Dynamics and Stability of Multiphase Flows
HW4

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1 Velocity Profile $U_{(z)}$ as a Funciton of z

Consider a liquid jet ejected from a circular orifice of radius a (point A). The jet has flux Q of fluid with density ρ that exits with velocity U_0. The surface tension between the fluid and the air with pressure P_0 is σ.
By assuming laminar, inviscid, incpmressible, axisymmetric and steady flow, we get the Bernoulli equation, between point A and arbitrary point downstream B:

$$\frac{1}{2}\rho U_0^2 + \rho g z + P_A = \frac{1}{2}\rho U_{(z)}^2 + P_B \tag{1}$$

The Laplace pressure for a jet:

$$\Delta P = \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \qquad \begin{cases} R_1 = R_{1(z)} = \text{inner radius (normal to the flow)} \\ R_2 = R_{2(z)} = \text{outer radius (parallel to the flow)} \end{cases}$$
$$\Downarrow$$
$$P_A = P_0 + \sigma\left(\frac{1}{a}\right) \qquad P_B = P_0 + \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \tag{2}$$

substituting into the Bernoulli equation:

$$\frac{1}{2}\rho U_0^2 + \rho g z + \cancel{P_0} + \frac{\sigma}{a} = \frac{1}{2}\rho U_{(z)}^2 + \cancel{P_0} + \sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \tag{3}$$

Isolating $U_{(z)}$ we get:

$$\frac{U_{(z)}}{U_0} = \sqrt{1 + \frac{2gz}{U_0^2} + \frac{2\sigma}{\rho U_0^2}\left(\frac{1}{a} - \frac{1}{R_1} - \frac{1}{R_2}\right)} \tag{4}$$

and in dimensionless form:

$$\frac{U_{(z)}}{U_0} = \sqrt{1 + \underbrace{\frac{2z}{Fr^2 a}}_{\substack{\text{acceleration}\\ \text{due to gravity}}} + \underbrace{\frac{2}{We}\left(1 - \frac{a}{R_1} - \frac{a}{R_2}\right)}_{\substack{\text{deceleration}\\ \text{due to surface tension}}}} \tag{5}$$

where:

- $We = \dfrac{\rho U_0^2 a}{\sigma}$ is the Webber #
- $Fr = \dfrac{U_0^2}{ga}$ is the Froud #

Let's consider the curvature of the wall of the get parallel to the flow:

$$\underbrace{\kappa}_{\substack{\text{principal}\\ \text{curvature}}} = -\frac{R_{1zz}}{\left(1+R_{1z}^2\right)^{\frac{3}{2}}} \underbrace{=}_{\text{linearization}} -R_{1zz} \tag{6}$$

The principal curvature equals one over the principal radius so:

$$\kappa = \frac{1}{R_2} = -\frac{\partial^2 R_1}{\partial z^2} \tag{7}$$

So, the velocity profile $U_{(z)}$ as a funciton of z is therefore:

$$\frac{U_{(z)}}{U_0} = \sqrt{1 + \frac{2z}{Fr^2 a} + \frac{2}{We}\left(1 - \frac{a}{R_1} + a\frac{\partial^2 R_1}{\partial z^2}\right)} \tag{8}$$

2 ODE of Jet Radius $r_{(z)}$

2.1 Formulate the ODE

From conservation of flux:

$$Q = 2\pi \int_0^{R_1} U_{(z)} r dr = \pi R_1^2 U_{(z)} \underbrace{=}_{z=0} \pi a^2 U_0$$

$$Q_{(z)} = \pi U_{(z)} R_1^2 \underbrace{=}_{\substack{\text{conservation}\\ \text{of mass}}} \pi a^2 U_0 \tag{9}$$

$$\Downarrow$$

$$\frac{R_1}{a} = \sqrt{\frac{U_0}{U_{(z)}}}$$

After substituting $U_{(z)}$:

$$\boxed{R_1 = \frac{a^2 U_0}{\sqrt{U_0^2 + 2gz + \frac{2\sigma}{\rho}\left(\frac{1}{a} - \frac{1}{R_1} + \frac{\partial^2 R_1}{\partial z^2}\right)}}} \tag{10}$$

2.2 Boundary Conditions

Assume BC:

1. $R_{1(z=0)} = a$
2. $\left.\frac{\partial R_1}{\partial z}\right|_{z=0} = 0$

The first conditions is given.
The second condition is derived from the fact that at the begining, the jet looks like a cylinder.

3 Numerical Solver

The selected method is the Runge-Kutta-Merson method.
Isolating the second derivitive of R_1:

$$\begin{aligned}
\frac{a^4}{R_1^4} &= 1 + \frac{2z}{\mathrm{Fr}^2 a} + \frac{2}{\mathrm{We}}\left(1 - \frac{a}{R_1} + a\frac{\partial^2 R_1}{\partial z^2}\right) \\
\frac{\mathrm{We}\cdot a^4}{2R_1^4} &= \frac{\mathrm{We}}{2} + \frac{\mathrm{We}z}{\mathrm{Fr}^2 a} + 1 - \frac{a}{R_1} + a\frac{\partial^2 R_1}{\partial z^2} \\
\frac{\partial^2 R_1}{\partial z^2} &= \frac{\mathrm{We}\cdot a^3}{2R_1^4} - \frac{\mathrm{We}}{2a} - \frac{\mathrm{We}\cdot z}{\mathrm{Fr}^2 a^2} - \frac{1}{a} + \frac{1}{R_1}
\end{aligned} \tag{11}$$

Defining the system of equations:

$$x = R_1 \qquad y = \frac{\partial R_1}{\partial z} \tag{12}$$

$$\begin{cases} \dfrac{\partial x}{\partial z} = g_{(z,x,y)} \\ \dfrac{\partial y}{\partial z} = f_{(z,x,y)} \end{cases} \quad \text{where} \quad \begin{aligned} g &= y \\ f &= \frac{\mathrm{We}\cdot a^3}{2x^4} - \frac{\mathrm{We}}{2a} - \frac{\mathrm{We}\cdot z}{\mathrm{Fr}^2 a^2} - \frac{1}{a} + \frac{1}{x} \end{aligned} \tag{13}$$

The Runge-Kutta-Merson method is defined as:

$$\begin{aligned}
&\begin{cases} x_{i+1} = x_i + \frac{1}{6}(m_1 + 4m_4 + m_5) \\ y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_4 + k_5) \end{cases} \\
&\begin{cases} m_1 = hg_{(z_i,x_i,y_i)} \\ k_1 = hf_{(z_i,x_i,y_i)} \end{cases} \\
&\begin{cases} m_2 = hg_{\left(z_i+\frac{1}{3}h,x_i+\frac{1}{3}m_1,y_i+\frac{1}{3}k_1\right)} \\ k_2 = hf_{\left(z_i+\frac{1}{3}h,x_i+\frac{1}{3}m_1,y_i+\frac{1}{3}k_1\right)} \end{cases} \\
&\begin{cases} m_3 = hg_{\left(z_i+\frac{1}{3}h,x_i+\frac{1}{6}(m_1+m_2),y_i+\frac{1}{6}(k_1+k_2)\right)} \\ k_3 = hf_{\left(z_i+\frac{1}{3}h,x_i+\frac{1}{6}(m_1+m_2),y_i+\frac{1}{6}(k_1+k_2)\right)} \end{cases} \\
&\begin{cases} m_4 = hg_{\left(z_i+\frac{1}{2}h,x_i+\frac{1}{8}(m_1+3m_3),y_i+\frac{1}{8}(k_1+3k_3)\right)} \\ k_4 = hf_{\left(z_i+\frac{1}{2}h,x_i+\frac{1}{8}(m_1+3m_3),y_i+\frac{1}{8}(k_1+3k_3)\right)} \end{cases} \\
&\begin{cases} m_5 = hg_{\left(z_i+h,x_i+\frac{1}{2}(m_1-3m_3+4m_4),y_i+\frac{1}{2}(k_1-3k_3+4k_4)\right)} \\ k_5 = hf_{\left(z_i+h,x_i+\frac{1}{2}(m_1-3m_3+4m_4),y_i+\frac{1}{2}(k_1-3k_3+4k_4)\right)} \end{cases}
\end{aligned} \tag{14}$$

4 Solution

$$U_0 = 1\left[\frac{\mathrm{m}}{\mathrm{sec}}\right] \quad a = 0.05\,[\mathrm{m}] \quad \rho = 998\left[\frac{\mathrm{kg}}{\mathrm{m}^3}\right] \quad \sigma = 0.0728\left[\frac{\mathrm{N}}{\mathrm{m}}\right] \quad g = 9.81\left[\frac{\mathrm{m}}{\mathrm{sec}^2}\right]$$
$$h = 10^{-5}\,[m] \qquad z = [0, h, 2h, \cdots, 5]\,[m] \tag{15}$$

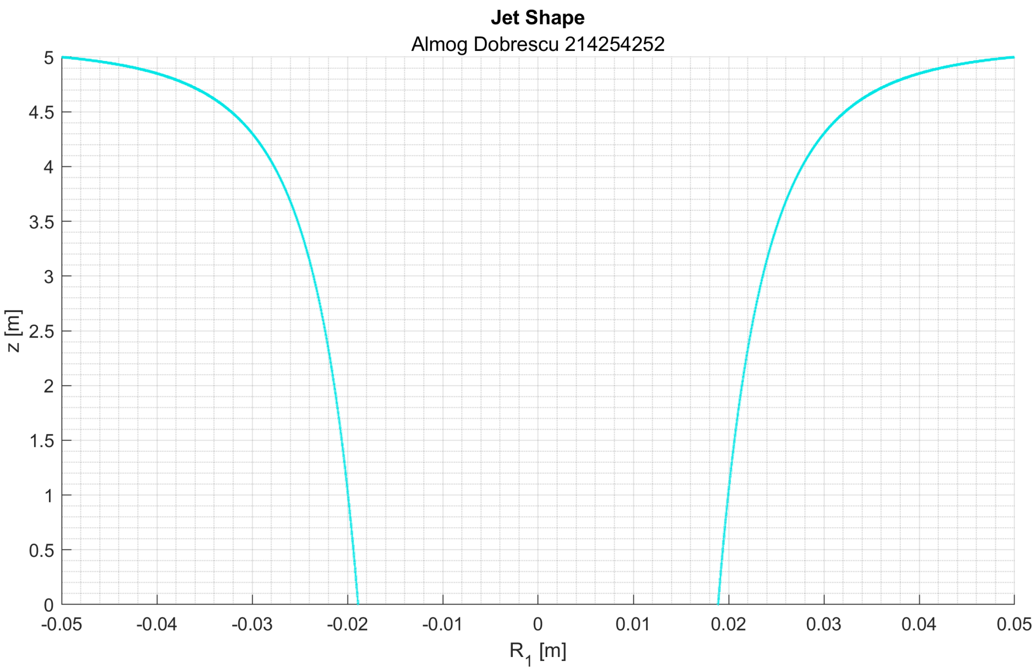

Figure 1: The jet shape

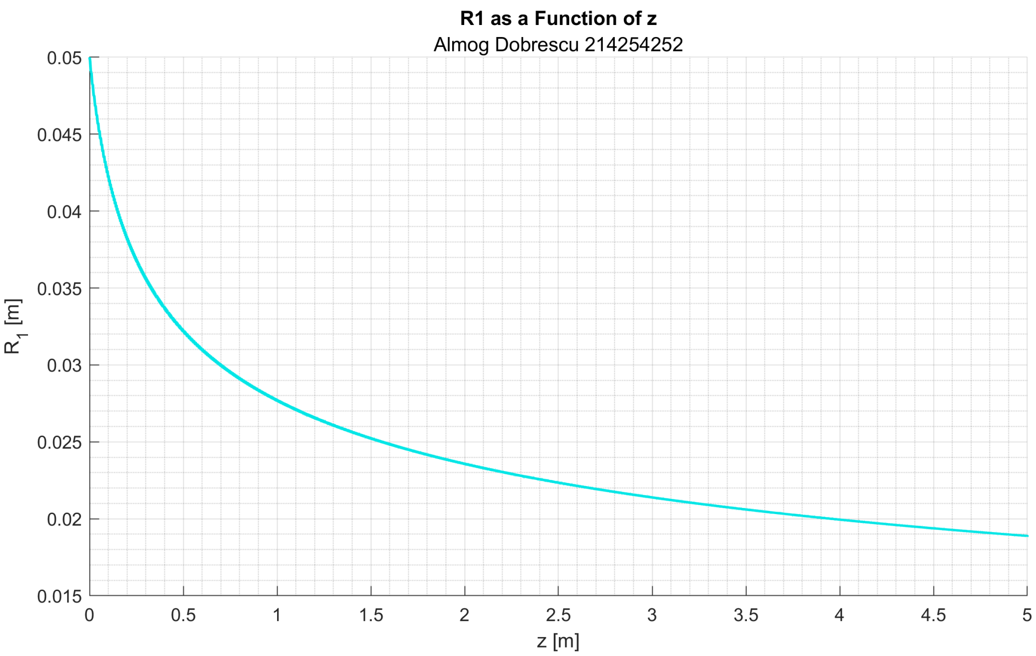

Figure 2: R_1 as a function of z

5 Compering The Results