Computational Fluid Dynamics
HW1

Almog Dobrescu

ID 214254252

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1 Inviscid Burgers Equation

The Inviscid Burgers equation, in conservation law form, is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad F = F_{(u)} = \frac{u^2}{2} \tag{1}$$

In non-conservation law form, is given by:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0 \quad A = \frac{\partial F}{\partial u} = u \tag{2}$$

The equation is obtained by neglecting the viscous term from the viscous Burger equation.

1.1 Boundary and Initial Conditions

$$\begin{aligned} u_{(x=0,t)} &= 1.0 \\ u_{(x=1,t)} &= u_1 \\ u_{(x,t=0)} &= 1-(1-u_1)\cdot x \end{aligned} \tag{3}$$

1.2 First Order Roe Method ($u_1 = 0.0$)

Roe scheme is based on the solution of the linear problem:

$$\frac{\partial u}{\partial t} + \bar{A}\frac{\partial u}{\partial x} = 0 \tag{4}$$

Where $\bar{A}$ is a constant matrix that is dependent on local conditions. The matrix is constructed in a way to guarantee unifrom validity across discontinuities:

1. For any u_i, u_{i+1}:
$$F_{i+1} - F_i = \bar{A}\cdot(u_{i+1} - u_i)$$

2. When $u = u_i = u_{i+1}$ then:
$$\bar{A}_{(u_i,u_{i+1})} = \bar{A}_{(u,u)} = \frac{\partial F}{\partial u} = u$$

In case of the Burgers equation, the matrix $\bar{A}$ is a scalar, namely, $\bar{A} = \bar{u}$. The equation becomes:

$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} = 0 \tag{5}$$

The value of $\bar{u}$ for the cell face between i and $i+1$ is determined from the first conditions:

$$\bar{u} = \bar{u}_{i+\frac{1}{2}} = \frac{F_{i+1} - F_i}{u_{i+1} - u_i} = \frac{\frac{1}{2}u_{i+1}^2 - \frac{1}{2}u_i^2}{u_{i+1} - u_i} = \begin{cases} \dfrac{u_i + u_{i+1}}{2} & u_i \neq u_{i+1} \\ u_i & u_i = u_{i+1} \end{cases} \tag{6}$$

The single wave that emanates from the cell interface travels either in the positive or negative direction, depending upon the sighn of $\bar{u}_{i+\frac{1}{2}}$. Define:

$$\begin{cases} \bar{u}^+_{i+\frac{1}{2}} \triangleq \frac{1}{2}\left(\bar{u}_{i+\frac{1}{2}} + \left|\bar{u}_{i+\frac{1}{2}}\right|\right) \geq 0 \\ \bar{u}^-_{i+\frac{1}{2}} \triangleq \frac{1}{2}\left(\bar{u}_{i+\frac{1}{2}} - \left|\bar{u}_{i+\frac{1}{2}}\right|\right) \leq 0 \end{cases} \quad \bar{u}_{i+\frac{1}{2}} = \bar{u}^+_{i+\frac{1}{2}} + \bar{u}^-_{i+\frac{1}{2}} \tag{7}$$

Using the jump relation, the numerical flux at the cell interface can be evaluated by one of the following:

$$\begin{cases} f_{i+\frac{1}{2}} - F_i = \bar{u}^-_{i+\frac{1}{2}} \cdot (u_{i+1} - u_i) \\ \\ F_{i+1} - f_{i+\frac{1}{2}} = \bar{u}^+_{i+\frac{1}{2}} \cdot (u_{i+1} - u_i) \end{cases} \tag{8}$$

The numerical flux may then be written in the following symmetric from:

$$f_{i+\frac{1}{2}} = \frac{F_i + F_{i+1}}{2} + \frac{1}{2}\left(\bar{u}^+_{i+\frac{1}{2}} - \bar{u}^-_{i+\frac{1}{2}}\right)(u_{i+1} - u_i) \tag{9}$$

Since Roe's scheme can't distinguish between the types of discontinuity, it may result in an expansion shock where the analytical solution is an expansion wave. To guarantee a physical solution the scheme will be modified like so:
Define

$$\varepsilon = \max\left(0, \frac{u_{i+1} - u_i}{2}\right)$$

The interface wave speed becomes

$$\bar{u}_{i+\frac{1}{2}} = \begin{cases} \bar{u}_{i+\frac{1}{2}} & \bar{u}_{i+\frac{1}{2}} \geq \varepsilon \quad \text{compression} \\ \varepsilon & \bar{u}_{i+\frac{1}{2}} < \varepsilon \quad \text{expansion} \end{cases} \tag{10}$$

1.3 Second Order Roe ($u_1 = 0.5$)

1.3.1 With Limiters

1.3.2 Without Limiters

2 Generalized Burgers Equation