = F + V$.

This work was printed. Legendre thought highly of it and encouraged Cauchy to continue, which Cauchy did in a second memoir (January 1812). Legendre and Malus (1775–1812) were the referees. Legendre was enthusiastic and predicted great things for the young author. But Malus was more reserved.

Étienne-Louis Malus was not a professional mathematician but an ex-officer of engineers in Napoleon's campaigns in Germany and Egypt, who made himself famous by his accidental discovery of the polarization of light by reflexion. So possibly his objections struck young Cauchy as just the sort of captious criticisms to be expected from an obstinate physicist. In proving his most important theorems Cauchy had used the 'indirect method' familiar to all beginners in geometry. It was to this method of proof that Malus objected.

In proving a proposition by the indirect method, a contradiction is deduced from the assumed falsity of the proposition; whence it follows, in Aristotelian logic, that the proposition is true. Cauchy could not meet the objection by supplying direct proofs, and Malus gave in – still unconvinced that Cauchy had proved anything. When we come to the conclusion of the whole story (in the last chapter) we shall see the same objection being raised in other connexions by the intuitionists. If Malus failed to make Cauchy see the point in 1812, Malus was avenged by Brouwer in 1912 and thereafter when Brouwer succeeded in making some of Cauchy's successors in mathematical analysis at least see that there is a point to be seen. Aristotelian logic, as Malus was trying to tell Cauchy, is not always a safe method of reasoning in mathematics.

Passing to the *theory of substitutions*, begun systematically by Cauchy, and elaborated by him in a long series of papers in the middle 1840's, which developed into the *theory of finite groups*, we shall presently illustrate the underlying notions by a simple example. First, however, the leading properties of a *group of operations* may be described informally.

Operations will be denoted by capital letters, A, B, C, D, ...; and the performance of two operations *in succession*, say A *first*, B *second*, will be indicated by juxtaposition thus, AB. Note that BA, by what has just been said, means that B is performed first, A second; so that AB and BA are *not necessarily* the same operation. For example, if A is the operation 'add 10 to a given number', and B is the operation 'divide a given number by 10', AB applied to x gives $\frac{x + 10}{10}$, while BA gives $\frac{x}{10} + 10$, or $\frac{x + 100}{10}$, and the resulting fractions are unequal; hence AB and BA are distinct.

If the effects of two operations X, Y are the same, X and Y

are said to be *equal* (or *equivalent*), and this is expressed by writing $X = Y$.

The next fundamental notion is that of *associativity*. Let U, V, W be any triple whatsoever of operations in the set. Then, if $(UV)W = U(VW)$, the set is said to satisfy the *associative law*. By $(UV)W$ is meant that UV is performed first, then, on the result, W is performed; by $U(VW)$ is meant that U is performed first, then, on the result of this VW is performed.

The last fundamental notion is that of an *identical operation*, or an *identity*: an operation I which leaves unchanged whatever it operates on is called an *identity*.

With these notions we can now state the simple postulates which define a group of operations.

A set of operations $I, A, B, C, \ldots, X, Y, \ldots$ is said to form a *group* if the postulates (1) – (4) are satisfied.

(1) There is a rule of combination applicable to *any* pair X, Y of operations* in the set such that the result, denoted by XY, of combining X, Y, in this order, according to the rule of combination, is a uniquely determined operation in the set.

(2) For *any three* operations X, Y, Z in the set, the rule in (1) is associative, namely $(XY)Z = X(YZ)$.

(3) There is a unique identity I in the set, such that, for every operation X in the set, $IX = XI = X$.

(4) If X is *any* operation in the set, there is in the set a *unique* operation, say X', such that $XX' = I$ (it can be easily proved that $X'X = I$ also).

These postulates contain redundancies deducible from other statements in (1) – (4), but in the form given the postulates are easier to grasp. To illustrate a group we shall take a very simple example relating to *permutations* (arrangements) of letters. This may seem trivial, but such *permutation* or *substitution* groups were found to be the long-sought clue to the algebraic solvability of equations.

There are precisely six orders in which the three letters a,b,c can be written, namely, *abc*, *acb*, *bca*, *bac*, *cab*, *cba*. Take any one of these, say the first *abc*, as the initial order. By what permutations of the letters can we pass from this to the remaining five

* The operations in a pair may be the same, thus X, X.

arrangements? To pass from *abc* to *acb* it is sufficient to *interchange*, or *permute*, b and c. To indicate the *operation* of permuting b and c, we write (bc), which is read, 'b into c, and c into b'. From *abc* to *bca* we pass by a into b, b into c, and c into a, which is written (abc). The order *abc* itself is obtained from *abc* by *no* change, namely a into a, b into b, c into c, which is the *identity* substitution and is denoted by I. Proceeding similarly with all six orders

$$abc,\ acb,\ bca,\ bac,\ cab,\ cba,$$

we get the corresponding *substitutions*,

$$I,\ (bc),\ (abc),\ (ab),\ (acb),\ (ac).$$

The 'rule of combination' in the postulates is here as follows. Take any two of the substitutions, say (bc) and (acb), and consider the effect of these applied successively in the order stated, namely (bc) first and (acb) second: (bc) carries b into c, then (acb) carries c into b. Thus b is left as it was. Take the next letter, c, in (bc): by (bc), c is carried into b, which, by (acb) is carried into a; thus c is carried into a. Continuing, we see what a is now carried into: (bc) leaves a as it was, but (acb) carries a into c. Finally then the total effect of (bc) followed by (acb) is seen to be (ca), which we indicate by writing $(bc)\,(acb) = (ca) = (ac)$.

In the same way it is easily verified that

$$(acb)\,(abc) = (abc)\,(acb) = I;$$
$$(abc)\,(ac) = (ab);\ (bc)\,(ac) = (acb),$$

and so on for all possible pairs. Thus postulate (1) is satisfied for these six substitutions, and it can be checked that (2), (3), (4) are also satisfied.

All this is summed up in the 'multiplication table' of the group, which we shall write out, denoting the substitutions by the letters under them (to save space),

$$\begin{matrix} I, & (bc), & (abc), & (ab), & (acb), & (ac) \\ I, & A, & B, & C, & D, & E. \end{matrix}$$

In reading the table any letter, say C, is taken from the left-hand *column*, and any letter, say D, from the top *row*, and the entry, here A, where the corresponding *row* and *column* intersect is the result of CD. Thus $CD = A$, $DC = E$, $EA = B$, and so on.

As an example we may verify the associative law for $(AB)C$ and $A(BC)$, which should be equal. First, $AB = C$; hence $(AB)C = CC = I$. Again $BC = A$; hence $A(BC) = AA = I$. In the same way $A(DB) = AI = A$; $(AD)B = EB = A$; thus $(AD)B = A(DB)$.

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	I	C	B	E	D
B	B	E	D	A	I	C
C	C	D	E	I	A	B
D	D	C	I	E	B	A
E	E	B	A	D	C	I

The total number of distinct operations in a group is called its *order*. Here six is the order of the group. By inspection of the table we pick out several *subgroups*, for example,

	I
I	I

	I	A
I	I	A
A	A	I

	I	B	D
I	I	B	D
B	B	D	I
D	D	I	B

which are of the respective orders one, two, three. This illustrates one of the fundamental theorems proved by Cauchy: *the order of any sub-group is a divisor of the order of the group.*

The reader may find it amusing to try to construct groups of orders other than six. For any given order the number of distinct groups (having different multiplication tables) is finite, but what this number may be for *any* given order (the *general* order n) is not known – nor likely to be in our lifetime. So at the very beginning of a theory which on its surface is as simple as dominoes we run into unsolved problems.

Having constructed the 'multiplication table' of a group, we forget about its derivation from substitutions (if that happens to be the way the table was made), and regard the table as defining an *abstract group*; that is, the symbols $I, A, B, \ldots$ are given no interpretation beyond that implied by the rule of combination, as in $CD = A$, $DC = E$, etc. This abstract point of view is that now current. It was not Cauchy's but was introduced by Cayley in 1854. Nor were completely satisfactory sets of postulates for groups stated till the first decade of the twentieth century.

When the operations of a group are interpreted as substitutions, or as the rotations of a rigid body, or in any other department of mathematics to which groups are applicable, the interpretation is called a *realization* of the *abstract* group defined by the multiplication table. A given abstract group may have many diverse realizations. This is one of the reasons that groups are of *fundamental* importance in modern mathematics: one abstract, *underlying structure* (that summarized in the multiplication table) of one and the same group is the essence of several apparently unrelated theories, and by an intensive study of the properties of the abstract group, a knowledge of the theories in question and their mutual relationships is obtained by one investigation instead of several.

To give but one instance, the set of all rotations which twirl a regular icosahedron (twenty-sided regular solid) about its axes of symmetry, so that after any rotation of the set the volume of the solid occupies the same space as before, forms a group, and this group of rotations, when expressed abstractly, is the same group as that which appears, under permutations of the roots, when we attempt to solve the general equation of the fifth degree. Further, this same group turns up (to anticipate slightly) in the theory of elliptic functions. This suggests that although it is impossible to solve the general quintic algebraically, the equation may be – and in fact is – solvable in terms of the functions mentioned. Finally, all this can be pictured geometrically by describing the rotations of an icosahedron already mentioned. This beautiful unification was the work of Felix Klein (1849–1925) in his book on the icosahedron (1884).

Cauchy was one of the great pioneers in the theory of substitution groups. Since his day an immense amount of work has been done in the subject, and the theory itself has been vastly extended by the accession of *infinite groups* – groups having an infinity of operations which can be counted off 1,2,3, ... , and further, to groups of *continuous* motions. In the latter an operation of the group shifts a body into another position by *infinitesimal* (arbitrarily small) displacements – not like the icosahedral group described above, where the rotations shift the whole body round by a finite amount. This is but one category of infinite groups (the terminology here is not exact, but is sufficient to bring out the point of importance – the distinction between *discrete* and *continuous* groups). Just as the theory of finite discrete groups is the structure underlying the theory of algebraic equations, so is the theory of infinite, continuous groups of great service in theory of differential equations – those of the greatest importance in mathematical physics. So in playing with groups Cauchy was not idling.

To close this description of groups we may indicate how the groups of substitutions discussed by Cauchy have entered the modern theory of atomic structure. A substitution, say (xy), containing precisely two letters in its symbol, is called a *transposition*. It is easily proved that any substitution is a combination of transpositions. For example,

$$(abcdef) = (ab)\,(ac)\,(ad)\,(ae)\,(af),$$

from which the rule for writing out any substitution in terms of transpositions is evident.

Now, it is an entirely reasonable hypothesis to assume that the electrons in an atom are identical, that is, one electron is indistinguishable from another. Hence, if in an atom two electrons are interchanged, the atom will remain unchanged. Suppose for simplicity that the atom contains precisely three electrons, say a,b,c. To the *group of substitutions* on a,b,c (the one whose multiplication table we gave) will correspond all interchanges of electrons leaving the atom *invariant* – as it was. From this to the spectral lines in the light emitted by an excited gas consisting of atoms may seem a long step, but it has been taken, and one school of experts in quantum mechanics finds a satisfactory background for the elucidation of spectra (and other phenomena associated with atomic structure) in the theory of substitution groups. Cauchy of course foresaw no such applications of the theory which he developed for its own fascinations, nor did he foresee its application to the outstanding riddle of algebraic equations. That triumph was reserved for a boy in his teens whom we shall meet later.

By the age of twenty-seven (in 1816) Cauchy had raised himself to the front rank of living mathematicians. His only serious rival was the reticent Gauss, twelve years older than himself. Cauchy's memoir of 1814 on definite integrals with complex-number limits inaugurated his great career as the independent creator and unequalled developer of the theory of functions of a complex variable. For the technical terms we must refer to the chapter on Gauss – who had reached the fundamental theorem in 1811, three years before Cauchy. Cauchy's luxuriantly detailed memoir on the subject was published only in 1827. The delay was due possibly to the length of the work – about 180 pages. Cauchy thought nothing of hurling massive works of from 80 to 300 pages at the Academy or the Polytechnique to be printed out of their stinted funds.

The following year (1815) Cauchy created a sensation by proving one of the great theorems which Fermat had bequeathed to a baffled posterity: every positive integer is a sum of three 'triangles', four 'squares', five 'pentagons', six 'hexagons', and so on, zero in each case being counted as a number of the kind concerned. A 'triangle' is one of the numbers

0,1,3,6,10,15,21, ... got by building up *regular* (equilateral) triangles out of dots,

. . . .·, etc.;

'squares' are built up similarly,

. ,   . . ,   . . . , ... , etc.,

where the 'bordering' by which one square is obtained from its predecessor is evident. Similarly 'pentagons' are *regular* pentagons built up by dots; and so on for 'hexagons' and the rest. This was not easy to prove. In fact it had been too much for Euler, Lagrange, and Legendre. Gauss had early proved the case of 'triangles'.

As if to show that he was not limited to first-rate work in pure mathematics Cauchy next captured the Grand Prize offered by the Academy in 1816 for a 'theory of the propagation of waves on the surface of a heavy fluid of indefinite depth' – ocean waves are close enough to this type for mathematical treatment. This finally (when printed) ran to more than 300 pages. At the age of twenty-seven Cauchy found himself being strongly 'rushed' for membership in the Academy of Sciences – a most unusual honour for so young a man. The very first vacancy in the mathematical section would fall to him, he was assured on the quiet. So far as popularity is concerned this was the highwater mark of Cauchy's career.

In 1816, then, Cauchy was ripe for election to the Academy. But there were no vacancies. Two of the seats, however, might soon be expected to be empty owing to the age of the incumbents: Monge was seventy, L. M. N. Carnot sixty-three. Monge we have already met; Carnot was a precursor of Poncelet. Carnot held his seat in the Academy on account of his researches which restored and extended the synthetic geometry of Pascal and Desargues, and for his heroic attempt to put the calculus on a firm logical foundation. Outside mathematics Carnot made an enviable name for himself in French history, being the genius who in 1793 organized fourteen armies to defeat the half million troops hurled against France by the united anti-democratic reactionaries of Europe. When Napoleon seized the power for himself in 1796, Carnot was banished for opposing the tyrant: 'I am an irreconcilable enemy of all kings', said Carnot. After the Russian campaign of 1812 Carnot offered his services as a soldier, but with one stipulation. He would fight for France, not for the French Empire of Napoleon.

In the reorganization of the Academy of Sciences during the political upheaval after Napoleon's glorious 'Hundred Days' following his escape from Elba, Carnot and Monge were expelled. Carnot's successor took his seat without much being said, but when young Cauchy calmly sat down in Monge's chair the storm broke. The expulsion of Monge was sheer political indecency, and whoever profited by it showed at least that he lacked the finer sensibilities. Cauchy of course was well within his rights and his conscience.

The hippopotamus is said to have a tender heart by those who have eaten that delicacy baked, so a thick skin is not necessarily a reliable index to what is inside a man. Worshipping the Bourbons as he did, and believing the dynasty to be the direct representatives of Heaven sent to govern France – even when Heaven sent an incompetent clown like Charles X – Cauchy was merely doing his loyal duty to Heaven and to France when he slipped into Monge's chair. That he was sincere and not merely self-seeking will appear from his subsequent devotion to the sanctified Charles.

Honourable and important positions now came thick and fast to the greatest mathematician in France – still well under thirty. Since 1815 (when he was twenty-six) Cauchy had been lecturing on analysis at the Polytechnique. He was now made Professor, and before long was appointed also at the Collège de France and the Sorbonne. Everything began coming his way. His mathematical activity was incredible; sometimes two full-length papers would be laid before the Academy in the same week. In addition to his own research he drew up innumerable reports on the memoirs of others submitted to the Academy, and found time to emit an almost constant stream of short papers on practically all branches of mathematics, pure and applied. He became better known than Gauss to the mathematicians of Europe. Savants as well as students came to hear his beautifully clear expositions of the new theories he was developing, particularly in analysis and mathematical physics. His auditors included well-known mathematicians from Berlin, Madrid, and St Petersburg.

In the midst of all this work Cauchy found time to do his courting. His fancy, Aloise de Bure, whom he married in 1818 and with whom he lived for nearly forty years, was the daughter of a cultured old family and, like himself, an ardent Catholic. They had two daughters, who were brought up as Cauchy had been.

One great work of this period may be noted. Encouraged by Laplace and others, Cauchy in 1821 wrote up for publication the course of lectures on analysis he had been giving at the Polytechnique. This is the work which for long set the standard in rigour. Even to-day Cauchy's definitions of limit and continuity, and much of what he wrote on the convergence of infinite series in this course of lectures, will be found in any carefully written book on the calculus. An extract from the preface will show what he had in mind and what he accomplished.

'I have sought to give to the methods [of analysis] all the rigour which is demanded in geometry, in such a way as never to refer to reasons drawn from the generality of algebra. [As it would be put to-day, the *formalism* of algebra.] Reasons of this kind, although commonly enough admitted, above all in the passage from convergent to divergent series, and from real quantities to imaginary, cannot be considered, it seems to me, as anything more than inductions which occasionally suggest the truth, but which agree but little with the boasted exactitude of mathematics. We must also observe that they tend to cause an indefinite validity to be attributed to algebraical formulae,*

* For example $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ to infinity, obtained by dividing 1 by $1 - x$, is nonsense if x is a positive number equal to or greater than 1.

while, in reality, the majority of these formulae subsist only under certain conditions, and for certain values of the quantities which they contain. By determining these conditions and values, and by fixing precisely the meaning of the notations I make use of, I shall dispel all uncertainty.'

Cauchy's productivity was so prodigious that he had to found a sort of journal of his own, the *Exercices de Mathématiques* (1826–30), continued in a second series as *Exercices d'Analyse Mathématique et de Physique*, for the publication of his expository and original work in pure and applied mathematics. These works were eagerly bought and studied, and did much to reform mathematical taste before 1860.

One aspect of Cauchy's terrific activity is rather amusing. In 1835 the Academy of Sciences began publishing its weekly bulletin (the *Comptes rendus*). Here was a virgin dumping ground for Cauchy, and he began swamping the new publication with notes and lengthy memoirs – sometimes more than one a week. Dismayed at the rapidly mounting bill for printing, the Academy passed a rule, in force to-day, prohibiting the publication of an article over four pages long. This cramped Cauchy's luxuriant style, and his longer memoirs, including a great one of 300 pages on the theory of numbers, were published elsewhere.

Happily married and as prolific in his research as a spawning salmon, Cauchy was ripe for the jester when the revolution of 1830 unseated his beloved Charles. Fate never enjoyed a heartier laugh than it did when it motioned Cauchy to rise from Monge's chair in the Academy and follow his anointed King into exile. Cauchy could not refuse; he had sworn a solemn oath of allegiance to Charles, and to Cauchy an oath was an oath, even if sworn to a deaf donkey. To his credit, Cauchy, at the age of forty, gave up all his positions and went into voluntary exile.

He was not sorry to go. The bloodied streets of Paris had turned his sensitive stomach. He firmly believed that good King Charles was in no way responsible for the gory mess.

Leaving his family in Paris, but not resigning his seat in the Academy, Cauchy went first to Switzerland, where he sought distraction in scientific conferences and research. He never asked the slightest favour from Charles and did not even know that the exiled king was aware of his voluntary sacrifice for a matter of principle. Shortly a more enlightened Charles, Charles Albert, King of Sardinia, heard that the renowned Cauchy was out of a job and made one for him as Professor of Mathematical Physics at Turin. Cauchy was perfectly happy. He quickly learned Italian and delivered his lectures in that language.

Presently overwork and excitement made him ill, and to his regret (as he wrote to his wife) he was forced to abandon evening work for a time. A vacation in Italy, with a visit to the Pope for good measure, completely restored him, and he returned to Turin, eagerly anticipating a long life devoted to teaching and research. But presently the obtuse Charles X butted into the retiring mathematician's life like a brainless goat and, in seeking to reward his loyal follower, did him a singular disservice. In 1833 Cauchy was entrusted with the education of Charles' heir, the thirteen-year-old Duke of Bordeaux. The job of male nurse and elementary tutor was the last thing on earth that Cauchy desired. Nevertheless he dutifully reported to Charles at Prague and took up the cross of loyalty. The following year he was joined by his family.

The education of the heir to the Bourbons proved no sinecure. From early morning to late evening, with barely time out for meals, Cauchy was pestered by the royal brat. Not only the elementary lessons of an ordinary school course had to be hammered somehow or another into the pampered boy, but Cauchy was detailed to see that his charge did not fall down and skin his knees on his gambols in the park. Needless to say the major part of Cauchy's instruction consisted in intimate talks on the peculiar brand of moral philosophy to which he was addicted; so perhaps it is as well that France finally decided not to take the Bourbons back to its heart, but to leave them and their innumerable descendants as prizes to be raffled off to the daughters of millionaires in the international marriage bureau.

In spite of almost constant attendance on his pupil Cauchy somehow managed to keep his mathematics going, dashing into his private quarters at odd moments to jot down a formula or scribble a hasty paragraph. The most impressive work of this period was the long memoir on the dispersion of light, in which Cauchy attempted to explain the phenomenon of dispersion (the separation of white light into colours owing to different refrangibilities of the coloured lights composing the white) on the hypothesis that light is caused by the vibrations of an elastic solid. This work is of great interest in the history of physics, as it exemplified the tendency of the nineteenth century to try to account for physical phenomena in terms of mechanical models instead of merely constructing an abstract, mathematical theory to correlate observations. This was a departure from the prevailing practice of Newton and his successors – although there had been attempts to 'explain' gravitation mechanically.

To-day the tendency is in the opposite direction of a purely mathematical correlation and a complete abandonment of ethers, elastic solids, or other mechanical 'explanations' more difficult to grasp than the thing explained. Physicists at present seem to have heeded Byron's query. 'Who will then explain the explanation?' The elastic solid theory had a long and brilliant success, and even to-day some of the formulae Cauchy derived from his false hypothesis are in use. But the theory itself was abandoned when, as not infrequently happens, refined experimental technique and unsuspected phenomena (anomalous dispersion in this case) failed to accord with the predictions of the theory.

Cauchy escaped from his pupil in 1833 (he was then almost fifty). Friends in Paris had been urging him for some time to return, and Cauchy seized the excuse of his parents' golden wedding to bid adieu to Charles and all his entourage. By a special dispensation members of the Institut (of which the Academy of Sciences was, and is, a part) were not required to take an oath of allegiance to the Government, so Cauchy resumed his seat. His mathematical activity now became greater than ever. During the last nineteen years of his life he produced over 500 papers on all branches of mathematics, including mechanics, physics, and astronomy. Many of these works were long treatises.

His troubles were not yet over. When a vacancy occurred at the Collège de France Cauchy was unanimously elected to fill the place. But here there was no dispensation and before he could step into the position Cauchy would have to take the oath of allegiance. Believing the Government to be usurping the divine rights of his master, Cauchy stiffened his neck and refused to take the oath. Once more he was out of a job. But the Bureau des Longitudes could use a mathematician of his calibre. Again he was unanimously elected.

Then began an amusing tug-of-war between Baron Cauchy and the Bureau at one end of the rope and the unsanctified Government at the other. Conscious for once that it was making a fool of itself the Government let go and Cauchy was shot backwards into the Bureau without an oath. Defiance of the Government was grossly illegal, not to say treasonable, but Cauchy stuck to his job. His colleagues at the Bureau embarrassed the Government by politely ignoring its request to elect someone legally. For four years Cauchy turned his obstinate back on the Government and went on with his work.

To this period belong some of Cauchy's most important contributions to mathematical astronomy. Leverrier had unwittingly started Cauchy off with his memoir of 1840 on Pallas. This was a lengthy work packed with numerical calculations which it would take any referee as long to check as it had taken the author to perform them in the first place. When the memoir was presented to the Academy the officers began looking about for someone willing to undertake the inhuman task of verifying the correctness of the conclusions. Cauchy volunteered. Instead of following Leverrier's footsteps he quickly found short cuts and invented new methods which enabled him to verify and extend the work in a remarkably short time.

The tussle with the Government reached its crisis in 1843 when Cauchy was fifty-four. The Minister declined to be made a public laughing stock any longer and demanded that the Bureau hold an election to fill the position Cauchy refused to vacate. On the advice of his friends Cauchy laid his case before the people in an open letter. This letter is one of the finest things Cauchy ever wrote.

Whatever we may think of his quixotic championship of a cause which all but flyblown reactionaries knew had been well lost for ever, we cannot help respecting Cauchy's fearlessness in stating his own case, with dignity and without passion, and in fighting for the freedom of his conscience. It was the old fight for free thought in a guise that was not very familiar then but is common enough now.

In the time of Galileo, Cauchy no doubt would have gone to the stake to maintain the freedom of his beliefs; under Louis Philippe he denied the right of any government to exact an oath of allegiance which traversed his conscience, and he suffered for his courage. His stand earned him the respect even of his enemies, and brought the Government into contempt, even in the eyes of its supporters. Presently the stupidity of repression was brought home to the Government in a way it could understand – street fighting, riots, strikes, civil war, and an unanswerable order to get out and stay out. Louis Philippe and all his gang were ousted in 1848. One of the first acts of the Provisional Government was to abolish the oath of allegiance. With rare good sense the politicians realized that all such oaths are either unnecessary or worthless.

In 1852, when Napoleon III took charge, the oath was restored. But by this time Cauchy had won his battle. Word was quietly passed to him that he might resume his lectures without taking the oath. It was understood on both sides that no fuss was to be made. The Government asked no thanks for its liberality, nor did Cauchy tender any, but went on with his lectures as if nothing had happened. From then to the end of his life he was the chief glory of the Sorbonne.

In the interim between official instability and unofficial stability Cauchy had taken time out to splinter a lance in défence of the Jesuits. The trouble was the old one – the State educational authorities insisting that the Jesuit trainings incurred a divided allegiance, the Jesuits defending religious instruction as the only sound basis for any education. It was a fight up Cauchy's own alley and he sailed into it with eloquent gusto. His defence of his friends was touching and sincere but unconvincing. Whenever Cauchy got off mathematics he substituted emotion for reason.

The Crimean War afforded Cauchy his last opportunity for getting himself disliked by his harder-headed colleagues. He became an enthusiastic propagandist for a singular enterprise known as Work of the Schools of the Orient. 'Work' here is intended in the sense of a particular 'good work'.

'It was necessary', according to the sponsors of the Work in 1855, 'to remedy the disorders of the past and at the same time impose a double check on Muscovite ambition and Mohammedan fanaticism: above all to prepare the regeneration of peoples brutalized by the Koran. ... ' In short the Crimean War had been the customary bayonet preparing the way for the Cross. Deeply impressed by the obvious necessity of replacing the brutalizing Koran by something more humane, Cauchy threw himself into the project, 'completing and consolidating ... the work of emancipation so admirably begun by the arms of France'.

The Jesuit Council, grateful for Cauchy's expert help, gave him full credit for many of the details (including the collection of subscriptions) which were to accomplish 'the moral regeneration of peoples enslaved to the law of the Koran, the triumph of the Gospel round the cradle and the sepulchre of Jesus Christ being the sole acceptable compensation for these billows of blood that have been shed' by the Christian French, English, Russians, Sardinians, and the Mohammedan Turks in the Crimean War.

It was good works of this character that caused some of Cauchy's friends, out of sympathy with the pious spirit of the orthodox religion of the time, to call him a smug hypocrite. The epithet was wholly undeserved. Cauchy was one of the sincerest bigots that ever lived.

The net result of the Work was the particularly revolting massacre of May 1860. Cauchy did not live to see his labours crowned.

Reputations of great mathematicians are subject to the same vicissitudes as those of other great men. For long after his death

–and even to-day – Cauchy was severely criticized for over-production and hasty composition. His total output is 789 papers (many of them very extensive works) filling twenty-four large quarto volumes. Criticism of this sort always seems rather beside the point if a man has put out a mass of first-rate work in addition to some that is not of high quality, and is usually indulged in by men who themselves have done comparatively little, and that little not of the highest order of originality. Cauchy's part in modern mathematics is somewhere not far from the centre of the stage. This is now almost universally admitted, if grudgingly in some quarters. Since his death, especially in recent decades, Cauchy's reputation as a mathematician has risen steadily. The methods he introduced, his whole programme inaugurating the first period of modern rigour, and his almost unequalled inventiveness have made a mark on mathematics that is, so far as we can now see, destined to be visible for many years to come.

One apparently unimportant detail out of all the mass of new things Cauchy did may be mentioned as an illustration of his prophetic originality. Instead of using the 'imaginary' i ($= \sqrt{-1}$) Cauchy proposed to accomplish all that complex numbers do in mathematics by operating with congruences to the modulus $i^2 + 1$. This was done in 1847. The paper – a short one – attracted but little attention. Yet it is the germ of something – Kronecker's programme – that is on its way to revolutionizing some of the fundamental concepts of mathematics. This matter will reappear frequently in later chapters, so we may pass it here with this allusion.

In social contacts Cauchy was extremely polite, not to say oily on occasion as when, for example, he was soliciting subscriptions for one of his jousts. His habits were temperate, and in all things except mathematics and religion he was moderate. On the last he lacked ordinary common sense. Everyone who came near him was a prospect for conversion. When William Thomson (Lord Kelvin) as a young man of twenty-one called on Cauchy to discuss mathematics, Cauchy spent the time trying to convert his visitor – then a staunch adherent of the Scottish Free Church – to Catholicism.

Cauchy had his share of rows over priority in which his enemies accused him of greed and unfair play. His last year was marred by one such dispute wherein it would seem that Cauchy had no case. But with his usual stubbornness where a matter of principle was involved he braved the outcry and stuck to his point with invincible sweetness and pertinacity.

Another peculiarity added to Cauchy's unpopularity with his scientific colleagues. In scientific academies and societies a man is supposed to base his vote for a candidate only on the candidate's scientific merits; any other procedure is considered bad ethics. Whether rightly or wrongly Cauchy was accused of voting in accordance with his religious and political views. His last years were embittered by what he considered a lack of understanding among his colleagues on this and similar foibles. Neither side could get the point of view of the other.

Cauchy died rather unexpectedly in his sixty-eighth year on 23 May 1857. Hoping to benefit a bronchial trouble he retired to the country to recuperate, only to be smitten with a fever which proved fatal. A few hours before his death he was talking animatedly with the Archbishop of Paris of the charitable works he had in view – charity was one of Cauchy's lifelong interests. His last words were addressed to the Archbishop: 'Men pass away but their deeds abide'.

FACTS FROM FIGURES

M. J. MORONEY

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The enormous success and rapid expansion of statistical techniques in recent years is ample proof of the need for them. They are not a cure-all, but many a headache persists because the research worker, production inspector, or executive imagines them as being too mathematical for him to apply. But there is nothing magical or mysterious in them. Statistical tools have been developed by practical men to deal with practical problems as simply as possible. Common sense and simple arithmetic will carry the reader through this book. Every symbol, every principle is explained and illustrated with examples drawn from a wide variety of subjects. The reader will find here a comprehensive introduction to the possibilities of the subject; he is given the how and the why and the wherefore by which he can recognize the kind of problem where statistics pay dividends. The author writes from experience, for he knows the limitations to the usefulness of statistical technique, and appreciates the difficulties of the non-mathematician. The book ranges from purely descriptive statistics, through probability theory, the game of Crown and Anchor, the design of sampling schemes, production quality control, correlation and ranking methods, to the analysis of variance and covariance.

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