

# Computational Fluid Dynamics

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# 1 Mathematical Problem

The Burgers Equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \quad (1)$$

By neglecting the convective term, the parabolic model equation is obtained:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) \quad (2)$$

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} \quad (3)$$

$$\mu = 1.0 \quad (4)$$

Boundary and initial conditions:

$$u_{(y_0,t)} = u_0 \quad u_{(y_1,t)} = u_1 \quad u_{(y,t=0)} = f_{(y)} \quad (5)$$

# 2 Numerical Scheme

A general explicit-implicit scheme for constant  $\mu$  is given by:

$$u_i^{n+1} = u_i^n + \frac{\mu \Delta t}{\Delta y^2} \delta_{yy} [(1 - \alpha) u_i^n + \alpha u_i^{n+1}] \quad (6)$$

where:

$$\alpha = \begin{cases} 0 & \text{Explicit} \\ \frac{1}{2} & \text{Crank-Nicolson} \\ 1 & \text{Implicit} \end{cases} \quad (7)$$

and the order is:

$$\left[ \Delta x^2, \Delta \left( \frac{1}{2} - \alpha \right) \right] \quad (8)$$

In delta form:

$$\left( I - \alpha \frac{\mu \Delta t}{\Delta y^2} \delta_{yy} \right) \Delta u_i^n = \frac{\mu \Delta t}{\Delta y^2} \delta_{yy} u_i^n \quad (9)$$

Applying the operators:

$$\Delta u_i^n - \alpha \frac{\mu \Delta t}{\Delta y^2} \delta_{yy} (\Delta u_i^n) = \frac{\mu \Delta t}{\Delta y^2} \delta_{yy} (u_i^n) \quad (10)$$

$$\Delta u_i^n - \alpha \frac{\mu \Delta t}{\Delta y^2} \delta_y \left( \Delta u_{(y_i + \frac{\Delta y}{2})}^n - \Delta u_{(y_i - \frac{\Delta y}{2})}^n \right) = \frac{\mu \Delta t}{\Delta y^2} \delta_y \left( u_{(y_i + \frac{\Delta y}{2})}^n - u_{(y_i - \frac{\Delta y}{2})}^n \right) \quad (11)$$

$$\Delta u_i^n - \alpha \frac{\mu \Delta t}{\Delta y^2} \left( \Delta u_{(y_i + \Delta y)}^n - 2\Delta u_{(y_i)}^n + \Delta u_{(y_i - \Delta y)}^n \right) = \frac{\mu \Delta t}{\Delta y^2} \left( u_{(y_i + \Delta y)}^n - 2u_{(y_i)}^n + u_{(y_i - \Delta y)}^n \right) \quad (12)$$

$$\Delta u_i^n - \alpha \frac{\mu \Delta t}{\Delta y^2} (\Delta u_{i+1}^n - 2\Delta u_i^n + \Delta u_{i-1}^n) = \frac{\mu \Delta t}{\Delta y^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad (13)$$

$$A_i \Delta u_{i-1}^n + B_i \Delta u_i^n + C_i \Delta u_{i+1}^n = D_i \quad (14)$$

where:

$$A'_i = -\alpha \frac{\mu \Delta t}{\Delta y^2} \quad (15)$$

$$B'_i = 1 + 2\alpha \frac{\mu \Delta t}{\Delta y^2} \quad (16)$$

$$C'_i = -\alpha \frac{\mu \Delta t}{\Delta y^2} \quad (17)$$

$$D'_i = \frac{\mu \Delta t}{\Delta y^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad (18)$$

and advancing the solution with:

$$u_i^{n+1} = u_i^n + \Delta u_i^n \quad (19)$$

To reduce problems of big floating point numbers, define  $r$ :

$$r \triangleq \frac{\mu \Delta t}{\Delta y^2} \quad (20)$$

After dividing by  $r$ :

$$A_i = -\alpha \quad (21)$$

$$B_i = \frac{1}{r} + 2\alpha \quad (22)$$

$$C_i = -\alpha \quad (23)$$

$$D_i = (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad (24)$$

### 2.1 For $\alpha = 0$

When  $\alpha = 0$  The scheme simplifies to:

$$\Delta u_i^n = r D_i \quad (25)$$

### 2.2 For $\alpha \neq 0$

When  $\alpha \neq 0$  in order to calculate  $\Delta u_i^n$  it is needed to invert matrix as follows:

$$\begin{pmatrix} B_1 & C_1 & 0 & \dots & \dots & \dots & 0 \\ A_2 & B_2 & C_2 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ 0 & 0 & A_i & B_i & C_i & 0 & 0 \\ 0 & \dots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & A_{N-2} & B_{N-2} & C_{N-2} \\ 0 & 0 & \dots & \dots & 0 & A_{N-1} & B_{N-1} \end{pmatrix} \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \\ \dots \\ \dots \\ \dots \\ \Delta u_{N-2} \\ \Delta u_{N-1} \end{pmatrix} = \begin{pmatrix} D_1 - A_1 \cdot u_0 \\ D_2 \\ \dots \\ \dots \\ \dots \\ D_{N-2} \\ D_{N-1} - C_{N-1} \cdot u_N \end{pmatrix} \quad (26)$$

### 2.3 In Our Case

$$y_0 = 0 \quad u_0 = 0 \quad y_1 = 1 \quad u_1 = 1 \quad f_{(y)} = 1 \quad N = 11 \quad (27)$$

The scheme considered converged when the size of the RHS drops by 6 order of magnitude.

### 3 Stability Analysis

According to the "von Neumann Stability Analysis", the error between the differential and numerical equation is given by:

$$\varepsilon = N - D \quad (28)$$

where:

$$\varepsilon(y, t) = \sum_m c_m(t) e^{ik_m y} \quad (29)$$

since the analysis is linear, one can choose one element from the series. Moreover, the error becomes:

$$\varepsilon^n = Z^n e^{iky} \quad (30)$$

The constant "k" is the wave number that is related to the initial error.

$$Z = e^{\lambda \Delta t} \Rightarrow Z^n = e^{\lambda n \Delta t} = e^{\lambda t} \quad (31)$$

Any element of the error therefore takes the form:

$$\varepsilon^n = e^{\lambda t} e^{iky} \quad (32)$$

The ratio between consecutive errors is the *amplification factor*, names "G":

$$G \triangleq \frac{\varepsilon^{n+1}}{\varepsilon^n} = e^{\lambda \Delta t} \quad (33)$$

The condition for Stability:

$$|G| < 1 \quad (34)$$

The error must satisfies the finite difference equation:

$$\Delta \varepsilon_i^n - \alpha \frac{\mu \Delta t}{\Delta y^2} (\Delta \varepsilon_{i+1}^n - 2\Delta \varepsilon_i^n + \Delta \varepsilon_{i-1}^n) = \frac{\mu \Delta t}{\Delta y^2} (\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n) \quad (35)$$

Where:

$$\Delta \varepsilon_i^n = \varepsilon_i^{n+1} - \varepsilon_i^n = e^{\lambda(t+\Delta t)} e^{iky} - e^{\lambda t} e^{iky} = e^{\lambda t} e^{iky} (e^{\lambda \Delta t} - 1) \quad (36)$$

$$\Delta \varepsilon_{i+1}^n = \varepsilon_{i+1}^{n+1} - \varepsilon_{i+1}^n = e^{\lambda(t+\Delta t)} e^{ik(y+\Delta y)} - e^{\lambda t} e^{ik(y+\Delta y)} = e^{\lambda t} e^{ik(y+\Delta y)} (e^{\lambda \Delta t} - 1) \quad (37)$$

$$\Delta \varepsilon_{i-1}^n = \varepsilon_{i-1}^{n+1} - \varepsilon_{i-1}^n = e^{\lambda(t+\Delta t)} e^{ik(y-\Delta y)} - e^{\lambda t} e^{ik(y-\Delta y)} = e^{\lambda t} e^{ik(y-\Delta y)} (e^{\lambda \Delta t} - 1) \quad (38)$$

Substitut the values:

$$\begin{aligned} & \left( e^{\lambda t} e^{iky} - \alpha \frac{\mu \Delta t}{\Delta y^2} (e^{\lambda t} e^{ik(y+\Delta y)} - 2e^{\lambda t} e^{iky} + e^{\lambda t} e^{ik(y-\Delta y)}) \right) (e^{\lambda \Delta t} - 1) = \\ & = \frac{\mu \Delta t}{\Delta y^2} (e^{\lambda t} e^{ik(y+\Delta y)} - 2e^{\lambda t} e^{iky} + e^{\lambda t} e^{ik(y-\Delta y)}) \end{aligned} \quad (39)$$

Dividing by  $e^{\lambda t} e^{iky}$ :

$$\left( 1 - \alpha \frac{\mu \Delta t}{\Delta y^2} (e^{ik\Delta y} - 2 + e^{-ik\Delta y}) \right) (e^{\lambda \Delta t} - 1) = \frac{\mu \Delta t}{\Delta y^2} (e^{ik\Delta y} - 2 + e^{-ik\Delta y}) \quad (40)$$

Define:

$$r \triangleq \frac{\mu \Delta t}{\Delta y^2}, \quad A \triangleq e^{ik\Delta y} - 2 + e^{-ik\Delta y} = 2 \cos \beta - 2, \quad \beta \triangleq k\Delta y \quad (41)$$

After substitution and rearranging we get:

$$e^{\lambda t} - 1 = \frac{rA}{1 - \alpha rA} \quad (42)$$

$\Downarrow$

$$G = e^{\lambda t} = 1 + \frac{rA}{1 - \alpha rA} \quad (43)$$

For stability:

$$|G| < 1 \Rightarrow \left| 1 + \frac{rA}{1 - \alpha rA} \right| < 1 \quad (44)$$

For  $\alpha = 0$ :

$$\begin{aligned} G &= 1 + rA \\ |1 - 4r| &< 1 \\ -1 &< 1 - 4r < 1 \\ -2 &< -4r < 0 \\ 0 &< 4r < 2 \\ 0 &< r < \frac{1}{2} \end{aligned}$$

$$\Delta t < \frac{\Delta y^2}{2\mu} \quad (45)$$

For  $\alpha = 1$ :

$$G = \frac{1}{1 - rA}$$

$$\left| \frac{1}{1 - rA} \right| < 1 \quad A \leq 0 \quad (46)$$

$\Downarrow$

smaller for every  $r$  ( $\Delta t$ )

For every  $\alpha$  and  $\Delta t \rightarrow \infty$ :

$$\lim_{\Delta t \rightarrow \infty} |G| = \frac{1 - \alpha}{\alpha} \quad (47)$$

Which means that stability for infinite time step:

$$\alpha \begin{cases} < \frac{1}{2} & \text{Unstable} \\ = \frac{1}{2} & \text{Neutrally stable} \\ > \frac{1}{2} & \text{Stable} \end{cases} \quad (48)$$

## 4 The Computer Program

## 5 Results

### 5.1 $\alpha = 0$ - Explicit Scheme

From Eq.45 we get the stability limit of  $\Delta t$ :

$$\Delta t < \frac{\Delta y^2}{2\mu} = \frac{0.1^2}{2 \cdot 1} = 5 \cdot 10^{-3} \quad (49)$$

### 5.2 $\alpha = \frac{1}{2}$ - Crank-Nicolson Scheme

### 5.3 $\alpha = 1$ - Implicit Scheme



## 6 Conclusions