

Computational Fluid Dynamics

HW1

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1 Inviscid Burgers Equation

The Inviscid Burgers equation, in conservation law form, is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad F = F(u) = \frac{u^2}{2} \quad (1)$$

In non-conservation law form, is given by:

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0 \quad A = \frac{\partial F}{\partial u} = u \quad (2)$$

The equation is obtained by neglecting the viscous term from the viscous Burger equation.

1.1 Boundary and Initial Conditions

$$\begin{aligned} u_{(x=0,t)} &= 1.0 \\ u_{(x=1,t)} &= u_1 \\ u_{(x,t=0)} &= 1 - (1 - u_1) \cdot x \end{aligned} \quad (3)$$

1.2 Finite Volume Formulation

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left(f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n \right) \quad (4)$$

- For first-order schemes, there is no variation within a cell, and the value there is constant.
- For second-order schemes, the variation within the cell is linear.

1.3 First Order Roe Method ($u_1 = 0.0$)

Roe scheme is based on the solution of the linear problem:

$$\frac{\partial u}{\partial t} + \bar{A} \frac{\partial u}{\partial x} = 0 \quad (5)$$

Where \bar{A} is a constant matrix that is dependent on local conditions. The matrix is constructed in a way to guarantee uniform validity across discontinuities:

1. For any u_i, u_{i+1} :

$$F_{i+1} - F_i = \bar{A} \cdot (u_{i+1} - u_i)$$

2. When $u = u_i = u_{i+1}$ then:

$$\bar{A}_{(u_i, u_{i+1})} = \bar{A}_{(u, u)} = \frac{\partial F}{\partial u} = u$$

In case of the Burgers equation, the matrix \bar{A} is a scalar, namely, $\bar{A} = \bar{u}$. The equation becomes:

$$\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} = 0 \quad (6)$$

The value of \bar{u} for the cell face between i and $i+1$ is determined from the first conditions:

$$\bar{u} = \bar{u}_{i+\frac{1}{2}} = \frac{F_{i+1} - F_i}{u_{i+1} - u_i} = \frac{\frac{1}{2}u_{i+1}^2 - \frac{1}{2}u_i^2}{u_{i+1} - u_i} = \begin{cases} \frac{u_i + u_{i+1}}{2} & u_i \neq u_{i+1} \\ u_i & u_i = u_{i+1} \end{cases} \quad (7)$$

The single wave that emanates from the cell interface travels either in the positive or negative direction, depending upon the sign of $\bar{u}_{i+\frac{1}{2}}$. Define:

$$\begin{cases} \bar{u}_{i+\frac{1}{2}}^+ \triangleq \frac{1}{2} \left(\bar{u}_{i+\frac{1}{2}} + \left| \bar{u}_{i+\frac{1}{2}} \right| \right) \geq 0 \\ \bar{u}_{i+\frac{1}{2}}^- \triangleq \frac{1}{2} \left(\bar{u}_{i+\frac{1}{2}} - \left| \bar{u}_{i+\frac{1}{2}} \right| \right) \leq 0 \end{cases} \quad \bar{u}_{i+\frac{1}{2}} = \bar{u}_{i+\frac{1}{2}}^+ + \bar{u}_{i+\frac{1}{2}}^- \quad (8)$$

Using the jump relation, the numerical flux at the cell interface can be evaluated by one of the following:

$$\begin{cases} f_{i+\frac{1}{2}} - F_i = \bar{u}_{i+\frac{1}{2}}^- \cdot (u_{i+1} - u_i) \\ F_{i+1} - f_{i+\frac{1}{2}} = \bar{u}_{i+\frac{1}{2}}^+ \cdot (u_{i+1} - u_i) \end{cases} \quad (9)$$

The numerical flux may then be written in the following symmetric form:

$$\begin{aligned} f_{i+\frac{1}{2}} &= \frac{F_i + F_{i+1}}{2} - \frac{1}{2} \left(\bar{u}_{i+\frac{1}{2}}^+ - \bar{u}_{i+\frac{1}{2}}^- \right) (u_{i+1} - u_i) \\ \text{OR :} & \\ f_{i+\frac{1}{2}} &= \frac{F_i + F_{i+1}}{2} - \frac{1}{2} \left| \bar{u}_{i+\frac{1}{2}} \right| (u_{i+1} - u_i) \end{aligned} \quad (10)$$

Since Roe's scheme can't distinguish between the types of discontinuity, it may result in an expansion shock where the analytical solution is an expansion wave. To guarantee a physical solution the scheme will be modified like so:

Define

$$\varepsilon = \max \left(0, \frac{u_{i+1} - u_i}{2} \right)$$

The interface wave speed becomes

$$\bar{u}_{i+\frac{1}{2}} = \begin{cases} \bar{u}_{i+\frac{1}{2}} & \bar{u}_{i+\frac{1}{2}} \geq \varepsilon \quad \text{compression} \\ \varepsilon & \bar{u}_{i+\frac{1}{2}} < \varepsilon \quad \text{expansion} \end{cases} \quad (11)$$

1.3.1 CFL number

For the Roe method, the CFL number is defined as:

$$\text{CFL} = \frac{u \Delta t}{\Delta x} \quad (12)$$

We will want to set the maximal value of the CFL number. We will find the Δt at each cell and (Δt_i) and set the Δt of the current step as:

$$\Delta t = \min (\Delta t_i) \quad \forall i \quad (13)$$

1.4 Second Order Roe ($u_1 = 0.5$)

The first-order accurate Roe method interface flux function will be denoted like this:

$$f_{i+\frac{1}{2}}^{\text{Roe},1} = f_{(u_i, u_{i+1})}$$

The second order accurate Roe takes the form:

$$f_{i+\frac{1}{2}}^{\text{Roe},2} = f_{(u'_{i+1}, u^r_{i+1})}$$

Hence:

$$f_{i+\frac{1}{2}}^{\text{Roe},2} = \frac{1}{2} \left(F(u_{1+\frac{1}{2}}^l) + F(u_{1+\frac{1}{2}}^r) - |\bar{u}_{i+\frac{1}{2}}| (u_{1+\frac{1}{2}}^r - u_{1+\frac{1}{2}}^l) \right) \quad (14)$$

$$\bar{u}_{1+\frac{1}{2}} = \frac{F(u_{1+\frac{1}{2}}^r) - F(u_{1+\frac{1}{2}}^l)}{u_{1+\frac{1}{2}}^r - u_{1+\frac{1}{2}}^l}$$

1.4.1 Without Limiters

The interface values without limiters are evaluated as:

$$\begin{cases} u_{i+\frac{1}{2}}^l &= u_i + \frac{1-k}{4} \delta u_{i-\frac{1}{2}} + \frac{1+k}{4} \delta u_{i+\frac{1}{2}} \\ u_{i+\frac{1}{2}}^r &= u_{i+1} - \frac{1+k}{4} \delta u_{i+\frac{1}{2}} - \frac{1-k}{4} \delta u_{i+\frac{3}{2}} \end{cases} \quad \delta u_i \triangleq u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} \quad (15)$$

The parameter k determines the scheme:

$$k = \begin{cases} -1 & \text{upwind} \\ 1 & \text{central} \end{cases}$$

1.4.2 With Limiters

The interface values with limiters are evaluated as:

$$\begin{cases} u_{i+\frac{1}{2}}^l &= u_i + \frac{1-k}{4} \bar{\delta}^+ u_{i-\frac{1}{2}} + \frac{1+k}{4} \bar{\delta}^- u_{i+\frac{1}{2}} \\ u_{i+\frac{1}{2}}^r &= u_{i+1} - \frac{1+k}{4} \bar{\delta}^+ u_{i+\frac{1}{2}} - \frac{1-k}{4} \bar{\delta}^- u_{i+\frac{3}{2}} \end{cases} \quad \bar{\delta}^\pm u \text{ are limited slopes} \quad (16)$$

$\bar{\delta}$ is an operator such that $\bar{\delta} u_i = \psi \delta u_i$, where $\psi(r)$ is a limiter function and:

$$r^\pm = \begin{cases} r_{1+\frac{1}{2}}^+ & \triangleq \frac{u_{i+2} - u_{i+1}}{u_{i+1} - u_i} = \frac{\Delta u_{i+1}}{\Delta u_i} \\ r_{1+\frac{1}{2}}^- & \triangleq \frac{u_i - u_{i-1}}{u_{i+1} - u_i} = \frac{\nabla u_i}{\nabla u_{i+1}} \end{cases} \quad (17)$$

There are many types of limiters. For example, the van Albada limiter:

$$\psi(r) = \frac{r + r^2}{1 + r^2} \quad (18)$$

2 Generalized Burgers Equation