Computational Fluid Dynamics

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Contents

1 Mathematical Problem

The Burgers Equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{1}$$

By neglection the convective term, the parabolic model equation is obtained:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} \tag{3}$$

$$\mu = 1.0 \tag{4}$$

Boundary and initial conditions:

$$u_{(y_0,t)} = u_0 \qquad u_{(y_1,t)} = u_1 \qquad u_{(y,t=0)} = f_{(y)} \tag{5}$$

2 Numerical Scheme

A general explicit-implicit scheme for constant μ is given by:

$$u_i^{n+1} = u_i^n + \frac{\mu\Delta t}{\Delta y^2}\delta_{yy}\left[(1-\alpha)\,u_i^n + \alpha u_i^{n+1}\right] \tag{6}$$

where:

$$\alpha = \begin{cases} 0 & \text{Explicit} \\ \frac{1}{2} & \text{Crank-Nicolson} \\ 1 & \text{Implicit} \end{cases} \tag{7}$$

and the order is:

$$\left[\Delta x^2, \Delta\left(\frac{1}{2}-\alpha\right)\right] \tag{8}$$

In delta form:

$$\left(I - \alpha\frac{\mu\Delta t}{\Delta y^2}\delta_{yy}\right)\Delta u_i^n = \frac{\mu\Delta t}{\Delta y^2}\delta_{yy}u_i^n \tag{9}$$

Applying the operators:

$$\Delta u_i^n - \alpha\frac{\mu\Delta t}{\Delta y^2}\delta_{yy}\left(\Delta u_i^n\right) = \frac{\mu\Delta t}{\Delta y^2}\delta_{yy}\left(u_i^n\right) \tag{10}$$

$$\Delta u_i^n - \alpha\frac{\mu\Delta t}{\Delta y^2}\delta_y\left(\Delta u^n_{\left(y_i+\frac{\Delta y}{2}\right)} - \Delta u^n_{\left(y_i-\frac{\Delta y}{2}\right)}\right) = \frac{\mu\Delta t}{\Delta y^2}\delta_y\left(u^n_{\left(y_i+\frac{\Delta y}{2}\right)} - u^n_{\left(y_i-\frac{\Delta y}{2}\right)}\right) \tag{11}$$

$$\Delta u_i^n - \alpha\frac{\mu\Delta t}{\Delta y^2}\left(\Delta u^n_{(y_i+\Delta y)} - 2\Delta u^n_{(y_i)} + \Delta u^n_{(y_i-\Delta y)}\right) = \frac{\mu\Delta t}{\Delta y^2}\left(u^n_{(y_i+\Delta y)} - 2u^n_{(y_i)} + u^n_{(y_i-\Delta y)}\right) \tag{12}$$

$$\Delta u_i^n - \alpha\frac{\mu\Delta t}{\Delta y^2}\left(\Delta u_{i+1}^n - 2\Delta u_i^n + \Delta u_{i-1}^n\right) = \frac{\mu\Delta t}{\Delta y^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \tag{13}$$

$$A_i\Delta u_{i-1}^n + B_i\Delta u_i^n + C_i\Delta u_{i+1}^n = D_i \tag{14}$$

where:

$$A_i = -\alpha \frac{\mu \Delta t}{\Delta y^2} \tag{15}$$

$$B_i = 1 + 2\alpha \frac{\mu \Delta t}{\Delta y^2} \tag{16}$$

$$C_i = -\alpha \frac{\mu \Delta t}{\Delta y^2} \tag{17}$$

$$D_i = \frac{\mu \Delta t}{\Delta y^2} \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n \right) \tag{18}$$

and

$$u_i^{n+1} = u_i^n + \Delta u_i^n \tag{19}$$

In matrix from:

$$\begin{pmatrix} B_1 & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ A_2 & B_2 & C_2 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & 0 & A_i & B_i & C_i & 0 & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & A_{N-2} & B_{N-2} & C_{N-2} \\ 0 & 0 & \cdots & \cdots & 0 & A_{N-1} & B_{N-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \cdots \\ \cdots \\ \cdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix} = \begin{pmatrix} D_1 - A_1 \cdot u_0 \\ D_2 \\ \cdots \\ \cdots \\ \cdots \\ D_{N-2} \\ D_{N-1} - C_{N-1} \cdot u_N \end{pmatrix} \tag{20}$$

To reduce problems of big flouting point numbers, define r:

$$r \triangleq \frac{\mu \Delta t}{\Delta y^2} \tag{21}$$

After dividing by r:

$$A_i = -\alpha \tag{22}$$

$$B_i = \frac{1}{r} + 2\alpha \tag{23}$$

$$C_i = -\alpha \tag{24}$$

$$D_i = \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n \right) \tag{25}$$

3 Stability Analysis

4 The Computer Program

5 Results

6 Conclusions