Computational Fluid Dynamics
HW1

Almog Dobrescu

ID 214254252

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1 Inviscid Burgers Equation

The Inviscid Burgers equation, in conservation law form, is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad F = F_{(u)} = \frac{u^2}{2} \tag{1}$$

In non-conservation law form, is given by:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \quad A = \frac{\partial F}{\partial u} = u \tag{2}$$

The equation is obtained by neglecting the viscous term from the viscous Burger equation.

1.1 Boundary and Initial Conditions

$$\begin{aligned} u_{(x=0,t)} &= 1.0 \\ u_{(x=1,t)} &= u_1 \\ u_{(x,t=0)} &= 1-(1-u_1)\cdot x \end{aligned} \tag{3}$$

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated like so:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} + 2\cdot u0 \\ u_{(i=N+1)} &= -u_{(i=N)} + 2\cdot u1 \end{aligned} \tag{4}$$

1.2 Finite Volume Formulation

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}\right) \tag{5}$$

- For first-order schemes, there is no variation within a cell, and the value there is constant.
- For second-order schemes, the variation within the cell is linear.

1.3 CFL number

For the Roe method, the CFL number is defined as:

$$\text{CFL} = \frac{u\Delta t}{\Delta x} \tag{6}$$

We will want to set the maximal value of the *CFL* number. We will find the Δt at each cell and (Δt_i) and set the Δt of the current step as:

$$\Delta t = \min(\Delta t_i) \ \ \forall i \tag{7}$$

1.4 First Order Roe Method $(u_1 = 0.0)$

Roe scheme is based on the solution of the linear problem:

$$\frac{\partial u}{\partial t} + \bar{A}\frac{\partial u}{\partial x} = 0 \tag{8}$$

Where $\bar{A}$ is a constant matrix that is dependent on local conditions. The matrix is constructed in a way that guarantees uniform validity across discontinuities:

1. For any u_i, u_{i+1}:
$$F_{i+1} - F_i = \bar{A} \cdot (u_{i+1} - u_i)$$
2. When $u = u_i = u_{i+1}$ then:
$$\bar{A}_{(u_i, ui+1)} = \bar{A}_{(u,u)} = \frac{\partial F}{\partial u} = u$$

In the case of the Burgers equation, the matrix $\bar{A}$ is a scalar, namely, $\bar{A} = \bar{u}$. The equation becomes:
$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} = 0 \tag{9}$$
The value of $\bar{u}$ for the cell face between i and $i+1$ is determined from the first conditions:
$$\bar{u} = \bar{u}_{i+\frac{1}{2}} = \frac{F_{i+1} - F_i}{u_{i+1} - u_i} = \frac{\frac{1}{2}u_{i+1}^2 - \frac{1}{2}u_i^2}{u_{i+1} - u_i} = \begin{cases} \frac{u_i + u_{i+1}}{2} & u_i \neq u_{i+1} \\ u_i & u_i = u_{i+1} \end{cases} \tag{10}$$
The single wave that emanates from the cell interface travels either in the positive or negative direction, depending upon the sign of $\bar{u}_{i+\frac{1}{2}}$. Define:
$$\begin{cases} \bar{u}^+_{i+\frac{1}{2}} \triangleq \frac{1}{2}\left(\bar{u}_{i+\frac{1}{2}} + \left|\bar{u}_{i+\frac{1}{2}}\right|\right) \geq 0 \\ \bar{u}^-_{i+\frac{1}{2}} \triangleq \frac{1}{2}\left(\bar{u}_{i+\frac{1}{2}} - \left|\bar{u}_{i+\frac{1}{2}}\right|\right) \leq 0 \end{cases} \qquad \bar{u}_{i+\frac{1}{2}} = \bar{u}^+_{i+\frac{1}{2}} + \bar{u}^-_{i+\frac{1}{2}} \tag{11}$$
Using the jump relation, the numerical flux at the cell interface can be evaluated by one of the following:
$$\begin{cases} f_{i+\frac{1}{2}} - F_i = \bar{u}^-_{i+\frac{1}{2}} \cdot (u_{i+1} - u_i) \\ F_{i+1} - f_{i+\frac{1}{2}} = \bar{u}^+_{i+\frac{1}{2}} \cdot (u_{i+1} - u_i) \end{cases} \tag{12}$$
The numerical flux may then be written in the following symmetric form:
$$\begin{aligned} f_{i+\frac{1}{2}} &= \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\left(\bar{u}^+_{i+\frac{1}{2}} - \bar{u}^-_{i+\frac{1}{2}}\right)(u_{i+1} - u_i) \\ \text{OR}&: \\ f_{i+\frac{1}{2}} &= \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\left|\bar{u}_{i+\frac{1}{2}}\right|(u_{i+1} - u_i) \end{aligned} \tag{13}$$

1.4.1 Effect of CFL

1.5 Second Order Roe ($u_1 = 0.5$)

The first-order accurate Roe method interface flux function will be denoted like this:
$$f^{\text{Roe},1}_{i+\frac{1}{2}} = f_{(u_i, u_{i+1})}$$

The second order accurate Roe takes the form:
$$f^{\text{Roe},2}_{i+\frac{1}{2}} = f_{(u^l_{i+1}, u^r_{i+1})}$$

Hence:

$$f^{\mathrm{Roe},2}_{i+\frac{1}{2}} = \frac{1}{2}\left(F_{\left(u^l_{1+\frac{1}{2}}\right)} + F_{\left(u^r_{1+\frac{1}{2}}\right)} - \left|\bar{u}_{i+\frac{1}{2}}\right|\left(u^r_{1+\frac{1}{2}} - u^l_{1+\frac{1}{2}}\right)\right)$$
$$\bar{u}_{1+\frac{1}{2}} = \frac{F_{\left(u^r_{1+\frac{1}{2}}\right)} - F_{\left(u^l_{1+\frac{1}{2}}\right)}}{u^r_{1+\frac{1}{2}} - u^l_{1+\frac{1}{2}}} = \frac{u^l_{i+\frac{1}{2}} + u^r_{i+\frac{1}{2}}}{2} \tag{14}$$

1.5.1 Without Limiters

The interface values without limiters are evaluated as:

$$\begin{cases} u^l_{i+\frac{1}{2}} &= u_i + \frac{1-k}{4}\delta u_{i-\frac{1}{2}} + \frac{1+k}{4}\delta u_{i+\frac{1}{2}} \\ u^r_{i+\frac{1}{2}} &= u_{i+1} - \frac{1+k}{4}\delta u_{i+\frac{1}{2}} - \frac{1-k}{4}\delta u_{i+\frac{3}{2}} \end{cases} \qquad \delta u_i \triangleq u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} \tag{15}$$

The parameter k determines the scheme:

$$k = \begin{cases} -1 & \text{upwind} \\ 1 & \text{central} \end{cases}$$

1.5.2 With Limiters

The interface values with limiters are evaluated as:

$$\begin{cases} u^l_{i+\frac{1}{2}} &= u_i + \tfrac{1-k}{4}\overline{\delta^+}u_{i-\frac{1}{2}} + \tfrac{1+k}{4}\overline{\delta^-}u_{i+\frac{1}{2}} \\ u^r_{i+\frac{1}{2}} &= u_{i+1} - \tfrac{1+k}{4}\overline{\delta^+}u_{i+\frac{1}{2}} - \tfrac{1-k}{4}\overline{\delta^-}u_{i+\frac{3}{2}} \end{cases} \qquad \overline{\delta^\pm}u \text{ are limited slopes} \tag{16}$$

$\overline{\delta}$ is an operator such that $\overline{\delta}u_i = \psi\delta u_i$, where $\psi(r)$ is a limiter function and:

$$r^\pm = \begin{cases} r^+_{1+\frac{1}{2}} & \triangleq & \dfrac{u_{i+2} - u_{i+1}}{u_{i+1} - u_i} = \dfrac{\Delta u_{i+1}}{\Delta u_i} \\ r^-_{1+\frac{1}{2}} & \triangleq & \dfrac{u_i - u_{i-1}}{u_{i+1} - u_i} = \dfrac{\nabla u_i}{\nabla u_{i+1}} \end{cases} \tag{17}$$

There are many types of limiters. For example, the van Albada limiter:

$$\psi(r) = \frac{r + r^2}{1 + r^2} \tag{18}$$

1.5.3 Effect of CFL

1.5.4 Effect of Limiter

2 Generalized Burgers Equation

The generalized Burgers equation is given by:

$$\frac{\partial u}{\partial t} + (c + bu)\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \tag{19}$$

Where:

$$c = \tfrac{1}{2} \qquad b = -1 \qquad \mu = [0.001, 0.25]$$

The equation can also be presented as:

$$\frac{\partial u}{\partial t} + \frac{\partial \bar{F}}{\partial x} = 0 \qquad \bar{F} = \underbrace{cu + \frac{bu^2}{2}}_{F} - \underbrace{\mu \frac{\partial u}{\partial x}}_{F_\nu} \tag{20}$$

In non-conservation law form, is given by:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \qquad A = \frac{\partial \bar{F}}{\partial u} = c + bu - \mu \frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\right) \tag{21}$$

The generalized Burgers equation has a stationary solution:

$$u = -\frac{c}{b}\left(1 + \tanh\left(\frac{c\,(x - x_0)}{2\mu}\right)\right) \tag{22}$$

2.1 Domain and Computational Mesh

Using 41 grid points with $\Delta x = 1$ and computing until $t = 18.0$. $\Delta t = [0.5, 1.0]$.

2.2 Boundary and Initial Conditions

2.2.1 Initial Conditions

$$u_{(x,t=0)} = \frac{1}{2}\left(1 + \tanh\left(250\,(x - 20)\right)\right) \tag{23}$$

2.2.2 Boundary Conditions

Using Dirichlet boundary conditions:

$$u_{(x=0,t)} = 0 \qquad u_{(x=40,t)} = 1 \tag{24}$$

2.3 First Order Roe Method (explicit)

As written above for the inviscid Burgers equation (1.4), Roes scheme is based on the solution of the linear problem:

$$\frac{\partial u}{\partial t} + \bar{A}\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \qquad \bar{A} = \frac{\partial F}{\partial u} \tag{25}$$

In the case of the Burgers equation, the matrix $\bar{A}$ is a scalar.

$$\bar{A} = \bar{A}_{i+\frac{1}{2}} = \frac{F_{i+1} - F_i}{u_{i+1} - u_i} = \begin{cases} \dfrac{c\,(u_{i+1} - u_i) + \dfrac{b}{2}\left(u_{i+1}^2 - u_i^2\right)}{u_{i+1} - u_i} & u_i \neq u_{i+1} \\ A_i & u_i = u_{i+1} \end{cases} \tag{26}$$

The numerical flux at the cell interface:

$$\bar{f}_{i+\frac{1}{2}} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\left(\bar{A}^{+}_{i+\frac{1}{2}} - \bar{A}^{-}_{i+\frac{1}{2}}\right)(u_{i+1} - u_i) \tag{27}$$

Where:

$$\begin{cases} \bar{A}^{+}_{i+\frac{1}{2}} \triangleq \frac{1}{2}\left(\bar{A}i + \frac{1}{2} + \left|\bar{A}_{i+\frac{1}{2}}\right|\right) \geq 0 \\ \bar{A}^{-}_{i+\frac{1}{2}} \triangleq \frac{1}{2}\left(\bar{A}_{i+\frac{1}{2}} - \left|\bar{A}_{i+\frac{1}{2}}\right|\right) \leq 0 \end{cases} \qquad \bar{A}_{i+\frac{1}{2}} = \bar{A}^{+}_{i+\frac{1}{2}} + \bar{A}^{-}_{i+\frac{1}{2}} \tag{28}$$

And finally:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left(\bar{f}^n_{i+\frac{1}{2}} - \bar{f}^n_{i-\frac{1}{2}}\right) + \mu\frac{\Delta t}{(\Delta x)^2}\left(u^n_{i+1} - 2u^n_i + u^n_{i-1}\right) \tag{29}$$

2.4 MacCormack Method

The original MacCormack method applied to Burgers equaiton results in:

$$\begin{aligned} &\text{Predictor :} && u_i^{\overline{n+1}} = u_i^n - \Delta t\frac{\Delta F_i^n}{\Delta x} + r\delta^2 u_i^n \\ &\text{Corrector :} && u_i^{n+1} = \frac{1}{2}\left(u_i^n + u_i^{\overline{n+1}} - \Delta t\frac{\nabla F_i^{\overline{n+1}}}{\Delta x}\right) + r\delta^2 u_i^{\overline{n+1}} \end{aligned} \tag{30}$$

Where:

- $r = \frac{\mu\Delta t}{(\Delta x)^2}$
- $\delta^2 u_i = u_{i+1} - 2u_i + u_{i-1}$
- $\Delta f = f_{i+1} - f_i$
- $\nabla f = f_i - f_{i-1}$

2.4.1 Stability

The stability condition for MacCormack's method is:

$$\Delta t \leq \frac{(\Delta x)^2}{|u|\,\Delta x + \mu} \tag{31}$$

2.5 Beam and Warming

Beam and Warming inroduced the Delta form and their method is more efficient:

$$\left(I + \theta\left(\Delta t\frac{D_0 A_i^n}{2\Delta x} - \frac{\Delta t\mu}{(\Delta x)^2}\delta^2\right)\right)\Delta u_i^n = \underbrace{-\Delta t\frac{F^n_{i+1} - F^n_{i-1}}{2\Delta x} + \Delta t\mu\frac{u^n_{i+1} - 2u^n_i + u^n_{i-1}}{(\Delta x)^2}}_{\mathrm{RHS}_i^n}$$

$$\begin{cases} \theta = 1 & \text{first order} \\ \theta = 0.5 & \text{second order} \end{cases} \tag{32}$$

Deriving the matrix to invert:

$$\left(I + \theta\Delta t \frac{D_0 A_i^n}{2\Delta x}\right)\Delta u_i^n - \theta\frac{\Delta t\mu}{(\Delta x)^2}\delta^2\Delta u_i^n = \mathrm{RHS}_i^n$$
$$\left(I + \theta\Delta t \frac{D_0 A_i^n}{2\Delta x}\right)\Delta u_i^n - \theta\frac{\Delta t\mu}{(\Delta x)^2}\left(\Delta u_{i+1}^n - 2\Delta u_i^n + \Delta u_{i-1}^n\right) = \mathrm{RHS_i^n}$$
$$\Delta u_i^n + \theta\frac{\Delta t}{2\Delta x}\left(A_{i-1}^n\Delta u_{i-1}^n - A_{i+1}^n\Delta u_{i+1}^n\right) - \theta\frac{\Delta t\mu}{(\Delta x)^2}\left(\Delta u_{i+1}^n - 2\Delta u_i^n + \Delta u_{i-1}^n\right) = \mathrm{RHS_i^n}$$
$$A_i\Delta u_{i-1}^n + B_i\Delta u_i^n + C_i\Delta u_{i+1}^n = D_i$$

$$\Downarrow$$

$$\begin{aligned}
A_i &= -\theta\frac{\mu\Delta t}{(\Delta x)^2} + \theta\frac{\Delta t}{2\Delta x}A_{i-1}^n \\
B_i &= 1 + 2\theta\frac{\mu\Delta t}{(\Delta x)^2} \\
C_i &= -\alpha\frac{\mu\Delta t}{(\Delta x)^2} - \theta\frac{\Delta t}{2\Delta x}A_{i-1}^n \\
D_i &= RHS_i^n
\end{aligned} \tag{33}$$

and advancing the solution with:

$$u_i^{n+1} = u_i^n + \Delta u_i^n \tag{34}$$

In order to calculate Δu_i^n it is needed to invert matrix as follows:

$$\begin{pmatrix}
B_1 & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\
A_2 & B_2 & C_2 & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\
0 & 0 & A_i & B_i & C_i & 0 & 0 \\
0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & A_{N-2} & B_{N-2} & C_{N-2} \\
0 & 0 & \cdots & \cdots & 0 & A_{N-1} & B_{N-1}
\end{pmatrix}
\begin{pmatrix}
\Delta u_1^n \\ \Delta u_2^n \\ \cdots \\ \cdots \\ \cdots \\ \Delta u_{N-2}^n \\ \Delta u_{N-1}^n
\end{pmatrix}
=
\begin{pmatrix}
D_1 - A_1 \cdot u_0 \\ D_2 \\ \cdots \\ \cdots \\ \cdots \\ D_{N-2} \\ D_{N-1} - C_{N-1}\cdot u_N
\end{pmatrix} \tag{35}$$

The Beam and Warming method is extremely dispersive and therefore artiricial viscousity must be explicitly added. Beam and Warming used the following artificial viscosity term:

$$-\frac{w}{8}\left(u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n\right) \qquad 0 < w \leq 1 \tag{36}$$

which can be added to the RHS with no change in accuracy