Computational Fluid Dynamics
HW2

Almog Dobrescu

ID 214254252

March 13, 2025

Contents

List of Figures

Nomenclature

Δx	size of each cell in the domain
$\Delta\tilde{t}$	normalized step size in time
$\Delta\tilde{x}$	normalized step size in space
γ	ratio of specific heats
κ	coefficient of thermal conductivity
μ	coefficient of viscosity
ρ	fluid density
c_p	constant specific heat capacity for a constant pressure
c_v	constant specific heat capacity for a constant volume
E	inviscid convective vector
e	total energy
E_ν	viscous convective vector
L	characteristic length
p	pressure
Q	conservation state space
R	gas constant
T	temperature
t	time
u	fluid velocity
x	spatial coordinate
x_F	x coordinate of the end of the domain
x_i	x coordinate of the i-th cell

Diagonal

Φ	main diagonal
Ψ	upper off-diagonal
Θ	lower off-diagonal

Far–Away Properties

κ_∞	coefficient of thermal conductivity far away
μ_∞	coefficient of viscosity far away

ρ_∞	density far away
a_∞	speed of sound far away
M_∞	mach number far away
p_∞	pressure far away
T_∞	temperature far away

Matrices

$\hat{\tilde{A}}$	Roe's average matrix
$\tilde{\Lambda}$	normalized eigenvalues matrix
$\tilde{A}$	normalized jacobian matrix of E w.r.t. Q
$\tilde{P}$	normalized jacobian matrix of E_ν w.r.t. Q
$\tilde{R}$	normalized jacobian matrix of E_ν w.r.t. Q_x
$\tilde{T}$	normalized eigenvectors matrix

Dimensionless Numbers

Pr_∞	Prandtl number far away
$Re_{L\infty}$	Reynolds number with respect to L far away

1 Problem Definition

1.1 Governing Equations

Consider the one-dimensional Navier-Stokes Equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = \frac{\partial E_\nu}{\partial x} \tag{1}$$

Where:

$$Q = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad E = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ (e+p)\,u \end{pmatrix}, \quad E_\nu = \begin{pmatrix} 0 \\ \tau_{xx} \\ u\tau_{xx} - q_x \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu \frac{\partial u}{\partial x} \\ \frac{4}{3}\mu u \frac{\partial u}{\partial x} + \kappa \frac{\partial T}{\partial x} \end{pmatrix}$$

$$p = (\gamma - 1)\left(e - \frac{1}{2}\rho u^2\right), \qquad T = \frac{p}{\rho R}, \tag{2}$$

$$\mu = 1.458 \cdot 10^{-6} \frac{T^{\frac{3}{2}}}{T + 110.4}, \quad \kappa = 2.495 \cdot 10^{-3} \frac{T^{\frac{3}{2}}}{T + 194}$$

$$R = c_p - c_v, \quad \gamma = \frac{c_p}{c_v}$$

The constants are:

- $\gamma = 1.4$ for air under standard atmospheric conditions
- $R = 287.0$ for air

1.2 Physical Domain

The physical domain is a tube extended between $x = 0.2$ and $x = 1.0$. At both ends there are impermeable walls.

1.3 Initial Conditions

The initial conditions are shown in Fig.1:

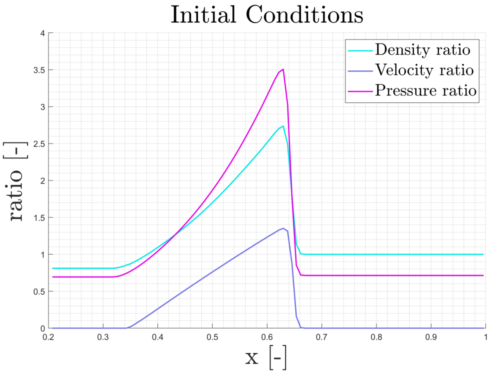

Figure 1: Initial conditions

1.4 Boundary Conditions

On each side of the tube there is an adiabatic, solid wall boundary conditions.

$$u_{(x=0.2)} = u_{(x=1.0)} = 0 \quad \Bigg\| \quad \left.\frac{\partial p}{\partial x}\right|_{x=0.2} = \left.\frac{\partial p}{\partial x}\right|_{x=1.0} = 0 \quad \Bigg\| \quad \left.\frac{\partial T}{\partial x}\right|_{x=0.2} = \left.\frac{\partial T}{\partial x}\right|_{x=1.0} = 0$$

2 Normalizing The Navier-Stokes Equations

Since the initial conditions are normalized, there is a need to normalize the N-S equations. We will use the following normalizations:

$$\rho = \rho_\infty \tilde{\rho}, \quad u = a_\infty \tilde{u}, \quad p = \gamma p_\infty \tilde{p}, \quad T = \gamma T_\infty \tilde{T}, \quad x = L\tilde{x}, \quad t = \frac{L}{a_\infty}\tilde{t}, \quad \mu = \mu_\infty \tilde{\mu}, \quad \kappa = \kappa_\infty \tilde{\kappa} \tag{3}$$

The normalization of the temperature was chosen to cancel out the γ in the normalization of the pressure:

$$\begin{aligned} p &= \rho R T \\ \gamma p_\infty \tilde{p} &= \rho_\infty \tilde{\rho} R \gamma T_\infty \tilde{T} \\ \tilde{p} &= \tilde{\rho}\tilde{T} \end{aligned} \tag{4}$$

The pressure normalization can be written also as:

$$p = \gamma p_\infty \tilde{p} = \gamma \rho_\infty R T_\infty \tilde{p} = \rho_\infty a_\infty^2 \tilde{p} \tag{5}$$

From equations 2 and 5 we can derive the normalization for the energy:

$$\begin{aligned} e &= \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \\ e &= \frac{\rho_\infty a_\infty^2 \tilde{p}}{\gamma - 1} + \frac{1}{2}\rho_\infty \tilde{\rho} a_\infty^2 \tilde{a}^2 \\ e &= \rho_\infty a_\infty^2 \left(\frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\tilde{a}^2 \right) \\ e &= \rho_\infty a_\infty^2 \tilde{e} \end{aligned} \tag{6}$$

After substituting the normalizations in the N-S equations we get:

$$\frac{\partial}{\partial \frac{L}{a_\infty}\tilde{t}} \begin{pmatrix} \rho_\infty \tilde{\rho} \\ \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{p} + \rho_\infty a_\infty^2 \tilde{\rho}\tilde{u}^2 \\ \rho_\infty a_\infty^3 (\tilde{e} + \tilde{p})\tilde{u} \end{pmatrix} = \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}\mu_\infty a_\infty \tilde{\mu} \frac{\partial \tilde{u}}{\partial L\tilde{x}} \\ \frac{4}{3}\mu_\infty a_\infty^2 \tilde{\mu}\tilde{u} \frac{\partial \tilde{u}}{\partial L\tilde{x}} + \frac{\kappa_\infty a_\infty^2}{R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial L\tilde{x}} \end{pmatrix} \tag{7}$$

Rearranging:

$$\frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ a_\infty \tilde{\rho}\tilde{u} \\ a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho}\tilde{u} \\ a_\infty \tilde{p} + a_\infty \tilde{\rho}\tilde{u}^2 \\ a_\infty^2 (\tilde{e} + \tilde{p})\tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L^2}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}a_\infty \tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3}a_\infty^2 \tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\kappa_\infty a_\infty^2}{\mu_\infty R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \tag{8}$$

Dividing the second equation by a_∞, the third equation by a_∞^2, and the whole set of equations by $\frac{\rho_\infty a_\infty}{L}$ we get:

$$\frac{\partial}{\partial \tilde{t}}\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix}+\frac{\partial}{\partial \tilde{x}}\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix}=\frac{\mu_\infty}{L\rho_\infty a_\infty}\frac{\partial}{\partial \tilde{x}}\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\kappa_\infty}{\mu_\infty R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}}\end{pmatrix} \tag{9}$$

The Reynolds number and the mach number far away are defined as:

$$\begin{gathered}M_\infty=\frac{u_\infty}{a_\infty}\quad Re_{L\infty}=\frac{\rho_\infty u_\infty L}{\mu_\infty}\\ \Downarrow\\ \frac{\mu_\infty}{L\rho_\infty a_\infty}=\frac{M_\infty}{Re_{L\infty}}\end{gathered} \tag{10}$$

The Prandtl number far away is defined as:

$$\begin{gathered}Pr_\infty=\frac{c_p\mu_\infty}{\kappa_\infty}\\ \Downarrow\\ \frac{\kappa_\infty}{\mu_\infty R}=\frac{c_p}{Pr_\infty\,(c_p-c_v)}=\frac{\gamma}{Pr_\infty\,(\gamma-1)}\end{gathered} \tag{11}$$

Substituting into the normalized N-S equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}}+\frac{\partial \tilde{E}}{\partial \tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{E}_\nu}{\partial \tilde{x}} \tag{12}$$

Where:

$$\tilde{Q}=\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix},\quad \tilde{E}=\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix},\quad \tilde{E}_\nu=\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\gamma}{Pr_\infty\,(\gamma-1)}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}}\end{pmatrix} \tag{13}$$

The normalized Navier-Stokes equations are:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}}+\frac{\partial \tilde{E}}{\partial \tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{14}$$

Where:

$$\tilde{V}_1=\tilde{V}_{1(\tilde{Q},\tilde{Q}_x)}=\tilde{E}_\nu$$

3 The Computational Domain

3.1 Discretization

The physical domain $[x_I, x_F]$ is discretized into N equispaced cells. The size of each cell is there for:

$$\Delta x=\frac{x_F-x_I}{N}=\frac{L}{N} \tag{15}$$

so the x coordinate of the i-th cell x_i is:

$$x_i=x_I+\frac{1}{2}\Delta x+\Delta x\cdot(i-1)\qquad \text{when starting from } i=1 \tag{16}$$

3.2 Boundary Conditions

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated as follows:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} \\ u_{(i=N+1)} &= -u_{(i=N)} \end{aligned} \tag{17}$$

in order to maintain velocity zero on the boundary and like so:

$$\begin{aligned} T_{(i=0)} &= T_{(i=1)} \\ T_{(i=N+1)} &= T_{(i=N)} \end{aligned} \tag{18}$$

in order to maintain adiabatic boundary conditions. Since the gradient of the pressure on the wall is zero, we get:

$$\begin{aligned} p_{(i=0)} &= p_{(i=1)} \\ p_{(i=N+1)} &= p_{(i=N)} \end{aligned} \tag{19}$$

From equations 2, 18, and 19 we can conclude:

$$\begin{aligned} \rho_{(i=0)} &= \rho_{(i=1)} \\ \rho_{(i=N+1)} &= \rho_{(i=N)} \end{aligned} \tag{20}$$

and from equations 2, 17, 19, and 20 we can conclude:

$$\begin{aligned} e_{(i=0)} &= e_{(i=1)} \\ e_{(i=N+1)} &= e_{(i=N)} \end{aligned} \tag{21}$$

4 The Numerical Schemes

4.1 Jacobian Matrices of The Navier-Stokes Equations

We can rewrite Eq.14 as:

$$\begin{aligned}
\frac{\partial \tilde{Q}}{\partial \tilde{t}} &= -\frac{\partial \tilde{E}}{\partial \tilde{x}} - \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \\
\frac{\partial \tilde{Q}}{\partial \tilde{t}} &= -\underbrace{\frac{\partial \tilde{E}}{\partial \tilde{Q}}}_{\tilde{A}} \frac{\partial \tilde{Q}}{\partial \tilde{x}} - \frac{M_\infty}{Re_{L\infty}} \left(\underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}}_{\tilde{P}} \frac{\partial \tilde{Q}}{\partial \tilde{x}} + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}}_{\tilde{R}} \frac{\partial \tilde{Q}_x}{\partial \tilde{x}} \right)
\end{aligned} \tag{22}$$

Where:

$$\begin{aligned}
\tilde{A} &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma - 3}{2}\tilde{u}^2 & (3-\gamma)\,\tilde{u} & \gamma - 1 \\ -\frac{\gamma \tilde{e} \tilde{u}}{\tilde{\rho}} - (\gamma - 1)\,\tilde{u}^3 & \frac{\gamma \tilde{e}}{\tilde{\rho}} - \frac{3\,(\gamma-1)\,\tilde{u}^2}{2} & \gamma \tilde{u} \end{pmatrix} \\
\tilde{P} - \tilde{R}_x &= -\frac{1}{\rho} \begin{pmatrix} 0 & 0 & 0 \\ -\tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & \left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \\ -\tilde{u}^2\left(\frac{4}{3}\tilde{\mu}\right)_x & \tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \end{pmatrix} \\
\tilde{R} &= -\frac{1}{\rho} \begin{pmatrix} 0 & 0 & 0 \\ \frac{4}{3}\tilde{u}\tilde{\mu} & -\frac{4}{3}\tilde{\mu} & 0 \\ \left(\frac{4}{3}\tilde{\mu} - \alpha\frac{\tilde{\kappa}}{c_v}\right)\tilde{u}^2 + \alpha\frac{\tilde{\kappa}}{c_v}\frac{\tilde{e}}{\tilde{\rho}} & -\left(\frac{4}{3}\tilde{\mu} - \alpha\frac{\tilde{\kappa}}{c_v}\right)\tilde{u} & -\alpha\frac{\tilde{\kappa}}{c_v} \end{pmatrix}
\end{aligned} \tag{23}$$

and α is:

$$\alpha = \frac{\gamma}{Pr_\infty\,(\gamma - 1)}$$

4.2 Linearization In Time

4.2.1 $\tilde{E}_i^{n+1}$ Estimation

$$\begin{aligned}
\tilde{E}_i^{n+1} &= \tilde{E}_i^n + \underbrace{\left.\frac{\partial \tilde{E}}{\partial \tilde{Q}}\right|_i^n}_{\tilde{A}_i^n} \Delta\tilde{Q}_i^n + \mathrm{H.O.T} \\
\tilde{E}_i^{n+1} &= \tilde{E}_i^n + \tilde{A}_i^n \Delta\tilde{Q}_i^n
\end{aligned} \tag{24}$$

4.2.2 $\tilde{V_1}_i^{n+1}$ Estimation

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^{n} + \underbrace{\left.\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}\right|_i^n}_{\tilde{P}_i^n} \Delta \tilde{Q}_i^n + \underbrace{\left.\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}\right|_i^n}_{\tilde{R}_i^n} \Delta \tilde{Q}_{x i}^{\ n} + \text{H.O.T} \tag{25}$$

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^{n} + \tilde{P}_i^n \Delta \tilde{Q}_i^n + \tilde{R}_i^n \Delta \tilde{Q}_{x i}^{\ n}$$

The difficulty stems from the fact that the solution vector is $\Delta\tilde{Q}$ and not $\Delta\tilde{Q}_x$. This can be solved by a linearization of the term $\Delta\tilde{Q}_x$ which can be conducted using the following relation:

$$\frac{\partial\left(\tilde{R}\Delta\tilde{Q}\right)_i^n}{\partial \tilde{x}} = \frac{\partial \tilde{R}_i^n}{\partial \tilde{x}}\Delta\tilde{Q}_i^n + \tilde{R}_i^n \frac{\partial \Delta\tilde{Q}_i^n}{\partial \tilde{x}} = \frac{\partial \tilde{R}}{\partial \tilde{x}}\Delta\tilde{Q} + \tilde{R}_i^n \Delta\tilde{Q}_{x i}^{\ n}$$

$$\Downarrow$$

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^{n} + \left(\tilde{P} - \tilde{R}_x\right)_i^n \Delta\tilde{Q}_i^n + \frac{\partial}{\partial \tilde{x}}\left(\tilde{R}\Delta\tilde{Q}\right)_i^n \tag{26}$$

4.3 First Order Approximate Riemann Roe Method

The normalized Navier-Stokes equations as written in Eq.14:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \frac{\partial \tilde{E}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{27}$$

In linearized form:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \tilde{A}\frac{\partial \tilde{Q}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{28}$$

The initial conditions are:

$$\tilde{Q}_{(x,0)} = \begin{cases} \tilde{Q}_L & \tilde{x} < \tilde{x}_0 \\ \\ \tilde{Q}_R & \tilde{x} > \tilde{x}_0 \end{cases}$$

Roe's linear approximation to the 1-D Riemann problem is expressed as:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \hat{\tilde{A}}\frac{\partial \tilde{Q}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{29}$$

Where $\hat{\tilde{A}}$ replaces the original jacobian matrix $\tilde{A}$ and referred to as Roe's average matrix. Roe's average matrix is assumed constant in this formulation and therefore the problem is linear. The components of Roe's average matrix are evaluated using average values of $\tilde{Q}$ at the interface separating the two states, L and R, namely:

$$\hat{\tilde{A}} = \hat{\tilde{A}}_{(\tilde{Q}_L,\tilde{Q}_R)}$$

By setting certain conditions on the matrix $\hat{\tilde{A}}$ the aforementioned "Property U" is obtained for the system of equations.

- A linear mapping relates the vector $\tilde{Q}$ to the vector $\tilde{E}$
- $\hat{\tilde{A}}_{(\tilde{Q}_L,\tilde{Q}_R)} \xrightarrow[\tilde{Q}_L\to\tilde{Q}_R\to\tilde{Q}]{} \tilde{A}_{(\tilde{Q})}$

4.4 First Order Steger-Warming – Explicit

The A matrix (from Eq.23) is a diagonalizable matrix and can be written as:

$$\tilde{A} = \tilde{T}\tilde{\Lambda}\tilde{T}^{-1}$$

$$\tilde{T} = \begin{pmatrix} 1 & \frac{\tilde{\rho}}{2\tilde{a}} & -\frac{\tilde{\rho}}{2\tilde{a}} \\ \tilde{u} & \frac{\tilde{\rho}}{2\tilde{a}}\left(\tilde{u}+\tilde{a}\right) & -\frac{\tilde{\rho}}{2\tilde{a}}\left(\tilde{u}-\tilde{a}\right) \\ \frac{\tilde{u}^2}{2} & \frac{\tilde{\rho}}{2\tilde{a}}\left(\frac{\tilde{u}^2}{2}+\tilde{u}\tilde{a}+\frac{\tilde{a}^2}{\gamma-1}\right) & -\frac{\tilde{\rho}}{2\tilde{a}}\left(\frac{\tilde{u}^2}{2}-\tilde{u}\tilde{a}+\frac{\tilde{a}^2}{\gamma-1}\right) \end{pmatrix}$$

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{u} & 0 & 0 \\ 0 & \tilde{u}+\tilde{a} & 0 \\ 0 & 0 & \tilde{u}-\tilde{a} \end{pmatrix} \tag{30}$$

$$\tilde{T}^{-1} = \begin{pmatrix} 1-\frac{\gamma-1}{2}\frac{\tilde{u}^2}{\tilde{a}^2} & \left(\gamma-1\right)\frac{\tilde{u}^2}{\tilde{a}^2} & -\frac{\gamma-1}{\tilde{a}^2} \\ \frac{1}{\tilde{\rho}\tilde{a}}\left(\left(\gamma-1\right)\tilde{u}^2-\tilde{u}\tilde{a}\right) & \frac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a}-\left(\gamma-1\right)\tilde{u}\right) & \frac{\gamma-1}{\tilde{\rho}\tilde{a}} \\ -\frac{1}{\tilde{\rho}\tilde{a}}\left(\left(\gamma-1\right)\tilde{u}^2+\tilde{u}\tilde{a}\right) & \frac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a}+\left(\gamma-1\right)\tilde{u}\right) & -\frac{\gamma-1}{\tilde{\rho}\tilde{a}} \end{pmatrix}$$

Where:

$$\tilde{a} = \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}}$$

Let the $\tilde{\Lambda}^{\pm}$ matrix be defined as:

$$\tilde{\Lambda}^{\pm} = \begin{pmatrix} \frac{\tilde{u}\pm|\tilde{u}|}{2} & 0 & 0 \\ 0 & \frac{\tilde{u}+\tilde{a}\pm|\tilde{u}+\tilde{a}|}{2} & 0 \\ 0 & 0 & \frac{\tilde{u}-\tilde{a}\pm|\tilde{u}-\tilde{a}|}{2} \end{pmatrix} \tag{31}$$

Where the matrix $\tilde{\Lambda}^{+}$ contains only positive eigenvalues and the matrix $\tilde{\Lambda}^{-}$ contains only negative eigenvalues.
Define:

$$\begin{array}{rcl} \tilde{A}^{+} & \triangleq & \tilde{T}\tilde{\Lambda}^{+}\tilde{T}^{-1} \\ \tilde{A}^{-} & \triangleq & \tilde{T}\tilde{\Lambda}^{-}\tilde{T}^{-1} \end{array} \quad \Rightarrow \quad \begin{array}{rcl} \tilde{A} & = & \tilde{A}^{+}+\tilde{A}^{-} \\ \left|\tilde{A}\right| & \triangleq & \tilde{A}^{+}-\tilde{A}^{-} \end{array} \tag{32}$$

Assuming a perfect gas, the flux vector $\tilde{E}_{(Q)}$ is a homogeneous function of degree one in $\tilde{Q}$, meaning:

$$\forall\alpha \quad \tilde{E}_{(\alpha\tilde{Q})} = \alpha\tilde{E}_{(\tilde{Q})}$$

The homogeneity allows to rewrite the flux vector $\tilde{E}$ using Eq.22 as:

$$\tilde{E} = \tilde{A}\tilde{Q} = \left(\tilde{A}^{+}+\tilde{A}^{-}\right)\tilde{Q} = \underbrace{\tilde{A}^{+}\tilde{Q}}_{\tilde{E}^{+}} + \underbrace{\tilde{A}^{-}\tilde{Q}}_{\tilde{E}^{-}} = \tilde{E}^{+}+\tilde{E}^{-} \tag{33}$$

There is a discontinuity and deference between $\tilde{E}^{+}, \tilde{E}^{-}$. To eliminate the discontinuities and guarantee a smooth transition through critical points (sonic points or stagnation points), a blending function is introduced together with a blending parameter ε. An appropriate choice of the blending parameter has to be chosen.

$$\begin{aligned} \tilde{\lambda}^{+} &= \frac{\tilde{\lambda}+\left|\tilde{\lambda}\right|}{2} & & \tilde{\lambda}^{+'} = \frac{\tilde{\lambda}+\sqrt{\tilde{\lambda}^{2}+\varepsilon^{2}}}{2} \\ & & \Rightarrow & \\ \tilde{\lambda}^{-} &= \frac{\tilde{\lambda}-\left|\tilde{\lambda}\right|}{2} & & \tilde{\lambda}^{-'} = \frac{\tilde{\lambda}-\sqrt{\tilde{\lambda}^{2}+\varepsilon^{2}}}{2} \end{aligned} \tag{34}$$

Rewriting the conservation law form of the N-S equations Eq.14 using Eq.33:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} = -\frac{\partial \tilde{E}^{+}}{\partial \tilde{x}} - \frac{\partial \tilde{E}^{-}}{\partial \tilde{x}} + \frac{M_{\infty}}{Re_{L\infty}} \frac{\partial \tilde{V}_{1}}{\partial \tilde{x}} \tag{35}$$

A simple, explicit, first order (in space and time) scheme in delta form is obtained using:

$$\Delta \tilde{Q}_{i}^{n} = -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\nabla \tilde{E}_{i}^{+n} + \Delta \tilde{E}_{i}^{-n} - \frac{M_{\infty}}{Re_{L\infty}} \delta \tilde{V}_{1,i}^{n} \right) \tag{36}$$

And advancing the solution by:

$$\tilde{Q}_{i}^{n+1} = \Delta \tilde{Q}_{i}^{n} + \tilde{Q}_{i}^{n} \tag{37}$$

4.4.1 Finite Volume Formulation

Rearranging Eq.36 using the finite volume notation:

$$\begin{aligned} \Delta \tilde{Q}_{i}^{n} &= -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\tilde{E}_{i}^{+n} - \tilde{E}_{i-1}^{+n} + \tilde{E}_{i+1}^{-n} - \tilde{E}_{i}^{-n} - \frac{M_{\infty}}{Re_{L\infty}} \left(\tilde{V}_{1,i+\frac{1}{2}}^{n} - \tilde{V}_{1,i-\frac{1}{2}}^{n} \right) \right) \\ \Delta \tilde{Q}_{i}^{n} &= -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\left(\tilde{E}_{i}^{+n} + \tilde{E}_{i+1}^{-n} \right) - \left(\tilde{E}_{i-1}^{+n} + \tilde{E}_{i}^{-n} \right) - \frac{M_{\infty}}{Re_{L\infty}} \left(\tilde{V}_{1,i+\frac{1}{2}}^{n} - \tilde{V}_{1,i-\frac{1}{2}}^{n} \right) \right) \end{aligned} \tag{38}$$

Define:

$$\begin{aligned} \tilde{E}_{i+\frac{1}{2}} &\triangleq \tilde{E}_{i}^{+} + E_{i+1}^{-} \\ &= \tilde{A}_{i}^{+} \tilde{Q}_{i} + \tilde{A}_{i+1}^{-} \tilde{Q}_{i+1} \\ &\equiv \tilde{\tilde{E}}_{i+\frac{1}{2}} \end{aligned} \tag{39}$$

Finally we get:

$$\Delta \tilde{Q}_{i}^{n} = -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\tilde{\tilde{E}}_{i+\frac{1}{2}}^{n} - \tilde{\tilde{E}}_{i-\frac{1}{2}}^{n} - \frac{M_{\infty}}{Re_{L\infty}} \left(\tilde{V}_{1,i+\frac{1}{2}}^{n} - \tilde{V}_{1,i-\frac{1}{2}}^{n} \right) \right) \tag{40}$$

4.4.2 Calculating $\tilde{V}_{1,i+\frac{1}{2}}$

$$\tilde{V}_{1,i+\frac{1}{2}} = \begin{pmatrix} 0 \\ \frac{4}{3} \left.\tilde{\mu}\right|_{i+\frac{1}{2}} \left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} \\ \frac{4}{3} \left.\tilde{\mu}\right|_{i+\frac{1}{2}} \left.\tilde{u}\right|_{i+\frac{1}{2}} \left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} + \frac{\gamma}{Pr_{\infty}(\gamma-1)} \left.\tilde{\kappa}\right|_{i+\frac{1}{2}} \left.\frac{\partial \tilde{T}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} \end{pmatrix} \tag{41}$$

Where:

$$\left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} = \frac{\tilde{u}_{i+1} - \tilde{u}_i}{\Delta \tilde{x}}, \qquad \tilde{\mu}|_{i+\frac{1}{2}} = \frac{\tilde{\mu}_{i+1} + \tilde{\mu}_i}{2}$$
$$\left.\frac{\partial \tilde{T}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} = \frac{\tilde{T}_{i+1} - \tilde{T}_i}{\Delta \tilde{x}}, \qquad \tilde{\kappa}|_{i+\frac{1}{2}} = \frac{\tilde{\kappa}_{i+1} + \tilde{\kappa}_i}{2} \tag{42}$$
$$\tilde{u}|_{i+\frac{1}{2}} = \frac{\tilde{u}_{i+1} + \tilde{u}_i}{2}$$

4.5 First Order Steger-Warming – Implicit

The Implicit Steger-Warming scheme starts from:

$$\begin{aligned}\frac{\Delta \tilde{Q}_i^n}{\Delta \tilde{t}} &= -\frac{\partial \tilde{E}_i^{n+1}}{\partial \tilde{x}} + \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_{1,i}^{n+1}}{\partial \tilde{x}} \\ \Delta \tilde{Q}_i^n &= -\Delta \tilde{t} \frac{\partial}{\partial \tilde{x}} \left(\tilde{E}_i^n + \tilde{A}_i^n \Delta \tilde{Q}_i^n \right) + \Delta \tilde{t} \frac{M_\infty}{Re_{L\infty}} \frac{\partial}{\partial \tilde{x}} \left(\tilde{V}_{1,i}^n + \left(\tilde{P} - \tilde{R}_x \right)_i^n \Delta \tilde{Q}_i^n + \frac{\partial}{\partial \tilde{x}} \left(\tilde{R} \Delta \tilde{Q} \right)_i^n \right)\end{aligned} \tag{43}$$

Rearranging in delta form:

$$\begin{gathered}\left(I + \Delta \tilde{t} \left(\frac{\partial}{\partial \tilde{x}} \left[\tilde{A} - \frac{M_\infty}{Re_{L\infty}} \left(\tilde{P} - \tilde{R}_x \right) \right]_i^n - \frac{M_\infty}{Re_{L\infty}} \frac{\partial^2 \tilde{R}_i^n}{\partial \tilde{x}^2} \right) \right) \Delta \tilde{Q}_i^n = \mathrm{RHS}_i^n \\ \mathrm{RHS}_i^n = -\Delta \tilde{t} \frac{\partial}{\partial \tilde{x}} \left(\tilde{E}^+ + \tilde{E}^- - \frac{M_\infty}{Re_{L\infty}} \tilde{V}_1 \right)_i^n\end{gathered} \tag{44}$$

$$\begin{gathered}\left(I + \frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\left[\nabla \tilde{A}^+ + \Delta \tilde{A}^- - \frac{M_\infty}{Re_{L\infty}} \frac{D_0}{2} \left(\tilde{P} - \tilde{R}_x \right) \right]_i^n - \frac{M_\infty}{Re_{L\infty}} \frac{\delta^2}{\Delta \tilde{x}} \tilde{R}_i^n \right) \right) \Delta \tilde{Q}_i^n = \mathrm{RHS}_i^n \\ \mathrm{RHS}_i^n = -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\nabla \tilde{E}^+ + \Delta \tilde{E}^- - \frac{M_\infty}{Re_{L\infty}} \delta \tilde{V}_1 \right)_i^n\end{gathered} \tag{45}$$

Rewrite using the finite volume notation for the inviscid terms like in Eq.40:

$$\begin{gathered}\left(I + \frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\left[\nabla \tilde{A}^+ + \Delta \tilde{A}^- - \frac{M_\infty}{Re_{L\infty}} \frac{D_0}{2} \left(\tilde{P} - \tilde{R}_x \right) \right]_i^n - \frac{M_\infty}{Re_{L\infty}} \frac{\delta^2}{\Delta \tilde{x}} \tilde{R}_i^n \right) \right) \Delta \tilde{Q}_i^n = \mathrm{RHS}_i^n \\ \mathrm{RHS}_i^n = -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\tilde{\tilde{E}}_{i+\frac{1}{2}}^n - \tilde{\tilde{E}}_{i-\frac{1}{2}}^n - \frac{M_\infty}{Re_{L\infty}} \left(\tilde{V}_{1,i+\frac{1}{2}}^n - \tilde{V}_{1,i-\frac{1}{2}}^n \right) \right)\end{gathered} \tag{46}$$

and opening the delta form on the LHS:

$$\begin{aligned}
\mathrm{LHS}_i^n &= \Delta\tilde{Q}_i^n + \frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\nabla\tilde{A}_i^{+n}\Delta\tilde{Q}_i^n + \Delta\tilde{A}_i^{-n}\Delta\tilde{Q}_i^n - \frac{M_\infty}{Re_{L\infty}}\frac{D_0}{2}\left(\tilde{P}-\tilde{R}_x\right)_i^n\Delta\tilde{Q}_i^n - \frac{M_\infty}{Re_{L\infty}}\frac{\delta^2}{\Delta\tilde{x}}\tilde{R}_i^n\Delta\tilde{Q}_i^n\right) \\
&= \Delta\tilde{Q}_i^n + \frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\tilde{A}_i^{+n}\Delta\tilde{Q}_i^n - \tilde{A}_{i-1}^{+n}\Delta\tilde{Q}_{i-1}^n + \tilde{A}_{i+1}^{-n}\Delta\tilde{Q}_{i+1}^n - \tilde{A}_i^{-n}\Delta\tilde{Q}_i^n - \right. \\
&\quad -\frac{1}{2}\frac{M_\infty}{Re_{L\infty}}\left[\left(\tilde{P}-\tilde{R}_x\right)_{i+1}^n\Delta\tilde{Q}_{i+1}^n - \left(\tilde{P}-\tilde{R}_x\right)_{i-1}^n\Delta\tilde{Q}_{i-1}^n\right] - \\
&\quad \left. -\frac{1}{\Delta\tilde{x}}\frac{M_\infty}{Re_{L\infty}}\left[\tilde{R}_{i+1}^n\Delta\tilde{Q}_{i+1}^n - 2\tilde{R}_i^n\Delta\tilde{Q}_i^n + \tilde{R}_{i-1}^n\Delta\tilde{Q}_{i-1}^n\right]\right) \\
&= \Theta_i^n\Delta\tilde{Q}_{i-1}^n + \Phi_i^n\Delta\tilde{Q}_i^n + \Psi_i^n\Delta\tilde{Q}_{i+1}^n
\end{aligned} \tag{47}$$

Where:

$$\begin{aligned}
\Theta_i^n &= -\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\tilde{A}_{i-1}^{+n} + \frac{\Delta\tilde{t}}{2\Delta\tilde{x}}\frac{M_\infty}{Re_{L\infty}}\left(\tilde{P}-\tilde{R}_x\right)_{i-1}^n - \frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\frac{M_\infty}{Re_{L\infty}}\tilde{R}_{i-1}^n \\
\Phi_i^n &= I + \frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\tilde{A}_i^{+n} - \tilde{A}_i^{-n}\right) + 2\frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\frac{M_\infty}{Re_{L\infty}}\tilde{R}_i^n \\
\Psi_i^n &= \frac{\Delta\tilde{t}}{\Delta\tilde{x}}\tilde{A}_{i+1}^{-n} - \frac{\Delta\tilde{t}}{2\Delta\tilde{x}}\frac{M_\infty}{Re_{L\infty}}\left(\tilde{P}-\tilde{R}_x\right)_{i+1}^n - \frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\frac{M_\infty}{Re_{L\infty}}\tilde{R}_{i+1}^n
\end{aligned} \tag{48}$$

For the implicit Steger-Warming scheme a matrix inversion is needed as follows:

$$\begin{pmatrix}
\Phi_1^n & \Psi_1^n & 0 & \cdots & \cdots & \cdots & 0 \\
\Theta_2^n & \Phi_2^n & \Psi_2^n & 0 & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\
0 & 0 & \Theta_i^n & \Phi_i^n & \Psi_i^n & 0 & 0 \\
0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & \Theta_{N-1}^n & \Phi_{N-1}^n & \Psi_{N-1}^n \\
0 & \cdots & \cdots & \cdots & 0 & \Theta_N^n & \Phi_N^n
\end{pmatrix}
\begin{pmatrix}
\Delta\tilde{Q}_1^n \\ \Delta\tilde{Q}_2^n \\ \cdots \\ \cdots \\ \cdots \\ \Delta\tilde{Q}_{N-1}^n \\ \Delta\tilde{Q}_N^n
\end{pmatrix}
=
\begin{pmatrix}
\mathrm{RHS}_1^n \\ \mathrm{RHS}_2^n \\ \cdots \\ \cdots \\ \cdots \\ \mathrm{RHS}_{N-1}^n \\ \mathrm{RHS}_N^n
\end{pmatrix} \tag{49}$$

4.5.1 Calculating $\left(\tilde{P}-\tilde{R}_x\right)_i$

$$\left(\tilde{P}-\tilde{R}_x\right)_i = \frac{1}{\rho}\begin{pmatrix}
0 & 0 & 0 \\
\tilde{u}\big|_i\frac{4}{3}\frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i & -\frac{4}{3}\frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i & 0 \\
\tilde{u}^2\big|_i\frac{4}{3}\frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i & -\tilde{u}\big|_i\frac{4}{3}\frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i & 0
\end{pmatrix}, \qquad \frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i = \frac{\tilde{\mu}_{i+1}-\tilde{\mu}_{i-1}}{2\Delta\tilde{x}} \tag{50}$$