

# Computational Fluid Dynamics

## HW2

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## Nomenclature

$\Delta x$	size of each cell in the domain
$\Delta \tilde{t}$	normalized step size in time
$\Delta \tilde{x}$	normalized step size in space
$\gamma$	ratio of specific heats
$\kappa$	coefficient of thermal conductivity
$\mu$	coefficient of viscosity
$\rho$	fluid density
$c_p$	constant specific heat capacity for a constant pressure
$c_v$	constant specific heat capacity for a constant volume
$E$	inviscid convective vector
$e$	total energy
$E_\nu$	viscous convective vector
$L$	characteristic length
$p$	pressure
$Q$	conservation state space
$R$	gas constant
$T$	temperature
$t$	time
$u$	fluid velocity
$x$	spatial coordinate
$x_F$	x coordinate of the end of the domain
$x_i$	x coordinate of the i-th cell

### Far-Away Properties

$\kappa_\infty$	coefficient of thermal conductivity far away
$\mu_\infty$	coefficient of viscosity far away
$\rho_\infty$	density far away
$a_\infty$	speed of sound far away
$M_\infty$	mach number far away
$p_\infty$	pressure far away



$T_\infty$       temperature far away

### Matrices

$\tilde{\Lambda}$       normalized eigenvalues matrix

$\tilde{A}$       normalized jacobian matrix of E w.r.t. Q

$\tilde{P}$       normalized jacobian matrix of  $E_\nu$  w.r.t. Q

$\tilde{R}$       normalized jacobian matrix of  $E_\nu$  w.r.t.  $Q_x$

$\tilde{T}$       normalized eigenvectors matrix

### Dimensionless Numbers

$Pr_\infty$       Prandtl number far away

$Re_{L\infty}$       Reynolds number with respect to L far away



# 1 Problem Definition

## 1.1 Governing Equations

Consider the one-dimensional Navier-Stokes Equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = \frac{\partial E_\nu}{\partial x} \quad (1)$$

Where:

$$Q = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad E = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ (e + p)u \end{pmatrix}, \quad E_\nu = \begin{pmatrix} 0 \\ \tau_{xx} \\ u\tau_{xx} - q_x \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu \frac{\partial u}{\partial x} \\ \frac{4}{3}\mu u \frac{\partial u}{\partial x} + \kappa \frac{\partial T}{\partial x} \end{pmatrix} \quad (2)$$

$$p = (\gamma - 1) \left( e - \frac{1}{2} \rho u^2 \right), \quad T = \frac{p}{\rho R},$$

$$\mu = 1.458 \cdot 10^{-6} \frac{T^{\frac{3}{2}}}{T + 110.4}, \quad \kappa = 2.495 \cdot 10^{-3} \frac{T^{\frac{3}{2}}}{T + 194}$$

$$R = c_p - c_v, \quad \gamma = \frac{c_p}{c_v}$$

The constants are:

- $\gamma = 1.4$  for air under standard atmospheric conditions
- $R = 287.0$  for air

## 1.2 Physical Domain

The physical domain is a tube extended between  $x = 0.2$  and  $x = 1.0$ . At both ends there are impermeable walls.

## 1.3 Initial Conditions

The initial conditions are shown in Fig.1:

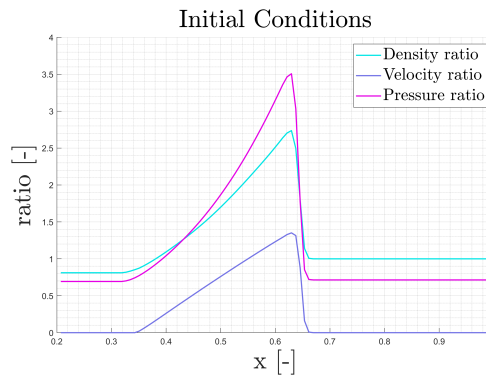


Figure 1: Initial conditions



## 1.4 Boundary Conditions

On each side of the tube there is an adiabatic, solid wall boundary conditions.

$$u_{(x=0.2)} = u_{(x=1.0)} = 0 \quad \left\| \quad \frac{\partial p}{\partial x} \Big|_{x=0.2} = \frac{\partial p}{\partial x} \Big|_{x=1.0} = 0 \quad \right\| \quad \frac{\partial T}{\partial x} \Big|_{x=0.2} = \frac{\partial T}{\partial x} \Big|_{x=1.0} = 0$$

## 2 Normalizing The Navier-Stokes Equations

Since the initial conditions are normalized, there is a need to normalize the N-S equations. We will use the following normalizations:

$$\rho = \rho_\infty \tilde{\rho}, \quad u = a_\infty \tilde{u}, \quad p = \gamma p_\infty \tilde{p}, \quad T = \gamma T_\infty \tilde{T}, \quad x = L \tilde{x}, \quad t = \frac{L}{a_\infty} \tilde{t}, \quad \mu = \mu_\infty \tilde{\mu}, \quad \kappa = \kappa_\infty \tilde{\kappa} \quad (3)$$

The normalization of the temperature was chosen to cancel out the  $\gamma$  in the normalization of the pressure:

$$\begin{aligned} p &= \rho R T \\ \gamma p_\infty \tilde{p} &= \rho_\infty \tilde{\rho} R \gamma T_\infty \tilde{T} \\ \tilde{p} &= \tilde{\rho} \tilde{T} \end{aligned} \quad (4)$$

The pressure normalization can be written also as:

$$p = \gamma p_\infty \tilde{p} = \gamma \rho_\infty R T_\infty \tilde{p} = \rho_\infty a_\infty^2 \tilde{p} \quad (5)$$

From equations 2 and 5 we can derive the normalization for the energy:

$$\begin{aligned} e &= \frac{p}{\gamma - 1} + \frac{1}{2} \rho u^2 \\ e &= \frac{\rho_\infty a_\infty^2 \tilde{p}}{\gamma - 1} + \frac{1}{2} \rho_\infty \tilde{\rho} a_\infty^2 \tilde{u}^2 \\ e &= \rho_\infty a_\infty^2 \left( \frac{\tilde{p}}{\gamma - 1} + \frac{1}{2} \tilde{\rho} \tilde{u}^2 \right) \\ e &= \rho_\infty a_\infty^2 \tilde{e} \end{aligned} \quad (6)$$

After substituting the normalizations in the N-S equations we get:

$$\frac{\partial}{\partial \frac{L}{a_\infty} \tilde{t}} \begin{pmatrix} \rho_\infty \tilde{\rho} \\ \rho_\infty a_\infty \tilde{\rho} \tilde{u} \\ \rho_\infty a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\partial}{\partial L \tilde{x}} \begin{pmatrix} \rho_\infty a_\infty \tilde{\rho} \tilde{u} \\ \rho_\infty a_\infty^2 \tilde{p} + \rho_\infty a_\infty^2 \tilde{\rho} \tilde{u}^2 \\ \rho_\infty a_\infty^3 (\tilde{e} + \tilde{p}) \tilde{u} \end{pmatrix} = \frac{\partial}{\partial L \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3} \mu_\infty a_\infty \tilde{\mu} \frac{\partial \tilde{u}}{\partial L \tilde{x}} \\ \frac{4}{3} \mu_\infty a_\infty^2 \tilde{\mu} \tilde{u} \frac{\partial \tilde{u}}{\partial L \tilde{x}} + \frac{\kappa_\infty a_\infty^2}{R} \tilde{\kappa} \frac{\partial \tilde{T}}{\partial L \tilde{x}} \end{pmatrix} \quad (7)$$

Rearranging:

$$\frac{\rho_\infty a_\infty}{L} \frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ a_\infty \tilde{\rho} \tilde{u} \\ a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\rho_\infty a_\infty}{L} \frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho} \tilde{u} \\ a_\infty \tilde{p} + a_\infty \tilde{\rho} \tilde{u}^2 \\ a_\infty^2 (\tilde{e} + \tilde{p}) \tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L^2} \frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3} a_\infty \tilde{\mu} \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3} a_\infty^2 \tilde{\mu} \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\kappa_\infty a_\infty^2}{\mu_\infty R} \tilde{\kappa} \frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \quad (8)$$

Dividing the second equation by  $a_\infty$ , the third equation by  $a_\infty^2$ , and the whole set of equations by  $\frac{\rho_\infty a_\infty}{L}$  we get:

$$\frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ \tilde{\rho} \tilde{u} \\ \tilde{e} \end{pmatrix} + \frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho} \tilde{u} \\ \tilde{p} + \tilde{\rho} \tilde{u}^2 \\ (\tilde{e} + \tilde{p}) \tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L \rho_\infty a_\infty} \frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3} \tilde{\mu} \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3} \tilde{\mu} \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\kappa_\infty}{\mu_\infty R} \tilde{\kappa} \frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \quad (9)$$

The Reynolds number and the mach number far away are defined as:

$$\begin{aligned} M_\infty &= \frac{u_\infty}{a_\infty} & Re_{L\infty} &= \frac{\rho_\infty u_\infty L}{\mu_\infty} \\ &\Downarrow & & \\ \frac{\mu_\infty}{L \rho_\infty a_\infty} &= \frac{M_\infty}{Re_{L\infty}} \end{aligned} \quad (10)$$

The Prandtl number far away is defined as:

$$\begin{aligned} Pr_\infty &= \frac{c_p \mu_\infty}{\kappa_\infty} \\ &\Downarrow \\ \frac{\kappa_\infty}{\mu_\infty R} &= \frac{c_p}{Pr_\infty (c_p - c_v)} = \frac{\gamma}{Pr_\infty (\gamma - 1)} \end{aligned} \quad (11)$$

Substituting into the normalized N-S equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \frac{\partial \tilde{E}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{E}_\nu}{\partial \tilde{x}} \quad (12)$$

Where:

$$\tilde{Q} = \begin{pmatrix} \tilde{\rho} \\ \tilde{\rho} \tilde{u} \\ \tilde{e} \end{pmatrix}, \quad \tilde{E} = \begin{pmatrix} \tilde{\rho} \tilde{u} \\ \tilde{p} + \tilde{\rho} \tilde{u}^2 \\ (\tilde{e} + \tilde{p}) \tilde{u} \end{pmatrix}, \quad \tilde{E}_\nu = \begin{pmatrix} 0 \\ \frac{4}{3} \tilde{\mu} \frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3} \tilde{\mu} \tilde{u} \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\gamma}{Pr_\infty (\gamma - 1)} \tilde{\kappa} \frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \quad (13)$$

The normalized Navier-Stokes equations are:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} + \frac{\partial \tilde{E}}{\partial \tilde{x}} = \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \quad (14)$$

Where:

$$\tilde{V}_1 = \tilde{V}_1(\tilde{Q}, \tilde{Q}_x) = \tilde{E}_\nu$$

### 3 The Computational Domain

#### 3.1 Discretization

The physical domain  $[x_I, x_F]$  is discretized into N equispaced cells. The size of each cell is there for:

$$\Delta x = \frac{x_F - x_I}{N} = \frac{L}{N} \quad (15)$$

so the x coordinate of the i-th cell  $x_i$  is:

$$x_i = x_I + \frac{1}{2} \Delta x + \Delta x \cdot (i - 1) \quad \text{when starting from } i = 1 \quad (16)$$



### 3.2 Boundary Conditions

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated like so:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} \\ u_{(i=N+1)} &= -u_{(i=N)} \end{aligned} \quad (17)$$

in order to maintain velocity zero on the boundary and like so:

$$\begin{aligned} T_{(i=0)} &= T_{(i=1)} \\ T_{(i=N+1)} &= T_{(i=N)} \end{aligned} \quad (18)$$

In order to maintain adiabatic boundary conditions. Since the gradient of the pressure on the wall is zero, we get:

$$\begin{aligned} p_{(i=0)} &= p_{(i=1)} \\ p_{(i=N+1)} &= p_{(i=N)} \end{aligned} \quad (19)$$

From equations 2, 18, and 19 we can conclude:

$$\begin{aligned} \rho_{(i=0)} &= \rho_{(i=1)} \\ \rho_{(i=N+1)} &= \rho_{(i=N)} \end{aligned} \quad (20)$$

and from equations 2, 17, 19, and 20 we can conclude:

$$\begin{aligned} e_{(i=0)} &= e_{(i=1)} \\ e_{(i=N+1)} &= e_{(i=N)} \end{aligned} \quad (21)$$

## 4 The Numerical Schemes

### 4.1 Jacobian Matrices of The Navier-Stokes Equations

$$\begin{aligned} \frac{\Delta Q}{\Delta t} &= - \left( \frac{\partial \tilde{E}}{\partial \tilde{x}} - \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \right)^{n+1} \\ \frac{\Delta Q}{\Delta t} &= - \left( \underbrace{\frac{\partial \tilde{E}}{\partial \tilde{Q}}}_{\tilde{A}} \frac{\partial \tilde{Q}}{\partial \tilde{x}} - \frac{M_\infty}{Re_{L\infty}} \left( \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}}_{\tilde{P}} \frac{\partial \tilde{Q}}{\partial \tilde{x}} + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}}_{\tilde{R}} \frac{\partial \tilde{Q}_x}{\partial \tilde{x}} \right) \right)^{n+1} \end{aligned} \quad (22)$$





Where:

$$\begin{aligned}
 \tilde{A} &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\tilde{u}^2 & (3-\gamma)\tilde{u} & \gamma-1 \\ -\frac{\gamma\tilde{e}\tilde{u}}{\tilde{\rho}} - (\gamma-1)\tilde{u}^3 & \frac{\gamma\tilde{e}}{\tilde{\rho}} - \frac{3(\gamma-1)\tilde{u}^2}{2} & \gamma\tilde{u} \end{pmatrix} \\
 \tilde{P} - \tilde{R}_x &= -\frac{1}{\rho} \begin{pmatrix} 0 & 0 & 0 \\ -\tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & \left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \\ -\tilde{u}^2\left(\frac{4}{3}\tilde{\mu}\right)_x & \tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \end{pmatrix} \\
 \tilde{R} &= -\frac{1}{\rho} \begin{pmatrix} 0 & 0 & 0 \\ \frac{4}{3}\tilde{u}\tilde{\mu} & -\frac{4}{3}\tilde{\mu} & 0 \\ \left(\frac{4}{3}\tilde{\mu} - \alpha\frac{\tilde{\kappa}}{c_v}\right)\tilde{u}^2 + \alpha\frac{\tilde{\kappa}}{c_v}\frac{\tilde{e}}{\tilde{\rho}} & -\left(\frac{4}{3}\tilde{\mu} - \alpha\frac{\tilde{\kappa}}{c_v}\right)\tilde{u} & -\alpha\frac{\tilde{\kappa}}{c_v} \end{pmatrix}
 \end{aligned} \tag{23}$$

and  $\alpha$  is:

$$\alpha = \frac{\gamma}{Pr_\infty(\gamma-1)}$$

## 4.2 Linearization In Time

### 4.2.1 $\tilde{E}_i^{n+1}$ Estimation

$$\tilde{E}_i^{n+1} = \tilde{E}_i^n + \underbrace{\frac{\partial \tilde{E}}{\partial \tilde{Q}} \Big|_i^n}_{\tilde{A}_i^n} \Delta \tilde{Q}_i^n + \text{H.O.T} \tag{24}$$

$$\tilde{E}_i^{n+1} = \tilde{E}_i^n + \tilde{A}_i^n \Delta \tilde{Q}_i^n$$

### 4.2.2 $\tilde{V}_{1,i}^{n+1}$ Estimation

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^n + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}} \Big|_i^n}_{\tilde{P}_i^n} \Delta \tilde{Q}_i^n + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x} \Big|_i^n}_{\tilde{R}_i^n} \Delta \tilde{Q}_{x_i}^n + \text{H.O.T} \tag{25}$$

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^n + \tilde{P}_i^n \Delta \tilde{Q}_i^n + \tilde{R}_i^n \Delta \tilde{Q}_{x_i}^n$$

The difficulty stems from the fact that the solution vector is  $\Delta \tilde{Q}$  and not  $\Delta \tilde{Q}_x$ . This can be solved by a linearization of the term  $\Delta \tilde{Q}_x$  which can be conducted using the following relation:

$$\frac{\partial \left( \tilde{R} \Delta \tilde{Q} \right)_i^n}{\partial \tilde{x}} = \frac{\partial \tilde{R}_i^n}{\partial \tilde{x}} \Delta \tilde{Q}_i^n + \tilde{R}_i^n \frac{\partial \Delta \tilde{Q}_i^n}{\partial \tilde{x}} = \frac{\partial \tilde{R}}{\partial \tilde{x}} \Delta \tilde{Q} + \tilde{R}_i^n \Delta \tilde{Q}_{x_i}^n$$

↓

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^n + \left( \tilde{P} - \tilde{R}_x \right)_i^n \Delta \tilde{Q}_i^n + \frac{\partial}{\partial \tilde{x}} \left( \tilde{R} \Delta \tilde{Q} \right)_i^n \quad (26)$$

### 4.3 First Order Approximate Riemann Roe Method

### 4.4 First Order Steger-Warming – Explicit

A is a Diagonalizable matrix and can be written as:

$$\tilde{A} = \tilde{T} \tilde{\Lambda} \tilde{T}^{-1}$$

$$\tilde{T} = \begin{pmatrix} 1 & \frac{\tilde{\rho}}{2\tilde{a}} & -\frac{\tilde{\rho}}{2\tilde{a}} \\ \tilde{u} & \frac{\tilde{\rho}}{2\tilde{a}} (\tilde{u} + \tilde{a}) & -\frac{\tilde{\rho}}{2\tilde{a}} (\tilde{u} - \tilde{a}) \\ \frac{\tilde{u}^2}{2} & \frac{\tilde{\rho}}{2\tilde{a}} \left( \frac{\tilde{u}^2}{2} + \tilde{u}\tilde{a} + \frac{\tilde{a}^2}{\gamma-1} \right) & -\frac{\tilde{\rho}}{2\tilde{a}} \left( \frac{\tilde{u}^2}{2} - \tilde{u}\tilde{a} + \frac{\tilde{a}^2}{\gamma-1} \right) \end{pmatrix}$$

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{u} & 0 & 0 \\ 0 & \tilde{u} + \tilde{a} & 0 \\ 0 & 0 & \tilde{u} - \tilde{a} \end{pmatrix} \quad (27)$$

$$\tilde{T}^{-1} = \begin{pmatrix} 1 - \frac{\gamma-1}{2} \frac{\tilde{u}^2}{\tilde{a}^2} & (\gamma-1) \frac{\tilde{u}^2}{\tilde{a}^2} & -\frac{\gamma-1}{\tilde{a}^2} \\ \frac{1}{\tilde{\rho}\tilde{a}} ((\gamma-1)\tilde{u}^2 - \tilde{u}\tilde{a}) & \frac{1}{\tilde{\rho}\tilde{a}} (\tilde{a} - (\gamma-1)\tilde{u}) & \frac{\gamma-1}{\tilde{\rho}\tilde{a}} \\ -\frac{1}{\tilde{\rho}\tilde{a}} ((\gamma-1)\tilde{u}^2 + \tilde{u}\tilde{a}) & \frac{1}{\tilde{\rho}\tilde{a}} (\tilde{a} + (\gamma-1)\tilde{u}) & -\frac{\gamma-1}{\tilde{\rho}\tilde{a}} \end{pmatrix}$$

Where:

$$\tilde{a} = \sqrt{\frac{\gamma \tilde{P}}{\tilde{\rho}}}$$

Let the  $\tilde{\Lambda}^\pm$  matrix be defined as:

$$\tilde{\Lambda}^\pm = \begin{pmatrix} \frac{\tilde{u} \pm |\tilde{u}|}{2} & 0 & 0 \\ 0 & \frac{\tilde{u} + \tilde{a} \pm |\tilde{u} + \tilde{a}|}{2} & 0 \\ 0 & 0 & \frac{\tilde{u} - \tilde{a} \pm |\tilde{u} - \tilde{a}|}{2} \end{pmatrix} \quad (28)$$

Where the matrix  $\tilde{\Lambda}^+$  contains only positive eigenvalues and the matrix  $\tilde{\Lambda}^-$  contains only negative eigenvalues.

Define:

$$\begin{aligned} \tilde{A}^+ &\triangleq \tilde{T} \tilde{\Lambda}^+ \tilde{T}^{-1} \\ \tilde{A}^- &\triangleq \tilde{T} \tilde{\Lambda}^- \tilde{T}^{-1} \end{aligned} \Rightarrow \begin{aligned} \tilde{A} &= \tilde{A}^+ + \tilde{A}^- \\ |\tilde{A}| &\triangleq \tilde{A}^+ - \tilde{A}^- \end{aligned} \quad (29)$$

Assuming a perfect gas, the flux vector  $\tilde{E}_{(Q)}$  is a homogeneous function of degree one in  $\tilde{Q}$ , meaning:

$$\forall \alpha \quad \tilde{E}_{(\alpha \tilde{Q})} = \alpha \tilde{E}_{(\tilde{Q})}$$



The homogeneity allows to rewrite the flux vector  $\tilde{E}$  as:

$$\tilde{E} = \tilde{A}\tilde{Q} = (\tilde{A}^+ + \tilde{A}^-) \tilde{Q} = \underbrace{\tilde{A}^+ \tilde{Q}}_{\tilde{E}^+} + \underbrace{\tilde{A}^- \tilde{Q}}_{\tilde{E}^-} = \tilde{E}^+ + \tilde{E}^- \quad (30)$$

There is a discontinuities and deference between  $\tilde{E}^+, \tilde{E}^-$ . To eliminate the discontinuities and guarantee a smooth transition through critical points (sonic points or stagnation points), a blending function is introduced together with a blending parameter  $\varepsilon$ . An appropriate choice of the blending parameter has to be chosen.

$$\begin{aligned} \tilde{\lambda}^+ &= \frac{\tilde{\lambda} + |\tilde{\lambda}|}{2} & \tilde{\lambda}^{+'} &= \frac{\tilde{\lambda} + \sqrt{\tilde{\lambda}^2 + \varepsilon^2}}{2} \\ \tilde{\lambda}^- &= \frac{\tilde{\lambda} - |\tilde{\lambda}|}{2} & \tilde{\lambda}^{-'} &= \frac{\tilde{\lambda} - \sqrt{\tilde{\lambda}^2 + \varepsilon^2}}{2} \end{aligned} \Rightarrow \quad (31)$$

Rewriting the conservation law from of the N-S equations:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} = -\frac{\partial \tilde{E}^+}{\partial \tilde{x}} - \frac{\partial \tilde{E}^-}{\partial \tilde{x}} + \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \quad (32)$$

A simple, explicit, first order (in space and time) scheme in delta from is obtained using:

$$\Delta \tilde{Q}_i^n = -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left( \nabla \tilde{E}_i^{+n} + \Delta \tilde{E}_i^{-n} - \frac{M_\infty}{Re_{L\infty}} \delta \tilde{V}_{1,i}^n \right) \quad (33)$$

And advancing the solution by:

$$\tilde{Q}_i^{n+1} = \Delta \tilde{Q}_i^n + \tilde{Q}_i^n \quad (34)$$

#### 4.5 First Order Steger-Warming – Implicit

The Implicit Steger-Warming scheme starts from:

$$\begin{aligned} \frac{\Delta \tilde{Q}_i^n}{\Delta \tilde{t}} &= -\frac{\partial \tilde{E}_i^{n+1}}{\partial \tilde{x}} + \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_{1,i}^{n+1}}{\partial \tilde{x}} \\ \Delta \tilde{Q}_i^n &= -\Delta \tilde{t} \frac{\partial}{\partial \tilde{x}} \left( \tilde{E}_i^n + \tilde{A}_i^n \Delta \tilde{Q}_i^n \right) + \Delta \tilde{t} \frac{M_\infty}{Re_{L\infty}} \frac{\partial}{\partial \tilde{x}} \left( \tilde{V}_{1,i}^n + \left( \tilde{P} - \tilde{R}_x \right)_i^n \Delta \tilde{Q}_i^n + \frac{\partial}{\partial \tilde{x}} \left( \tilde{R} \Delta \tilde{Q} \right)_i^n \right) \end{aligned} \quad (35)$$

Rearranging in delta form:

$$\left( I + \Delta \tilde{t} \left( \frac{\partial}{\partial \tilde{x}} \left[ \tilde{A} - \left( \tilde{P} - \tilde{R}_x \right) \right]_i^n - \frac{\partial^2 \tilde{R}_i^n}{\partial \tilde{x}^2} \right) \right) \Delta \tilde{Q}_i^n = \text{RHS}_i^n \quad (36)$$

$$\begin{aligned} \text{RHS}_i^n &= -\Delta \tilde{t} \frac{\partial}{\partial \tilde{x}} \left( \tilde{E}^+ + \tilde{E}^- - \frac{M_\infty}{Re_{L\infty}} \tilde{V}_1 \right)_i^n \\ \left( I + \frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left( \left[ \nabla \tilde{A}_i^{+n} + \Delta \tilde{A}_i^{-n} - \delta \left( \tilde{P} - \tilde{R}_x \right) \right]_i^n - \frac{\delta^2}{\Delta \tilde{x}} \tilde{R}_i^n \right) \right) \Delta \tilde{Q}_i^n &= \text{RHS}_i^n \\ \text{RHS}_i^n &= -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left( \nabla \tilde{E}^+ + \Delta \tilde{E}^- - \frac{M_\infty}{Re_{L\infty}} \delta \tilde{V}_1 \right)_i^n \end{aligned} \quad (37)$$