Computational Fluid Dynamics
HW2

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Nomenclature

β	entropy fixed eigenvalues
Δx	size of each cell in the domain
$\Delta \tilde{t}$	normalized step size in time
$\Delta \tilde{x}$	normalized step size in space
γ	ratio of specific heats
κ	coefficient of thermal conductivity
μ	coefficient of viscosity
ρ	fluid density
$\tilde{H}$	normalized total enthalpy
c_p	constant specific heat capacity for a constant pressure
c_v	constant specific heat capacity for a constant volume
e	total energy
L	characteristic length
p	pressure
R	gas constant
T	temperature
t	time
u	fluid velocity
x	spatial coordinate
x_F	x coordinate of the end of the domain
x_I	x coordinate of the start of the domain
x_i	x coordinate of the i-th cell

Diagonals

Φ	main diagonal
Ψ	upper off-diagonal
Θ	lower off-diagonal

Far–Away Properties

κ_∞	coefficient of thermal conductivity far away
μ_∞	coefficient of viscosity far away

ρ_∞	density far away
a_∞	speed of sound far away
M_∞	mach number far away
p_∞	pressure far away
T_∞	temperature far away

Matrices

$\hat{\tilde{\Lambda}}$	normalized eigenvalues matrix of Roe's average matrix
$\hat{\tilde{A}}$	normalized Roe's average matrix
$\hat{\tilde{T}}$	normalized eigenvectors matrix of Roe's average matrix
$\tilde{\Lambda}$	normalized eigenvalues matrix
$\tilde{A}$	normalized jacobian matrix of E w.r.t. Q
$\tilde{B}$, $\tilde{C}$	deconstruction of Roe's average matrix
$\tilde{P}$	normalized jacobian matrix of E_ν w.r.t. Q
$\tilde{R}$	normalized jacobian matrix of E_ν w.r.t. Q_x
$\tilde{T}$	normalized eigenvectors matrix

Dimensionless Numbers

Pr_∞	Prandtl number far away
$Re_{L\infty}$	Reynolds number with respect to L far away

Normalized Average Properties

$\hat{\tilde{\rho}}$	normalized average density
$\hat{\tilde{H}}$	normalized average total enthalpy
$\hat{\tilde{u}}$	normalized average velocity

Vectors

$\tilde{\mathfrak{R}}$	residual of first order explicit Roe
$\tilde{\tilde{E}}$	normalized average flux vector
E	inviscid convective vector
E_ν	viscous convective vector
Q	conservation state space vector

1 Problem Definition

1.1 Governing Equations

Consider the one-dimensional Navier-Stokes Equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = \frac{\partial E_\nu}{\partial x} \tag{1}$$

Where:

$$Q = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad E = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ (e+p)\,u \end{pmatrix}, \quad E_\nu = \begin{pmatrix} 0 \\ \tau_{xx} \\ u\tau_{xx} - q_x \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{4}{3}\mu\frac{\partial u}{\partial x} \\ \frac{4}{3}\mu u\frac{\partial u}{\partial x} + \kappa\frac{\partial T}{\partial x} \end{pmatrix}$$

$$p = (\gamma - 1)\left(e - \frac{1}{2}\rho u^2\right), \qquad T = \frac{p}{\rho R}, \tag{2}$$

$$\mu = 1.458 \cdot 10^{-6}\frac{T^{\frac{3}{2}}}{T + 110.4}, \quad \kappa = 2.495 \cdot 10^{-3}\frac{T^{\frac{3}{2}}}{T + 194}$$

$$R = c_p - c_v, \quad \gamma = \frac{c_p}{c_v}$$

The constants are:

- $\gamma = 1.4$ for air under standard atmospheric conditions
- $R = 287.0$ for air

1.2 Physical Domain

The physical domain is a tube extended between $x = 0.2$ and $x = 1.0$. At both ends there are impermeable walls.

1.3 Initial Conditions

The initial conditions are shown in Fig.1:

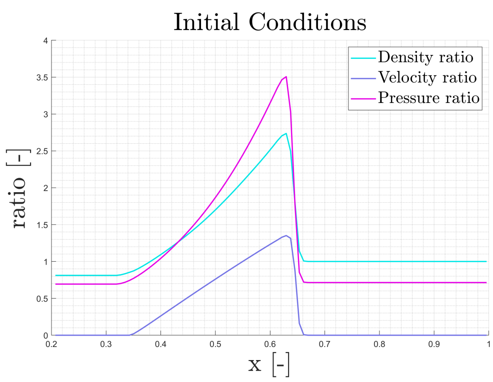

Figure 1: Initial conditions

1.4 Boundary Conditions

On each side of the tube there is an adiabatic, solid wall boundary conditions.

$$u_{(x=0.2)} = u_{(x=1.0)} = 0 \quad \Big\| \quad \left.\frac{\partial p}{\partial x}\right|_{x=0.2} = \left.\frac{\partial p}{\partial x}\right|_{x=1.0} = 0 \quad \Big\| \quad \left.\frac{\partial T}{\partial x}\right|_{x=0.2} = \left.\frac{\partial T}{\partial x}\right|_{x=1.0} = 0$$

2 Normalizing The Navier-Stokes Equations

Since the initial conditions are normalized, there is a need to normalize the N-S equations. We will use the following normalizations:

$$\rho = \rho_\infty \tilde{\rho}, \quad u = a_\infty \tilde{u}, \quad p = \gamma p_\infty \tilde{p}, \quad T = \gamma T_\infty \tilde{T}, \quad x = L\tilde{x}, \quad t = \frac{L}{a_\infty}\tilde{t}, \quad \mu = \mu_\infty \tilde{\mu}, \quad \kappa = \kappa_\infty \tilde{\kappa} \tag{3}$$

The normalization of the temperature was chosen to cancel out the γ in the normalization of the pressure:

$$\begin{aligned} p &= \rho R T \\ \gamma p_\infty \tilde{p} &= \rho_\infty \tilde{\rho} R \gamma T_\infty \tilde{T} \\ \tilde{p} &= \tilde{\rho}\tilde{T} \end{aligned} \tag{4}$$

The pressure normalization can be written also as:

$$p = \gamma p_\infty \tilde{p} = \gamma \rho_\infty R T_\infty \tilde{p} = \rho_\infty a_\infty^2 \tilde{p} \tag{5}$$

From equations 2 and 5 we can derive the normalization for the energy:

$$\begin{aligned} e &= \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \\ e &= \frac{\rho_\infty a_\infty^2 \tilde{p}}{\gamma - 1} + \frac{1}{2}\rho_\infty \tilde{\rho} a_\infty^2 \tilde{a}^2 \\ e &= \rho_\infty a_\infty^2 \left(\frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\tilde{a}^2\right) \\ e &= \rho_\infty a_\infty^2 \tilde{e} \end{aligned} \tag{6}$$

After substituting the normalizations in the N-S equations we get:

$$\frac{\partial}{\partial \frac{L}{a_\infty}\tilde{t}} \begin{pmatrix} \rho_\infty \tilde{\rho} \\ \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} \rho_\infty a_\infty \tilde{\rho}\tilde{u} \\ \rho_\infty a_\infty^2 \tilde{p} + \rho_\infty a_\infty^2 \tilde{\rho}\tilde{u}^2 \\ \rho_\infty a_\infty^3 \left(\tilde{e} + \tilde{p}\right)\tilde{u} \end{pmatrix} = \frac{\partial}{\partial L\tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}\mu_\infty a_\infty \tilde{\mu}\frac{\partial \tilde{u}}{\partial L\tilde{x}} \\ \frac{4}{3}\mu_\infty a_\infty^2 \tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial L\tilde{x}} + \frac{\kappa_\infty a_\infty^2}{R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial L\tilde{x}} \end{pmatrix} \tag{7}$$

Rearranging:

$$\frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{t}} \begin{pmatrix} \tilde{\rho} \\ a_\infty \tilde{\rho}\tilde{u} \\ a_\infty^2 \tilde{e} \end{pmatrix} + \frac{\rho_\infty a_\infty}{L}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} \tilde{\rho}\tilde{u} \\ a_\infty \tilde{p} + a_\infty \tilde{\rho}\tilde{u}^2 \\ a_\infty^2 \left(\tilde{e} + \tilde{p}\right)\tilde{u} \end{pmatrix} = \frac{\mu_\infty}{L^2}\frac{\partial}{\partial \tilde{x}} \begin{pmatrix} 0 \\ \frac{4}{3}a_\infty \tilde{\mu}\frac{\partial \tilde{u}}{\partial \tilde{x}} \\ \frac{4}{3}a_\infty^2 \tilde{\mu}\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\kappa_\infty a_\infty^2}{\mu_\infty R}\tilde{\kappa}\frac{\partial \tilde{T}}{\partial \tilde{x}} \end{pmatrix} \tag{8}$$

Dividing the second equation by a_∞, the third equation by a_∞^2, and the whole set of equations by $\dfrac{\rho_\infty a_\infty}{L}$ we get:

$$\frac{\partial}{\partial\tilde{t}}\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix}+\frac{\partial}{\partial\tilde{x}}\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix}=\frac{\mu_\infty}{L\rho_\infty a_\infty}\frac{\partial}{\partial\tilde{x}}\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial\tilde{u}}{\partial\tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\frac{\kappa_\infty}{\mu_\infty R}\tilde{\kappa}\frac{\partial\tilde{T}}{\partial\tilde{x}}\end{pmatrix} \tag{9}$$

The Reynolds number and the mach number far away are defined as:

$$\begin{gathered}M_\infty=\frac{u_\infty}{a_\infty}\quad Re_{L\infty}=\frac{\rho_\infty u_\infty L}{\mu_\infty}\\ \Downarrow\\ \frac{\mu_\infty}{L\rho_\infty a_\infty}=\frac{M_\infty}{Re_{L\infty}}\end{gathered} \tag{10}$$

The Prandtl number far away is defined as:

$$\begin{gathered}Pr_\infty=\frac{c_p\mu_\infty}{\kappa_\infty}\\ \Downarrow\\ \frac{\kappa_\infty}{\mu_\infty R}=\frac{c_p}{Pr_\infty\left(c_p-c_v\right)}=\frac{\gamma}{Pr_\infty\left(\gamma-1\right)}\end{gathered} \tag{11}$$

Substituting into the normalized N-S equations:

$$\frac{\partial\tilde{Q}}{\partial\tilde{t}}+\frac{\partial\tilde{E}}{\partial\tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial\tilde{E}_\nu}{\partial\tilde{x}} \tag{12}$$

Where:

$$\tilde{Q}=\begin{pmatrix}\tilde{\rho}\\ \tilde{\rho}\tilde{u}\\ \tilde{e}\end{pmatrix},\quad \tilde{E}=\begin{pmatrix}\tilde{\rho}\tilde{u}\\ \tilde{p}+\tilde{\rho}\tilde{u}^2\\ (\tilde{e}+\tilde{p})\,\tilde{u}\end{pmatrix},\quad \tilde{E}_\nu=\begin{pmatrix}0\\ \frac{4}{3}\tilde{\mu}\frac{\partial\tilde{u}}{\partial\tilde{x}}\\ \frac{4}{3}\tilde{\mu}\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\frac{\gamma}{Pr_\infty(\gamma-1)}\tilde{\kappa}\frac{\partial\tilde{T}}{\partial\tilde{x}}\end{pmatrix} \tag{13}$$

The normalized Navier-Stokes equations are:

$$\frac{\partial\tilde{Q}}{\partial\tilde{t}}+\frac{\partial\tilde{E}}{\partial\tilde{x}}=\frac{M_\infty}{Re_{L\infty}}\frac{\partial\tilde{V}_1}{\partial\tilde{x}} \tag{14}$$

Where:

$$\tilde{V}_1=\tilde{V}_{1\left(\tilde{Q},\tilde{Q}_x\right)}=\tilde{E}_\nu$$

3 The Computational Domain

3.1 Discretization

The physical domain $[x_I, x_F]$ is discretized into N equispaced cells. The size of each cell is there for:

$$\Delta x=\frac{x_F-x_I}{N}=\frac{L}{N} \tag{15}$$

so the x coordinate of the i-th cell x_i is:

$$x_i=x_I+\frac{1}{2}\Delta x+\Delta x\cdot(i-1)\qquad\text{when starting from } i=1 \tag{16}$$

3.2 Boundary Conditions

In order to set the boundary conditions on the edge faces we will define ghost cells that will be calculated as follows:

$$\begin{aligned} u_{(i=0)} &= -u_{(i=1)} \\ u_{(i=N+1)} &= -u_{(i=N)} \end{aligned} \tag{17}$$

in order to maintain velocity zero on the boundary and like so:

$$\begin{aligned} T_{(i=0)} &= T_{(i=1)} \\ T_{(i=N+1)} &= T_{(i=N)} \end{aligned} \tag{18}$$

in order to maintain adiabatic boundary conditions. Since the gradient of the pressure on the wall is zero, we get:

$$\begin{aligned} p_{(i=0)} &= p_{(i=1)} \\ p_{(i=N+1)} &= p_{(i=N)} \end{aligned} \tag{19}$$

From equations 2, 18, and 19 we can conclude:

$$\begin{aligned} \rho_{(i=0)} &= \rho_{(i=1)} \\ \rho_{(i=N+1)} &= \rho_{(i=N)} \end{aligned} \tag{20}$$

and from equations 2, 17, 19, and 20 we can conclude:

$$\begin{aligned} e_{(i=0)} &= e_{(i=1)} \\ e_{(i=N+1)} &= e_{(i=N)} \end{aligned} \tag{21}$$

4 The Numerical Schemes

4.1 Jacobian Matrices of The Navier-Stokes Equations

We can rewrite Eq.14 as:

$$\begin{aligned}
\frac{\partial \tilde{Q}}{\partial \tilde{t}} &= -\frac{\partial \tilde{E}}{\partial \tilde{x}} - \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \\
\frac{\partial \tilde{Q}}{\partial \tilde{t}} &= -\underbrace{\frac{\partial \tilde{E}}{\partial \tilde{Q}}}_{\tilde{A}} \frac{\partial \tilde{Q}}{\partial \tilde{x}} - \frac{M_\infty}{Re_{L\infty}} \left(\underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}}_{\tilde{P}} \frac{\partial \tilde{Q}}{\partial \tilde{x}} + \underbrace{\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}}_{\tilde{R}} \frac{\partial \tilde{Q}_x}{\partial \tilde{x}} \right)
\end{aligned} \tag{22}$$

Where:

$$\begin{aligned}
\tilde{A} &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\tilde{u}^2 & (3-\gamma)\,\tilde{u} & \gamma-1 \\ -\frac{\gamma\tilde{e}\tilde{u}}{\tilde{\rho}} - (\gamma-1)\,\tilde{u}^3 & \frac{\gamma\tilde{e}}{\tilde{\rho}} - \frac{3\,(\gamma-1)\,\tilde{u}^2}{2} & \gamma\tilde{u} \end{pmatrix} \\
\tilde{P} - \tilde{R}_x &= -\frac{1}{\rho} \begin{pmatrix} 0 & 0 & 0 \\ -\tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & \left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \\ -\tilde{u}^2\left(\frac{4}{3}\tilde{\mu}\right)_x & \tilde{u}\left(\frac{4}{3}\tilde{\mu}\right)_x & 0 \end{pmatrix} \\
\tilde{R} &= -\frac{1}{\rho} \begin{pmatrix} 0 & 0 & 0 \\ \frac{4}{3}\tilde{u}\tilde{\mu} & -\frac{4}{3}\tilde{\mu} & 0 \\ \left(\frac{4}{3}\tilde{\mu} - \alpha\frac{\tilde{\kappa}}{c_v}\right)\tilde{u}^2 + \alpha\frac{\tilde{\kappa}}{c_v}\frac{\tilde{e}}{\tilde{\rho}} & -\left(\frac{4}{3}\tilde{\mu} - \alpha\frac{\tilde{\kappa}}{c_v}\right)\tilde{u} & -\alpha\frac{\tilde{\kappa}}{c_v} \end{pmatrix}
\end{aligned} \tag{23}$$

and α is:

$$\alpha = \frac{\gamma}{Pr_\infty\,(\gamma-1)}$$

4.2 FVS – First Order Explicit Steger-Warming

Since the inviscid N-S equations (Euler equations) is a hyperbolic system of equations, the A matrix (from Eq.23) is a diagonalizable matrix and can be written as:

$$\tilde{A} = \tilde{T}\tilde{\Lambda}\tilde{T}^{-1}$$

$$\tilde{T} = \begin{pmatrix} 1 & \frac{\tilde{\rho}}{2\tilde{a}} & -\frac{\tilde{\rho}}{2\tilde{a}} \\ \tilde{u} & \frac{\tilde{\rho}}{2\tilde{a}}\left(\tilde{u}+\tilde{a}\right) & -\frac{\tilde{\rho}}{2\tilde{a}}\left(\tilde{u}-\tilde{a}\right) \\ \frac{\tilde{u}^2}{2} & \frac{\tilde{\rho}}{2\tilde{a}}\left(\frac{\tilde{u}^2}{2}+\tilde{u}\tilde{a}+\frac{\tilde{a}^2}{\gamma-1}\right) & -\frac{\tilde{\rho}}{2\tilde{a}}\left(\frac{\tilde{u}^2}{2}-\tilde{u}\tilde{a}+\frac{\tilde{a}^2}{\gamma-1}\right) \end{pmatrix}$$

$$\tilde{\Lambda} = \begin{pmatrix} \tilde{u} & 0 & 0 \\ 0 & \tilde{u}+\tilde{a} & 0 \\ 0 & 0 & \tilde{u}-\tilde{a} \end{pmatrix} \tag{24}$$

$$\tilde{T}^{-1} = \begin{pmatrix} 1-\frac{\gamma-1}{2}\frac{\tilde{u}^2}{\tilde{a}^2} & (\gamma-1)\frac{\tilde{u}^2}{\tilde{a}^2} & -\frac{\gamma-1}{\tilde{a}^2} \\ \frac{1}{\tilde{\rho}\tilde{a}}\left((\gamma-1)\tilde{u}^2-\tilde{u}\tilde{a}\right) & \frac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a}-(\gamma-1)\tilde{u}\right) & \frac{\gamma-1}{\tilde{\rho}\tilde{a}} \\ -\frac{1}{\tilde{\rho}\tilde{a}}\left((\gamma-1)\tilde{u}^2+\tilde{u}\tilde{a}\right) & \frac{1}{\tilde{\rho}\tilde{a}}\left(\tilde{a}+(\gamma-1)\tilde{u}\right) & -\frac{\gamma-1}{\tilde{\rho}\tilde{a}} \end{pmatrix}$$

Where:

$$\tilde{a} = \sqrt{\frac{\gamma\tilde{p}}{\tilde{\rho}}}$$

Let the $\tilde{\Lambda}^{\pm}$ matrix be defined as:

$$\tilde{\Lambda}^{\pm} = \begin{pmatrix} \frac{\tilde{u}\pm|\tilde{u}|}{2} & 0 & 0 \\ 0 & \frac{\tilde{u}+\tilde{a}\pm|\tilde{u}+\tilde{a}|}{2} & 0 \\ 0 & 0 & \frac{\tilde{u}-\tilde{a}\pm|\tilde{u}-\tilde{a}|}{2} \end{pmatrix} \tag{25}$$

Where the matrix $\tilde{\Lambda}^{+}$ contains only positive eigenvalues and the matrix $\tilde{\Lambda}^{-}$ contains only negative eigenvalues. As in Roe's scheme, one can define:

$$\begin{matrix} \tilde{A}^{+} & \triangleq & \tilde{T}\tilde{\Lambda}^{+}\tilde{T}^{-1} \\ \tilde{A}^{-} & \triangleq & \tilde{T}\tilde{\Lambda}^{-}\tilde{T}^{-1} \end{matrix} \quad \Rightarrow \quad \begin{matrix} \tilde{A} & = & \tilde{A}^{+}+\tilde{A}^{-} \\ \left|\tilde{A}\right| & \triangleq & \tilde{A}^{+}-\tilde{A}^{-} \end{matrix} \tag{26}$$

Assuming a perfect gas, the flux vector $\tilde{E}_{(Q)}$ is a homogeneous function of degree one in $\tilde{Q}$, meaning:

$$\forall\alpha \quad \tilde{E}_{\left(\alpha\tilde{Q}\right)} = \alpha\tilde{E}_{\left(\tilde{Q}\right)}$$

The homogeneity allows to rewrite the flux vector $\tilde{E}$ using Eq.22 as:

$$\tilde{E} = \tilde{A}\tilde{Q} = \left(\tilde{A}^{+}+\tilde{A}^{-}\right)\tilde{Q} = \underbrace{\tilde{A}^{+}\tilde{Q}}_{\tilde{E}^{+}} + \underbrace{\tilde{A}^{-}\tilde{Q}}_{\tilde{E}^{-}} = \tilde{E}^{+}+\tilde{E}^{-} \tag{27}$$

There is a discontinuity and deference between $\tilde{E}^{+}, \tilde{E}^{-}$. To eliminate the discontinuities and guarantee a smooth transition through critical points (sonic points or stagnation points), a blending function is introduced together with a blending parameter ε. An appropriate choice of the blending parameter has to be chosen.

$$\begin{aligned} \tilde{\lambda}^{+} &= \frac{\tilde{\lambda} + \left|\tilde{\lambda}\right|}{2} & & \tilde{\lambda}^{+'} = \frac{\tilde{\lambda} + \sqrt{\tilde{\lambda}^2 + \varepsilon^2}}{2} \\ & & \Rightarrow & \\ \tilde{\lambda}^{-} &= \frac{\tilde{\lambda} - \left|\tilde{\lambda}\right|}{2} & & \tilde{\lambda}^{-'} = \frac{\tilde{\lambda} - \sqrt{\tilde{\lambda}^2 + \varepsilon^2}}{2} \end{aligned} \tag{28}$$

Rewriting the conservation law form of the N-S equations Eq.14 using Eq.27:

$$\frac{\partial \tilde{Q}}{\partial \tilde{t}} = -\frac{\partial \tilde{E}^{+}}{\partial \tilde{x}} - \frac{\partial \tilde{E}^{-}}{\partial \tilde{x}} + \frac{M_\infty}{Re_{L\infty}} \frac{\partial \tilde{V}_1}{\partial \tilde{x}} \tag{29}$$

A simple, explicit, first order (in space and time) scheme in delta form is obtained using:

$$\Delta \tilde{Q}_i^n = -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\nabla \tilde{E}_i^{+n} + \Delta \tilde{E}_i^{-n} - \frac{M_\infty}{Re_{L\infty}} \delta \tilde{V}_{1,i}^n \right) \tag{30}$$

And advancing the solution by:

$$\tilde{Q}_i^{n+1} = \Delta \tilde{Q}_i^n + \tilde{Q}_i^n \tag{31}$$

4.2.1 Finite Volume Formulation

Rearranging Eq.30 using the finite volume notation:

$$\begin{aligned} \Delta \tilde{Q}_i^n &= -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\tilde{E}_i^{+n} - \tilde{E}_{i-1}^{+n} + \tilde{E}_{i+1}^{-n} - \tilde{E}_i^{-n} - \frac{M_\infty}{Re_{L\infty}} \left(\tilde{V}_{1,i+\frac{1}{2}}^n - \tilde{V}_{1,i-\frac{1}{2}}^n \right) \right) \\ \Delta \tilde{Q}_i^n &= -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\left(\tilde{E}_i^{+n} + \tilde{E}_{i+1}^{-n} \right) - \left(\tilde{E}_{i-1}^{+n} + \tilde{E}_i^{-n} \right) - \frac{M_\infty}{Re_{L\infty}} \left(\tilde{V}_{1,i+\frac{1}{2}}^n - \tilde{V}_{1,i-\frac{1}{2}}^n \right) \right) \end{aligned} \tag{32}$$

Define:

$$\begin{aligned} \tilde{E}_{i+\frac{1}{2}} &\triangleq \tilde{E}_i^{+} + E_{i+1}^{-} \\ &= \tilde{A}_i^{+} \tilde{Q}_i + \tilde{A}_{i+1}^{-} \tilde{Q}_{i+1} \\ &\equiv \tilde{\tilde{E}}_{i+\frac{1}{2}} \end{aligned} \tag{33}$$

Finally we get:

$$\Delta \tilde{Q}_i^n = -\frac{\Delta \tilde{t}}{\Delta \tilde{x}} \left(\tilde{\tilde{E}}_{i+\frac{1}{2}}^n - \tilde{\tilde{E}}_{i-\frac{1}{2}}^n - \frac{M_\infty}{Re_{L\infty}} \left(\tilde{V}_{1,i+\frac{1}{2}}^n - \tilde{V}_{1,i-\frac{1}{2}}^n \right) \right) \tag{34}$$

4.2.2 Calculating $\tilde{V}_{1,i+\frac{1}{2}}$

$$\tilde{V}_{1,i+\frac{1}{2}} = \begin{pmatrix} 0 \\ \frac{4}{3} \left.\tilde{\mu}\right|_{i+\frac{1}{2}} \left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} \\ \frac{4}{3} \left.\tilde{\mu}\right|_{i+\frac{1}{2}} \left.\tilde{u}\right|_{i+\frac{1}{2}} \left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} + \frac{\gamma}{Pr_\infty (\gamma - 1)} \left.\tilde{\kappa}\right|_{i+\frac{1}{2}} \left.\frac{\partial \tilde{T}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} \end{pmatrix} \tag{35}$$

Where:

$$\begin{aligned}
\left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} &= \frac{\tilde{u}_{i+1}-\tilde{u}_i}{\Delta \tilde{x}}, & \tilde{\mu}|_{i+\frac{1}{2}} &= \frac{\tilde{\mu}_{i+1}+\tilde{\mu}_i}{2} \\
\left.\frac{\partial \tilde{T}}{\partial \tilde{x}}\right|_{i+\frac{1}{2}} &= \frac{\tilde{T}_{i+1}-\tilde{T}_i}{\Delta \tilde{x}}, & \tilde{\kappa}|_{i+\frac{1}{2}} &= \frac{\tilde{\kappa}_{i+1}+\tilde{\kappa}_i}{2}
\end{aligned} \tag{36}$$

$$\tilde{u}|_{i+\frac{1}{2}} = \frac{\tilde{u}_{i+1}+\tilde{u}_i}{2}$$

4.3 FVS – First Order Implicit Steger-Warming

4.3.1 Linearization In Time

- $\tilde{E}_i^{n+1}$ Estimation

$$\tilde{E}_i^{n+1} = \tilde{E}_i^n + \underbrace{\left.\frac{\partial \tilde{E}}{\partial \tilde{Q}}\right|_i^n}_{\tilde{A}_i^n} \Delta \tilde{Q}_i^n + \text{H.O.T} \tag{37}$$

$$\tilde{E}_i^{n+1} = \tilde{E}_i^n + \tilde{A}_i^n \Delta \tilde{Q}_i^n$$

- ${\tilde{V}_1}_i^{n+1}$ Estimation

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^n + \underbrace{\left.\frac{\partial \tilde{V}_1}{\partial \tilde{Q}}\right|_i^n}_{\tilde{P}_i^n} \Delta \tilde{Q}_i^n + \underbrace{\left.\frac{\partial \tilde{V}_1}{\partial \tilde{Q}_x}\right|_i^n}_{\tilde{R}_i^n} \Delta {\tilde{Q}_x}_i^n + \text{H.O.T} \tag{38}$$

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^n + \tilde{P}_i^n \Delta \tilde{Q}_i^n + \tilde{R}_i^n \Delta {\tilde{Q}_x}_i^n$$

The difficulty stems from the fact that the solution vector is $\Delta\tilde{Q}$ and not $\Delta\tilde{Q}_x$. This can be solved by a linearization of the term $\Delta\tilde{Q}_x$ which can be conducted using the following relation:

$$\frac{\partial\left(\tilde{R}\Delta\tilde{Q}\right)_i^n}{\partial \tilde{x}} = \frac{\partial \tilde{R}_i^n}{\partial \tilde{x}}\Delta\tilde{Q}_i^n + \tilde{R}_i^n \frac{\partial \Delta\tilde{Q}_i^n}{\partial \tilde{x}} = \frac{\partial \tilde{R}}{\partial \tilde{x}}\Delta\tilde{Q} + \tilde{R}_i^n \Delta {\tilde{Q}_x}_i^n$$

$$\Downarrow$$

$$\tilde{V}_{1,i}^{n+1} = \tilde{V}_{1,i}^n + \left(\tilde{P} - \tilde{R}_x\right)_i^n \Delta\tilde{Q}_i^n + \frac{\partial}{\partial \tilde{x}}\left(\tilde{R}\Delta\tilde{Q}\right)_i^n \tag{39}$$

4.3.2 The Scheme

The Implicit Steger-Warming scheme starts from:

$$\begin{aligned}
\frac{\Delta\tilde{Q}_i^n}{\Delta\tilde{t}} &= -\frac{\partial \tilde{E}_i^{n+1}}{\partial \tilde{x}} + \frac{M_\infty}{Re_{L\infty}}\frac{\partial \tilde{V}_{1,i}^{n+1}}{\partial \tilde{x}} \\
\Delta\tilde{Q}_i^n &= -\Delta\tilde{t}\frac{\partial}{\partial \tilde{x}}\left(\tilde{E}_i^n + \tilde{A}_i^n\Delta\tilde{Q}_i^n\right) + \Delta\tilde{t}\frac{M_\infty}{Re_{L\infty}}\frac{\partial}{\partial \tilde{x}}\left(\tilde{V}_{1,i}^n + \left(\tilde{P}-\tilde{R}_x\right)_i^n\Delta\tilde{Q}_i^n + \frac{\partial}{\partial \tilde{x}}\left(\tilde{R}\Delta\tilde{Q}\right)_i^n\right)
\end{aligned} \tag{40}$$

Rearranging in delta form:

$$\left(I+\Delta\tilde{t}\left(\frac{\partial}{\partial\tilde{x}}\left[\tilde{A}-\frac{M_\infty}{Re_{L\infty}}\left(\tilde{P}-\tilde{R}_x\right)\right]_i^n-\frac{M_\infty}{Re_{L\infty}}\frac{\partial^2\tilde{R}_i^n}{\partial\tilde{x}^2}\right)\right)\Delta\tilde{Q}_i^n=\mathrm{RHS}_i^n$$
$$\mathrm{RHS}_i^n=-\Delta\tilde{t}\frac{\partial}{\partial\tilde{x}}\left(\tilde{E}^++\tilde{E}^--\frac{M_\infty}{Re_{L\infty}}\tilde{V}_1\right)_i^n \tag{41}$$

$$\left(I+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\left[\nabla\tilde{A}^++\Delta\tilde{A}^--\frac{M_\infty}{Re_{L\infty}}\frac{D_0}{2}\left(\tilde{P}-\tilde{R}_x\right)\right]_i^n-\frac{M_\infty}{Re_{L\infty}}\frac{\delta^2}{\Delta\tilde{x}}\tilde{R}_i^n\right)\right)\Delta\tilde{Q}_i^n=\mathrm{RHS}_i^n$$
$$\mathrm{RHS}_i^n=-\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\nabla\tilde{E}^++\Delta\tilde{E}^--\frac{M_\infty}{Re_{L\infty}}\delta\tilde{V}_1\right)_i^n \tag{42}$$

Rewrite using the finite volume notation for the inviscid terms like in Eq.34:

$$\left(I+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\left[\nabla\tilde{A}^++\Delta\tilde{A}^--\frac{M_\infty}{Re_{L\infty}}\frac{D_0}{2}\left(\tilde{P}-\tilde{R}_x\right)\right]_i^n-\frac{M_\infty}{Re_{L\infty}}\frac{\delta^2}{\Delta\tilde{x}}\tilde{R}_i^n\right)\right)\Delta\tilde{Q}_i^n=\mathrm{RHS}_i^n$$
$$\mathrm{RHS}_i^n=-\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\tilde{\tilde{E}}_{i+\frac{1}{2}}^n-\tilde{\tilde{E}}_{i-\frac{1}{2}}^n-\frac{M_\infty}{Re_{L\infty}}\left(\tilde{V}_{1,i+\frac{1}{2}}^n-\tilde{V}_{1,i-\frac{1}{2}}^n\right)\right) \tag{43}$$

and opening the delta form on the LHS:

$$\begin{aligned}
\mathrm{LHS}_i^n &= \Delta\tilde{Q}_i^n+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\nabla\tilde{A}_i^{+n}\Delta\tilde{Q}_i^n+\Delta\tilde{A}_i^{-n}\Delta\tilde{Q}_i^n-\frac{M_\infty}{Re_{L\infty}}\frac{D_0}{2}\left(\tilde{P}-\tilde{R}_x\right)_i^n\Delta\tilde{Q}_i^n-\frac{M_\infty}{Re_{L\infty}}\frac{\delta^2}{\Delta\tilde{x}}\tilde{R}_i^n\Delta\tilde{Q}_i^n\right)\\
&= \Delta\tilde{Q}_i^n+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\Big(\tilde{A}_i^{+n}\Delta\tilde{Q}_i^n-\tilde{A}_{i-1}^{+n}\Delta\tilde{Q}_{i-1}^n+\tilde{A}_{i+1}^{-n}\Delta\tilde{Q}_{i+1}^n-\tilde{A}_i^{-n}\Delta\tilde{Q}_i^n-\\
&\quad -\frac{1}{2}\frac{M_\infty}{Re_{L\infty}}\left[\left(\tilde{P}-\tilde{R}_x\right)_{i+1}^n\Delta\tilde{Q}_{i+1}^n-\left(\tilde{P}-\tilde{R}_x\right)_{i-1}^n\Delta\tilde{Q}_{i-1}^n\right]-\\
&\quad -\frac{1}{\Delta\tilde{x}}\frac{M_\infty}{Re_{L\infty}}\left[\tilde{R}_{i+1}^n\Delta\tilde{Q}_{i+1}^n-2\tilde{R}_i^n\Delta\tilde{Q}_i^n+\tilde{R}_{i-1}^n\Delta\tilde{Q}_{i-1}^n\right]\Big)\\
&= \Theta_i^n\Delta\tilde{Q}_{i-1}^n+\Phi_i^n\Delta\tilde{Q}_i^n+\Psi_i^n\Delta\tilde{Q}_{i+1}^n
\end{aligned} \tag{44}$$

Where:

$$\begin{aligned}
\Theta_i^n &= -\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\tilde{A}_{i-1}^{+n}+\frac{\Delta\tilde{t}}{2\Delta\tilde{x}}\frac{M_\infty}{Re_{L\infty}}\left(\tilde{P}-\tilde{R}_x\right)_{i-1}^n-\frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\frac{M_\infty}{Re_{L\infty}}\tilde{R}_{i-1}^n\\
\Phi_i^n &= I+\frac{\Delta\tilde{t}}{\Delta\tilde{x}}\left(\tilde{A}_i^{+n}-\tilde{A}_i^{-n}\right)+2\frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\frac{M_\infty}{Re_{L\infty}}\tilde{R}_i^n\\
\Psi_i^n &= \frac{\Delta\tilde{t}}{\Delta\tilde{x}}\tilde{A}_{i+1}^{-n}-\frac{\Delta\tilde{t}}{2\Delta\tilde{x}}\frac{M_\infty}{Re_{L\infty}}\left(\tilde{P}-\tilde{R}_x\right)_{i+1}^n-\frac{\Delta\tilde{t}}{\Delta\tilde{x}^2}\frac{M_\infty}{Re_{L\infty}}\tilde{R}_{i+1}^n
\end{aligned} \tag{45}$$

For the implicit Steger-Warming scheme a matrix inversion is needed as follows:

$$\begin{pmatrix}
\Phi_1^n & \Psi_1^n & 0 & \cdots & \cdots & \cdots & 0\\
\Theta_2^n & \Phi_2^n & \Psi_2^n & 0 & \cdots & \cdots & 0\\
0 & \ddots & \ddots & \ddots & 0 & \cdots & 0\\
0 & 0 & \Theta_i^n & \Phi_i^n & \Psi_i^n & 0 & 0\\
0 & \cdots & 0 & \ddots & \ddots & \ddots & 0\\
0 & \cdots & \cdots & 0 & \Theta_{N-1}^n & \Phi_{N-1}^n & \Psi_{N-1}^n\\
0 & \cdots & \cdots & \cdots & 0 & \Theta_N^n & \Phi_N^n
\end{pmatrix}
\begin{pmatrix}
\Delta\tilde{Q}_1^n\\ \Delta\tilde{Q}_2^n\\ \cdots\\ \cdots\\ \cdots\\ \Delta\tilde{Q}_{N-1}^n\\ \Delta\tilde{Q}_N^n
\end{pmatrix}
=
\begin{pmatrix}
\mathrm{RHS}_1^n\\ \mathrm{RHS}_2^n\\ \cdots\\ \cdots\\ \cdots\\ \mathrm{RHS}_{N-1}^n\\ \mathrm{RHS}_N^n
\end{pmatrix} \tag{46}$$

4.3.3 Calculating $\left(\tilde{P}-\tilde{R_x}\right)_i$

$$\left(\tilde{P}-\tilde{R_x}\right)_i = \frac{1}{\rho}\begin{pmatrix} 0 & 0 & 0 \\ \tilde{u}\big|_i \frac{4}{3}\frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i & -\frac{4}{3}\frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i & 0 \\ \tilde{u}^2\big|_i \frac{4}{3}\frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i & -\tilde{u}\big|_i \frac{4}{3}\frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i & 0 \end{pmatrix}, \qquad \frac{\partial\tilde{\mu}}{\partial\tilde{x}}\Big|_i = \frac{\tilde{\mu}_{i+1}-\tilde{\mu}_{i-1}}{2\Delta\tilde{x}} \tag{47}$$

4.4 FDS – First Order Explicit Roe

The normalized Navier-Stokes equations as written in Eq.14:

$$\frac{\partial\tilde{Q}}{\partial\tilde{t}} + \frac{\partial\tilde{E}}{\partial\tilde{x}} = \frac{M_\infty}{Re_{L\infty}}\frac{\partial\tilde{V}_1}{\partial\tilde{x}} \tag{48}$$

In linearized form:

$$\frac{\partial\tilde{Q}}{\partial\tilde{t}} + \tilde{A}\frac{\partial\tilde{Q}}{\partial\tilde{x}} = \frac{M_\infty}{Re_{L\infty}}\frac{\partial\tilde{V}_1}{\partial\tilde{x}} \tag{49}$$

The initial conditions are:

$$\tilde{Q}_{(x,0)} = \begin{cases} \tilde{Q}_L & \tilde{x} < \tilde{x}_0 \\ \tilde{Q}_R & \tilde{x} > \tilde{x}_0 \end{cases}$$

Roe's linear approximation to the 1-D Riemann problem is expressed as:

$$\frac{\partial\tilde{Q}}{\partial\tilde{t}} + \hat{\tilde{A}}\frac{\partial\tilde{Q}}{\partial\tilde{x}} = \frac{M_\infty}{Re_{L\infty}}\frac{\partial\tilde{V}_1}{\partial\tilde{x}} \tag{50}$$

Where $\hat{\tilde{A}}$ replaces the original jacobian matrix $\tilde{A}$ and referred to as Roe's average matrix. Roe's average matrix is assumed constant in this formulation and therefore the problem is linear. The components of Roe's average matrix are evaluated using average values of $\tilde{Q}$ at the interface separating the two states, L and R, namely:

$$\hat{\tilde{A}} = \hat{\tilde{A}}_{(\tilde{Q}_L,\tilde{Q}_R)}$$

By setting certain conditions on the matrix $\hat{\tilde{A}}$ the aforementioned "Property U" is obtained for the system of equations:

- A linear mapping relates the vector $\tilde{Q}$ to the vector $\tilde{E}$
- $\hat{\tilde{A}}_{(\tilde{Q}_L,\tilde{Q}_R)} \xrightarrow[\tilde{Q}_L\to\tilde{Q}_R\to\tilde{Q}]{} \tilde{A}_{(\tilde{Q})}$
- $\tilde{E}_R - \tilde{E}_L = \hat{\tilde{A}}\left(\tilde{Q}_R - \tilde{Q}_L\right)$
- The eigenvalues of Roe's average matrix are real and linearly independent

The linear approximate problem is then hyperbolic and therefore Roe's average matrix may be diagonalized as follows:

$$\hat{\tilde{A}} = \hat{\tilde{T}}\hat{\tilde{\Lambda}}\hat{\tilde{T}}^{-1} \tag{51}$$

One can define now the following:

$$\hat{\tilde{A}}^{+} = \hat{\tilde{T}}\hat{\tilde{\Lambda}}^{+}\hat{\tilde{T}}^{-1} \quad \Big\| \quad \hat{\tilde{A}}^{-} = \hat{\tilde{T}}\hat{\tilde{\Lambda}}^{-}\hat{\tilde{T}}^{-1} \quad \Big\| \quad \left|\hat{\tilde{A}}\right| = \hat{\tilde{T}}\left|\hat{\tilde{\Lambda}}\right|\hat{\tilde{T}}^{-1}$$

Since Roe's average matrix can be split based on negative and positive waves (eigenvalues), the calculation of the fluxes may be split into contributions across negative and positive waves to determine appropriate formulae for the cell-face fluxes in the linear Riemann problem.
Each interface has a series of waves emanating from it and traveling left and right as follows:

Starting from the left state, one has:	Starting from the right state results in:
$\tilde{\tilde{E}}_{i+\frac{1}{2}} = \tilde{E}_L + \hat{\tilde{A}}^{-}\left(\tilde{Q}_R - \tilde{Q}_L\right)$	$\tilde{E}_R = \tilde{\tilde{E}}_{i+\frac{1}{2}} + \hat{\tilde{A}}^{+}\left(\tilde{Q}_R - \tilde{Q}_L\right)$

$$\Downarrow$$

$$\begin{cases} \tilde{\tilde{E}}_{i+\frac{1}{2}} & = & \tilde{E}_L + \hat{\tilde{A}}^{-}\left(\tilde{Q}_R - \tilde{Q}_L\right) \\ \tilde{\tilde{E}}_{i+\frac{1}{2}} & = & \tilde{E}_R - \hat{\tilde{A}}^{+}\left(\tilde{Q}_R - \tilde{Q}_L\right) \end{cases} \tag{52}$$

By way of averaging, the interface flux becomes:

$$\tilde{\tilde{E}}_{i+\frac{1}{2}} = \frac{1}{2}\left(\tilde{E}_L + \tilde{E}_R\right) - \frac{1}{2}\left|\hat{\tilde{A}}\right|\left(\tilde{Q}_R - \tilde{Q}_L\right) \tag{53}$$

4.4.1 Constructing The Roe Matrix

Let $\tilde{H}$ be the normalized total enthalpy:

$$\tilde{H} = \tilde{h} + \frac{1}{2}\tilde{u}^2 = \frac{\tilde{e} + \tilde{p}}{\tilde{\rho}}$$

One can rewrite the vectors $\tilde{Q}$ and $\tilde{E}$ in terms of the normalized total enthalpy instead of the normalized total energy $\tilde{e}$ as follows:

$$\tilde{Q} = \begin{pmatrix} \tilde{\rho} \\ \tilde{\rho}\tilde{u} \\ \dfrac{\tilde{\rho}\tilde{H}}{\gamma} + \dfrac{\gamma - 1}{2\gamma}\tilde{\rho}\tilde{u}^2 \end{pmatrix}, \qquad \tilde{E} = \begin{pmatrix} \tilde{\rho}\tilde{u} \\ \dfrac{\gamma - 1}{\gamma}\tilde{\rho}\tilde{H} + \dfrac{\gamma + 1}{2\gamma}\tilde{\rho}\tilde{u}^2 \\ \tilde{\rho}\tilde{u}\tilde{H} \end{pmatrix} \tag{54}$$

The jacobian matrix $\tilde{A}$ can also be expressed in terms of the total enthalpy:

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{\gamma - 3}{2}\tilde{u}^2 & (3 - \gamma)\,\tilde{u} & \gamma - 1 \\ \dfrac{1}{2}(\gamma - 1)\,\tilde{u}^3 - \tilde{u}\tilde{H} & \tilde{H} - (\gamma - 1)\,\tilde{u}^2 & \gamma\tilde{u} \end{pmatrix} \tag{55}$$

Let the vector $\tilde{Z}$ be defined as:

$$\tilde{Z} = \sqrt{\tilde{\rho}}\begin{pmatrix} 1 \\ \tilde{u} \\ \tilde{H} \end{pmatrix}$$

The vectors $\tilde{Q}$ and $\tilde{E}$ can be expressed as quadratic functions of the variable $\tilde{Z}$:

$$\tilde{Q} = \begin{pmatrix} \tilde{z}_1^2 \\ \\ \tilde{z}_1 \tilde{z}_2 \\ \\ \dfrac{\tilde{z}_1 \tilde{z}_3}{\gamma} + \dfrac{\gamma - 1}{2\gamma} \tilde{z}_2^2 \end{pmatrix}, \qquad \tilde{E} = \begin{pmatrix} \tilde{z}_1 \tilde{z}_2 \\ \\ \dfrac{\gamma - 1}{\gamma} \tilde{z}_1 \tilde{z}_3 + \dfrac{\gamma + 1}{2\gamma} \tilde{z}_2^2 \\ \\ \tilde{z}_2 \tilde{z}_3 \end{pmatrix} \tag{56}$$

Define:

$$\bar{\tilde{x}} \triangleq \frac{1}{2} \left(\tilde{x}_L + \tilde{x}_R \right) \tag{57}$$

Applying the above formula results in:

$$\left\{ \begin{array}{lcl} \tilde{Q}_R - \tilde{Q}_L & = & \tilde{B} \left(\tilde{z}_R - \tilde{z}_L \right) \\ \\ \tilde{E}_R - \tilde{E}_L & = & \tilde{C} \left(\tilde{z}_R - \tilde{z}_L \right) \end{array} \right. \Rightarrow \quad \tilde{E}_R - \tilde{E}_L = \tilde{C} \tilde{B}^{-1} \left(\tilde{Q}_R - \tilde{Q}_L \right) \tag{58}$$

Where the matrices $\tilde{B}$ and $\tilde{C}$:

$$\tilde{B} = \begin{pmatrix} 2\bar{\tilde{z}}_1 & 0 & 0 \\ \\ \bar{\tilde{z}}_2 & \bar{\tilde{z}}_1 & 0 \\ \\ \dfrac{\bar{\tilde{z}}_3}{\gamma} & \dfrac{\gamma - 1}{\gamma} \bar{\tilde{z}}_2 & \dfrac{\bar{\tilde{z}}_1}{\gamma} \end{pmatrix}, \qquad \tilde{C} = \begin{pmatrix} \bar{\tilde{z}}_2 & \bar{\tilde{z}}_1 & 0 \\ \\ \dfrac{\gamma - 1}{\gamma} \bar{\tilde{z}}_3 & \dfrac{\gamma + 1}{\gamma} \bar{\tilde{z}}_2 & \dfrac{\gamma - 1}{\gamma} \bar{\tilde{z}}_1 \\ \\ 0 & \bar{\tilde{z}}_3 & \bar{\tilde{z}}_2 \end{pmatrix} \tag{59}$$

The matrix $\hat{\tilde{A}} = \tilde{C} \tilde{B}^{-1}$ is identical to the matrix $\tilde{A}$ if the original variables $\left(\tilde{\rho},\ \tilde{u},\ \text{and } \tilde{H} \right)$ are replaced by an average weighted by the square root of the density, namely:

$$\left\{ \begin{array}{lclcl} \hat{\tilde{\rho}}_{1+\frac{1}{2}} & = & \sqrt{\tilde{\rho}_L \tilde{\rho}_R} & = & \tilde{\mathcal{R}}_{i+\frac{1}{2}} \tilde{\rho}_L \\ \\ \hat{\tilde{u}}_{i+\frac{1}{2}} & = & \dfrac{\sqrt{\tilde{\rho}_L} \tilde{u}_L + \sqrt{\tilde{\rho}_R} \tilde{u}_R}{\sqrt{\tilde{\rho}_L} + \sqrt{\tilde{\rho}_R}} & = & \dfrac{\tilde{u}_L + \tilde{\mathcal{R}}_{i+\frac{1}{2}} \tilde{u}_R}{1 + \tilde{\mathcal{R}}_{i+\frac{1}{2}}} \\ \\ \hat{\tilde{H}}_{i+\frac{1}{2}} & = & \dfrac{\sqrt{\tilde{\rho}_L} \tilde{H}_L + \sqrt{\tilde{\rho}_R} \tilde{H}_R}{\sqrt{\tilde{\rho}_L} + \sqrt{\tilde{\rho}_R}} & = & \dfrac{\tilde{H}_L + \tilde{\mathcal{R}}_{i+\frac{1}{2}} \tilde{H}_R}{1 + \tilde{\mathcal{R}}_{i+\frac{1}{2}}} \end{array} \right. \tag{60}$$

Where:

$$\tilde{\mathcal{R}}_{i+\frac{1}{2}} = \sqrt{\frac{\tilde{\rho}_R}{\tilde{\rho}_L}}$$

4.4.2 Roe's Average Matrix

The Roe average matrix $\hat{\tilde{A}}$ is therefore given by:

$$\hat{\tilde{A}}_{i+\frac{1}{2}} = \begin{pmatrix} 0 & 1 & 0 \\ \\ \dfrac{\gamma - 3}{2} \hat{\tilde{u}}^2 & (3 - \gamma) \hat{\tilde{u}} & \gamma - 1 \\ \\ \dfrac{1}{2} (\gamma - 1) \hat{\tilde{u}}^3 - \hat{\tilde{u}} \hat{\tilde{H}} & \hat{\tilde{H}} - (\gamma - 1) \hat{\tilde{u}}^2 & \gamma \hat{\tilde{u}} \end{pmatrix} \tag{61}$$

The matrices $\hat{\tilde{T}}$, $\hat{\tilde{\Lambda}}$, and $\hat{\tilde{T}}^{-1}$ are obtained in the same manner as in Eq.24:

$$\hat{\tilde{T}} = \begin{pmatrix} 1 & \frac{\hat{\tilde{\rho}}}{2\hat{\tilde{a}}} & -\frac{\hat{\tilde{\rho}}}{2\hat{\tilde{a}}} \\ \hat{\tilde{u}} & \frac{\hat{\tilde{\rho}}}{2\hat{\tilde{a}}}\left(\hat{\tilde{u}}+\hat{\tilde{a}}\right) & -\frac{\hat{\tilde{\rho}}}{2\hat{\tilde{a}}}\left(\hat{\tilde{u}}-\hat{\tilde{a}}\right) \\ \frac{\hat{\tilde{u}}^2}{2} & \frac{\hat{\tilde{\rho}}}{2\hat{\tilde{a}}}\left(\hat{\tilde{H}}+\hat{\tilde{u}}\hat{\tilde{a}}\right) & -\frac{\hat{\tilde{\rho}}}{2\hat{\tilde{a}}}\left(\hat{\tilde{H}}-\hat{\tilde{u}}\hat{\tilde{a}}\right) \end{pmatrix}$$

$$\hat{\tilde{\Lambda}} = \begin{pmatrix} \hat{\tilde{u}} & 0 & 0 \\ 0 & \hat{\tilde{u}}+\hat{\tilde{a}} & 0 \\ 0 & 0 & \hat{\tilde{u}}-\hat{\tilde{a}} \end{pmatrix} \tag{62}$$

$$\hat{\tilde{T}}^{-1} = \begin{pmatrix} 1-\frac{\gamma-1}{2}\frac{\hat{\tilde{u}}^2}{\hat{\tilde{a}}^2} & (\gamma-1)\frac{\hat{\tilde{u}}^2}{\hat{\tilde{a}}^2} & -\frac{\gamma-1}{\hat{\tilde{a}}^2} \\ \frac{1}{\hat{\tilde{\rho}}\hat{\tilde{a}}}\left((\gamma-1)\hat{\tilde{u}}^2-\hat{\tilde{u}}\hat{\tilde{a}}\right) & \frac{1}{\hat{\tilde{\rho}}\hat{\tilde{a}}}\left(\hat{\tilde{a}}-(\gamma-1)\hat{\tilde{u}}\right) & \frac{\gamma-1}{\hat{\tilde{\rho}}\hat{\tilde{a}}} \\ -\frac{1}{\hat{\tilde{\rho}}\hat{\tilde{a}}}\left((\gamma-1)\hat{\tilde{u}}^2+\hat{\tilde{u}}\hat{\tilde{a}}\right) & \frac{1}{\hat{\tilde{\rho}}\hat{\tilde{a}}}\left(\hat{\tilde{a}}+(\gamma-1)\hat{\tilde{u}}\right) & -\frac{\gamma-1}{\hat{\tilde{\rho}}\hat{\tilde{a}}} \end{pmatrix}$$

Where:

$$\hat{\tilde{a}} = \sqrt{(\gamma-1)\left(\hat{\tilde{H}}-\frac{1}{2}\hat{\tilde{u}}^2\right)}$$

4.4.3 Entropy Fix

The formulation of the Roe's scheme admits an expansion shock as a perfectly appropriate solution of the approximate problem. As a consequence, stationary expansion shocks are not dissipated by Roe's scheme. An appropriate entropy fix, but one that does not distinguish between shocks and expansions, is obtained by replacing the eigenvalues by:

$$\left|\hat{\tilde{\lambda}}_{i+\frac{1}{2}}\right| \rightarrow \beta\left(\hat{\tilde{\lambda}}_{i+\frac{1}{2}}\right) = \begin{cases} \left|\hat{\tilde{\lambda}}_{i+\frac{1}{2}}\right| & \left|\hat{\tilde{\lambda}}_{i+\frac{1}{2}}\right| \geq \varepsilon \\ \sqrt{\hat{\tilde{\lambda}}^2_{i+\frac{1}{2}}+\varepsilon^2} & \left|\hat{\tilde{\lambda}}_{i+\frac{1}{2}}\right| < \varepsilon \end{cases} \tag{63}$$

4.4.4 The Scheme

A first order (in space), finite volume scheme is easily realized by the following steps:

- Let the residual be defined as: $\tilde{\mathfrak{R}}_i^n = -\frac{1}{\Delta\tilde{x}}\left(\tilde{\tilde{E}}^n_{i+\frac{1}{2}} - \tilde{\tilde{E}}^n_{i-\frac{1}{2}} - \frac{M_\infty}{Re_{L\infty}}\left(\tilde{V}^n_{1,i+\frac{1}{2}} - \tilde{V}^n_{1,i-\frac{1}{2}}\right)\right)$
- The numerical flux is: $\tilde{\tilde{E}}_{i+\frac{1}{2}} = \frac{1}{2}\left(\tilde{E}_i+\tilde{E}_{i+1}\right) - \frac{1}{2}\hat{\tilde{T}}_{i+\frac{1}{2}}\left|\hat{\tilde{\Lambda}}_{i+\frac{1}{2}}\right|\hat{\tilde{T}}^{-1}_{i+\frac{1}{2}}\left(\tilde{Q}_{i+1}-\tilde{Q}_i\right)$
- A first order (in time) explicit scheme is given by: $\Delta\tilde{Q}_i^n = \Delta\tilde{t}\cdot\tilde{\mathfrak{R}}_i^n$

5 The Results

5.1 FVS – First Order Explicit Steger-Warming

5.2 FVS – First Order Implicit Steger-Warming

5.3 FDS – First Order Explicit Roe