Computational Fluid Dynamics
HW2

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Nomenclature

Δx	size of each cell in the domain
$\Delta \tilde{t}$	normalized step size in time
$\Delta \tilde{x}$	normalized step size in space
γ	ratio of specific heats
κ	coefficient of thermal conductivity
μ	coefficient of viscosity
ρ	fluid density
c_p	constant specific heat capacity for a constant pressure
c_v	constant specific heat capacity for a constant volume
E	inviscid convective vector
e	total energy
E_ν	viscous convective vector
L	characteristic length
p	pressure
Q	conservation state space
R	gas constant
T	temperature
t	time
u	fluid velocity
x	spatial coordinate
x_F	x coordinate of the end of the domain
x_i	x coordinate of the i-th cell

Diagonal

Φ	main diagonal
Ψ	upper off-diagonal
Θ	lower off-diagonal

Far–Away Properties

κ_∞	coefficient of thermal conductivity far away
μ_∞	coefficient of viscosity far away

ρ_∞ density far away

a_∞ speed of sound far away

M_∞ mach number far away

p_∞ pressure far away

T_∞ temperature far away

Matrices

$\tilde{\Lambda}$ normalized eigenvalues matrix

$\tilde{A}$ normalized jacobian matrix of E w.r.t. Q

$\tilde{P}$ normalized jacobian matrix of E_ν w.r.t. Q

$\tilde{R}$ normalized jacobian matrix of E_ν w.r.t. Q_x

$\tilde{T}$ normalized eigenvectors matrix

Dimensionless Numbers

Pr_∞ Prandtl number far away

$Re_{L\infty}$ Reynolds number with respect to L far away

1 Problem Definition

1.1 Governing Equations

Consider the one-dimensional Navier-Stokes Equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = \frac{\partial E_\nu}{\partial x} \tag{1}$$

Where:

$$Q = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad E = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ (e+p)\,u \end{pmatrix}, \quad E_\nu = \begin{pmatrix} 0 \\ \tau_{xx} \\ u\tau_{xx} - q_x \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu\frac{\partial u}{\partial x} \\ \frac{4}{3}\mu u\frac{\partial u}{\partial x} + \kappa\frac{\partial T}{\partial x} \end{pmatrix}$$

$$p = (\gamma - 1)\left(e - \frac{1}{2}\rho u^2\right), \qquad T = \frac{p}{\rho R}, \tag{2}$$

$$\mu = 1.458 \cdot 10^{-6}\frac{T^{\frac{3}{2}}}{T + 110.4}, \quad \kappa = 2.495 \cdot 10^{-3}\frac{T^{\frac{3}{2}}}{T + 194}$$

$$R = c_p - c_v, \quad \gamma = \frac{c_p}{c_v}$$

The constants are:

- $\gamma = 1.4$ for air under standard atmospheric conditions
- $R = 287.0$ for air

1.2 Physical Domain

The physical domain is a tube extended between $x = 0.2$ and $x = 1.0$. At both ends there are impermeable walls.

1.3 Initial Conditions

The initial conditions are shown in Fig.1:

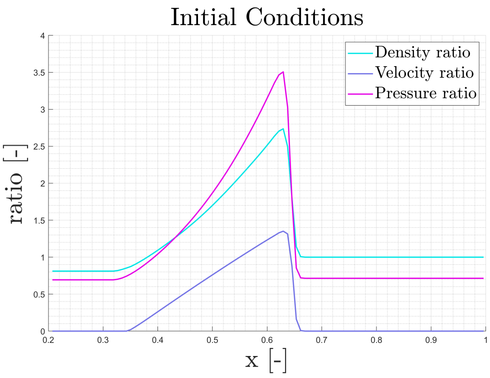

Figure 1: Initial conditions

1.4 Boundary Conditions

On each side of the tube there is an adiabatic, solid wall boundary conditions.

$$u_{(x=0.2)} = u_{(x=1.0)} = 0 \quad \Big\| \quad \left.\frac{\partial p}{\partial x}\right|_{x=0.2} = \left.\frac{\partial p}{\partial x}\right|_{x=1.0} = 0 \quad \Big\| \quad \left.\frac{\partial T}{\partial x}\right|_{x=0.2} = \left.\frac{\partial T}{\partial x}\right|_{x=1.0} = 0$$

2 Normalizing The Navier-Stokes Equations

Since the initial conditions are normalized, there is a need to normalize the N-S equations. We will use the following normalizations:

$$\rho = \rho_\infty \tilde{\rho}, \quad u = a_\infty \tilde{u}, \quad p = \gamma p_\infty \tilde{p}, \quad T = \gamma T_\infty \tilde{T}, \quad x = L\tilde{x}, \quad t = \frac{L}{a_\infty}\tilde{t}, \quad \mu = \mu_\infty \tilde{\mu}, \quad \kappa = \kappa_\infty \tilde{\kappa} \tag{3}$$

The normalization of the temperature was chosen to cancel out the γ in the normalization of the pressure:

$$\begin{aligned} p &= \rho R T \\ \gamma p_\infty \tilde{p} &= \rho_\infty \tilde{\rho} R \gamma T_\infty \tilde{T} \\ \tilde{p} &= \tilde{\rho}\tilde{T} \end{aligned} \tag{4}$$

The pressure normalization can be written also as:

$$p = \gamma p_\infty \tilde{p} = \gamma \rho_\infty R T_\infty \tilde{p} = \rho_\infty a_\infty^2 \tilde{p} \tag{5}$$

From equations 2 and 5 we can derive the normalization for the energy:

$$\begin{aligned} e &= \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \\ e &= \frac{\rho_\infty a_\infty^2 \tilde{p}}{\gamma - 1} + \frac{1}{2}\rho_\infty \tilde{\rho} a_\infty^2 \tilde{a}^2 \\ e &= \rho_\infty a_\infty^2 \left(\frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\tilde{a}^2 \right) \\ e &= \rho_\infty a_\infty^2 \tilde{e} \end{aligned} \tag{6}$$

After substituting the normalizations in the N-S equations we get:

$$\frac{\partial}{\partial \frac{L}{a_\infty}\tilde{t}} \begin{pmatrix} \rho_\infty \tilde{\rho} \\ \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{p} + \rho_\infty a_\infty^2 \tilde{\rho}\tilde{u}^2 \\ \rho_\infty a_\infty^3 (\tilde{e} + \tilde{p})\,\tilde{u} \end{pmatrix} = \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}\mu_\infty a_\infty \tilde{\mu} \frac{\partial \tilde{u}}{\partial L\tilde{x}} \\ \frac{4}{3}\mu_\infty a_\infty^2 \tilde{\mu}\tilde{u} \frac{\partial \tilde{u}}{\partial L\tilde{x}} + \frac{\kappa_\infty a_\infty^2}{R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial L\tilde{x}} \end{pmatrix} \tag{7}$$

Rearranging:

$$\frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ a_\infty \tilde{\rho}\tilde{u} \\ a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho}\tilde{u} \\ a_\infty \tilde{p} + a_\infty \tilde{\rho}\tilde{u}^2 \\ a_\infty^2 (\tilde{e} + \tilde{p})\,\tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L^2}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}a_\infty \tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3}a_\infty^2 \tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\kappa_\infty a_\infty^2}{\mu_\infty R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \tag{8}$$

Dividing the second equation by a_∞, the third equation by a_∞^2, and the whole set of equations by $\frac{\rho_\infty a_\infty}{L}$ we get:

$$\frac{\partial}{\partial \tilde{t}}\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix}+\frac{\partial}{\partial \tilde{x}}\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix}=\frac{\mu_\infty}{L\rho_\infty a_\infty}\frac{\partial}{\partial \tilde{x}}\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\kappa_\infty}{\mu_\infty R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}}\end{pmatrix} \tag{9}$$

The Reynolds number and the mach number far away are defined as:

$$\begin{gathered} M_\infty=\frac{u_\infty}{a_\infty}\quad Re_{L\infty}=\frac{\rho_\infty u_\infty L}{\mu_\infty}\\ \Downarrow\\ \frac{\mu_\infty}{L\rho_\infty a_\infty}=\frac{M_\infty}{Re_{L\infty}} \end{gathered} \tag{10}$$

The Prandtl number far away is defined as:

$$\begin{gathered} Pr_\infty=\frac{c_p\mu_\infty}{\kappa_\infty}\\ \Downarrow\\ \frac{\kappa_\infty}{\mu_\infty R}=\frac{c_p}{Pr_\infty\,(c_p-c_v)}=\frac{\gamma}{Pr_\infty\,(\gamma-1)} \end{gathered} \tag{11}$$

Substituting into the normalized N-S equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}}+\frac{\partial \tilde{E}}{\partial \tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{E}_\nu}{\partial \tilde{x}} \tag{12}$$

Where:

$$\tilde{Q}=\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix},\quad \tilde{E}=\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix},\quad \tilde{E}_\nu=\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\gamma}{Pr_\infty\,(\gamma-1)}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}}\end{pmatrix} \tag{13}$$

The normalized Navier-Stokes equations are:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}}+\frac{\partial \tilde{E}}{\partial \tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{14}$$

Where:

$$\tilde{V}_1=\tilde{V}_{1(\tilde{Q},\tilde{Q}_x)}=\tilde{E}_\nu$$

3 The Computational Domain

3.1 Discretization

The physical domain $[x_I, x_F]$ is discretized into N equispaced cells. The size of each cell is there for:

$$\Delta x=\frac{x_F-x_I}{N}=\frac{L}{N} \tag{15}$$

so the x coordinate of the i-th cell x_i is:

$$x_i=x_I+\frac{1}{2}\Delta x+\Delta x\cdot(i-1)\qquad \text{when starting from } i=1 \tag{16}$$

3.2 Boundary Conditions

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated as follows:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} \\ u_{(i=N+1)} &= -u_{(i=N)} \end{aligned} \tag{17}$$

in order to maintain velocity zero on the boundary and like so:

$$\begin{aligned} T_{(i=0)} &= T_{(i=1)} \\ T_{(i=N+1)} &= T_{(i=N)} \end{aligned} \tag{18}$$

in order to maintain adiabatic boundary conditions. Since the gradient of the pressure on the wall is zero, we get:

$$\begin{aligned} p_{(i=0)} &= p_{(i=1)} \\ p_{(i=N+1)} &= p_{(i=N)} \end{aligned} \tag{19}$$

From equations 2, 18, and 19 we can conclude:

$$\begin{aligned} \rho_{(i=0)} &= \rho_{(i=1)} \\ \rho_{(i=N+1)} &= \rho_{(i=N)} \end{aligned} \tag{20}$$

and from equations 2, 17, 19, and 20 we can conclude:

$$\begin{aligned} e_{(i=0)} &= e_{(i=1)} \\ e_{(i=N+1)} &= e_{(i=N)} \end{aligned} \tag{21}$$

4 The Numerical Schemes

4.1 Jacobian Matrices of The Navier-Stokes Equations

We can rewrite Eq.14 as:

$$\begin{aligned}
\frac{\partial \tilde{Q}}{\partial \tilde{t}} &= -\frac{\partial \tilde{E}}{\partial \tilde{x}} - \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \\
\frac{\partial \tilde{Q}}{\partial \tilde{t}} &= -\underbrace{\frac{\partial \tilde{E}}{\partial \tilde{Q}}}_{\tilde{A}} \frac{\partial \tilde{Q}}{\partial \tilde{x}} - \frac{M_\infty}{Re_{L\infty}} \left(\underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}}_{\tilde{P}} \frac{\partial \tilde{Q}}{\partial \tilde{x}} + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}}_{\tilde{R}} \frac{\partial \tilde{Q}_x}{\partial \tilde{x}} \right)
\end{aligned} \tag{22}$$

Where:

$$\begin{aligned}
\tilde{A} &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma - 3}{2}\tilde{u}^2 & (3-\gamma)\,\tilde{u} & \gamma - 1 \\ -\frac{\gamma \tilde{e} \tilde{u}}{\tilde{\rho}} - (\gamma - 1)\,\tilde{u}^3 & \frac{\gamma \tilde{e}}{\tilde{\rho}} - \frac{3(\gamma-1)\,\tilde{u}^2}{2} & \gamma \tilde{u} \end{pmatrix} \\
\tilde{P} - \tilde{R}_x &= -\frac{1}{\rho} \begin{pmatrix} 0 & 0 & 0 \\ -\tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & \left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \\ -\tilde{u}^2\left(\frac{4}{3}\tilde{\mu}\right)_x & \tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \end{pmatrix} \\
\tilde{R} &= -\frac{1}{\rho} \begin{pmatrix} 0 & 0 & 0 \\ \frac{4}{3}\tilde{u}\tilde{\mu} & -\frac{4}{3}\tilde{\mu} & 0 \\ \left(\frac{4}{3}\tilde{\mu} - \alpha \frac{\tilde{\kappa}}{c_v}\right)\tilde{u}^2 + \alpha \frac{\tilde{\kappa}}{c_v}\frac{\tilde{e}}{\tilde{\rho}} & -\left(\frac{4}{3}\tilde{\mu} - \alpha \frac{\tilde{\kappa}}{c_v}\right)\tilde{u} & -\alpha \frac{\tilde{\kappa}}{c_v} \end{pmatrix}
\end{aligned} \tag{23}$$

and α is:

$$\alpha = \frac{\gamma}{Pr_\infty (\gamma - 1)}$$

4.2 Linearization In Time

4.2.1 $\tilde{E}_i^{n+1}$ Estimation

$$\begin{aligned}
\tilde{E}_i^{n+1} &= \tilde{E}_i^n + \underbrace{\left.\frac{\partial \tilde{E}}{\partial \tilde{Q}}\right|_i^n}_{\tilde{A}_i^n} \Delta \tilde{Q}_i^n + \text{H.O.T} \\
\tilde{E}_i^{n+1} &= \tilde{E}_i^n + \tilde{A}_i^n \Delta \tilde{Q}_i^n
\end{aligned} \tag{24}$$

4.2.2 $\tilde{V}_{1i}^{n+1}$ Estimation

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^{n} + \underbrace{\left.\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}\right|_i^n}_{\tilde{P}_i^n} \Delta \tilde{Q}_i^n + \underbrace{\left.\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}\right|_i^n}_{\tilde{R}_i^n} \Delta \tilde{Q}_{xi}^{\ n} + \text{H.O.T} \tag{25}$$

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^{n} + \tilde{P}_i^n \Delta \tilde{Q}_i^n + \tilde{R}_i^n \Delta \tilde{Q}_{xi}^{\ n}$$

The difficulty stems from the fact that the solution vector is $\Delta\tilde{Q}$ and not $\Delta\tilde{Q}_x$. This can be solved by a linearization of the term $\Delta\tilde{Q}_x$ which can be conducted using the following relation:

$$\frac{\partial\left(\tilde{R}\Delta\tilde{Q}\right)_i^n}{\partial\tilde{x}} = \frac{\partial \tilde{R}_i^n}{\partial \tilde{x}}\Delta\tilde{Q}_i^n + \tilde{R}_i^n\frac{\partial\Delta\tilde{Q}_i^n}{\partial\tilde{x}} = \frac{\partial\tilde{R}}{\partial\tilde{x}}\Delta\tilde{Q} + \tilde{R}_i^n\Delta\tilde{Q}_{xi}^{\ n}$$

$$\Downarrow$$

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^{n} + \left(\tilde{P} - \tilde{R}_x\right)_i^n \Delta\tilde{Q}_i^n + \frac{\partial}{\partial\tilde{x}}\left(\tilde{R}\Delta\tilde{Q}\right)_i^n \tag{26}$$

4.3 First Order Approximate Riemann Roe Method

4.4 First Order Steger-Warming – Explicit

The A matrix (from Eq.23) is a diagonalizable matrix and can be written as:

$$\tilde{A} = \tilde{T}\tilde{\Lambda}\tilde{T}^{-1}$$

$$\tilde{T} = \begin{pmatrix} 1 & \frac{\tilde{\rho}}{2\tilde{a}} & -\frac{\tilde{\rho}}{2\tilde{a}} \\ \tilde{u} & \frac{\tilde{\rho}}{2\tilde{a}}\left(\tilde{u}+\tilde{a}\right) & -\frac{\tilde{\rho}}{2\tilde{a}}\left(\tilde{u}-\tilde{a}\right) \\ \frac{\tilde{u}^2}{2} & \frac{\tilde{\rho}}{2\tilde{a}}\left(\frac{\tilde{u}^2}{2}+\tilde{u}\tilde{a}+\frac{\tilde{a}^2}{\gamma-1}\right) & -\frac{\tilde{\rho}}{2\tilde{a}}\left(\frac{\tilde{u}^2}{2}-\tilde{u}\tilde{a}+\frac{\tilde{a}^2}{\gamma-1}\right) \end{pmatrix}$$

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{u} & 0 & 0 \\ 0 & \tilde{u}+\tilde{a} & 0 \\ 0 & 0 & \tilde{u}-\tilde{a} \end{pmatrix} \tag{27}$$

$$\tilde{T}^{-1} = \begin{pmatrix} 1-\frac{\gamma-1}{2}\frac{\tilde{u}^2}{\tilde{a}^2} & \left(\gamma-1\right)\frac{\tilde{u}^2}{\tilde{a}^2} & -\frac{\gamma-1}{\tilde{a}^2} \\ \frac{1}{\tilde{\rho}\tilde{a}}\left(\left(\gamma-1\right)\tilde{u}^2-\tilde{u}\tilde{a}\right) & \frac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a}-\left(\gamma-1\right)\tilde{u}\right) & \frac{\gamma-1}{\tilde{\rho}\tilde{a}} \\ -\frac{1}{\tilde{\rho}\tilde{a}}\left(\left(\gamma-1\right)\tilde{u}^2+\tilde{u}\tilde{a}\right) & \frac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a}+\left(\gamma-1\right)\tilde{u}\right) & -\frac{\gamma-1}{\tilde{\rho}\tilde{a}} \end{pmatrix}$$

Where:

$$\tilde{a} = \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}}$$

Let the $\tilde{\Lambda}^{\pm}$ matrix be defined as:

$$\tilde{\Lambda}^{\pm} = \begin{pmatrix} \dfrac{\tilde{u} \pm |\tilde{u}|}{2} & 0 & 0 \\ 0 & \dfrac{\tilde{u}+\tilde{a} \pm |\tilde{u}+\tilde{a}|}{2} & 0 \\ 0 & 0 & \dfrac{\tilde{u}-\tilde{a} \pm |\tilde{u}-\tilde{a}|}{2} \end{pmatrix} \tag{28}$$

Where the matrix $\tilde{\Lambda}^{+}$ contains only positive eigenvalues and the matrix $\tilde{\Lambda}^{-}$ contains only negative eigenvalues.
Define:

$$\begin{array}{lcl} \tilde{A}^{+} & \triangleq & \tilde{T}\tilde{\Lambda}^{+}\tilde{T}^{-1} \\ \tilde{A}^{-} & \triangleq & \tilde{T}\tilde{\Lambda}^{-}\tilde{T}^{-1} \end{array} \Rightarrow \begin{array}{lcl} \tilde{A} & = & \tilde{A}^{+}+\tilde{A}^{-} \\ \left|\tilde{A}\right| & \triangleq & \tilde{A}^{+}-\tilde{A}^{-} \end{array} \tag{29}$$

Assuming a perfect gas, the flux vector $\tilde{E}_{(Q)}$ is a homogeneous function of degree one in $\tilde{Q}$, meaning:

$$\forall\alpha \quad \tilde{E}_{\left(\alpha\tilde{Q}\right)} = \alpha\tilde{E}_{\left(\tilde{Q}\right)}$$

The homogeneity allows to rewrite the flux vector $\tilde{E}$ using Eq.22 as:

$$\tilde{E} = \tilde{A}\tilde{Q} = \left(\tilde{A}^{+}+\tilde{A}^{-}\right)\tilde{Q} = \underbrace{\tilde{A}^{+}\tilde{Q}}_{\tilde{E}^{+}} + \underbrace{\tilde{A}^{-}\tilde{Q}}_{\tilde{E}^{-}} = \tilde{E}^{+}+\tilde{E}^{-} \tag{30}$$

There is a discontinuity and deference between $\tilde{E}^{+}, \tilde{E}^{-}$. To eliminate the discontinuities and guarantee a smooth transition through critical points (sonic points or stagnation points), a blending function is introduced together with a blending parameter ε. An appropriate choice of the blending parameter has to be chosen.

$$\begin{array}{lcl} \tilde{\lambda}^{+} & = & \dfrac{\tilde{\lambda}+\left|\tilde{\lambda}\right|}{2} \\ \tilde{\lambda}^{-} & = & \dfrac{\tilde{\lambda}-\left|\tilde{\lambda}\right|}{2} \end{array} \Rightarrow \begin{array}{lcl} \tilde{\lambda}^{+\prime} & = & \dfrac{\tilde{\lambda}+\sqrt{\tilde{\lambda}^2+\varepsilon^2}}{2} \\ \tilde{\lambda}^{-\prime} & = & \dfrac{\tilde{\lambda}-\sqrt{\tilde{\lambda}^2+\varepsilon^2}}{2} \end{array} \tag{31}$$

Rewriting the conservation law form of the N-S equations Eq.14 using Eq.30:

$$\frac{\partial\tilde{Q}}{\partial\tilde{t}} = -\frac{\partial\tilde{E}^{+}}{\partial\tilde{x}} - \frac{\partial\tilde{E}^{-}}{\partial\tilde{x}} + \frac{M_\infty}{Re_{L\infty}}\frac{\partial\tilde{V}_1}{\partial\tilde{x}} \tag{32}$$

A simple, explicit, first order (in space and time) scheme in delta form is obtained using:

$$\Delta\tilde{Q}_i^n = -\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\nabla\tilde{E}_i^{+n} + \Delta\tilde{E}_i^{-n} - \frac{M_\infty}{Re_{L\infty}}\delta\tilde{V}_{1,i}^n\right) \tag{33}$$

And advancing the solution by:

$$\tilde{Q}_i^{n+1} = \Delta\tilde{Q}_i^n + \tilde{Q}_i^n \tag{34}$$

4.4.1 Finite Volume Formulation

Rearranging Eq.33 using the finite volume notation:

$$\begin{array}{lcl} \Delta\tilde{Q}_i^n & = & -\dfrac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\tilde{E}_i^{+n} - \tilde{E}_{i-1}^{+n} + \tilde{E}_{i+1}^{-n} - \tilde{E}_i^{-n} - \dfrac{M_\infty}{Re_{L\infty}}\left(\tilde{V}_{1,i+\frac{1}{2}}^n - \tilde{V}_{1,i-\frac{1}{2}}^n\right)\right) \\ \Delta\tilde{Q}_i^n & = & -\dfrac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\left(\tilde{E}_i^{+n} + \tilde{E}_{i+1}^{-n}\right) - \left(\tilde{E}_{i-1}^{+n} + \tilde{E}_i^{-n}\right) - \dfrac{M_\infty}{Re_{L\infty}}\left(\tilde{V}_{1,i+\frac{1}{2}}^n - \tilde{V}_{1,i-\frac{1}{2}}^n\right)\right) \end{array} \tag{35}$$

Define:

$$\begin{aligned}\tilde{E}_{i+\frac{1}{2}} &\triangleq \tilde{E}_i^{+} + E_{i+1}^{-} \\ &= \tilde{A}_i^{+}\tilde{Q}_i + \tilde{A}_{i+1}^{-}\tilde{Q}_{i+1} \\ &\equiv \tilde{\tilde{E}}_{i+\frac{1}{2}}\end{aligned} \tag{36}$$

Finally we get:

$$\Delta\tilde{Q}_i^n = -\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\tilde{\tilde{E}}^n_{i+\frac{1}{2}} - \tilde{\tilde{E}}^n_{i-\frac{1}{2}} - \frac{M_\infty}{Re_{L\infty}}\left(\tilde{V}^n_{1,i+\frac{1}{2}} - \tilde{V}^n_{1,i-\frac{1}{2}}\right)\right) \tag{37}$$

4.4.2 Calculating $\tilde{V}_{1,i+\frac{1}{2}}$

$$\tilde{V}_{1,i+\frac{1}{2}} = \begin{pmatrix} 0 \\ \frac{4}{3}\,\tilde{\mu}\big|_{i+\frac{1}{2}}\,\frac{\partial\tilde{u}}{\partial\tilde{x}}\Big|_{i+\frac{1}{2}} \\ \frac{4}{3}\,\tilde{\mu}\big|_{i+\frac{1}{2}}\,\tilde{u}\big|_{i+\frac{1}{2}}\,\frac{\partial\tilde{u}}{\partial\tilde{x}}\Big|_{i+\frac{1}{2}} + \frac{\gamma}{Pr_\infty(\gamma-1)}\,\tilde{\kappa}\big|_{i+\frac{1}{2}}\,\frac{\partial\tilde{T}}{\partial\tilde{x}}\Big|_{i+\frac{1}{2}} \end{pmatrix} \tag{38}$$

Where:

$$\begin{aligned}\frac{\partial\tilde{u}}{\partial\tilde{x}}\Big|_{i+\frac{1}{2}} &= \frac{\tilde{u}_{i+1}-\tilde{u}_i}{\Delta\tilde{x}}, & \tilde{\mu}\big|_{i+\frac{1}{2}} &= \frac{\tilde{\mu}_{i+1}+\tilde{\mu}_i}{2} \\ \frac{\partial\tilde{T}}{\partial\tilde{x}}\Big|_{i+\frac{1}{2}} &= \frac{\tilde{T}_{i+1}-\tilde{T}_i}{\Delta\tilde{x}}, & \tilde{\kappa}\big|_{i+\frac{1}{2}} &= \frac{\tilde{\kappa}_{i+1}+\tilde{\kappa}_i}{2} \\ & & \tilde{u}\big|_{i+\frac{1}{2}} &= \frac{\tilde{u}_{i+1}+\tilde{u}_i}{2}\end{aligned} \tag{39}$$

4.5 First Order Steger-Warming – Implicit

The Implicit Steger-Warming scheme starts from:

$$\begin{aligned}\frac{\Delta\tilde{Q}_i^n}{\Delta\tilde{t}} &= -\frac{\partial\tilde{E}_i^{n+1}}{\partial\tilde{x}} + \frac{M_\infty}{Re_{L\infty}}\frac{\partial\tilde{V}_{1,i}^{n+1}}{\partial\tilde{x}} \\ \Delta\tilde{Q}_i^n &= -\Delta\tilde{t}\frac{\partial}{\partial\tilde{x}}\left(\tilde{E}_i^n + \tilde{A}_i^n\Delta\tilde{Q}_i^n\right) + \Delta\tilde{t}\frac{M_\infty}{Re_{L\infty}}\frac{\partial}{\partial\tilde{x}}\left(\tilde{V}_{1,i}^n + \left(\tilde{P}-\tilde{R}_x\right)_i^n\Delta\tilde{Q}_i^n + \frac{\partial}{\partial\tilde{x}}\left(\tilde{R}\Delta\tilde{Q}\right)_i^n\right)\end{aligned} \tag{40}$$

Rearranging in delta form:

$$\begin{aligned}\left(I + \Delta\tilde{t}\left(\frac{\partial}{\partial\tilde{x}}\left[\tilde{A} - \left(\tilde{P}-\tilde{R}_x\right)\right]_i^n - \frac{\partial^2\tilde{R}_i^n}{\partial\tilde{x}^2}\right)\right)\Delta\tilde{Q}_i^n &= \mathrm{RHS}_i^n \\ \mathrm{RHS}_i^n = -\Delta\tilde{t}\frac{\partial}{\partial\tilde{x}}\left(\tilde{E}^+ + \tilde{E}^- - \frac{M_\infty}{Re_{L\infty}}\tilde{V}_1\right)_i^n\end{aligned} \tag{41}$$

$$\begin{gathered}\left(I+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\left[\nabla\tilde{A}^{+}+\Delta\tilde{A}^{-}-\frac{D_0}{2}\left(\tilde{P}-\tilde{R}_x\right)\right]_i^n-\frac{\delta^2}{\Delta\tilde{x}}\tilde{R}_i^n\right)\right)\Delta\tilde{Q}_i^n=\mathrm{RHS}_i^n\\ \mathrm{RHS}_i^n=-\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\nabla\tilde{E}^{+}+\Delta\tilde{E}^{-}-\frac{M_\infty}{Re_{L\infty}}\delta\tilde{V}_1\right)_i^n\end{gathered}\tag{42}$$

Rewrite using the finite volume notation for the inviscid terms like in Eq.37:

$$\begin{gathered}\left(I+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\left[\nabla\tilde{A}^{+}+\Delta\tilde{A}^{-}-\frac{D_0}{2}\left(\tilde{P}-\tilde{R}_x\right)\right]_i^n-\frac{\delta^2}{\Delta\tilde{x}}\tilde{R}_i^n\right)\right)\Delta\tilde{Q}_i^n=\mathrm{RHS}_i^n\\ \mathrm{RHS}_i^n=-\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\tilde{\tilde{E}}^n_{i+\frac{1}{2}}-\tilde{\tilde{E}}^n_{i-\frac{1}{2}}-\frac{M_\infty}{Re_{L\infty}}\left(\tilde{V}^n_{1,i+\frac{1}{2}}-\tilde{V}^n_{1,i-\frac{1}{2}}\right)\right)\end{gathered}\tag{43}$$

and opening the delta form on the LHS:

$$\begin{aligned}\mathrm{LHS}_i^n &= \Delta\tilde{Q}_i^n+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\nabla\tilde{A}_i^{+n}\Delta\tilde{Q}_i^n+\Delta\tilde{A}_i^{-n}\Delta\tilde{Q}_i^n-\frac{D_0}{2}\left(\tilde{P}-\tilde{R}_x\right)_i^n\Delta\tilde{Q}_i^n-\frac{\delta^2}{\Delta\tilde{x}}\tilde{R}_i^n\Delta\tilde{Q}_i^n\right)\\ &= \Delta\tilde{Q}_i^n+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\Big(\tilde{A}_i^{+n}\Delta\tilde{Q}_i^n-\tilde{A}_{i-1}^{+n}\Delta\tilde{Q}_{i-1}^n+\tilde{A}_{i+1}^{-n}\Delta\tilde{Q}_{i+1}^n-\tilde{A}_i^{-n}\Delta\tilde{Q}_i^n-\\ &\quad -\frac{1}{2}\left[\left(\tilde{P}-\tilde{R}_x\right)_{i+1}^n\Delta\tilde{Q}_{i+1}^n-\left(\tilde{P}-\tilde{R}_x\right)_{i-1}^n\Delta\tilde{Q}_{i-1}^n\right]-\\ &\quad -\frac{1}{\Delta\tilde{x}}\left[\tilde{R}_{i+1}^n\Delta\tilde{Q}_{i+1}^n-2\tilde{R}_i^n\Delta\tilde{Q}_i^n+\tilde{R}_{i-1}^n\Delta\tilde{Q}_{i-1}^n\right]\Big)\\ &= \Theta_i^n\Delta\tilde{Q}_{i-1}^n+\Phi_i^n\Delta\tilde{Q}_i^n+\Psi_i^n\Delta\tilde{Q}_{i+1}^n\end{aligned}\tag{44}$$

Where:

$$\begin{aligned}\Theta_i^n &= -\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\tilde{A}_{i-1}^{+n}+\frac{\Delta\tilde{t}}{2\Delta\tilde{x}}\left(\tilde{P}-\tilde{R}_x\right)_{i-1}^n-\frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\tilde{R}_{i-1}^n\\ \Phi_i^n &= I+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\tilde{A}_i^{+n}-\tilde{A}_i^{-n}\right)+2\frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\tilde{R}_i^n\\ \Psi_i^n &= \frac{\Delta\tilde{t}}{\Delta\tilde{x}}\tilde{A}_{i+1}^{-n}-\frac{\Delta\tilde{t}}{2\Delta\tilde{x}}\left(\tilde{P}-\tilde{R}_x\right)_{i+1}^n-\frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\tilde{R}_{i+1}^n\end{aligned}\tag{45}$$

For the implicit Steger-Warming scheme a matrix inversion is needed as follows:

$$\begin{pmatrix}\Phi_1^n & \Psi_1^n & 0 & \cdots & \cdots & \cdots & 0\\ \Theta_2^n & \Phi_2^n & \Psi_2^n & 0 & \cdots & \cdots & 0\\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0\\ 0 & 0 & \Theta_i^n & \Phi_i^n & \Psi_i^n & 0 & 0\\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & \cdots & 0 & \Theta_{N-1}^n & \Phi_{N-1}^n & \Psi_{N-1}^n\\ 0 & \cdots & \cdots & \cdots & 0 & \Theta_N^n & \Phi_N^n\end{pmatrix}\begin{pmatrix}\Delta\tilde{Q}_1^n\\ \Delta\tilde{Q}_2^n\\ \cdots\\ \cdots\\ \cdots\\ \Delta\tilde{Q}_{N-1}^n\\ \Delta\tilde{Q}_N^n\end{pmatrix}=\begin{pmatrix}\mathrm{RHS}_1^n\\ \mathrm{RHS}_2^n\\ \cdots\\ \cdots\\ \cdots\\ \mathrm{RHS}_{N-1}^n\\ \mathrm{RHS}_N^n\end{pmatrix}\tag{46}$$