Computational Fluid Dynamics
HW2

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Nomenclature

Dimensionless Numbers

Δx size of each cell in the domain

γ ratio of specific heats

κ coefficient of thermal conductivity

μ coefficient of viscosity

ρ fluid density

c_p constant specific heat capacity for a constant pressure

c_v constant specific heat capacity for a constant volume

E inviscid convective vector

e total energy

E_ν viscous convective vector

L characteristic length

p pressure

Q conservation state space

R gas constant

T temperature

t time

u fluid velocity

x spatial coordinate

x_F x coordinate of the end of the domain

x_i x coordinate of the i-th cell

Far–Away Properties

κ_∞ coefficient of thermal conductivity far away

μ_∞ coefficient of viscosity far away

ρ_∞ density far away

a_∞ speed of sound far away

M_∞ mach number far away

p_∞ pressure far away

T_∞ temperature far away

Dimensionless Numbers

Pr_∞ Prandtl number far away

$Re_{L\infty}$ Reynolds number with respect to L far away

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1   Problem Definition

1.1   Governing Equations

Consider the one-dimensional Navier-Stokes Equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = \frac{\partial E_\nu}{\partial x} \tag{1}$$

Where:

$$Q = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad E = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ (e + p)\,u \end{pmatrix}, \quad E_\nu = \begin{pmatrix} 0 \\ \tau_{xx} \\ u\tau_{xx} - q_x \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{4}{3}\mu\dfrac{\partial u}{\partial x} \\ \dfrac{4}{3}\mu u\dfrac{\partial u}{\partial x} + \kappa\dfrac{\partial T}{\partial x} \end{pmatrix}$$

$$p = (\gamma - 1)\left(e - \frac{1}{2}\rho u^2\right), \qquad T = \frac{p}{\rho R}, \tag{2}$$

$$\mu = 1.458 \cdot 10^{-6}\frac{T^{\frac{3}{2}}}{T + 110.4}, \quad \mu = 2.495 \cdot 10^{-3}\frac{T^{\frac{3}{2}}}{T + 194}$$

$$R = c_p - c_v, \quad \gamma = \frac{c_p}{c_v}$$

The constants are:

- $\gamma = 1.4$ for air under standard atmospheric conditions

- $R = 287.0$ for air

1.2   Physical Domain

The physical domain is a tube extended between $x = 0.2$ and $x = 1.0$. At both ends there are impermeable walls.

1.3   Initial Conditions

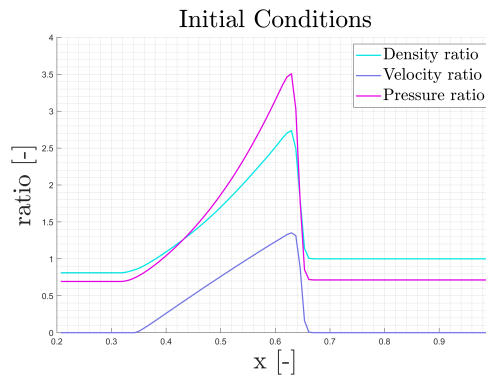The initial conditions are shown in Fig.1:

Figure 1: Initial conditions

1.4   Boundary Conditions

On each side of the tube there is an adiabatic, solid wall boundary conditions.

$$u_{(x=0.2)} = u_{(x=1.0)} = 0 \quad \bigg\| \quad \frac{\partial p}{\partial x}\bigg|_{x=0.2} = \frac{\partial p}{\partial x}\bigg|_{x=1.0} = 0 \quad \bigg\| \quad \frac{\partial T}{\partial x}\bigg|_{x=0.2} = \frac{\partial T}{\partial x}\bigg|_{x=1.0} = 0$$

2   Normalizing The Navier-Stokes Equations

Since the initial conditions are normalized, there is a need to normalize the N-S equations. We will use the following normalizations:

$$\rho = \rho_\infty \tilde{\rho}, \quad u = a_\infty \tilde{u}, \quad p = \gamma p_\infty \tilde{p}, \quad T = \gamma T_\infty \tilde{T}, \quad x = L\tilde{x}, \quad t = \frac{L}{a_\infty}\tilde{t}, \quad \mu = \mu_\infty \tilde{\mu}, \quad \kappa = \kappa_\infty \tilde{\kappa}$$

(3)

The normalization of the temperature was chosen to cancel out the γ in the normalization of the pressure:

$$\begin{array}{rcl} p & = & \rho R T \\ \gamma p_\infty \tilde{p} & = & \rho_\infty \tilde{\rho} R \gamma T_\infty \tilde{T} \\ \tilde{p} & = & \tilde{\rho}\tilde{T} \end{array}$$

(4)

The pressure normalization can be written also as:

$$p = \gamma p_\infty \tilde{p} = \gamma \rho_\infty R T_\infty \tilde{p} = \rho_\infty a_\infty^2 \tilde{p}$$

(5)

From equations 2 and 5 we can derive the normalization for the energy:

$$\begin{array}{rcl} e & = & \dfrac{p}{\gamma - 1} + \dfrac{1}{2}\rho u^2 \\[3mm] e & = & \dfrac{\rho_\infty a_\infty^2 \tilde{p}}{\gamma - 1} + \dfrac{1}{2}\rho_\infty \tilde{\rho} a_\infty^2 \tilde{a}^2 \\[3mm] e & = & \rho_\infty a_\infty^2 \left(\dfrac{\tilde{p}}{\gamma - 1} + \dfrac{1}{2}\tilde{\rho}\tilde{a}^2 \right) \\[3mm] e & = & \rho_\infty a_\infty^2 \tilde{e} \end{array}$$

(6)

After substituting the normalizations in the N-S equations we get:

$$\frac{\partial}{\partial \frac{L}{a_\infty}\tilde{t}} \begin{pmatrix} \rho_\infty \tilde{\rho} \\ \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{p} + \rho_\infty a_\infty^2 \tilde{\rho}\tilde{u}^2 \\ \rho_\infty a_\infty^3 (\tilde{e} + \tilde{p})\tilde{u} \end{pmatrix} = \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}\mu_\infty a_\infty \tilde{\mu}\frac{\partial \tilde{u}}{\partial L\tilde{x}} \\ \frac{4}{3}\mu_\infty a_\infty^2 \tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial L\tilde{x}} + \frac{\kappa_\infty a_\infty^2}{R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial L\tilde{x}} \end{pmatrix}$$

(7)

Rearranging:

$$\frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ a_\infty \tilde{\rho}\tilde{u} \\ a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho}\tilde{u} \\ a_\infty \tilde{p} + a_\infty \tilde{\rho}\tilde{u}^2 \\ a_\infty^2 (\tilde{e} + \tilde{p})\tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L^2}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}a_\infty \tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3}a_\infty^2 \tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\kappa_\infty a_\infty^2}{\mu_\infty R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix}$$

(8)

Dividing the second equation by a_∞, the third equation by a_∞^2, and the whole set of equations by $\dfrac{\rho_\infty a_\infty}{L}$ we get:

$$\frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ \tilde{\rho}\tilde{u} \\ \tilde{e} \end{pmatrix} + \frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho}\tilde{u} \\ \tilde{p} + \tilde{\rho}\tilde{u}^2 \\ (\tilde{e} + \tilde{p})\tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L\rho_\infty a_\infty} \frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \dfrac{4}{3}\tilde{\mu}\dfrac{\partial \tilde{u}}{\partial \tilde{x}} \\ \dfrac{4}{3}\tilde{\mu}\tilde{u}\dfrac{\partial \tilde{u}}{\partial \tilde{x}} + \dfrac{\kappa_\infty}{\mu_\infty R}\tilde{\kappa}\dfrac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \tag{9}$$

The Reynolds number and the mach number far away are defined as:

$$M_\infty = \frac{u_\infty}{a_\infty} \quad Re_{L\infty} = \frac{\rho_\infty u_\infty L}{\mu_\infty}$$
$$\Downarrow$$
$$\frac{\mu_\infty}{L\rho_\infty a_\infty} = \frac{M_\infty}{Re_{L\infty}} \tag{10}$$

The Prandtl number far away is defined as:

$$Pr_\infty = \frac{c_p \mu_\infty}{\kappa_\infty}$$
$$\Downarrow$$
$$\frac{\kappa_\infty}{\mu_\infty R} = \frac{c_p}{Pr_\infty (c_p - c_v)} = \frac{\gamma}{Pr_\infty (\gamma - 1)} \tag{11}$$

Substituting into the normalized N-S equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \frac{\partial \tilde{E}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{E}_\nu}{\partial \tilde{x}} \tag{12}$$

Where:

$$\tilde{Q} = \begin{pmatrix} \tilde{\rho} \\ \tilde{\rho}\tilde{u} \\ \tilde{e} \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} \tilde{\rho}\tilde{u} \\ \tilde{p} + \tilde{\rho}\tilde{u}^2 \\ (\tilde{e} + \tilde{p})\tilde{u} \end{pmatrix}, \quad \tilde{E}_\nu = \begin{pmatrix} 0 \\ \dfrac{4}{3}\tilde{\mu}\dfrac{\partial \tilde{u}}{\partial \tilde{x}} \\ \dfrac{4}{3}\tilde{\mu}\tilde{u}\dfrac{\partial \tilde{u}}{\partial \tilde{x}} + \dfrac{\gamma}{Pr_\infty (\gamma - 1)}\tilde{\kappa}\dfrac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \tag{13}$$

The normalized Navier-Stokes equations are:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \frac{\partial \tilde{E}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{14}$$

Where:

$$\tilde{V}_1 = \tilde{V}_{1(\tilde{Q},\tilde{Q}_x)} = \tilde{E}_\nu$$

3   Linearizing The Navier-Stokes Equations

To linearize the N-S equations we need to estimate the derivatives $\dfrac{\partial \tilde{E}}{\partial \tilde{x}}$ and $\dfrac{\partial \tilde{V}_1}{\partial \tilde{x}}$.

3.1 $\frac{\partial \tilde{E}}{\partial \tilde{x}}$ Estimation

$$\frac{\partial \tilde{E}}{\partial \tilde{x}} = \underbrace{\frac{\partial \tilde{E}}{\partial \tilde{Q}}}_{\tilde{A}} \frac{\partial \tilde{Q}}{\partial \tilde{x}}$$

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{\gamma - 3}{2}\tilde{u}^2 & (3 - \gamma)\,\tilde{u} & \gamma - 1 \\ (\gamma - 1)\,\tilde{u}^3 \dfrac{\gamma \tilde{e}\tilde{u}}{\tilde{\rho}} & \dfrac{\gamma \tilde{e}}{\tilde{\rho}} - \dfrac{3\,(\gamma - 1)\,\tilde{u}^2}{2} & \gamma \tilde{u} \end{pmatrix} \tag{15}$$

3.2 $\frac{\partial \tilde{V}_1}{\partial \tilde{x}}$ Estimation

$$\frac{\partial \tilde{V}_1}{\partial \tilde{x}} = \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}} \frac{\partial \tilde{Q}}{\partial \tilde{x}}}_{P} + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x} \frac{\partial \tilde{Q}_x}{\partial \tilde{x}}}_{R} \tag{16}$$

4   The Computational Domain

4.1   Discretization

The physical domain $[x_I, x_F]$ is discretized into N equispaced cells. The size of each cell is there for:

$$\Delta x = \frac{x_F - x_I}{N} = \frac{L}{N} \tag{17}$$

so the x coordinate of the i-th cell x_i is:

$$x_i = x_I + \frac{1}{2}\Delta x + \Delta x \cdot (i - 1) \qquad \text{when starting from } i = 1 \tag{18}$$

4.2   Boundary Conditions

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated like so:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} \\ u_{(i=N+1)} &= -u_{(i=N)} \end{aligned} \tag{19}$$

in order to maintain velocity zero on the boundary and like so:

$$\begin{aligned} T_{(i=0)} &= T_{(i=1)} \\ T_{(i=N+1)} &= T_{(i=N)} \end{aligned} \tag{20}$$

In order to maintain adiabatic boundary conditions. Since the gradient of the pressure on the wall is zero, we get:

$$\begin{aligned} p_{(i=0)} &= p_{(i=1)} \\ p_{(i=N+1)} &= p_{(i=N)} \end{aligned} \tag{21}$$

From equations 2, 20, and 21 we can conclude:

$$\begin{aligned} \rho_{(i=0)} &= \rho_{(i=1)} \\ \rho_{(i=N+1)} &= \rho_{(i=N)} \end{aligned} \tag{22}$$

and from equations 2, 19, 21, and 22 we can conclude:

$$\begin{aligned} e_{(i=0)} &= e_{(i=1)} \\ e_{(i=N+1)} &= e_{(i=N)} \end{aligned} \tag{23}$$

5   The Numerical Schemes

The linearized Navier-Stokes equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \underbrace{\frac{\partial \tilde{E}}{\partial \tilde{Q}}}_{A} \frac{\partial \tilde{Q}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}} \left(\underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}}_{P} \frac{\partial \tilde{Q}}{\partial \tilde{x}} + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}}_{R} \frac{\partial \tilde{Q}_x}{\partial \tilde{x}} \right) \tag{24}$$

5.1   First Order Approximate Riemann Roe Method

5.2   First Order Steger-Warming – Explicit

A is a Diagonalizable matrix and can be written as:

$$A = T\Lambda T^{-1}$$

$$T = \begin{pmatrix} 1 & \dfrac{\tilde{\rho}}{2\tilde{a}} & -\dfrac{\tilde{\rho}}{2\tilde{a}} \\[2ex] \tilde{u} & \dfrac{\tilde{\rho}}{2\tilde{a}}\left(\tilde{u}+\tilde{a}\right) & -\dfrac{\tilde{\rho}}{2\tilde{a}}\left(\tilde{u}-\tilde{a}\right) \\[2ex] \dfrac{\tilde{u}^2}{2} & \dfrac{\tilde{\rho}}{2\tilde{a}}\left(\dfrac{\tilde{u}^2}{2}+\tilde{u}\tilde{a}+\dfrac{\tilde{a}^2}{\gamma-1}\right) & -\dfrac{\tilde{\rho}}{2\tilde{a}}\left(\dfrac{\tilde{u}^2}{2}-\tilde{u}\tilde{a}+\dfrac{\tilde{a}^2}{\gamma-1}\right) \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \tilde{u} & 0 & 0 \\ 0 & \tilde{u}+\tilde{a} & 0 \\ 0 & 0 & \tilde{u}-\tilde{a} \end{pmatrix} \tag{25}$$

$$T^{-1} = \begin{pmatrix} 1-\dfrac{\gamma-1}{2}\dfrac{\tilde{u}^2}{\tilde{a}^2} & (\gamma-1)\dfrac{\tilde{u}^2}{\tilde{a}^2} & -\dfrac{\gamma-1}{\tilde{a}^2} \\[2ex] \dfrac{1}{\tilde{\rho}\tilde{a}}\left((\gamma-1)\tilde{u}^2-\tilde{u}\tilde{a}\right) & \dfrac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a}-(\gamma-1)\tilde{u}\right) & \dfrac{\gamma-1}{\tilde{\rho}\tilde{a}} \\[2ex] -\dfrac{1}{\tilde{\rho}\tilde{a}}\left((\gamma-1)\tilde{u}^2+\tilde{u}\tilde{a}\right) & \dfrac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a}+(\gamma-1)\tilde{u}\right) & -\dfrac{\gamma-1}{\tilde{\rho}\tilde{a}} \end{pmatrix}$$

Where:

$$\tilde{a} = \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}}$$

Let the $\Lambda^\pm$ matrix be defined as:

$$\Lambda^\pm = \begin{pmatrix} \dfrac{\tilde{u}\pm|\tilde{u}|}{2} & 0 & 0 \\[2ex] 0 & \dfrac{\tilde{u}+\tilde{a}\pm|\tilde{u}+\tilde{a}|}{2} & 0 \\[2ex] 0 & 0 & \dfrac{\tilde{u}-\tilde{a}\pm|\tilde{u}-\tilde{a}|}{2} \end{pmatrix} \tag{26}$$

Where the matrix Λ^+ contains only positive eigenvalues and the matrix Λ^- contains only negative eigenvalues.

5.3   First Order Steger-Warming – Implicit