Computational Fluid Dynamics
HW2

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Nomenclature

Δx	size of each cell in the domain
$\Delta\tilde{t}$	normalized step size in time
$\Delta\tilde{x}$	normalized step size in space
γ	ratio of specific heats
κ	coefficient of thermal conductivity
μ	coefficient of viscosity
ρ	fluid density
c_p	constant specific heat capacity for a constant pressure
c_v	constant specific heat capacity for a constant volume
E	inviscid convective vector
e	total energy
E_ν	viscous convective vector
L	characteristic length
p	pressure
Q	conservation state space
R	gas constant
T	temperature
t	time
u	fluid velocity
x	spatial coordinate
x_F	x coordinate of the end of the domain
x_i	x coordinate of the i-th cell

Far–Away Properties

κ_∞	coefficient of thermal conductivity far away
μ_∞	coefficient of viscosity far away
ρ_∞	density far away
a_∞	speed of sound far away
M_∞	mach number far away
p_∞	pressure far away

T_∞ temperature far away

Matrices

Λ normalized eigenvalues matrix

$\tilde{A}$ normalized jacobian matrix of E w.r.t. Q

$\tilde{P}$ normalized jacobian matrix of E_ν w.r.t. Q

$\tilde{R}$ normalized jacobian matrix of E_ν w.r.t. Q_x

T normalized eigenvectors matrix

Dimensionless Numbers

Pr_∞ Prandtl number far away

$Re_{L\infty}$ Reynolds number with respect to L far away

1 Problem Definition

1.1 Governing Equations

Consider the one-dimensional Navier-Stokes Equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = \frac{\partial E_\nu}{\partial x} \tag{1}$$

Where:

$$Q = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad E = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ (e+p)\,u \end{pmatrix}, \quad E_\nu = \begin{pmatrix} 0 \\ \tau_{xx} \\ u\tau_{xx} - q_x \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu\frac{\partial u}{\partial x} \\ \frac{4}{3}\mu u\frac{\partial u}{\partial x} + \kappa\frac{\partial T}{\partial x} \end{pmatrix}$$

$$p = (\gamma - 1)\left(e - \frac{1}{2}\rho u^2\right), \qquad T = \frac{p}{\rho R}, \tag{2}$$

$$\mu = 1.458 \cdot 10^{-6}\frac{T^{\frac{3}{2}}}{T + 110.4}, \quad \kappa = 2.495 \cdot 10^{-3}\frac{T^{\frac{3}{2}}}{T + 194}$$

$$R = c_p - c_v, \quad \gamma = \frac{c_p}{c_v}$$

The constants are:

- $\gamma = 1.4$ for air under standard atmospheric conditions
- $R = 287.0$ for air

1.2 Physical Domain

The physical domain is a tube extended between $x = 0.2$ and $x = 1.0$. At both ends there are impermeable walls.

1.3 Initial Conditions

The initial conditions are shown in Fig.1:

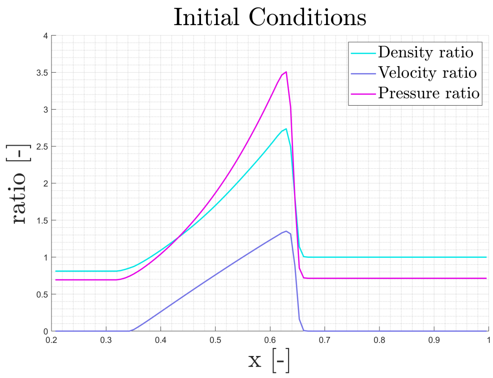

Figure 1: Initial conditions

1.4 Boundary Conditions

On each side of the tube there is an adiabatic, solid wall boundary conditions.

$$u_{(x=0.2)} = u_{(x=1.0)} = 0 \quad \Bigg\| \quad \left.\frac{\partial p}{\partial x}\right|_{x=0.2} = \left.\frac{\partial p}{\partial x}\right|_{x=1.0} = 0 \quad \Bigg\| \quad \left.\frac{\partial T}{\partial x}\right|_{x=0.2} = \left.\frac{\partial T}{\partial x}\right|_{x=1.0} = 0$$

2 Normalizing The Navier-Stokes Equations

Since the initial conditions are normalized, there is a need to normalize the N-S equations. We will use the following normalizations:

$$\rho = \rho_\infty \tilde{\rho}, \quad u = a_\infty \tilde{u}, \quad p = \gamma p_\infty \tilde{p}, \quad T = \gamma T_\infty \tilde{T}, \quad x = L\tilde{x}, \quad t = \frac{L}{a_\infty}\tilde{t}, \quad \mu = \mu_\infty \tilde{\mu}, \quad \kappa = \kappa_\infty \tilde{\kappa} \tag{3}$$

The normalization of the temperature was chosen to cancel out the γ in the normalization of the pressure:

$$\begin{array}{lcl} p & = & \rho R T \\ \gamma p_\infty \tilde{p} & = & \rho_\infty \tilde{\rho} R \gamma T_\infty \tilde{T} \\ \tilde{p} & = & \tilde{\rho}\tilde{T} \end{array} \tag{4}$$

The pressure normalization can be written also as:

$$p = \gamma p_\infty \tilde{p} = \gamma \rho_\infty R T_\infty \tilde{p} = \rho_\infty a_\infty^2 \tilde{p} \tag{5}$$

From equations 2 and 5 we can derive the normalization for the energy:

$$\begin{aligned} e &= \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \\ e &= \frac{\rho_\infty a_\infty^2 \tilde{p}}{\gamma - 1} + \frac{1}{2}\rho_\infty \tilde{\rho} a_\infty^2 \tilde{a}^2 \\ e &= \rho_\infty a_\infty^2 \left(\frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\tilde{a}^2 \right) \\ e &= \rho_\infty a_\infty^2 \tilde{e} \end{aligned} \tag{6}$$

After substituting the normalizations in the N-S equations we get:

$$\frac{\partial}{\partial \frac{L}{a_\infty}\tilde{t}} \begin{pmatrix} \rho_\infty \tilde{\rho} \\ \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{p} + \rho_\infty a_\infty^2 \tilde{\rho}\tilde{u}^2 \\ \rho_\infty a_\infty^3 (\tilde{e} + \tilde{p})\, \tilde{u} \end{pmatrix} = \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}\mu_\infty a_\infty \tilde{\mu} \frac{\partial \tilde{u}}{\partial L\tilde{x}} \\ \frac{4}{3}\mu_\infty a_\infty^2 \tilde{\mu}\tilde{u} \frac{\partial \tilde{u}}{\partial L\tilde{x}} + \frac{\kappa_\infty a_\infty^2}{R}\tilde{\kappa} \frac{\partial \tilde{T}}{\partial L\tilde{x}} \end{pmatrix} \tag{7}$$

Rearranging:

$$\frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ a_\infty \tilde{\rho}\tilde{u} \\ a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho}\tilde{u} \\ a_\infty \tilde{p} + a_\infty \tilde{\rho}\tilde{u}^2 \\ a_\infty^2 (\tilde{e} + \tilde{p})\, \tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L^2}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3} a_\infty \tilde{\mu} \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3} a_\infty^2 \tilde{\mu}\tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\kappa_\infty a_\infty^2}{\mu_\infty R}\tilde{\kappa} \frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \tag{8}$$

Dividing the second equation by a_∞, the third equation by a_∞^2, and the whole set of equations by $\frac{\rho_\infty a_\infty}{L}$ we get:

$$\frac{\partial}{\partial \tilde{t}}\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix}+\frac{\partial}{\partial \tilde{x}}\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix}=\frac{\mu_\infty}{L\rho_\infty a_\infty}\frac{\partial}{\partial \tilde{x}}\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\kappa_\infty}{\mu_\infty R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}}\end{pmatrix} \tag{9}$$

The Reynolds number and the mach number far away are defined as:

$$\begin{gathered}M_\infty=\frac{u_\infty}{a_\infty}\quad Re_{L\infty}=\frac{\rho_\infty u_\infty L}{\mu_\infty}\\ \Downarrow\\ \frac{\mu_\infty}{L\rho_\infty a_\infty}=\frac{M_\infty}{Re_{L\infty}}\end{gathered} \tag{10}$$

The Prandtl number far away is defined as:

$$\begin{gathered}Pr_\infty=\frac{c_p\mu_\infty}{\kappa_\infty}\\ \Downarrow\\ \frac{\kappa_\infty}{\mu_\infty R}=\frac{c_p}{Pr_\infty\,(c_p-c_v)}=\frac{\gamma}{Pr_\infty\,(\gamma-1)}\end{gathered} \tag{11}$$

Substituting into the normalized N-S equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}}+\frac{\partial \tilde{E}}{\partial \tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{E}_\nu}{\partial \tilde{x}} \tag{12}$$

Where:

$$\tilde{Q}=\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix},\quad \tilde{E}=\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix},\quad \tilde{E}_\nu=\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}}+\frac{\gamma}{Pr_\infty\,(\gamma-1)}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}}\end{pmatrix} \tag{13}$$

The normalized Navier-Stokes equations are:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}}+\frac{\partial \tilde{E}}{\partial \tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{14}$$

Where:

$$\tilde{V}_1=\tilde{V}_{1(\tilde{Q},\tilde{Q}_x)}=\tilde{E}_\nu$$

3 The Computational Domain

3.1 Discretization

The physical domain $[x_I, x_F]$ is discretized into N equispaced cells. The size of each cell is there for:

$$\Delta x=\frac{x_F-x_I}{N}=\frac{L}{N} \tag{15}$$

so the x coordinate of the i-th cell x_i is:

$$x_i=x_I+\frac{1}{2}\Delta x+\Delta x\cdot(i-1)\qquad \text{when starting from } i=1 \tag{16}$$

3.2 Boundary Conditions

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated like so:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} \\ u_{(i=N+1)} &= -u_{(i=N)} \end{aligned} \tag{17}$$

in order to maintain velocity zero on the boundary and like so:

$$\begin{aligned} T_{(i=0)} &= T_{(i=1)} \\ T_{(i=N+1)} &= T_{(i=N)} \end{aligned} \tag{18}$$

In order to maintain adiabatic boundary conditions. Since the gradient of the pressure on the wall is zero, we get:

$$\begin{aligned} p_{(i=0)} &= p_{(i=1)} \\ p_{(i=N+1)} &= p_{(i=N)} \end{aligned} \tag{19}$$

From equations 2, 18, and 19 we can conclude:

$$\begin{aligned} \rho_{(i=0)} &= \rho_{(i=1)} \\ \rho_{(i=N+1)} &= \rho_{(i=N)} \end{aligned} \tag{20}$$

and from equations 2, 17, 19, and 20 we can conclude:

$$\begin{aligned} e_{(i=0)} &= e_{(i=1)} \\ e_{(i=N+1)} &= e_{(i=N)} \end{aligned} \tag{21}$$

4 The Numerical Schemes

4.1 Linearizing The Navier-Stokes Equations In Time

$$\begin{aligned} \frac{\Delta Q}{\Delta t} &= -\left(\frac{\partial \tilde{E}}{\partial \tilde{x}} - \frac{\partial \tilde{V}_1}{\partial \tilde{x}}\right)^{n+1} \\ \frac{\Delta Q}{\Delta t} &= -\left(\underbrace{\frac{\partial \tilde{E}}{\partial \tilde{Q}}}_{\tilde{A}}\frac{\partial \tilde{Q}}{\partial \tilde{x}} - \frac{M_\infty}{Re_{L\infty}}\left(\underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}}_{\tilde{P}}\frac{\partial \tilde{Q}}{\partial \tilde{x}} + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}}_{\tilde{R}}\frac{\partial \tilde{Q}_x}{\partial \tilde{x}}\right)\right)^{n+1} \end{aligned} \tag{22}$$

Where:

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\tilde{u}^2 & (3-\gamma)\,\tilde{u} & \gamma-1 \\ -\frac{\gamma\tilde{e}\tilde{u}}{\tilde{\rho}} - (\gamma-1)\,\tilde{u}^3 & \frac{\gamma\tilde{e}}{\tilde{\rho}} - \frac{3\,(\gamma-1)\,\tilde{u}^2}{2} & \gamma\tilde{u} \end{pmatrix}$$

$$\tilde{P} - \tilde{R}_x = -\frac{1}{\rho}\begin{pmatrix} 0 & 0 & 0 \\ -\tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & \left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \\ -\tilde{u}^2\left(\frac{4}{3}\tilde{\mu}\right)_x & \tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \end{pmatrix} \tag{23}$$

$$\tilde{R} = -\frac{1}{\rho}\begin{pmatrix} 0 & 0 & 0 \\ \frac{4}{3}\tilde{u}\tilde{\mu} & -\frac{4}{3}\tilde{\mu} & 0 \\ \left(\frac{4}{3}\tilde{\mu} - \alpha\frac{\tilde{\kappa}}{c_v}\right)\tilde{u}^2 + \alpha\frac{\tilde{\kappa}}{c_v}\frac{\tilde{e}}{\tilde{\rho}} & -\left(\frac{4}{3}\tilde{\mu} - \alpha\frac{\tilde{\kappa}}{c_v}\right)\tilde{u} & -\alpha\frac{\tilde{\kappa}}{c_v} \end{pmatrix}$$

and α is:

$$\alpha = \frac{\gamma}{Pr_\infty\,(\gamma-1)}$$

4.2 First Order Approximate Riemann Roe Method

4.3 First Order Steger-Warming – Explicit

A is a Diagonalizable matrix and can be written as:

$$\tilde{A} = \tilde{T}\tilde{\Lambda}\tilde{T}^{-1}$$

$$\tilde{T} = \begin{pmatrix} 1 & \frac{\tilde{\rho}}{2\tilde{a}} & -\frac{\tilde{\rho}}{2\tilde{a}} \\ \tilde{u} & \frac{\tilde{\rho}}{2\tilde{a}}\,(\tilde{u}+\tilde{a}) & -\frac{\tilde{\rho}}{2\tilde{a}}\,(\tilde{u}-\tilde{a}) \\ \frac{\tilde{u}^2}{2} & \frac{\tilde{\rho}}{2\tilde{a}}\left(\frac{\tilde{u}^2}{2} + \tilde{u}\tilde{a} + \frac{\tilde{a}^2}{\gamma-1}\right) & -\frac{\tilde{\rho}}{2\tilde{a}}\left(\frac{\tilde{u}^2}{2} - \tilde{u}\tilde{a} + \frac{\tilde{a}^2}{\gamma-1}\right) \end{pmatrix}$$

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{u} & 0 & 0 \\ 0 & \tilde{u}+\tilde{a} & 0 \\ 0 & 0 & \tilde{u}-\tilde{a} \end{pmatrix} \tag{24}$$

$$\tilde{T}^{-1} = \begin{pmatrix} 1 - \frac{\gamma-1}{2}\frac{\tilde{u}^2}{\tilde{a}^2} & (\gamma-1)\,\frac{\tilde{u}^2}{\tilde{a}^2} & -\frac{\gamma-1}{\tilde{a}^2} \\ \frac{1}{\tilde{\rho}\tilde{a}}\left((\gamma-1)\,\tilde{u}^2 - \tilde{u}\tilde{a}\right) & \frac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a} - (\gamma-1)\,\tilde{u}\right) & \frac{\gamma-1}{\tilde{\rho}\tilde{a}} \\ -\frac{1}{\tilde{\rho}\tilde{a}}\left((\gamma-1)\,\tilde{u}^2 + \tilde{u}\tilde{a}\right) & \frac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a} + (\gamma-1)\,\tilde{u}\right) & -\frac{\gamma-1}{\tilde{\rho}\tilde{a}} \end{pmatrix}$$

Where:

$$\tilde{a} = \sqrt{\frac{\gamma \tilde{p}}{\tilde{\rho}}}$$

Let the $\Lambda^{\pm}$ matrix be defined as:

$$\tilde{\Lambda}^{\pm} = \begin{pmatrix} \frac{\tilde{u} \pm |\tilde{u}|}{2} & 0 & 0 \\ 0 & \frac{\tilde{u} + \tilde{a} \pm |\tilde{u} + \tilde{a}|}{2} & 0 \\ 0 & 0 & \frac{\tilde{u} - \tilde{a} \pm |\tilde{u} - \tilde{a}|}{2} \end{pmatrix} \tag{25}$$

Where the matrix $\tilde{\Lambda}^{+}$ contains only positive eigenvalues and the matrix $\tilde{\Lambda}^{-}$ contains only negative eigenvalues.
Define:

$$\begin{array}{rcl} \tilde{A}^{+} & \triangleq & \tilde{T}\tilde{\Lambda}^{+}\tilde{T}^{-1} \\ \tilde{A}^{-} & \triangleq & \tilde{T}\tilde{\Lambda}^{-}\tilde{T}^{-1} \end{array} \Rightarrow \begin{array}{rcl} \tilde{A} & = & \tilde{A}^{+} + \tilde{A}^{-} \\ \left|\tilde{A}\right| & \triangleq & \tilde{A}^{+} - \tilde{A}^{-} \end{array} \tag{26}$$

Assuming a perfect gas, the flux vector $\tilde{E}_{(Q)}$ is a homogeneous function of degree one in $\tilde{Q}$, meaning:

$$\forall \alpha \quad \tilde{E}_{(\alpha\tilde{Q})} = \alpha \tilde{E}_{(\tilde{Q})}$$

The homogeneity allows to rewrite the flux vector $\tilde{E}$ as:

$$\tilde{E} = \tilde{A}\tilde{Q} = \left(\tilde{A}^{+} + \tilde{A}^{-}\right)\tilde{Q} = \underbrace{\tilde{A}^{+}\tilde{Q}}_{\tilde{E}^{+}} + \underbrace{\tilde{A}^{-}\tilde{Q}}_{\tilde{E}^{-}} = \tilde{E}^{+} + \tilde{E}^{-} \tag{27}$$

There is a discontinuities and deference between $\tilde{E}^{+}, \tilde{E}^{-}$. To eliminate the discontinuities and guarantee a smooth transition through critical points (sonic points or stagnation points), a blending function is introduced together with a blending parameter ε. An appropriate choice of the blending parameter has to be chosen.

$$\begin{array}{rcl} \tilde{\lambda}^{+} & = & \frac{\tilde{\lambda} + \left|\tilde{\lambda}\right|}{2} \\ \tilde{\lambda}^{-} & = & \frac{\tilde{\lambda} - \left|\tilde{\lambda}\right|}{2} \end{array} \Rightarrow \begin{array}{rcl} \tilde{\lambda}^{+'} & = & \frac{\tilde{\lambda} + \sqrt{\tilde{\lambda}^{2} + \varepsilon^{2}}}{2} \\ \tilde{\lambda}^{-'} & = & \frac{\tilde{\lambda} - \sqrt{\tilde{\lambda}^{2} + \varepsilon^{2}}}{2} \end{array} \tag{28}$$

Rewriting the conservation law from of the N-S equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} = -\frac{\partial \tilde{E}^{+}}{\partial \tilde{x}} - \frac{\partial \tilde{E}^{-}}{\partial \tilde{x}} + \frac{M_{\infty}}{Re_{L\infty}} \frac{\partial \tilde{V}_{1}}{\partial \tilde{x}} \tag{29}$$

A simple, explicit, first order (in space and time) scheme is obtained using:

$$\Delta \tilde{Q}_{i}^{n} = -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\nabla \tilde{E}_{i}^{+n} + \Delta \tilde{E}_{i}^{-n} + \frac{M_{\infty}}{Re_{L\infty}} \text{WTF TO DO?} \right) \tag{30}$$

And advancing the solution by:

$$\tilde{Q}_{i}^{n+1} = \Delta \tilde{Q}_{i}^{n} + \tilde{Q}_{i}^{n} \tag{31}$$

4.4 First Order Steger-Warming – Implicit