Computational Fluid Dynamics
HW2

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Nomenclature

Dimensionless Numbers

Δx	size of each cell in the domain
γ	ratio of specific heats
κ	coefficient of thermal conductivity
μ	coefficient of viscosity
ρ	fluid density
c_p	constant specific heat capacity for a constant pressure
c_v	constant specific heat capacity for a constant volume
E	inviscid convective vector
e	total energy
E_ν	viscous convective vector
L	characteristic length
p	pressure
Q	conservation state space
R	gas constant
T	temperature
t	time
u	fluid velocity
x	spatial coordinate
x_F	x coordinate of the end of the domain
x_i	x coordinate of the i-th cell

Far–Away Properties

κ_∞	coefficient of thermal conductivity far away
μ_∞	coefficient of viscosity far away
ρ_∞	density far away
a_∞	speed of sound far away
M_∞	mach number far away
p_∞	pressure far away
T_∞	temperature far away

Dimensionless Numbers

Pr_∞	Prandtl number far away
$Re_{L\infty}$	Reynolds number with respect to L far away

1 Problem Definition

1.1 Governing Equations

Consider the one-dimensional Navier-Stokes Equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = \frac{\partial E_\nu}{\partial x} \tag{1}$$

Where:

$$Q = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad E = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ (e+p)\,u \end{pmatrix}, \quad E_\nu = \begin{pmatrix} 0 \\ \tau_{xx} \\ u\tau_{xx} - q_x \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu\frac{\partial u}{\partial x} \\ \frac{4}{3}\mu u\frac{\partial u}{\partial x} + \kappa\frac{\partial T}{\partial x} \end{pmatrix}$$

$$p = (\gamma - 1)\left(e - \frac{1}{2}\rho u^2\right), \qquad T = \frac{p}{\rho R}, \tag{2}$$

$$\mu = 1.458 \cdot 10^{-6}\frac{T^{\frac{3}{2}}}{T + 110.4}, \quad \mu = 2.495 \cdot 10^{-3}\frac{T^{\frac{3}{2}}}{T + 194}$$

$$R = c_p - c_v, \quad \gamma = \frac{c_p}{c_v}$$

The constants are:

- $\gamma = 1.4$ for air under standard atmospheric conditions
- $R = 287.0$ for air

1.2 Physical Domain

The physical domain is a tube extended between $x = 0.2$ and $x = 1.0$. At both ends there are impermeable walls.

1.3 Initial Conditions

The initial conditions are shown in Fig.1:

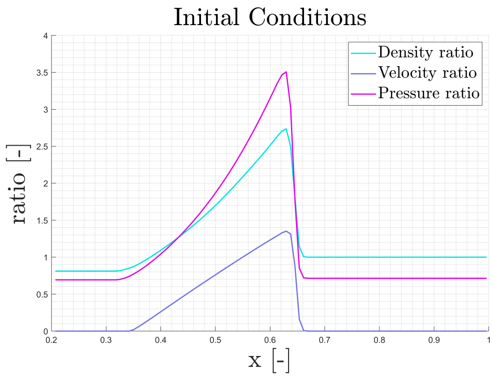

Figure 1: Initial conditions

1.4 Boundary Conditions

On each side of the tube there is an adiabatic, solid wall boundary conditions.

$$u_{(x=0.2)} = u_{(x=1.0)} = 0 \quad \Big\| \quad \left.\frac{\partial p}{\partial x}\right|_{x=0.2} = \left.\frac{\partial p}{\partial x}\right|_{x=1.0} = 0 \quad \Big\| \quad \left.\frac{\partial T}{\partial x}\right|_{x=0.2} = \left.\frac{\partial T}{\partial x}\right|_{x=1.0} = 0$$

2 Normalizing The Navier-Stokes Equations

Since the initial conditions are normalized, there is a need to normalize the N-S equations. We will use the following normalizations:

$$\rho = \rho_\infty \tilde{\rho}, \quad u = a_\infty \tilde{u}, \quad p = \gamma p_\infty \tilde{p}, \quad T = \gamma T_\infty \tilde{T}, \quad x = L\tilde{x}, \quad t = \frac{L}{a_\infty}\tilde{t}, \quad \mu = \mu_\infty \tilde{\mu}, \quad \kappa = \kappa_\infty \tilde{\kappa} \tag{3}$$

The normalization of the temperature was chosen to cancel out the γ in the normalization of the pressure:

$$\begin{aligned} p &= \rho R T \\ \gamma p_\infty \tilde{p} &= \rho_\infty \tilde{\rho} R \gamma T_\infty \tilde{T} \\ \tilde{p} &= \tilde{\rho}\tilde{T} \end{aligned} \tag{4}$$

The pressure normalization can be written also as:

$$p = \gamma p_\infty \tilde{p} = \gamma \rho_\infty R T_\infty \tilde{p} = \rho_\infty a_\infty^2 \tilde{p} \tag{5}$$

From equations 2 and 5 we can derive the normalization for the energy:

$$\begin{aligned} e &= \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \\ e &= \frac{\rho_\infty a_\infty^2 \tilde{p}}{\gamma - 1} + \frac{1}{2}\rho_\infty \tilde{\rho} a_\infty^2 \tilde{a}^2 \\ e &= \rho_\infty a_\infty^2 \left(\frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\tilde{a}^2 \right) \\ e &= \rho_\infty a_\infty^2 \tilde{e} \end{aligned} \tag{6}$$

After substituting the normalizations in the N-S equations we get:

$$\frac{\partial}{\partial \frac{L}{a_\infty}\tilde{t}} \begin{pmatrix} \rho_\infty \tilde{\rho} \\ \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{p} + \rho_\infty a_\infty^2 \tilde{\rho}\tilde{u}^2 \\ \rho_\infty a_\infty^3 \left(\tilde{e} + \tilde{p}\right)\tilde{u} \end{pmatrix} = \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}\mu_\infty a_\infty \tilde{\mu} \frac{\partial \tilde{u}}{\partial L\tilde{x}} \\ \frac{4}{3}\mu_\infty a_\infty^2 \tilde{\mu}\tilde{u} \frac{\partial \tilde{u}}{\partial L\tilde{x}} + \frac{\kappa_\infty a_\infty^2}{R}\tilde{\kappa} \frac{\partial \tilde{T}}{\partial L\tilde{x}} \end{pmatrix} \tag{7}$$

Rearranging:

$$\frac{\rho_\infty a_\infty}{L} \frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ a_\infty \tilde{\rho}\tilde{u} \\ a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\rho_\infty a_\infty}{L} \frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho}\tilde{u} \\ a_\infty \tilde{p} + a_\infty \tilde{\rho}\tilde{u}^2 \\ a_\infty^2 \left(\tilde{e} + \tilde{p}\right)\tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L^2} \frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}a_\infty \tilde{\mu} \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3}a_\infty^2 \tilde{\mu}\tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\kappa_\infty a_\infty^2}{\mu_\infty R}\tilde{\kappa} \frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \tag{8}$$

Dividing the second equation by a_∞, the third equation by a_∞^2, and the whole set of equations by $\frac{\rho_\infty a_\infty}{L}$ we get:

$$\frac{\partial}{\partial \tilde{t}}\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix}+\frac{\partial}{\partial \tilde{x}}\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+a_\infty\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix}=\frac{\mu_\infty}{L\rho_\infty a_\infty}\frac{\partial}{\partial \tilde{x}}\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\kappa_\infty}{\mu_\infty R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}}\end{pmatrix} \tag{9}$$

The Reynolds number and the mach number far away are defined as:

$$\begin{gathered} M_\infty=\frac{u_\infty}{a_\infty}\quad Re_{L\infty}=\frac{\rho_\infty u_\infty L}{\mu_\infty}\\ \Downarrow\\ \frac{\mu_\infty}{L\rho_\infty a_\infty}=\frac{M_\infty}{Re_{L\infty}}\end{gathered} \tag{10}$$

The Prandtl number far away is defined as:

$$\begin{gathered} Pr_\infty=\frac{c_p\mu_\infty}{\kappa_\infty}\\ \Downarrow\\ \frac{\kappa_\infty}{\mu_\infty R}=\frac{c_p}{Pr_\infty\,(c_p-c_v)}=\frac{\gamma}{Pr_\infty\,(\gamma-1)}\end{gathered} \tag{11}$$

Substituting into the normalized N-S equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}}+\frac{\partial \tilde{E}}{\partial \tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{E}_\nu}{\partial \tilde{x}} \tag{12}$$

Where:

$$\tilde{Q}=\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix},\quad \tilde{E}=\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+a_\infty\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix},\quad \tilde{E_\nu}=\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\gamma}{Pr_\infty\,(\gamma-1)}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}}\end{pmatrix} \tag{13}$$

3 The Computational Domain

3.1 Discretization

The physical domain $[x_I, x_F]$ is discretized into N equispaced cells. The size of each cell is there for:

$$\Delta x=\frac{x_F-x_I}{N}=\frac{L}{N} \tag{14}$$

so the x coordinate of the i-th cell x_i is:

$$x_i=x_I+\frac{1}{2}\Delta x+\Delta x\cdot(i-1)\qquad \text{when starting from } i=1 \tag{15}$$

3.2 Boundary Conditions

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated like so:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} \\ u_{(i=N+1)} &= -u_{(i=N)} \end{aligned} \tag{16}$$

in order to maintain velocity zero on the boundary and like so:

$$\begin{aligned} T_{(i=0)} &= T_{(i=1)} \\ T_{(i=N+1)} &= T_{(i=N)} \end{aligned} \tag{17}$$

In order to maintain adiabatic boundary conditions. Since the gradient of the pressure on the wall is zero, we get:

$$\begin{aligned} p_{(i=0)} &= p_{(i=1)} \\ p_{(i=N+1)} &= p_{(i=N)} \end{aligned} \tag{18}$$

From equations 2, 17, and 18 we can conclude:

$$\begin{aligned} \rho_{(i=0)} &= \rho_{(i=1)} \\ \rho_{(i=N+1)} &= \rho_{(i=N)} \end{aligned} \tag{19}$$

and from equations 2, 16, 18, and 19 we can conclude:

$$\begin{aligned} e_{(i=0)} &= e_{(i=1)} \\ e_{(i=N+1)} &= e_{(i=N)} \end{aligned} \tag{20}$$

4 The Numerical Schemes

The normalized Navier-Stokes equations are:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \frac{\partial \tilde{E}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{21}$$

Where:

$$\tilde{V}_1 = \tilde{V}_{1(Q,Q_x)} = \tilde{E}_\nu$$

4.1 First Order Approximate Riemann Roe Method

4.2 First Order Steger-Warming – Explicit

4.3 First Order Steger-Warming – Implicit