Computational Fluid Dynamics

Almog Dobrescu

ID 214254252

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Contents

1 Mathematical Problem

The Burgers Equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{1}$$

By neglection the convective term, the parabolic model equation is obtained:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} \tag{3}$$

$$\mu = 1.0 \tag{4}$$

Boundary and initial conditions:

$$u_{(y_0,t)} = u_0 \qquad u_{(y_1,t)} = u_1 \qquad u_{(y,t=0)} = f_{(y)} \tag{5}$$

2 Numerical Scheme

A general explicit-implicit scheme for constant μ is given by:

$$u_i^{n+1} = u_i^n + \frac{\mu\Delta t}{\Delta y^2}\delta_{yy}\left[(1-\alpha)\,u_i^n + \alpha u_i^{n+1}\right] \tag{6}$$

where:

$$\alpha = \begin{cases} 0 & \text{Explicit} \\ \frac{1}{2} & \text{Crank-Nicolson} \\ 1 & \text{Implicit} \end{cases} \tag{7}$$

and the order is:

$$\left[\Delta x^2, \Delta\left(\frac{1}{2}-\alpha\right)\right] \tag{8}$$

In delta form:

$$\left(I - \alpha\frac{\mu\Delta t}{\Delta y^2}\delta_{yy}\right)\Delta u_i^n = \frac{\mu\Delta t}{\Delta y^2}\delta_{yy}u_i^n \tag{9}$$

Applying the operators:

$$\Delta u_i^n - \alpha\frac{\mu\Delta t}{\Delta y^2}\delta_{yy}\left(\Delta u_i^n\right) = \frac{\mu\Delta t}{\Delta y^2}\delta_{yy}\left(u_i^n\right) \tag{10}$$

$$\Delta u_i^n - \alpha\frac{\mu\Delta t}{\Delta y^2}\delta_y\left(\Delta u^n_{\left(y_i+\frac{\Delta y}{2}\right)} - \Delta u^n_{\left(y_i-\frac{\Delta y}{2}\right)}\right) = \frac{\mu\Delta t}{\Delta y^2}\delta_y\left(u^n_{\left(y_i+\frac{\Delta y}{2}\right)} - u^n_{\left(y_i-\frac{\Delta y}{2}\right)}\right) \tag{11}$$

$$\Delta u_i^n - \alpha\frac{\mu\Delta t}{\Delta y^2}\left(\Delta u^n_{(y_i+\Delta y)} - 2\Delta u^n_{(y_i)} + \Delta u^n_{(y_i-\Delta y)}\right) = \frac{\mu\Delta t}{\Delta y^2}\left(u^n_{(y_i+\Delta y)} - 2u^n_{(y_i)} + u^n_{(y_i-\Delta y)}\right) \tag{12}$$

$$\Delta u_i^n - \alpha\frac{\mu\Delta t}{\Delta y^2}\left(\Delta u_{i+1}^n - 2\Delta u_i^n + \Delta u_{i-1}^n\right) = \frac{\mu\Delta t}{\Delta y^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \tag{13}$$

$$A_i\Delta u_{i-1}^n + B_i\Delta u_i^n + C_i\Delta u_{i+1}^n = D_i \tag{14}$$

where:

$$A_i = -\alpha \frac{\mu \Delta t}{\Delta y^2} \tag{15}$$

$$B_i = 1 + 2\alpha \frac{\mu \Delta t}{\Delta y^2} \tag{16}$$

$$C_i = -\alpha \frac{\mu \Delta t}{\Delta y^2} \tag{17}$$

$$D_i = \frac{\mu \Delta t}{\Delta y^2} \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n \right) \tag{18}$$

and

$$u_i^{n+1} = u_i^n + \Delta u_i^n \tag{19}$$

In matrix from:

$$\begin{pmatrix} B_1 & C_1 & 0 & \cdots & \cdots & \cdots & 0 \\ A_2 & B_2 & C_2 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & 0 & A_i & B_i & C_i & 0 & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & A_{N-2} & B_{N-2} & C_{N-2} \\ 0 & 0 & \cdots & \cdots & 0 & A_{N-1} & B_{N-1} \end{pmatrix} \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \cdots \\ \cdots \\ \cdots \\ \Delta u_{N-2} \\ \Delta u_{N-1} \end{pmatrix} = \begin{pmatrix} D_1 - A_1 \cdot u_0 \\ D_2 \\ \cdots \\ \cdots \\ \cdots \\ D_{N-2} \\ D_{N-1} - C_{N-1} \cdot u_N \end{pmatrix} \tag{20}$$

To reduce problems of big flouting point numbers, define r:

$$r \triangleq \frac{\mu \Delta t}{\Delta y^2} \tag{21}$$

After dividing by r:

$$A_i = -\alpha \tag{22}$$

$$B_i = \frac{1}{r} + 2\alpha \tag{23}$$

$$C_i = -\alpha \tag{24}$$

$$D_i = \left(u_{i+1}^n - 2u_i^n + u_{i-1}^n \right) \tag{25}$$

In our case:

$$y_0 = 0 \qquad u_0 = 0 \qquad y_1 = 1 \qquad u_1 = 1 \qquad f_{(y)} = 1 \tag{26}$$

The scheme considered converged when the size of the RHS drops by 6 order of magnitude.

3 Stability Analysis

According to the "von Neumann Stability Analysis", the error between the differential and numerical equation is given by:

$$\varepsilon = N - D \tag{27}$$

where:

$$\varepsilon(y,t) = \sum_m c_m(t) e^{ik_m y} \tag{28}$$

since the analysis is linear, one can choose one element from the series. Morover, the error becomes:

$$\varepsilon^n = Z^n e^{iky} \tag{29}$$

The constant"k" is the wave number theat is related to the initial error.

$$Z = e^{\lambda \Delta t} \Rightarrow Z^n = e^{\lambda n \Delta t} = e^{\lambda t} \tag{30}$$

Any element of the error therefore takes the form:

$$\varepsilon^n = e^{\lambda t} e^{iky} \tag{31}$$

The ratio between consecutive errors is the *amplification factor*, names "G":

$$G \triangleq \frac{\varepsilon^{n+1}}{\varepsilon^n} = e^{\lambda \Delta t} \tag{32}$$

The condition for Stability:

$$|G| < 1 \tag{33}$$

The error must satisfies the finite difference equation:

$$\Delta\varepsilon_i^n - \alpha \frac{\mu \Delta t}{\Delta y^2} \left(\Delta\varepsilon_{i+1}^n - 2\Delta\varepsilon_i^n + \Delta\varepsilon_{i-1}^n \right) = \frac{\mu \Delta t}{\Delta y^2} \left(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n \right) \tag{34}$$

Where:

$$\Delta\varepsilon_i^n = \varepsilon_i^{n+1} - \varepsilon_i^n = e^{\lambda(t+\Delta t)} e^{iky} - e^{\lambda t} e^{iky} = e^{\lambda t} e^{iky} \left(e^{\lambda \Delta t} - 1 \right) \tag{35}$$

$$\Delta\varepsilon_{i+1}^n = \varepsilon_{i+1}^{n+1} - \varepsilon_{i+1}^n = e^{\lambda(t+\Delta t)} e^{ik(y+\Delta y)} - e^{\lambda t} e^{ik(y+\Delta y)} = e^{\lambda t} e^{ik(y+\Delta y)} \left(e^{\lambda \Delta t} - 1 \right) \tag{36}$$

$$\Delta\varepsilon_{i-1}^n = \varepsilon_{i-1}^{n+1} - \varepsilon_{i-1}^n = e^{\lambda(t+\Delta t)} e^{ik(y-\Delta y)} - e^{\lambda t} e^{ik(y-\Delta y)} = e^{\lambda t} e^{ik(y-\Delta y)} \left(e^{\lambda \Delta t} - 1 \right) \tag{37}$$

Substitut the values:

$$\begin{aligned} \left(e^{\lambda t} e^{iky} - \alpha \frac{\mu \Delta t}{\Delta y^2} \left(e^{\lambda t} e^{ik(y+\Delta y)} - 2 e^{\lambda t} e^{iky} + e^{\lambda t} e^{ik(y-\Delta y)} \right) \right) \left(e^{\lambda \Delta t} - 1 \right) = \\ = \frac{\mu \Delta t}{\Delta y^2} \left(e^{\lambda t} e^{ik(y+\Delta y)} - 2 e^{\lambda t} e^{iky} + e^{\lambda t} e^{ik(y-\Delta y)} \right) \end{aligned} \tag{38}$$

Dividing by $e^{\lambda t} e^{iky}$:

$$\left(1 - \alpha \frac{\mu \Delta t}{\Delta y^2} \left(e^{ik\Delta y} - 2 + e^{-ik\Delta y} \right) \right) \left(e^{\lambda t} - 1 \right) = \frac{\mu \Delta t}{\Delta y^2} \left(e^{ik\Delta y} - 2 + e^{-ik\Delta y} \right) \tag{39}$$

Define:

$$r \triangleq \frac{\mu \Delta t}{\Delta y^2}, \quad A \triangleq e^{ik\Delta y} - 2 + e^{ik\Delta y} = 2\cos\beta - 2, \quad \beta \triangleq k\Delta y \tag{40}$$

After substitution and rearranging we get:

$$e^{\lambda t} - 1 = \frac{rA}{1 - \alpha r A} \tag{41}$$

$$\Downarrow$$

$$G = e^{\lambda t} = 1 + \frac{rA}{1 - \alpha r A} \tag{42}$$

For stability:

$$|G| < 1 \Rightarrow \left|1 + \frac{rA}{1 - \alpha r A}\right| < 1 \tag{43}$$

For $\alpha = 0$:

$$G = 1 + rA \tag{44}$$

$$|1 - 4r| < 1 \tag{45}$$

$$-1 < 1 - 4r < 1 \tag{46}$$

$$-2 < -4r < 0 \tag{47}$$

$$0 < 4r < 2 \tag{48}$$

$$0 < r < \frac{1}{2} \tag{49}$$

$$\Delta t < \frac{\Delta y^2}{2\mu} \tag{50}$$

For $\alpha = 1$:

$$G = \frac{1}{1 - rA} \tag{51}$$

$$\left|\frac{1}{1 - rA}\right| < 1 \quad A \leq 0 \tag{52}$$

$$\Downarrow$$

smaller for every r (Δt)

For every α and $\Delta t \to \infty$:

$$\lim_{\Delta t \to \infty} |G| = \frac{1 - \alpha}{\alpha} \tag{53}$$

Which means that stability for infinite time step:

$$\alpha \begin{cases} < \dfrac{1}{2} & \text{Unstable} \\ = \dfrac{1}{2} & \text{Neutrally stable} \\ > \dfrac{1}{2} & \textit{Stable} \end{cases} \tag{54}$$

4 The Computer Program

5 Results

6 Conclusions