

Dynamics and Stability of Multiphase Flows
HW4

Almog Dobrescu

ID 214254252

February 1, 2025



Contents

1	Velocity Profile $U_{(z)}$ as a Function of z	2
2	ODE of Jet Radius $r_{(z)}$	3
2.1	Formulate the ODE	3
2.2	Boundary Conditions	3
3	Numerical Solver	4
4	Solution	5
5	Compering The Results	7
6	Oscillatory Behavior	9
A	The Code	10

List of Figures

1	The jet shape	5
2	R_1 as a function of z	5
3	R_1 as a function of z - zoomed	6
4	R_1 as a function of z when $R_2 \rightarrow \infty$	7
5	R_1 as a function of z when $We \rightarrow \infty$	8



1 Velocity Profile $U_{(z)}$ as a Function of z

Consider a liquid jet ejected from a circular orifice of radius a (point A). The jet has flux Q of fluid with density ρ that exits with velocity U_0 . The surface tension between the fluid and the air with pressure P_0 is σ .

By assuming laminar, inviscid, incompressible, axisymmetric and steady flow, we get the Bernoulli equation, between point A and arbitrary point downstream B:

$$\frac{1}{2}\rho U_0^2 + \rho g z + P_A = \frac{1}{2}\rho U_{(z)}^2 + P_B \quad (1)$$

The Laplace pressure for a jet:

$$\Delta P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad \left\{ \begin{array}{l} R_1 = R_{1(z)} = \text{inner radius (normal to the flow)} \\ R_2 = R_{2(z)} = \text{outer radius (parallel to the flow)} \end{array} \right. \quad (2)$$

$$\Downarrow$$

$$P_A = P_0 + \sigma \left(\frac{1}{a} \right) \quad P_B = P_0 + \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

substituting into the Bernoulli equation:

$$\frac{1}{2}\rho U_0^2 + \rho g z + P_0 + \frac{\sigma}{a} = \frac{1}{2}\rho U_{(z)}^2 + P_0 + \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad (3)$$

Isolating $U_{(z)}$ we get:

$$\frac{U_{(z)}}{U_0} = \sqrt{1 + \frac{2gz}{U_0^2} + \frac{2\sigma}{\rho U_0^2} \left(\frac{1}{a} - \frac{1}{R_1} - \frac{1}{R_2} \right)} \quad (4)$$

and in dimensionless form:

$$\frac{U_{(z)}}{U_0} = \sqrt{1 + \underbrace{\frac{2z}{Fr^2 a}}_{\text{acceleration due to gravity}} + \underbrace{\frac{2}{We} \left(1 - \frac{a}{R_1} - \frac{a}{R_2} \right)}_{\text{deceleration due to surface tension}}} \quad (5)$$

where:

- $We = \frac{\rho U_0^2 a}{\sigma}$ is the Webber #
- $Fr = \frac{U_0^2}{ga}$ is the Froude #

Let's consider the curvature of the wall of the jet parallel to the flow:

$$\underbrace{\kappa}_{\text{principal curvature}} = -\frac{R_{1zz}}{(1 + R_{1z}^2)^{\frac{3}{2}}} \quad \underbrace{=}_{\text{linearization}} -R_{1zz} \quad (6)$$

The principal curvature equals one over the principal radius so:

$$\kappa = \frac{1}{R_2} = -\frac{\partial^2 R_1}{\partial z^2} \quad (7)$$

So, the velocity profile $U_{(z)}$ as a function of z is therefore:

$$\frac{U_{(z)}}{U_0} = \sqrt{1 + \frac{2z}{Fr^2 a} + \frac{2}{We} \left(1 - \frac{a}{R_1} + a \frac{\partial^2 R_1}{\partial z^2} \right)} \quad (8)$$



2 ODE of Jet Radius $r_{(z)}$

2.1 Formulate the ODE

From conservation of flux:

$$Q = 2\pi \int_0^{R_1} U_{(z)} r dr = \pi R_1^2 U_{(z)} \underbrace{=}_{z=0} \pi a^2 U_0$$

$$Q_{(z)} = \pi U_{(z)} R_1^2 \underbrace{=}_{\substack{\text{conservation} \\ \text{of mass}}} \pi a^2 U_0 \quad (9)$$

$$\Downarrow$$

$$\frac{R_1}{a} = \sqrt{\frac{U_0}{U_{(z)}}}$$

After substituting $U_{(z)}$:

$$R_1 = \frac{a^2 U_0}{\sqrt{U_0^2 + 2gz + \frac{2\sigma}{\rho} \left(\frac{1}{a} - \frac{1}{R_1} + \frac{\partial^2 R_1}{\partial z^2} \right)}} \quad (10)$$

2.2 Boundary Conditions

Assume BC:

1. $R_{1(z=0)} = a$
2. $\left. \frac{\partial R_1}{\partial z} \right|_{z=0} = 0$

The first conditions is given.

The second condition is derived from the fact that at the begining, the jet looks like a cylinder.

3 Numerical Solver

The selected method is the Runge-Kutta-Merson method.

Isolating the second derivative of R_1 :

$$\begin{aligned}\frac{a^4}{R_1^4} &= 1 + \frac{2z}{\text{Fr}^2 a} + \frac{2}{\text{We}} \left(1 - \frac{a}{R_1} + a \frac{\partial^2 R_1}{\partial z^2} \right) \\ \frac{\text{We} \cdot a^4}{2R_1^4} &= \frac{\text{We}}{2} + \frac{\text{We}z}{\text{Fr}^2 a} + 1 - \frac{a}{R_1} + a \frac{\partial^2 R_1}{\partial z^2} \\ \frac{\partial^2 R_1}{\partial z^2} &= \frac{\text{We} \cdot a^3}{2R_1^4} - \frac{\text{We}}{2a} - \frac{\text{We} \cdot z}{\text{Fr}^2 a^2} - \frac{1}{a} + \frac{1}{R_1}\end{aligned}\tag{11}$$

Defining the system of equations:

$$x = R_1 \quad y = \frac{\partial R_1}{\partial z}\tag{12}$$

$$\begin{cases} \frac{\partial x}{\partial z} = g(z, x, y) \\ \frac{\partial y}{\partial z} = f(z, x, y) \end{cases} \quad \text{where} \quad \begin{cases} g = y \\ f = \frac{\text{We} \cdot a^3}{2x^4} - \frac{\text{We}}{2a} - \frac{\text{We} \cdot z}{\text{Fr}^2 a^2} - \frac{1}{a} + \frac{1}{x} \end{cases}\tag{13}$$

The Runge-Kutta-Merson method is defined as:

$$\begin{aligned}\begin{cases} x_{i+1} &= x_i + \frac{1}{6} (m_1 + 4m_4 + m_5) \\ y_{i+1} &= y_i + \frac{1}{6} (k_1 + 4k_4 + k_5) \end{cases} \\ \begin{cases} m_1 &= hg(z_i, x_i, y_i) \\ k_1 &= hf(z_i, x_i, y_i) \end{cases} \\ \begin{cases} m_2 &= hg(z_i + \frac{1}{3}h, x_i + \frac{1}{3}m_1, y_i + \frac{1}{3}k_1) \\ k_2 &= hf(z_i + \frac{1}{3}h, x_i + \frac{1}{3}m_1, y_i + \frac{1}{3}k_1) \end{cases} \\ \begin{cases} m_3 &= hg(z_i + \frac{1}{3}h, x_i + \frac{1}{6}(m_1+m_2), y_i + \frac{1}{6}(k_1+k_2)) \\ k_3 &= hf(z_i + \frac{1}{3}h, x_i + \frac{1}{6}(m_1+m_2), y_i + \frac{1}{6}(k_1+k_2)) \end{cases} \\ \begin{cases} m_4 &= hg(z_i + \frac{1}{2}h, x_i + \frac{1}{8}(m_1+3m_3), y_i + \frac{1}{8}(k_1+3k_3)) \\ k_4 &= hf(z_i + \frac{1}{2}h, x_i + \frac{1}{8}(m_1+3m_3), y_i + \frac{1}{8}(k_1+3k_3)) \end{cases} \\ \begin{cases} m_5 &= hg(z_i + h, x_i + \frac{1}{2}(m_1-3m_3+4m_4), y_i + \frac{1}{2}(k_1-3k_3+4k_4)) \\ k_5 &= hf(z_i + h, x_i + \frac{1}{2}(m_1-3m_3+4m_4), y_i + \frac{1}{2}(k_1-3k_3+4k_4)) \end{cases}\end{aligned}\tag{14}$$

4 Solution

$$\begin{aligned}
 U_0 &= 1 \left[\frac{\text{m}}{\text{sec}} \right] & a &= 0.05 \text{ [m]} & \rho &= 998 \left[\frac{\text{kg}}{\text{m}^3} \right] & \sigma &= 0.0728 \left[\frac{\text{N}}{\text{m}} \right] & g &= 9.81 \left[\frac{\text{m}}{\text{sec}^2} \right] \\
 h &= 10^{-5} \text{ [m]} & z &= [0, h, 2h, \dots, 5] \text{ [m]}
 \end{aligned} \tag{15}$$

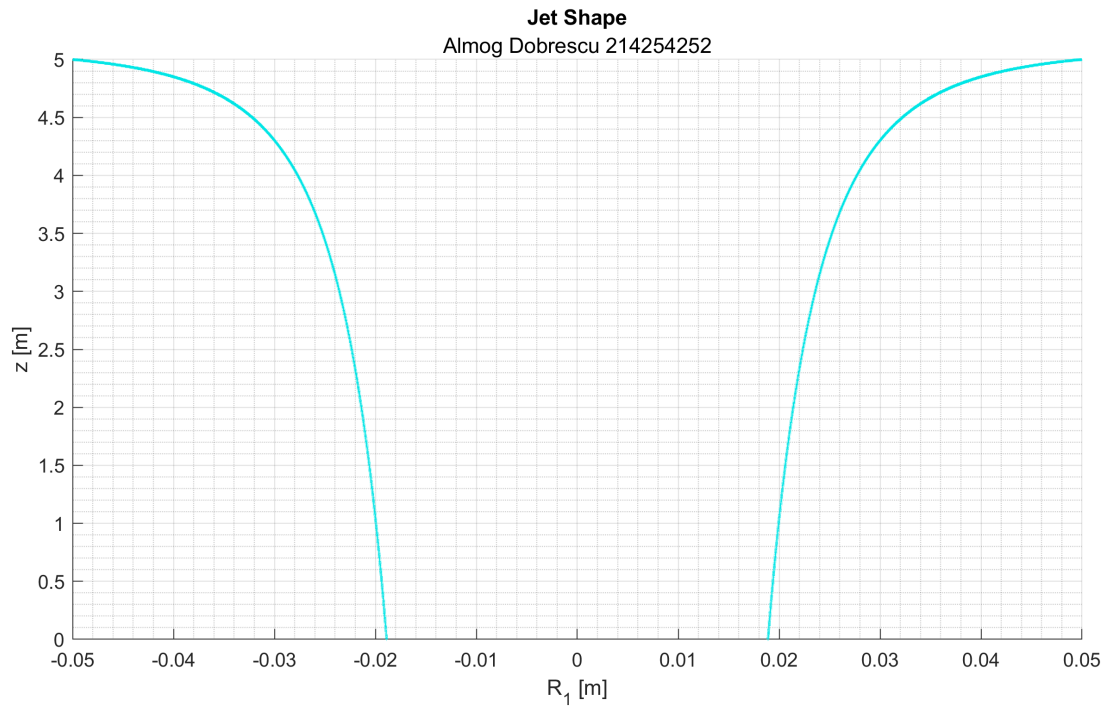
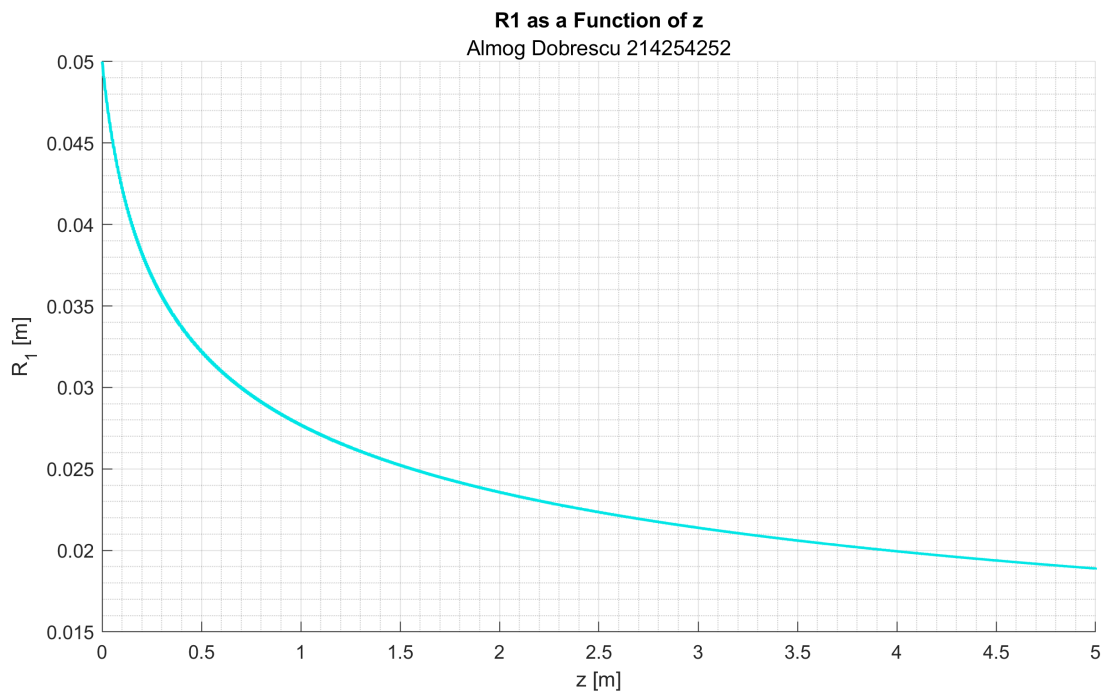
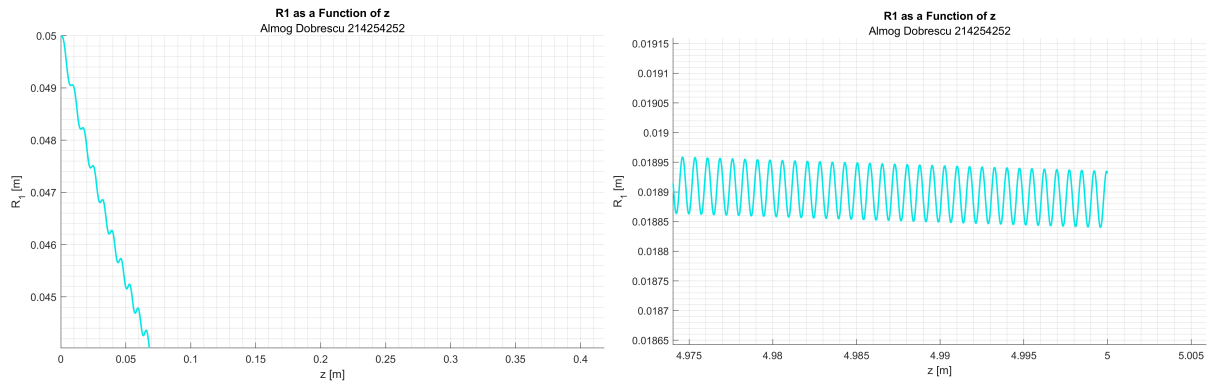


Figure 1: The jet shape

Figure 2: R_1 as a function of z

(a) R_1 as a function of z - zoomed on begining(b) R_1 as a function of z - zoomed on endFigure 3: R_1 as a functon of z - zoomed

5 Compering The Results

One can that the principal radius parallel to the flow is much larger then the principal radius normal to the flow, which means $R_2 \rightarrow \infty$. The derived equation for the radius is therefore:

$$\begin{aligned}
 \frac{a^2}{R_1^2} &= \sqrt{1 + \frac{2z}{Fr^2 a} + \frac{2}{We} \left(1 - \frac{a}{R_1}\right)} \\
 \frac{a^4}{R_1^4} &= 1 + \frac{2z}{Fr^2 a} + \frac{2}{We} - \frac{2}{We} \frac{a}{R_1} \\
 a^4 &= R^4 + \frac{2zR_1^4}{Fr^2 a} + \frac{2R_1^4}{We} - \frac{2aR_1^3}{We} \\
 0 &= \left(1 + \frac{2z}{Fr^2 a} + \frac{2}{We}\right) R_1^4 - \frac{2aR_1^3}{We} - a^4
 \end{aligned} \tag{16}$$

Using the *roots* function in *MatLab* we can claculate the radius R_1 as a function of z

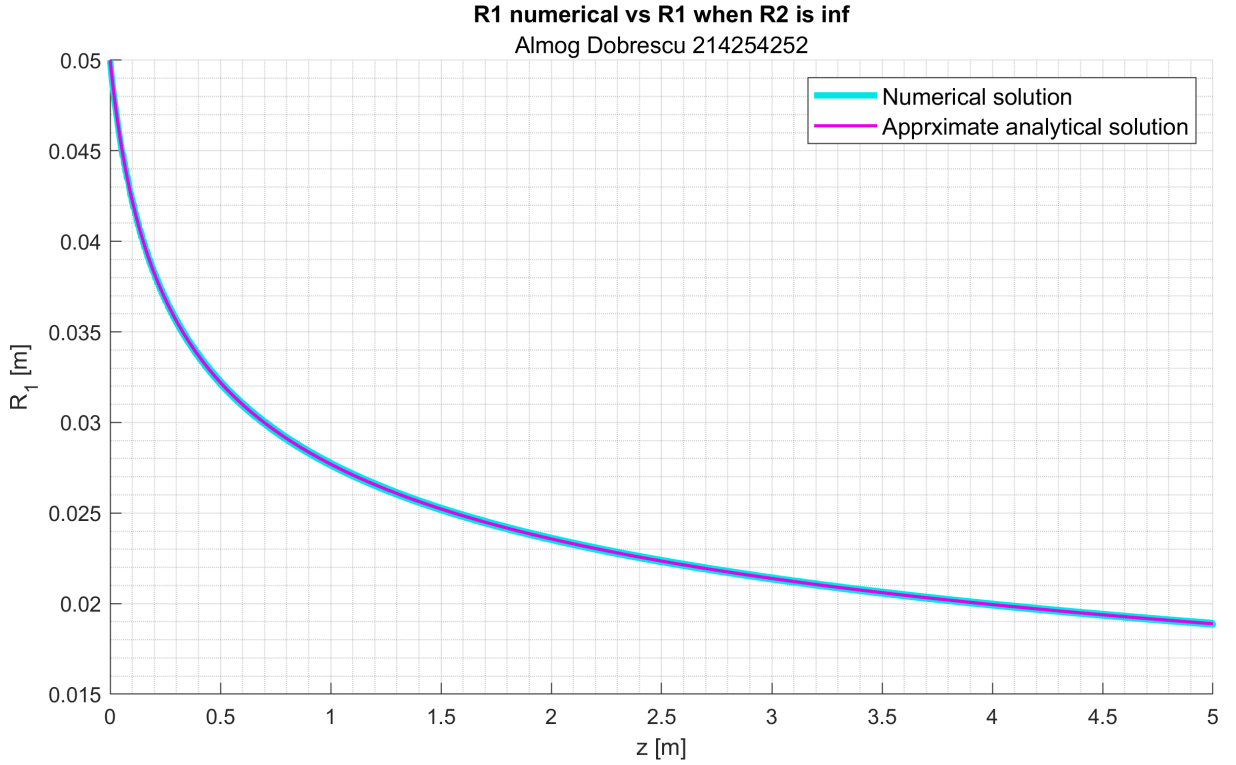


Figure 4: R_1 as a function of z when $R_2 \rightarrow \infty$

We can see that the approximate analytical solution follows closely the numerical solution.

Another assumption one can make is to assume that the Webber number is large so $\frac{1}{We} \rightarrow 0$

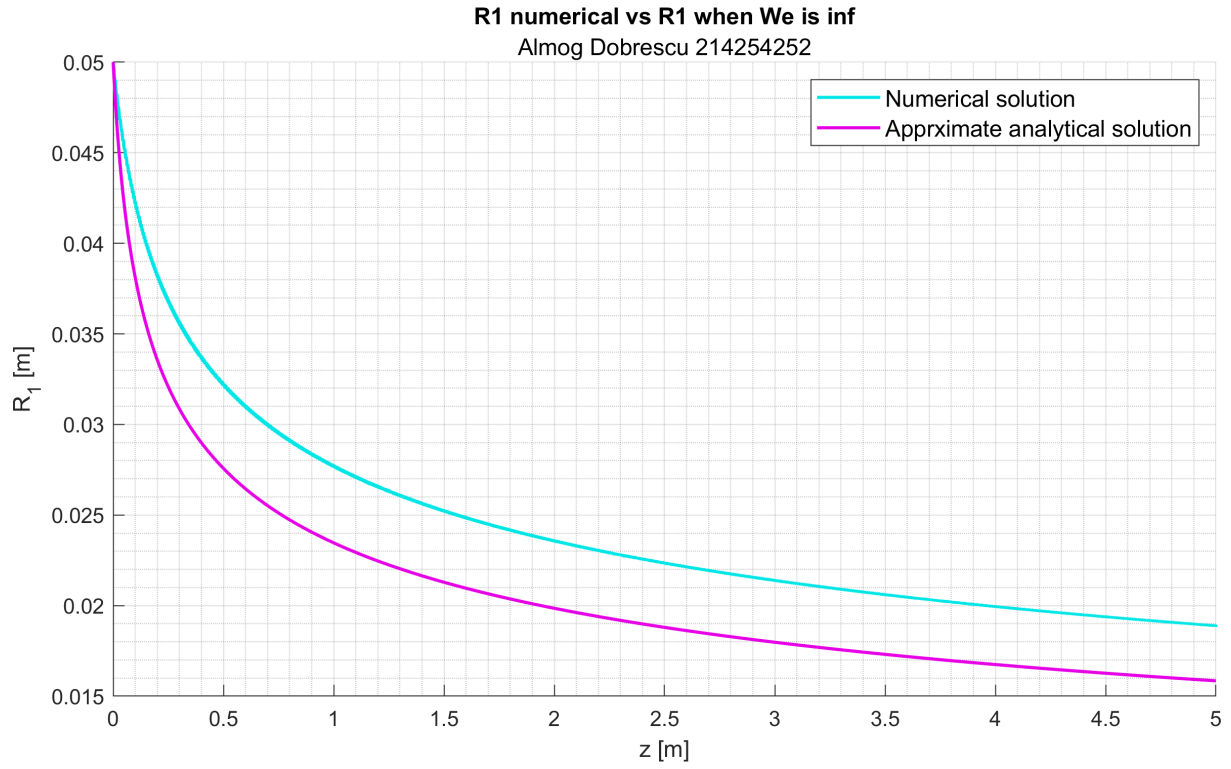


Figure 5: R_1 as a function of z when $We \rightarrow \infty$

We can see that this assumption is not good and falls apart when z (U) is large enough.



6 Oscillatory Behavior

We can see oscillations around the solution. The oscillations occur no matter the step size or the length of the solution. I think that the oscillations are not a result of the numerical method but they are a part of the actual solution. They describe the balance between the acceleration due to gravity and the deceleration due to surface tension. Moreover, the oscillations don't occur when the principal radius parallel to the flow is neglected, which strengthens the fact that the oscillations are the results of the balance between the curvature and gravity.

A The Code

```

1  clc; clear; close all;
2
3  U0 = 1; % [m/sec]
4  a = 0.05; % [m]
5  rho = 998; % [kg/m^3]
6  sigma = 0.0728; % [N/m]
7  g = 9.81; % [m/sec^2]
8  z_max = 5; % [m]
9
10 We = rho * U0^2 * a / sigma;
11 Fr = U0^2 / g / a;
12
13 h = 1e-5; % [m]
14
15 zs_temp = 0:h:z_max;
16 for i = 1:length(zs_temp)
17     zs(i,1) = zs_temp(i);
18 end
19 xs = zeros(length(zs), 1);
20 ys = zeros(length(zs), 1);
21
22 xs(1) = a;
23 ys(1) = 0;
24
25 for i = 1:length(zs)-1
26     m1 = h * calc_g(zs(i), xs(i), ys(i));
27     k1 = h * calc_f(zs(i), xs(i), ys(i), We, Fr, a);
28     m2 = h * calc_g(zs(i) + 1/3*h, xs(i) + 1/3*m1, ys(i) + 1/3*k1
29         );
30     k2 = h * calc_f(zs(i) + 1/3*h, xs(i) + 1/3*m1, ys(i) + 1/3*k1
31         , We, Fr, a);
32     m3 = h * calc_g(zs(i) + 1/3*h, xs(i) + 1/6*(m1+m2), ys(i) +
33         1/6*(k1+k2));
34     k3 = h * calc_f(zs(i) + 1/3*h, xs(i) + 1/6*(m1+m2), ys(i) +
35         1/6*(k1+k2), We, Fr, a);
36     m4 = h * calc_g(zs(i) + 1/2*h, xs(i) + 1/8*(m1+3*m3), ys(i) +
37         1/8*(k1+3*k3));
38     k4 = h * calc_f(zs(i) + 1/2*h, xs(i) + 1/8*(m1+3*m3), ys(i) +
39         1/8*(k1+3*k3), We, Fr, a);
40     m5 = h * calc_g(zs(i) + h, xs(i) + 1/2*(m1-3*m3+4*m4), ys(i)
41         + 1/2*(k1-3*k3+4*k4));
42     k5 = h * calc_f(zs(i) + h, xs(i) + 1/2*(m1-3*m3+4*m4), ys(i)
43         + 1/2*(k1-3*k3+4*k4), We, Fr, a);
44
45     xs(i+1) = xs(i) + 1/6*(m1 + 4*m4 + m5);
46     ys(i+1) = ys(i) + 1/6*(k1 + 4*k4 + k5);
47 end
48
49 R1s = xs;

```



```

42
43 %% Q4
44 fig1 = figure ("Name","Jet Shape",'Position',[100 300 900 500]);
45 colors = cool(4)*0.9;
46 hold all
47
48 plot(R1s, z_max-zs, -R1s, z_max-zs, "--", "LineWidth", 1.5, "Color
    ", colors(1,:))
49
50 xlabel('R_1 [m]')
51 ylabel('z [m]')
52 grid on
53 grid minor
54 title("Jet Shape")
55 subtitle("Almog Dobrescu 214254252")
56 % legend({},'FontSize',11,'Location','northwest')
57 % exportgraphics(fig1, 'graph1.png','Resolution',300);
58
59 fig2 = figure ("Name","R1 as a Function of z",'Position',[250 300
    900 500]);
60 colors = cool(4)*0.9;
61 hold all
62
63 plot(zs, R1s, "--", "LineWidth", 1.5, "Color", colors(1,:))
64
65 xlabel('z [m]')
66 ylabel('R_1 [m]')
67 grid on
68 grid minor
69 title("R1 as a Function of z")
70 subtitle("Almog Dobrescu 214254252")
71 % legend({},'FontSize',11,'Location','northwest')
72 % exportgraphics(fig2, 'graph2.png','Resolution',300);
73
74 %% Q5
75
76 R1s_when_R2_is_inf = zeros(length(zs), 1);
77 for i = 1:length(zs)
78     radiuses = roots([1+2*zs(i)/Fr^2/a+2/We, -2*a/We, 0, 0, -a
        ^4]);
79     for index = 1:length(radiuses)
80         if (isreal(radiuses(index)) && 0<radiuses(index))
81             R1s_when_R2_is_inf(i,1) = radiuses(index);
82         end
83     end
84 end
85
86 fig3 = figure ("Name","R1 numerical vs R1 when R2 is inf",'
    Position',[400 300 900 500]);
87 colors = cool(4)*0.9;
88 hold all

```



```

89
90 plot(zs, R1s, "-", "LineWidth", 3, "Color", colors(1,:))
91 plot(zs, R1s_when_R2_is_inf, "-", "LineWidth", 1.5, "Color",
    colors(4,:))
92
93 xlabel('z [m]')
94 ylabel('R_1 [m]')
95 grid on
96 grid minor
97 title("R1 numerical vs R1 when R2 is inf")
98 subtitle("Almog Dobrescu 214254252")
99 legend({'Numerical solution','Apprximate analytical solution'},'
    FontSize',11,'Location','northeast')
100 % exportgraphics(fig3, 'graph3.png','Resolution',300);
101
102 fig4 = figure ("Name","R1 numerical vs R1 when We is inf",'
    Position',[550 300 900 500]);
103 colors = cool(4)*0.9;
104 hold all
105
106 plot(zs, R1s, "-", "LineWidth", 1.5, "Color", colors(1,:))
107 plot(zs, a.*(1+2.*g.*zs./U0^2).^(-1/4), "-", "LineWidth", 1.5, "
    Color", colors(4,:))
108
109 xlabel('z [m]')
110 ylabel('R_1 [m]')
111 grid on
112 grid minor
113 title("R1 numerical vs R1 when We is inf")
114 subtitle("Almog Dobrescu 214254252")
115 legend({'Numerical solution','Apprximate analytical solution'},'
    FontSize',11,'Location','northeast')
116 % exportgraphics(fig4, 'graph4.png','Resolution',300);
117
118 %% functions
119 function g = calc_g(z, x, y)
120     g = y;
121 end
122
123 function f = calc_f(z, x, y, We, Fr, a)
124     f = We*a^3/2/x^4 - We/2/a - We*z/Fr^2/a^2 - 1/a + 1/x;
125 end

```