Computational Fluid Dynamics
HW1

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1 Inviscid Burgers Equation

The Inviscid Burgers equation, in conservation law form, is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad F = F_{(u)} = \frac{u^2}{2} \tag{1}$$

In non-conservation law form, is given by:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \quad A = \frac{\partial F}{\partial u} = u \tag{2}$$

The equation is obtained by neglecting the viscous term from the viscous Burger equation.

1.1 Boundary and Initial Conditions

$$\begin{aligned} u_{(x=0,t)} &= 1.0 \\ u_{(x=1,t)} &= u_1 \\ u_{(x,t=0)} &= 1 - (1-u_1)\cdot x \end{aligned} \tag{3}$$

1.2 Finite Volume Formulation

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n\right) \tag{4}$$

- For first-order schemes, there is no variation within a cell, and the value there is constant.
- For second-order schemes, the variation within the cell is linear.

1.3 First Order Roe Method ($u_1 = 0.0$)

Roe scheme is based on the solution of the linear problem:

$$\frac{\partial u}{\partial t} + \bar{A}\frac{\partial u}{\partial x} = 0 \tag{5}$$

Where $\bar{A}$ is a constant matrix that is dependent on local conditions. The matrix is constructed in a way to guarantee unifrom validity across discontinuities:

1. For any u_i, u_{i+1}:
$$F_{i+1} - F_i = \bar{A}\cdot(u_{i+1} - u_i)$$
2. When $u = u_i = u_{i+1}$ then:
$$\bar{A}_{(u_i,u_{i+1})} = \bar{A}_{(u,u)} = \frac{\partial F}{\partial u} = u$$

In case of the Burgers equation, the matrix $\bar{A}$ is a scalar, namely, $\bar{A} = \bar{u}$. The equation becomes:

$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} = 0 \tag{6}$$

The value of $\bar{u}$ for the cell face between i and $i+1$ is determined from the first conditions:

$$\bar{u} = \bar{u}_{i+\frac{1}{2}} = \frac{F_{i+1} - F_i}{u_{i+1} - u_i} = \frac{\frac{1}{2}u_{i+1}^2 - \frac{1}{2}u_i^2}{u_{i+1} - u_i} = \begin{cases} \dfrac{u_i + u_{i+1}}{2} & u_i \neq u_{i+1} \\ u_i & u_i = u_{i+1} \end{cases} \tag{7}$$

The single wave that emanates from the cell interface travels either in the positive or negative direction, depending upon the sighn of $\bar{u}_{i+\frac{1}{2}}$. Define:

$$\begin{cases} \bar{u}^+_{i+\frac{1}{2}} \triangleq \frac{1}{2}\left(\bar{u}_{i+\frac{1}{2}} + \left|\bar{u}_{i+\frac{1}{2}}\right|\right) \geq 0 \\ \\ \bar{u}^-_{i+\frac{1}{2}} \triangleq \frac{1}{2}\left(\bar{u}_{i+\frac{1}{2}} - \left|\bar{u}_{i+\frac{1}{2}}\right|\right) \leq 0 \end{cases} \qquad \bar{u}_{i+\frac{1}{2}} = \bar{u}^+_{i+\frac{1}{2}} + \bar{u}^-_{i+\frac{1}{2}} \tag{8}$$

Using the jump relation, the numerical flux at the cell interface can be evaluated by one of the following:

$$\begin{cases} f_{i+\frac{1}{2}} - F_i = \bar{u}^-_{i+\frac{1}{2}} \cdot (u_{i+1} - u_i) \\ \\ F_{i+1} - f_{i+\frac{1}{2}} = \bar{u}^+_{i+\frac{1}{2}} \cdot (u_{i+1} - u_i) \end{cases} \tag{9}$$

The numerical flux may then be written in the following symmetric from:

$$\begin{aligned} f_{i+\frac{1}{2}} &= \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\left(\bar{u}^+_{i+\frac{1}{2}} - \bar{u}^-_{i+\frac{1}{2}}\right)(u_{i+1} - u_i) \\ \text{OR} &: \\ f_{i+\frac{1}{2}} &= \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\left|\bar{u}_{i+\frac{1}{2}}\right|(u_{i+1} - u_i) \end{aligned} \tag{10}$$

Since Roe's scheme can't distinguish between the types of discontinuity, it may result in an expansion shock where the analytical solution is an expansion wave. To guarantee a physical solution the scheme will be modified like so:
Define

$$\varepsilon = \max\left(0, \frac{u_{i+1} - u_i}{2}\right)$$

The interface wave speed becomes

$$\bar{u}_{i+\frac{1}{2}} = \begin{cases} \bar{u}_{i+\frac{1}{2}} & \bar{u}_{i+\frac{1}{2}} \geq \varepsilon \quad \text{compression} \\ \varepsilon & \bar{u}_{i+\frac{1}{2}} < \varepsilon \quad \text{expansion} \end{cases} \tag{11}$$

1.3.1 CFL number

For the Roe method, the CFL number is defined as:

$$\text{CFL} = \frac{u\Delta t}{\Delta x} \tag{12}$$

We will want to set the maximal value of the *CFL* number. We will find the Δt at each cell and (Δt_i) and set the Δt of the current step as:

$$\Delta t = \min(\Delta t_i) \ \ \forall i \tag{13}$$

1.4 Second Order Roe ($u_1 = 0.5$)

The first-order accurate Roe method interface flux function will be denoted like this:

$$f^{\text{Roe},1}_{i+\frac{1}{2}} = f_{(u_i, u_{i+1})}$$

The second order accurate Roe takes the form:

$$f^{\text{Roe},2}_{i+\frac{1}{2}} = f_{(u^l_{i+1}, u^r_{i+1})}$$

Hence:

$$
\begin{aligned}
f^{\text{Roe},2}_{i+\frac{1}{2}} &= \frac{1}{2}\left(F_{\left(u^l_{1+\frac{1}{2}}\right)} + F_{\left(u^r_{1+\frac{1}{2}}\right)} - \left|\bar{u}_{i+\frac{1}{2}}\right|\left(u^r_{1+\frac{1}{2}} - u^l_{1+\frac{1}{2}}\right)\right) \\
\bar{u}_{1+\frac{1}{2}} &= \frac{F_{\left(u^r_{1+\frac{1}{2}}\right)} - F_{\left(u^l_{1+\frac{1}{2}}\right)}}{u^r_{1+\frac{1}{2}} - u^l_{1+\frac{1}{2}}} = \frac{u^l_{i+\frac{1}{2}} + u^r_{i+\frac{1}{2}}}{2}
\end{aligned} \tag{14}
$$

1.4.1 Without Limiters

The interface values without limiters are evaluated as:

$$
\begin{cases} u^l_{i+\frac{1}{2}} &= u_i + \frac{1-k}{4}\delta u_{i-\frac{1}{2}} + \frac{1+k}{4}\delta u_{i+\frac{1}{2}} \\ u^r_{i+\frac{1}{2}} &= u_{i+1} - \frac{1+k}{4}\delta u_{i+\frac{1}{2}} - \frac{1-k}{4}\delta u_{i+\frac{3}{2}} \end{cases} \qquad \delta u_i \triangleq u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} \tag{15}
$$

The parameter k determines the scheme:

$$
k = \begin{cases} -1 & \text{upwind} \\ 1 & \text{central} \end{cases}
$$

1.4.2 With Limiters

The interface values with limiters are evaluated as:

$$
\begin{cases} u^l_{i+\frac{1}{2}} &= u_i + \frac{1-k}{4}\overline{\delta^+} u_{i-\frac{1}{2}} + \frac{1+k}{4}\overline{\delta^-} u_{i+\frac{1}{2}} \\ u^r_{i+\frac{1}{2}} &= u_{i+1} - \frac{1+k}{4}\overline{\delta^+} u_{i+\frac{1}{2}} - \frac{1-k}{4}\overline{\delta^-} u_{i+\frac{3}{2}} \end{cases} \qquad \overline{\delta^\pm} u \text{ are limited slopes} \tag{16}
$$

$\overline{\delta}$ is an operator such that $\overline{\delta} u_i = \psi \delta u_i$, where $\psi\left(r\right)$ is a limiter function and:

$$
r^\pm = \begin{cases} r^+_{1+\frac{1}{2}} & \triangleq & \dfrac{u_{i+2} - u_{i+1}}{u_{i+1} - u_i} = \dfrac{\Delta u_{i+1}}{\Delta u_i} \\ r^-_{1+\frac{1}{2}} & \triangleq & \dfrac{u_i - u_{i-1}}{u_{i+1} - u_i} = \dfrac{\nabla u_i}{\nabla u_{i+1}} \end{cases} \tag{17}
$$

There are many types of limiters. For example, the van Albada limiter:

$$
\psi\left(r\right) = \frac{r + r^2}{1 + r^2} \tag{18}
$$

2 Generalized Burgers Equation

The generalized Burgers equation is given by:

$$\frac{\partial u}{\partial t} + (c + bu)\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} \tag{19}$$

2.1 Boundary and Initial Conditions

2.1.1 Initial Conditions

$$u_{x,t=0} = \frac{1}{2}\left(1 + \tanh\left(250\left(x - 20\right)\right)\right) \tag{20}$$

2.1.2 Boundary Conditions

Using Dirichlet boundary conditions:

$$u_{x=0,t} = \tag{21}$$