Intro to Turbulent Flow
HW2

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Contents

List of Figures

Listings

1 Stationarity and Ergodicity

For a random process to be stationary, it needs to fulfill the following conditions:

1. $\mathbb{E}\{x_{k(t)}\} = \text{constant}$
2. $Q_{(t,t+s)} = \mathbb{E}\left\{x_{k(t)}x_{k(t+s)}\right\} = Q_{(s)}$

To determine if a random process is ergodic the following condition needs to be met:

1. $\lim_{T\to\infty} \frac{1}{T}\int_0^T Q_{(\hat{s})}d\hat{s} = 0$

1.1 Random Process #1

$$x_k = a_k \sin(2\pi f t + \theta) \tag{1}$$

Where:

$$a_k \sim U[0,1] \tag{2}$$

1.1.1 Statinarity

The expectation operator for x_k is given by:

$$\begin{aligned}
\mathbb{E}\{x_{k(t)}\} &= \int_0^1 a\sin(2\pi f t+\theta)da \\
&= \left[\frac{1}{2}a^2\sin(2\pi f t+\theta)\right]_0^1 \\
&= \frac{1}{2}\sin(2\pi t f+\theta) \neq \text{const}
\end{aligned} \tag{3}$$

$$\Downarrow$$

$$\boxed{\text{Not stationary}}$$

1.1.2 Ergodicity

The covariance function is given by:

$$\begin{aligned}
Q_{(t,t+s)} &= \mathbb{E}\left\{x_{k(t)}x_{k(t+s)}\right\} \\
&= \mathbb{E}\left\{a_k\sin(2\pi f t+\theta)a_k\sin(2\pi f(t+s)+\theta)\right\} \\
&\Downarrow \ \sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right] \\
&= \mathbb{E}\left\{a_k^2\frac{1}{2}\left[\cos(2\pi f t+\theta-2\pi f(t+s)-\theta)-\cos(2\pi f t+\theta+2\pi f(t+s)+\theta)\right]\right\} \\
&= \int_0^1 a^2\frac{1}{2}\left[\cos(2\pi f s)-\cos(4\pi f t+2\pi f s+2\theta)\right]da \\
&= \int_0^1 a^3\frac{1}{6}\left[\cos(2\pi f s)-\cos(4\pi f t+2\pi f s+2\theta)\right]da \neq Q_{(s)}
\end{aligned} \tag{4}$$

$$\Downarrow$$

$$\boxed{\text{Not ergodic}}$$

1.2 Random Process #2

$$x_k = a\sin(2\pi ft + \theta_k) \tag{5}$$

Where:

$$\theta_k \sim U[0, 2\pi] \tag{6}$$

1.2.1 Statinarity

The expectation operator for x_k is given by:

$$\begin{aligned}
\mathbb{E}\{x_{k(t)}\} &= \int_0^{2\pi} \frac{1}{2\pi}\alpha \sin(2\pi ft + \theta)d\theta \\
&= \left[-\frac{1}{2\pi}\alpha\cos(2\pi ft + \theta)\right]_0^{2\pi} \\
&= -\frac{1}{2\pi}\alpha\cos(2\pi ft + 2\pi) + \frac{1}{2\pi}\alpha\cos(2\pi ft + 0) \\
&= 0 = \text{const}
\end{aligned} \tag{7}$$

The covariance function is given by:

$$\begin{aligned}
Q_{(t,t+s)} &= \mathbb{E}\left\{x_{k(t)}x_{k(t+s)}\right\} \\
Q_{(t,t+s)} &= \mathbb{E}\{a\sin(2\pi ft + \theta_k)a\sin(2\pi f(t+s) + \theta_k)\} \\
&\Downarrow \ \sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \\
&= \mathbb{E}\left\{a^2\frac{1}{2}[\cos(2\pi ft + \theta_k - 2\pi f(t+s) - \theta_k) - \cos(2\pi ft + \theta_k + 2\pi f(t+s) + \theta_k)]\right\} \\
&= \int_0^{2\pi} a^2\frac{1}{4\pi}[\cos(2\pi ft + \theta - 2\pi f(t+s) - \theta) - \cos(2\pi ft + \theta + 2\pi f(t+s) + \theta)]\,d\theta \\
&= \int_0^{2\pi}\left[a^2\frac{1}{4\pi}\cos(2\pi fs) - a^2\frac{1}{4\pi}\cos(4\pi ft + 2\theta + 2\pi fs)\right]d\theta \\
&= \left[a^2\frac{1}{4\pi}\cos(2\pi fs)\,\theta - a^2\frac{1}{8\pi}\sin(4\pi ft + 2\theta + 2\pi fs)\right]_0^{2\pi} \\
&= a^2\frac{1}{2}\cos(2\pi fs) - a^2\frac{1}{8\pi}\sin(4\pi ft + 4\pi + 2\pi fs) + a^2\frac{1}{8\pi}\sin(4\pi ft + 2\pi fs) \\
&= \frac{a^2}{2}\cos(2\pi fs) = Q_{(s)} \qquad \blacksquare
\end{aligned} \tag{8}$$

$$\Downarrow$$
$$\boxed{\text{Stationary}}$$

1.2.2 Ergodicity

To prove an ergodic process we need to check the next limit:

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\int_0^T Q_{(\hat{s})}d\hat{s} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{a^2}{2}\cos\left(2\pi f\hat{s}\right)d\hat{s} \\
&= \lim_{T\to\infty}\frac{1}{T}\left[\frac{a^2}{4\pi f}\sin\left(2\pi f\hat{s}\right)\right]_0^T \\
&= \lim_{T\to\infty}\frac{1}{T}\frac{a^2}{4\pi f}\sin\left(2\pi fT\right) = 0 \qquad \blacksquare \\
&\Downarrow \\
&\boxed{\text{Ergodic}}
\end{aligned} \tag{9}$$

1.3 Random Process #3

$$x_k = a_k\sin\left(2\pi ft+\theta_k\right) \tag{10}$$

Where:

$$a_k \sim U\left[0,1\right] \qquad \theta_k \sim U\left[0,2\pi\right] \tag{11}$$

1.3.1 Statinarity

The expectation operator for x_k is given by:

$$\begin{aligned}
\mathbb{E}\{x_{k(t)}\} &= \int_0^{2\pi}\frac{1}{2\pi}\int_0^1 a\sin\left(2\pi ft+\theta\right)dad\theta \\
&= \int_0^{2\pi}\frac{1}{2\pi}\left[\frac{a^2}{2}\sin\left(2\pi ft+\theta\right)\right]_0^1 d\theta \\
&= \int_0^{2\pi}\frac{1}{4\pi}\sin\left(2\pi ft+\theta\right)d\theta \\
&= \left[-\frac{1}{4\pi}\cos\left(2\pi ft+\theta\right)\right]_0^{2\pi} = 0 = \text{const}
\end{aligned} \tag{12}$$

The covariance function is given by:

$$
\begin{aligned}
Q_{(t,t+s)} &= \mathbb{E}\left\{x_{k(t)}x_{k(t+s)}\right\} \\
Q_{(t,t+s)} &= \mathbb{E}\left\{a_k \sin(2\pi f t + \theta_k) a_k \sin(2\pi f (t+s) + \theta_k)\right\} \\
&\Downarrow \quad \sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right] \\
&= \mathbb{E}\left\{a_k^2 \frac{1}{2}\left[\cos(2\pi f t + \theta_k - 2\pi f(t+s) - \theta_k) - \cos(2\pi f t + \theta_k + 2\pi f(t+s) + \theta_k)\right]\right\} \\
&= \mathbb{E}\left\{a_k^2 \frac{1}{2}\left[\cos(2\pi f s) - \cos(4\pi f t + 2\theta_k + 2\pi f s)\right]\right\} \\
&= \int_0^{2\pi} \frac{1}{2\pi} \int_0^1 a^2 \frac{1}{2}\left[\cos(2\pi f s) - \cos(4\pi f t + 2\theta + 2\pi f s)\right] da d\theta \\
&= \int_0^{2\pi} \frac{1}{2\pi} \left[a^3 \frac{1}{6}\left[\cos(2\pi f s) - \cos(4\pi f t + 2\theta + 2\pi f s)\right]\right]_0^1 d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{12\pi}\cos(2\pi f s) - \frac{1}{12\pi}\cos(4\pi f t + 2\theta + 2\pi f s)\right] d\theta \\
&= \left[\frac{1}{12\pi}\cos(2\pi f s)\,\theta - \frac{1}{24\pi}\sin(4\pi f t + 2\theta + 2\pi f s)\right]_0^{2\pi} \\
&= \frac{1}{6}\cos(2\pi f s) = Q_{(s)} \qquad \blacksquare
\end{aligned} \tag{13}
$$

$$\Downarrow$$

$$\boxed{\text{Stationary}}$$

1.3.2 Ergodicity

To prove an ergodic process we need to check the next limit:

$$
\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\int_0^T Q_{(\hat{s})} d\hat{s} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{6}\cos(2\pi f \hat{s})\, d\hat{s} \\
&= \lim_{T\to\infty} \frac{1}{T}\left[\frac{1}{12\pi f}\sin(2\pi f \hat{s})\right]_0^T \\
&= \lim_{T\to\infty} \frac{1}{T}\frac{1}{12\pi f}\sin(2\pi f T) = 0 \qquad \blacksquare
\end{aligned} \tag{14}
$$

$$\Downarrow$$

$$\boxed{\text{Ergodic}}$$

2 Getting Closure

Considering the differencing equation:

$$u_{n+1} = ru_n\left(1 - u_n\right) \tag{15}$$

2.1 Ensemble Average $\langle u \rangle$

In order to calculate the ensemble average $\langle u \rangle$ we will assume that the limit $\lim_{n\to\infty} u_n$ exists. This means there is n at which $u_{n+1} = u_n = u$. Hence:

$$\langle u \rangle = \langle u_n + 1 \rangle = \langle ru - ru^2 \rangle \tag{16}$$

3 Scalar TKE Equation

Considering the instantaneous transport equation for a passive scalar $\tilde{T}$:

$$\frac{\partial \tilde{T}}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{T}\tilde{u}_j\right) = \alpha \frac{\partial^2 \tilde{T}}{\partial x_j^2} \tag{17}$$

Where:

- $\tilde{T} = \bar{T} + T$
- $\tilde{u} = \bar{u} + u$
- $\tilde{(\cdot)}$ - instantaneous quantity
- $\bar{(\cdot)}$ - ensemble average
- $(\cdot)$ - fluctuating component representing the turbulence

In order to describe the magnitude of the fluctuations we will derive an equivalent to the TKE for the temperature fluctuations, $\frac{1}{2}\bar{T}^2$:

$$\begin{aligned}
\frac{\partial \bar{T} + T}{\partial t} + \frac{\partial}{\partial x_j}\left(\left(\bar{T} + T\right)\left(\bar{u}_j + u_j\right)\right) &= \alpha \frac{\partial^2}{\partial x_j^2}\left(\bar{T} + T\right) \\
\frac{\partial \bar{T}}{\partial t} + \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{T}\bar{u}_j + \bar{T}u_j + T\bar{u}_j + Tu_j\right) &= \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2} + \alpha \frac{\partial^2 T}{\partial x_j^2}
\end{aligned} \tag{18}$$

Let's ensemble average:

$$\begin{gathered}
\overline{\frac{\partial \bar{T}}{\partial t} + \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{T}\bar{u}_j + \bar{T}u_j + T\bar{u}_j + Tu_j\right) = \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2} + \alpha \frac{\partial^2 T}{\partial x_j^2}} \\
\overline{\frac{\partial \bar{T}}{\partial t}} + \overline{\frac{\partial T}{\partial t}} + \overline{\frac{\partial}{\partial x_j}\left(\bar{T}\bar{u}_j + \bar{T}u_j + T\bar{u}_j + Tu_j\right)} = \alpha \overline{\frac{\partial^2 \bar{T}}{\partial x_j^2}} + \alpha \overline{\frac{\partial^2 T}{\partial x_j^2}} \\
\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\bar{T}\bar{u}_j + \bar{T}u_j + T\bar{u}_j + Tu_j}\right) = \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2} \\
\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{T}\bar{u}_j + \overline{Tu_j}\right) = \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2} \\
\frac{\partial \bar{T}}{\partial t} + \frac{\partial \bar{T}\bar{u}_j}{\partial x_j} + \frac{\partial \overline{Tu_j}}{\partial x_j} = \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2}
\end{gathered} \tag{19}$$

By subtracting Eq.19 from Eq.18 we get:

$$\begin{gathered}
\frac{\cancel{\partial \bar{T}}}{\cancel{\partial t}} + \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}\left(\cancel{\bar{T}\bar{u}_j} + \bar{T}u_j + T\bar{u}_j + Tu_j\right) - \frac{\cancel{\partial \bar{T}}}{\cancel{\partial t}} - \cancel{\frac{\partial \bar{T}\bar{u}_j}{\partial x_j}} - \frac{\partial \overline{Tu_j}}{\partial x_j} = \alpha \cancel{\frac{\partial^2 \bar{T}}{\partial x_j^2}} + \alpha \frac{\partial^2 T}{\partial x_j^2} - \alpha \cancel{\frac{\partial^2 \bar{T}}{\partial x_j^2}} \\
\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{T}u_j + T\bar{u}_j + Tu_j\right) - \frac{\partial \overline{Tu_j}}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j^2}
\end{gathered} \tag{20}$$

By assuming incompressible flow and applying the ensemble average and substraction to the overall continuity equation, $\dfrac{\partial \tilde{u}}{\partial x_j} = 0$ yields:

$$\frac{\partial}{\partial x_j}\bar{u}_j = 0 \quad \text{and} \quad \frac{\partial}{\partial x_j}u_j = 0 \quad \text{and} \quad \frac{\partial \overline{Tu_j}}{\partial x_j} = 0 \tag{21}$$

Hence we can rewrite Eq.20 as:

$$\frac{\partial T}{\partial t} + \frac{\partial \bar{T} u_j}{\partial x_j} + \frac{\partial T \bar{u}_j}{\partial x_j} + \frac{\partial T u_j}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j^2}$$

$$\frac{\partial T}{\partial t} + \bar{T}\cancel{\frac{\partial u_j}{\partial x_j}} + u_j \frac{\partial \bar{T}}{\partial x_j} + T\cancel{\frac{\partial \bar{u}_j}{\partial x_j}} + \bar{u}_j \frac{\partial T}{\partial x_j} + T\cancel{\frac{\partial u_j}{\partial x_j}} + u_j \frac{\partial T}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j^2} \tag{22}$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial \bar{T}}{\partial x_j} + \bar{u}_j \frac{\partial T}{\partial x_j} + u_j \frac{\partial T}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j^2}$$

By multiplying by T we get:

$$T\frac{\partial T}{\partial t} + T u_j \frac{\partial \bar{T}}{\partial x_j} + T\bar{u}_j \frac{\partial T}{\partial x_j} + T u_j \frac{\partial T}{\partial x_j} = T\alpha \frac{\partial^2 T}{\partial x_j^2} \tag{23}$$

and ensemble averaging the equation:

$$\overline{\frac{1}{2}\frac{\partial T^2}{\partial t} + T u_j \frac{\partial \bar{T}}{\partial x_j} + T\bar{u}_j \frac{\partial T}{\partial x_j} + T u_j \frac{\partial T}{\partial x_j} = T\alpha \frac{\partial^2 T}{\partial x_j^2}} \tag{24}$$