

Computational Fluid Dynamics

HW1

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January 16, 2025

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1 Inviscid Burgers Equation

The Inviscid Burgers equation, in conservation law form, is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad F = F(u) = \frac{u^2}{2} \quad (1)$$

In non-conservation law form, is given by:

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0 \quad A = \frac{\partial F}{\partial u} = u \quad (2)$$

The equation is obtained by neglecting the viscous term from the viscous Burger equation.

1.1 Boundary and Initial Conditions

$$\begin{aligned} u_{(x=0,t)} &= 1.0 \\ u_{(x=1,t)} &= u_1 \\ u_{(x,t=0)} &= 1 - (1 - u_1) \cdot x \end{aligned} \quad (3)$$

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated like so:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} + 2 \cdot u_0 \\ u_{(i=N+1)} &= -u_{(i=N)} + 2 \cdot u_1 \end{aligned} \quad (4)$$

1.2 Finite Volume Formulation

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left(f_{i+\frac{1}{2}}^n - f_{i-\frac{1}{2}}^n \right) \quad (5)$$

- For first-order schemes, there is no variation within a cell, and the value there is constant.
- For second-order schemes, the variation within the cell is linear.

1.3 CFL number

For the Roe method, the CFL number is defined as:

$$\text{CFL} = \frac{u \Delta t}{\Delta x} \quad (6)$$

We will want to set the maximal value of the *CFL* number. We will find the Δt at each cell and (Δt_i) and set the Δt of the current step as:

$$\Delta t = \min(\Delta t_i) \quad \forall i \quad (7)$$

1.4 First Order Roe Method ($u_1 = 0.0$)

Roe scheme is based on the solution of the linear problem:

$$\frac{\partial u}{\partial t} + \bar{A} \frac{\partial u}{\partial x} = 0 \quad (8)$$

Where \bar{A} is a constant matrix that is dependent on local conditions. The matrix is constructed in a way that guarantees uniform validity across discontinuities:

1. For any u_i, u_{i+1} :

$$F_{i+1} - F_i = \bar{A} \cdot (u_{i+1} - u_i)$$

2. When $u = u_i = u_{i+1}$ then:

$$\bar{A}_{(u_i, u_{i+1})} = \bar{A}_{(u, u)} = \frac{\partial F}{\partial u} = u$$

In the case of the Burgers equation, the matrix \bar{A} is a scalar, namely, $\bar{A} = \bar{u}$. The equation becomes:

$$\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} = 0 \quad (9)$$

The value of \bar{u} for the cell face between i and $i+1$ is determined from the first conditions:

$$\bar{u} = \bar{u}_{i+\frac{1}{2}} = \frac{F_{i+1} - F_i}{u_{i+1} - u_i} = \frac{\frac{1}{2}u_{i+1}^2 - \frac{1}{2}u_i^2}{u_{i+1} - u_i} = \begin{cases} \frac{u_i + u_{i+1}}{2} & u_i \neq u_{i+1} \\ u_i & u_i = u_{i+1} \end{cases} \quad (10)$$

The single wave that emanates from the cell interface travels either in the positive or negative direction, depending upon the sign of $\bar{u}_{i+\frac{1}{2}}$. Define:

$$\begin{cases} \bar{u}_{i+\frac{1}{2}}^+ \triangleq \frac{1}{2} \left(\bar{u}_{i+\frac{1}{2}} + \left| \bar{u}_{i+\frac{1}{2}} \right| \right) \geq 0 \\ \bar{u}_{i+\frac{1}{2}}^- \triangleq \frac{1}{2} \left(\bar{u}_{i+\frac{1}{2}} - \left| \bar{u}_{i+\frac{1}{2}} \right| \right) \leq 0 \end{cases} \quad \bar{u}_{i+\frac{1}{2}} = \bar{u}_{i+\frac{1}{2}}^+ + \bar{u}_{i+\frac{1}{2}}^- \quad (11)$$

Using the jump relation, the numerical flux at the cell interface can be evaluated by one of the following:

$$\begin{cases} f_{i+\frac{1}{2}} - F_i = \bar{u}_{i+\frac{1}{2}}^- \cdot (u_{i+1} - u_i) \\ F_{i+1} - f_{i+\frac{1}{2}} = \bar{u}_{i+\frac{1}{2}}^+ \cdot (u_{i+1} - u_i) \end{cases} \quad (12)$$

The numerical flux may then be written in the following symmetric form:

$$\begin{aligned} f_{i+\frac{1}{2}} &= \frac{F_i + F_{i+1}}{2} - \frac{1}{2} \left(\bar{u}_{i+\frac{1}{2}}^+ - \bar{u}_{i+\frac{1}{2}}^- \right) (u_{i+1} - u_i) \\ \text{OR :} \\ f_{i+\frac{1}{2}} &= \frac{F_i + F_{i+1}}{2} - \frac{1}{2} \left| \bar{u}_{i+\frac{1}{2}} \right| (u_{i+1} - u_i) \end{aligned} \quad (13)$$

1.4.1 Effect of CFL

1.5 Second Order Roe ($u_1 = 0.5$)

The first-order accurate Roe method interface flux function will be denoted like this:

$$f_{i+\frac{1}{2}}^{\text{Roe},1} = f_{(u_i, u_{i+1})}$$

The second order accurate Roe takes the form:

$$f_{i+\frac{1}{2}}^{\text{Roe},2} = f_{(u_{i+1}^l, u_{i+1}^r)}$$

Hence:

$$f_{i+\frac{1}{2}}^{\text{Roe},2} = \frac{1}{2} \left(F(u_{1+\frac{1}{2}}^l) + F(u_{1+\frac{1}{2}}^r) - |\bar{u}_{i+\frac{1}{2}}| (u_{1+\frac{1}{2}}^r - u_{1+\frac{1}{2}}^l) \right) \quad (14)$$

$$\bar{u}_{1+\frac{1}{2}} = \frac{F(u_{1+\frac{1}{2}}^r) - F(u_{1+\frac{1}{2}}^l)}{u_{1+\frac{1}{2}}^r - u_{1+\frac{1}{2}}^l} = \frac{u_{i+\frac{1}{2}}^l + u_{i+\frac{1}{2}}^r}{2}$$

1.5.1 Without Limiters

The interface values without limiters are evaluated as:

$$\begin{cases} u_{i+\frac{1}{2}}^l &= u_i + \frac{1-k}{4} \delta u_{i-\frac{1}{2}} + \frac{1+k}{4} \delta u_{i+\frac{1}{2}} \\ u_{i+\frac{1}{2}}^r &= u_{i+1} - \frac{1+k}{4} \delta u_{i+\frac{1}{2}} - \frac{1-k}{4} \delta u_{i+\frac{3}{2}} \end{cases} \quad \delta u_i \triangleq u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}} \quad (15)$$

The parameter k determines the scheme:

$$k = \begin{cases} -1 & \text{upwind} \\ 1 & \text{central} \end{cases}$$

1.5.2 With Limiters

The interface values with limiters are evaluated as:

$$\begin{cases} u_{i+\frac{1}{2}}^l &= u_i + \frac{1-k}{4} \bar{\delta}^+ u_{i-\frac{1}{2}} + \frac{1+k}{4} \bar{\delta}^- u_{i+\frac{1}{2}} \\ u_{i+\frac{1}{2}}^r &= u_{i+1} - \frac{1+k}{4} \bar{\delta}^+ u_{i+\frac{1}{2}} - \frac{1-k}{4} \bar{\delta}^- u_{i+\frac{3}{2}} \end{cases} \quad \bar{\delta}^\pm u \text{ are limited slopes} \quad (16)$$

$\bar{\delta}$ is an operator such that $\bar{\delta} u_i = \psi \delta u_i$, where $\psi(r)$ is a limiter function and:

$$r^\pm = \begin{cases} r_{1+\frac{1}{2}}^+ & \triangleq \frac{u_{i+2} - u_{i+1}}{u_{i+1} - u_i} = \frac{\Delta u_{i+1}}{\Delta u_i} \\ r_{1+\frac{1}{2}}^- & \triangleq \frac{u_i - u_{i-1}}{u_{i+1} - u_i} = \frac{\nabla u_i}{\nabla u_{i+1}} \end{cases} \quad (17)$$

There are many types of limiters. For example, the van Albada limiter:

$$\psi(r) = \frac{r + r^2}{1 + r^2} \quad (18)$$

1.5.3 Effect of CFL

1.5.4 Effect of Limiter

2 Generalized Burgers Equation

The generalized Burgers equation is given by:

$$\frac{\partial u}{\partial t} + (c + bu) \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \quad (19)$$

Where:

$$c = \frac{1}{2} \quad b = -1 \quad \mu = [0.001, 0.25]$$

The equation can also be presented as:

$$\frac{\partial u}{\partial t} + \frac{\partial \bar{F}}{\partial x} = 0 \quad \bar{F} = \underbrace{cu + \frac{bu^2}{2}}_F - \underbrace{\mu \frac{\partial u}{\partial x}}_{F_\nu} \quad (20)$$

In non-conservation law form, is given by:

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = 0 \quad A = \frac{\partial \bar{F}}{\partial u} = c + bu - \mu \frac{\partial}{\partial u} \left(\frac{\partial u}{\partial x} \right) \quad (21)$$

The generalized Burgers equation has a stationary solution:

$$u = -\frac{c}{b} \left(1 + \tanh \left(\frac{c(x - x_0)}{2\mu} \right) \right) \quad (22)$$

2.1 Domain and Computational Mesh

Using 41 grid points with $\Delta x = 1$ and computing until $t = 18.0$. $\Delta t = [0.5, 1.0]$.

2.2 Boundary and Initial Conditions

2.2.1 Initial Conditions

$$u_{(x,t=0)} = \frac{1}{2} (1 + \tanh(250(x - 20))) \quad (23)$$

2.2.2 Boundary Conditions

Using Dirichlet boundary conditions:

$$u_{(x=0,t)} = 0 \quad u_{(x=40,t)} = 1 \quad (24)$$

2.3 First Order Roe Method (explicit)

As written above for the inviscid Burgers equation (1.4), Roes scheme is based on the solution of the linear problem:

$$\frac{\partial u}{\partial t} + \bar{A} \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \quad \bar{A} = \frac{\partial F}{\partial u} \quad (25)$$

In the case of the Burgers equation, the matrix \bar{A} is a scalar.

$$\bar{A} = \bar{A}_{i+\frac{1}{2}} = \frac{F_{i+1} - F_i}{u_{i+1} - u_i} = \begin{cases} \frac{c(u_{i+1} - u_i) + \frac{b}{2}(u_{i+1}^2 - u_i^2)}{u_{i+1} - u_i} & u_i \neq u_{i+1} \\ A_i & u_i = u_{i+1} \end{cases} \quad (26)$$

The numerical flux at the cell interface:

$$\bar{f}_{i+\frac{1}{2}} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2} \left(\bar{A}_{i+\frac{1}{2}}^+ - \bar{A}_{i+\frac{1}{2}}^- \right) (u_{i+1} - u_i) \quad (27)$$

Where:

$$\begin{cases} \bar{A}_{i+\frac{1}{2}}^+ \triangleq \frac{1}{2} \left(\bar{A}_i + \frac{1}{2} + \left| \bar{A}_{i+\frac{1}{2}} \right| \right) \geq 0 \\ \bar{A}_{i+\frac{1}{2}}^- \triangleq \frac{1}{2} \left(\bar{A}_{i+\frac{1}{2}} - \left| \bar{A}_{i+\frac{1}{2}} \right| \right) \leq 0 \end{cases} \quad \bar{A}_{i+\frac{1}{2}} = \bar{A}_{i+\frac{1}{2}}^+ + \bar{A}_{i+\frac{1}{2}}^- \quad (28)$$

And finally:

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} \left(\bar{f}_{i+\frac{1}{2}}^n - \bar{f}_{i-\frac{1}{2}}^n \right) + \mu \frac{\Delta t}{(\Delta x)^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad (29)$$

2.4 MacCormack Method

The original MacCormack method applied to Burgers equation results in:

$$\begin{aligned} \text{Predictor :} \quad u_i^{\overline{n+1}} &= u_i^n - \Delta t \frac{\Delta F_i^n}{\Delta x} + r \delta^2 u_i^n \\ \text{Corrector :} \quad u_i^{n+1} &= \frac{1}{2} \left(u_i^n + u_i^{\overline{n+1}} - \Delta t \frac{\nabla F_i^{\overline{n+1}}}{\Delta x} \right) + r \delta^2 u_i^{\overline{n+1}} \end{aligned} \quad (30)$$

Where:

- $r = \frac{\mu \Delta t}{(\Delta x)^2}$
- $\delta^2 u_i = u_{i+1} - 2u_i + u_{i-1}$
- $\Delta f = f_{i+1} - f_i$
- $\nabla f = f_i - f_{i-1}$

2.4.1 Stability

The stability condition for MacCormack's method is:

$$\Delta t \leq \frac{(\Delta x)^2}{|u| \Delta x + \mu} \quad (31)$$

2.5 Beam and Warming

Beam and Warming introduced the Delta form and their method is more efficient:

$$\begin{aligned} \left(I + \theta \left(\Delta t \frac{D_0 A_i^n}{2\Delta x} - \frac{\Delta t \mu}{(\Delta x)^2} \delta^2 \right) \right) \Delta u_i^n &= \underbrace{-\Delta t \frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x} + \Delta t \mu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}}_{\text{RHS}_i^n} \\ \begin{cases} \theta = 1 & \text{first order} \\ \theta = 0.5 & \text{second order} \end{cases} \end{aligned} \quad (32)$$

Deriving the matrix to invert:

$$\begin{aligned}
& \left(I + \theta \Delta t \frac{D_0 A_i^n}{2 \Delta x} \right) \Delta u_i^n - \theta \frac{\Delta t \mu}{(\Delta x)^2} \delta^2 \Delta u_i^n = \text{RHS}_i^n \\
& \left(I + \theta \Delta t \frac{D_0 A_i^n}{2 \Delta x} \right) \Delta u_i^n - \theta \frac{\Delta t \mu}{(\Delta x)^2} (\Delta u_{i+1}^n - 2 \Delta u_i^n + \Delta u_{i-1}^n) = \text{RHS}_i^n \\
& \Delta u_i^n + \theta \frac{\Delta t}{2 \Delta x} (A_{i+1}^n \Delta u_{i+1}^n - A_{i-1}^n \Delta u_{i-1}^n) - \theta \frac{\Delta t \mu}{(\Delta x)^2} (\Delta u_{i+1}^n - 2 \Delta u_i^n + \Delta u_{i-1}^n) = \text{RHS}_i^n \\
& A_i \Delta u_{i-1}^n + B_i \Delta u_i^n + C_i \Delta u_{i+1}^n = D_i
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
A'_i &= -\theta \frac{\mu \Delta t}{(\Delta x)^2} - \theta \frac{\Delta t}{2 \Delta x} A_{i-1}^n \\
B'_i &= 1 + 2\theta \frac{\mu \Delta t}{(\Delta x)^2} \\
C'_i &= -\theta \frac{\mu \Delta t}{(\Delta x)^2} + \theta \frac{\Delta t}{2 \Delta x} A_{i+1}^n \\
D'_i &= \text{RHS}_i^n
\end{aligned} \tag{33}$$

and advancing the solution with:

$$u_i^{n+1} = u_i^n + \Delta u_i^n \tag{34}$$

In order to calculate Δu_i^n it is needed to invert matrix as follows:

$$\begin{pmatrix} B'_1 & C'_1 & 0 & \cdots & \cdots & \cdots & 0 \\ A'_2 & B'_2 & C'_2 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & 0 & A'_i & B'_i & C'_i & 0 & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & A'_{N-1} & B'_{N-1} & C'_{N-1} \\ 0 & 0 & \cdots & \cdots & 0 & A'_{N+1} & B'_{N+1} \end{pmatrix} \begin{pmatrix} \Delta u_1^n \\ \Delta u_2^n \\ \cdots \\ \cdots \\ \cdots \\ \Delta u_N^n \\ \Delta u_{N+1}^n \end{pmatrix} = \begin{pmatrix} D'_1 \\ D'_2 \\ \cdots \\ \cdots \\ \cdots \\ D'_N \\ D'_{N+1} \end{pmatrix} \tag{35}$$

- $(N + 1)$ because of the ghost cells

The Beam and Warming method is extremely dispersive and therefore artificial viscosity must be explicitly added. Beam and Warming used the following artificial viscosity term:

$$-\frac{w}{8} (u_{i+2}^n - 4u_{i+1}^n + 6u_i^n - 4u_{i-1}^n + u_{i-2}^n) \quad 0 < w \leq 1 \tag{36}$$

which can be added to the RHS with no change in accuracy