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AN INTRODUCTION TO TURBULENT FLOW

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Contents

What is turbulence?

The bad news is that there is no good definition of turbulence. Indeed, there may be no real comprehensive theory for turbulence, in the way we have come to expect theory for other parts of physics. The field of turbulence might just be a collection of entirely different flow problems in which we have tried to identify some common elements.

Basic Features

For a start, let's try to identify what these common elements are. The items on this list should be considered necessary but not sufficient conditions for turbulence.

1. Irregular/random/chaotic behavior: Turbulent flows exhibit a high sensitivity to initial conditions, fluctuations in time, disorder in space, and consist of a wide range of characteristic length and time scales.
2. High Reynolds number: The Reynolds number represents the ratio of advective to diffusive forces in a flow. As Reynolds number goes up, the relative strength of diffusive forces goes down, until viscous diffusion is not capable of dampening disturbances in the flow, leading to the development of turbulence.
3. Dissipation: Turbulence loses energy by viscous dissipation: shear stresses do work on the flow, converting kinetic energy to thermal energy. This dissipation means that turbulence requires a continuous supply of energy to sustain itself, and that the extent of dissipation is crucially important for describing the behavior of the turbulence.
4. Mixing: Turbulent flows exhibit strong mixing, which appears as a non-viscous diffusive process. This mixing enhances transport of momentum, kinetic energy, as well as heat. Examples include mixing creamer into coffee versus viscous diffusion of the coffee and turbulent combustion versus laminar flames.
5. Complex vorticity distribution: turbulence is composed of fluctuating vortices in three-dimensions; they serve as the mechanism for energy transfer and thus are intimately connected to the idea of dissipation (whereby kinetic energy is transfered to thermal energy)

Let's consider these properties and attempt to make some basic deductions about how turbulent flows behave. We begin with the first property of randomness. There are two ways to describe an essentially random process: by means of stastical descriptions or by means of scaling arguments. We will start with scaling arguments in order to build physical intuition and return to a formal statistical picture later.

Scales of Turbulence

We begin with a turbulent, instantaneous velocity field, $\vec{u}^*$, which can be decomposed into some average field, $\vec{U}$ and some fluctuating field $\vec{u}$. . Now, we know that this fluctuating field is composed of many different motions with corresponding length/time scales. We can therefore think of the field as a superposition of a collection of different motions, each with its own length/time scale. And let's call each of these 'motions' an **eddy**.

Don't worry about how we define this average field just yet, but let's assume that the fluctuating field contains all of the randomness of interest, i.e. the stuff of turbulence

What is an eddy? In 1959, Corrsin defined an eddy as a "spatially local blob of fluid, usually swirling". In 1991, Robinson got a bit more specific when he wrote that a "coherent motion is a three-dimensional region of flow over which one fundamental flow variable (velocity, density, temperature, etc.) exhibits significant correlation with itself or another variable over a range of space and time that is much larger than the smallest scales in the flow". In other words, an eddy is some identifiable motion inside a flow. A vortex is just another name for an eddy (to be precise, a vortex is an eddy in which vorticity is the correlated flow variable). And we will assume that our fluctuating velocity field, $\vec{u}$, is a superposition of eddies of varies sizes and shapes.

What is the largest eddy in our flow? If our flow domain has boundaries, then the largest eddy is characterized by the size of the flow domain itself. If the flow is unbounded, then we can argue that the largest eddy is characterized by the size of the instability which caused it to develop in the first place. In either case, let's call this largest size ℓ and assume that it has a characteristic velocity given by $u \sim |\vec{u}|$. This velocity scale represents the magnitude of the variation in the velocity. We can then construct a lifespan for this largest eddy, $\frac{\ell}{u}$, called the **eddy turnover time**. The eddy turnover time is assumed to be much longer than the timescale of the average flow, $\frac{\ell}{U}$, such that the eddy satisfies the definition of 'coherence' proposed by Robinson, above.

Note that u should not be confused with the average velocity $U \sim |\vec{U}|$; rather u represents the largest scale within the fluctuations themselves

For turbulent flows, we assume the Reynolds number defined by the geometry of the flow, Re is large. If the largest eddies are the same size as the length scale of the fluid domain geometry (for a bounded flow), then the Reynolds number for these largest eddies is also very large,

What does $\gg 1$ mean? How much bigger than 1? These are crucially important questions which will be discussed in the detailed analysis

$$\mathrm{Re}_\ell = \frac{u\ell}{\nu} \gg 1 \tag{1}$$

For large Reynolds numbers, the viscous forces are small compared to inertial forces, and thus viscous dissipation is negligible. But we argued above that the dissipation is a key feature of turbulence! When does viscous dissipation become relatively significant, in comparison to inertial effects? Only when Re $\sim$ 1. In order to satisfy the condition that dissipation is important, there must be smaller scales within turbulence, with characteristic length η and velocity v, such that

$$\mathrm{Re}_\eta = \frac{v\eta}{\nu} \sim 1 \tag{2}$$

We see that the combination of the two assumptions, large Reynolds number and viscous dissipation, means that there must be energy transfer between different eddies. Large eddies don't experience dissipation and small eddies do, and since the energy is fed into the flow at large scales, it must somehow get transferred from the large scales to the small scales. This is our first hint of what Richardson called a **cascade**. Energy is passed down from large to small scales until it is eventually dissipated as thermal energy. If the energy enters turbulence through the large scales, and exists through the small scales, and if the system is in energetic equilibrium, then there must be a balance between the production of turbulent energy and its dissipation. In other words, the rate of change of kinetic energy coming into the system at the large scales is given by

How do we know the energy is passed down through a cascade? Maybe it just jumps directly from large scales to small scales?

$$\frac{d}{dt}u^2 \sim \frac{u^2}{u/\ell} \sim \varepsilon \tag{3}$$

u^2 is the energy of the large eddy per unit mass and u/ℓ is its eddy turnover time; ε is the dissipation rate per unit mass, dimensionally $\left[\frac{L^2}{T^3}\right]$

and this input energy must balance the energy dissipated to heat, which we call ε. It is somewhat surprising that the dissipation rate, ε, does not depend on the kinematic viscosity, ν. The dissipation is a viscous effect so we would have expected a dependence, and yet the dissipation here is controlled entirely by the energy transfer for large eddies. So what is the effect of changing the viscosity? In order to preserve the relation, $\frac{v\eta}{\nu} \sim 1$, we see that a reduction in viscosity actually reduces the dissipative length scale, η, such that the same energy is being dissipated, it's just being dissipated at yet small scales.

Even though the rate of dissipation seems independent of viscosity, the physical mechanism of dissipation is surely viscous. Therefore, we expect that we can represent ε in terms of the relevant dissipative scales, v and η, as well as the viscosity, ν. We know that dissipation is driven by shear, so we expect ε to be expressed as a function

Technically, we should use the dynamic viscosity μ to describe the smallest scales, since for viscous dominated flow the density ρ is not important; however, in turbulence we traditionally represent ε per unit mass.

$$\varepsilon\left(\nu, \frac{v}{\eta}\right) \tag{4}$$

and by dimensional analysis we find that

Compare to the exact definition of dissipation rate for incompressible, laminar flows, $\varepsilon = 2\nu s_{ij}s_{ij}$

$$\varepsilon \sim \nu\left(\frac{v}{\eta}\right)^2 \tag{5}$$

Combining our two definitions of dissipation, in terms of (u, ℓ) and (v, η), we can write the dissipative scales in terms of the largest eddy scales by solving

$$\frac{u^3}{\ell} \sim \nu \left(\frac{v}{\eta}\right)^2 \quad \text{and} \quad \frac{v\eta}{\nu} \sim 1 \tag{6}$$

to yield

$$\frac{\ell}{\eta} \sim \mathrm{Re}_\ell^{3/4} \quad \text{and} \quad \frac{v}{u} \sim \mathrm{Re}_\ell^{-1/4} \tag{7}$$

or combining to describe the ratio of timescales

$$\frac{\ell/u}{\eta/v} \sim \mathrm{Re}_\ell^{1/2} \tag{8}$$

Physically, we see that the dissipative scales get smaller relative to the inertial scales as Reynolds number increases, and the the timescale of the dissipation gets much smaller, such that the dissipation occurs very quickly. Indeed, from the perspective of the small, dissipative scales, the large-scale motions appear almost frozen in time, as Reynolds number is very large. Or, said another way, the large scales appear to be in equilibrium from the point of view of the small scales. Therefore, the details of the large scales don't matter to the small scales; all the small scales care about is how much energy they are disspating, ε. This means we should be able to write the small scales in terms of only ν and ε, independent of the particular large scale motions, u and ℓ. And we find that we can write

This is the idea behind Kolmogorov's "Universal Equilibrium Theory" of 1941, where the word 'universal' indicates that it is not dependent on the particular flow parameters u and ℓ

$$\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \quad \text{and} \quad v \sim (\nu\varepsilon)^{1/4} \tag{9}$$

We've now addressed the first three properties of turbulence, and learned that there must be a wide rande of characteristic eddy scales in order to allow for dissipation at high Reynolds number. We now turn to the fourth characteristic of turbulence: mixing. How does mixing in turbulence differ from typical mixing by diffusion?

Turbulent Mixing

Let's consider a cloud of particles in a turbulent flow, e.g. smokes particles in the atmosphere. Let's track two particles in this cloud. How fast do the particles separate from each other relative to the rate of motion of their joint center of mass, i.e. neglecting the net advection of the pair, or working from a moving reference frame fixed on their joint center of mass. If the particles spread by Fickian diffusion, then we can describe their spread by a diffusivity constant, D. We know that the squared separation distance between

the particles, r^2, under the effect of Fickian diffusion alone, is then given by

$$r^2 \sim Dt \tag{10}$$

This follows from dimensional analysis, where the dimensions of the diffusivity, D, are $\left[\frac{L^2}{T}\right]$ from the non-dimensional form of the diffusion equation.

This is referred to as normal diffusion and is a very slow process — the diffusivity of carbon dioxide in air is $D \approx 16 \times 10^{-6}\ \frac{\mathrm{m}^2}{\mathrm{s}}$ so you can estimate the velocity $\frac{dr}{dt}$.

What happens when the mixing process is turbulent mixing instead of molecular diffusion? Let's estimate the spreading rate in this case. Based on the argument above, the energy exchange between large and small scales in turbulence can be described entirely by the dissipation rate, ε. The molecular effects, like viscosity did not matter. So we assume here too that molecular diffusivity doesn't matter and that we can write the spreading of particules in our turbulent cloud as a function of only

$$r^2(\varepsilon, t) \tag{11}$$

By dimensional analysis, we see that

$$r^2 \sim \varepsilon t^3 \tag{12}$$

This is therefore classified as 'super-diffusion' since the exponent of time is greater than one. We can also translate this 'super-diffusion' in terms of an effective diffusivity for turbulence, D_{turb}, by taking the time derivative of the mean squared separation

$$\frac{dr^2}{dt} \sim \varepsilon t^2 \sim \varepsilon^{1/3} r^{4/3} \tag{13}$$

Similarly, we can use dimensional analysis to define a turbulent kinematic viscosity, $\nu_{\mathrm{turb}} \sim u\ell$, which can be compared to the molecular viscosity by $\frac{\nu_{\mathrm{turb}}}{\nu} \sim \mathrm{Re}_\ell$, just like $\frac{D_{\mathrm{turb}}}{D} \sim \frac{u\ell}{D} \equiv \mathrm{Pe}$ where Pe is the Peclet number

So we find that instead of a constant diffusivity like in the case of Fick's law, we have a scale-dependent effective diffusivity given by

$$D_{\mathrm{turb}} \sim \varepsilon^{1/3} r^{4/3} \tag{14}$$

which is known as Richardson's 4/3's Law. As the particles spread apart farther, their rate of spreading accelerates according to a power law of the mean separation to the 4/3. This super-diffusion is a consequence of turbulent mixing, and thus we expect that turbulence substantially accelerates the mixing process comapred to molecular diffusion.

The final property of turbulence to discuss is its relationship to vorticity. The motions which constitute turbulence can be described as eddies, and these eddies can be thought of as interacting vortices. Therefore, we need to develop the tools to describe vortex dynamics in order to understand eddy dynamics.

Vortex Vector Notation

Vorticity is a vector defined as the curl of velocity. Because we will be dealing with the complexities of vector-valued partial-differential-equations, it is sensible to choose a notation which

simplifies writing these equations as much as possible. We adopt the use of index (or Einstein) notation, where tensor elements are denoted by a subscript and repeated indices denote summation over the repeated index. So the velocity vector $\vec{u}$ is written as u_i where $i = \{1,2,3\}$ for the three spatial dimensions over which the vector is defined. Calculating the divergence of the velocity is simple since we just need to employ the summation rule. We write the gradient operator in Cartesian coordinates as $\frac{\partial}{\partial x_i}$ and use the identical index for the velocity vector to indicate summation

See Aris's book, 'Vectors, Tensors, and the Basic Equations of Fluid Mechanics' for a thorough introduction to index notation, or books by Kundu & Cohen or Panton for a shorter overview

$$\vec{\nabla} \cdot \vec{u} \quad \Rightarrow \quad \frac{\partial}{\partial x_i} u_i \equiv \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \tag{15}$$

The vorticity vector, $\vec{\omega}$, is a bit tricker, because it is not given by a simple summation. Rather it is written according to its determinant definition, as

$$\vec{\omega} = \vec{\nabla} \wedge \vec{u} = \begin{vmatrix} \hat{x_1} & \hat{x_2} & \hat{x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} \tag{16}$$

Expanding the determinant we find

$$\vec{\omega} = \hat{x_1}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - \hat{x_2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) + \hat{x_3}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{17}$$

and therefore we need a way of expressing the unusual summation with alternating signs. We introduce the Levi-Civita operator, which is defined as

Write the numbers $\{1,2,3\}$ like the numbers on a clock face. Combinations read in a clockwise direction yield $+1$, in a counter-clockwise direction -1, and any combination that cannot be read from either direction is 0.

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}. \\ -1, & \text{if } (i,j,k) \in \{(3,2,1),(1,3,2),(2,1,3)\}. \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

and this allows us to write the summation for the cross-product in the form

$$\omega_i = \left(\vec{\nabla} \wedge \vec{u}\right)_i = \epsilon_{ijk} \nabla_j u_k \tag{19}$$

where we have a double sum over j and k yielding a vector with index i. We can verify one of the components of vorticity, for example, by

$$\omega_3 = \epsilon_{3jk} \nabla_j u_k = \epsilon_{312} \frac{\partial}{\partial x_1} u_2 + \epsilon_{321} \frac{\partial}{\partial x_2} u_1 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \tag{20}$$

Switching the order of indices in the Levi-Civita operator changes the sign, such that $\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} = -\epsilon_{ikj} \frac{\partial u_k}{\partial x_j}$ and thus $\omega_i = \frac{1}{2} \epsilon_{ijk} \left(\frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} \right)$

which matches the $\hat{x_3}$ component listed above in the determinant. A similarly useful operator is the Kronecker delta, δ_{ij}, which represents the identity tensor and is defined as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j. \\ 0, & \text{if } i \neq j. \end{cases} \tag{21}$$

where

$$\epsilon_{imn}\epsilon_{ijk} = \delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj} \tag{22}$$

We can now easily write and manipulate the vorticity field given the velocity field by the use of index notation. It is also possible to write the reverse relationship: the velocity field given a vorticity field. Let's assume for the moment that the velocity field is solenoidal, such that $\nabla \cdot \vec{u} = 0$. We can therefore write the velocity field as the curl of a vector potential, such that

The component of the vector potential ψ_3, in 2D flows, is called the stream function. The relationship between $\vec{u}$ and $\vec{\psi}$ is not unique, since we can always add a second potential, $\vec{\psi} + \nabla\phi$ and obtain the same $\vec{u}$. Thus, for uniqueness we require that $\nabla \cdot \vec{\psi} = 0$, called 'gauge fixing'

$$\vec{u} = \nabla \wedge \vec{\psi} \tag{23}$$

because we know that such a potential automatically satisfies the solenoidal condition, $\nabla \cdot \nabla \wedge \vec{\psi} = 0$. Then we obtain Poisson's equation relating vorticity and velocity

$$-\vec{\omega} = \nabla^2\vec{\psi} \tag{24}$$

which can be solved by Green's functions, where in three-dimensions the Green's function is given by

$$G(\vec{x}, \vec{x}') = -\frac{1}{4\pi}\frac{1}{|\vec{x} - \vec{x}'|} \tag{25}$$

and thus the velocity potential is given by

$$\vec{\psi}(\vec{x}) = -\int G(\vec{x}, \vec{x}')\vec{\omega}(\vec{x}')d^3\vec{x}' \tag{26}$$

from which we can calculate the velocity by taking the curl, such that

$$\vec{u} = \frac{1}{4\pi}\nabla_{\vec{x}} \wedge \int \frac{\vec{\omega}(\vec{x}')}{|\vec{x} - \vec{x}'|}d^3\vec{x}' \tag{27}$$

Then simplifying we obtain

$$\nabla_{\vec{x}} \wedge \left(\frac{\vec{\omega}(\vec{x}')}{|\vec{x} - \vec{x}'|}\right) = \frac{1}{|\vec{x} - \vec{x}'|}\underbrace{\nabla_{\vec{x}} \wedge \vec{\omega}(\vec{x}')}_{=0} + \nabla_{\vec{x}}\left(\frac{1}{|\vec{x} - \vec{x}'|}\right) \wedge \vec{\omega}(\vec{x}') \tag{28}$$

and finally we obtain

This is the familiar Biot-Savart Law in 3D. For 2D, we can make use of the fact that $\int_\infty^\infty \frac{dz}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{x^2+y^2}$ to integrate away one dimension, leaving a factor of 2 difference for the 2D case.

$$\vec{u} = -\frac{1}{4\pi}\int \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \wedge \vec{\omega}(\vec{x}')d^3\vec{x}' \tag{29}$$

or, defining $\vec{r} \equiv \vec{x} - \vec{x}'$ we obtain

$$\vec{u} = -\frac{1}{4\pi}\int \frac{\vec{r} \wedge \vec{\omega}}{|\vec{r}|^3}d^3\vec{x}' \tag{30}$$

Now that we have fully described the mathematical notation for vorticity, we need to examine its physical meaning. What does the presence of the vorticity field represent?

Circulation & Vortices

Let's consider three velocity fields and calculate the vorticity within each. The first field is a plane shear flow, with $(u, v) \sim (y, 0)$ that has corresponding vorticity, $\omega_z \sim -1$ everywhere. Although physically we associate vorticity with rotational motion, here is a flow with straight, parallel streamlines that exhibits uniform vorticity everywhere. (You an think of the vorticity in this shear flow as little rollers which allow layers of fluid moving at different velocities to roll past each other, like ball-bearings.)

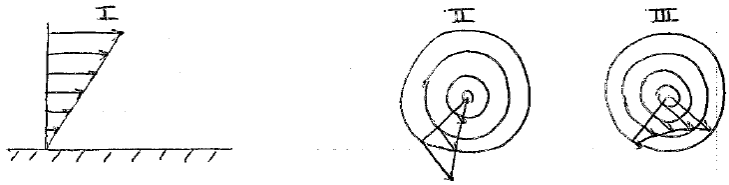


Figure 1: Three vortical velocity fields with velocity distributions given by (I) $u \sim y$, (II) $u_\theta \sim r$ and (III) $u_\theta \sim \frac{1}{r}$.

The second velocity field is described by $u_\theta \sim r$ and has circular streamlines, with angular velocity increasing linearly away from the origin and corresponding vorticity is $\omega_z \sim 2$. This vortex represents solid body rotation — the individual fluid elements experience no viscous stresses as they move around because they are not being deformed, only rotated. The third velocity field is described by $u_\theta \sim \frac{1}{r}$ and has identical streamlines as the solid body rotation case. However, the vorticity in this case is zero everywhere except at the origin, where it is undefined. This is called an irrotational of 'free' vortex, and is the most common found in nature (sink drains, tornados). In this case, the elements experience shear stresses and are thus deformed, but the elements themselves do not rotate. We therefore see that vorticity describes the rotation but not the shear-deformation of fluid elements.

In cylindrical coordinates, $\vec{\nabla} \wedge \vec{u} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & r u_\theta & u_z \end{vmatrix}$ and the viscous stress is given by $\tau_{r\theta} = \mu\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right]$

The 'free vortex' doesn't require external forcing but decays viscously; the solid body vortex cannot exist without external forcing, e.g. a rotating cylinder of fluid

This leads us to the awkward conclusion that not all vortices have vorticity. The presence of vorticity does not imply a thing shaped like a vortex,with circular streamlines, and circular streamlines do not imply measurable vorticity. Vorticity is defined pointwise in a fluid field, but it seems from these sketches that if we want to define a coherent vortex, we need a measurement that is more global than a single point, or else we will miss irrotational vortices altogether. What we really need to consider is the vorticity integrated over an area, denoted ∂V with edge C. The integrated vorticity is called the circulation, Γ and can be written as

$$\Gamma \equiv \iint_{\partial V} \vec{\omega} \cdot \hat{n} dS = \iint_{\partial V} (\nabla \wedge \vec{u}) \cdot \hat{n} dS = \oint_C \vec{u} \cdot d\vec{\ell} \tag{31}$$

where we have used Stokes Theorem to rewrite the surface integral as a contour integral. The surface integral of vorticty represents the flux of vorticity through a surface , in the same way that the integral of an incompressible velocity represents the volumetric flux of fluid.

It is easy to show that $\nabla \cdot \vec{\omega} = 0$ by means of index notation and thus vorticity is solenoidal, thus the number of vortex lines flowing through a vortex tube is conserved

With this measurement tool, we can now consider the circulation about the irrotational vortex and we find that

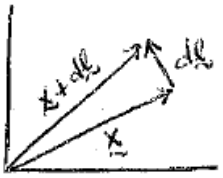
$$\Gamma \sim \oint_{C=|r|} \frac{1}{r} r d\theta \sim 2\pi \tag{32}$$

the circulation is non-zero when measured over a surface containing the origin, even though the vorticity everywhere on that surface (except the origin) is zero. Circulation therefore seems to be a powerful tool for tracking vortical motions via the distribution of vorticity even in the absence of vorticity at every point. Let's see what we can say about how this circulation evolves dynamically. In other words, lets calculate the material derivative of circulation, $\frac{D\Gamma}{Dt}$, allowing Γ to evolve in time and space.

The material derivative is given by $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$, where the time derivative is for fixed spatial location and the spatial derivative is evaluated in the direction of the motion of the material volume, at fixed time.

$$\frac{D}{Dt}\Gamma = \oint_C \frac{D}{Dt}\vec{u} \cdot d\vec{\ell} + \oint_C \vec{u} \cdot \frac{D}{Dt} d\vec{\ell} \tag{33}$$

To find the dynamics of a material line element, $d\vec{\ell}$, consider the evolution of its two endpoints, $\vec{x}$ and $\vec{x} + d\vec{\ell}$ moving in a field with velocity $\vec{u}$. In time dt,



$$\vec{x} \rightarrow \vec{x} + \vec{u}(\vec{x}, t)\delta t \tag{34}$$

$$\vec{x} + d\vec{\ell} \rightarrow \vec{x} + d\vec{\ell} + \vec{u}(\vec{x} + d\vec{\ell}, t)\delta t \tag{35}$$

and thus the material change in the line element is given by

$$\frac{D\vec{\ell}}{Dt} = \vec{u}(\vec{x} + d\vec{\ell}, t) - \vec{u}(\vec{x}, t) = \left(d\vec{\ell} \cdot \nabla\right)\vec{u} \tag{36}$$

Plugging this into the integral yields

$$\oint_C \vec{u} \cdot \frac{D}{Dt} d\vec{\ell} = \oint_C \vec{u} \cdot \left(d\vec{\ell} \cdot \nabla\right)\vec{u} = \frac{1}{2}\oint_C d\vec{\ell} \cdot \nabla(\vec{u} \cdot \vec{u}) = \frac{1}{2}\iint_{\partial V} \nabla \wedge \nabla(\vec{u} \cdot \vec{u}) \cdot \hat{n} dS = 0 \tag{37}$$

and therefore

$$\frac{D\Gamma}{Dt} = \oint_C \frac{D\vec{u}}{Dt} \cdot d\vec{\ell} \tag{38}$$

If we assume that the flow is incompressible and barotropic (i.e. that $p = p(\rho)$ only) and any body forces are conservative (i.e. $\nabla \wedge \vec{B} = 0$), then we can substitute the momentum equation for $\frac{D\vec{u}}{Dt}$ to yield

If the flow is baroclinic, i.e. that $\nabla p \wedge \nabla\rho \neq 0$, then there is an additional source of circulation due to baroclinic forces

$$\frac{D\Gamma}{Dt} = -\nu \oint_C \nabla \wedge \vec{\omega} \cdot d\vec{\ell} \tag{39}$$

which is the general form of Kelvin's Circulation Theorem. If at any point in time, the flow field is irrotational ($\vec{\omega} = 0$ everywhere), then it will remain irrotational for all time, since $\frac{D\Gamma}{Dt} = 0$. If there is vorticity in the flow field, it can diffuse via viscous diffusion across the material bounday C. However, if the Reynolds number if very large, then viscous diffusion is relatively weak compared

to advection, and we can neglect the diffusion, in which case the circulation in a material volume is approximately conserved .

$$\frac{D\Gamma}{Dt} \approx 0 + \mathcal{O}\left(\mathrm{Re}^{-1}\right) \tag{40}$$

Of course near a wall, where the no-slip condition applies, the Reynolds number will decrease, diffusive effects will become more important, and vorticity diffusion cannot be neglected: this is called the boundary layer

In other words, at very high Reynolds number, vorticity moves with the fluid. So if we define an eddy in the flow as some coherent motion of fluid, the eddy's vorticity content moves along with it at sufficiently high Reynolds number. The question is then how does the eddy move?

Vortex Stretching

Let's think about the behavior of an eddy in terms of a more familiar concept: angular momentum, $\vec{H}$. The angular momentum can be written in terms of the moment of inertia, I, and the angular velocity, $\vec{\Omega}$ accordint o

$$\vec{H} = I\vec{\Omega} \tag{41}$$

and we've already seen that the angular velocity can be related to the vorticity in a very simple way, in the case of solid body rotation. Using Stokes Theorem again, we write

$$\iint_{\partial V} \vec{\omega} \cdot \hat{n} dS = \oint_C \vec{u} \cdot d\vec{\ell} \tag{42}$$

and for solid body rotation in two dimensions, $u_\theta = \Omega r$ we can evaluate the integrals over a circular contour centered at the origin to yield

$$(\Omega r) 2\pi r = \omega_z \pi r^2 \tag{43}$$

This result can be shown rigorously in multiple dimensions by writing the general form of rigid body rotation as $\vec{u} = \vec{\Omega} \wedge \vec{r}$ and then calculating $\nabla \wedge \vec{u}$, using index notation

and finally, $\Omega = \frac{1}{2}\omega_z$, or more generally in three-dimensions, $\vec{\Omega} = \frac{1}{2}\vec{\omega}$. Then we obtain the angular momentum of our eddy as

$$\vec{H} = \frac{1}{2} I \vec{\omega} \tag{44}$$

and again we want to understand the evolution of this angular momentum as the eddy moves and deforms in the flow. The material derivative of the angular momentum is, by conservation of angular momentum, equal to the net torque acting on the eddy. Because pressure is isotropic, there is no net torque from pressure forces in the flow and thus the net torque is just viscous torque exerted via shear stress.

Isotropic means identical in all directions; the momentum flux due to pressure is represented as $-p\delta_{ij}$ and has no net contribution when integrated over a volume

$$\frac{D\vec{H}}{Dt} = \sum(\text{viscous torque}) \tag{45}$$

Expanding the material derivative yields

$$2\frac{D\vec{H}}{Dt} = \vec{\omega}\frac{DI}{Dt} + I\frac{D\vec{\omega}}{Dt} = 2\sum(\text{viscous torque}) \tag{46}$$

and re-arranging yields

$$I\frac{D\vec{\omega}}{Dt} = -\vec{\omega}\frac{DI}{Dt} + 2\sum(\text{viscous torque}) \tag{47}$$

This equation describes the evolution of the vorticity the moves with the eddy. We conclude:

1. The evolution of $\vec{\omega}$ does not depend on the pressure field
2. If $\vec{\omega} = 0$ initially, then it should remain 0 in the limit that viscous forces are negligible (another form of Kelvin's Theorem)
3. If $\frac{DI}{Dt} < 0$ (i.e. the moment of inertia of the eddy decreases), then the vorticity in the eddy increases ($\frac{D\vec{\omega}}{Dt} > 0$) and vice versa. In other words, stretching eddies increases the intensity of their vorticity; squashing eddies decreases the intensity of their vorticity.

We see that vortex stretching is therefore crucial to turbulence because it is the only mechanism at high Reynolds numbers by which the intensity of vorticity can fluctuate. Of course, we have been a bit vague by simply writing 'viscous torques' since at high Reynolds number they should be negligible. But we can be more precise about the evolution of vorticity in an eddy by simply writing the exact dynamics of $\vec{\omega}$ from the momentum equation itself.

Vorticity Dynamics

We start with the momentum equation for an constant-density, Newtonian fluid

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}u_i u_j + \nu\frac{\partial^2 u_i}{\partial x_j^2} \tag{48}$$

and we can rewrite this using the identities $\vec{u}\cdot\nabla\vec{u} = -\vec{u}\wedge\vec{\omega} + \frac{1}{2}\nabla|\vec{u}|^2$ and $\nabla^2\vec{u} = -\nabla\wedge\vec{\omega} + \nabla(\nabla\cdot\vec{u})$ to yield

Note the close connection between the viscous diffusion of momentum, $\nu\nabla^2\vec{u}$ and the viscous diffusion of vorticity, $-\nu\nabla\wedge\vec{\omega}$; in an irrotational flow, momentum diffusion cannot occur

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i}\left(\frac{p}{\rho} + \frac{1}{2}|\vec{u}|^2\right) + \epsilon_{ijk}u_j\omega_k - \nu\epsilon_{ijk}\frac{\partial\omega_k}{\partial x_j} \tag{49}$$

Then we can take the curl, $\nabla\wedge$ (or $\epsilon pqi\frac{\partial}{\partial x_q}$) of this equation to yield the vorticity dynamics

$$\frac{\partial\omega_p}{\partial t} + u_k\frac{\partial\omega_p}{\partial x_k} = \omega_k\frac{\partial u_p}{\partial x_k} + \nu\frac{\partial^2\omega_p}{\partial x_k^2} \tag{50}$$

which allows us to compare our exact result to our heuristic result above in terms of angular momentum

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\nabla)\vec{u} + \nu\nabla^2\vec{\omega} \tag{51}$$

$$I\frac{D\vec{\omega}}{Dt} = -\vec{\omega}\frac{DI}{Dt} + 2\sum(\text{viscous torque}) \tag{52}$$

We can immediately recognize that the term $(\vec{\omega}\cdot\nabla)\vec{u}$ represents vortex stretching (i.e. the change in the moment of inertia of the

eddy). We also recognize that this term is identically zero in two-dimensional flows, so in two-dimensions vorticity can diffuse but not stretch. However, we can say even more about this stretching term, since it depends on the gradient of the velocity field.

In 2D flow, $\vec{u} = (u, v, 0)$, the vorticity is entirely out of plane, $\vec{\omega} = (0, 0, \omega_z)$

The velocity gradient tensor can be written as the sum of symmetric and anti-symmetric parts

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{symmetric, } s_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{anti-symmetric, } r_{ij}} \tag{53}$$

where the symmetric part, s_{ij} is called the rate of strain tensor, and r_{ij} is the rate of rotation tensor, where the rate of rotation tensor is related to the vorticity by $r_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$. We can then evaluate

$$\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j s_{ij} + \omega_j r_{ij} \tag{54}$$

Note that $-\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k = -\frac{1}{2}\epsilon_{ijk}\omega_k\omega_j = \frac{1}{2}\epsilon_{ikj}\omega_k\omega_j = \frac{1}{2}\epsilon_{ijk}\omega_j\omega_k$ and thus the quantity must be equal to zero, since it is equal to its own negative

and

$$\omega_j r_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_j\omega_k = 0 \tag{55}$$

Therefore, we can rewrite the vortex stretching term in the vorticity dynamics as

$$\frac{D\omega_i}{Dt} = \omega_k s_{ik} + \nu\frac{\partial^2\omega_i}{\partial x_k^2} \tag{56}$$

and we can identify the source of the stretching: vortices are stretches by the local rate of strain, not by the local rotation of the flow. How does local straining motion stretch vortices?

Enstrophy

Let's consider a simple, two-dimensional shear flow described by $\vec{u} = (\alpha x, -\alpha y, 0)$ such that its local rate of strain tensor has two components, $s_{xx} = -s_{yy} = \alpha$. We place a vortex tube inside this flow and observe how it responds. As the straining flow acts on the vortex tube, we want to measure the vorticity content in the tube as it evolves over time in the form of a scalar quantity. For a two-dimensional surface, we can use circulation, but for a three-dimensional tube, we need another measure, called enstrophy. Enstrophy is defined as $\frac{1}{2}|\vec{\omega}|^2 \equiv \frac{1}{2}\omega^2$ and thus is related to vorticity as kinetic energy density is related to momentum density. Enstrophy is, in fact, related directly to the rate of viscous dissipation of kinetic energy ε, according to the Bobyleff-Forsyth formula

A vortex tube is defined as a surface composed of vortex lines, like a stream tube is a surface composed of streamlines

$$\int_V \varepsilon dV = \nu \int_V \omega^2 dV + 2\nu \iint_{\partial V} \frac{D\vec{u}}{Dt} \cdot \hat{n} dS \tag{57}$$

Both dissipation and vortex stretching are essential elements to turbulence and we see how they are directly connected by the Bobyleff-Forsyth formula. For a control volume in which the velocity is zero along the boundaries, the surface integral drops out and we have $\varepsilon = \nu\omega^2$ exactly, for incompressible flow

and thus when we measure the enstrophy, we are measuring the dissipation rate as well as the intensity of vorticity. By taking the

scalar product of $\frac{D\vec{\omega}}{Dt}$ and $\vec{\omega}$ we can write the dynamics of the enstrophy as

$$\frac{D}{Dt}\left(\frac{\omega^2}{2}\right) = \omega_i \omega_j s_{ij} - \nu\left[|\nabla \wedge \vec{\omega}|^2 + \nabla \cdot (\vec{\omega} \wedge \nabla \wedge \vec{\omega})\right] \tag{58}$$

For simplicity, we assume the Reynolds number is sufficiently high to neglect the viscous diffusion terms, to yield

$$\frac{D}{Dt}\left(\frac{\omega^2}{2}\right) \approx \omega_i \omega_j s_{ij} \tag{59}$$

Now let's consider two possible orientations of the vortex tube inside the straining flow to determine the effect of each, as measured by the enstrophy, figure 4.

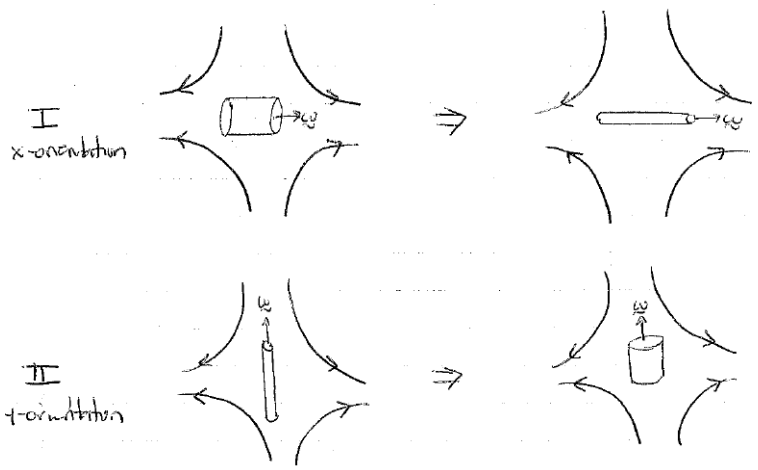


Figure 2: Two orientations of a vortex tube in a simple straining flow. (I) The vortex tube is oriented in the x-direction, along the direction of tension. (II) The tube is oriented in the y-direction along the direction of compression.

If the tube is oriented in the direction of tension (x-direction), then we can write

$$\frac{D}{Dt}\left(\frac{\omega^2}{2}\right) \approx \omega_x^2 s_{xx} \approx \omega_x^2 \alpha \tag{60}$$

and if the vortex is moving in a material volume, we can treat the material derivative like a regular temporal derivative and write

Saying the tube is a material volume is equivalent to saying that we are moving the coordinate system with the tube, such that there is no mean advection of the tube, and thus the material derivative reduces to a time derivative only

$$\frac{1}{\omega_x^2} d\omega_x^2 = 2\alpha dt \Rightarrow \ln \frac{\omega_x^2}{\omega_{x0}^2} = 2\alpha t \tag{61}$$

or, integrated over the material volume, we obtain the total enstrophy in the tube as a function of time as

$$\int \omega_x^2 dV = \int \omega_{x0}^2 e^{2\alpha t} dV \tag{62}$$

Repeating this calculation for the tube oriented in the direction of compression (y-direction) we obtain

$$\int \omega_y^2 dV = \int \omega_{y0}^2 e^{-2\alpha t} dV \tag{63}$$

In a turbulent flow, of course both orientations would exist, along with all the other possible random orientations in between. So

what we want to know is the net effect of equal numbers of vortices oriented in both directions. What is the net impact of the stretching and compression processes on the enstrophy? The sum of the two directional contributions, in equal measure, is

$\cosh(2\alpha t) > 0$ for all real t and α

$$\int \frac{\omega^2}{2} dV = \int \omega_0^2 \cosh(2\alpha t) dV \tag{64}$$

The net effect of the stretching and compression is always an increase in enstrophy; said another way, the stretching is the dominant process. Similarly, the dissipation rate is always increased by the stretching of vortices, or, in other words, the straining flow is losing kinetic energy by means of the vortex stretching. We can illustrate this transfer of kinetic energy directly.

Energy Cascade

So far, we've been describing the outer, straining flow, $\vec{u}$. Let's also describe the velocity field associated with the vortex tube itself, and call it $\vec{u}'$. . The total momentum balance for the outer flow and the vortex tube is then given by

Given the vorticity within the tube, we can calculate the corresponding velocity field by means of the Biot-Savart formula derived earlier

$$\rho \frac{D}{Dt}(\vec{u} + \vec{u}') = -\nabla(p + p') + \mathcal{O}(\mathrm{Re}^{-1}) \tag{65}$$

where the material derivative is defined in terms of the total flow velocity, $\vec{u} + \vec{u}'$. Neglecting terms of $\mathcal{O}(\vec{u}'^2)$ and substracting the outer flow dynamics yields a linearized approximate equation for the momentum of the vortex tube flow

$$\overline{\frac{D}{Dt}}(\vec{u}') \approx -\vec{u}' \cdot \nabla \vec{u} - \frac{1}{\rho} \nabla p' \tag{66}$$

where the material derivative $\overline{\frac{D}{Dt}}$ is defined in terms of the outer flow velocity only, $\vec{u}$. Multiplying through by $\vec{u}'$ yields

$$\overline{\frac{D}{Dt}}\left(\frac{1}{2}\vec{u}'^2\right) \approx -u_i' u_j' s_{ij} - \frac{1}{\rho} \nabla \cdot (\vec{u}' p') \tag{67}$$

and integrating over the material volume, far away from the vortex tube, the final divergence term drops out, since the velocity induced by the vortex dies off far away from the vortex, leaving

$$\frac{D}{Dt} \int \frac{1}{2} \vec{u}'^2 dV = \int \alpha \left({u_y'}^2 - {u_x'}^2 \right) dV \tag{68}$$

Now, when we stretch in the x-direction, enstrophy increases which implies that u_y' and u_z' (the off-axis velocity components, the ones that contribute to u_θ) increase also. When we compress in the y-direction, enstrophy decreases, which means u_x' and u_z' decrease also. When we combine both of these movements, stretching and compressing, the next effect is that u_y' increases, u_x' decreases, and u_z' varies slowly. This implies that $\frac{D}{Dt} \int \frac{1}{2} \vec{u}'^2 dV > 0$, or, in other words, the energy of the vortex tube increases due to the cumulative effect of stretching and compressing. But where is this energy

coming from? Obviously, from the outer straining flow. We already showed that the dissipation of the outer straining flow is increased due to the vortex dynamics, so the large-scale outer flow is losing energy and the small-scale eddy inside is gaining that energy. This is another hint to the process of energy transfer from large to small scales that we referred to earlier as the energy cascade.

Statistics of Random Processes

We said that one of the key features of turbulence is randomness, by which we mean that the instantaneous measurements from turbulent flows are highly irregular, and vary greatly even for very similar initial and boundary conditions. Therefore we need some sort of averaging technique in order to avoid these irregular, instantaneous fields and to work with more useful, smooth, average values that are amenable to analysis. But to do this, we have to define what we mean by the word 'average'.

Reynolds Averaging

What are the qualities we want in an average? The overall goal is, of course, to obtain sufficiently simple equations of mean quantities of all of the relevant fluid dynamics variables. Let's consider a toy equation to illustrate what we want. In this toy system, we have an instantaneous flow variable, $u(x,t)$ that obeys the following equation

The key feature of this toy equation is the non-linearity in u. Without the non-linearity, the problem of properly defining an averaging operation becomes substantially easier

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} u^2 = 0 \tag{1}$$

Then we suppose that each of these variables can be decomposed into some 'average' variable, denoted with an overbar, $\overline{(\cdot)}$, and some fluctuating quantity, denoted by the prime, $(\cdot)'$, according to

$$u = \overline{u} + u' \tag{2}$$

We don't yet know what this 'averaging' process is, but we can substitute these definitions into the governing dynamics to obtain

$$\frac{\partial}{\partial t}(\overline{u} + u') + \frac{\partial}{\partial x}\left(\overline{u}^2 + 2\overline{u}u' + u'^2\right) = 0 \tag{3}$$

Now, we apply our averaging operation to this equation.

$$\overline{\frac{\partial}{\partial t}(\overline{u} + u') + \frac{\partial}{\partial x}\left(\overline{u}^2 + 2\overline{u}u' + u'^2\right)} = 0 \tag{4}$$

What do we want the result of the averaging to look like? Obviously we want it to be as simple as possible, but no simpler. For instance, a form like

$$\frac{\partial}{\partial t}(\overline{u}) + \frac{\partial}{\partial x}\left(\overline{u}^2 + \overline{u'^2}\right) = 0 \tag{5}$$

would be very nice. It would eliminate the averages of the fluctuating quantities. What else does this form imply about the averaging operation? It implies some distributive property, which allowed the average of the sum to be the sum of averages. It also implied that averaging could pass through differentiation operations. In 1894, Reynolds defined five properties that this averaging process should ideally possess and which will produce the sort of results we desired in our toy problem. Consider functions f and g, constant a, and spatial or temporal distance s.

$$\overline{f+g} = \overline{f} + \overline{g} \tag{6}$$

$$\overline{af} = a\overline{f} \tag{7}$$

$$\overline{a} = a \tag{8}$$

$$\overline{\left(\frac{\partial f}{\partial s}\right)} = \frac{\partial \overline{f}}{\partial s} \tag{9}$$

This is really a specific instance of the general requirement that $\overline{\lim_{s\to\infty} f(s)} = \lim_{s\to\infty} \overline{f(s)}$ since limits are part of the definition of derivatives; in other words, limits and averages commute.

$$\overline{\overline{f}g} = \overline{f}\,\overline{g} \tag{10}$$

These properties can't be 'proved'; rather they are requirements that we choose to enforce in order to obtain a useful definition of averaging. Most of them probably seem trivial or obvious, but that is only because we are used to the averaging process already. Let's consider an instantaneous signal, f which is composed of some average quantity, $\overline{f}$ and a fluctuating component, f', according to

$$\underbrace{f}_{\text{instantaneous}} = \underbrace{\overline{f}}_{\text{average}} + \underbrace{f'}_{\text{fluctuating}} \tag{11}$$

Now, let's ensemble average this equation:

$$\overline{f = \overline{f} + f'} \tag{12}$$

Using the first (distributive) property, we can rewrite this as

$$\overline{f} = \overline{\overline{f}} + \overline{f'} \tag{13}$$

And using the last property, with $g = 1$, we find that $\overline{\overline{f}} = \overline{f}$ and thus we find

$$\overline{f} = \overline{f} + \overline{f'} \tag{14}$$

or, in other words, the average of the fluctuating quantity is identically zero, $\overline{f'} = 0$. The other properties of the averaging (commutativity with limits, distributive over addition, etc.) are already explicit in the rules laid out by Reynolds. The only problem is that we haven't defined, mathematically, what the averaging operation, $\overline{\cdot}$, actually is.

'Averaging'

Let's say we define our averaging operation as the familiar temporal or spatial mean operation that we recognize from other domains of science, which we can write in the general space-time form as

$$\overline{f(x_1, x_2, x_3, t)} = \int_{\infty}^{\infty} f(x_1 - \zeta_1, x_2 - \zeta_2, x_3 - \zeta_3, t - \tau) w(\zeta_1, \zeta_2, \zeta_3, \tau) d\zeta_1 d\zeta_2 d\zeta_3 d\tau \tag{15}$$

where w is some weighting function, which satisfies the normalization condition that when $f(x_1 - \zeta_1, x_2 - \zeta_2, x_3 - \zeta_3, t - \tau) = 1$ the above integral is identically 1.

Another way of thinking about this is that the weighted average integral is really just a convolution integral and the weighting function is a convolution kernel

Does this operation satisfy our rules of averaging? It certainly satisfies the first four of Reynolds averaging rules, no matter what choice of weighting function w we choose. But it does not necessarily satisfy $\overline{\overline{f}\overline{g}} = \overline{f}\overline{g}$, and thus it does not necessarily yield the property that the average of the fluctuation is identically zero.

Imagine, for instance, that our weighting function w is shaped like a box-car function or a Gaussian distribution, with some definable width in space or time, such that averages are performed over some interval. And suppose that our signal of interest, f, varies over at least two different intervals, a large scale (associated with $\overline{f}$) and a small scale (associated with f').

Now, if the interval of our averaging (i.e. the interval of the weighting function) is large compared to the scale of $\overline{f}$, then all of the variation in $\overline{f}$ will appear as if it were part of f' and thus $\overline{f'} \neq 0$. Similarly, if the averaging interval were small compared to the scale of f', then variations in f' will get incorporated in $\overline{f}$. So the only way to satisfy $\overline{f'} = 0$ is if the interval of the averaging is simultaneously much larger than the interval of f' and much smaller than the interval of $\overline{f}$. This assumption is usually satisfied automatically, but not always – it depends on the type of signal being measured.

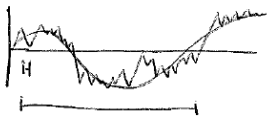


Figure 3: The underbar represents an averaging interval length, which is on the scale of the large-scale fluctuations representing $\overline{f}$

Even though we use time and space averaging in the lab, they are not the best definitions for developing an analytical theory of turbulence, both for the reason above, that they depend on assumptions regarding the nature of a given signal, and because we would need to specify the particular form of the weighting function in order to proceed, and we would rather leave that decision arbitrary. Therefore we turn to probability theory to develop a more general approach to averaging turbulent quantitites.

Ensemble Averaging

The basic idea of the probability theory approach to turbulence is that we conceptualize the problem not in terms of a single flow realization that we are trying to measure, but rather as a 'statistical ensemble of all similar flow realizations created by some fixed (common) set of external conditions'. The high sensitivity to initial

conditions and perturbations within turbulence means that each of the realizations within this ensemble will be different. So when we speak of an average, we no longer mean a time or space average within a single realization; rather we mean the arithmetical mean of all of the realizations in the ensemble, which we call the ensemble mean.

The ensemble mean is thus an a function of time and space, averaged over all possible realizations for a given flow

Let's start with a measurable quantity, say the velocity in a flow at a particular time and location and with a particular initial condition. We can call a scalar component of this velocity, u. And we want to determine the probability that this velocity $u < c$ where c is some constant. We can write that

$$\Pr\{u < c\} = \frac{\text{number of realizations in which } u < c}{\text{total number of realizations}} \tag{16}$$

e.g. Each realization could be obtained from a separate experimental run in a wind-tunnel

Without loss of generality, we can assign every realization (i.e. every experimental run) an ID number, α, with $\alpha \in [0, 1]$. So for each realization α we have a corresponding measurement u which is either $>$ or $< c$. If we plot all of the realizations of u, arranged by their ID number, we can see

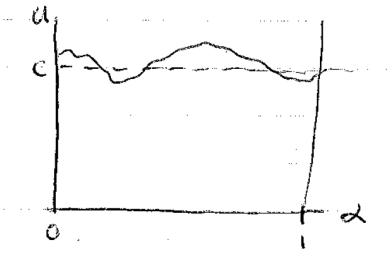


Figure 4: Velocity realizations arranged by ID number

Then $\Pr\{u < c\}$ is the fraction of $[0, 1]$ over which $u < c$. But this is a rather awkward way of describing a probability, by having to inspect a graph every time. We need a way to express this mathematically. We can define an indicator function, $\phi(\alpha)$, such that

$$\phi(\alpha) = \begin{cases} 1, & \text{if } u < c. \\ 0, & \text{else.} \end{cases} \tag{17}$$

and then we can write the probability as

$$\Pr\{u < c\} = \int_0^1 \phi(\alpha) d\alpha \tag{18}$$

The problem with this definition is that the indicator function, $\phi(\alpha)$, is not necessarily continuous. Indeed, because there is no order to the realizations (they are assumed independent), we could arrange them any way we want. Imagine two possible representations of the indicator function: one continuous almost everywhere and one discontinuous. The discontinuous indicator function is not actually integrable in the Riemann sense. Riemann says that we divide the α-axis into narrow segments of width $\Delta\alpha$, multiply the width of each segment by the average value of that segment's

We can allow only occasional discontinuities and still retain something that can usefully be integrated by Riemann's method

indicator ϕ, and sum over all segments. If the indicator is not continuous almost everywhere, then there is no way to draw segments which span more than a single point.

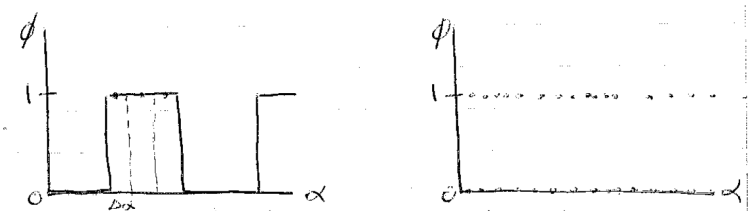


Figure 5: Plot of the indicator function for an ensemble: (Left) continuous almost everywhere; (Right) discontinuous.

If we can't draw vertical segments of width $\Delta\alpha$, maybe a better approach is to draw horizontal segments of width $\Delta\phi$. By doing so, we no longer require a continuous indicator function. All we require is that the sum over all of the segments is ultimately finite in some sense. This sort of integration with horizontal segments is called Lebesgue integration.

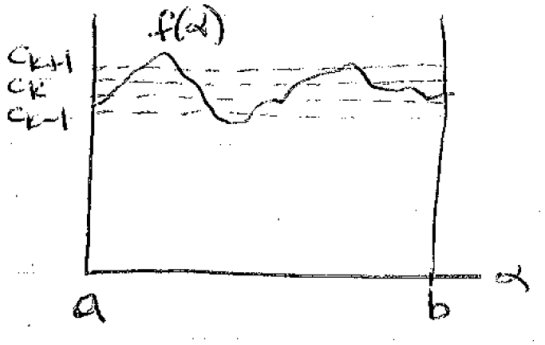


Figure 6: The general case of Lebesgue integration of a function $f(\alpha)$ over the domain $\alpha \in [a, b]$ with segments denoted by c_{k-1}, c_k, c_{k+1}, etc.

In order to apply Lebesgue integration to a general function $f(\alpha)$, we need to define some expression for the total length of all the intervals of $f(\alpha)$ that fall between the edges of a given segment, $[c_{k-1}, c_k)$. We call this expression (which is really just a generalized form of the length of the function) a measure, μ, and we write it as

$$\mu(\{\alpha | c_{k-1} \leq f(\alpha) < c_k\}) \tag{19}$$

and now we can easily write the integral of the function $f(\alpha)$ as

$$\int_a^b f(\alpha) d\alpha \approx \sum_k \hat{c}_k \cdot \mu(\{\alpha | c_{k-1} \leq f < c_k\}) \tag{20}$$

where $\hat{c}_k$ is some representaive value of the segment spanning between $[c_{k-1}, c_k)$. As with Riemann integration, we make these segments narrorwer and narrower until we reach a limit.

$\hat{c}_k$ could be equal to c_{k-1} in which case we have something akin to the left-hand Riemann sum, i.e. we approximate the value of the segment by the value of one of its endpoints.

Returning to our original problem with indicator function $\phi(\alpha)$, we can substitute into the above definition to obtain

$$\int_0^1 \phi(\alpha) d\alpha \approx 0 \cdot \mu(\{\alpha | -\infty \leq \phi < 1\}) + 1 \cdot \mu(\{\alpha | 1 \leq \phi < \infty\}) = \mu(\{\alpha | u < c\}) \tag{21}$$

where the sum includes just two segments, $(-\infty, 1)$ and $[1, \infty)$. Now it might seem like this Lebesgue integration is just a formality, but in fact it is very important as we develop the concept of statistical moments.

Ultimately, we are interested in the probabalistic distribution of u, which means considering its probable occurence with respect to various values of c, not just a single value. So we can define the probability as a function of c, such that

$$\Pr\{u < c\} = F(c) \quad (= \mu(\{\alpha | u < c\})) \tag{22}$$

where this function, $F(c)$ is called the cumulative distribution function (CDF). We know by definition that

$$0 \leq F(x) \leq 1 \tag{23}$$

and that the CDF is monotonic, such that

$$F(c_1) \leq F(c_2) \quad \text{if} \quad c_1 \leq c_2 \tag{24}$$

and that the limits of the function are well-defined as

$$F(-\infty) = 0 \quad \text{and} \quad F(\infty) = 1 \tag{25}$$

With this in mind, we can define a more sophisticated version of the indicator function which indicates the presence of a value within a range (as opposed to a simple one-sided comparison). We start with the traditional indicator and define a modified form with the cutoff $c + \Delta$

$$\phi(\alpha) = \begin{cases} 1, & \text{if } u < c. \\ 0, & \text{else.} \end{cases} \quad \text{and} \quad \phi'(\alpha) = \begin{cases} 1, & \text{if } u < c + \Delta. \\ 0, & \text{else.} \end{cases} \tag{26}$$

and then we can define a new indicator function $\phi = \phi'(1 - \phi)$ which behaves as

We note that $(1 - \phi)$ basically reverses the behavior of the old indicator function; and that $\phi\phi' = \phi$ since everywhere that is ϕ is true, ϕ' is automatically true also.

$$\psi(\alpha) = \begin{cases} 1, & \text{if } c \leq u < (c + \Delta) \text{ for } \Delta > 0. \\ 0, & \text{else.} \end{cases} \tag{27}$$

This new indicator function, ϕ, allows us to define the probability over a range of c as

$$\Pr\{c \leq u < (c + \Delta)\} = \int_0^1 \psi(\alpha) d\alpha \tag{28}$$

If we write this integral out fully, we obtain

$$\int_0^1 [\phi' - \phi'\phi] d\alpha = \int_0^1 [\phi' - \phi] d\alpha = F(c + \Delta) - F(c) \tag{29}$$

So the probability of finding u over fixed range $[c, c + \Delta)$ can be calculated by simply substracting the CDF at the two endpoints of

the given range. If we want to find the probabilty at a single value of c (not over a range of c values), then we simply need to take the limit as $\Delta \to 0$.

$$\lim_{\Delta \to 0} = \frac{F(c+\Delta) - F(c)}{\Delta} = \frac{dF(c)}{dc} \equiv B(c) \tag{30}$$

This limit is defined as the probability density function (PDF). It is called a density because it shows the probabilty per unit extent of the range of possible values. The integral of the probability density yields the probability itself, according to

And of course we recognize that $\int_{-\infty}^{\infty} B(c)dc = 1$ from the known limits of the CDF, $F(c)$.

$$\Pr\{c \leq u < (c+\Delta)\} = \int_{c}^{c+\Delta} B(c')dc' = F(c+\Delta) - F(c) \tag{31}$$

Now we need to remind ourselves what we are looking for: we are trying to represent an ensemble average of u. So we don't care about the probability that $u = c$ but rather we want to know the expected (ensemble-averaged) value of u given all of the possible probabilities of its being equal to different values (in different realizations). Said more generally, we want to know the expected value of some function of our random variable u, call it $f(u)$. We denote the expected value as $\mathbb{E}\{f(u)\}$ defined as

For our definition of α from earlier, the limits $[a, b] = [0, 1]$ which we substitute automatically from here on.

$$\mathbb{E}\{f(u)\} = \frac{1}{b-a} \int_{a}^{b} f(u(\alpha))d\alpha \tag{32}$$

and here is where the Lebesgue integration becomes very useful. We can rewrite this integral in terms of the measure of $f(u)$ such that

$$\int_{0}^{1} f(u(\alpha))d\alpha \approx \sum_{k} f(\hat{c}_k) \cdot \mu(\{\alpha | c_k \leq f < c_{k+1}\}) \tag{33}$$

but we know that the measure μ is given by

$$\mu(\{\alpha | c_k \leq f < c_{k+1}\}) = F(c_{k+1}) - F(c_k) \tag{34}$$

and in the limit as $\Delta c \to 0$ this is given by $dF(c) = B(c)dc$, so that we can then write

We replace the summation over k with an integral over c in the limit that the segments become infinitessimally narrow

$$\int_{0}^{1} f(u(\alpha))d\alpha = \int_{-\infty}^{\infty} f(c)B(c)dc = \mathbb{E}\{f(u)\} \tag{35}$$

By using the probability density function of a random variable u, we can calculate the expected value (ensemble average) of any function of that random variable, $f(u)$. We can develop the same result for a pair of random variables u and v. Then we have the joint CDF

$$\Pr\{u < c, v < c'\} = F(c, c') \tag{36}$$

and the joint PDF

$$\Pr\{a \leq u < b, m \leq v < n\} = \int_a^b \int_m^n B(c, c')dcdc' \tag{37}$$

and the joint expectation value

$$\mathbb{E}\{f(u, v)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(c, c')B(c, c')dcdc' \tag{38}$$

Moments of Random Variables

We developed the tools to calculate the expected value of any function of a random variable. One of the most useful expected values of interest is $\mathbb{E}\{u^n\}$, called the nth raw moment of u.

For the nth raw moment, the function $f(u) = u^n$ so $f(c) = c^n$

$$\mathbb{E}\{u^n\} = \int_{-\infty}^{\infty} c^n B(c)dc \tag{39}$$

We can also calculate the 'central moment', i.e. the moment in which the random variable is re-centered about its mean (its first raw moment, $n = 1$) yielding

Note that $\mathbb{E}\{u\}$ is not a function of c, and thus there is no interchange of variables when substituting into $f(c)$.

$$\mathbb{E}\{(u - \mathbb{E}\{u\})^n\} = \int_{-\infty}^{\infty} (c - \mathbb{E}\{u\})^n B(c)dc \tag{40}$$

These central moments are so famous that they are given specific names when they are written in normalized form.

Central Moment	Definition
Mean	$\mu = \mathbb{E}\{u\}$
Variance	$\sigma^2 = \mathbb{E}\{(u-\mu)^2\}$
Standard Deviation	σ
Skewness	$S = \frac{1}{\sigma^3}\mathbb{E}\{(u-\mu)^3\}$
Kurtosis	$K = \frac{1}{\sigma^4}\mathbb{E}\{(u-\mu)^4\}$
Covariance	$Q_{uv} = \mathbb{E}\{(u-\mu_u)(v-\mu_v)\}$
Correlation Coefficient	$R_{uv} = \frac{1}{\sigma_u\sigma_v}\mathbb{E}\{(u-\mu_u)(v-\mu_v)\}$

Table 1: Common central moments, where subscripts refer to random variable names. Without the normalization prefactor, $\frac{1}{\sigma^n}$, the moments are referred to by number only, e.g. nth central moment

What do these moments represent (physically) about the distribution of a random variable?

The first thing to note is that as the order of the moment increases, the contribution from the tails of the PDF, $B(c)$, to the value of the moment, increases. The higher the order of the moment, the more accurately the tails of the PDF need to be represented to resolve the moment correctly. Because the tails of the PDF represent rare events, large amounts of data (including many, many realizations) are then needed to fully represent the tails of the PDF, which means resolving higher order moments is difficult.

Essentially, higher moments weight larger values of c by a power n, thus inflating the importance of the tails, where c is largest.

The moments also convey information based on whether they are even or odd. Even moments include the contribution from only even parts of the PDF, so that they are essentially measures

The integral $\int f(c)B(c)dc$ can be divided into $\int f(c)B_{\text{even}}(c)dc + \int f(c)B_{\text{odd}}(c)dc$ and we know that the integral is zero if the integrand is odd, hence, e.g. odd moment functions $f(c)$ include the contribution from $B_{\text{odd}}(c)$ only, and similarly with even moments.

of the width of the PDF. Odd moments include only contributions from odd parts of the PDF, and thus measure the asymmetry of the distribution tails. So the skewness, S, based on the third moment, represents the extent that the tails of the PDF are biased: positive skewness means the positive (right) tail is longer than the negative (left) tail; negative skewness represents the reverse.

Figure 7: PDFs, $B(c)$, for (left) symmetric, (center) negative skew, and (right) positive skew distributions. The tail is drawn over top of the symmetric distribution to emphasize the direction of the bias.

The covariance matrix, Q_{uv}, is of particular interest to the study of turbulence, since it will help us define relationship between two different quantities (e.g. velocity and pressure, or velocity at two different locations, or velocity at two different times). We already wrote that coherent motions, according to Robinson, are defined by correlation of flow variables over space or time, so the covariance matrix provides a tool for studying the coherent motions of turbulence. We can plot the level sets of joint PDFs in order to intepret magnitude and form of the correlation between different variables.

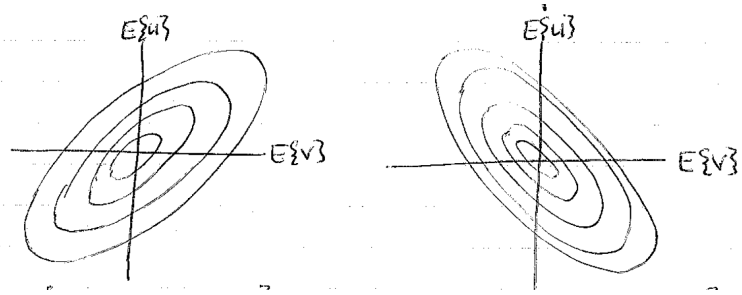


Figure 8: Two-dimensional covariance matrix level sets: (Left) $Q_{uv} = \mathbb{E}\{(u - \mu_u)(v - \mu_v)\} > 0$ and (Right) $Q_{uv} = \mathbb{E}\{(u - \mu_u)(v - \mu_v)\} < 0$

Now that we have defined the whole statistical apparatus for describing averages and higher order moments of the fluctuating quantities of fluid mechanics, we have the practical question: is this relevant to what we actually do in a laboratory? Do we actually measure all possible realizations of a given flow in order to describe it statistically? Of course not. Usually, we do a small snumber of experiments (sometimes just one) and then perform temporal or spatial averaging within that single realization to obtain averages. So we have gone in a circle in our analysis of ensemble averages: we wanted to avoid defining averages in terms of time/space averages, because we didn't want out definitions to depend on any single averaging technique, so intead we defined ensemble averages. But now we have to ask under what circumstances can we use standard time/space averaging in place of ensembles? How close to regular averages get to our ideal of an ensemble average of all realizations?

Stationary Random Processes

The basic idea of using time/space averages in place of ensemble averages rests on the hope that the averaging interval becomes large enough that somehow it converges to the ensemble average, such that we can use one as the proxy for the other. This is called the ergodic hypothesis. But before we can get to checking whether this hypothesis is useful, we need to note that all of the PDF's we have written above can actually be written as function of time. Thus the problem of turbulence is really just the problem of tracking the time-evolution of PDF's of flow variables (instead of the flow variables themselves). The only difference from our earlier definitions is to write the flow variable $u(t)$ explicitly as a function of time. We can then employ the same definitions for CDF

i.e. the PDF represents the distribution of some flow quantity across all possible realizations, evaluated at a particular time, t

u is called a random variable; $u(t)$ is called a random process.

$$\Pr\{u(t) < c\} = F(c,t) \tag{41}$$

and PDF

$$\frac{\partial F(c,t)}{\partial c} = B(c,t) \tag{42}$$

as earlier. We can also define a joint-distribution between a single flow variable at N different times (instead of two or more flow variables at the same instant, like we wrote above)

$$\Pr\{u(t_1) < c_1, u(t_2) < c_2, \ldots, u(t_N) < c_N\} = F(c_1,t_1;c_2,t_2;\ldots;c_N,t_N) \tag{43}$$

and this could be expanded for multiple variables at multiple times. But these joint distributions in time are really quite unwieldy, since they explicitly depend on N different time points. Ideally, we would like the distributions to depend on a time-interval (i.e. the difference between each of the N time points), instead of the actual times themselves. In other words, we would like to be able to assume that multi-time statistical quantities calculated from a given random process, $u(t)$, were invariant under a shift in time, s. Such a process is called statistically stationary. The PDF of a stationary process satisfies

$$B(c_1,t_1;c_2,t_2;\ldots;c_N,t_N) = B(c_1,t_1+s;c_2,t_2+s;\ldots;c_N,t_N+s) \tag{44}$$

We can then rewrite the covariance for a process $u(t)$ with zero mean ($\mu = 0$) as

Note: from here on, u without a prime $(\cdot)'$ will always represent a zero-mean random process. The averate is denoted with $\overline{(\cdot)}$ and instantaneous values with a tilde, $\tilde{(\cdot)}$.

$$Q(t,t') = \mathbb{E}\{u(t)u(t')\} \tag{45}$$

where the two time points, t and t' are separated by shift s, according to $t' = t + s$. If the process is stationary, then the statistics depend on only s and we can write

Note that the dependence on t is integrated out during the expectation operation, $\mathbb{E}$, hence the dependence is on s alone.

$$Q(s) = \mathbb{E}\{u(t)u(t+s)\} \tag{46}$$

which is usually called the autocovariance. The autocorrelation

function is just the normalized version, written as

$$R(s) = \frac{\mathbb{E}\{u(t)u(t+s)\}}{\mathbb{E}\{u(t)u(t)\}} \tag{47}$$

We can also see that the autocovariance can be written with either a positive or negative lag, s, according to the stationary assumption, and thus

$$\underbrace{\mathbb{E}\{u(t)u(t+s)\}}_{Q(s)} = \underbrace{\mathbb{E}\{u(t-s)u(t)\}}_{Q(-s)} \tag{48}$$

from which we conclude that the covariance function of a stationary process is even.

We could easily rewrite these results for a spatial process, $u(x)$, and a spatial lag, r, with $x' = x + r$. We will see this type of spatial covariance more as we discussion the structure of turbulence.

What happens to $R(s)$ as s increases? For a physical process, we expect the correlation between flow variables to drop off as spatial or temporal spacing increases (since the physical forces linking the variables tend to decay with distance or time).

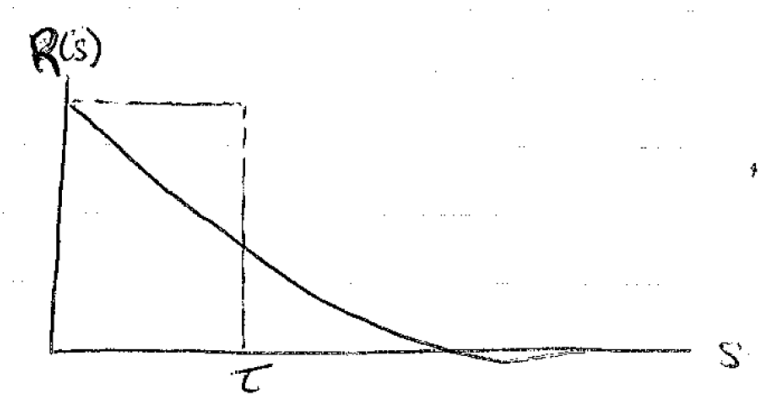


Figure 9: The auto-cross-correlation function, $R(s)$, for a stationary random process.

Because the even function of $R(s)$ decays, we can define a characteristic length scale of that decay, called the integral time scale, τ, as

$$\tau = \int_0^\infty R(s)ds \tag{49}$$

which we expect to converge. This scale thus characterizes the largest scales of space or time over which there is still some discernable correlation. We can also use $R(s)$ to develop a characteristic scale for the smallest scale coherence, in the limit as $s \to 0$. We write the Taylor series about this limit, knowing that $R(0) = 1$ and that the function is even

There is a missing factor of 2 in the definition of Taylor's microscale given in Tennekes & Lumley eq. 6.4.6

$$R(s) = 1 - \frac{s^2}{2\lambda^2} + \mathcal{O}(s^4) \tag{50}$$

Solving for the characteristic length scale of the second term in the Taylor expansion, λ, by means of differentiation, yields

$$\left.\frac{d^2R(s)}{ds^2}\right|_{s=0} = -\frac{1}{\lambda^2} \tag{51}$$

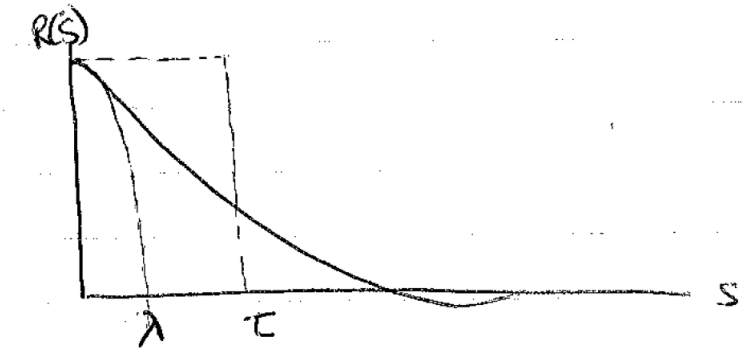


Figure 10: The auto-cross-correlation function, $R(s)$, for a stationary random process with both the integral length scale, τ and the Taylor microscale, λ, marked. The Taylor scale is found via a parabolic fit to $R(s)$ about the origin.

and we call this scale, λ, the Taylor microscale.

All of the above analyis applies to both temporal and spatial covariance functions. When working in spatial coordinates, instead of using the word stationary we describe the random process as homogeneous, meaning that all statistics are invariant to a shift in space, r. For simultaneous time measurements in a stationary and homogenous process, we can write the covariance between velocity vector u_i at location $\vec{x}$ and velocity vector u_j at $\vec{x}+\vec{r}$ as

Imagine the bark of a tree as an example of a homogeneous field, where measurements of the bark roughness can be taken in the vicinity of any spatial origin on the tree's surface.

$$Q_{ij}(r) = \mathbb{E}\{u_i(\vec{x},t)u_j(\vec{x}+\vec{r},t)\} \tag{52}$$

where we have used index notation to indentify the vectors and we write the covariance as a two-dimensional tensor. . The integral scale for the u_1 component of velocity, L_{11}, is given by

Because we have written the covariance as a general tensor, there are 9 different characteristic length scales for both the integral and micro-scales, so we need to identify the component of the random process for the characteristic scale of interest

$$L_{11} = \int_0^\infty R_{11}(r)dr \tag{53}$$

With this background, we can now return to our original question about the ergodic hypothesis: what must be assume in order to treat time or space averaging as equivalent to ensemble averaging?

The Ergodic Hypothesis

Let's define the experimental mean of a velocity component $u(t)$ over some averaging period T as

$$\overline{u}(t;T) = \frac{1}{T}\int_{-T/2}^{T/2} u(t+s)ds \quad \stackrel{?}{=} \quad \mathbb{E}\{u\}(t) \tag{54}$$

and we want to know if it converges to the ensemble average value (the expectation value over all realizations), called $\mathbb{E}\{u\}(t)$. Because $\overline{u}$ is based on just one realization, and $\mathbb{E}\{u\}(t)$ is based on all realizations, we need some way to relate a given value within a distribution to the average of all values in the distribution. Such a relation is called Chebyshev's Inequality and is written as

The probability that the difference between a realization and the ensemble average is bigger than some fixed amount, ε, is bounded.

$$\Pr\{|\overline{u} - \mathbb{E}\{u\}|^2 \geq \varepsilon^2\} \leq \frac{\mathbb{E}\{|\overline{u} - \mathbb{E}\{u\}|^2\}}{\varepsilon^2} \tag{55}$$

This means that, in any random distribution, 'nearly all' of the individual values (realizations) are 'close' to the average. Now, for the ergodic hypothesis to be true, we need to require that $\mathbb{E}\{|\overline{u} - \mathbb{E}\{u\}|^2\} = 0$ somehow. In particular we must require that, for a sufficiently long time-average, as $T \to \infty$ that

$$\lim_{T\to\infty} \mathbb{E}\{|\overline{u}(T) - \mathbb{E}\{u\}|^2\} = 0 \tag{56}$$

This is called the condition of convergence in the quadratic mean. If this convergence is true, then we can conclude that

$$\lim_{T\to\infty} \Pr\{|\overline{u}(T) - \mathbb{E}\{u\}| \geq \varepsilon\} = 0 \tag{57}$$

which is a very powerful statement. It tells us that the probability that the experimental average from one realization deviates from the ensemble average by more than ε tends to zero as $T \to \infty$, which is essentially the ergodic hypothesis: we can use time averages of sufficient length in place of ensemble averages. Of course, this result depends on the condition of convergence in the quadratic mean, so we now need to establish exactly when $\mathbb{E}\{|\overline{u} - \mathbb{E}\{u\}|^2\} = 0$.

We can write the variance as

$$\mathbb{E}\{|\overline{u} - \mathbb{E}\{u\}|^2\} = \mathbb{E}\left\{ \left| \frac{1}{T} \int_{-T/2}^{T/2} [u(t+s) - \mathbb{E}\{u\}] ds \right|^2 \right\} \tag{58}$$

We can change the order of operations because $\mathbb{E}$ is a function of t and the integration is over s; we can then rewrite the squared integral in terms of two independent integrals in s and s'.

and then exchanging the time-integration and ensemble-averaging operations

$$\mathbb{E}\{|\overline{u} - \mathbb{E}\{u\}|^2\} = \frac{1}{T^2} \left| \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \underbrace{\mathbb{E}\big\{ \big(u(t+s) - \mathbb{E}\{u\}\big)\big(u(t+s') - \mathbb{E}\{u\}\big) \big\}}_{Q(s-s')} ds ds' \right| \tag{59}$$

and we recognize the integrand of this double integral as the covariance function, $Q(s - s')$. We can rewrite this time difference as $\hat{s} = s - s'$ for a stationary process and then, since the covariance is even, it can be shown that a necessary and sufficient condition of convergence is equivalent to

For the formal proof, see Monin & Yaglom, vol 1, p254, where the limit of this double integral is shown to be equivalent to the limit of the single integral shown below

$$\lim_{T\to\infty} \mathbb{E}\{|\overline{u}(T) - \mathbb{E}\{u\}|^2\} = 0 \quad \Longleftrightarrow \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T Q(\hat{s}) d\hat{s} = 0 \tag{60}$$

In other words, if the integral of the covariance is zero for sufficiently large time time samples (as it usually is), then this is a necessary and sufficient condition for convergence of experimental and ensemble averages. We can also approximate the error associated with the ergodic assumption.

This still assumes that the random process is stationary, which may not be satisfied in turbulence measurements

If $Q(s)$ decays quickly enough such that

$$\int_0^\infty Q(s)ds < \infty \tag{61}$$

i.e. that the integral converges to a finite value

then we can calculate a correlation time associated with the decay (the integral timescale)

$$\tau = \frac{\int_0^\infty Q(s)ds}{Q(0)} \tag{62}$$

and then we can write, asymptotically

$$\mathbb{E}\{|\overline{u}(T) - \mathbb{E}\{u\}|^2\} \sim Q(0)\frac{\tau}{T} \tag{63}$$

We take $\frac{1}{T}\int_0^T (\cdot)ds' \sim 1$ when writing the order of magnitude of definition of the condition of convergence.

and thus, for a desired level of mean-square error, we can approximate the length of the time sample necessary in order to satisfy the ergodic hypothesis. Of course, we need to know the covariance function first, so this ergodic check is usually performed post-facto.

Now we have the tools to decompose turbulent flows into average and fluctuating components, where the average is defined in the ensemble sense. This decomposition depends on the ability to distinguish between large and small scale motions in turbulence, and we have shown how that can be done in spacce and time by means of the covariance function. But, turbulence contains a range of scales, not just large and small. So the next goal is to develop techniques to describe the full range of scales within turbulence, besides the integral and micro-scales we already defined.

Describing the Eddy Population

We've now seen how the covariance function can be used to describe the extent of correlation between different fluctuations in a turbulence signal. We can define a characteristic scale for the largest times or distances over which correlations exist, the integral scale, τ or L, which represents the largest coherent structures in turbulence. We can also define a characteristic scale for the smallest times and distances, the Taylor micro-scale, λ. But what we really want is a systematic way of describing coherence in turbulent signals at all different length/time scales, because it is within these intermediate scales that energy is transferred, and if we want to understand the physics of turbulence, we need to understand how energy gets transferred from the largest scales down to the smallest scales.

Structure Functions

The first technique for describing the distribution of eddies in a turbulent field was developed by Kolmogorov. The basic idea is that we want to construct a function of argument r where the argument is directly related to the energetic content of eddies with characteristic size r. We already have a auto-covariance function, $Q(\vec{r})$, which we can write in its spatial form as

i.e. eddies which have a coherent structure with characteristic dimension r as measured through measurements of the flow field. Typically we thick of this dimension as the radius or diameter of an eddy.

$$Q(\vec{r}) = \overline{u(\vec{x})u(\vec{x}+\vec{r})} \tag{1}$$

Note that we are using the ensemble averaging operation and the time/space averaging interchangeably from here on in the text, assuming that the ergodic hypothesis is always satisfied. We are also assuming that u represents the zero-mean fluctuation, as usual.

This covariance tensor is actually a specific case of a more general way of describing the correlation between two random processes. To introduce this more general method, let's think about what it is we actually want to represent in terms of an eddy. Consider a turbulent signal (in space) as shown in figure 11

What we want is some way of describing the energy of coherent motions with a particular size, r. This means our measure should receive contributions from coherent motions separated by no greater than distance r. Consider two measurement points, A and B separated by r. If both of these points stood within a very large coherent structure (much bigger than r), then the difference in velocity measurements between A and B would seem insignificant (the velocities at the two points would be measured in roughly the same part of this very large eddy). But if A and B stood within an

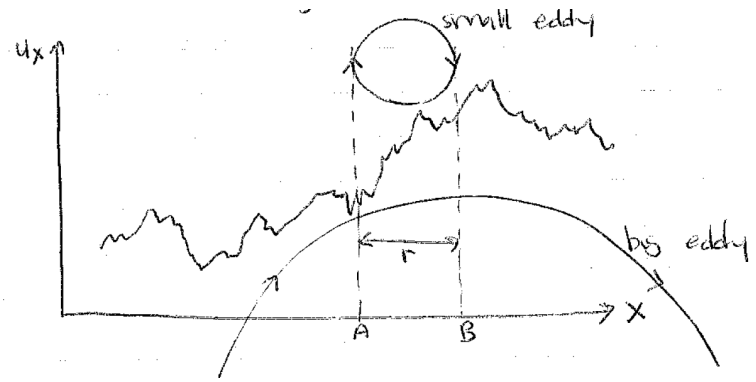


Figure 11: A sketch of a spatial random process $u(x)$ and two circular eddies. Measurements are recorded at two points, A and B separated by distane r, which is small compared to the large eddy and approximately equivalent to the size of the small eddy.

eddy with coherence scale r, then the difference between the two velocity measurements would be very significnat (the velocities would be measured at opposite sides of the eddy). Notice that we are describing the presence of large or small eddies in terms of the velocity **difference**, defined as $(u_A - u_B)$, such that this difference is very small (≈ 0) for eddies much larger than r (i.e. no contribution) and the difference is significant for eddies of size r (i.e. measurable contribution). This is very different from the covariance definition we previously discussed, which is based on the product of the two measurements. Why do we prefer a difference instead of a product?

The problem with using a product of the form $u_A u_B$ is that the product is sensitive to the absolute magnitude of the two components. Imagine shifting the two measurement points to a new origin (with r fixed). There are random signals for which this shift will result in the product changing substatially while the difference remains roughly the same. Slide the segment A-B in the figure slightly to the left to observe this effect. What kind of signal has statistics which depend on the origin of measurements and not just the distance between them? Non-stationary signals. If the signal is stationary, then the origin of measurement doesn't matter, and both products and differences can be used equivalently. To use the difference measure, we need assume only that the signal composed of velocity differences is stationary, but not that the velocity itself is stationary.

There are also signals for which both will change! But there are no signals for which the difference changes and the product does not, so the difference is the more robust measure

The most general (and intuitive) way of defining the correlation is therefore via differences (since this requires less restrictive assumptions of stationarity). In 1941, Kolmogorov began the development of his turbulence theory by defining an nth-order structure function, $D(\vec{r})$ as

$$D(\vec{r}) = \overline{(u(\vec{x}+\vec{r}) - u(\vec{x}))^n} \tag{2}$$

For the purposes of meauring energetic content of eddies, we use the second-order structure function, defined as

$$D(\vec{r}) = \overline{(u(\vec{x}+\vec{r}) - u(\vec{x}))^2} \tag{3}$$

This should represent the cumulative kinetic energy of eddies of characteristic size r or less

Throughout the text, we refer to kinetic energy on a per-unit-mass basis and thus we exclude the factor of ρ. The energy density (i.e. energy per unit size r of an eddy) is given by $\frac{dD}{dr}$ and thus the energy contained in a particular eddy of size r is just $r\frac{dD}{dr}$

If the structure function, D, is the most general way of representing energy content at specific scales of coherence, then we expect that we can relate the structure function to the more particular measure of coherence, the covariance function, Q. Let's write the structure function in full form, for vector velocities

$$D_{ij}(\vec{r}) = \overline{(u_i(\vec{x}+\vec{r}) - u_i(\vec{x}))(u_j(\vec{x}+\vec{r}) - u_j(\vec{x}))} \tag{4}$$

and expanding, with the assumption that the signal of differences, Δu_i is stationary, but not the signal itself, yields

Recall that $Q_{ij}(\vec{r}) = \overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})}$

$$D_{ij}(\vec{r}) = \underbrace{\overline{u_i(\vec{x}+\vec{r})u_j(\vec{x}+\vec{r})}}_{Q_{ij}(\vec{x}+\vec{r},0)} - \underbrace{\overline{u_i(\vec{x}+\vec{r})u_j(\vec{x})}}_{Q_{ij}(\vec{x},-\vec{r})} - \underbrace{\overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})}}_{Q_{ij}(\vec{x},\vec{r})} + \underbrace{\overline{u_i(\vec{x})u_j(\vec{x})}}_{Q_{ij}(\vec{x},0)} \tag{5}$$

Now, if we assume that the signal is, itself, stationary, then we lose the $\vec{x}$ dependence in all of the covariance matrices (which are also now even) and thus

$$D_{ij}(\vec{r}) = 2\big[Q_{ij}(0) - Q_{ij}(\vec{r})\big] \tag{6}$$

or, in other words, if the process is stationary, then we can express the structure function explicitly in terms of the covariance matrices. And by differentiating it is easy to show that

$$\frac{d}{d\vec{r}}D_{ij}(\vec{r}) = -2\frac{d}{d\vec{r}}Q_{ij}(\vec{r}) \tag{7}$$

so the energy densities of the two measures are multiples of each other. But what about calculating the covariance given the structure function? Because there are two unknowns in the covariance ($\vec{r} = 0$ and $\vec{r}$), it is not possible to write the covariance from the structure function without an additional assumption. It is often reasonable to assume that $\lim_{\vec{r}\to\infty} Q(\vec{r}) = 0$ based on physical arguments, in which case we obtain

$$\lim_{\vec{r}\to\infty} D_{ij}(\vec{r}) = 2Q_{ij}(0) \tag{8}$$

and therefore we can write

$$D_{ij}(\vec{r}) = D_{ij}(\infty) - 2Q_{ij}(\vec{r}) \tag{9}$$

The problem here is that we need to calculate $D_{ij}(\infty)$ but the structure function typically converges very slowly, so it is very difficult to calculate this quantity accurately

and thus we can finally write the covariance in terms of the structure function as

$$Q_{ij}(\vec{r}) = \frac{1}{2}\big[D_{ij}(\infty) - D_{ij}(\vec{r})\big] \tag{10}$$

Therefore, for stationary processes with reasonable convergence assumptions, the structure function and covariance descriptions are interchangeable. Without such assumptions, the structure function is the more basic and robust tool for characterizing the coherent motions of an arbitrary random signal.

Indeed, the structure function is widely used in financial modeling for non-stationary price signals

Energy Spectra

We now have two tools for describing the energy content within coherent motions of arbitrary size. However, we haven't considered one of the most natural ways of thinking about the decomposition of signals into individual scales of motion: a spectral decomposition. Kolmogorov (1940) showed that for a stationary, random process $u(t)$, we can write

$$u(t) = \int_{-\infty}^{\infty} e^{ikt}\hat{u}(k)dk \tag{11}$$

Wavenumber k is related to wavelength λ by $k = \frac{2\pi}{\lambda}$. Frequency f is given by $\frac{1}{\lambda}$. Radial frequency is given by $\omega = 2\pi f$.

where $\hat{u}(k)$ is a complex function of wavenumber k. This means that 'any stationary random process can be replaced (to any required approximation) by the sum of uncorrelated harmonic oscillations with random amplitudes and phases'.

See Monin & Yaglom, vol 2, p 5. The phase information is contained implicitly within $\hat{u}(k)$

This means that we can isolate different spectral components of a random process by means of filtering. Let's say we have a filter of bandwidth, Δk which spans wavenumbers $[k_1, k_2]$. The output of the filter is then

$$u(\Delta k, t) = \int_{k_1}^{k_2} e^{ikt}\hat{u}(k)dk + \int_{-k_2}^{-k_1} e^{ikt}\hat{u}(k)dk = 2\mathbb{R}\left\{\int_{k_1}^{k_2} e^{ikt}\hat{u}(k)dk\right\} \tag{12}$$

$\mathbb{R}\{\cdot\}$ denotes the real part of a complex number.

where '$u(\Delta k, t)$ is the spectral component of the process $u(t)$ corresponding to the wavenumber range Δk'. Now, if our signal were periodic, each of these contributions would be entirely independent, i.e. there would be no overlap between frequency intervals. The spectrum would be discrete. But if our signal is not periodic, i.e. its spectrum is continuous, then we need to ask the question: how well can we actually separate different spectral contributions? In other words, by writing

$$u(t) = \int_{-\Delta k/2}^{\Delta k/2} e^{ikt}\hat{u}(k)dk = \int_{-\infty}^{\infty} e^{ikt}\hat{H}(k)\hat{u}(k)dk \tag{13}$$

we are really applying a box-filter in the frequency domain, where the box filter is defined as

$$\hat{H}(k) = \begin{cases} 1, & -\frac{\Delta k}{2} < k < \frac{\Delta k}{2}. \\ 0, & \text{else.} \end{cases} \tag{14}$$

We can rewrite this integral by using the covolution theorem which states that

$$\mathcal{F}\{\hat{H}(k)\hat{u}(k)\} = \mathcal{F}\{\hat{H}(k)\} * \mathcal{F}\{\hat{u}(k)\} \tag{15}$$

where convolution is denoted by $*$, and the Fourier transform operation is denoted by $\mathcal{F}$.. The convolution is then given by

$\mathcal{F}\{\hat{u}(k)\} = u(t)$ and $\mathcal{F}\{\hat{H}(k)\} = \frac{\sin\left(\frac{t}{2}\Delta k\right)}{t/2}$

$$\mathcal{F}\{\hat{H}(k)\hat{u}(k)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin\left(\frac{s}{2}\Delta k\right)}{s/2}u(t-s)ds \tag{16}$$

which represents the convolution of the random process with a sinc function, i.e. a smoothing process. If we take our independent variable t to be spatial x, and write the wavenumber as wavelength λ, then the smoothing kernel is given by

$$\frac{\sin\left(\frac{x\pi}{\lambda}\right)}{x\pi} \tag{17}$$

which admits scales less than λ (in physical space). Meaning that, even though we are making a sharp cut in spectral space, this does not translate to a sharp cut in physical space — there is no sharp distinction between physical eddies. Similarly, if we made a sharp cut between different sizes of physical eddies, the spectral space will show only a blurred distinction between different wavenumbers. So spectrally, there is no way to discriminate among different physical eddies without having some overlap between the different categories of eddy size. Despite this problem, the spectral approach is a very convenient and powerful tool for describing the population of different sized eddies in a turbulent velocity field. Writing the spectral decomposition in the spatial domain in three dimensions (for a single component of velocity) yields

$$u(\vec{r}) = \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}}\hat{u}(\vec{k})d^3\vec{k} \tag{18}$$

We always combine the factor of $\frac{1}{2\pi}$ per dimension to the integral over space, for consistency. Also note that $d^3\vec{x}$ is equivalent to the triple integral over all space, $dx_1dx_2dx_3$ and similarly for wavenumber.

and

$$\hat{u}(\vec{k}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{-i\vec{k}\cdot\vec{r}}u(\vec{r})d^3\vec{x} \tag{19}$$

where we are integrating over all of physical or wavenumber space. Now, we can decompose a single random process into its constituent scales, but what we really want to know is how the kinetic energy is distributed across those scales, not the velocity fluctuation. In other words, we want to know how $\overline{u_iu_i}$ is distributed. This reminds us of the covariance tensor, $Q_{ij} = \overline{u_i(\vec{r}_1)u_j(\vec{r}_2)}$ where $\vec{r} = \vec{r}_2 - \vec{r}_1$, so let's consider calculating the spectral decomposition of the covariance.

We write

$$Q_{ij}(\vec{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\vec{k}_1\cdot\vec{r}_1+\vec{k}_2\cdot\vec{r}_2)}\overline{\hat{u}_i(\vec{k}_1)\hat{u}_j(\vec{k}_2)}d^3\vec{k}_1d^3\vec{k}_2 \tag{20}$$

and, to enforce that Q is a function of $\vec{r}$ only (i.e homogeneity), we multiply by the Dirac delta function, $\delta(\vec{k}_1 + \vec{k}_2)$, such that the result

is nonzero only when $\vec{k}_1 = -\vec{k}_2$, which yields

$$Q_{ij}(\vec{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\vec{k}_1\cdot\vec{r}_1+\vec{k}_2\cdot\vec{r}_2)}\overline{\hat{u}_i(\vec{k}_1)\hat{u}_j(\vec{k}_2)}d^3\vec{k}_1 d^3\vec{k}_2\delta(\vec{k}_1+\vec{k}_2) \quad (21)$$

Then, integrating once, over $\vec{k}_1$, with the use of the Dirac delta, yields

We make use of the fact that the Fourier Transform is symmetric, such that $\hat{u}(\vec{k}_2) = \hat{u}(-\vec{k}_2)$

$$Q_{ij}(\vec{r}) = \int_{-\infty}^{\infty} e^{i(\vec{k}_2\cdot\vec{r})}\overline{\hat{u}_i(\vec{k}_2)\hat{u}_j(\vec{k}_2)}d^3\vec{k}_2 = \int_{-\infty}^{\infty} e^{i(\vec{k}\cdot\vec{r})}\overline{\hat{u}_i(\vec{k})\hat{u}_j(\vec{k})}d^3\vec{k} \quad (22)$$

and we can give a name to the Fourier transform of the covariance matrix, $Q_{ij}(\vec{r})$: the spectrum tensor, $\Phi_{ij}(\vec{k}) = \overline{\hat{u}_i(\vec{k})\hat{u}_j(\vec{k})}$, which satisfies the following transform pair

This result is really just the Wiener-Khinchin Theorem (or, more generally, the cross-correlation theorem) which relates the auto-correlation of a function to the absolute square of the Fourier transform of that same function.

$$\Phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{-i(\vec{k}\cdot\vec{r})}Q_{ij}(\vec{r})d^3\vec{r} \quad (23)$$

$$Q_{ij}(\vec{r}) = \int_{-\infty}^{\infty} e^{i(\vec{k}\cdot\vec{r})}\Phi_{ij}(\vec{k})d^3\vec{k} \quad (24)$$

We note immediately that the diagonal components of the spectrum tensor, Φ_{ij}, represent the spectral energy of the different directional velocity components. . Therefore, in order to think about the kinetic energy at a given point in space, it is useful to think about the contracted tensor, Φ_{ii} with $\vec{r} = 0$, which is written as

The off-diagonal componets represent the energy content in the non-normal Reynolds stresses, as we will discuss later

$$Q_{ii}(0) = \overline{u_i u_i} = \int_{-\infty}^{\infty}\Phi_{ii}(\vec{k})d^3\vec{k} \quad (25)$$

and the left-hand side is just twice the average kinetic energy. We can integrate over spherical shells of differential thickness centered at wavenumber $k = |\vec{k}|$ and write the kinetic energy for all fluctuations with wavenumber k as

$$E_{ii}(k) = \frac{1}{2}\int_{k=|\vec{k}|}\Phi_{ii}(\vec{k})4\pi k^2 dk \quad (26)$$

$E(k)$ is called the energy density because $E(k)dk$ represents the contribution to $\frac{1}{2}\overline{|\vec{u}|^2}$ from all eddies with wavenumbers in the range of $[k, k+dk)$.

or, we could define an energy density function as $E(k) = 2\pi k^2\Phi_{ii}(k)$ such that

$$\frac{1}{2}\overline{u_i u_i} = \int_0^{\infty} E(k)dk \quad (27)$$

Although we have written the energy density as if it were uniformly distributed in all directions (and hence shell integration would be meaningful), we haven't actually assumed any kind of symmetry in the distribution of kinetic energy. In other words, it is possible that more energy will be distributed in wavenumbers

in a particular direction than in other directions, and we will return to the topic of symmetries soon. But we now have the tools to describe the extent of correlation and the distribution of energy across different scales of turbulence, in three different ways: covariance functions (or PDFs), structure functions, and spectral densities. But we haven't discusses the dynamics of turbulence: how can we use these tools to describe the evolution of turbulence in space and time.

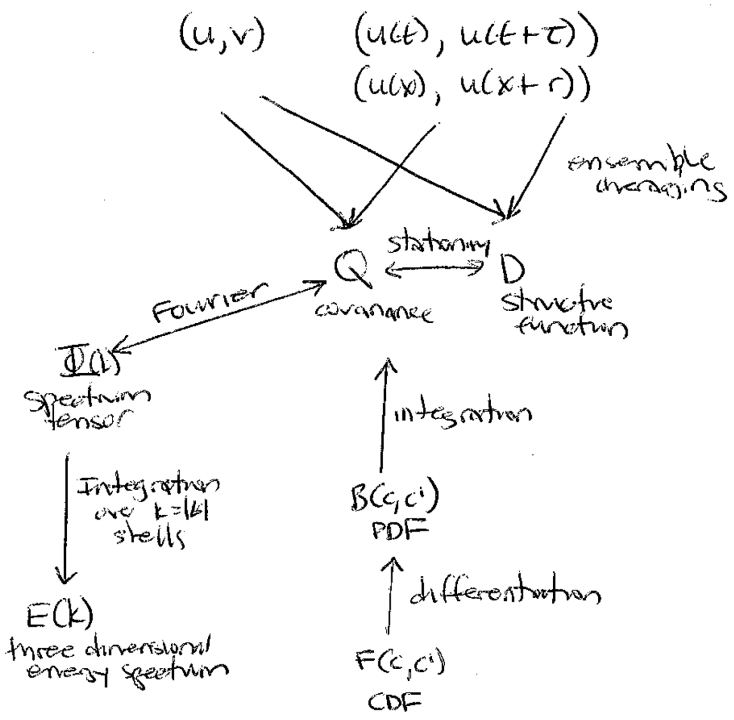


Figure 12: A summary of the different statistical measures we have developed to analyze velocity signals of different components, or with separation in time or space.

Turbulence Dynamics

We have learned a number of ways to describe the statistics of turbulence: the correlations between turbulent fluctuations over time and space, the distribution of the different scales of turbulence, and the distribution of kinetic energy across those scales. But what we need now are evolution equations which describe how these descriptions evolve in time. In order to understand the physics of turbulence, we need to understand how the kinetic energy within turbulence is distributed among different scales, how it shifts between those scales in time, and what ranges of scales are important energetically to the dynamical processes within turbulence. We therefore need to write dynamics equations.

Instantaneous Governing Equations

We begin by reminding ourselves of the governing equations of fluid mechanics, which govern the instantaneous velocity in a fluid. The basic tool for writing these governing equations is the Reynolds Transport Theorem, which expresses the time derivative of a material volume of some substance ψ evolving over time, t, in terms of its status at a fixed time, t^*, and the mean flow, $\tilde{u}_j$, in which the material volume is moving, written as

Reynolds Transport Theorem is nothing more than the three-dimensional version of the Leibniz Theorem from integral calculus

$$\left[\frac{d}{dt}\int\limits_{V(t)}\psi dV\right]_{t=t^*} = \int\limits_{V(t^*)}\frac{\partial\psi}{\partial t}dV + \int\limits_{\partial V(t^*)}\psi\tilde{u}_j n_j dS \tag{1}$$

From this we can quickly derive the conservation of mass, which is stated generally for a fluid with density ρ and no sources, sinks, or diffusive fluxes as:

$$\frac{d}{dt}\int\limits_{V(t)}\rho dV = 0 \tag{2}$$

Application of Reynolds Transport Theorem yields

$$\int\limits_{V(t)}\frac{\partial}{\partial t}\rho dV + \int\limits_{\partial V(t^*)}\rho\tilde{u}_j n_j dS = 0 \tag{3}$$

The divergence theorem reads: $\int\limits_V \nabla\cdot\psi dV = \int\limits_{\partial V}\psi\hat{n}dS$ where $\hat{n}$ is the unit-normal facing out of the bounding surface ∂V of the material volume V.

and use of the divergence theorem yields the continuity equation

$$\underbrace{\frac{\partial \rho}{\partial t} + \tilde{u}_j \frac{\partial \rho}{\partial x_j}}_{\frac{D\rho}{Dt}} = -\rho \frac{\partial \tilde{u}_j}{\partial x_j} \tag{4}$$

The material derivative represents the time derivative at fixed position plus the spatial derivative (in the direction of the mean flow) at fixed time, and thus represents the total rate of change of a material element.

where the material derivative is given by $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$. The equation of state for an incompressible fluid is $\frac{D\rho}{Dt} = 0$ and therefore, equivalently, incompressible fluids satisfy $\nabla \cdot \vec{u} = 0$.

The linear momentum balance is just

$$\frac{d}{dt} \int\limits_{V(t)} \rho \tilde{u}_i dV = \int\limits_{\partial V(t)} n_j \tilde{\sigma}_{ij} dS \tag{5}$$

where $\tilde{\sigma}_{ij}$ is the total flux of momentum through the surface of the material volume. Applying Reynolds Transport and the divergence theorem yields the linear momentum balance

$$\rho \frac{D\tilde{u}_i}{Dt} = \frac{\partial}{\partial x_j} \tilde{\sigma}_{ij} \quad (+\text{Body Forces}) \tag{6}$$

The momentum flux tensor $\tilde{\sigma}_{ij}$ can be decomposed into isotropic (pressure, p) and deviatoric (strain) parts, according to

We assume that the fluid is incompressible and thus the mechanical pressure and thermodynamics pressure are equal

$$\tilde{\sigma}_{ij} = -\tilde{p}\delta_{ij} + \tilde{\tau}_{ij} \tag{7}$$

For a Newtonian fluid, we assume that the deviatoric stress tensor, $\tilde{\tau}_{ij}$ is a simple linear function of local velocity gradients, $\frac{\partial \tilde{u}_i}{\partial x_j}$ in the fluid, and that the fluid properties are homogeneous and isotropic (uniform and identical in all directions). The local velocity gradient tensor can be decomposed into symmetric and anti-symmetric parts according to:

$$\frac{\partial \tilde{u}_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right)}_{\text{symmetric, } \tilde{s}_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} - \frac{\partial \tilde{u}_j}{\partial x_i}\right)}_{\text{anti-symmetric, } r_{ij}} \tag{8}$$

$\tilde{s}_{ij}$ is called the 'rate of strain tensor' and r_{ij} is called the 'rate of rotation tensor'. r_{ij} is related directly to the vorticity.

and from conservation of angular momentum we know that $\tilde{\sigma}_{ij}$ must be symmetric, hence all of its dependencies must also be symmetric. We then write the Newtonian stress tensor for a fluid with dynamic viscosity μ as

$$\tilde{\tau}_{ij} = 2\mu\left(\tilde{s}_{ij} - \frac{1}{3}\frac{\partial \tilde{u}_j}{\partial x_j}\delta_{ij}\right) \tag{9}$$

Note that $\frac{\partial \tilde{u}_j}{\partial x_j} = 0$ for an incompressible fluid

and then for an incompressible fluid with constant μ we have

$$\rho \frac{D\tilde{u}_i}{Dt} = -\frac{\partial \tilde{p}}{\partial x_i} + \mu \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} \quad (+\text{Body Forces}) \tag{10}$$

In order to solve for the instantaneous velocity field, $\tilde{u}_i$, we need to solve a system of equations with 5 unknowns: $\tilde{u}_i$, p, and ρ; and 5 equations: continuity, equation of state, and three momentum

equations. The system of all 5 equations is called the Navier-Stokes equations, and it is 'closed' in the sense that the number of unknowns equals the number of equations. However, because the momentum equations are non-linear PDEs , there is no guarantee of a general solution. Even if we could solve the equation, it wouldn't be terribly useful, since we expect the velocity field to fluctuate wildly and we are interested not in the instantaneous velocity, $\tilde{u}_i$ but rather an average velocity $\bar{u}_i$.

The non-linearity is the advection term within the material derivative, $\tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}$; in the limit that advection is negligible, we are left with creeping flow (Stokes flow) which is a linear, second-order system

Because we are interested in the kinetic energy of our turbulent system, we can write an equation for the instantaneous kinetic energy by multiplying the momentum equation by $\tilde{u}_i$ to obtain

$$\underbrace{\frac{D}{Dt}\left(\frac{1}{2}\tilde{u}_i\tilde{u}_i\right)}_{\substack{\text{Rate of change of KE}\\ \text{per unit mass}}} = \underbrace{-\frac{1}{\rho}\frac{\partial}{\partial x_i}(\tilde{u}_i\tilde{p})}_{\substack{\text{Work done by}\\ \text{hydrostatic pressure}}} + \underbrace{\frac{1}{\rho}\frac{\partial}{\partial x_j}(\tilde{u}_i\tilde{\tau}_{ij})}_{\substack{\text{Work done by}\\ \text{viscous stresses}}} - \underbrace{\frac{1}{\rho}\tilde{\tau}_{ij}\frac{\partial \tilde{u}_i}{\partial x_j}}_{\substack{\text{Dissipation rate}\\ \text{per unit mass, } \varepsilon}} \tag{11}$$

We've already seen the dissipation rate, ε, in our dimensional analysis of turbulence, and we realize how important it is for the exchange of energy between the mean flow and turbulence at equilibrium. For a Newtonian fluid, we can express the dissipation rate as

$$\frac{1}{\rho}\tilde{\tau}_{ij}\frac{\partial \tilde{u}_i}{\partial x_j} = \frac{1}{\rho}\tilde{\tau}_{ij}\left(\tilde{s}_{ij} + r_{ij}\right) = \frac{1}{\rho}\tilde{\tau}_{ij}\tilde{s}_{ij} \tag{12}$$

The product $\tilde{\tau}_{ij}r_{ij} = 0$ because the former is symmetric and the latter is anti-symmetric. Expanding $\tilde{\tau}_{ij}\tilde{s}_{ij}$ yields $s_{11}^2 + s_{12}^2 + s_{13}^2 + \cdots + s_{33}^2 - \frac{1}{3}\left(s_{11}^2 + s_{22}^2 + s_{33}^2\right) > 0$

where we make use of the fact that $\tilde{\tau}_{ij}$ is symmetric. Then we can substitute for the Newtonian stress tensor to find

$$\frac{1}{\rho}\tilde{\tau}_{ij}\tilde{s}_{ij} = 2\nu\left(\tilde{s}_{ij}\tilde{s}_{ij} - \frac{1}{3}s_{kk}^2\right) > 0 \tag{13}$$

or, in other words, the dissipation rate is always positive, $\varepsilon > 0$ for $\mu > 0$. For simplicity, in the case of an incompressible flow, we can write this dissipation rate as $\varepsilon = 2\nu\tilde{s}_{ij}\tilde{s}_{ij}$.

The kinetic energy equation is often coupled with the advection-diffusion equation for thermal energy density, $\rho C_p\tilde{T}$ by means of this dissipation term: energy dissipated by viscosity appears as a temperature increase of the fluid. This equation can be developed using the Reynolds Transport Theorem again, for the balance of $\rho C_p\tilde{T}$, including both heat diffusive flux, $\vec{q} = -k\nabla\tilde{T}$ (Fourier's Law) and a volumetric dissipation term, $\rho\varepsilon$.

C_p is the specific heat capacity of the fluid at constant pressure; k is the thermal conductivity, $\alpha = \frac{k}{\rho C_p}$ is the thermal diffusivity

Another useful governing equation is the advection-diffusion equation for the transport of mass concentration (since we know that turbulence also affects the rate at which particles of mass mix in a flow). We can write the concentration per unit volume as c, with a diffusive flux given by Fick's Law as $\vec{j} = -D\nabla\tilde{c}$, where D is the mass diffusivity. Then, using Reynolds Transport Theorem again, we obtain

Note that the flux of mass, j_j, is directed against the concentration gradient, hence the minus sign, and the flux integral is written as $\int_{\partial V} -n_j j_j dS$ with a negative sign before the outward facing unit normal, n_j, such that flux into the material volume increases the total mass in the volume

$$\frac{D\tilde{c}}{Dt} = \frac{\partial}{\partial x_j}\left(D\frac{\partial \tilde{c}}{\partial x_j}\right) \tag{14}$$

All of these governing equations have been written for the instantaneous flow quantities, which we denote with a tilde, as $\tilde{u}_i$ and $\tilde{p}$ and $\tilde{\tau}_{ij}$, etc. In the study of turbulence, we are interested in studying the turbulent fluctuations within the instantaneous fields, fluctuations that have zero average and can be used to construct ensemble statistics. From the point of view of engineering, we want to know the average flow field quantities, without the fluctuations, in order to understand the practical behavior of the flow. So we need to employ the Reynolds averaging rules and decompose all of our instantaneous fields in average and fluctuating quantities.

Reynolds Averaged Navier Stokes Equations

According to the Reynolds averaging rules, we can write any instantaneous quantity as

$$\tilde{u}_i = \bar{u}_i + u_i \tag{15}$$

where $\bar{u}_i$ represents the ensemble average of the instantaneous quantity $\tilde{u}_i$ and u_i is the fluctuating component representing the turbulence. Because the overbar represents an ensemble average (and not a spatial or temporal average), it has no affect on quantities like $\frac{\partial}{\partial t}$, but when we think about practical experiments we have to be more careful about how we define averaging for time-varying flows.

Other authors use u_i' to represent the fluctuating component, as we wrote earlier with f', but we will drop that notation in order to avoid writing the prime over and over again. However, we must remember that $\bar{u}_i$ is the mean flow, whereas the operator $\overline{()}$ acting on u_i is identically zero, so this can be a bit confusing at first.

Let's start our decomposition of the governing equations with continuity, assuming for the moment that the instantaneous density, $\tilde{\rho}$, also fluctuates. Continuity can be written as

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{\rho}\tilde{u}_j\right) = 0 \tag{16}$$

and we substitute the Reynolds decomposition for both density and velocity to obtain

$$\frac{\partial}{\partial t}(\bar{\rho}+\rho) + \frac{\partial}{\partial x_j}\left((\bar{\rho}+\rho)(\bar{u}+u)_j\right) = 0 \tag{17}$$

and then we ensemble average the entire equation

$$\overline{\frac{\partial}{\partial t}(\bar{\rho}+\rho) + \frac{\partial}{\partial x_j}\left((\bar{\rho}+\rho)(\bar{u}+u)_j\right)} = 0 \tag{18}$$

and apply all of the Reynolds averaging rules (distributivity, etc) to simplify this to

$$\frac{\partial}{\partial t}\bar{\rho} + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j\right) + \frac{\partial}{\partial x_j}\overline{\rho u_j} = 0 \tag{19}$$

We see that there is an additional term in the continuity equation which describes the covariance between the density and velocity fluctuations. This correlation may be important in incompressible flows in which density fluctuations are important, but in general

throughout this course we will assume that the flow is incompressible, in which case the overall continuity equation can be written as

$$\frac{\partial}{\partial x_j}\tilde{u}_j = \frac{\partial}{\partial x_j}(\bar{u}_j + u_j) = 0 \tag{20}$$

Solenoidal comes from the Greek word for pipe, and reminds us that for an incompressible flow, a streamtube composed of streamlines has equal in-flow and out-flow, like a pipe of water. It is just another word for 'divergenceless' but the physical interpretation is important when thinking about vorticity, $\vec{\omega} \equiv \nabla \wedge \vec{u}$, which is another solenoidal field and thus has the same property in terms of a vortex tube with constant circulation — the density of vortex lines in the tube represents the intensity of the vortcity (for fixed circulation) just like the density of streamlines in an incompressible streamtube represents the intensity of velocity (for fixed volumetric flow rate).

Applying the ensemble average and substraction yields

$$\frac{\partial}{\partial x_j}\bar{u}_j = 0 \quad \text{and} \quad \frac{\partial}{\partial x_j}u_j = 0 \tag{21}$$

such that, for incompressible flow, both the mean velocity field and the fluctuating field are solenoidal. Now we can apply this same procedure to the incompressible momentum equation, starting from the instantaneous form with no body forces (denoting the mean density as ρ)

$$\rho\frac{D\tilde{u}_i}{Dt} = -\frac{\partial\tilde{p}}{\partial x_i} + \mu\frac{\partial^2\tilde{u}_i}{\partial x_j^2} \tag{22}$$

and substituting the Reynolds decomposition and averaging yields

$$\overline{\rho\left[\frac{\partial}{\partial t}(\bar{u}+u)_i + (\bar{u}+u)_j\frac{\partial}{\partial x_j}(\bar{u}+u)_i\right]} = -\frac{\partial}{\partial x_i}(\bar{p}+p) + \mu\frac{\partial^2}{\partial x_j^2}(\bar{u}+u)_i \tag{23}$$

After simplifying the ensemble average, we obtain

$$\rho\left[\underbrace{\frac{\partial}{\partial t}\bar{u}_i + \bar{u}_j\frac{\partial}{\partial x_j}\bar{u}_i}_{\frac{D\bar{u}_i}{Dt}} + \underbrace{\overline{u_j\frac{\partial}{\partial x_j}u_i}}_{\frac{\partial}{\partial x_j}\overline{u_ju_i}}\right] = -\frac{\partial}{\partial x_i}\bar{p} + \mu\frac{\partial^2}{\partial x_j^2}\bar{u}_i \tag{24}$$

where we define the material derivative, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}\cdot\nabla$, in terms of the mean velocity, from here on, and we use the continuity equation to simplify the remaining advective term.

$$\rho\left[\frac{D\bar{u}_i}{Dt} + \frac{\partial}{\partial x_j}\overline{u_ju_i}\right] = -\frac{\partial}{\partial x_i}\bar{p} + \mu\frac{\partial^2}{\partial x_j^2}\bar{u}_i \tag{25}$$

The resulting average momentum equation has one addition term more than the instantaneous momentum equation: $\overline{u_j\frac{\partial}{\partial x_j}u_i}$. This vector is an advective term which includes a covariance between turbulent fluctuations. Because it is an advective term, it represents the movement of momentum through the material volume, just like the other advective term, $\bar{u}_j\frac{\partial}{\partial x_j}\bar{u}_i$. However, the usual advective term represents that movement due to the average flow; here the momentum flux is due to the fluctuations of the flow.

But there is another interpretation of this additional term, when we write it as $\rho\frac{\partial}{\partial x_j}\overline{u_ju_i}$. If we assume that ρ is constant, then we

There is a change of sign between the two interpretations: as an advective term, the sign is positive since it represents the movement of momentum into the material volume; as a stress term, the sign is negative because the stress is acting opposite the unit normal of the material volume. In the stress formulation, it is often defined as $\tau_{ij}^R \equiv -\rho\overline{u_ju_i}$ where the R represents the Reynolds decomposition.

can write this as $\frac{\partial}{\partial x_j}\rho\overline{u_j u_i}$ and therefore group it with the other divergence term on the right hand side of the equation

$$\rho\frac{D\bar{u}_i}{Dt} = -\frac{\partial}{\partial x_i}\bar{p} + \frac{\partial}{\partial x_j}\left[\bar{\tau}_{ij} - \rho\overline{u_j u_i}\right] \tag{26}$$

where we have written $\bar{\tau}_{ij} = 2\mu\bar{s}_{ij}$ and $\bar{s}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{u}_i}{\partial x_j} + \frac{\partial\bar{u}_j}{\partial x_i}\right)$. We now recognize that this additional term is not merely a turbulent advection term, but it also represents a non-viscous stress term, much like the viscous Newtonian stress, $\bar{\tau}_{ij}$. Written in the form of a 'stress tensor', it is convenient to refer to $-\rho\overline{u_j u_i}$ as the Reynolds stress (although we commonly refer to just the tensor $\overline{u_j u_i}$ as the Reynolds stress). Moreover, it is clear that $\overline{u_j u_i}$ must be symmetric in order to satisfy conservation of angular momentum, just like σ_{ij}. The symmetry means that $\overline{u_j u_i}$ contains at most 6 independent unknowns.

Just as with σ_{ij}, we can also decompose $\overline{u_i u_j}$ into isotropic and deviatoric parts, such that $\overline{u_i u_j} = \frac{1}{3}\overline{u_k u_k}\delta_{ij} + \left(\overline{u_i u_j} - \frac{1}{3}\overline{u_k u_k}\delta_{ij}\right)$

After writing the instantaneous equations, we showed that the system was 'closed' in the sense that the number of equations equaled the number the unknowns. Is the system still closed after ensemble averaging? We still have 5 equations (three average momentum equations, one average continuity equation, and an average equation of state, in this case $\rho = \text{const}$). But we have many more unknowns: the three average velocity components, $\bar{u}_i$, the average pressure $\bar{p}$, the average density ρ, and the 6 independent components of $\overline{u_i u_j}$. So we have 5 equations and 11 unknowns, and therefore the averaged system is not closed. This result is somewhat counter-intuitive: the instantaneous equation, with all of the random fluctuations, can be written in closed form, whereas it seems the averaged equation, cannot.

The Closure Problem

In order to close the ensemble-averaged equations, we need additional equations to describe the dynamics of $\overline{u_i u_j}$. Therefore, we need to write the dynamics of the fluctuations themselves, u_i and u_j, and then average them appropriately. To obtain the dynamics for u_i, we can simply subtract the dynamical equation for $\bar{u}_i$ from the instantaneous dynamics of $\tilde{u}_i$ to yield

$$\rho\left[\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k} + u_k\frac{\partial\bar{u}_i}{\partial x_k} + \bar{u}_k\frac{\partial u_i}{\partial x_k}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\tau_{ij} + \rho\overline{u_j u_i}\right] \tag{27}$$

Note that τ_{ij} is defined in terms of the fluctuating velocity gradients, $\frac{\partial u_i}{\partial x_j}$

We can write this again for u_j. Then multiply the equation for $\frac{\partial u_i}{\partial t}$ by u_j and the equation for $\frac{\partial u_j}{\partial t}$ by u_i. Adding them together results in an equation for $\frac{\partial}{\partial t}u_i u_j$. Then take the ensemble average to yield

$$\frac{D}{Dt}\overline{u_i u_j} = -\frac{\partial}{\partial x_k}\overline{u_i u_j u_k} - \left(\overline{u_j u_k}\frac{\partial\bar{u}_i}{\partial x_k} + \overline{u_i u_k}\frac{\partial\bar{u}_j}{\partial x_k}\right) \underbrace{-\frac{1}{\rho}\overline{\left(u_j\frac{\partial p}{\partial x_i} + u_i\frac{\partial p}{\partial x_j}\right)}}_{\Pi_{ij}} + \nu\frac{\partial^2}{\partial x_k^2}\overline{u_i u_j} - 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{28}$$

where we define the material derivative with the mean convective velocity, $\bar{u}_k$. So we now have a set of 6 new equations which describes the evolution of $\overline{u_j u_i}$, but this set depends on a number of other quantities that we don't know. In other words, we would now need to write new equations to describe these higher order moments like $\overline{u_i u_j u_k}$ and the mixed pressure-velocity moments. And these equations, in turn, will depend on still higher moments, in a unending chain. This is called the **closure problem** and is a result of the ensemble averaging of the advective non-linearity. If we can't close the system of momentum equations, perhaps we can at least interpret them to provide some physical insights. In particular, we can transform the dynamics of $\overline{u_i u_j}$ into an equation describing the dynamics of turbulent kinetic energy, in order to better understand the physical processes control energy flow in turbulence.

These higher order moments contribute less and less information to the system as their order increases: recall that the mean and variance provide a very good description of a distribution, whereas higher order moments only 'tweak' that description in small ways. So technically, we might be able to write a system of equations out to the nth moment and achieve a reasonable approximation of closure — this approach was taken by Millionshtchikov 1941.

Turbulent Kinetic Energy

We start from the dynamical equation for the Reynolds stress tensor and we multiply through by $\frac{1}{2}\delta_{ij}$ to obtain an equation for the evolution of the kinetic energy of the turbulent fluctuations, the TKE, $\frac{1}{2}\overline{u_i u_i}$

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{u_i u_i}\right) = -\frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i u_i u_k}\right) - \left(\overline{u_i u_k}\frac{\partial \bar{u}_i}{\partial x_k}\right) - \frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{u_i p} + \nu\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{2}\overline{u_i u_i}\right) - \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} \tag{29}$$

Note that we can expand the final viscous term according to

$$\underbrace{\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}}}_{\hat{\varepsilon}} = \underbrace{2\nu\overline{s_{ik}s_{ik}}}_{\varepsilon} - \underbrace{\nu\frac{\partial^2}{\partial x_i \partial x_k}\overline{u_i u_k}}_{\ll \varepsilon} \tag{30}$$

where we see the regular viscous dissipation associated with the turbulent fluctuations, which we call ε, as well as a second term which is much small than the dissipation and is usually negligible. Therefore the average of the two gradients is referred to as the pseudo-dissipation, $\hat{\varepsilon}$, because it is usually within a few percent of the actual dissipation, ε. We can re-group the viscous terms to yield

$$\nu\frac{\partial^2}{\partial x_k^2}\left(\frac{1}{2}\overline{u_i u_i}\right) - \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} = -\varepsilon + \nu\left[\frac{\partial^2}{\partial x_i \partial x_k}\overline{u_i u_k} + \frac{\partial^2}{\partial x_k^2}\left(\frac{1}{2}\overline{u_i u_i}\right)\right] = -\varepsilon + 2\nu\frac{\partial}{\partial x_k}\overline{u_i s_{ki}} \tag{31}$$

and then we can rewrite the TKE balance as

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{u_i u_i}\right) + \underbrace{\frac{\partial}{\partial x_k}\left[\left(\frac{1}{2}\overline{u_i u_i u_k}\right) + \frac{1}{\rho}\overline{u_k p} - 2\nu\overline{u_i s_{ki}}\right]}_{\text{Transport}} = \underbrace{-\overline{u_i u_k}\frac{\partial \bar{u}_i}{\partial x_k}}_{\mathcal{P}} - \varepsilon \tag{32}$$

We see that three terms are all written within a divergence operation, $\frac{\partial}{\partial x_k}$. This means that these terms are important for local transport of energy within turbulence, but none of these terms can contribute, in an integrated sense, to the total energy balance. If we integrate the energy balance over our control volume V and apply

the divergence theorem and assume that at the boundaries, ∂V, the turbulence is zero, then this term is also zero. So these terms cannot have a net effect on the total energy in the volume – they simply 'transport' energy around within the volume from place to place.

All terms that appear within a divergence are, in fact, transport terms, including the advection terms in the instantaneous momentum dynamics which describe the movement of a material volume in flow. There, too, the advection only rearranges momentum within the material volume, but does not change the total amount of momentum density inside, unlike, for instance, a diffusive term.

These three transport terms represent three different mechanisms of energy transport within turbulence. The first is advective transport, since we can rewrite the first term as

$$\frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i u_i u_k}\right) = \overline{u_k \frac{\partial}{\partial x_k}\left(\frac{1}{2}u_i u_i\right)} \tag{33}$$

which looks just like an advection of the TKE by the turbulent fluctuations themselves. The third term

$$-2\nu \frac{\partial}{\partial x_k}\overline{u_i s_{ki}} = -\frac{1}{\rho}\frac{\partial}{\partial x_k}\overline{u_i \tau_{ki}} \tag{34}$$

is just a viscous transport term. The second term,

$$\frac{\partial}{\partial x_k}\overline{u_k p} \tag{35}$$

however, is a bit trickier. This is the only term that we needed to use the incompressible continuity equation to simplify. So we need to be careful in interpreting it, because we need to consider what information was hidden by the use of continuity. This term originates from the contraction of the tensor derived above, Π_{ij}, which relates the pressure and velocity strain rate. We can rewrite this tensor, assuming an incompressible fluid, as

We used the fact that $\frac{\partial u_k}{\partial x_k} = 0$ to pass the velocity through the derivative when writing the $\overline{u_k p}$ term, so we need to think about what energetic terms are actually hidden within this divergence operation.

$$\Pi_{ij} = -\frac{1}{\rho}\overline{\left(u_j \frac{\partial p}{\partial x_i} + u_i \frac{\partial p}{\partial x_j}\right)} = \underbrace{\frac{2}{\rho}\overline{p s_{ij}}}_{\substack{\text{Pressure strain-}\\ \text{rate tensor}}} \underbrace{- \frac{1}{\rho}\frac{\partial}{\partial x_k}\left(\overline{u_i p}\delta_{jk} + \overline{u_j p}\delta_{ik}\right)}_{\text{Transport}} \tag{36}$$

and identify two contributions: one is called the pressure strain rate tensor, the other a transport term associated with pressure and velocity correlations. When we contracted Π_{ij} to obtain the TKE equation (setting $i = j$), the pressure strain-rate tensor is identically zero , and the transport term became the pressure transport term of the TKE. What are we actually doing when we perform the contraction operation and let $i = j$? Because the index is repeated, we are summing over all of the components of velocity, $i = \{1, 2, 3\}$, to obtain the total kinetic energy of all of the velocity components. But the fact that the total sums to zero (in the case of the $\frac{2}{\rho}\overline{p s_{ij}}$) doesn't mean that the kinetic energy in each of the individual components is identically zero. Let's write out the sum explicitly

$s_{ii} = 0$ for an incompressible fluid

$$\frac{2}{\rho}\overline{p s_{ii}} = \frac{2}{\rho}(\overline{p s_{11}} + \overline{p s_{22}} + \overline{p s_{33}}) = 0 \tag{37}$$

The assumption of incompressibility fixes that, for instance,

$$\overline{p s_{11}} = -(\overline{p s_{22}} + \overline{p s_{33}}) \tag{38}$$

In other words, that whatever kinetic energy is lost from one velocity component by means of pressure-strain is gained by the other velocity components. The pressure-strain rate serves to shift kinetic energy between the different velocity components of turbulence. Even if the flow field is directed dominantly in a single direction, the turbulent kinetic energy will be spread to all velocity components by means of this term. But of course, this term has no net effect on the total kinetic energy of turbulence, hence it was not visible in the form of the TKE we wrote above. So the pressure strain rate has two effects: a shift of total kinetic energy from one place to another, which appears as a transport term, but also a shift of kinetic energy between different velocity components, which does not appear in the net TKE balance.

This can be illustrated nicely by writing the component-wise TKE balances for a homogeneous shear flow, $\overline{u}_1(x_2)$, an idea suggested in W.K. George's 'Lectures in Turbulence for the 21st Century', p75; we will discuss what that homogeneous assumption means later.

We have now explained how TKE moves around within a flow, and we have already identified how it is dissipated by means of viscosity. The question is: where does the TKE come from — what is the source of energy in turbulence?

Production

The remaining term in the TKE balance is

$$-\overline{u_i u_k}\frac{\partial \overline{u}_i}{\partial x_k} = \frac{1}{\rho}\tau^R_{ik}\frac{\partial \overline{u}_i}{\partial x_k} \equiv \mathcal{P} \tag{39}$$

which is the product of the Reynolds stress and velocity gradient tensors. This is the only term in the TKE balance which could behave as a source of energy, assuming its sign is positive, hence we call it the production term, $\mathcal{P}$. But if this is some sort of source term of TKE, then where is that energy coming from? Because both $\overline{u_i u_k}$ and $\frac{\partial \overline{u}_i}{\partial x_k}$ appear in the momentum equation for the mean flow, it seems likely that the mean flow is the source for the TKE. So let's write the kinetic energy balance for the mean flow, $\frac{1}{2}\rho\overline{u}_i\overline{u}_i$. We do this by contracting the mean momentum equation with $\overline{u}_i$ to find

There is an analogy between this term and viscous dissipation: both take the form of $\tau_{ik}\frac{\partial u_i}{\partial x_k}$. In the case of viscous dissipation, this dissipation term appears in both the instantaneous kinetic energy balance and the thermal energy balance, with opposite signs, representing a shift of energy from kinetic to thermal. In the TKE, this term also represents an exchange of energy, this time with the average flow.

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{u}_i\overline{u}_i\right) + \frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i u_k}\overline{u}_i + \frac{1}{\rho}\overline{p}\overline{u}_k - 2\nu\overline{u}_i\bar{s}_{ki}\right) = -\mathcal{P} - \bar{\varepsilon} \tag{40}$$

The mean flow viscous dissipation is given by $\bar{\varepsilon} = 2\nu\bar{s}_{ik}\bar{s}_{ik}$, compared to the TKE dissipation denoted $\varepsilon = 2\nu\overline{s_{ik}s_{ik}}$

We see, indeed, that this same production term does show up in the mean kinetic energy balance, although the sign is opposite, as we expected. However, if $\mathcal{P}$ is really a source term, then $\mathcal{P} > 0$, such that energy flows from the mean flow into the turbulence. But we don't know the sign of $\mathcal{P}$ a priori, because we don't know the sign of $\overline{u_i u_k}$. However, we can make a reasonable argument for the sign of the Reynolds stress in a general shear flow.

Imagine that we have a small blob of fluid in the flow field and we relocate it to a new position. If the blob moves upwards in this streamwise flow, that means $u_y > 0$, since $\overline{u}_y = 0$. We assume that the blob carries its original $\hat{x}$-momentum with it as it moves. So when the blob arrives at its new position, its $\hat{x}$-momentum is less

than the surrounding $\hat{x}$-momentum; in other words, $u_x < 0$ with respect to the mean velocity at the new position. Hence, $u_x u_y < 0$. If the blob moves down the velocity gradient, all of the trends are opposite and still $u_x u_y < 0$.

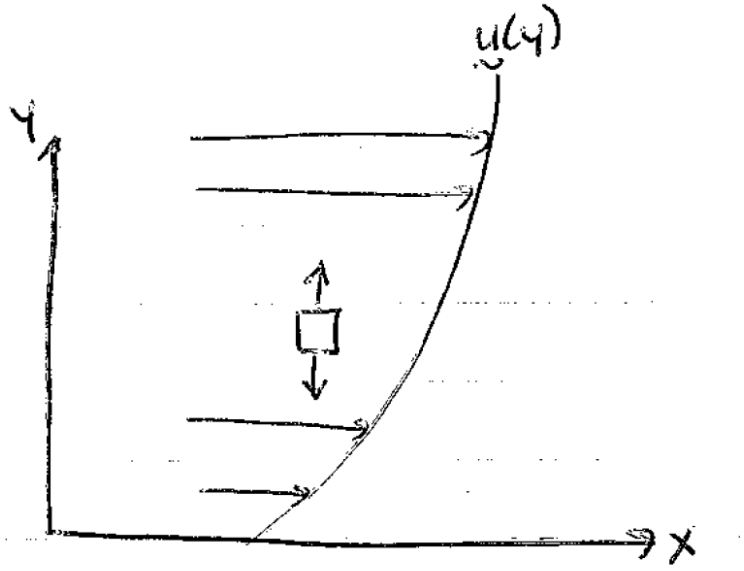


Figure 13: A blob of fluid within an arbitrary, two-dimensional shear flow. The blob retains its initial $\hat{x}$-momentum as it shifts its $\hat{y}$-location in the flow.

Therefore, in general we can assume with reasonable accuracy that

$$\mathrm{sgn}\left\{\tau_{ik}^R\right\} = \mathrm{sgn}\left\{\frac{\partial \overline{u}_i}{\partial x_k}\right\} \tag{41}$$

This was pointed out already in Tennekes & Lumley, p43

and thus we find that $\mathcal{P} = \tau_{ik}^R \frac{\partial \overline{u}_i}{\partial x_k} > 0$ almost always. Turbulent kinetic energy is generated by the mean velocity gradient and is transferred from the mean flow to the turbulence, and $\mathcal{P}$ is, indeed, a source term in the TKE balance.

Of course, this heuristic argument is only meant to be approximate — the Reynolds stress can appear with both signs for a given shear flow. Moreover, the argument itself is somewhat dubious . We need to simultaneously assume (1) that the original $\hat{x}$-momentum is retained as the blob of fluid moves in the vertical direction and (2) that there is complete mixing of the momentum after this vertical displacement, such that the blob causes a defect or surplus of momentum in its new location. Despite this internal contradiction, the heuristic argument works remarkably well in a wide variety of flows, so much so that it can be used to form predictive models of the Reynolds stress that we will explore later.

This was pointed out already in Tennekes & Lumley, p43

If the argument is roughly correct, then there are essentially two dissipation-type terms in the mean kinetic energy balance, since both $-\mathcal{P}$ and $-\bar{\varepsilon}$ are negative quantities. Kinetic energy leaves the mean flow and enters either turbulent fluctuations or viscous dissipation to heat. Which pathway is dominant? We can non-dimensionalize the two terms using our inertial scaling (appropri-

ate for the mean flow) as

$$\mathcal{P} \sim \left[u^2 \frac{u}{\ell}\right] \qquad \bar{\varepsilon} \sim \left[\nu \frac{u^2}{\ell^2}\right] \tag{42}$$

and the ratio of the two is then

$$\frac{\mathcal{P}}{\bar{\varepsilon}} \sim \frac{u\ell}{\nu} \equiv \mathrm{Re} \tag{43}$$

Therefore, as Reynolds number increases, more of the kinetic energy of the mean flow goes into turbulence and less goes into viscous dissipation; for high enough Reynolds number, almost all of the energy feeds into the turbulence. But then, in the TKE balance, how does the energy ever get to be dissipated if production is always dominant? In order to have viscous dissipation, we need to have a low Reynolds number. Indeed, for viscous effects to be important, we require $\mathrm{Re} \sim 1$. But for turbulence, we need $\mathrm{Re} \gg 1$. So how do we reconcile this conflict: on the one hand, the Reynolds number of the flow must be very high in order to have a TKE balance at all; on the other hand, the Reynolds number must be very low in order for the turbulence to eventually dissipate viscously? The answer: a spectrum of different length scales of turbulence. In other words, we cannot use the same scales to non-dimensionalize the TKE (and specifically the dissipation term) that we used for the mean kinetic energy balance. We need different scales for dissipation that satisfy $\frac{\eta v}{\nu} \sim 1$, which we recall as the Kolmogorov dissipative scales.

We have now explained the energy dynamics of turbulence for any point in a turbulent field. However, we already explained that turbulence is composed of coherent motions of different length scales. So we really need to explain how the energy is distributed and evolves among these different scales, which are spread out in space, not at a single point. In other words, we need to write the dynamics in terms of our scale-based measures, likes covariance, structure functions, and spectral densities.

2-Point Dynamics

In the last chapter, we derived the basic dynamical equations governing turbulence and the average velocity field. We can now consider a chart of different representations of the scales of turbulence, and write these dynamics for coherent motions in turbulence, not merely single points in the flow field.

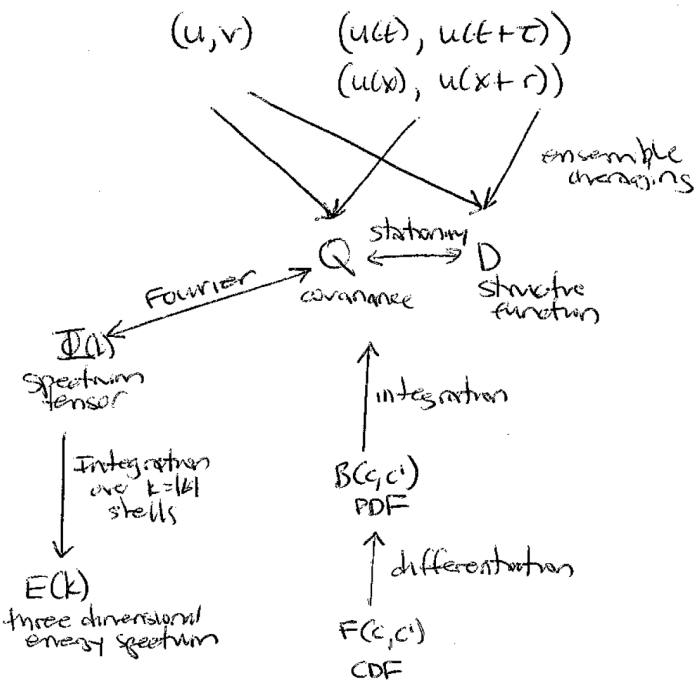


Figure 14: Map of the different representation tools for turbulence: Q, D, B, F, Φ, E, all based on one or two point velocity measurements. We have the tools to describe the dynamics of the velocity fields at a single point, (u, v) as show in the figure, but we need to translate those dynamics into the multi-point descriptions.

This is a problem of translating one-point equations into multi-point equations, in order to describe turbulent motions which span finite distances within the flow field.

Covariance Dynamics

Our goal is derive a dynamical equation for the two-point statistics, in other words the statistics of some flow quantity as measured at two independent points in space, A and A'.

By 'independent', we are assuming that the value of quantities measured at point A' is entirely independent of our choice of point A. Or, saying it another way, the choice of A is entirely independent of the choice of A'; when we choose A, that decision does not

Note that the prime notation here refers to an independent measurement at a second location, and has nothing to do with the Reynolds decomposition.

influence in any way the choice of A'. We assume that point A is located at position x_k and the instantaneous flow quantities there are denoted $\tilde{u}_i(x_k,t)$, $\tilde{p}(x_k,t)$. Point A' is located at position x'_k and the flow quantities there are denoted $\tilde{u}'_j(x'_k,t)$, $\tilde{p}'(x'_k,t)$. The two points are separated in space by $r_k = x'_k - x_k$. Now, let's say there is some spatial variation of $u'_j(x'_k,t)$ in the neighborhood of point A'. That spatial variation is the same, no matter what point A we choose. The variation measured at A' depends only on our choice of point A' itself, and vice versa. The fact that the choice of A is independent from the choice of A' does not mean that the quantities measured at the two locations aren't correlated.

For instance, let's say that velocity component u_1 is positively correlated with component u_2 everywhere in the flow field. Now, if we sample a value of u_1 at point A and a value of u_2 at point A', they will also be positively correlated. But we don't have to assume that the correlation is identical everywhere. Let's say that u_1 is positively correlated with u_2 only in region R. If we choose A and A' within R, then they will be correlated, if not, they won't be correlated.

The instantaneous momentum equation at point A is then given by

$$\frac{D\tilde{u}_i}{Dt} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} \tag{1}$$

which can be expanded, by substituting the Reynolds decomposition, as

$$\frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x_k}u_i u_k + \bar{u}_k\frac{\partial}{\partial x_k}u_i + \underbrace{\left(\frac{\partial}{\partial t}\bar{u}_i + \bar{u}_k\frac{\partial}{\partial x_k}\bar{u}_i + u_k\frac{\partial}{\partial x_k}\bar{u}_i - \nu\frac{\partial^2}{\partial x_k^2}\bar{u}_i + \frac{1}{\rho}\frac{\partial}{\partial x_i}\bar{p}\right)}_{\text{Terms with Mean Gradients}} = -\frac{1}{\rho}\frac{\partial}{\partial x_i}p + \nu\frac{\partial^2}{\partial x_k^2}u_i \tag{2}$$

In order to simplify this equation further, let's assume that there are no spatial or temporal gradients of any single-point, mean flow quantities in the region of the two points, and thus the terms in parenthesis are ignored and the momentum equation at point A became

This assumption neglecting mean gradients is not necessary and the 2-point dynamics can be derived in general form for flows with mean shear as well, see the appendix below and also Hinze's book for details. Indeed, ignoring the mean gradients is a bit surprising since we know that they appear in the production term, $\mathcal{P}$ and thus that assumption means that the equations we will derive can describe only the decay of current turbulence, not the production of new turbulence

$$\frac{\partial}{\partial t}u_i + \frac{\partial}{\partial x_k}u_i u_k + \bar{u}_k\frac{\partial}{\partial x_k}u_i = -\frac{1}{\rho}\frac{\partial}{\partial x_i}p + \nu\frac{\partial^2}{\partial x_k^2}u_i \tag{3}$$

Because we assume no gradients in the mean flow in the vicinity of the two points, the mean velocity will be the same at points A and A' (they share a common 'background' velocity, $\bar{u}_k$. Therefore, we can write the equation for the dynamics at point A' as

$$\frac{\partial}{\partial t}u'_j + \frac{\partial}{\partial x'_k}u'_j u'_k + \bar{u}_k\frac{\partial}{\partial x'_k}u'_j = -\frac{1}{\rho}\frac{\partial}{\partial x'_j}p' + \nu\frac{\partial^2}{\partial x_k'^2}u'_j \tag{4}$$

Now, we want to eventually write the dynamics relating these two points, so we can write our two-point dynamics, by cross-multiplying the above two equations at points A and A' by u'_j and u_i respectively and adding them together to yield :

Because the points are independent, we see that $\frac{\partial}{\partial x'_k}u_i(x_k) = \frac{\partial}{\partial x_k}u'_i(x'_k) = 0$ and hence we can combine the derivatives conveniently

$$\frac{\partial}{\partial t}u_i u'_j + \frac{\partial}{\partial x_k}u_i u_k u'_j + \frac{\partial}{\partial x'_k}u_i u'_k u'_j + \bar{u}_k\left(\frac{\partial}{\partial x_k}u_i u'_j + \frac{\partial}{\partial x'_k}u_i u'_j\right) = -\frac{1}{\rho}\left(\frac{\partial}{\partial x_i}pu'_j + \frac{\partial}{\partial x'_j}p'u_i\right) + \nu\left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial x_k'^2}\right)u_i u'_j \tag{5}$$

Finally, we can ensemble average this equation, to obtain

$$\frac{\partial}{\partial t}\overline{u_i u_j'} + \frac{\partial}{\partial x_k}\overline{u_i u_k u_j'} + \frac{\partial}{\partial x_k'}\overline{u_i u_k' u_j'} + \bar{u}_k\left(\frac{\partial}{\partial x_k}\overline{u_i u_j'} + \frac{\partial}{\partial x_k'}\overline{u_i u_j'}\right) = -\frac{1}{\rho}\left(\frac{\partial}{\partial x_i}\overline{p u_j'} + \frac{\partial}{\partial x_j'}\overline{p' u_i}\right) + \nu\left(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial x_k'^2}\right)\overline{u_i u_j'} \quad (6)$$

This is an equation which describes the evolution of the ensemble quantity $\overline{u_i u_j'}$ in terms of other ensemble quantities of the flow, as sampled from two points. Now that we have an equation in terms of ensemble quantities, we can make assumptions about the statistical properties of these quantities. In particular, we will assume that the statistical quantities are all homogeneous. This means that all of the turbulence statistics are invariant to a shift in origin. In other words, statistical quantities of fluctuations at two-points can depend only on the distance between the two points (r_k), not on the points themselves (x_k,x_k'). Now we consider correlations that can vary only with separation, but do not depend on the absolute measurement locations.

This assumption is entirely different from the assumption above that there are no single-point mean gradients in the flow field; homogeneity applies to statistics of fluctuating quantities only.

Homogeneous Assumption

The assumption of homogeneity means that the ensemble turbulence statistics can depend only on r_k. So all of the averaged quantities in the above equation are a function of r_k only, e.g. $\overline{u_i u_j'}(\vec{r})$. And we can easily relate the spatial derivatives in x_k and x_k' to r_k by

$$\frac{\partial}{\partial x_k'} = \frac{\partial}{\partial r_k} \quad \text{and} \quad \frac{\partial}{\partial x_k} = -\frac{\partial}{\partial r_k} \quad (7)$$

It is important to note: we assume homogeneity only after ensemble averaging since it applies only to statistical quantities!

Now, we can rewrite our two-point equation in terms of derivatives of r_k.

$$\frac{\partial}{\partial t}\overline{u_i u_j'} - \frac{\partial}{\partial r_k}\overline{u_i u_k u_j'} + \frac{\partial}{\partial r_k}\overline{u_i u_k' u_j'} + \underbrace{\bar{u}_k\left(-\frac{\partial}{\partial r_k}\overline{u_i u_j'} + \frac{\partial}{\partial r_k}\overline{u_i u_j'}\right)}_{\text{Mean Advection}} = -\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}\overline{p u_j'} + \frac{\partial}{\partial r_j}\overline{p' u_i}\right) + \nu\left(\frac{\partial^2}{\partial r_k^2} + \frac{\partial^2}{\partial r_k^2}\right)\overline{u_i u_j'} \quad (8)$$

Notice that the mean advective term has dropped out. This makes sense since because, by assuming homogeneity, we are claiming the origin of the coordinate system doesn't matter. So we could choose an origin moving with the flow, and thus there is no convective contribution. Simplifying, we have:

$$\frac{\partial}{\partial t}\overline{u_i u_j'} - \frac{\partial}{\partial r_k}\overline{u_i u_k u_j'} + \frac{\partial}{\partial r_k}\overline{u_i u_k' u_j'} = -\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}\overline{p u_j'} + \frac{\partial}{\partial r_j}\overline{p' u_i}\right) + 2\nu\frac{\partial^2}{\partial r_k^2}\overline{u_i u_j'} \quad (9)$$

We recognize immediately that these terms are just multi-point covariance quantities. For simplicity, we can rename these terms

using subscript notation, where the comma indicates the separation between measurements at point A and point A', such that

$$Q_{i,j}(\vec{r}) = \overline{u_i u'_j} \equiv \overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})} \tag{10}$$

$$Q_{i,jk}(\vec{r}) = \overline{u_i u'_j u'_k} \equiv \overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})u_k(\vec{x}+\vec{r})} \tag{11}$$

$$Q_{p^*,j}(\vec{r}) = \overline{p u'_j} = \overline{p(\vec{x})u_j(\vec{x}+\vec{r})} \tag{12}$$

where we use an asterisk to note that p represents pressure and is not an index. We can then rewrite the dynamics in the form of covariance tensors as

Therefore, $Q_{p*,j}(\vec{r})$ is a first order tensor, since it has only one index, whereas $Q_{i,j}(\vec{r})$ is a second-order tensor and $Q_{i,jk}(\vec{r})$ is third-order.

$$\frac{\partial}{\partial t}Q_{i,j} = \underbrace{\frac{\partial}{\partial r_k}Q_{ik,j} - \frac{\partial}{\partial r_k}Q_{i,kj}}_{T_{i,j}} \underbrace{-\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}Q_{p^*,j} + \frac{\partial}{\partial r_j}Q_{i,p^*}\right)}_{P_{i,j}} + 2\nu\frac{\partial^2}{\partial r_k^2}Q_{i,j} \tag{13}$$

And we can rewrite this again

$$\frac{\partial}{\partial t}Q_{i,j}(\vec{r},t) = T_{i,j}(\vec{r},t) + P_{i,j}(\vec{r},t) + 2\nu\frac{\partial^2}{\partial r_k^2}Q_{i,j}(\vec{r},t) \tag{14}$$

where $T_{i,j}$ represents the inertial (advective) effects, $P_{i,j}$ represents the pressure effects, and the viscous effects are: $2\nu\frac{\partial^2}{\partial r_k^2}Q_{i,j}$. As with the one-point Reynolds stress equations, the two-point equations also suffer from a closure problem. However, because the covariance functions can be written in terms of PDFs, we can think of this equation as an evolution equation for the PDFs of the fluctuating quantities, which means that if we can hypothesize some interrelationship between the different PDFs we might be able to write an approximate closure scheme for this system. We also know that we can translate the covariance functions into structure functions or spectral densities, which means this single equation is a very powerful tool for studying homogeneous turbulence .

Homogeneity means that all spatial gradients of averaged fluctuating quantities must vanish (i.e. gradients of turbulence statistical quantities that do not depend on r_k). Therefore, you should be able to convince yourself that $T_{i,j}(0) = 0$ and, from continuity, $P_{i,i}(r) = 0$.

Appendices

Continuity

We can also write the two-point, covariance form of the continuity equation for an incompressible fluid. We begin with the continuity equation at point A

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{15}$$

and we multiply by the velocity at point A' to obtain

$$u'_j\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}u_i u'_j = 0 \tag{16}$$

and then we perform the ensemble average

$$\frac{\partial}{\partial x_i}\overline{u_i u'_j} = 0 \tag{17}$$

and finally we assume that the ensemble quantity $\overline{u_i u_j'}$ is homogeneous, and therefore we can change the variables according to $r_i = x_i' - x_i$ to obtain

$$\frac{\partial}{\partial r_i}\overline{u_i u_j'} = \frac{\partial}{\partial r_i} Q_{i,j} = \frac{\partial}{\partial r_j} Q_{i,j} = 0 \tag{18}$$

where we have used symmetry to argue that the final equality is true.

Symmetry of Q

We have previously noted that the scalar covariance function, $Q = \overline{u(\vec{x})u(\vec{x}+\vec{r})}$ is an even function, $Q(\vec{r}) = Q(-\vec{r})$, because

$$\overline{u(\vec{x})u(\vec{x}+\vec{r})} = \overline{u(\vec{x})u(\vec{x}-\vec{r})} \tag{19}$$
$$= \overline{u(\vec{x}+\vec{r})u(\vec{x})} \tag{20}$$
$$= \overline{u(\vec{x})u(\vec{x}+\vec{r})} \tag{21}$$

We even made use of this fact when we expanded the covariance in a Taylor series in order to define the Taylor microscale. However, we need to be a bit more careful now that we are considering covariance functions that are tensors, of the form $Q_{i,j} = \overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})}$, where the covariance relates two vector quantities. In this case, we can say that

$$Q_{i,j}(\vec{r}) = Q_{j,i}(-\vec{r}) \tag{22}$$

We can show this by simply writing out the definition, as above

$$\overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})} = \overline{u_j(\vec{x})u_i(\vec{x}-\vec{r})} \tag{23}$$
$$= \overline{u_j(\vec{x}+\vec{r})u_i(\vec{x})} \tag{24}$$
$$= \overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})} \tag{25}$$

Order of Differentiation

When we take advantage of the fact that points A and A' are independent, we need to be a bit careful, because we can easily run into contradictions. For instance, consider the following sequence of transformations of the left-hand side of the equation

$$\frac{\partial}{\partial x_k} u_i u_j' = u_j' \frac{\partial}{\partial x_k} u_i \tag{26}$$

$$-\frac{\partial}{\partial r_k} u_i u_j' = \tag{27}$$

$$-u_i \frac{\partial}{\partial r_k} u_j' - u_j' \frac{\partial}{\partial r_k} u_i = \tag{28}$$

$$-u_i \frac{\partial}{\partial x_k'} u_j' + u_j' \frac{\partial}{\partial x_k} u_i = \tag{29}$$

$$-u_i \frac{\partial}{\partial x_k'} u_j' = 0 \tag{30}$$

How did this happen? When we replace the spatial derivatives in the last step, going from r_k back to x_k and x_k', we are required to replace them consistently, so that just like we went from $\frac{\partial}{\partial x_k}$ to $-\frac{\partial}{\partial r_k}$ originally, we need to replace both terms of $-\frac{\partial}{\partial r_k}$ with $\frac{\partial}{\partial x_k}$ (as opposed to one x_k' and the other x_k) and then we get:

$$\frac{\partial}{\partial x_k} u_i u_j' = u_j' \frac{\partial}{\partial x_k} u_i \tag{31}$$

$$-\frac{\partial}{\partial r_k} u_i u_j' = \tag{32}$$

$$-u_i \frac{\partial}{\partial r_k} u_j' - u_j' \frac{\partial}{\partial r_k} u_i = \tag{33}$$

$$u_i \frac{\partial}{\partial x_k} u_j' + u_j' \frac{\partial}{\partial x_k} u_i = \tag{34}$$

$$u_j' \frac{\partial}{\partial x_k} u_i = u_j' \frac{\partial}{\partial x_k} u_i \tag{35}$$

$$\tag{36}$$

The reason we have to be consistent is that we cheated in our very first line: that line should read formally: $\frac{\partial}{\partial x_k} u_i u_j' = u_j' \frac{\partial}{\partial x_k} u_i + u_i \frac{\partial}{\partial x_k} u_j'$. By not being careful about this, we essentially converted a value of 0 into a nonzero quantity which resulted in the contradiction. So going slowly and consistently is key, so as not to introduce artificial non-zero terms.

Two-point and One-point Equations

Also, do not confuse the equation for two-point homogeneous statistics (above) with the one-point equation for the Reynolds stress, because we employed different assumptions to obtain the two point statistics. Note that we wrote the Reynolds stress equation using the operator for the mean convective derivative, such that

$$\frac{D}{Dt}\overline{u_i u_j} = -\frac{\partial}{\partial x_k}\overline{u_i u_j u_k} - \underbrace{\left(\overline{u_j u_k}\frac{\partial \bar{u}_i}{\partial x_k} + \overline{u_i u_k}\frac{\partial \bar{u}_j}{\partial x_k}\right)}_{\text{Terms with Mean Gradients}} + \Pi_{ij} + \nu\frac{\partial^2}{\partial x_k^2}\overline{u_i u_j} - 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{37}$$

We could consider that in the limit that $\vec{r} \to 0$, the two point-equation and one-point equations should be very similar. Howeever, the key difference is that in the two-point equation, we assumed that there were no spatial gradients in the mean quantities in the region of interest and in the one-point Reynolds stress equation we did not make this assumption. Therefore, in the two-point equation, we will not have any terms like $\frac{\partial \overline{u}_i}{\partial x_k}$, but those terms will appear in the general one-point equation.

To say it another way, our two-point equation for homogeneous turbulence is not the most general form of a two-point equation for homogeneous turbulence. To be more precise, we should really call it the 'two-point equation for homogeneous turbulence without mean gradients'. If we include mean velocity gradients, then we can write the general two-point equation for homogeneous turbulence as:

$$\frac{\partial}{\partial t} Q_{i,j} + \underbrace{\left(Q_{k,j} \frac{\partial \overline{u}_i}{\partial x_k} + Q_{i,k} \frac{\partial \overline{u'}_j}{\partial x'_k} + \left(\overline{u'}_k - \overline{u}_k \right) \frac{\partial}{\partial r_k} Q_{i,j} \right)}_{\text{Terms with Mean Gradients}} = T_{i,j}(\vec{r}, t) + P_{i,j}(\vec{r}, t) + 2\nu \frac{\partial^2}{\partial r_k^2} Q_{i,j} \tag{38}$$

where the terms in brackets represent the effect of the mean velocity gradient. Homogeneity describes the turbulence itself (i.e. the fluctuating quantities) but does not describe the mean quantities, which can still have a spatial variation even if the fluctuations are homogeneous. This more general form of the equation can be called the 'two-point equation for homogeneous turbulence with mean gradients'. Now, in the limit that $\vec{r} \to 0$, this latter equation will indeed be equal to our one-point Reynolds stress equation.

Homogeneous Isotropic Turbulence

In the last chapter we derived the 2-point dynamics of turbulence, assuming no mean velocity gradients and homogeneous ensemble statistics. However, these dynamical equations continued to suffer from the same closure problem as the one-point statistics. Although we are able to learn about the general features of turbulence by careful examination of the equations, the equations remain too complicated for thorough analysis and incomplete for finding a solution. In order to make progress, we need to introduce additional assumptions about the statistical fields.

To motivate the necessity of additional simplifications, consider the dissipation rate and how many measurements we need in order to express it completely.

$$\varepsilon = 2\nu\overline{s_{ij}s_{ij}} \tag{1}$$

Recall the rate of strain tensor, $s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

If we write out all of the terms, we find 12 separate derivative correlations of the form

$$\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2}, \quad \overline{\left(\frac{\partial u_1}{\partial x_2}\right)^2}, \quad \overline{\left(\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}\right)}, \quad \dots \tag{2}$$

To measure all of these quantities accurately in a flow is nearly impossible. What happens, however, if we assume homogeneity? Well, let's write our dissipation in terms of the pseudo-dissipation, as we did when we derived the TKE balance

$$\underbrace{\nu\overline{\left(\frac{\partial u_i}{\partial x_k}\right)^2}}_{9\text{ Components}} = \underbrace{\varepsilon}_{12\text{ Components}} - \nu\frac{\partial^2}{\partial x_i \partial x_k}\overline{u_i u_k} \tag{3}$$

The pseudo-dissipation requires measuring only 9 independent gradients, compared to 12 with the actual dissipation. The difference between the two dissipations can be expressed as the gradient of an ensemble averaged quantity, $\overline{u_i u_k}$. If the flow is homogeneous, then this gradient of an ensemble averaged quantity is zero, and we can represent the dissipation with only 9 measurements. This is a significant simplification: we have reduced the complexity of the problem by 25%. Can we make additional assumptions to simplify things even more?

Now we realize why the pseudo-dissipation and dissipation are almost equal in most cases: any situation in which the flow is approximately homogeneous means the two dissipations are approximately equal.

The next least-restrictive assumption is to assume axisymmetry: that all ensemble quantities have rotational symmetry

about a single axis. However, we are going to proceed with a slightly more restrictive symmetry assumption, in order to make more progress. We will assume that the flow is isotropic: that all ensemble-averaged quantities are invariant to reflections and rotations about all axes. This is a very strong assumption, and we hope that it will result in a massive simplification. However, because it is such a strong assumption, we must be very careful when we try to apply results based on this assumption to real flows. In the case of the dissipation, if we assume that the ensemble statistics are the same in any direction, then we are left with only 3 terms to measure (one average gradient for each velocity component, independent of direction), which reduces the complexity of the original problem by 75%, but we shall see that isotropy combined with incompressibility simplify this even more.

See George & Hussein, JFM 1991

Note that the assumptions apply only to the ensemble statistical quantities, not to the velocity fields themselves.

Hypothesis of Local Isotropy

Isotropy is a good approximation for turbulence generated by passing a flow through a grid, such that the turbulence starts in an entirely symmetric way. However, in many other flows, the large scale motions are strongly anisotropic — consider the flow near any kind of wall, where the shape of the wall strongly influences large scale motions. Kolmogorov claimed that despite large-scale anisotropy, the flow can still be approximated as locally isotropic at the small scales (for sufficiently high Reynolds number). In 1941, Kolmogorov called this the hypothesis of local isotropy.

> **Kolmogorov's hypothesis of local isotropy**: At sufficiently high Reynolds number, small scale fluctuations are statistically isotropic.

What is the basis for such an assumption? To answer this, we need to understand the relative rate of evolution of small and large scales. From our dimensional analysis, we know that the turnover time of the smallest scales (dissipative scales) is given by $\frac{\eta}{v}$ and that of the largest scales is $\frac{\ell}{u}$, and their ratio is determined by

$$\frac{\eta}{v} \sim \frac{\ell}{u} \mathrm{Re}^{-1/2} \tag{4}$$

Therefore, as Reynolds number increases, the timescale of the smallest scales decreases, and thus the smallest scales evolve much more rapidly than the largest scales. In other words, from the point of view of the small scales, the large scales motions appear to be approximately in equilibrium. Moreover, these small scales are significantly smaller than the large scale

$$\frac{\eta}{\ell} \sim \mathrm{Re}^{-3/4} \tag{5}$$

so it seems reasonable that the smallest scales don't feel the large scale anistropy. But this argument holds only for the dissipative scales. What about other small scales, that aren't quite the very smallest? Local isotropy is still a useful assumption, and we can

illustrate this with a cartoon. Let's start by assuming that we have a large scale vortex tube, aligned with the $\hat{z}$-axis, and being stretched in the $\hat{z}$-direction by a mean shear. As it is stretched, its kinetic energy grows, and that kinetic energy growth is mostly in the $\hat{x}$- and $\hat{y}$-velocity components. These enhanced velocity components can themselves stretch smaller vortex tubes.

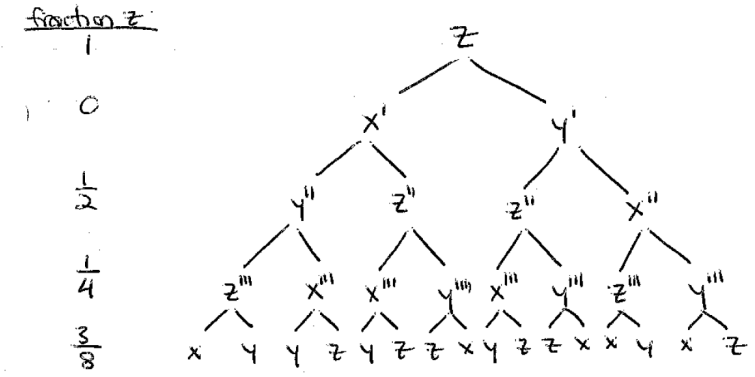


Figure 15: A tree of vortex tubes, each labeled by its axis of orientation. We begin with one large scale structure, oriented in the $\hat{z}$ direction, which induces stretching of ever small scale structures, and we track the fraction of those smaller structures that are oriented in the $\hat{z}$-direction. Isotropy would yield a fraction of 1/3.

With each generation, we see less and less of the original anisotropy ($\hat{z}$-orientation) and more isotropy of scales. If this hypothesis of local (small-scale) isotropy is believable, in the sense that we expect it to be applicable even in cases where the flow is anisotropic at the large scales, then perhaps we can use it to simplify the governing equations.

Isotropic Formulation

We have already made use of the assumption of homogeneity, which meant that the ensemble quantities were invariant to translations (change of origin). Now we will assume that the ensemble quantities are isotropic: invariant to rigid rotation and reflection in any plane. In the case of the homogeneous assumption, we wrote the general dynamics and then applied homogeneity by neglecting certain derivatives. In this case, because we already have the homogeneous dynamics, we will work backwards. We start with the question: what is the general mathematical form of a covariance tensor, Q, which satisfies isotropy? If we can answer that question, we can simply rewrite our homogeneous dynamics in terms of isotropic tensors. The general form of two-point, isotropic tensors is given by

The mathematical theory of general isotropic tensors is described in a paper by Robertson 1940

Tensor	Definition
$Q_i(\vec{r})$	$= Ar_i$
$Q_{ij}(\vec{r})$	$= Ar_ir_j + B\delta_{ij}$
$Q_{ijk}(\vec{r})$	$= Ar_ir_jr_k + Br_i\delta_{jk} + Cr_j\delta_{ki} + Dr_k\delta_{ij}$

Table 2: The general form of two-point isotropic tensors, where the two points, A and A', are separated by vector $\vec{r}$ and $\{A, B, C, D\}$ are all even functions of r with $r = |\vec{r}|$, such that $A = A(r^2, r^4, \ldots)$.

It is important to note that these tensors are formally functions of both distance between points, $\vec{r}$, and time, t, but for simplicity, we will not write the dependence on t explicitly. In addition to

the general form of 2-point isotropic tensors, we can also make sure of two additional properties when simplifying the governing equations.

1. Incompressibility: The covariance tensor is solenoidal in any solitary index. For example, we showed in the previous chapter's appendix that, for the second order covariance

$$\frac{\partial}{\partial r_i} Q_{i,j} = \frac{\partial}{\partial r_j} Q_{i,j} = 0 \tag{6}$$

   However, for the third-order, 2-point covariance

$$\frac{\partial}{\partial r_i} Q_{i,jk} = 0 \quad \text{whereas} \quad \frac{\partial}{\partial r_j} Q_{i,jk} \neq 0 \tag{7}$$

   because the index j is not alone at point A'.

2. Symmetry in (i, j): The indices for components of the covariance tensor can be re-ordered at either point, A or A', such that

$$Q_{i,jk} = Q_{i,kj} \tag{8}$$

   because $\overline{u_i(\vec{x})u_j(\vec{x}+\vec{r})u_k(\vec{x}+\vec{r})} = \overline{u_i(\vec{x})u_k(\vec{x}+\vec{r})u_j(\vec{x}+\vec{r})}$. But obviously, the indices cannot be switched across the comma without changing the meaning of the tensor.

If we write out the third order covariance, we see that $\overline{\frac{\partial}{\partial x_i} u_i u'_j u'_k} = \overline{\frac{\partial u_i}{\partial x_i} u'_j u'_k} = 0$ whereas $\overline{\frac{\partial}{\partial x'_j} u_i u'_j u'_k} = \overline{u_i \frac{\partial}{\partial x'_j} u'_j u'_k} \neq 0$

Now, let's apply these mathematical definitions for isotropic tensors to the tensors in our covariance dynamics. Recall that we have three orders of tensors in the covariance dynamics:

$$\frac{\partial}{\partial t}\overline{u_i u'_j} - \frac{\partial}{\partial r_k}\overline{u_i u_k u'_j} + \frac{\partial}{\partial r_k}\overline{u_i u'_k u'_j} = -\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}\overline{p u'_j} + \frac{\partial}{\partial r_j}\overline{p' u_i}\right) + 2\nu\frac{\partial^2}{\partial r_k^2}\overline{u_i u'_j} \tag{9}$$

first order tensors, like $\overline{pu'_j}$, second order $\overline{u_i u'_j}$ and third order $\overline{u_i u_k u'_j}$.

First Order Isotropic Tensor

Consider first the rank 1 tensor, whose general form above is Ar_i where A is an arbitrary, even function of r. Therefore, we should be able to express the isotropic form of $\overline{pu'_j}$ as Ar_i. We just need to determine the function $A(r^2, r^4, \ldots)$. Let's apply continuity

The derivative $\frac{\partial}{\partial r_i}$ is the directed derivative in the i direction. It can be expressed as $r_i \frac{\partial A(r)}{\partial r_i} = r_i \frac{\partial r}{\partial r_i}\frac{\partial A(r)}{\partial r}$ and $\frac{\partial r^2}{\partial r_i} = \frac{\partial}{\partial r_i} r_m r_m = 2r_m \frac{\partial r_m}{\partial r_i} = 2r_m \delta_{mi} = 2r_i = 2r\frac{\partial r}{\partial r_i}$ and therefore $\frac{\partial r}{\partial r_i} = \frac{r_i}{r}$ and finally $r_i \frac{\partial A(r)}{\partial r_i} = r\frac{\partial A(r)}{\partial r}$

$$\frac{\partial}{\partial r_i} Q_i = \frac{\partial}{\partial r_i}[A(r) r_i] = A(r)\frac{\partial r_i}{\partial r_i} + r_i \frac{\partial A(r)}{\partial r_i} \tag{10}$$

$$= A(r)\delta_{ii} + r\frac{\partial A(r)}{\partial r} \tag{11}$$

$$= 3A(r) + r\frac{\partial A(r)}{\partial r} = 0 \tag{12}$$

Solving the resulting differential equation for $A(r)$ yields

$$A(r) = \frac{C_1(t)}{r^3} \tag{13}$$

where $C_1(t)$ is a constant of integration. Because we expect that the covariances be defined at $r = 0$, the constant must be 0, and we conclude that rank 1, two-point, isotropic tensors are all identically zero. Hence $\overline{pu'_j} = 0$ assuming isotropy.

We can illustrate this same conclusion geometrically as well. Without loss of generality, let the vector $\vec{r}$ between the two points A and A′ coincide with the $\hat{x}_1$ axis. The tensor $\overline{pu'_j}$ has three components

$$\overline{p\vec{u}'} = \left(\overline{pu'_1}, \overline{pu'_2}, \overline{pu'_3}\right) \tag{14}$$

and we want them to be invariant under reflection. If we reflect about the $\hat{x}_1$-$\hat{x}_2$ plane, then

$$\overline{pu'_3}(\vec{r}) = -\overline{pu'_3}(\vec{r}) = 0 \tag{15}$$

and similarly if we reflect about the $\hat{x}_1$-$\hat{x}_3$ plane, then

$$\overline{pu'_2}(\vec{r}) = -\overline{pu'_2}(\vec{r}) = 0 \tag{16}$$

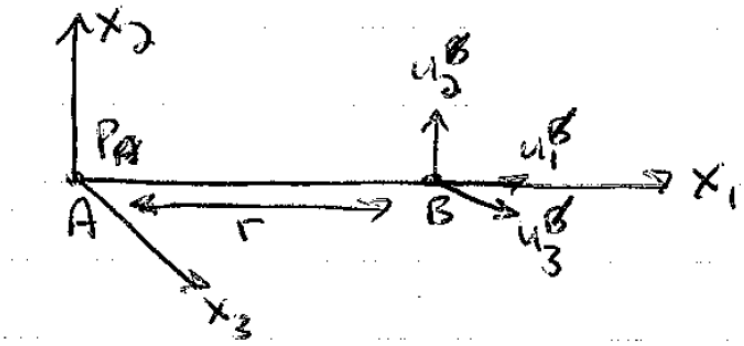


Figure 16: Two points, labeled A and B (instead of A and A′ as in our notation), with $\vec{r}$ aligned along the $\hat{x}_1$ axis.

If we reflect about $\hat{x}_2$-$\hat{x}_3$, then not only does the value of the sign of the tensor change, but the direction of the vector $\vec{r}$ is reversed, because we chose $\vec{r}$ to coincide with the $\hat{x}_1$ axis. So when we reflect across the plane, we are measuring the two point in the opposite direction, and hence

$$\overline{pu'_1}(\vec{r}) = \overline{p[-u'_1]}(-\vec{r}) \quad \Rightarrow \quad \overline{pu'_1}(\vec{r}) \quad \text{odd} \tag{17}$$

Therefore we conclude that $\overline{pu'_1}(\vec{r})$ is an odd function of r. Therefore, the general form of the rank 1 isotropic tensor must be the product of a vector r_i and an even function of r, called $A(r)$, which is exactly the form we wrote above, $Q_i = A(r)r_i$. Combining this with the requirement of continuity yields the result above that $\overline{pu'_1}(\vec{r}) = 0$.

Second Order Isotropic Tensor

We now consider the rank 2 tensor from the dynamics equation, $\overline{u_i u_j'}$ or $Q_{i,j}$. Starting with the mathematical formulation

$$Q_{i,j}(\vec{r}) = F r_i r_j + G\delta_{ij} \tag{18}$$

where F and G are even functions of r. As before, we apply continuity in order to further simplify this expression.

Again, formally speaking, $F(r,t)$ and $G(r,t)$ as are all subsequent scalar functions a function of time.

$$\frac{\partial}{\partial r_i} Q_{i,j} = \frac{\partial}{\partial r_i}\left[F(r) r_i r_j\right] + \frac{\partial}{\partial r_i}\left[G(r)\delta_{ij}\right] \tag{19}$$

$$= F\frac{\partial}{\partial r_i} r_i r_j + r_i r_j \frac{\partial F}{\partial r_i} + \frac{\partial G}{\partial r_j} \tag{20}$$

$$= F\left(\frac{\partial r_i}{\partial r_i} r_j + \frac{\partial r_j}{\partial r_i} r_i\right) + r r_j \frac{\partial F}{\partial r} + \frac{r_j}{r}\frac{\partial G}{\partial r} \tag{21}$$

$$= F\left(\delta_{ii} r_j + \delta_{ij} r_i\right) + r r_j \frac{\partial F}{\partial r} + \frac{r_j}{r}\frac{\partial G}{\partial r} \tag{22}$$

$$= F\left(3 r_j + r_j\right) + r r_j \frac{\partial F}{\partial r} + \frac{r_j}{r}\frac{\partial G}{\partial r} \tag{23}$$

$$= 4F r_j + r r_j \frac{\partial F}{\partial r} + \frac{r_j}{r}\frac{\partial G}{\partial r} \tag{24}$$

$$= r_j\left(4F + r\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial G}{\partial r}\right) = 0 \tag{25}$$

And we conclude that the two scalar functions are related by the differential equation

$$4F + r\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial G}{\partial r} = 0 \tag{26}$$

which means that the rank 2 isotropic tensor can be completely defined in terms of a single scalar function (and the other function can be found by means of this differential equation). Let's choose this scalar function to have some simple physical significance, such that we could actually measure it easily in a laboratory. Consider two points, A and A', position along the longitudinal access of flow (in the direction $\vec{r}$), i.e. the dominant flow direction or the direction of the average flow. Then imagine measuring two components of velocity at each of these two points: a component of velocity in the longitudinal direction, $u_L(\vec{x})$, and a component in the transverse direction, $u_T(\vec{x})$.

There are, in fact, 2 transverse directions, so we just choose one of them for the purposes of these definitions. For instance, if the laboratory mean flow were in the $\hat{x}_1$ direction and it was convenient to measure velocity in that direction and the $\hat{x}_2$ direction, we would simply choose u_2 to be the transverse component, u_T.

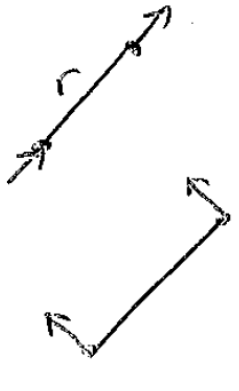


Figure 17: The longitudinal and transverse, 2-point, second-order covariances.

We can then define two scalar correlation functions for these measurements

$$f(r) = \frac{\overline{u_L(\vec{x}) u_L(\vec{x}+\vec{r})}}{\overline{u_L^2}} \quad \text{and} \quad g(r) = \frac{\overline{u_T(\vec{x}) u_T(\vec{x}+\vec{r})}}{\overline{u_T^2}} \tag{27}$$

where we note that $f(r)$ and $g(r)$ are both even functions with $f(0) = g(0) = 1$. Moreover, if the flow is isotropic, then the variance of the velocity at a single point must be identical in both directions,

and can be defined as a third of the total kinetic energy, which for convenient we call u^2.

$$\overline{u_T^2} = \overline{u_L^2} = \frac{1}{3}\overline{u_i u_i} \equiv u^2 \tag{28}$$

We can now relate our two experimental (measurable) scalar functions to the general definition of an rank 2 isotropic tensor

$$Q_{i,j}(\vec{r}) = F r_i r_j + G\delta_{ij} \tag{29}$$

In this coordinate system, we can write the vector between the two points as

$$\vec{r} = r\hat{x}_L + 0\hat{x}_T + 0\hat{x}_3 \tag{30}$$

We know that the correlation is $f(r)$ when $i = L$ and $j = L$, so it follows that

$$u^2 f(r) = F r_L r_L + G\delta_{LL} \tag{31}$$
$$u^2 f(r) = F r^2 + G \tag{32}$$

and the correlation is $g(r)$ when $i = T$ and $j = T$, so then

$$u^2 g(r) = F r_T r_T + G\delta_{TT} \tag{33}$$
$$u^2 g(r) = G \tag{34}$$

Therefore we can relate our experimental correlation functions, f and g, to the general scalar functions for an isotropic tensor, F and G, to rewrite our statement of continuity

$G(r) = u^2 g(r)$ and $F(r) = \frac{u^2}{r^2}[f(r) - g(r)]$ and we note that because F and G are even functions of r, so too f and g are even functions of r

$$4F + r\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial g}{\partial r} = 0 \tag{35}$$
$$4\left(\frac{f-g}{r^2}\right) + r\frac{\partial}{\partial r}\left(\frac{f-g}{r^2}\right) + \frac{1}{r}\frac{\partial g}{\partial r} = 0 \tag{36}$$
$$f - g + \frac{r}{2}\frac{\partial f}{\partial r} = 0 \tag{37}$$

The relationship betwen f and g is known as the von Karman relationship for isotropic turbulence

And finally we can rewrite the rank 2 isotropic tensor in terms of a single scalar function, $f(r)$, as

$$Q_{i,j}(\vec{r}) = u^2\left[\left(-\frac{1}{2r}\frac{\partial f}{\partial r}\right) r_i r_j + \left(f + \frac{r}{2}\frac{\partial f}{\partial r}\right)\delta_{ij}\right] \tag{38}$$

Unlike the rank 1 tensor, the rank 2 tensor does not drop out of our dynamical equations when we assume isotropic flow. However, it can be written in terms of a single, scalar, measurable quantity, $f(r)$, that is easy to obtain in the laboratory. It is important to emphasize the amount of simplification achieved here: we started with a second order tensor with 9 unknown components, and with the assumption of isotropy we can express this entire tensor in terms of a single scalar function.

We can also obtain this same result geometrically, as before. Let's consider the rank 2 tensor written as

$$Q_{i,j} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix} \tag{39}$$

If we rotate our coordinates about the $\hat{x}_1$ axis (the longitudinal axis of $\vec{r}$), then we conclude that

$$Q_{12} = Q_{21} = Q_{13} = Q_{31} = 0 \tag{40}$$

because the orientation of $\vec{r}$ remains the same and the $\hat{x}_2$ and $\hat{x}_3$ direction can be negated by rotation. By means of this same rotation, we can also turn Q_{22} into Q_{33} so we conclude that

$$Q_{22} = Q_{33} \tag{41}$$

If we reflect across the $\hat{x}_1$-$\hat{x}_3$ plane, then the $\hat{x}_3$ axis remains fixed but the $\hat{x}_2$ axis is negated, hence we conclude that

$$Q_{23} = Q_{32} = 0 \tag{42}$$

Finally, if we reflect across the $\hat{x}_2$-$\hat{x}_3$ plane, then the $\hat{x}_1$ longitudinal axis is negated and we conclude that

$$Q_{11}(\vec{r}) = \overline{u_1(\vec{x})u_1(\vec{x}+\vec{r})} = \overline{[-u_1(\vec{x})][-u_1(\vec{x}-\vec{r})]} = Q_{11}(-\vec{r}) \tag{43}$$

and therefore Q_{11} is an even function of r. Similarly, if we rotate about the $\hat{x}_3$ axis, we can negate both the $\hat{x}_1$ and $\hat{x}_2$ axes to obtain

$$Q_{22}(\vec{r}) = \overline{u_2(\vec{x})u_2(\vec{x}+\vec{r})} = \overline{[-u_2(\vec{x})][-u_2(\vec{x}-\vec{r})]} = Q_{22}(-\vec{r}) \tag{44}$$

and therefore Q_{22} is also an even function. We can then rewrite our tensor in terms of just two, even, scalar functions, as

$$Q_{i,j} = \begin{pmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{22} \end{pmatrix} \tag{45}$$

which is the same result we obtained mathematically above. And we could relate these two function by means of continuity.

Third Order Isotropic Tensor

Finally, we need to describe the rank 3, two-point covariance matrix, $Q_{ij,k} = \overline{u_i u_j u_k'}$ in isotropic form. We know the general form already

$$Q_{ij,k}(\vec{r}) = Ar_i r_j r_k + Br_i\delta_{jk} + Cr_j\delta_{ki} + Dr_k\delta_{ij} \tag{46}$$

and we know that we have symmetry in (i, j) which means that $B = C$, leaving

$$Q_{ij,k}(\vec{r}) = Ar_i r_j r_k + B(r_i\delta_{jk} + r_j\delta_{ki}) + Dr_k\delta_{ij} \tag{47}$$

and again we can apply continuity

$$\frac{\partial}{\partial r_k}Q_{ij,k} = \underbrace{\frac{\partial}{\partial r_k}\left(Ar_i r_j r_k\right)}_{r_i r_j\left(r\frac{\partial A}{\partial r}+5A\right)} + \underbrace{\frac{\partial}{\partial r_k}(Br_i)\delta_{jk} + \frac{\partial}{\partial r_k}\left(Br_j\right)\delta_{ki}}_{r_i r_j\left(\frac{2}{r}\frac{\partial B}{\partial r}\right)+2B\delta_{ij}} + \underbrace{\frac{\partial}{\partial r_k}(Dr_k)\delta_{ij}}_{\left(r\frac{\partial D}{\partial r}+3D\right)\delta_{ij}} = 0 \tag{48}$$

and simplifying yields

$$r_i r_j\left(r\frac{\partial A}{\partial r} + 5A + \frac{2}{r}\frac{\partial B}{\partial r}\right) + \left(r\frac{\partial D}{\partial r} + 3D + 2B\right)\delta_{ij} = 0 \tag{49}$$

This equation must be satisfied for all r, hence both parts must be satisfied independently, so we have

$$r\frac{\partial A}{\partial r} + 5A + \frac{2}{r}\frac{\partial B}{\partial r} = 0 \quad \text{and} \quad r\frac{\partial D}{\partial r} + 3D + 2B = 0 \tag{50}$$

which is a system of 2 equations in 3 unknowns (A,B,D), meaning that only one scalar function can independently specify the whole tensor. Just like we did in the case of the rank 2 tensor, let's introduce scalar functions that are easily measured in the laboratory and that have simple physical significance, and associate those functions with the three unknown scalar functions of the isotropic tensor.

Consider the case where $\vec{r}$ is along the $\hat{x}_1$ axis, in which case $r_3 r_3 = 0$ and $\delta_{33} = 1$ and the case where $\vec{r} = \sqrt{\frac{r}{2}}(1, 1, 0)$, in which case $r_1 r_2 = \frac{r}{2}$ and $\delta_{12} = 0$

We again define the experimental correlation functions in terms of longitudinal and transverse velocity components, normalized by the mean kinetic energy measured at a single point, $u^2 = \frac{1}{3}\overline{u_i u_i}$.

$$h(r) = \frac{\overline{u_L^2(\vec{x})u_L(\vec{x}+\vec{r})}}{u^3} \tag{51}$$

$$k(r) = \frac{\overline{u_T^2(\vec{x})u_L(\vec{x}+\vec{r})}}{u^3} \tag{52}$$

$$q(r) = \frac{\overline{u_L(\vec{x})u_T(\vec{x})u_T(\vec{x}+\vec{r})}}{u^3} \tag{53}$$

Each of these functions corresponds to a particular configuration of the isotropic tensor. The first, $u^3 h(r) = Q_{LL,L}$, so substituting yields

$$u^3 h(r) = Ar_L r_L r_L + B(r_L\delta_{LL} + r_L\delta_{LL}) + Dr_L\delta_{LL} \tag{54}$$

$$u^3 h(r) = Ar^3 + 2Br + Dr \tag{55}$$

We make similar substitutions with the remaining functions, taking note that $r_T = 0$ because there is no transverse component of the 2-point vector, $\vec{r}$. Finally we obtain

$$u^3 h(r) = Ar^3 + 2Br + Dr = -2rD \tag{56}$$

$$u^3 k(r) = Dr \tag{57}$$

$$u^3 q(r) = Br = -\frac{3}{2}rD - \frac{1}{2}r^2\frac{\partial D}{\partial r} \tag{58}$$

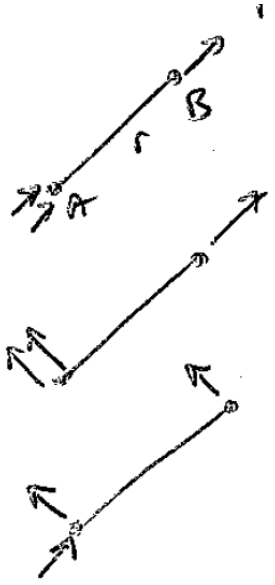


Figure 19: The longitudinal and transverse, 2-point, third-order covariances. (Again, point B represents point A′ in our notation)

where we have made use of the continuity equation above to relate the scalar functions, A, B, and D. Therefore, we are able to write all three scalar function in terms of just a single scalar function, D. As before, this is an amazing degree of simplification from the original 27 components of this third order tensor, down to a single component. In terms of experimental measurement, it is often easiest to measure the streamwise (longitudinal) velocity, hence we would like to write the isotropic tensor in terms of just $h(r)$.

$$Q_{ij,k} = u^3 \left[\left(\frac{h - r\frac{\partial h}{\partial r}}{2r^3} \right) r_i r_j r_k + \left(\frac{2h + r\frac{\partial h}{\partial r}}{4r} \right) \left(r_i \delta_{jk} + r_j \delta_{ik} \right) - \left(\frac{h}{2r} \right) r_k \delta_{ij} \right] \tag{59}$$

At this point, we have expressed all of the 2-point, isotropic covariance tensors in terms of simple (measurable) scalar functions: $f(r)$ for the rank 2 tensor and $h(r)$ for the rank 3 tensor, and we showed that the rank 1 tensor must be zero. We are now ready to substitute our isotropic tensor back into our governing dynamics.

Isotropic Dynamics

Recall the equation for homogenous, 2-point dynamics assuming no mean velocity gradients

$$\frac{\partial}{\partial t} Q_{i,j} = \frac{\partial}{\partial r_k} Q_{ik,j} - \frac{\partial}{\partial r_k} Q_{i,kj} - \frac{1}{\rho} \left(-\frac{\partial}{\partial r_i} Q_{p^*,j} + \frac{\partial}{\partial r_j} Q_{i,p^*} \right) + 2\nu \frac{\partial^2}{\partial r_k^2} Q_{i,j} \tag{60}$$

We have shown that the rank 1 tensors like $Q_{p^*,j}$ must be identically zero for statistically isotropic flow, so we can simplify the equation to

$$\frac{\partial}{\partial t} Q_{i,j} = \frac{\partial}{\partial r_k} Q_{ik,j} - \frac{\partial}{\partial r_k} Q_{i,kj} + 2\nu \frac{\partial^2}{\partial r_k^2} Q_{i,j} \tag{61}$$

In isotropic flow, the pressure strain-rate tensor, $\Pi_{ij} = 0$. This is consistent with our physical explanation of pressure strain rate as a mechanism for shifting kinetic energy between different velocity components; in an isotropic flow, the kinetic energy is already divided evenly among the different components, in an ensemble sense, hence this term must be zero

And we recognize that the remaining equation is a scalar equation in terms of only $f(r)$ and $h(r)$. We could substitute our earlier definitions directly, but because we know the resulting equation is scalar anyway, we might as well contract the i and j indices first to make the substitution easier. Letting $i = j$ yields

$$\frac{\partial}{\partial t} Q_{i,i} = \frac{\partial}{\partial r_k} Q_{ik,i} - \frac{\partial}{\partial r_k} Q_{i,ki} + 2\nu \frac{\partial^2}{\partial r_k^2} Q_{i,i} \tag{62}$$

and we recognize that $Q_{ik,i}(\vec{r}) = Q_{i,ik}(-\vec{r})$. However, for a 2-point isotropic tensor, we also know that $Q_{ij,k}$ is an odd function of r, in which case $Q_{i,ik}(-\vec{r}) = -Q_{i,ik}(\vec{r})$ and thus $Q_{ik,i}(\vec{r}) = -Q_{i,ik}(\vec{r})$ and we can simplify

$$\frac{\partial}{\partial t} Q_{i,i} = 2\frac{\partial}{\partial r_k} Q_{ik,i} + 2\nu \frac{\partial^2}{\partial r_k^2} Q_{i,i} \tag{63}$$

Now we just need to substitute our experimental scalar functions into the governing equation. Rewriting the rank 2 tensor we obtain

$$Q_{i,i}(\vec{r}) = u^2\left[\left(-\frac{1}{2r}\frac{\partial f}{\partial r}\right)r_i r_i + \left(f + \frac{r}{2}\frac{\partial f}{\partial r}\right)\delta_{ii}\right] = u^2\left[r\frac{\partial f}{\partial r} + 3f\right] = \frac{u^2}{r^2}\frac{\partial}{\partial r}\left(r^3 f\right) \equiv Q(r) \tag{64}$$

and similarly for the rank 3 tensor. However, here let's let $i = k$ and then afterwards exchange the dummy index j for k.

$$Q_{ij,i} = u^3\left[\left(\frac{h - r\frac{\partial h}{\partial r}}{2r^3}\right)r_i r_j r_i + \left(\frac{2h + r\frac{\partial h}{\partial r}}{4r}\right)\left(r_i\delta_{ji} + r_j\delta_{ii}\right) - \left(\frac{h}{2r}\right)r_i\delta_{ij}\right] \tag{65}$$

$$= u^3\frac{r_j}{2r}\left[4h + r\frac{\partial h}{\partial r}\right] \tag{66}$$

$$= \frac{1}{2}r_j\left[\frac{u^3}{r^4}\frac{\partial}{\partial r}(r^4 h)\right] \equiv \frac{1}{2}r_j H(r) \tag{67}$$

And we can exchange the dummy index to write $Q_{ik,i} = \frac{1}{2}r_k H(r)$. Finally, we can substitute our measurable functions back into the governing dynamics to obtain

We can make use of the result that $\frac{\partial^2}{\partial r_k^2}s(r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial s}{\partial r})$, which is found by repeated use of our earlier rule that $\frac{\partial r}{\partial r_k} = \frac{r_k}{r}$ and similarly we can show that $\frac{\partial}{\partial r_k}(r_k s(r)) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 s(r))$

$$\frac{\partial}{\partial t}Q_{i,i} = 2\frac{\partial}{\partial r_k}Q_{ik,i} + 2\nu\frac{\partial^2}{\partial r_k^2}Q_{i,i} \tag{68}$$

$$\frac{\partial}{\partial t}Q(r) = \frac{\partial}{\partial r_k}(r_k H(r)) + 2\nu\frac{\partial^2}{\partial r_k^2}Q(r) \tag{69}$$

$$\frac{\partial}{\partial t}Q(r) = \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^3 H)\right] + 2\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial Q}{\partial r}\right)\right] \tag{70}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial t}\left(u^2 r^3 f\right)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[(r^3 H)\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[2\nu\left(r^2\frac{\partial Q}{\partial r}\right)\right] \tag{71}$$

We note that we defined $H(r) \equiv \frac{u^3}{r^4}\frac{\partial}{\partial r}(r^4 h)$ and $Q(r) \equiv \frac{u^2}{r^2}\frac{\partial}{\partial r}(r^3 f)$ above

and after integrating once over r we obtain

$$\frac{\partial}{\partial t}\left(u^2 r^3 f\right) = r^3 H + 2\nu r^2\frac{\partial Q}{\partial r} \tag{72}$$

and finally we can substitute for our definitions of H and Q to obtain the full evolution equation for $f(r,t)$, the single scalar function which defines the 2-point, second order covariance for isotropic turbulence. Of course, this equation still depends on a third order covariance function, $h(r,t)$, so we have not avoided the closure problem even in the case of isotropic ensemble statistics.

$$\frac{\partial}{\partial t}\left(u^2 f\right) = u^3\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)h + 2\nu u^2\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)f \tag{73}$$

This equation is known as the Karman-Howarth equation. We can also write this in the form of structure functions or spectral densities, by means of the relationships we developed earlier, and the general form of isotropic tensors. This equation is very significant for the understanding of homogeneous, isotropic turbulence in the absence of mean shear, as we will see in the next chapter.

The Karman-Howarth Equation

In the last chapter, we simplified the covariance dynamics for a general homogeneous flow (without mean shear) to the particular case of an isotropic flow. We showed that the dynamics can be written entirely in terms of two scalar functions, the longitudinal second-order correlation function $f(r)$, and the longitudinal third-order correlation, $h(r)$, both of which can be measured in the laboratory. Now that we have the dynamical equation, we can explore its physical consequences.

Kinetic Energy Formulation

We wrote the Karman-Howarth equation in the form

$$\frac{\partial}{\partial t}\left(u^2 f\right) = u^3\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)h + 2\nu u^2\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)f \tag{1}$$

and we know that in the limit that $r \to 0$, $f(0) = 1$ and thus we have a dynamical equation for u^2. Recall that we defined $u^2 = \frac{1}{3}\overline{u_i u_i}$, and therefore the turbulent kinetic energy is given by $\frac{1}{2}\overline{u_i u_i} = \frac{3}{2}u^2$. Therefore we can evaluate the Karman-Howarth equation at $r = 0$, with the fact that $h'(0) = 0$, to obtain

$$\frac{d}{dt}u^2 = u^3 \underbrace{\left[\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)h\right]_{r=0}}_{\lim_{r\to 0} 5\frac{\partial h}{\partial r}} + 2\nu u^2 \underbrace{\left[\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)f\right]_{r=0}}_{\lim_{r\to 0} 5\frac{\partial^2 f}{\partial r^2}} \tag{2}$$

which we can rewrite in terms of the turbulent kinetic energy as

$$\frac{d}{dt}\left(\frac{3}{2}u^2\right) = 15\nu u^2 f''(0) \tag{3}$$

Now we need to find a representation for the value of $\left.\frac{\partial^2 f}{\partial r^2}\right|_{r=0} \equiv f''(0)$. We know that $f(r)$ is the longitudinal correlation function, which is just the regular correlation function evaluate for longitudinal velocity components in the longitudinal direction. And we know that we can relate the regular correlation function, $R(s)$ to the Taylor microscale, by writing a Taylor series

$$R(s) = 1 - \frac{s^2}{2\lambda^2} + \mathcal{O}(s^4) \tag{4}$$

The function h is an odd function, therefore $h(0) = 0$. The limit $\lim_{r\to 0}\frac{4h}{r} = \lim_{r\to 0}\frac{4\frac{\partial h}{\partial r}}{1}$ by l'Hopital's Rule. Therefore $\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)h \to 5\left.\frac{\partial h}{\partial r}\right|_{r=0}$. Similarly, the function $f(r)$ is even, hence its derivative $\frac{\partial f}{\partial r}$ is odd. The limit $\lim_{r\to 0}\frac{4\frac{\partial f}{\partial r}}{r} = \lim_{r\to 0}\frac{4\frac{\partial^2 f}{\partial r^2}}{1}$. We need to determine the value of $h'(0)$, so let's write out the full form $u^3 h(r_L) = \overline{u_L^2(\vec{x})u_L(\vec{x}+r_L)}$ where we have explicitly written the scalar r_L in the longitudinal direction, for clarity. The derivative is given by $u^3\frac{\partial}{\partial r_L}h(r_L) = \overline{u_L^2(\vec{x})\frac{\partial}{\partial x_L}u_1(\vec{x}+x_L)}$. In the limit as $r_L \to 0$, we have $\lim_{r_L\to 0} u^3\frac{\partial}{\partial r_L}h(r_L) = \overline{u_L^2(\vec{x})\frac{\partial}{\partial x_L}u_L(\vec{x})} = \frac{1}{3}\overline{\frac{\partial}{\partial x_L}u_L^3(\vec{x})} = \frac{1}{3}\frac{\partial}{\partial x_L}\overline{u_L^3(\vec{x})} = 0$ by homogeneity. Repeating the same process for $f''(0)$ shows that it is non-zero.

Therefore, for $R(s) = f(r)$ we have $f''(0) = -\frac{1}{\lambda^2}$, which we can insert back into our equation to obtain an equation for the rate of change of TKE in homogeneous, isotropic turbulence.

$$\frac{d}{dt}\left(\frac{1}{2}\overline{u_i u_i}\right) = -15\frac{\nu u^2}{\lambda^2} \equiv -\varepsilon \tag{5}$$

This rate of change of TKE is, by definition, the dissipation rate, ε, and therefore we can relate the dissipation rate and Taylor microscale directly. (There is no production because we assumed that no mean velocity gradients exist, hence this form of the Karman-Howarth equation describes the decay of homogeous, isotropic turbulence)

$15\frac{\nu u^2}{\lambda^2} = \varepsilon$ is a useful, ballpark approximation for the dissipation anywhere the assumption of local isotropy applies

The fact that we can express the dissipation in terms of a single measurement, $f''(0)$, means that we should be able to express all of the different averaged velocity gradients that constitute ε. We can indeed express this, although the derivation is a bit complicated

$$\overline{\frac{\partial u_i}{\partial x_n}\frac{\partial u_j}{\partial x_m}} = -2u^2\left[\delta_{ij}\delta_{mn} - \frac{1}{4}\left(\delta_{in}\delta_{jm} + \delta_{im}\delta_{jn}\right)\right]f''(0) \tag{6}$$

Start with $\frac{\partial^2}{\partial r_n \partial r_m} Q_{i,j}$, substitute the definition of $Q_{i,j}$ in terms of $f(r)$ written above, expand all of the derivatives, and finally take the limit as $r \to 0$; see Hinze's book, eq 3-19

Using this formula, there are only four possible outcomes

$\frac{1}{u^2}\overline{\frac{\partial u_i}{\partial x_n}\frac{\partial u_j}{\partial x_m}}$	Case
$f''(0)$	$\{i = j = m = n\}$
$-\frac{1}{2}f''(0)$	$\{i = m, j = n, i \neq j\}$ or $\{i = n, j = m, i \neq j\}$
$2f''(0)$	$\{i = j, m = n, i \neq m\}$
0	else

Table 3: All of the ensemble averaged velocity gradients in a homogeneous, isotropic flow, expressed in terms of the longitudinal correlation function, $f(r)$.

This means that we know that, for instance,

$$\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^2} = \overline{\left(\frac{\partial u_2}{\partial x_2}\right)^2} = \overline{\left(\frac{\partial u_3}{\partial x_3}\right)^2} = u^2 f''(0) \tag{7}$$

for isotropic flow, and we could use the ratio of these different averaged gradients as a measure of how close the flow is to actually behaving as isotropic.

Physical Consequences of Karman Howarth

Turbulence is Non-Gaussian

If we consider again the standard form of the Karman-Howarth equation, we have already identified the final viscous term as representing dissipative processes, from the point of view of TKE. In our derivation of the homogeneous dynamics, we had assumed that there are no mean velocity gradients in the flow, and therefore we know that there is no production of TKE via a term like $\mathcal{P}$. So the remaining term, which depends on $h(r)$, must represent transport processes in turbulence, like we saw in our one-point TKE equation. These transport processes occur at the large-scale, or what we call inertial scales, in turbulence, and they have no net effect on the total TKE in a closed system (this term dropped out of out TKE balance);

transport only moves kinetic energy around, from large scales to smaller scales and eventually down to dissipative scales.

$$\frac{\partial}{\partial t}\left(u^2 f\right) = \underbrace{u^3\left(\frac{\partial}{\partial r} + \frac{4}{r}\right)h}_{\text{Inertial Transport}} + \underbrace{2\nu u^2\left(\frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r}\right)f}_{\text{Dissipation}} \tag{8}$$

Imagine that all of our ensemble quantities had Gaussian distributions. We know that we can write any covariance moment in terms of a join PDF of fluctuations, $B(c, c')$, so let's assume that this PDF is normally distributed, such that it has zero skewness. Thus the third-order moments are zero, i.e. $h(r) = 0$ everywhere. If there is no inertial transport of energy, then energy cannot move from large scales down to dissipative scales. The entire idea of an energy cascade breaks down. So the first conclusion we can draw from the Karman-Howarth equation is that turbulence must be non-Gaussian, or else the energy cascade is not possible.

Landau & Lifshitz §34 show that Loitsyansky's integral, which can be written as the volume integral $I = -\int_0^\infty r^2 Q_{i,i} d^3\vec{r}$, represents the total angular momentum content in a material volume.

Kolmogorov's Decay Law

We can rewrite the Karman Howarth equation in a simplified form, be grouping the derivatives

$$\frac{\partial}{\partial t}\left(u^2 r^4 f\right) = u^3 \frac{\partial}{\partial r}(r^4 h) + 2\nu u^2 \frac{\partial}{\partial r}\left(r^4 \frac{\partial f}{\partial r}\right) \tag{9}$$

and then we might be tempted to try to integrate this equation once, over r from 0 to ∞ to yield

$$\frac{d}{dt}\int_0^\infty u^2 r^4 f dr = \left[u^3(r^4 h) + 2\nu u^2\left(r^4 \frac{\partial f}{\partial r}\right)\right]_{r=0}^{\infty} \tag{10}$$

Landau & Lifshitz §34 show that Loitsyansky's integral, which can be written as the volume integral $I = -\int_0^\infty r^2 Q_{i,i} d^3\vec{r}$, represents the total angular momentum content in a material volume.

This integral is called Loitsyansky's integral. Now we know that $h(0) = 0$ because h is an odd function, and $\frac{\partial f}{\partial r}$ is also odd, since f is an even function, so we just need to evaluate the right hand side in the limit as $r \to \infty$. Let's assume that very distance points are statistically independent, i.e. that $h(r)$ and $f'(r)$ both decay to zero at a rate faster than r^4. If these assumption are valid (they aren't!), then we can conclude

$$u^2 \int_0^\infty r^4 f dr = \text{constant} \tag{11}$$

which indicates, dimensionally, that for freely decaying turbulence, $u^2 \ell^5 \sim$ constant. If we combine this conclusion with our earlier dimensional result that the approximate turnover time for a large-scale eddy is $t \sim \frac{\ell}{u}$, we can write

$$u^2 \sim t^{-10/7} \quad \text{and} \quad \ell \sim t^{2/7} \tag{12}$$

This is a remarkable prediction, which is called Kolmogorov's Decay Law. It allows us to predict the rate at which free turbulence decays in time, and similarly it predicts that the characteristic size

of the eddies in free turbulence tends to grow as the turbulence decays, because the smallest eddies decay first, thereby shifting the 'average' to larger eddies. And, in fact, earlier experiments were able to verify this prediction, albeit in limited circumstances. Why could they not verify the results in general?

The key problem with this integral analysis is that the assumption that distant measurement points are uncorrelated is not true. In an incompressible flow, the pressure field can transmit information instantaneously over any distance. Moreover, Saffman showed that Loitsyansky's integral is not well defined: it actually diverges in certain cases. Therefore, this analysis holds only when long-range correlations are especially weak, but in this cases it remains a powerful result.

Recall that the speed of sound is defined by the isentropic derivative, $c = \sqrt{\frac{\partial p}{\partial \rho}\Big|_s}$. We can define this in terms of the compressibility, β (inverse of the bulk modulus) to obtain $c = \sqrt{\frac{1}{\beta\rho}}$, so as compressibility $\beta \to 0$, the speed of sound, $c \to \infty$. Thus pressure is considered a non-local effect, in comparison to, say, vorticity, which decays rapidly and thus cannot induce velocity at great distance.

Final Period of Decay

The Karman-Howarth equation describes the decay of free (unforced) turbulence. We saw that it describes both inertial and viscous effects. Let's ask what happens when viscosity becomes the dominant effect, i.e. inertial effects are negligible. We could describe the time period in which this assumption applies as the final period of decay, when all of the energy has already been transferred to the smallest scales and is about to be dissipated into heat. Let's start with the Karman-Howarth equation in the more general form

$$\frac{\partial}{\partial t}Q(r) = 2\nu\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial Q}{\partial r}\right) \tag{13}$$

This looks like the diffusion equation for symmetrical, spherical coordinates, which is written in terms of the temperature T and the thermal diffusivity α as

$$\frac{\partial}{\partial t}T(r) = \alpha\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \tag{14}$$

The solution is written in terms of a similarity variable, $\frac{r^2}{\alpha t}$, but in our case we assume incomplete similarity and thus we write the solution in the form of

If we assume complete similarity we find that $f(r)$ is undefined at $r = 0$, hence we need to assume that a similarity variable alone is not sufficient

$$Q(r,t) \sim \phi(t)\psi\left(\frac{r^2}{\nu t}\right) \tag{15}$$

If we plug this solution back into our definition of $Q(r) = \frac{u^2}{r^2}\frac{\partial}{\partial r}(r^3 f)$, we find

$$f(r,t) = \exp\left(-\frac{r^2}{8\nu t}\right) \tag{16}$$

This means that, in its final period of decay, the correlation function has a Gaussian form. In other words, the non-Gaussian behavior is most important at inertial small scales; once the flow is entirely dominated by viscous effects, the flow can be characterized as Gaussian in character.

All of these simple physical conclusions were drawn from the correlation form of the Karman-Howarth equation. However, we can write this equation using structure functions or spectral densities in order to obtain more insights about the physics of decay turbulence.

Structure Function Formulation

To formulate the Karman Howarth equation in terms of structure functions, we first start with the definition of two measurable, longitudinal structure functions, and then attempt to relate them to our two measurable, longitudinal correlation functions, $f(r)$ and $h(r)$. We start with the second order structure function describing two points aligned in the longitudinal access, x, separated by distance r on that axis.

We drop the L subscripts for simplicity, although we could formally write the position as x_L and separation as r_L to reflect the longitudinal component of the $\vec{x}$ and $\vec{r}$ vectors.

$$D_{LL} = \overline{[u_L(x+r) - u_L(x)]^2} \tag{17}$$

We can then expand the product to yield

$$D_{LL} = \underbrace{\overline{u_L(x+r)u_L(x+r)}}_{u^2 f(0)} - \underbrace{\overline{u_L(x+r)u_L(x)}}_{u^2 f(r)} - \underbrace{\overline{u_L(x)u_L(x+r)}}_{u^2 f(r)} + \underbrace{\overline{u_L(x)u_L(x)}}_{u^2 f(0)} \tag{18}$$

and simplify to obtain

$$D_{LL} = 2u^2(1-f) \tag{19}$$

We can perform the same operation on the third order structure function

$$D_{LLL} = \overline{[u_L(x+r) - u_L(x)]^3} \tag{20}$$

using the fact that $h(r)$ is an odd function, to obtain

We make use of $f = 1 - \frac{D_{LL}}{2u^2}$, $h = \frac{D_{LLL}}{6u^3}$, and $\frac{d}{dt}u^2 = -\frac{2}{3}\varepsilon$

$$D_{LLL} = 6u^3 h \tag{21}$$

Finally, we substitute into the standard form of the Karman-Howarth equation to obtain

Called so because Kolmogorov derived essentially the same result as Karman and Howarth, but using structure functions instead of correlation functions

$$\frac{\partial}{\partial t}D_{LL} = -\frac{1}{3r^4}\frac{\partial}{\partial r}\left(r^4 D_{LLL}\right) + \frac{2\nu}{r^4}\left(r^4\frac{\partial}{\partial r}D_{LL}\right) - \frac{4}{3}\varepsilon \tag{22}$$

and multiplying through by r^4, integrating once over r, and then dividing by r^4, we obtain the Kolmogorov equation

$$\frac{3}{r^4}\frac{d}{dt}\int_0^r s^4 D_{LL} ds = -D_{LLL} + 6\nu\frac{\partial}{\partial r}D_{LL} - \frac{4}{5}\varepsilon r \tag{23}$$

The most useful formulation of Karman Howarth, however, is the spectral formulation.

Spectral Formulation

Homogeneous Spectral Dynamics

Let's begin with the homogeneous dynamics first, before we assume isotropic behavior.

$$\frac{\partial}{\partial t}Q_{i,j} = \underbrace{\frac{\partial}{\partial r_k}Q_{ik,j} - \frac{\partial}{\partial r_k}Q_{i,kj}}_{\Rightarrow \Gamma_{i,j}} \underbrace{-\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}Q_{p^*,j} + \frac{\partial}{\partial r_j}Q_{i,p^*}\right)}_{\Rightarrow \Pi_{i,j}} + 2\nu\frac{\partial^2}{\partial r_k^2}Q_{i,j} \tag{24}$$

We already learned that the Fourier transform of the covariance tensor $Q_{i,j}(\vec{r})$ is $\Phi_{i,j}(\vec{k})$ where $\vec{k}$ is the three-dimensional wavenumber. We can similarly define the transform of $Q_{ij,k}(\vec{r})$ as $\Phi_{ij,k}(\vec{k})$ and $Q_{p^*,j}(\vec{r})$ is $\Phi_{p^*,j}(\vec{k})$. We can label all of Fourier-transformed transport terms as $\Gamma_{ij}(\vec{k})$ and the transformed pressure rate of strain terms as $\Pi_{ij}(\vec{k})$ and then write the dynamical equation as

See Problem 3.2

$$\frac{\partial}{\partial t}\Phi_{i,j} = \Gamma_{i,j} + \Pi_{i,j} - 2\nu k^2\Phi_{i,j} \tag{25}$$

where $k = |\vec{k}|$ is the magnitude of the wavenumber vector. We can interpret this equation in two ways.

1. We can think about the total spectral energy across all wavenumbers $\vec{k}$ for fixed velocity components. In this case, we integrate the above equation over all wavenumbers

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Phi_{i,j}d^3\vec{k} = \underbrace{\int_{-\infty}^{\infty}\Gamma_{i,j}d^3\vec{k}}_{=0} + \int_{-\infty}^{\infty}\Pi_{i,j}d^3\vec{k} - 2\nu\int_{-\infty}^{\infty}k^2\Phi_{i,j}d^3\vec{k} \tag{26}$$

   where it is left as an exercise to show that the integrated transport term is identically zero. Thus we conclude that the transport term redistributes energy between different scales (i.e. different values of $\vec{k}$) of the same velocity components (i, j) but has no net effect on the total energy over all scales. This is consistent with our understanding that the third-order terms are important for the energy cascade — the movement of energy from large scales down to the dissipative scales.

2. Alternatively, we can think about the energy at fixed wavenumbers, across all velocity components. To consider all velocity components, we need to sum over the velocity components (i.e. let $i = j$) to obtain

$$\frac{\partial}{\partial t}\Phi_{i,i} = \Gamma_{i,i} + \underbrace{\Pi_{i,i}}_{=0} - 2\nu k^2\Phi_{i,i} \tag{27}$$

   where again it is left as an exercise to show that the contracted pressure rate of strain spectral density is identically zero. Thus

we conclude that the pressure forces spread out energy among different components of velocity, for a fixed wavenumber, but have no net impact on the total energy at a given wavenumber. Of course, we've already learned that for isotropic flows, the pressure-strain tensor must be identically zero (the energy has already been divided up evenly in the different directions).

The spreading of energy across different velocity components is referred to as the 'return to isotropy'

Isotropic Spectral Dynamics

Now, if we assume isotropic behavior, this means that all of the spectral tensors must obey the isotropic rules just like the correlations tensors did. We can then write the Karman-Howarth equation in terms of spectral isotropic tensors, $\Phi_{i,j}(k)$ and $\Phi_{ij,k}(k)$ which are functions of $k = |\vec{k}|$, and we can contract the indices $i = j$ in order to write the dynamics in terms of simpler, scalar functions called $E(k)$ and $T(k)$. These scalar spectral functions $E(k)$ and $T(k)$ can also be related to the measurable correlation functions $f(r)$ and $h(r)$ by integral transforms.

Recall that $\frac{1}{2}\overline{u_i u_i} = \int_0^\infty E(k)dk$ and thus $E(k)$ is the energy density (per unit wavenumber)

Covariance	Spectral Density	Structure Function
$Q_{i,i}(r) = u^2 \frac{(r^3 f)'}{r^2}$	$E(k) = 2\pi k^2 \Phi_{i,i}(k)$	$D_{LL} = 2u^2(1-f)$
$Q_{ij,i}(r) = u^3 \frac{1}{2} r_j \frac{(r^4 h)'}{r^4}$	$T(k) = 4\pi i k^2 k_j \Phi_{ij,i}(k)$	$D_{LLL} = 6u^3 h$

Table 4: Isotropic 2-point functions in terms of covariance, spectral density, and structure functions.

Then we can rewrite the dynamical covariance equation

$$\frac{\partial}{\partial t} Q_{i,i} = 2\frac{\partial}{\partial r_k} Q_{ik,i} + 2\nu \frac{\partial^2}{\partial r_k^2} Q_{i,i} \tag{28}$$

We defined the Fourier transform pairs above as: $\Phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{-i(\vec{k}\cdot\vec{r})} Q_{ij}(\vec{r}) d^3\vec{r}$ and $Q_{ij}(\vec{r}) = \int_{-\infty}^{\infty} e^{i(\vec{k}\cdot\vec{r})} \Phi_{ij}(\vec{k}) d^3\vec{k}$

in spectral form, by using the Fourier transform pairs and the above definitions, to write

$$\frac{\partial}{\partial t} E(k,t) = T(k,t) - 2\nu k^2 E(k,t) \tag{29}$$

The function $T(k)$ is the isotropic counterpart to the homogeneous function $\Gamma_{i,i}$, which we showed above represents the transport of kinetic energy between different wavenumbers, k, within a given velocity component. We have already shown that energy tends to spread from larger scales (small k) to smaller scales (high k). As the energy moves to smaller and smaller scales, viscous dissipation becomes more important, and we see that in the equation: for very large k, the dissipation term is dominant, since it scales with a prefactor of k^2. Where does the energy come from originally? This equation describes only the decay (since there are no mean velocity gradients), but we assume that the energy originated with very large scale motions associated with the boundaries of the flow itself, which we had previously called ℓ.

With this picture in mind, we should be able to sketch a picture of the energy density, $E(k)$. First, let's consider the behavior for large k (very small scale fluctuations, assuming inertial transport is

weak). Then we have

$$\frac{\partial}{\partial t}E(k,t) \approx -2\nu k^2 E(k,t) \tag{30}$$

which we can integrate to obtain

$$E(k,t) \approx E(k,0)\exp\left[-2\nu k^2(t-t_0)\right] \tag{31}$$

Thus, for large wavenumbers, there is an exponential drop-off in the energy density, $E(k)$. What happens at small wavenumbers (large scales)? We can write the energy density in a Taylor series for small k as

$$E(k) \approx L\frac{k^2}{4\pi^2} + I\frac{k^4}{24\pi^2} + \cdots \tag{32}$$

where L and I are constants given by integrals of $Q_{i,i}$ and we assume that L is neglible. Therefore we see that initially, $E(k)$ grows quickly as a power of k^4. Putting together the low and high wavenumber behavior, we can draw

$I = -\int_0^\infty r^2 Q_{i,i} d^3\vec{r}$, which is just Loitsyansky's integral, discussed above, and $L = \int_0^\infty Q_{i,i} d^3\vec{r}$, where these are volume integrals, and we can define $d^3\vec{r} = 4\pi r^2 dr$. L is called Saffman's integral, and can be evaluated as $4\pi u^2(rf)\big|_\infty$ which is assumed to be negligible.

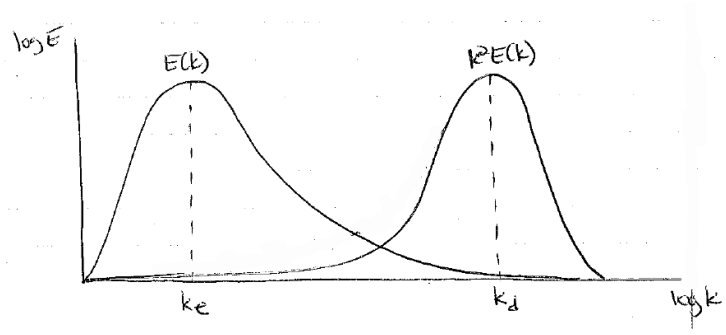


Figure 20: A sketch of the spectral energy density, $E(k)$ and the distribution of the dissipation term, $2\nu k^2 E(k)$, each with its peak labeled by wavenumber k_e for the peak energetic scale and k_d for the peak dissipative scale.

The peak of the energy distribution occurs at a wavenumber k_e which corresponds roughly to the large, energy containing eddies of size ℓ. We can also draw the distribution of energy from the perspective of viscous dissipation, which appears in the governing equation premulitplied by k^2. The peak of dissipation occurs at k_d.

Dissipation Rate & Transport Rate

If we integrate the governing equation over the entire spectrum (all wavenumbers), we obtain

$$\underbrace{\frac{d}{dt}\int_k^\infty E(k,t)dk}_{\frac{d}{dt}\left(\frac{1}{2}\overline{u_i u_i}\right)} = \underbrace{-2\nu\int_0^\infty k^2 E(k,t)dk}_{\varepsilon} \tag{33}$$

The equation $2\nu\int_0^\infty k^2 E(k,t)dk \equiv \varepsilon$ can be used as a 'normalization condition' in the subsequent mathematical development of the spectral density

because we have already established that $T(k)$ represents transport between scales, so its net effect over all scales is zero. We see that the dissipation rate, ε, is defined from the integral over all wavenumbers, and is thus not a function of wavenumber. However, the rate of energy transfer between two eddies is dependent on the

eddy size, hence $T(k)$ is a function of wavenumber. Let's integrate our dynamics over a narrower range of wavenumbers, $k \in [k1, k2]$ such that

$$\frac{d}{dt}\int_{k_1}^{k_2} E(k,t)dk = \int_{k_1}^{k_2} T(k,t)dk - 2\nu\int_{k_1}^{k_2} k^2 E(k,t)dk \tag{34}$$

If we choose our range at low wavenumbers, in the vicinity of the energetic peak, $[k_1, k_2] = [k_e - \Delta k, k_e + \Delta k]$], then the dissipation term is neglible, and most of the change in TKE within this range of wavenumbers is occurring by means of transport, $T(k)$ to higher wavenumbers, $k > k_2$, outside this range. If we choose our range at high wavenumbers, in the vicinity of the dissipative peak, $[k_1, k_2] = [k_d - \Delta k, k + d + \Delta k]$, then the transport term is negligible, and most of the change in TKE in this range of wavenumbers is occuring by means of viscous dissipation to heat. However, note that the rate of energy exchange between scales is not constant because the energy and dissipation spectra overlap. If the energy and distribution spectra were, instead, two discrete, Dirac $\delta(k)$ functions, then all of the energy transferred out of k_e would be dissipated at k_d, and that rate would be identically ϵ. However, because the spectra are continuous with some overlap, the amount of energy exchanged between each pair of successively smaller scales decreases.

We assume that the transport of TKE moves from large eddies to small eddies (in the direction of increasing wavenumber). At the lowest wavenumbers, there is only one direction to move energy in an isolated system, so this seems like a reasonable assumption, although for the special case of so-called 2D turbulence and strongly rotating flows, the cascade has been observed to operate in reverse.

Imagine a line of people (say 100 of them) holding buckets and they are trying to transfer water from a large well, down the line. But the buckets are all a bit leaky – some water drips out of the buckets as water is poured from one person's bucket to the next. And as the line gets longer, the buckets get leakier. Meaning the buckets at the end of the line tend to leak more than the buckets at the beginning of the line. Now, all the water removed out of the well eventually leaks onto the ground. So the total rate of extraction of water from the well is equal to the total amount of water pouring onto the ground, i.e. the rate at which the well water enters the first bucket is equal to the sum of all of the water leaking from all the buckets. However, most of the water that spills to the ground very far down the line, from the leakier buckets. Very little water is lost in the first dozen buckets in line. So now let's measure the rate of transfer of water between two buckets. It's obviously not equal to the rate of transfer of water from the well to the first bucket, because a little bit of water is lost from each bucket down the line. However, the rate of transfer is almost equal to the initial rate for most of the buckets, until we are very far down the line, at which point the buckets start to become very leaky, and then most of the water is lost from just a few buckets down the line. If there were only two buckets, the first one without any leaks and the seconds one completely leaky, then the rate of water transfer would be exactly the same across all (both) buckets.

Kolmogorov Turbulence

In the last chapter, we wrote the dynamical equation for decaying isotropic turbulence in covariance, structure function, and spectral forms, and we derived the basic shape of the spectral density. Because the spectral density tells us how the energy in turbulence is distributed across different sized coherent structures, it is important to provide a detailed picture of this distribution.

Universal Equilibrium Range

How do we determine the exact shape of the energy density? We expect the exact shape to depend on the Reynolds number because Reynolds number describes the relative strength of inertial and viscous processes in a flow. If we non-dimensionalize the governing equation, we obtain

$$\mathrm{St}\frac{\partial}{\partial t}E(k,t) = T(k,t) - \frac{1}{\mathrm{Re}}2k^2E(k,t) \tag{1}$$

The non-dimensionalization is performed in terms of inertial scales, u and ℓ, and arbitrary timescale τ. The Strouhal number $\mathrm{St} = \frac{L/U}{\tau}$ is a non-dimensional timescale, relating the ratio of the dimensional scales of the equation, (U, L), to the relevant timescale of the dynamics, τ.

As the Reynolds number increases, for a fixed wavenumber, the relative strength of the dissipative term decreases. Dissipation then becomes important only at high wavenumbers. $T(k,t)$ represents the energy being transferred down the energy cascade, from the inertial scales to dissipative scales. As Reynolds number increases, so do the corresponding wavenumbers at which dissipation occurs. In other words, the higher the Reynolds number, the more inertia dominates over viscosity, the greater the ability of $T(k)$ to shift energy to ever higher wavenumbers before viscosity dissipates it. Stronger viscosity can dissipate energy at lower wavenumbers (larger structures), weaker viscosity acts only on higher wavernumbers (smaller structures).

At small Reynolds number, the inertial and viscous forces are similar in magnitude, and thus viscosity is able to dissipate a wide range of scales, including relatively large scales. This means that the inertially dominante range of wavenumbers is constrained. As Reynolds number increases, the separation between the energetic (inertial) and dissipative scales increases. The inertial scales spread to higher and higher wavesnumbers before dissipation kicks in. As the two peaks become separated, a region of energetic scales emerges wherein dissipation is negligible, meaning that roughly the

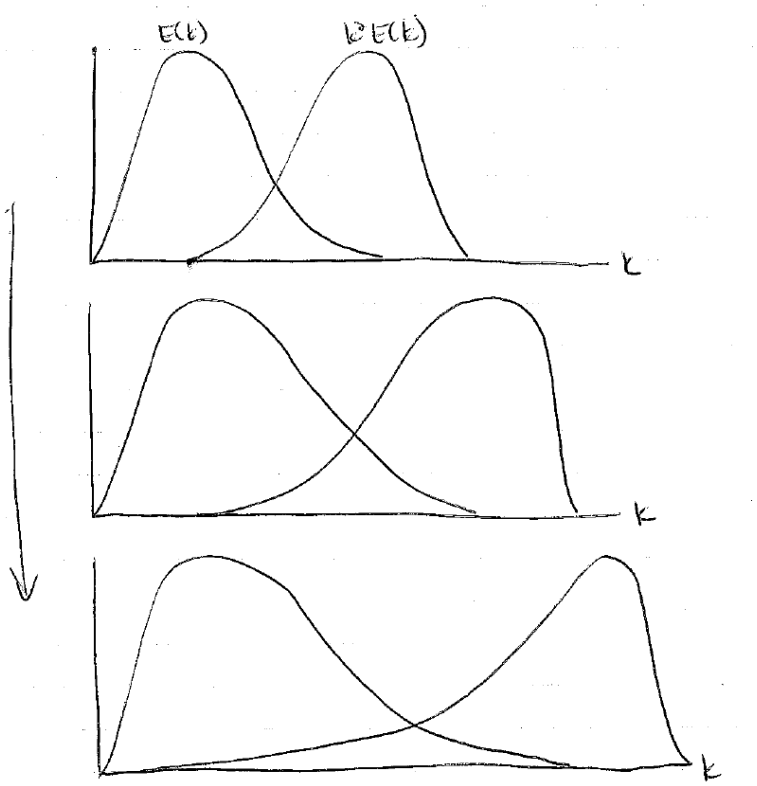


Figure 21: A sketch of the spectral energy density, $E(k)$ and the distribution of the dissipation term, $2\nu k^2 E(k)$ as Reynolds number increases, showing greater separation between the two peaks, as dissipation shifts to higher wavenumbers.

same amount of energy is transferred from scale to scale (ε) until the energy approaches the much narrower dissipative peak, where it is dissipated. In other words, we can say (approximately) that the rate of energy entering the dissipative region is approximately ε.

As the inertial (energetic) scales become smaller and smaller, we recall our hypothesis of local isotropy, which we used to justify this entire analytical framework to begin with. The basis of local isotropy is the idea that information about initial conditions (which may be strongly anisotropic) is lost as the energy is transferred down the cascade to smaller scales. We justified this idea by arguing that the length and time scales of the smallest eddies will eventually become independent of the largest eddies; i.e. the smallest eddies evolve so much faster than the large eddies that the large eddies appear to be in equilibrium from the point of view of the small eddies. We can now define a sufficient condition for this local isotropy/equilibrium condition: $k_e \ll k_d$. When there exist a wide range of eddies with wavenumbers much higher than the energetic (input) eddies, then the large eddies can be considered in equilibrium, (spectrally) independent of the small eddies.

The idea that we can apply an isotropic equation to a general problem of turbulence is based on the idea that at sufficiently small scales, any turbulence behaves isotropically — recall the tree sketch.

We call this range of wavenumbers the equilibrium range. The equilibrium range depends on only the rate of energy addition at the large scales (defined by ε, because dissipation is negligible at the large scales due to the scale separation) and the rate of dissipation at the smallest scales (determined by ν), where these two rates are in balance due to equilibrium. Moreover, we hypothesize that

this equilibrium range is universal: it applies to any flow geometry or situation. In other words, we can describe the equilibrium range eddies of any kind of fluid in terms of just two parameters: ε and ν.

> **Kolmogorov's first similarity hypothesis**: At sufficiently high Reynolds number, the statistical properties of small scale velocity fluctuations, $k \gg k_e$, exhibit a universal form which depends on only ε and ν.

Recall the Kolmogorov (dissipative) scales for length, $\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ and velocity, $v \sim (\nu\varepsilon)^{1/4}$. Also recall that at equilibrium, the dissipation can be balanced with the rate of energy loss at inertial scales according to $\varepsilon \sim \frac{u^3}{\ell}$

Of course, we need to know how high the Reynolds number must be in order to establish a sufficient range of small inertial scales in an equilibrium region (i.e. to satisfy Kolmogorov's first similarity hypothesis). We already stated the sufficient condition as

$$k_e \ll k_d \tag{2}$$

and we know that $k_d \sim \frac{1}{\eta}$ and $k_e \sim \frac{1}{\ell}$. We can rewrite this condition as

$$\frac{1}{\ell} \ll \frac{1}{\eta} \tag{3}$$

$$\frac{1}{\ell} \ll \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \tag{4}$$

$$\frac{1}{\ell} \ll \left(\frac{u^3}{\ell\nu^3}\right)^{1/4} \tag{5}$$

$$1 \ll \left(\frac{u^3\ell^3}{\nu^3}\right)^{1/4} \tag{6}$$

$$1 \ll \mathrm{Re}^{3/4} \tag{7}$$

Thus the condition for the existance of an equilibrium range (sufficient scale separation) is $\mathrm{Re}^{3/4} \gg 1$, which tells us that energy containing eddies are inertially dominated. And then the eddies within that equilibrium range must satisfy $k \gg \frac{1}{\ell}$, such that they are independent of the large-scale energetic eddies.

If we non-dimensionalize the integrated govering equation using the dissipative scales (η, v), which are expressed in terms of only ε and ν, we obtain

$$\mathrm{St}\frac{\partial}{\partial t}E(k,t) = T(k,t) - 2k^2E(k,t) \tag{8}$$

$$\frac{1}{\sqrt{\mathrm{Re}}}\frac{\partial}{\partial t}E(k,t) = T(k,t) - 2k^2E(k,t) \tag{9}$$

Where Re_η based on dissipative scales is, by definition, unity. The Strouhal number is then $\mathrm{St} = \frac{\eta/v}{\tau}$ For $\tau = \frac{\ell}{u}$, $\mathrm{St} = \mathrm{Re}^{-1/2}$ defined on the inertial scales. Note that the requirement for time-steadiness ($\mathrm{Re}^{1/2} \gg 1$) is more restrictive than the requirement for scale separation ($\mathrm{Re}^{3/4} \gg 1$) — both are technically requirements for universal equilibrium behavior, although this is rarely noted.

where the Strouhal number, St, represents the ratio of the timescale of the small wavenumbers, $\frac{\eta}{v}$, to the timescale of the energetic, large-scale dynamics, $\frac{\ell}{u}$. For sufficiently large inertial Reyolds number, $\mathrm{Re}^{1/2} \gg 1$, the time derivative is negligible, and the governing dynamics in the universal equilibrium region are

$$T(k) = 2\nu k^2 E(k) \tag{10}$$

where the balance is between spectral transfer and dissipation. Neglecting the time-derivative is often referred to as the local equilibrium hypothesis — it applies only for high wavenumbers (since we non-dimensionalized the governing equation in terms of dissipative scales). In other words, for sufficiently high Reynolds number, the flow appears to be time-independent from the perspective of the small scales. In integrated form, we can then conclude that for k in the universal equilibrium region,

$$\underbrace{\int_k^\infty T(k',t)dk'}_{\Pi_E(k)} \approx 2\nu \int_k^\infty k'^2 E(k',t)dk' \tag{11}$$

Note that the time-derivative in the dynamics is the derivative of an ensemble-averaged quantity. If the underlying turbulent process is stationary, i.e. invariant to shifts in time origin, then the time derivative of its ensemble statistics must necessarily be zero, because the ensemble can depend only on the time difference between two points, not on the absolute time at any pair of simultaneous measurements. In other words, stationary processes automatically satisfy the assumption of local equilibrium. If the process is not stationary, this assumption may be questionable.

where we call the left hand side, $\Pi_E(k)$, the spectral transfer function (which is very important for turbulence modeling, because knowing this function would provide closure to the spectral equations).

If the equilibrium range is universal, and thus dependent on only ε and ν, then we should be able to write a non-dimensional form of the energy density, $\hat{E}_{\text{eq}}$, in the equilibrium range in terms of just these two parameters.

$$E(k,t) = v^2 \eta \hat{E}_{\text{eq}}(\eta k) \tag{12}$$

$$= \nu^{5/4}\varepsilon^{1/4}\hat{E}_{\text{eq}}(\eta k) \tag{13}$$

Recall that $u^2 \sim \int E(k)dk$ so dimensionally we expect $v^2 \sim \int E(k)d\frac{1}{\eta}$ and thus the dimensions of $E(k)$ are $\left[v^2\eta\right]$

Because nearly all of the dissipation occurs within the equilibrium range, we can write

$$2\nu \int_k^\infty k'^2 E(k')dk' \approx 2\nu \int_0^\infty k^2 E(k)dk \tag{14}$$

and therefore integrating the governing dynamics over all wavenumbers yields

$$2\nu \int_0^\infty k^2 E(k)dk = \varepsilon \tag{15}$$

$$2\nu \int_0^\infty k^2 \left[v^2 \eta \hat{E}_{\text{eq}}(\eta k)\right] dk = \varepsilon \tag{16}$$

$$\int_0^\infty (\eta k)^2 \hat{E}_{\text{eq}}(\eta k) d(\eta k) = \frac{1}{2} \tag{17}$$

We recall that the integral of $T(k)$ over all wavenumbers is zero, because it reprepsents transfer between wavenumbers

where we obtain an actual prediction for a constraint on the energy density within the equilibrium region. The ability to write the energy density function in this non-dimensional form, satisfying the integral constraint, would then be strong evidence in favor of Kolmogorov's first similarity hypothesis (and the local equilibrium hypothesis).

We claim that $E(k,t)$ can be determined by a universal function dependent solely on the equilibrium range wavenumbers, but what about other quantities? For instance, can we express the ensemble-averaged value of $\vec{u}(\vec{x},t)$ within the equilibrium range in some universal form? What about the ensemble-average of the velocity difference, $\vec{u}(\vec{x}+\vec{r},t) - \vec{u}(\vec{x}+\vec{r},t)$? Assuming only one spatial dimension, x, for simplicity, we can write

$$u(x+r,t) - u(x,t) = \int_{-\infty}^{\infty} \left[e^{ik(x+r)} - e^{ikx} \right] \hat{u}(k,t) dk \tag{18}$$

and ensemble averaging yields

$$\overline{u(x+r,t) - u(x,t)} = \int_{-\infty}^{\infty} \left[e^{ikr} - 1 \right] e^{ikx} \overline{\hat{u}(k,t)} dk \tag{19}$$

The Fourier transfrom of the fluctuating velocity signal is denoted with the hat, $\hat{u}(k,t)$. The ensemble average is an average over space, which means that the ensemble average in Fourier space is an average over wave-number, k, which leaves the exponential unchanged after averaging, since it is not a random signal.

For small r, we can write the Taylor expansion of the exponential as

$$e^{ikr} - 1 \approx ikr + \mathcal{O}\left(k^2 r^2\right) \tag{20}$$

In the equilibrium range, we assume that the lengthscale of interest, r, is much smaller than the energy-containing scales, in order to satisfy $r \ll \frac{1}{k_e}$. Therefore, for k in the energy-containing scales, the prefactor $ikr \sim k_e \cdot \left(\ll \frac{1}{k_e} \right) \ll 1$. Therefore, the contribution to the integral is negligible over wavenumbers in the energy-containing region, and most of the contribution to the integral comes from the equilibrium region, where the prefactor $ikr \sim (\gg k_e) \cdot \left(\ll \frac{1}{k_e} \right) \sim 1$

$$\overline{u(x+r,t) - u(x,t)} \approx \int_{k \lesssim k_e} \underbrace{(ikr)}_{\ll 1} e^{ikx} \overline{\hat{u}(k,t)} dk + \int_{k \gg k_e} \underbrace{(ikr)}_{\sim 1} e^{ikx} \overline{\hat{u}(k,t)} dk \approx \int_{k \gg k_e} (ikr) e^{ikx} \overline{\hat{u}(k,t)} dk \tag{21}$$

Therefore, we expect that we can write the ensemble quantity in terms of an integral primarily over the equilibrium range of wavenumbers, $k \gg k_e$, and thus we expect that we can write this quantity in universal form. However, if, instead of the velocity difference, we had considered only the fluctuating velocity itself, then the prefactor would have taken the form $1 + e^{ikr}$ which is not negligible for any range of wavenumbers. We see that for velocity differences, the prefactor is roughly linear in k, which means it is surpressed for small wavenumbers and enhanced for large wavenumbers, at least as strongly as the dissipation term is. From this analysis, we conclude that any flow quantity which depends on velocity differences or gradients can be written in universal form, according to the equilibrium hypothesis.

In the limit as $r \to 0$, the prefactor is exactly linear, since the derivative within the Fourier transform yields a factor of k. Dissipation has two derivatives, and hence two factors of k.

One of the flow quantities for which this criterion applies is the second-order structure function, D_{LL}, since it is defined precisely in terms of velocity differences. Therefore, we expect that we can express this function in a universal form, in terms of ε and ν. We already wrote the isotropic structure function as

$$D_{LL} = 2u^2(1-f) \tag{22}$$

Recall that $f \approx 1 - \frac{r^2}{2\lambda^2} + \cdots$ and that $\lambda^2 \approx 15\frac{\nu u^2}{\varepsilon}$

which we can write, using our earlier definitions of f and λ, in the limit of small r, as

$$D_{LL} = \frac{\varepsilon r^2}{15\nu} \tag{23}$$

In other words, the structure function can be written entirely as a function of ε and ν in the equilibrium range, where $r \ll \ell$ (or $r \ll \frac{1}{k_e}$), because the contribution from large wavenumbers is negligible, as we showed above.

This is the form of the universal equilibrium prediction that Kolmogorov wrote in 1941, when he wrote his theory in terms of structure functions

Inertial Subrange

We argued above that the sufficient condition for the existence of a universal equilibrium range is $k_e \ll k_d$, and then if this condition is satisfied, the equilibrium wavenumbers, k, satisfy $k_e \ll k$. However, if the separation between the energy-containing and dissipation scales is large enough, it is possible to obtain a subrange of k within this equilibrium range, where $k_e \ll k$ but simultaneously $k \ll k_d$. In other words, there is a subrange of equilibrium wavenumbers in which inertial transfer of energy still matters, but where dissipation is not significant. We call this the inertial subrange, and it obeys

$$k_e \ll k \ll k_d \tag{24}$$

$$\frac{1}{\ell} \ll k \ll \frac{1}{\eta} \tag{25}$$

Recall that $\mathrm{Re}^{3/4} \gg 1$ for a universal equilibrium range. The geometric mean of n numbers is given by $\sqrt[n]{x_1 x_2 \cdots x_n}$

How large does the Reynolds number need to be to obtain this level of separation? Let's imagine that we define the wavenumber within the subrange, k, as the geometric mean of the two boundaries. Then we obtain

$$\frac{1}{\ell} \ll \left(\frac{1}{\eta\ell}\right)^{1/2} \ll \frac{1}{\eta} \tag{26}$$

$$1 \ll \left(\frac{\ell}{\eta}\right)^{1/2} \ll \frac{\ell}{\eta} \tag{27}$$

We already established that, for the universal equilibrium range, $\frac{\ell}{\eta} \equiv \mathrm{Re}^{3/4} \gg 1$. But now we have the stricter requirement that even $\left(\frac{\ell}{\eta}\right)^{1/2} \gg 1$, which means that $\mathrm{Re}^{3/8} \gg 1$ for an inertial subrange to exist.

Within this inertial subrange, the eddies are significantly larger than dissipation scales, and therefore are assumed to be independent of the viscosity, ν. The subrange begins at the lower boundary of the equilibrium range, and we know that the rate of energy flowing into this range is ε because dissipation is negligible at lower wavenumbers. But within the subrange, dissipation remains negligible, and thus the region is described entirely by the dissipation

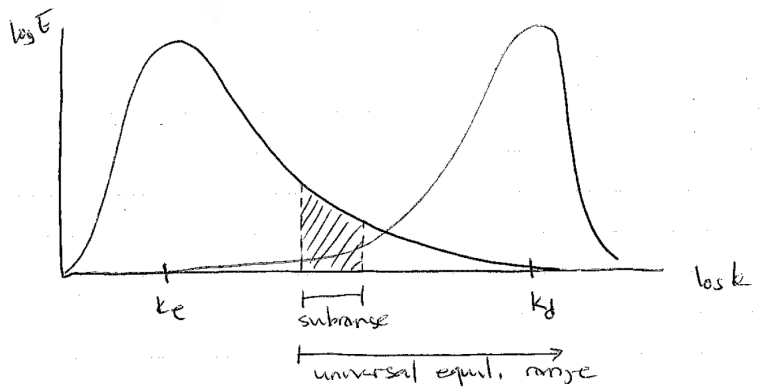


Figure 22: A sketch of the spectral energy density, $E(k)$ and the distribution of the dissipation term, $2\nu k^2 E(k)$ for sufficiently high Reynolds number to obtain an inertial subrange within the equilibrium range of wavenumbers.

rate. Returning to our governing dynamics, we have that the inertial transport term must be constant, and thus must equal the total dissipation rate

$$T(k) = \varepsilon \tag{28}$$

In the equilibrium range, we wrote

$$E(k,t) = v^2 \eta \hat{E}_{\text{eq}}(\eta k) \tag{29}$$

But now, we ought to write the spectral density within the inertial subrange in terms of only ε. Let's assume that the universal spectral density for this subrange can be written as a power law, in the form

$$\hat{E}_{\text{sub}}(\eta k) = \alpha(\eta k)^{\beta} \tag{30}$$

and then plugging back into the overall dimensional spectral density yields

$$E(k,t) = (\nu\varepsilon)^{1/2}\left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \alpha k^{\beta}\left(\frac{\nu^3}{\varepsilon}\right)^{\beta/4} \tag{31}$$

Recall that $\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ and $v = (\nu\varepsilon)^{1/4}$. Then, substituting and expanding, we find that the dependence on viscosity is $\nu^{\frac{1}{2}+\frac{3}{4}+\frac{3\beta}{4}}$, and setting this power to zero yields $\beta = -\frac{5}{3}$

Now, we solve for α and β such that $E(k,t)$ depends on only ε and k, to obtain

$$E(k,t) = \alpha\varepsilon^{2/3}k^{-5/3} \tag{32}$$

which is known as Kolmogorov's -5/3$^{\text{rd}}$ Law. This is a remarkably specific result: we are able to describe the precise shape of a region of the spectral density function for isotropic turbulence, and we obtained this prediction based, essentially, on dimensional analysis and physical assumptions about the nature of energy transport. It is important to reiterate that this prediction is expected to apply only for very high Reynolds numbers, and thus is unlikely to be easily verifiable in the laboratory.

We can similarly write this in physical space, in terms of the structure function. The criterion for the existence of an inertial subrange in spatial terms requires that spatial scales r must obey

$$k_e \ll k \ll k_d \tag{33}$$

and then we can write the second order structure function as a power-law in the form

$$D_{LL} = \alpha' v^2 \left(\frac{r}{\eta}\right)^{\beta'} = \alpha' (\nu\varepsilon)^{1/2} r^{\beta'} \left(\frac{\nu^3}{\varepsilon}\right)^{-\beta'/4} \tag{34}$$

The dependence on viscosity is of the form $\nu^{\frac{1}{2}-\frac{3\beta'}{4}}$ and eliminating this dependence for the inertial subrange yields $\beta' = \frac{2}{3}$

Finally, solving for the coefficient such that the structure function depends only on ε and r yields

$$D_{LL} = \alpha' (\varepsilon r)^{2/3} \tag{35}$$

which is known as Kolmogorov's $2/3^{\text{rd}}$ Law.

There is an additional implication of the inertial subrange to the structure function shape. Recall the structure function form of the Karman-Howarth equation

$$\frac{3}{r^4}\frac{d}{dt}\int_0^r s^4 D_{LL} ds = -D_{LLL} + 6\nu\frac{\partial}{\partial r} D_{LL} - \frac{4}{5}\varepsilon r \tag{36}$$

If we assume the local equilibrium hypothesis holds, then we can neglect the time-varying term to obtain, for the equlibrium range,

$$D_{LLL} = 6\nu\frac{\partial}{\partial r} D_{LL} - \frac{4}{5}\varepsilon r \tag{37}$$

and within the inertial subrange, we can neglect terms depending on viscosity, to obtain

$$D_{LLL} = -\frac{4}{5}\varepsilon r \tag{38}$$

which is called Kolmogorov's $4/5^{\text{th}}$ Law. This remarkable result describes the form of the inertial term, which heretofore was unknown. We can also combine this with the $2/3^{\text{rd}}$ Law (which holds throughout the equilibrium region) to obtain an expression for a measure of 'skewness' in terms of structure functions

$$\frac{D_{LLL}}{D_{LL^{3/2}}} = \text{constant} \tag{39}$$

which is predicted to be constant within the inertial subrange.

We have now explained major features of the 2-point statistics for high wavenumbers, in the equilibrium range and inertial subrange. Earlier, we explained that in the limit as $k \to 0$, the spectral density grows rapidly, as k^4. The eddies inhabiting this low wavenumber region develop very slowly with respect to the rest of the spectrum, and thus are assumed to have a roughly 'permanent' form, determined by their initial conditions. These eddies are associated with the large-scale diffusive action of turbulence that we encountered in the analysis of Richardson. We showed that according to Richardson, the turbulent diffusivity behaves according to

von Karman and Lin (1951) first proposed that the large, permanent eddies are associated with diffusive effects.

$$D_{\text{turb}} \sim \varepsilon^{1/3} \ell^{4/3} \tag{40}$$

for large, energetic scales, ℓ, and if we assume that these large scales are described by their diffusivity and the total rate of energy transfer, ε, we can write

$$E(k,t) \sim D_{\text{turb}}^2 k \tag{41}$$

such that we expect a linear growth in the spectral density over some region of large-scale motions. We can summarize all of these results in a table and sketch.

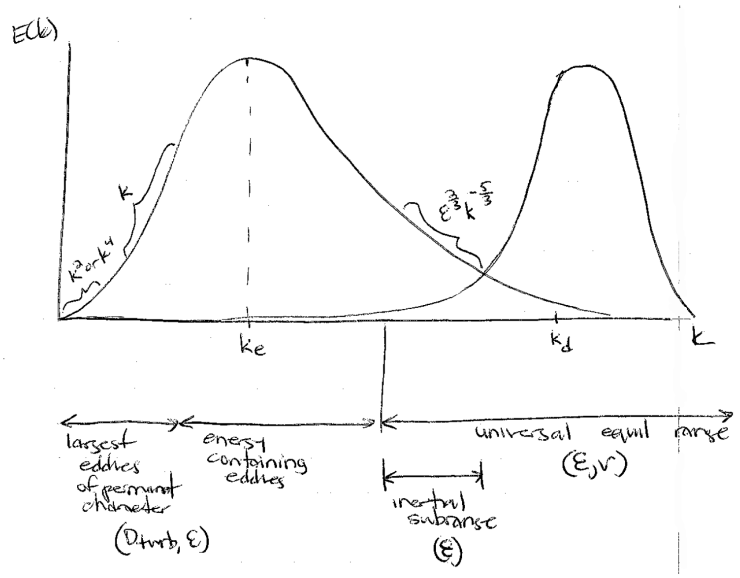


Figure 23: A sketch of the spectral energy density, $E(k)$ and the distribution of the dissipation term, $2\nu k^2 E(k)$ for sufficiently high Reynolds number to display all of the subranges predicted by Kolmogorov's theory.

	Spectral Density	Structure Function
Permanent Eddies	$E(k,t) \sim k^4$, $D_{\text{turb}}^2 k$	
Universal Equilibrium Range	$E(k,t) = \nu^{5/4}\varepsilon^{1/4}\hat{E}_{\text{eq}}(\eta k)$	$D_{LL} = \frac{\varepsilon r^2}{15\nu}$
Inertial Subrange	$E(k,t) = \alpha\varepsilon^{2/3}k^{-5/3}$	$D_{LL} = \alpha'(\varepsilon r)^{2/3}$, $D_{LLL} = -\frac{4}{5}\varepsilon r$

Table 5: Functional forms for spectral density and structures functions in the universal equilibrium and inertial subranges, according to Kolmogorov's theory.

Doubts about Kolmogorov

In order to derive this description of homogeneous, isotropic turbulence, we made a number of assumptions. Firstly, we assumed that these results are applicable universally, independent of the particular large scale form of the flow, because the small scales are approximately isotropic at sufficiently high Reynolds number (local isotropy). We then assumed that the small scale motions are equilibrium, such that we could ignore the time-rate of change of the energy-containing scales (local equilibrium). This local equilibrium assumption then provided for the universal equilibrium region, and also the inertial subrange, and all of the predictions associated with them. Therefore, we might ask whether this assumption is correct? Earlier, we argued that the time-derivative is negligible from the

point of view of small-scales by dimensional arguments

$$\frac{1}{\sqrt{\mathrm{Re}}}\frac{\partial}{\partial t}E(k,t) = T(k,t) - 2k^2E(k,t) \tag{42}$$

such that, for sufficient Reynolds number defined on the large-scales, we expect the Strouhal number to be very small and thus the time-variation to be negligible. We also noted that, if the underlying turbulence is stationary, this time-derivative is identically zero. However, for more general cases of non-stationary turbulence, we can test this assumption another way. Let's isolate the effect of the time-change only by considering the normalized time-derivative

$$\frac{1}{E}\frac{\partial E}{\partial t} \tag{43}$$

and ask whether this term is negligibly small at high wavenumbers, as assumed by local equilibrium. If it is negligibly small, then we already predicted an inertial subrange of the form $E(k,t) = \alpha\varepsilon^{2/3}k^{-5/3}$. So let's plug this is and check whether it satisfies the required assumption

This check was proposed by William K. George, who raises questions about the underlying assumptions of K41 in a number of his papers and talks.

$$\frac{1}{E}\frac{\partial E}{\partial t} = \frac{2}{3}\left(\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial t}\right) \tag{44}$$

We see that the magnitude of the normalized time-derivative does not depend on wavenumber, and thus is not negligible for small wavenumbers as predicted by the local equilibrium assumption. Moreover, we see that the magnitude of the time-derivative depends on the time-derivative of the dissipation itself. We had never even considered the possibility of a time variation in ε, because we assumed that the slow change of the large scales implied that the dissipative scales were also slowly changing. We see that there appears to be an internal contradiction within the local equilibrium hypothesis for non-stationary flows: the time-derivative is not actually dependent on wavenumber, is of $mathcal{O}(1)$ within the inertial subrange, and depends on the time-derivative of dissipation itself, which was never formally assumed negligible. So we should approach the predictions of Kolmogorov's theory with some caution, particularly in non-stationary flows (for instance, flows outside of a carefully controlled wind-tunnel).

Free Turbulent Flows

We have now studied the decay of homogeneous, isotropic turbulence from the point of view of 2-point ensemble statistics, and have made a number of predictions on the form of these statistics. In the process, we learned about the nature of the energy cascade in turbulence, how energy moves from large eddies down to small eddies. We also argued that this isotropic analysis may also provide insight into the energetic picture of small-scale motions even in flows which have a particular directionality at the large-scales (and thus are not isotropic). We turn now to a series of canonical turbulent flows in order to understand the momentum and energy balances associated with them, and how they differ from the decaying turbulence we have studied so far.

Laminar Jet

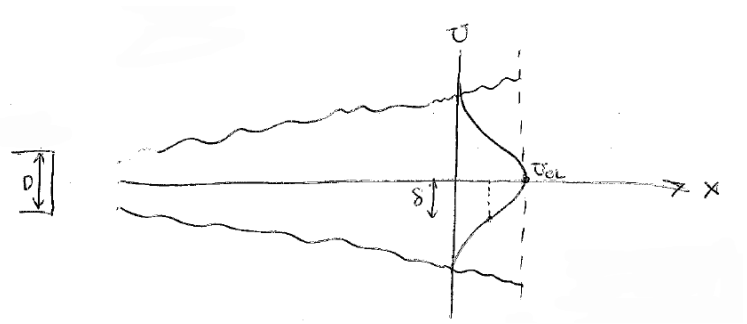


Figure 24: A sketch of the laminar, planar jet, emptying into a quiescent fluid (i.e. $u_\infty = 0$) with centerline velocity u_s, and characteristic width, $\delta(x)$

Let's start with the planar jet. We begin with the problem of a laminar jet, in order to review the basic ideas and build some intuition, then we turn to the turbulent jet. We assume that the jet is emitted from a rectangular slot, such that we can assume is it approximately two-dimensional, $\vec{u} = (u, v, 0)$ in $\vec{x} = (x, y, 0)$. We then write the continuity and momentum equations for constant density fluid as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

and let's define the initial momentum flux (per unit width of the jet) exiting the slot as

$$J_0 = \int_{-\infty}^{\infty} (\rho u) u dy \tag{4}$$

such that we have a measure of the 'strength' of the jet. We can also assume that the streamwise velocity very far from the jet is zero, $u(y \to \pm\infty) = 0$ because we expect the total momentum flux integral, $J(x)$, at any steamwise point, to be convergent.

Let's also define characteristic scales for our flow variables: δ represents the characteristic width of the jet in the y-direction; L a characteristic lengthscale in the x-direction; u_s the characteristic streamwise velocity; and v_s the characteristic transverse velocity. Π is the characteristic pressure scale. And we define the Reynolds number, $\mathrm{Re} = \frac{u_s L}{\nu}$. We assume that the Reynolds number is large, $\mathrm{Re} \gg 1$, although the flow remains laminar. To examine the consequence of this Reynolds number assumption, we non-dimensionalize the governing equations. Continuity yields

$$\left(\frac{u_s}{L}\right)\frac{\partial \tilde{u}}{\partial \tilde{x}} + \left(\frac{v_s}{\delta}\right)\frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \tag{5}$$

and thus we conclude that

$$\left(\frac{u_s}{L}\right) \sim \left(\frac{v_s}{\delta}\right) \quad \Rightarrow \quad v_s \sim \frac{\delta}{L} u_s \tag{6}$$

because we expect both streamwise variation in the streamwise velocity and transverse variation in the traverse velocity. Using this balance, we can non-dimensionalize the momentum equations as

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{\Pi}{\rho u_s^2}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\nu}{u_s L}\left[\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \left(\frac{L}{\delta}\right)^2 \frac{\partial^2 \tilde{u}}{\partial \tilde{y}^2}\right] \tag{7}$$

$$\left(\frac{\delta}{L}\right)^2\left[\tilde{u}\frac{\partial \tilde{v}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{v}}{\partial \tilde{y}}\right] = -\frac{\Pi}{\rho u_s^2}\frac{\partial \tilde{p}}{\partial \tilde{y}} + \frac{\nu}{u_s L}\left[\left(\frac{\delta}{L}\right)^2 \frac{\partial^2 \tilde{v}}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}}{\partial \tilde{y}^2}\right] \tag{8}$$

and then take the limit for large Reynolds number. For the x-momentum equation, if Re is large, then the first viscous term is negligible. However, the magnitude of the second viscous term depends on the behavior of $\frac{L}{\delta}$. By dimensional arguments, we expect that $\delta \ll L$ for $\mathrm{Re} \gg 1$; we can also argue this geometrically. Then

The jet is rotational flow, where the vorticity is generated at the two corners of the exit slot, because the flow experiences a pressure gradient at each sharp corner and pressure gradients are associated with vorticity flux, as can be shown from the x-momentum equation evaluated along a solid wall: $\frac{1}{\rho}\frac{\partial p}{\partial x}\Big|_{\text{wall}} = -\nu \frac{\partial \omega_z}{\partial y}\Big|_{\text{wall}}$. The fluid into which the jet empties is irrotational (quiescent). We can approximate the width of the rotational region, δ, by balancing the diffusive timescale of the vorticity, $t_{\text{diff}} \sim \frac{\delta^2}{\nu}$, and the advective timescale of vorticity, $t_{\text{adv}} \sim \frac{L}{u_s}$ to obtain $\frac{\delta}{L} \sim \frac{1}{\sqrt{\text{Re}}}$

we need to establish the limit

$$\lim_{\substack{\mathrm{Re}\to\infty \\ \delta/\ell\to 0}} \left[\frac{1/\mathrm{Re}}{(\delta/\ell)^2} \right] = 1 \tag{9}$$

We conclude this limit must be finite (in this case, we choose 1) in order to preserve at least some viscous diffusion of vorticity within the jet; if the limit were infinite then the jet would have a non-symmetric velocity profile in u; if the limit were zero, then the flow would be entirely irrotational, against our physical intuition that vorticity is generated at the outlet. Therefore, we find that

This distinguished limit yields the same scaling of the jet thickness δ as we found in the note above from dimensional arguments about vorticity diffusion and advection

$$\frac{\delta}{L} = \frac{1}{\sqrt{\mathrm{Re}}} \tag{10}$$

and we can write the x-momentum balance, to leading order, as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{11}$$

We can also rewrite the y-momentum equation as

$$\mathcal{O}\left(\frac{1}{\mathrm{Re}}\right) = -\frac{\Pi}{\rho u_s^2}\frac{\partial \tilde{p}}{\partial \tilde{y}} \tag{12}$$

and we therefore conclude that, to leading order, $p = p(x)$ only; or, in other words, there is no transverse pressure gradient within the jet. If $\frac{dp}{dx} = 0$ as $y \to \pm\infty$ where there is no streamwise flow, then we know that $\frac{dp}{dx} = 0$ for all y.

We must use the inertial scaling for pressure, $\Pi = \rho u_s^2$ to conclude that the transverse pressure gradient is $\mathcal{O}(1)$, because we assume that the Reynolds number is large.

By adding together the continuity and x-momentum equations,

$$u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{13}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{14}$$

we obtain

$$\frac{\partial}{\partial x}u^2 + \frac{\partial}{\partial y}uv = \nu\frac{\partial^2 u}{\partial y^2} \tag{15}$$

Then, integrate this momentum flux balance across the place of the jet to obtain

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2 dy + uv\Big|_{-\infty}^{\infty} = \nu\frac{\partial u}{\partial y}\Big|_{-\infty}^{\infty} \tag{16}$$

Because we assumed that the streamwise velocity u was zero far from the jet, both non-integral terms are equal to zero and we find that the total momentum flux is a constant, equal to the initial flux of the jet

$$\frac{d}{dx}\int_{-\infty}^{\infty} \rho u^2 dy = 0 \quad \Rightarrow \quad \frac{dJ}{dx} = 0 \quad \Rightarrow \quad J(x) = J_0 \tag{17}$$

In order to solve for the velocity field of the jet, we use a streamfunction formulation in order to satisfy continuity and eliminate v from the x-momentum equation, and then seek a similarity solution since the problem does not contain natural lengthscales (δ and L are fictional scales used for dimensional argument only). We define a similarity variable as

$$\eta = \frac{y}{\delta(x)} \tag{18}$$

and a streamfunction as

$$\psi(x,y) = u_s(x)\delta(x)f(\eta) \tag{19}$$

We then obtain all of the components of the momentum equation

Similarity Formulation
$u = u_s f'$
$v = -u_s'\delta f + u_s\delta'(\eta f' - f)$
$\frac{\partial u}{\partial x} = u_s' f' - u_s f''\eta\frac{\delta'}{\delta}$
$\frac{\partial u}{\partial y} = u_s f''\frac{1}{\delta}$
$\frac{\partial^2 u}{\partial y^2} = u_s f'''\frac{1}{\delta^2}$

Table 6: The similarity formulation for all of the components in the momentum equation, found by repeated differentation and the chain rule. The prime refers to differentiation with respect to the independent variable of a given function.

which we can rewrite as

$$-\left[\frac{1}{2}(u_s\delta^2)'\right]ff'' + \left[u_s'\delta^2\right]\left(f'^2 - \frac{1}{2}ff''\right) = [\nu]f''' \tag{20}$$

where we have grouped together all of the terms which depend on x. In order to obtain a similarity solution, these x-dependent terms must be proportional to each other, such that they can all be divided out of the governing equation. In other words, in order to obtain a similarity solution, we require

$$\left[\frac{1}{2}(u_s\delta^2)'\right] \propto \left[u_s'\delta^2\right] \propto [\nu] \tag{21}$$

Satisfying this proportionality means

$$\left[\frac{1}{2}(u_s\delta^2)'\right] \propto [\nu] \quad \Rightarrow \quad \delta^2 \propto \frac{\nu x}{u_s} \tag{22}$$

Note that the criterion for similarity implies the same scale of δ that we found by dimensional arguments earlier, such that $\frac{\delta}{x} = \frac{1}{\sqrt{\mathrm{Re}_x}}$ where $\mathrm{Re}_x = \frac{u_s x}{\nu}$

$$\left[u_s'\delta^2\right] \propto [\nu] \quad \Rightarrow \quad u \propto x^b \tag{23}$$

$$\left[\frac{1}{2}(u_s\delta^2)'\right] \propto \left[u_s'\delta^2\right] \quad \Rightarrow \quad u \propto \delta^a \tag{24}$$

If we choose the proportionality constant in the first relation to be 1 (i.e. rewrite the proportionality as equality, $\delta^2 = \frac{\nu x}{u_s}$), then we can rewrite the momentum equation as

$$-ff'' + \underbrace{\left[\frac{u_s'}{u_s}x\right]}_{\lambda}\left(f'^2 - \frac{1}{2}ff''\right) = f''' \tag{25}$$

Note that the criterion for similarity implies the same scale of δ that we found by dimensional arguments earlier, such that $\frac{\delta}{x} = \frac{1}{\sqrt{\mathrm{Re}_x}}$ where $\mathrm{Re}_x = \frac{u_s x}{\nu}$

where the final bracketed term must be a constant in order to satisfy similarity; we call that constant λ because it is actually an

eigenvalue to this ODE. Solving for the centerline velocity in terms of this eigenvalue yields

$$u_s(x) = cx^\lambda \tag{26}$$

Consider the version of this problem for a boundary layer with zero-pressure gradient: then $u_s' = 0$ and the eigenvalue $\lambda = 0$. Or consider a boundary layer with a power-law pressure gradient, i.e. the Falkner-Skan problem, where $u_s \propto x^n$ so that $\lambda = n$, etc. The whole family of boundary-layer type problems, including the jet and wake and wedge and convergent channel, constitute a single eigenproblem – see Schlichting's book, Boundary Layer Theory, §7.2.1

As with most eigenvalue problems, we can use our initial condition to fix the eigenvalue. We already established that the total momentum flux is constant everywhere in the jet, so that

$$J_0 = \int_{-\infty}^{\infty} \rho u^2 dy \tag{27}$$

and we can simply substitute our velocity $u = u_s f'$ to obtain

$$J_0 = \rho u_s^2 \delta \underbrace{\int_{-\infty}^{\infty} f'^2 d\eta}_{C_0} = C_0 \rho u_s^2 \delta \tag{28}$$

The momentum flux has no dependence on x and thus we set $2\lambda + \frac{1}{2} - \lambda/2 = 0$ to find $\lambda = -\frac{1}{3}$.

and grouping the constants on the left-hand side

$$\frac{J_0}{\rho C_0} = u_s^2 \delta = c^{3/2} (\nu)^{1/2} x^{(2\lambda + \frac{1}{2} - \lambda/2)} \tag{29}$$

from which we conclude that $\lambda = -\frac{1}{3}$, and $c = \left(\frac{J^2}{\rho^2 C_0^2 \nu} \right)^{1/3}$. Then the governing momentum equation in similarity form is

$$3f''' + ff'' + f'^2 = 0 \tag{30}$$

and we need to translate the boundary conditions for this third-order, non-linear ODE.

Boundary Condition	Similarity Form
Symmetry in v	$f(0) = 0$
Symmetry in $\frac{\partial u}{\partial y}$	$f''(0) = 0$
Centerline $u = u_s$	$f'(0) = 1$
Decay in u as $y \to \pm\infty$	$f'(\pm\infty) = 0$

Table 7: Boundary conditions for the laminar jet, defined at $y = 0$ using symmetry.

Integrating by hand yields

$$f(\eta) = \sqrt{6} \tanh\left(\frac{\eta}{\sqrt{6}} \right) \tag{31}$$

and then we can calculate the integral $C_0 = \frac{4\sqrt{6}}{3}$. Finally, we can calculate the mass flux per unit width of the jet,

$$\dot{m} = \int_{-\infty}^{\infty} \rho u dy \quad = \rho u_s \delta \int_{-\infty}^{\infty} f' d\eta \quad = \rho u_s \delta f|_{-\infty}^{\infty} \quad = \rho u_s \delta (2\sqrt{6}) \quad = \left(36 J_0 \rho^2 \nu x \right)^{1/3} \tag{32}$$

We conclude that the laminar jet has a constant momentum flux, but its mass flux increases according to $m \propto x^{1/3}$. The jet width grows according to $\delta \propto x^{2/3}$ and the centerline velocity decreases as $u_s \propto x^{-1/3}$. The mass flux of the jet increases, indicating that the jet is entraining additional quiescent fluid. The additional mass makes

up for the decrease in velocity, thus maintaining a fixed momentum flux. In order to entrain mass, there must be some non-zero velocity in the transverse direction, i.e. $v(y \to \pm\infty) \neq 0$. From the above solution, we can calculate

$$v(y \to \pm\infty) = \mp\sqrt{6}(u_s\delta)' \quad = \mp\frac{\sqrt{6}u_s(x)}{3\sqrt{\mathrm{Re}_x}} \tag{33}$$

such that there is indeed a transverse inflow of fluid toward the jet, from quiescent environment, even though the streamwise velocity of the jet, u, decays as $y \to \pm\infty$. With this theoretical background on how to develop the similarity solution for the laminar jet, we can proceed to the turbulent jet. In some sense, the turbulent jet should really have been the starting point, since we had to assume large Reynolds number for the laminar analysis, and in most cases, at large Reynolds number, the flow is turbulent.

Turbulent Jet

For the turbulent jet analysis, we adopt the same assumptions as the laminar jet analysis, including the idea that the flow is, at least quasi two-dimensional Therefore, the governing equations for the average flow are given as

If the flow were perfectly two-dimensional, then the vortex stretching term from our vorticity dynamics, $\vec{\omega}\cdot\nabla\vec{u}$ would be identically zero, and thus there would be no physical mechanism for the energy cascade; thus we must assume only quasi two-dimensional flow.

$$\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y} = 0 \tag{34}$$

$$\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} + \frac{\partial}{\partial x}\overline{u^2} + \frac{\partial}{\partial y}\overline{uv} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x} + \nu\left(\frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial^2\bar{u}}{\partial y^2}\right) \tag{35}$$

$$\bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial\bar{v}}{\partial y} + \frac{\partial}{\partial x}\overline{uv} + \frac{\partial}{\partial y}\overline{v^2} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial y} + \nu\left(\frac{\partial^2\bar{v}}{\partial x^2} + \frac{\partial^2\bar{v}}{\partial y^2}\right) \tag{36}$$

As part of the quasi-2D assumption, we ignore Reynolds stress components in the out-of-plane direction, like $\overline{uw}$, because we assume that fluctuations in w are uncorrelated with u or v, because they have no preferential direction, i.e. w fluctuations are perfectly random.

Just like for the laminar jet, we adopt velocity scales for the mean flow, u_s and v_s, and mean-flow length scales L and δ, which are related by continuity according to

$$v_s \sim \frac{\delta}{L}u_s \tag{37}$$

However, the momentum equations contain an additional velocity scale within the Reynolds stress terms. So we need to decide how to scale terms like $\overline{u^2}$, $\overline{uv}$, and $\overline{v^2}$. Let's assume that these components are all within the same order, since we know that the pressure-strain rate terms in the Reynolds stress dynamics tend to push the flow towards isotropy. Then we can argue that all of these fluctuating quantities can be written as $\mathcal{O}(u^2)$. Then we can write the momentum equations in non-dimensional form as

$$\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}} + \tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}} + \left(\frac{u}{u_s}\right)^2\left[\frac{\partial}{\partial\tilde{x}}\tilde{\overline{u^2}} + \left(\frac{L}{\delta}\right)\frac{\partial}{\partial\tilde{y}}\tilde{\overline{uv}}\right] = -\frac{\Pi}{\rho u_s^2}\frac{\partial\tilde{p}}{\partial\tilde{x}} + \frac{\nu}{u_s L}\left[\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2} + \left(\frac{L}{\delta}\right)^2\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right] \tag{38}$$

$$\left(\frac{\delta}{L}\right)^2\left[\tilde{u}\frac{\partial\tilde{v}}{\partial\tilde{x}} + \tilde{v}\frac{\partial\tilde{v}}{\partial\tilde{y}}\right] + \left(\frac{u}{u_s}\right)^2\left[\left(\frac{\delta}{L}\right)\frac{\partial}{\partial\tilde{x}}\tilde{\overline{uv}} + \frac{\partial}{\partial\tilde{y}}\tilde{\overline{v^2}}\right] = -\frac{\Pi}{\rho u_s^2}\frac{\partial\tilde{p}}{\partial\tilde{y}} + \frac{\nu}{u_s L}\left[\left(\frac{\delta}{L}\right)^2\frac{\partial^2\tilde{v}}{\partial\tilde{x}^2} + \frac{\partial^2\tilde{v}}{\partial\tilde{y}^2}\right] \tag{39}$$

Recall that in the y-momentum equation, for the laminar flow, the leading order term was the pressure gradient, and thus we concluded that there was no transverse pressure gradient. Now, however, we have additional terms which may potentially be the same order as the pressure gradient, because we don't know the scaling of u^2 for the Reynolds stress components. The largest term on the lefthand side is $\left(\frac{u}{u_s}\right)^2$; if we assume this term is negligible, then both of the turbulent terms are negligible, and we return to the laminar momentum balance, with no turbulent terms. So this cannot be the correct choice. If we assume this term is dominant, then the leading order momentum balance indicates a constant value of $\overline{v^2}$ across the entire flow field, which is not consistent with quiescent flow far from the jet. Therefore, we must conclude that $\left(\frac{u}{u_s}\right)^2$ balances the pressure term

$$\left(\frac{u}{u_s}\right)^2 \sim \frac{\Pi}{\rho u_s^2} \tag{40}$$

Careful: note that Π here represents the characteristic pressure difference between the irrotational and rotational flow, $(\bar{p} - \bar{p}_\infty)$, which is order ρu^2. This should not be confused with the scale of $\bar{p}_\infty$ which is order ρu_s^2. The difference doesn't matter here because we assume $\bar{p}_\infty$ is constant, but it will matter in the scaling for the boundary layer equations.

This assumes that the jet width is sufficiently small and the Reynolds number is sufficiently large

$$\left(\frac{\delta}{L}\right)^2 \ll \left(\frac{u}{u_s}\right)^2 \quad \text{and} \quad \frac{1}{\mathrm{Re}} \ll \left(\frac{u}{u_s}\right)^2 \quad \text{and} \quad \frac{\delta}{L} \ll 1 \tag{41}$$

then we have a leading-order balance between a turbulent momentum flux and the transverse pressure gradient. The transverse momentum equation is

$$\frac{\partial}{\partial y}\overline{v^2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \mathcal{O}\left(\frac{\delta}{L}\right)^2 \tag{42}$$

Integrating over y, and applying our boundary conditions, including the assumption that there are no turbulent fluctuations at $y \to \pm\infty$, yields

We also assume that there is no mean streamwise pressure gradient, $\left.\frac{\partial \bar{p}}{\partial x}\right|_{y\to\pm\infty} = 0$, based on the assumption that $\bar{u}(y \to \pm\infty) = 0$. Then $\bar{p}(x, \infty) = \bar{p}_\infty$ constant.

$$\int_y^\infty \frac{\partial}{\partial y'}\overline{v^2}\,dy' = -\frac{1}{\rho}\int_y^\infty \frac{\partial \bar{p}}{\partial y'}\,dy' \tag{43}$$

$$0 - \overline{v^2} = -\frac{1}{\rho}[\bar{p}(x,\infty) - \bar{p}(x,y)] \tag{44}$$

$$\bar{p}(x,y) = \bar{p}(x,\infty) - \rho\overline{v^2} \tag{45}$$

$$\bar{p}(x,y) = \bar{p}_\infty - \rho\overline{v^2} \tag{46}$$

We can then solve for the streamwise pressure gradient by differentiating

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = \frac{\partial}{\partial x}\overline{v^2} \tag{47}$$

Note that the scaling of the pressure term is still consistent with our earlier assumption that $\Pi \sim \rho u_s^2$ because we have already established the very surprising result that $u_s \sim u$

and we see that the turbulent jet is very different from the laminar jet, due to the presence of a streamwise pressure gradient. Returning to the x-momentum equation,

$$\tilde{u}\frac{\partial\tilde{u}}{\partial\tilde{x}}+\tilde{v}\frac{\partial\tilde{u}}{\partial\tilde{y}}+\left(\frac{u}{u_s}\right)^2\left[\frac{\partial}{\partial\tilde{x}}\tilde{\overline{u^2}}+\left(\frac{L}{\delta}\right)\frac{\partial}{\partial\tilde{y}}\tilde{\overline{uv}}\right]=\left(\frac{u}{u_s}\right)^2\frac{\partial}{\partial\tilde{x}}\tilde{\overline{v^2}}+\frac{\nu}{u_sL}\left[\frac{\partial^2\tilde{u}}{\partial\tilde{x}^2}+\left(\frac{L}{\delta}\right)^2\frac{\partial^2\tilde{u}}{\partial\tilde{y}^2}\right] \tag{48}$$

and the leading order turbulence term on the left-hand side is

$$\left(\frac{u}{u_s}\right)^2\left(\frac{L}{\delta}\right) \tag{49}$$

and this must balance the advective terms, which are $\mathcal{O}(1)$. If it dominates the advective terms, then we have a constant turbulence distribution, which is inconsistent with the flow physics; if it is negligible, then there are no turbulence effects in the streamwise direction. Therefore, we conclude that

$$\left(\frac{u}{u_s}\right)^2\sim\frac{\delta}{L} \tag{50}$$

In addition to our assumptions above, we have an additional constraint on the Reynolds number in order to neglect the viscous effects

This is a stronger assumption than we had assumed based on the y-momentum, where we required only that $\frac{1}{\mathrm{Re}}\ll\frac{\delta}{L}$.

$$\frac{1}{\mathrm{Re}}\ll\left(\frac{\delta}{L}\right)^2 \tag{51}$$

Finally, we can write the x-momentum equation as

$$\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\frac{\partial}{\partial y}\overline{uv}=\mathcal{O}\left(\frac{\delta}{L}\right) \tag{52}$$

Combining the above, we can then write the momentum equations to leading order, with the second-order terms in curly braces, as

$$\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}+\frac{\partial}{\partial y}\overline{uv}=\underbrace{\left\{\frac{\partial}{\partial x}\left(\overline{v^2}-\overline{u^2}\right)\right\}}_{\mathcal{O}\left(\frac{\delta}{L}\right)} \tag{53}$$

$$\underbrace{\left\{\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}+\frac{\partial}{\partial x}\overline{uv}\right\}}_{\mathcal{O}\left(\frac{\delta}{L}\right)^2}+\frac{\partial}{\partial y}\overline{v^2}=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial y} \tag{54}$$

In order to solve this system to leading order, we need to eliminate $\bar{v}$ by means of continuity. For the laminar flow, we employed the streamfunction formulation to accomplish this; here we will take employ an integral approach – both techniques are equivalent. Integrating the continuity equation

$$\int_0^y\frac{\partial\bar{v}}{\partial y'}dy'=-\int_0^y\frac{\partial\bar{u}}{\partial x}dy' \tag{55}$$

and using the symmetry condition that $\bar{v}(y=0)=0$, we obtain

$$\bar{v}(y)=-\int_0^y\frac{\partial\bar{u}}{\partial x}dy'=-\frac{d}{dx}\int_0^y\bar{u}(x,y)dy' \tag{56}$$

Note that the entrainment velocity is just $\bar{v}(y\to\infty)$ and half the entrainment rate is just $-\frac{1}{2}\frac{dQ}{dx}$ where Q is the volumetric flow rate of fluid in the jet, defined as $Q=2\int_0^y\bar{u}(x,y)dy'$, so we see the relationship between entrainment and mass flux. Note also that $\bar{u}(x,y\to\pm\infty)=0$ for the flowrate integral to converge.

In addition to defining $\bar{v}$, we need to establish an integral constraint on the momentum flux, J, as we did with the laminar jet. Adding the x-momentum and continuity

Recall that the curly braces represents second order terms.

$$\bar{u}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) = 0 \tag{57}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \frac{\partial}{\partial y}\overline{uv} = \left\{\frac{\partial}{\partial x}\left(\overline{v^2} - \overline{u^2}\right)\right\} \tag{58}$$

we obtain

$$\frac{\partial}{\partial x}\bar{u}^2 + \frac{\partial}{\partial y}\bar{u}\bar{v} + \frac{\partial}{\partial y}\overline{uv} = \left\{\frac{\partial}{\partial x}\left(\overline{v^2} - \overline{u^2}\right)\right\} \tag{59}$$

and then integrating over y, employing the boundary conditions that both $\bar{u}$ and $\overline{uv}$ are zero as $y \to \pm\infty$, we obtain

$$0 = -\frac{d}{dx}\int_{-\infty}^{\infty}\left[\bar{u}^2 + \left\{\overline{u^2} - \overline{v^2}\right\}\right]dy' \tag{60}$$

or, in terms of the total momentum flux, J,

$$J = \rho\int_{-\infty}^{\infty}\left[\bar{u}^2 + \left\{\overline{u^2} - \overline{v^2}\right\}\right]dy' \quad = \underbrace{\rho\int_{-\infty}^{\infty}\left[\bar{u}^2 + \left\{\overline{u^2}\right\}\right]dy'}_{\text{Streamwise Flux}} - \underbrace{\rho\int_{-\infty}^{\infty}\left\{\overline{v^2}\right\}dy'}_{\text{Pressure Gradient Effect}} \tag{61}$$

The total streamwise momentum flux through a transverse plane is not constant, unlike the laminar jet, but is modified due to the existance of turbulent fluctuations which manifest in the form of a streamwise pressure gradient. . And the turbulent fluctuations will modify the laminar momentum conservation constraint by as much as $\mathcal{O}\left(\frac{\delta}{L}\right)$, which may not be insignificant.

Note also that J is not constant where there is a background flow, i.e. $u(x, y \to \pm\infty) \neq 0$. So laboratory experiments may be very sensitive to the presence of even slight flows in the lab

Now we can seek a similarity solution in terms of $\eta = y/\delta(x)$ in the form

$$\bar{u} = u_s(x)f(\eta) \tag{62}$$

$$\overline{uv} = r(x)g(\eta) \tag{63}$$

where we have two scaling functions: $u_s(x)$ for the mean velocity and $r(x)$ for the turbulent fluctuations. We need to establish the criteria under which a similarity solution can exist. We can write

$$\frac{\partial \bar{u}}{\partial x} = u_s f' - u_s f'\eta\frac{\delta'}{\delta} = -\frac{\partial \bar{v}}{\partial y} \tag{64}$$

and using continuity (like we did above, in place of a streamfunction) we can integrate over y to obtain

$$\bar{v}(x,y) - \bar{v}(x,0) = \int_0^y\left(-u_s f' + u_s f'\eta\frac{\delta'}{\delta}\right)dy' \tag{65}$$

and after integration by parts we obtain

$$\bar{v}(x,y) = u_s\delta'\eta f - (u_s\delta)' \int_0^\eta f d\eta \tag{66}$$

and we then write all of the other differentiated quantities in similarity form.

Similarity Formulation
$\frac{\partial \bar{u}}{\partial x} = u_s f' - u_s f' \eta \frac{\delta'}{\delta} = -\frac{\partial \bar{v}}{\partial y}$
$\frac{\partial \bar{u}}{\partial y} = u_s \frac{1}{\delta}$
$\frac{\partial}{\partial y}\overline{uv} = r(x)\frac{g'}{\delta}$

Table 8: The similarity formulation for all of the components in the momentum equation. The prime refers to differentiation with respect to the independent variable of a given function.

Plugging all of the similarity forms into the momentum equation, written to leading order, yields

$$\left[\frac{\delta u_s'}{u_s}\right] f^2 - \left[\frac{(u_s\delta)'}{u_s}\right] f' \int_0^\eta f d\eta' + \left[\frac{r}{u_s^2}\right] g' = 0 \tag{67}$$

where all of the bracketed quantities with explicit x-dependence must be proportional to each other, in order that the x-dependence can be divided out of the dynamics. We conclude for similarity to exist

$$\frac{\delta u_s' u_s}{r} \propto \frac{u_s^2 \delta'}{r} \propto 1 \tag{68}$$

Note that $\frac{(u_s\delta)'}{u_s} = \frac{u_s\delta'}{u_s} + \delta'$ and the first term is redundant with the first bracketed quantity, so we need to consider only the second term in the expanded derivative. Also note that $r \neq 0$ everywhere, except at the boundaries, so we can safely divide by r.

From this we can conclude that $u_s \propto \delta^a$ and $\delta^{2a}\delta' \propto r$ for some undetermined power a. But we can't determine the precise power from the momentum equation, just like in the laminar case, we could not fix λ without using the momentum flux constraint. If we rewrite the integral constraint, we obtain

$$J = \rho \int_{-\infty}^{\infty} \bar{u}^2 dy = \left[\rho u_s^2 \delta\right] \int_{-\infty}^{\infty} f^2 d\eta \quad \Rightarrow J \propto \delta^{2a+1} \tag{69}$$

so that for constant J we have $2a + 1 = 0$ or $a = -1/2$. But we are still left with two unknown functions of x, given by

$$u_s(x) \propto \delta^{-1/2} \quad \text{and} \quad \frac{u_s'}{u_s} \propto r(x) \tag{70}$$

and we are unable to resolve the ambiguity. In order to say anything about the actual scaling, presumably we will need an additional constraint to relate the similarity scaling functions. We could have expected this from the beginning, because we started with two unknown similarity functions (due to the Reynolds stress), instead of just one, as in the laminar jet. If a similarity solution exists, it must hold for both the momentum and the energy description of the system, so let's write the TKE balance, defining the total energy fluctuation as $q^2 = u_i u_i$, where the TKE is denoted by $k = \frac{1}{2}\overline{q^2}$. We apply all of the same boundary-layer type approximations to simplify the equations, to obtain

$$\underbrace{\left(\bar{u}\frac{\partial}{\partial x}+\bar{v}\frac{\partial}{\partial y}\right)k}_{\delta/L}+\underbrace{\frac{\partial}{\partial x}\left(\frac{1}{2}\overline{q^2u}+\frac{1}{\rho}\overline{p\upsilon}\right)}_{(\delta/L)^{3/2}}-\underbrace{\frac{\partial}{\partial x}2\nu\overline{u_i s_{xi}}}_{\mathrm{Re}^{-1}\sqrt{\delta/L}}+\underbrace{\frac{\partial}{\partial y}\left(\frac{1}{2}\overline{q^2v}+\frac{1}{\rho}\overline{pv}\right)}_{\sqrt{\delta/L}}-\underbrace{\frac{\partial}{\partial y}2\nu\overline{u_i s_{yi}}}_{\mathrm{Re}^{-1}\sqrt{L/\delta}}$$
$$\underbrace{\overline{u^2}\frac{\partial\bar{u}}{\partial x}+\overline{v^2}\frac{\partial\bar{v}}{\partial y}}_{\delta/L}+\underbrace{\overline{uv}\frac{\partial\bar{v}}{\partial x}}_{(\delta/L)^2}+\underbrace{\overline{uv}\frac{\partial\bar{u}}{\partial y}}_{1}+\underbrace{\varepsilon}_{\sqrt{\delta/L}}=0 \quad (71)$$

In the streamwise momentum balance, we retained terms of $\mathcal{O}(\delta/L)$, at least initially, because they affected the mean momentum flux. Therefore, we choose to retain terms to the same order of magnitude in the TKE, including $\mathcal{O}(1)$, $\mathcal{O}(\sqrt{\delta/L})$, and $\mathcal{O}(\delta/L)$, to obtain

$$\underbrace{\left(\bar{u}\frac{\partial}{\partial x}+\bar{v}\frac{\partial}{\partial y}\right)k}_{\text{Advection}}+\underbrace{\overbrace{\overline{uv}\frac{\partial\bar{u}}{\partial y}}^{\mathcal{O}(1)}+\overbrace{\overline{u^2}\frac{\partial\bar{u}}{\partial x}+\overline{v^2}\frac{\partial\bar{v}}{\partial y}}^{\mathcal{O}(\delta/L)}}_{\text{Production}}+\underbrace{\frac{\partial}{\partial y}\left(\frac{1}{\rho}\overline{pv}+\frac{1}{2}\overline{q^2v}\right)}_{\text{Transport}}+\underbrace{\varepsilon}_{\text{Dissipation}}=0 \quad (72)$$

where we typically neglect the two smaller production terms, which are quite small in comparison to the dominant production term. We will need a number of additional similarity functions in order to write the TKE balance in self-similar form.

Recall that $\varepsilon \sim \frac{u^3}{\ell}$ which we can use for our definition of the self-similar dissipation. Note, however, that we can also write $\varepsilon \sim \nu\left(\frac{v}{\eta}\right)^2$ in terms of dissipative scales, so there is some ambiguity in whether we should write dissipation with a dependence on viscosity or not — we will return to this point.

$$\overline{q^2} = r(x)g_1(\eta) \quad (73)$$
$$\overline{u^2} = r(x)g_2(\eta) \quad (74)$$
$$\overline{v^2} = r(x)g_3(\eta) \quad (75)$$
$$\frac{1}{\rho}\overline{pv}+\frac{1}{2}\overline{q^2v} = r(x)^{3/2}g_4(\eta) \quad (76)$$
$$\varepsilon = \frac{r(x)^{3/2}}{\delta(x)}g_5(\eta) \quad (77)$$

and then we can write out the needed derivatives in terms of these similarity functions.

Similarity Formulation
$\frac{\partial}{\partial y}\frac{1}{2}\overline{q^2} = \frac{1}{2}r\frac{g_1'}{\delta}$
$\frac{\partial}{\partial x}\frac{1}{2}\overline{q^2} = \frac{1}{2}\left(r'g_1 - \frac{r}{\delta}g_1'\delta'\eta\right)$
$\frac{\partial}{\partial y}\left(\frac{1}{\rho}\overline{pv}+\frac{1}{2}\overline{q^2v}\right) = r^{3/2}\frac{g_4'}{\delta}$

Table 9: The similarity formulation for all of the components in the TKE equation. The prime refers to differentiation with respect to the independent variable of a given function.

Finally, we can write the TKE balance in similarity form as

$$\left[\frac{1}{2}u_s r'\right]fg_1+\left[u_s r\frac{\delta'}{\delta}\right](\eta g_3 f'-\eta g_2 f')-\left[\frac{1}{2}(u_s\delta)'\frac{r}{\delta}\right]\left(g_1'\int_{-\infty}^{\infty}f\,d\eta\right)$$
$$+\left[\frac{u_s r}{\delta}\right]gf'+\underbrace{[ru_s']\left(g_2 f-g_3 f\right)}_{\mathcal{O}(\delta/L)}+\left[\frac{r^{3/2}}{\delta}\right](g_4'+g_5) \quad (78)$$

And we conclude that, for similarity to be obeyed, we need to satisfy

$$\delta \frac{r'}{r} \propto \delta' \propto \delta \frac{u_s'}{u_s} \propto \frac{r^{1/2}}{u_s} \propto 1 \tag{79}$$

Now, we can see that $\delta' \propto 1$, so we conclude that $\delta \propto x$. More importantly, we conclude that $u_s^2 \propto r$. This suprising result means that, for a similarity solution to exist, based on our assumed scaling, there is only a single velocity scale in the problem. The Reynolds stresses scale precisely according to the mean flow scaling. Using our ealier results from the momentum balance, we can then conclude that $u_s \propto x^{-1/2}$ for turbulent flow, compared to $u_s \propto x^{-1/3}$ for laminar flow. We also find that $\overline{uv} \propto x^{-1}$ and the mass flux is $\dot{m} \propto x^{1/2}$, so that the entrainment rate is much faster than the laminar entrainment rate, $m \propto x^{1/3}$.

Over-constrained Similarity

The problem with this analysis is that it is not verified in experiments. In experiments, the constant prefactors (proportionality constants) for these self-similar relations seem to vary by as much as 20% from experiment to experiment. In other words, each flow appears to be self-similar on its own, but the similarity breaks down when comparing across different experiments, and it becomes very difficult to 'collapse the data' from multiple experiments onto a single, self-similar set of variables. The question is: where is the flaw in our similarity analysis?

See W. K. George, 'The self-preservation of turbulent flows and its relation to initial conditions and coherent structures' 1989

The issue is our assumption of the form of the scaling functions, like $u_s(x)$. By writing the scale in this form, we are implicitly assuming that the only length scale that matters is the distance from the origin of the jet, x. However, what if there were another relevant lengthscale? For instance, the initial width of the jet at the outlet, h. If the initial conditions of the jet matter (i.e. we do not treat the jet as a point source but rather as a source with finite dimension h), then we could define our scaling functions in a more general form, making use of the previous results, to write

$$u_s(x,h) = x^{-1/2} F_1(x/h) \tag{80}$$

$$r(x,h) = x^{-1} F_2(x/h) \tag{81}$$

$$\delta(x,h) = x F_3(x/h) \tag{82}$$

where these addition functions, F_1, F_2, and F_3 are all unity in the limit that the jet behaves as a point sources (i.e. in the limit that the initial conditions are universal). Let's plug these new forms of the scaling functions into our momentum balance,

$$\left[\frac{\delta u_s'}{u_s}\right] f^2 - \left[\frac{(u_s \delta)'}{u_s}\right] f' \int_0^\eta f d\eta' + \left[\frac{r}{u_s^2}\right] g' = 0 \tag{83}$$

to obtain the criterion for similarity again, in terms of these initial condition functions

$$\left[F_3\left(-\frac{1}{2}+\frac{F_1'}{F_1}\frac{x}{h}\right)\right]f^2-\left[F_3\left(\frac{1}{2}+\frac{F_1'}{F_1}\frac{x}{h}\right)+\frac{x}{h}F_3'\right]f'\int_0^\eta f d\eta'+\left[\frac{F_2}{F_1^2}\right]g'=0 \tag{84}$$

which we can rewrite more simply, along with the momentum flux constraint, as

$$\frac{F_1'}{F_1}\left(\frac{x}{h}\right)\propto\frac{F_3'}{F_3}\left(\frac{x}{h}\right)\propto\frac{F_2}{F_3F_1^2}\propto 1 \quad \text{and} \quad F_3F_1^2\propto 1 \tag{85}$$

Then we conclude that similarity is satisfied if

$$F_1\propto F_3^{-1/2}\propto\left(\frac{x}{h}\right)^b \quad \text{and} \quad F_2\propto 1 \tag{86}$$

for some arbitrary constant b. In other words, the scaling we obtained earlier for u_s and δ may actually scale according to any power of x and still obey self-similarity. All of our earlier power laws can be rewritten with the addition of an arbitrary power, and we obtain our earlier results only if $b=0$. This explains why every experiment seems to behave in a self-similar way, but that comparisons between experiments are difficult: the self similarity constant, b, depends on the initial conditions of the jet, and thus will always vary and is not expected to be universal. We see that by assuming a particular form of the scaling functions, like $u_s(x)$ and $r(x)$, we have over-constrained our problem, and thus produced a similarity solution that is valid only in particular circumstances.

Have we over-constrained our problem in other ways? Let's consider the similarity form for the dissipation

Recall in our TKE balance we had $\cdots+\left[u_s r\frac{\delta'}{\delta}\right](\cdot)+\left[\frac{u_s r}{\delta}\right]gf'+\left[\frac{r^{3/2}}{\delta}\right]g_5$

$$\varepsilon=\frac{r(x)^{3/2}}{\delta(x)}g_5(\eta) \quad \Rightarrow \quad \delta'\propto\frac{r^{1/2}}{u_s}\propto 1 \tag{87}$$

which ended up resulting in the important conclusion that there was only a single velocity scale in our problem, for both turbulence and the mean flow. This idea of as single velocity scale is what ultimately over-constrains our problem, along with the requirement that $\delta'\propto 1$. However, now that we realize our earlier analysis was over-constrained, we might consider choosing an alternate scaling for the dissipation, based on our dimensional analysis for the dissipation scales. We can choose to write the dissipation dependent on the viscosity, such that

$$\varepsilon=\nu\frac{r(x)}{\delta(x)^2}g_5(\eta) \quad \Rightarrow \quad \delta'\propto\frac{\nu}{u_s\delta}\propto 1 \tag{88}$$

Under the dissipative scaling, $\delta'\propto\frac{\nu}{u_s\delta}$. Combining this with the momentum constraint, $u_s^2\delta\propto 1$ yields $\delta\propto x^{2/3}$, i.e. the laminar jet solution. In other words, if we choose the dissipative, low Reynolds number scaling, we can obtain the laminar similarity solution.

which no longer implies a single velocity scale. But we still retain $\delta'\propto 1$. We see that we can obtain similarity solutions even without a single velocity scale, which of course just reiterates the point that, if we do assume a single velocity scale, we are by definition over-constraining the problem. The choice of scaling for dissipation

can drastically change the exponents observed in the similarity solution, anywhere from the laminar solution ($\delta \propto x^{2/3}$) to the over-constrained turbulent solution ($\delta \propto x$). What is the difference between these two choices of dissipation scaling?

Recall that the first formulation for dissipation came about by writing

$$\frac{d}{dt}u^2 \sim \varepsilon \quad \Rightarrow \quad \varepsilon \sim \frac{u^3}{\ell} \tag{89}$$

but this expression makes sense only when there is a strong distinction between large, energy-containing scales u and the small scales of dissipation. In other words, only at very high Reynolds number can we imagine that nearly all of the large-scale energy enters dissipation. If the Reynolds number is smaller, then there is overlap between the large scales u and the small scales of dissipation, and we can't write this same simple equality. So in some sense, our choice of scaling for ε was appropriate only in the limit as $\mathrm{Re} \to \infty$, and that is the only case in which there is a single velocity scale for turbulence. For finite Reynolds numbers, dissipation will necessarily scale, at least partly, on viscosity, and thus we expect multiple velocity scales in the jet.

One final way we can picture the over-constrained nature of the jet problem is to rewrite the momentum balance by incorporating the momentum flux constraint, $u_s^2\delta \propto 1$

Note that $u_s^2\delta \propto 1$ implies that $u_s \propto \delta^{-1/2}$ which means we can write a power law relation $\frac{u_s'}{u_s} \propto \frac{\delta'}{\delta}$

$$\left[\delta'\right]f^2 - \left[2\delta'\right]f' \int_0^\eta f d\eta' + \left[\frac{r}{u_s^2}\right]g' = 0 \tag{90}$$

$$f^2 - 2f' \int_0^\eta f d\eta' + \underbrace{\left[\frac{1}{\delta'}\frac{r}{u_s^2}\right]}_{\beta} g' = 0 \tag{91}$$

In this form, we see that the functions $f(\eta)$ and $g(\eta)$ can have the same shape to satisfy the similarity equation for a variety of different jets; the only change in this functions is the amplitude, which we can describe in the governing equation as a scale factor, β. In our over-constrained analysis, we chose $\beta = 1$, but there is nothing to prevent us from obtaining other similarity solutions for difference values of β.

TKE in the Jet

Before concluding the analysis of the turbulent jet, we can make some additional comments on the TKE balance across the profile of the jet.

$$\underbrace{\left(\bar{u}\frac{\partial}{\partial x} + \bar{v}\frac{\partial}{\partial y}\right)k}_{\text{Advection}} + \underbrace{\overline{u}\overline{v}\frac{\partial \bar{u}}{\partial y} + \overline{u^2}\frac{\partial \bar{u}}{\partial x} + \overline{v^2}\frac{\partial \bar{v}}{\partial y}}_{\text{Production}} + \underbrace{\frac{\partial}{\partial y}\left(\frac{1}{\rho}\overline{p}\overline{v} + \frac{1}{2}\overline{q^2 v}\right)}_{\text{Transport}} + \underbrace{\varepsilon}_{\text{Dissipation}} = 0 \tag{92}$$

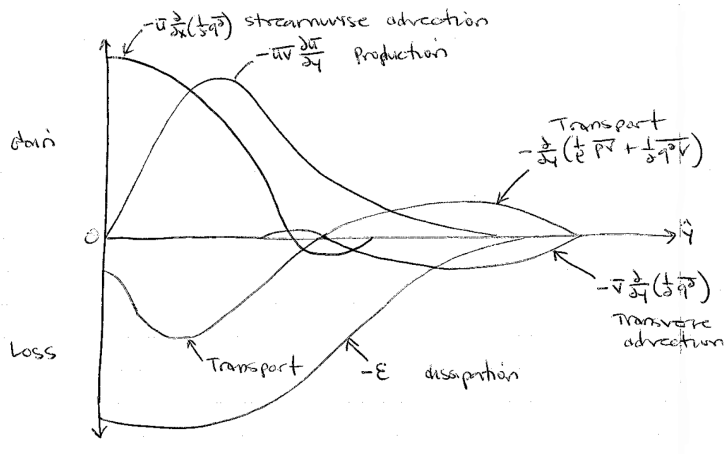


Figure 25: A sketch of the TKE balance across the transverse profile of the plane turbulent jet, based on empirical measurements. All quantities are negated in order that the production should appear as a positive term (gain in energy) and the dissipation as negative (loss in energy).

At the outer edge of the jet, we have an entrainment velocity in the transverse direction, and the TKE is decreasing with y. This means that the transverse advection, $\bar{v}\frac{\partial k}{\partial y} > 0$, and energy is being lost to the entrainment process. Where does the energy come from then? The lost energy of transverse advection is balanced by the transport, $\frac{\partial}{\partial y}\left(\frac{1}{\rho}\overline{pv} + \frac{1}{2}\overline{q^2 v}\right)$. The transport terms can be thought of as diffusive: they are positive towards the center of the jet (transporting energy outwards, against the local gradient in k) and negative near the outer edge (transporting energy inwards, against the local gradient in k). At this outer edge, transport and advection are roughly in balance, and thus dissipation is not significant.

Recall that $\bar{v} < 0$ for entrainment on the positive side of the jet, $y > 0$.

In the remainder of the jet, dissipation is the dominant energetic process. The dissipation is approximately balanced by production, although production is negligible at the center of the jet (because of symmetry), so the production peak is actually off-center, where the maximum Reynolds stress occurs. The streamwise advection term, $\bar{u}\frac{\partial k}{\partial x} < 0$, because the TKE decays downstream. The energy from streamwise advection is directed into the transport term moving energy outwards from the center of the jet – they are roughly balanced near the center of the jet.

Entrainment

We described the energetic balance between transverse advection and transport that occurs during the entrainment process at the outer edge of the jet. And from the similarity analysis, we concluded that

$$\delta' \propto \frac{r}{u_s^2} \tag{93}$$

which is really just a constraint on the rate at which the jet expands to envelop additional, non-turbulent fluid, i.e. the entrainment rate. Both of these descriptions of entrainment imply that the entrainment process is independent of viscosity. But we know from Kelvin's circulation theorem that for an incompressible, barotropic

Recall that $\frac{D\Gamma}{Dt} = -\nu \oint_C \nabla \wedge \vec{\omega} \cdot d\vec{\ell}$

fluid, the only way irrotational fluid can become rotational is by means of viscous diffusion. The quiescent fluid outside the jet is irrotational, so how does it become rotational in the entrainment process, if not by means of viscous action? It must be that viscous diffusion is, of course, responsible for the entrainment, but that this diffusion is dominated by interaction with large, energy-containing eddies, and can thus be described in terms of the energy-containing flow parameters, independent of viscosity.

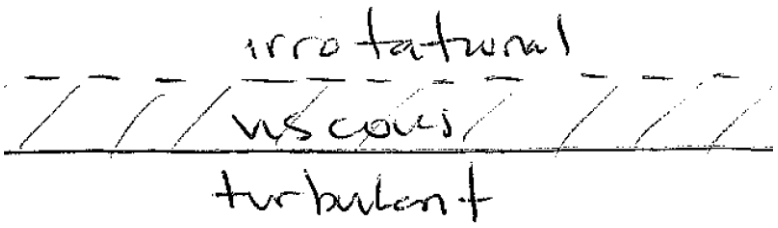


Figure 26: A sketch of the initially thin layer between rotational and irrotational flow, in which vorticity diffusion is important.

Let's start by considering a thin layer of fluid acting as an interface between the rotational flow within the jet and the irrotational flow outside. This thin layer represents the extent of viscous diffusion, which would occur even in the absence of advection. But, in turbulence, eddies will interact with this diffusive region. Because the region is very small initially, only the smallest eddies will interact with it, causing the region to distort under the straining motions of the eddies.

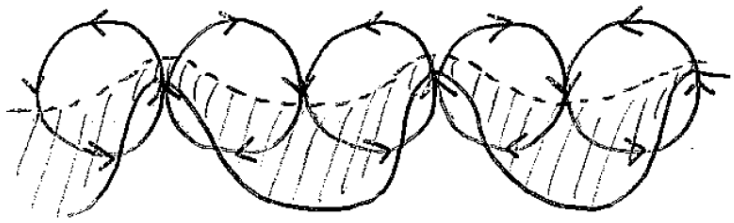


Figure 27: A sketch of the distorted diffusive layer, after the straining action of very small eddies causes it to be slightly distorted.

We can imagine a row of small eddies distorting the diffusive layer, compressing the layer at its crests and stretching it at its troughs, thereby creating an asymmetric, wavy pattern. Since the timescales of the eddies are much faster than the diffusive timescales, the effect of stretching the trough regions is to spread vorticity much faster than diffusion alone could. Moreover, the compression in the crests generates higher spatial gradients of vorticity, which also enhances the magnitude of the diffusion compared to the initial conditions. After the diffusive layer gets stretched a bit larger, the next larger scale of eddies is able to interact with it, causing further waviness (or wrinkling).

Again, troughs get stretched, the region of vortical flow expands, and the diffusive gradients are enhanced in the crests. Somewhat

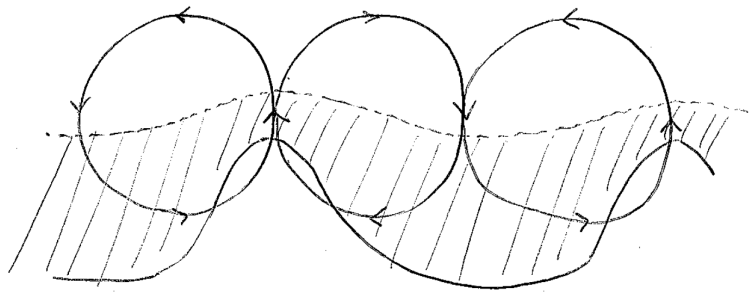


Figure 28: A sketch of the distorted diffusive layer, after the straining action of large eddies causes it to be distorted significantly.

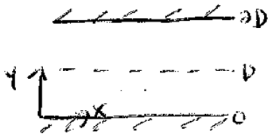
surprisingly, we observe a 'reverse cascade' of sorts, as progressively larger scales interact with the diffusive layer and distort it, up until the very largest coherent eddies which generate the largest regions of vortical flow at the edge of the jet. If the eddies and diffusive region are not compatible in size, then the diffusive region may become distorted without stretching or compression (just deformation). In this case, pockets of irrotational flow could become almost enveloped in the rotational flow, without the rotational flow actually spreading to fill in the gaps, a feature often observed at the edge of jets.

Wall-bounded Flows

Turbulent Channel

Now that we have considered a canonical, external flow (i.e. a flow without boundaries), we turn to the canonical internal flow: flow within a channel. Consider the flow between smooth, rigid wall as $y = 0$ and $y = 2h$.

We assume the flow is fully developed in the streamwise direction (ensemble statistics are also homogeneous in x) and steady (stationary). As in the case of the planar jet, we will assume that the flow is quasi two-dimensional as well. From the mean continuity equation and no-penetration boundary conditions, we conclude that $\bar{v} = 0$. Then we can write our momentum equations as



The assumption of homogeneity in x means we can write derivatives of the ensemble averaged components as regular (total) derivatives in y; this fully developed/homogeneous assumption does not apply to the pressure gradient, since the flow is driven by a pressure gradient in x.

$$\frac{d}{dy}\overline{uv} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\frac{d^2\bar{u}}{dy^2} \tag{1}$$

$$\frac{d}{dy}\overline{v^2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} \tag{2}$$

As with the jet, we can integrate the y-momentum equation from $y' : 0 \to y$ to obtain a relation for the pressure in terms of the pressure measured at the wall, $\bar{p}_w$.

$$\underbrace{\bar{p}(x,0)}_{\bar{p}_w(x)} = \bar{p}(x,y) + \rho\overline{v^2} \tag{3}$$

Because we assume the flow is homogeneous in x, we conclude that $\overline{v^2}$ is a function of y only, and therefore

$$\frac{d\bar{p}_w(x)}{dx} = \frac{\partial \bar{p}(x,y)}{\partial x} \tag{4}$$

The lefthand side is a function of x only; the righthand side a function of y only.

We can then rewrite the momentum equation as

$$\frac{d\bar{p}_w}{dx} = \frac{d}{dy}\left(\mu\frac{d\bar{u}}{dy} - \rho\overline{uv}\right) \tag{5}$$

and integrating over $y' : 0 \to y$ yields

$$\frac{d\bar{p}_w}{dx}y = \mu\frac{d\bar{u}}{dy} - \rho\overline{uv} - \underbrace{\mu\frac{d\bar{u}}{dy}\bigg|_{y=0}}_{\tau_w} \tag{6}$$

Using the boundary conditions that $\overline{uv} = 0$ at $y = 0$ because no-slip and no-penetration apply also to the turbulent fluctuations themselves. The wall shear stress is defined as $\vec{\tau}_w = \vec{\vec{\tau}} \cdot \hat{n}\Big|_w$ where $\hat{n}$ is outward facing unit normal on the wall, i.e. facing into the fluid. The wall stress generally is a vector, but for quasi 2D flow, it is a scalar quantity, $\tau_w = \mu\frac{d\bar{u}}{dy}$ in the $+\hat{y}$ direction. Therefore, on the bottom wall, $\tau_w > 0$ and on the top wall, both the gradient and the unit normal changes signs, so $\tau_w > 0$ also.

At $y = 2h$ (on the upper wall) we obtain

$$\frac{d\bar{p}_w}{dx}2h = \underbrace{\mu\frac{d\bar{u}}{dy}\bigg|_{y=2h}}_{-\tau_w} - \tau_w \tag{7}$$

where the upper wall shear stress has the opposite sign since it has an opposite facing wall-normal vector. We conclude that there is a simple relationship between the wall shear stress and the pressure-gradient driving the flow, given by

We can derive this from a force balance between the driving force per unit width of the channel, $2hd\bar{p}_w$, and the drag force opposing the driving force, $-\tau_w dx$. τ_w is the stress acting on the wall; $-\tau_w$ is the stress acting on the fluid. This is a consequence of the sign convention of the unit normal vector.

$$-h\frac{d\bar{p}_w}{dx} = \tau_w \tag{8}$$

The wall shear stress can be used to define a velocity scale relevant near the wall, known as the friction velocity

$$u_\tau \equiv \sqrt{\frac{\tau_w}{\rho}} \tag{9}$$

If one of the walls is moving (Couette flow), then there will be an asymmetry in the wall shear stresses, and two different friction velocities will be needed to characterize the flow.

which, in turn, can be used to define a Reynolds number that characterizes wall-bounded flows in terms of the frictional drag on the wall

$$\mathrm{Re}_\tau \equiv \frac{u_\tau h}{\nu} \tag{10}$$

The wall shear stress is purely viscous, because turbulent fluctuations are identically zero at the wall. However, throughout the rest of the channel we expect contributions from both viscous and turbulent stress toward the total stress, τ which we define as

$$\tau \equiv \mu\frac{d\bar{u}}{dy} - \rho\overline{uv} \tag{11}$$

Substituting into the integrated momentum balance yields

$$\frac{d\bar{p}_w}{dx}y = \tau - \tau_w \quad \text{with} \quad -h\frac{d\bar{p}_w}{dx} = \tau_w \tag{12}$$

from which we conclude that the total stress across the channel varies linearly

$$\tau = \tau_w\Big(1 - \frac{y}{h}\Big) \tag{13}$$

This linear stress relation applies to both laminar and turbulent flows. For a laminar flow, the total stress is just the viscous stress, and then we can easily integrate to solve for the velocity distribution.

The laminar profile in normalized form, $\frac{\bar{u}}{u_\tau} = \mathrm{Re}_\tau\big(\frac{y}{h}\big)\big(1 - \frac{y}{2h}\big)$

$$\tau = \mu\frac{d\bar{u}}{dy} = \tau_w\Big(1 - \frac{y}{h}\Big) \quad \Rightarrow \quad \bar{u} = \frac{\tau_w y}{\mu}\Big(1 - \frac{y}{2h}\Big) \tag{14}$$

In turbulent flow, without knowing the functional form of $\overline{uv}$, we cannot integrate the momentum balance.

Not surprisingly, the ratio of the two contributions to the total stress, $\frac{\rho\overline{uv}}{\mu\frac{d\bar{u}}{dy}} \sim \frac{\rho u^2}{\mu\frac{u}{\delta}} \sim \frac{u\delta}{\nu} \sim \mathrm{Re}$, so as Reynolds number increases in turbulence, the Reynolds stress becomes the dominant contribution to the total stress.

If we fix the Reynolds number defined using the wall-shear stress, Re_τ, then we expect the velocity gradient at the wall to be identical for both laminar and turbulent flows. And we know that

both laminar and turbulent flows have the same total stress (following a linear profile). However, because $-\rho\overline{u}\overline{v}$ has the same sign as the viscous shear stress, we expect the velocity gradient in the laminar flow to be much steeper than that of the turbulent flow. In other words, we expect the turbulent flow to have a blunt velocity profile, with smaller mean velocity gradient away from the wall, than the corresponding laminar flow at matched Reynolds number. This means that the volumetric flowrate in a turbulent flow well be significantly less than its equivalent laminar flow.

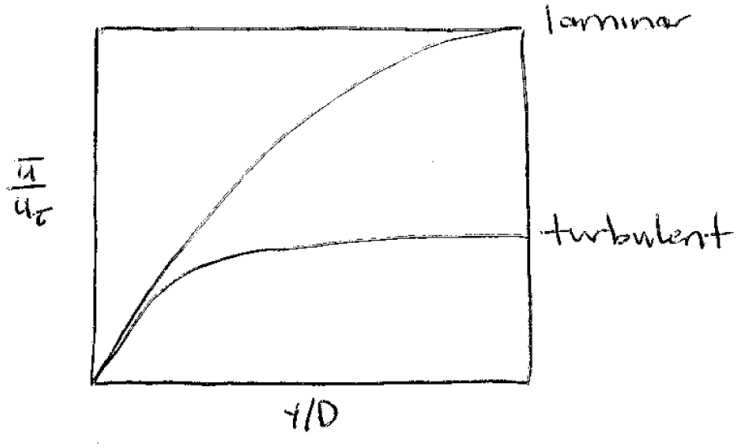


Figure 29: A sketch of laminar and turbulent velocity profiles over a channel (half-width D) for a fixed Reynolds number defined by the friction velocity, $\mathrm{Re}_\tau = \frac{u_\tau D}{\nu}$. Note that typically, the turbulent flow occurs at higher Reynolds number, so the comparison here is a bit artificial.

By the same argument above regarding the blunted velocity profile, we can also draw conclusions about the distribution of the Reynolds stress. The sum of the Reynolds stress and viscous shear stress decrease linearly moving away from the wall, towards the center of the channel. At the wall, the Reynolds stress is zero and thus all of the stress at the wall is the result of viscous stresses ($\tau = \tau_w$). The ratio between the two stresses is given by the local Reynolds number, which we call y^+

$$\frac{\rho\overline{u}\overline{v}}{\mu\frac{d\overline{u}}{dy}} \sim \frac{\rho u_\tau^2}{\mu\frac{u_\tau}{y}} \sim \frac{u_\tau y}{\nu} \equiv y^+ \tag{15}$$

At the wall itself, the local Reynolds number is zero and the flow is dominated by viscous diffusion. As we move away from the wall, viscous diffusion becomes less significant relative to turbulent advective effects, and the total stress becomes dominated by the Reynolds stress contribution. In fact, based on our earlier derivation of the total stress, we can conclude that very near the wall, as $y/h \to 0$, the total stress is approximately constant ($\tau \approx \tau_w$), and within this constant stress region, the viscous and Reynolds stress essentially switch magnitudes. Of course, the stress is not technically constant, since it is linearly decreasing, but very close to the wall the magnitude of the linear change is quite small. We expect that the maximum in the turbulence production, $\mathcal{P} = -\overline{u}\overline{v}\frac{\partial\overline{u}}{\partial y}$, should occur where these two stresses cross, in the middle of this constant stress region.

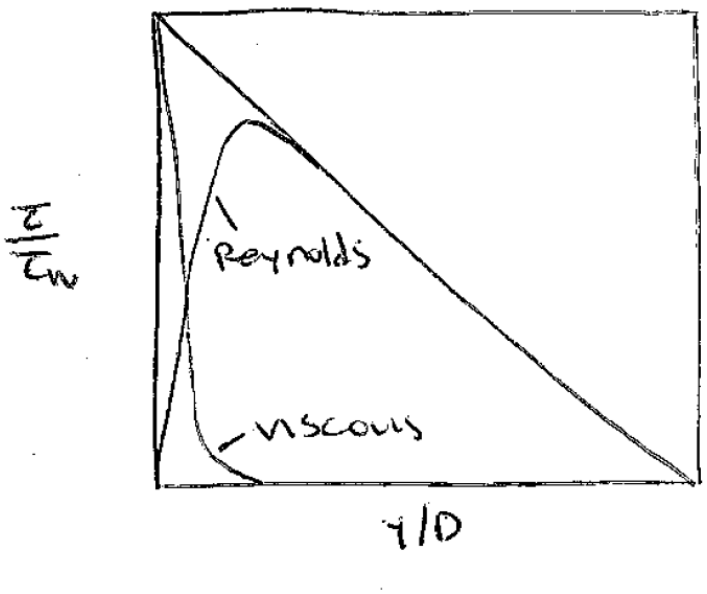


Figure 30: A sketch of the stress distribution across the half-channel, showing the decrease in viscous stress and increasing dominance of Reynolds stress with increasing wall-normal location, y^+.

This argument leads to the hypothesis that there are two distinct regions within wall-bounded turbulence: a near-wall, inner region in which viscous diffusion is important, and an outer region, farther from the wall, in which the effect of viscosity is negligible.

Townsend's Reynolds Number Similarity

Townsend (1976) proposed that if the the viscous layer is thin compared to the channel width, then the flow in the outer region should depend only on h and τ_w and the statistical description of this outer flow should be similar for all types of wall-bounded flows at all Reynolds numbers, independent of the specific nature or geometry of the wall. This is a much stronger claim than the usual claim of 'dynamical similarity' which states that geometrically similar flows should be statistically similar at equivalent Reynolds number, since Townsend's claim apply to all (sufficiently high) Reynolds numbers and all geometries and all wall conditions.

Recall that τ_w is directly related to the pressure drop driving the flow, $\frac{d\bar{p}_w}{dx}$, hence we expect it to determine the overall flow.

We can estimate the width of the viscous layer dimensionally from our local wall Reynolds number, $y^+ \equiv \frac{y}{\nu/u_\tau}$. The requirement that this thickness is small compared to the channel width is

$$\frac{\nu}{u_\tau} \ll h \quad \Rightarrow \quad \frac{u_\tau h}{\nu} \gg 1 \tag{16}$$

and therefore the condition for Townsend's similarity hypothesis is essentially that the Reynolds number is sufficiently high. Townsend's hypothesis predicts that, in the fully turbulent (outer) region of the flow, all mean relative motions and energy-containing components of the turbulent motions can be described independent of viscosity (except insofar as viscosity affects τ_w) or wall properties, including geometry and roughness.

Townsend's similarity hypothesis is very reminiscent of Kolmogorov's universal equilibrium theory. There also we required a sufficiently high Reynolds number, $\mathrm{Re}^{3/4} \gg 1$ such that there would be spectral separation between large and small scale motions. In Kolmogorov's theory, we claimed that all relevant dynamic quantities describing small-scale (i.e. viscous) motions could then be written in a universal form. Here, we are claiming that all of the energetic, large-scale motions far from the wall (beyond where viscous diffusion is important) can be written in a universal form. Townsend's hypothesis implies that we can write the mean streamwise velocity profile in the turbulent channel as a function of only u_τ and h (and also, more generally, some Galilean invariant translational velocity, $\bar{u}_0$)

We use u_τ to non-dimensionalize the outer flow, although perhaps it makes more sense to use $\bar{u}_0$ instead — see Zagarola & Smits 1998

$$\frac{\bar{u}}{u_\tau} = \frac{\bar{u}_0}{u_\tau} + f\left(\frac{y}{h}\right) \tag{17}$$

In this case, the non-dimensional function f is a universal, similarity function, and this velocity profile should apply for $y^+ \gg 1$. If we choose our velocity off-set $\bar{u}_0$ as the centerline velocity of the channel or the average velocity across the channel, we can write the defect between the centerline and local velocity as a universal function

$$\frac{\bar{u}_0 - \bar{u}}{u_\tau} = f_o\left(\frac{y}{h}\right) \tag{18}$$

which we call the 'velocity defect law'. Closer to the wall, the Reynolds number similarity hypothesis begins to break down as viscous forces becomes more important, and thus we have to write the general form of the defect law as

The similarity condition can be thought of as the asymptotic limit of this more general Reynolds number dependent formulation of the mean flow

$$\frac{\bar{u}_0 - \bar{u}}{u_\tau} = f_o\left(\frac{y}{h}, \mathrm{Re}_\tau\right) \tag{19}$$

If the inner region is sufficient small ($\frac{\nu}{u_\tau} \ll h$), then we might also expect to be able to describe the inner behavior in terms of the local (non-universal) wall properties,

$$\frac{\bar{u}}{u_\tau} = f_i\left(y^+, \mathrm{Re}_\tau, k^+, \ldots\right) \tag{20}$$

called the 'law of the wall'. Here we have an explicit dependent of viscosity, built into the wall scaling y^+, as well as a parameter k^+ which describes the magnitude of wall roughness, usually defined by the root mean square roughness, k. In the asymptotic limit of large Reynolds number and a smooth wall, the law of the wall can be written as

$$\frac{\bar{u}}{u_\tau} = f_i\left(y^+\right) \tag{21}$$

Very near the wall, we can write a Taylor series for the law of the wall

The velocity profile is obviously linear near the wall for laminar flows as well. Our only assumption was that the velocity near the wall can be defined in terms of a viscous length scale, which is surely true for a laminar flow.

$$\frac{\bar{u}}{u_\tau} = f_i(0) + \left.\frac{df_i}{dy^+}\right|_{y^+=0} y^+ + \cdots \quad = y^+ + \cdots \tag{22}$$

to show that in some vicinity of the wall itself the turbulent velocity profile has a linear variation with y. This vicinity is referred to as the linear sublayer, and extends from the wall out to around $y^+ \lesssim 5$.

Now that we have established two distinct regions of flow, each scaled by a different characteristic length scale, we can ask the question: what happens between these two regions of the flow, where flow features can be scaled according to both the inner (ν/u_τ) and outer (h) scales? Is there an overlap between them?

Equilibrium/Overlap Layer

Let's consider the TKE balance for the channel flow, where we apply the fully-developed assumption to the steady TKE balance, and also assume that the viscous diffusion term, $\frac{\partial}{\partial y}\left(2\nu\overline{u_i s_{yi}}\right)$ is negligible, for simplicity.

In the jet analysis we showed that this viscous diffusion term scales as $\mathrm{Re}^{-1}\sqrt{L\delta} \ll 1$ and we assume that the Reynolds number in the channel is also large.

$$\underbrace{\frac{\partial}{\partial y}\left(\frac{1}{\rho}\overline{pv} + \frac{1}{2}\overline{q^2 v}\right)}_{\text{Transverse Transport}} = \underbrace{-\overline{uv}\frac{\partial \overline{u}}{\partial y}}_{\mathcal{P}} - \underbrace{\varepsilon}_{\text{Dissipation}} \tag{23}$$

We noted above that very near the wall, the total stress is approximately constant ($\tau \approx \tau_w$) and the Reynolds and viscous stresses switch magnitudes. Therefore, at the outer edge of the constant stress region, nearly all of the stress is associated with the Reynolds stress (i.e. $-\rho\overline{uv} \approx \tau_w$) and the Reynolds stress is also near its maximum. The gradient in the Reynolds stress is thus quite small, and we assume that gradient in other fluctuating quantities are similarly small in this region, so we can neglect the transverse transport term. We conclude that at the outer edge of the constant stress region, production and dissipation are in balance

$$-\overline{uv}\frac{\partial \overline{u}}{\partial y} = \varepsilon \tag{24}$$

and we refer to this as the equilibrium layer, i.e. the layer of fluid at the outer edge of the constant stress region, or equivalently, at the outer edge of the viscous inner region, in which production and dissipation are in balance.

Throughout the constant stress region, we expect the flow to be described in terms of the parameters relevant to production and dissipation: ν and τ_w (or equivalently, u_τ), along with the local position of the layer itself, y. We can then write the components of the energy balance in non-dimensional form as

We include an unknown constant κ for the velocity gradient and dissipation; for the Reynolds stress we know that this constant is equal to one at the outer edge of the constant stress region

$$\frac{y}{u_\tau}\frac{\partial \overline{u}}{\partial y} = \frac{1}{\kappa}f\left(y^+\right) \to \frac{1}{\kappa} \tag{25}$$

$$\frac{\overline{uv}}{u_\tau^2} = g\left(y^+\right) \to 1 \tag{26}$$

$$\frac{y}{u_\tau^3}\varepsilon = \frac{1}{\kappa}h\left(y^+\right) \to \frac{1}{\kappa} \tag{27}$$

where κ is some arbitrary proportionality constant. Inside the equilibrium layer, we are at the outer edge of the constant stress region, where the flow no longer depends on viscosity, ν, although the Reynolds stress is approximately equal to the wall shear stress. In this asymptotic limit, as viscosity is negligible ($y^+ \gg 1$), then the functions f, g, and h should approach a constant value, and we conclude that, for the mean velocity profile

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_\tau}{\kappa}\frac{1}{y} \tag{28}$$

or, in other words, that in the equilibrium layer, at the outer edge of the constant stress region, the mean velocity profile of wall-bounded turbulence varies logarithmically with wall-normal distance. This assumes that (1) we are in the constant stress region ($y/h \ll 1$); (2) that the Reynolds number is asymptocially large such that the law of the wall applies (Re $\gg 1$); and (3) we are simultaneously far enough from the wall itself, such that viscosity is not important and the total stress is nearly all Reynolds stress ($y^+ \gg 1$). We therefore see that the equilibrium layer, where production and dissipation balance, can be called a 'log layer' because this balance at sufficiently high Reynolds number implies a logarithmic variation of the mean velocity. Integrating the velocity gradient yields

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa}\log y^+ + B_i \tag{29}$$

where κ is referred to as the Karman constant and B_i is an unknown integrating constant. This derivation for the 'log-law' was based on the constant stress layer and the TKE balance of $\mathcal{P}$ and ε.

There are many other ways of obtaining the same prediction for the velocity profile; a popular approach is that of Millikan (1938) and is know as the overlap argument. Recall that, in the limit of very large Reynolds number, we can write

$$\frac{\bar{u}}{u_\tau} = f_i(y^+) \tag{30}$$

$$\frac{\bar{u}_0 - \bar{u}}{u_\tau} = f_o\left(\frac{y}{h}\right) \tag{31}$$

Previously we are argued that the asymptotic form of the TKE balance in the inner region implied a log law. We implicitly made use of the fact that the only relevant length scale in the equilibrium region was y itself, not the inner or outer scaled y. Now, let's make that assumption more explicit and assume that there is some region of flow in which both the inner scalinng ν/u_τ and the outer scaling h are valid. In this case, we expect that for some values of y obeying $\nu/u_\tau \ll y \ll h$, that the mean velocity values predicted by the law of the wall and the defect law should match. Not only the mean velocity values, but also that the mean velocity gradients should match.

The reason for choosing to match the gradient is somewhat subtle; it is the result of formal asymptotic matching and a missing generation of asymptotic terms. We can think of the velocity gradient as representing a momentum flux, in which case we are really requiring matched momentum flux in both scalings.

$$\left.\frac{\partial \bar{u}}{\partial y}\right|_{y^+ \gg 1} = \left.\frac{\partial \bar{u}}{\partial y}\right|_{y/h \ll 1} \tag{32}$$

$$\frac{\partial \frac{y}{h}}{\partial y}\frac{d}{d\frac{y}{h}}\left[-u_\tau f_o\left(\frac{y}{h}\right) + \bar{u}_0\right] = \frac{\partial y^+}{\partial y}\frac{d}{dy^+}\left[u_\tau f_i(y^+)\right] \tag{33}$$

$$-\frac{u_\tau}{h} f_o' = \frac{u_\tau^2}{\nu} f_i' \tag{34}$$

$$-\left(\frac{y}{h}\right) f_o' = \left(\frac{u_\tau y}{\nu}\right) f_i' = \frac{1}{\kappa} \tag{35}$$

and since both sides are functions of y/h and y^+ independently, they much each equal some constant. We conclude that the condition for overlap between the inner and outer scalings is

$$\frac{df_i}{dy^+} = \frac{1}{\kappa y^+} \quad \text{and} \quad \frac{df_o}{d\frac{y}{h}} = \frac{1}{\kappa \frac{y}{h}} \tag{36}$$

or, in other words, logarithmic velocity scaling

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa}\log y^+ + B_i \quad \text{and} \quad \frac{\bar{u}}{u_\tau} = -\frac{1}{\kappa}\log\frac{y}{h} + B_o \tag{37}$$

It is important to reiterate that all of the same assumptions apply to this overlap derivation: we assumed that Reynolds number is sufficiently large such that separate inner and outer scaling can be used, justifying the law of the wall and defect law formulations. At lower Reynolds number, these assumptions are not correct and thus we do not expect a log layer for all wall-bounded turbulent flows. What is the form of the velocity profile if we relax the Reynolds number asymptotic assumption?

Power-Law Mean Velocity Scaling

Before any log law was ever derived, Prandtl already argued that the mean velocity profile in wall-bounded turbulence could be represented with reasonable accuracy by

$$\frac{\bar{u}}{u_\tau} = 8.7(y^+)^{1/7} \tag{38}$$

at least for $\mathrm{Re} < 10^5$. In other words, Prandtl thought that the best representation for the mean profile was not a log law, rather a power law.

Let's include the Reynolds number dependence explicitly in our earlier derivations, by relaxing our assumption of a law of the wall that is a function of y^+ only. In the most general form, we expect the non-dimensional velocity gradient to depend on the Reynolds number

$$\frac{y}{u_\tau}\frac{\partial \bar{u}}{\partial y} = \Phi(y^+, \mathrm{Re}_\tau) \tag{39}$$

In both the equilibrium argument and the overlap argument, we essentially assumed that $\Phi(y^+, \mathrm{Re}_\tau)$ obtains the asymptotic value of

$1/\kappa$ in the limit as $\mathrm{Re}_\tau \to \infty$ and $y^+ \gg 1$, in which case we obtain a log law. But what if Φ continues to depend on Re_τ even for very large Reynolds number? For simplicity, let's assume that Φ takes the form of a power law

$$\Phi(y^+, \mathrm{Re}_\tau) = A(\mathrm{Re}_\tau)\left[y^+\right]^{\alpha(\mathrm{Re}_\tau)} \tag{40}$$

where functions $A(\mathrm{Re}_\tau)$ and $\alpha(\mathrm{Re}_\tau)$ both depend on Reynolds number. Now, let's assume that these functions can be written as perturbations away from an infinite Reynolds number case

i.e. we cannot choose a simple inverse relation $\delta_1(\mathrm{Re}_\tau) = \mathrm{Re}_\tau^{-1}$ because the Reynolds number dependence would be negligible for most typical Reynolds numbers and our goal here is to include Reynolds number dependence

$$A(\mathrm{Re}_\tau) = A_\infty + A_1\delta_1(\mathrm{Re}_\tau) + \cdots \tag{41}$$
$$\alpha(\mathrm{Re}_\tau) = \alpha_\infty + \alpha_1\delta_1(\mathrm{Re}_\tau) + \cdots \tag{42}$$

where $\delta_1(\mathrm{Re}_\tau)$ is a gauge function of the asymptotic series and varies inversely with Reynolds number, but in such a way that second-order terms are not negligible at laboratory Reynolds numbers. Then we can write

$$\Phi(y^+, \mathrm{Re}_\tau) = \left[A_\infty + A_1\delta_1(\mathrm{Re}_\tau)\right]\left[y^+\right]^{\alpha_\infty + \alpha_1\delta_1(\mathrm{Re}_\tau)} \tag{43}$$
$$= \left[A_\infty + A_1\delta_1(\mathrm{Re}_\tau)\right]\exp\left[\alpha_\infty \log y^+ + \alpha_1\delta_1(\mathrm{Re}_\tau)\log y^+\right] \tag{44}$$

As $\mathrm{Re}_\tau \to \infty$ or equivalently, as $y^+ \to \infty$, the exponential should be defined, which means that $\alpha_\infty = 0$, or else the log term blows up. To prevent the second log term in the exponential from blowing up, we require the gauge function δ_1 to take the form

$$\delta_1(\mathrm{Re}_\tau) = \frac{1}{\log \mathrm{Re}_\tau} \tag{45}$$

which satisfies our earlier requirement that second order terms are not neglible even at large Reynolds number. Then we obtain

$\frac{1}{\log \mathrm{Re}_\tau}$ decays exponentially more slowly than $\frac{1}{\mathrm{Re}_\tau}$ with increasing Reynolds number

$$\Phi(y^+, \mathrm{Re}_\tau) = \left[A_\infty + \frac{A_1}{\log \mathrm{Re}_\tau}\right]\left(y^+\right)^{\frac{\alpha_1}{\log \mathrm{Re}_\tau}} \tag{46}$$

Now, we can plug that back in to our definition of the mean velocity gradient and integrate

$$\frac{\partial \bar{u}}{\partial y} = \frac{u_\tau}{y}\left[A_\infty + \frac{A_1}{\log \mathrm{Re}_\tau}\right]\left(y^+\right)^{\frac{\alpha_1}{\log \mathrm{Re}_\tau}} \tag{47}$$
$$\frac{\partial \bar{u}}{\partial y^+} = \frac{u_\tau}{y^+}\left[A_\infty + \frac{A_1}{\log \mathrm{Re}_\tau}\right]\left(y^+\right)^{\frac{\alpha_1}{\log \mathrm{Re}_\tau}} \tag{48}$$
$$\frac{\partial \bar{u}}{\partial y^+} = u_\tau\left[A_\infty + \frac{A_1}{\log \mathrm{Re}_\tau}\right]\left(y^+\right)^{\frac{\alpha_1}{\log \mathrm{Re}_\tau}-1} \tag{49}$$
$$\bar{u} = u_\tau\left[A_\infty + \frac{A_1}{\log \mathrm{Re}_\tau}\right]\left(y^+\right)^{\frac{\alpha_1}{\log \mathrm{Re}_\tau}}\frac{\log \mathrm{Re}_\tau}{\alpha_1} \tag{50}$$
$$\frac{\bar{u}}{u_\tau} = \left[C_\infty \log \mathrm{Re}_\tau + C_1\right]\left(y^+\right)^{\frac{\alpha_1}{\log \mathrm{Re}_\tau}} \tag{51}$$

See the papers of McKeon, et. al. (2004) using the Princeton Superpipe, the responses of Barenblatt, and the review by W. K. George 2007 for an example of this controversy based on very high Reynolds number laboratory experiments.

and we find a continued dependence on Reynolds number. Baren-

blatt (1997) also showed that the envelope of the different power law velocity curves approximates the log-law curve, even though the individual curves retain their Reynolds number dependence. Based on the assumptions involved in our derivations, we expect to see log-law behavior at only extreme Reynolds numbers and power-law behavior for large Reynolds.

Spectral Similarity Theory

We have described wall-bounded turbulence in physical space, by means of momentum and energy analysis. We know apply our spectral techniques to describe wall-bounded flows also. In particular, we seek to decribe the spectral picture of turbulence within the equilibrium layer (i.e. the log-layer) of flow.

We begin with a review of one-dimensional spectra, which are the most practical spectral functions from an experimental point of view, and thus will form the basis of our analysis. Recall the general spectral transform pair

$$\Phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{-i(\vec{k}\cdot\vec{r})} Q_{ij}(\vec{r}) d^3\vec{r} \tag{52}$$

$$Q_{ij}(\vec{r}) = \int_{-\infty}^{\infty} e^{i(\vec{k}\cdot\vec{r})} \Phi_{ij}(\vec{k}) d^3\vec{k} \tag{53}$$

and evaluated at $r = 0$ we obtain

$$Q_{ij}(0) = \overline{u_i u_j} = \int_{-\infty}^{\infty} \Phi_{ij}(\vec{k}) d^3\vec{k} \tag{54}$$

In the general definition, both $\vec{r} = (r_1, r_2, r_3)$ and $\vec{k} = (k_1, k_2, k_3)$ are vectors in three-dimensional space. But we can also definite one-dimensional spectra, in which we define the distance between our two-points A and A' by a single scalar component of the vector, r_1 and k_1. And for convenience, we can choose the direction of that component as the longitudinal direction, i.e. $r_L = r_1$. Then we recall our definitions for the longitudinal and transverse two-point correlation function, where we choose x_2 as the transverse direction.

The normalization coefficients, u_1^2 and u_2^2, are distinct because we are not assuming isotropy in the case of wall-bounded flows. However, we continue to assume that the statistics are homogeneous in $\vec{x}$.

$$u_1^2 f(r) = \overline{u_1(\vec{x}) u_1(\vec{x} + r_1)} \tag{55}$$

$$u_2^2 g(r) = \overline{u_2(\vec{x}) u_2(\vec{x} + r_1)} \tag{56}$$

For this one-dimensional correlation, we can write the transform into spectral space as

$$\phi_{11}(k_1) = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-ik_1 r_1} \overline{u_1(\vec{x}) u_1(\vec{x}+r_1)} dr_1 \tag{57}$$

$$= \frac{1}{(2\pi)} \int_{-\infty}^{\infty} e^{-ik_1 r_1} u_1^2 f(r_1) dr_1 \tag{58}$$

and given that $f(r_1)$ is an even function, we can simplify this as

$e^{-ik_1 r_1} = \cos(k_1 r_1) - i \sin(k_1 r_1)$ and retain only the even contribution in the integral

$$\phi_{11}(k_1) = \frac{1}{(\pi)} \int_{0}^{\infty} u_1^2 f(r_1) \cos(k_1 r_1) dr_1 \tag{59}$$

$$\phi_{22}(k_1) = \frac{1}{(\pi)} \int_{0}^{\infty} u_2^2 g(r_1) \cos(k_1 r_1) dr_1 \tag{60}$$

where we similarly wrote the transverse spectral density, ϕ_{22}. We can then write the inverse transforms in one-dimension as

$$\overline{u_1(\vec{x}) u_1(\vec{x}+r_1)} = \int_{-\infty}^{\infty} e^{ik_1 r_1} \phi_{11}(k_1) dk_1 \tag{61}$$

and taking $r = 0$ we obtain

More generally, we have $\overline{u_i u_j} = 2\int_0^\infty \phi_{ij}(k_1) dk_1$

$\int_{-\infty}^{\infty} \phi_{11}(k_1) dk_1 = \int_{-\infty}^{\infty} \phi_{11}(k_1 h) d(hk_1)$ by a simple change of variable, and thus $\frac{\phi_{11}(k_1)}{h} = \phi_{11}(k_1 h)$

$$\overline{u_1 u_1} = \int_{-\infty}^{\infty} \phi_{11}(k_1) dk_1 \tag{62}$$

$$\overline{u_2 u_2} = \int_{-\infty}^{\infty} \phi_{22}(k_1) dk_1 \tag{63}$$

We can now use these one-dimensional spectral densities in order to describe the turbulence near a wall, in particular in the physical vicinity of the log layer. We are interested in finding the form of the streamwise spectral density $\phi_{11}(k_1)$. If the Reynolds number is sufficiently high that we have scale separation, then we expect this spectral density to obtain a universal, non-dimensional form in the dissipative region of the spectrum, in the form $\frac{\phi_{11}(k_1\eta)}{v^2}$, and a universal, non-dimensional form in the energetic, large-scale region of the spectrum (according to Townsend's hypothesis) in the form $\frac{\phi_{11}(k_1 h)}{u_\tau^2}$. These scaling should hold in every spatial location, if the Reynolds number is sufficiently high. However, we also showed that there is a particular spatial location in wall-bounded turbulence called the equilibrium or log layer, in which the appropriate lengthscale is y. We showed by definition that $y \ll h$, and we expect that $y \gg \eta$, because the dissipative scales are the smallest scales in the flow. Therefore we can explore the possibility that we have spectral overlap as well as physical overlap, i.e. that a particular form of the spectral density can be used to describe both dissipative scales and log-layer scales; or both energetic scales and log-layer scales.

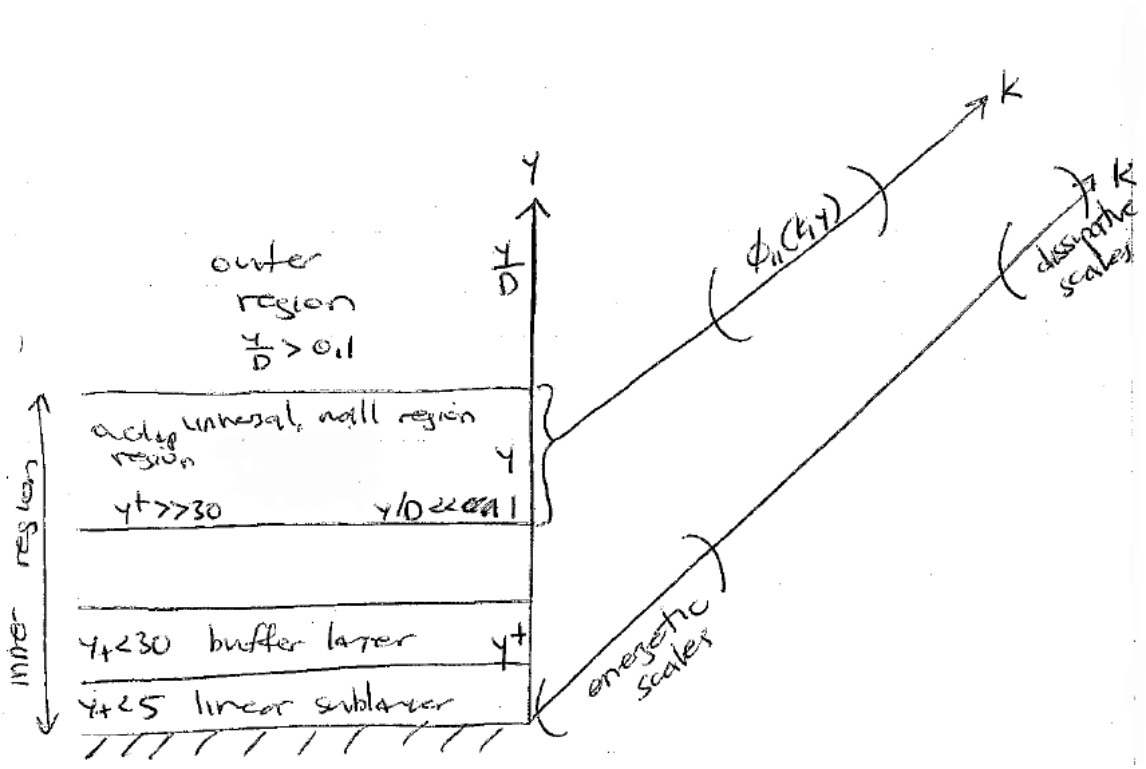


Figure 31: Wall-bounded turbulence near the wall, projected into spatial coordinates (y) and spectral coordinates (k). We examine the shape of the spectrum as measured in and around the log-layer, considering the spectral overlap between three possible length-scales for the wavenumber: the energetic scale h, the dissipative scale η, and the log-layer scale y.

Let's start by writing the outer-scaled spectral density in a universal (non-dimensional) form as

$$\frac{\phi_{11}(k_1 h)}{u_\tau^2} = g_1(k_1 h) = \frac{\phi_{11}(k_1)}{h u_\tau^2} \tag{64}$$

In a similar way, we can write the form of the spectral density for the log-layer, where the only length scale is y and thus

$$\frac{\phi_{11}(k_1 y)}{u_\tau^2} = g_2(k_1 y) = \frac{\phi_{11}(k_1)}{y u_\tau^2} \tag{65}$$

Now, let's consider the possibility of spectral overlap between the energetic region and the log layer, such that

$$\phi_{11}(k_1)|_{k_1 \sim \frac{1}{h}} = \phi_{11}(k_1)|_{k_1 \sim \frac{1}{y}} \tag{66}$$

$$g_1(k_1 h) h = g_2(k_1 y) y \tag{67}$$

For this overlap to occur, the two non-dimensional functions g_1 and g_2 must be defined as inverse functions of the form

$$g_1(k_1 h) = \frac{A_1}{k_1 h} \qquad g_2(k_1 y) = \frac{A_1}{k_1 y} \tag{68}$$

Imagine writing $g_1(k_1 h) k_1 h = g_2(k_1 y) k_1 y$ such that each side is a function of a single variable and thus both sides must be identically equal to some constant, A.

In other words, the presence of a log layer implies a k_1^{-1} range of the spectral density function. This k_1^{-1} is expected to be apparent in the spectrum when the spectrum is measured physically in the log layer, and iis expected to appear spectrally as an overlap between energetic and log-layer scale motions. So in addition to our predictions for the spectral shape in isotropic turbulence at the small

scales, we also have a prediction for wall-bounded turbulence in the presence of a log layer.

Now, let's consider the other end of the spectrum, at the dissipative scales. We can write the spectral density of wall-bounded turbulence in the dissipative range as

Recall the Kolmogorov dissipative scalings as $\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$ and $v = (\nu\varepsilon)^{1/4}$

$$\frac{\phi_{11}(k_1\eta)}{v^2} = g_3(k_1\eta) = \frac{\phi_{11}(k_1)}{\eta v^2} \tag{69}$$

If we consider the possibility of a spectral overlap between this dissipative region and the log region, we can rewrite the dissipative spectra using a number of facts about log-layer behavior that we already derived. In the log layer, we argued that production and dissipation are in balance,

$$\mathcal{P} = -\overline{u}\overline{v}\frac{\partial \overline{u}}{\partial y} \approx \varepsilon \tag{70}$$

and the mean velocity profile obeys a log law such that $\frac{\partial \overline{u}}{\partial y} = \frac{1}{\kappa}\frac{u_\tau}{y}$. Finally, at the outer edge of the constant stress region, we have $-\overline{u}\overline{v} \approx \frac{\tau_w}{\rho}$. Combining all of these results in the equilibrium layer yields

$$\mathcal{P} = \frac{u_\tau^3}{y\kappa} \approx \varepsilon \tag{71}$$

from which we conclude that the dissipative scales can be written in the equilibrium layer as

$$\eta = \left(\frac{\nu^3\kappa y}{u_\tau^3}\right)^{1/4} \qquad v = \left(\frac{\nu u_\tau^3}{y\kappa}\right)^{1/4} \tag{72}$$

Then finally, substituting into our spectral definition

$$\frac{\phi_{11}(k_1\eta)}{v^2} = g_3(k_1\eta) = \frac{\phi_{11}(k_1)}{\left(\frac{\nu^3\kappa y}{u_\tau^3}\right)^{1/4}\left(\frac{\nu u_\tau^3}{y\kappa}\right)^{1/2}} \tag{73}$$

$$= \frac{\phi_{11}(k_1)}{\left(\frac{\nu^5 u_\tau^5}{y\kappa}\right)^{1/4}} \tag{74}$$

We notice that $g_3(k_1\eta)$ contains an explicit dependence on viscosity, ν, whereas our log-layer spectrum $g_2(k_1 y)$ does not. Therefore, the effect of viscosity must enter the dissipative spectra through the independent variable, $k_1\eta$. We can use this fact to propose a functional form for $g_3(k_1\eta)$ that is compatible with spectral overlap. Assume that $g_3(k_1\eta)$ takes the form of a power law, such that the viscosity appears only in the independent variable

$$g_3(k_1\eta) = A(k_1\eta)^\beta = Ak_1^\beta\left(\frac{\nu^3\kappa y}{u_\tau^3}\right)^{\beta/4} = \frac{\phi_{11}(k_1)}{\left(\frac{\nu^5 u_\tau^5}{y\kappa}\right)^{1/4}} \tag{75}$$

In order to match the dependence on ν, we must assign $\beta = -\frac{5}{3}$ in which case

Note that we have inertial-subrange behavior in the spectra, because implicit in the log-layer scaling is the assumption of infinite Reynolds number, which, when combined with dissipative i.e. small scale motions yields the inertial subrange of Kolmogorov.

$$g_3(k_1\eta) = A_3(k_1\eta)^{-5/3} \tag{76}$$

Because this represents an overlap spectral density, we can write it in terms of $g_2(k_1y)$ as well, by writing the formal overlap argument

$$\phi_{11}(k_1)|_{k_1\sim\frac{1}{\eta}} = \phi_{11}(k_1)|_{k_1\sim\frac{1}{y}} \tag{77}$$

$$g_3(k_1\eta)\eta v^2 = g_2(k_1y)yu_\tau^2 \tag{78}$$

and plugging in for our power-law approximation of the dissipative spectrum, to obtain

$$g_2(k_1y) = \frac{A_3}{\kappa^{2/3}}(k_1y)^{-5/3} \tag{79}$$

Therefore, we expect a $-5/3$ spectral scaling in wall bounded turbulence at sufficient Reynolds numbers, when we physically measure the spectrum in the vicinity of the log law. Spectrally, the overlap occurs between log-layer sized scales and the dissipative scales. We can sketch these different regions within the streamwise spectral density for wall-bounded turbulence to illustrate the overlap arguments in spectral space, where we denote the different transitions between energetic, log, and dissipative scaling as F, P, N, and M.

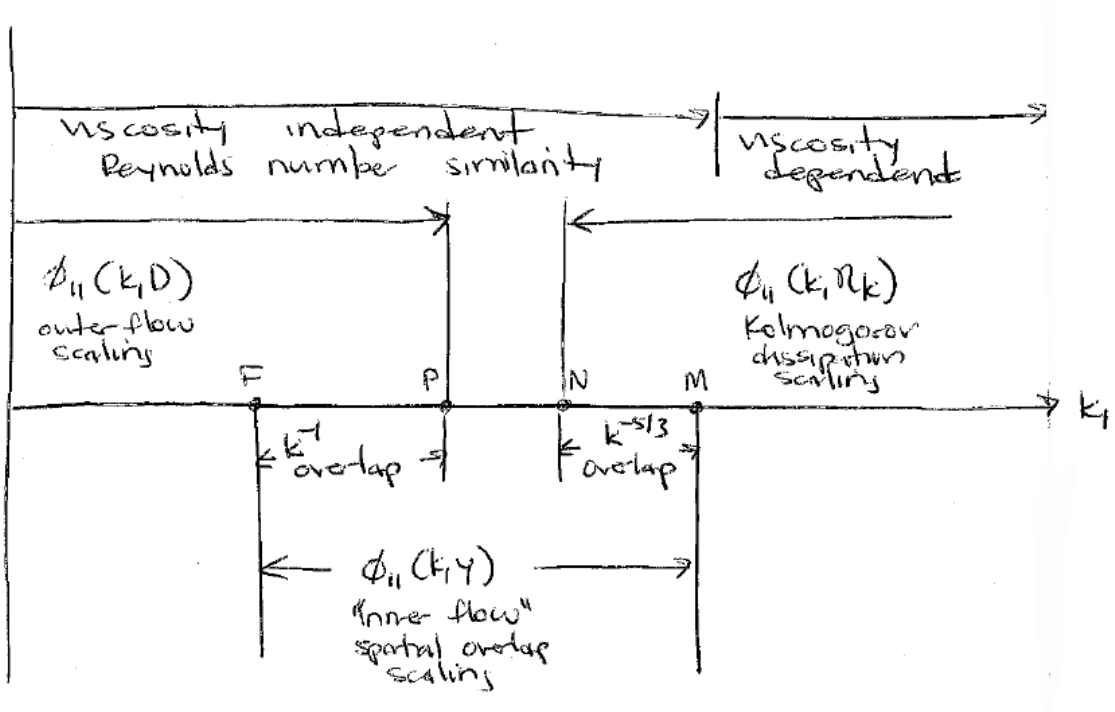


Figure 32: Spectral overlap for streamwise spectral density functions measured in the physical vicinity of the log-layer in wall-bounded flow. Points $P = k_1y$, $N = k_1y$ and $M = k_1\eta$ (or $k_1y = M\kappa^{-1/4}(y^+)$) are assumed to be universal constants representing the

Now we can apply the results of these overlap arguments to the actual streamwise, one-dimensional energy spectral density functions, which we can sketch using log-layer units (i.e. wavenumbers scaled as k_1y) or outer, energetic units (i.e. wavenumbers scaled as k_1h).

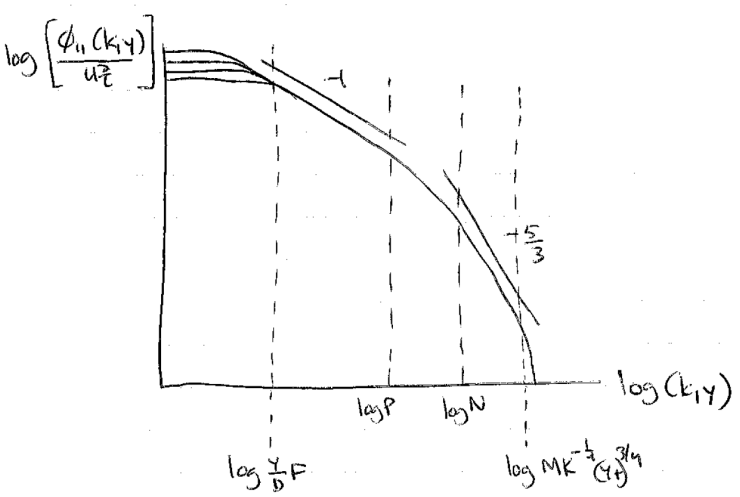


Figure 33: Spectral density functions for wall-bounded flows at different Reynolds number, plotted using log-layer scaling ($k_1 y$). The small scale motions exhibit universal behavior, but the large-scale, energetic motions do not, and thus the spectra do not collapse for low wavenumber in the $k_1 y$-scaling.

We expect the log-layer scaling to represent universal trends among high wavenumber, small-scale motions, according to Kolmogorov theory. Using outer, energetic scaling, we expect universal trends for large-scale, low-wavenumber structures according to Townsend's similarity theory.

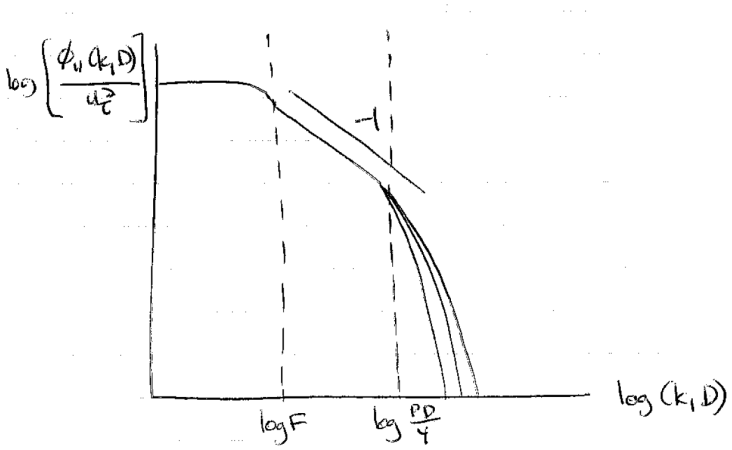


Figure 34: Spectral density functions for wall-bounded flows at different Reynolds number, plotted using energetic/outer scaling ($k_1 h$). Now, the large scale motions exhibit universal behavior, but the small-scale, dissipative motions do not, and thus the spectra do not collapse for high wavenumber in the $k_1 h$-scaling.

Recall that the integral of the spectral energy density over all wavenumbers yields the broadband turbulence intensity for the streamwise velocity component, $\overline{u_1 u_1}$, according to

$$\overline{u_1 u_1} = \int_{-\infty}^{\infty} \phi_{11}(k_1) dk_1 \tag{80}$$

so let's try to integrate over the full spectrum using the overal arguments above. We can write the total energy content as the sum of integrals

Spectral Range	Functional Form	Scaling	Functional Form
$0 < k_1 h < F$	$g_1(k_1 h)$	h	Constant
$\frac{y}{h}F < k_1 y < P$	$g_2(k_1 y)$	h-y Overlap	$\frac{A_1}{k_1 y}$
$P < k_1 y < N$	$g_2(k_1 y)$	y	Constant
$N < k_1 y < M\kappa^{-1/4}(y^+)^{3/4}$	$g_2(k_1 y)$	y-η Overlap	$\frac{A_3}{\kappa^{2/3}}(k_1 y)^{-5/3}$
$M < k_1 \eta < \infty$	$g_3(k_1 \eta)$	η	Decaying

Table 10: The functional descriptions of streamwise, one-dimensional spectral density for wall-bounded turbulence, using the overlap arguments of Perry.

$$\frac{1}{2}\frac{\overline{u_1^2}}{u_\tau^2} = \overbrace{\int_0^F g_1(k_1 h)d(k_1 h)}^{\text{Large-scale constant, } B_1} + \overbrace{\int_{\frac{y}{h}F}^P g_2(k_1 y)d(k_1 y)}^{B_2 - B_3 \log \frac{y}{h}} + \overbrace{\int_P^N g_2(k_1 y)d(k_1 y)}^{\text{Universal Constant, } B_6} + \underbrace{\int_N^{M\kappa^{-1/4}(y^+)^{3/4}} g_2(k_1 y)d(k_1 y)}_{B_4 - B_5(y^+)^{-1/2}} + \underbrace{\frac{v^2}{u_\tau^2}\int_M^\infty g_3(k_1 \eta)d(k_1 \eta)}_{\text{Negligible}} \quad (81)$$

and then evaluate the different integrals using the results from above, along with other physical arguments about the behavior of the spectra. The first integral is expected to be roughly constant, based on the spectral sketches above, and this constant, B_1 is characteristic of the large scale motions in the flow. The last integral is assumed to contribute negligibly to the energy in the spectral density. The second integral represents the first overlap region, so we can use the predicted functional form to evaluate the integral explicitly as

Recall that the energy decay at high wavenumbers is quite abrupt, and the sketches above are logarithmic.

$$\int_{\frac{y}{h}F}^P \left[\frac{A_1}{k_1 y}\right] d(k_1 y) = A_1 \log(k_1 y)\Big|_{\frac{y}{h}F}^P = B_2 - B_3 \log\frac{y}{h} \quad (82)$$

and we see that the k^{-1} scaling of wavenumber in the log region implies a logarithmic variation in the total energy content there, in analogy to the logarithmic variation in the mean velocity profile we found earlier. We can also use our overlap results to evaluate the fourth integral as

$$\int_N^{M\kappa^{-1/4}(y^+)^{3/4}} \left[\frac{A_3}{\kappa^{2/3}}(k_1 y)^{-5/3}\right] d(k_1 y) = -\frac{3}{2}\frac{A_3}{\kappa^{2/3}}(k_1 y)^{-2/3}\Big|_N^{M\kappa^{-1/4}(y^+)^{3/4}} = B_4 - B_5(y^+)^{-1/2} \quad (83)$$

The remaining integral within the log-scaling is assumed to describe a universal constant, B_6, for simplicitly, although it is somewhat difficult to justify this rigorously. Combining the above results we obtain the form of the total energy content within the physical log-layer of the wall

Note that we assume $y^+ \gg 1$ within the log law and thus the final term is a negligible correction to the logarithmic behavior

$$\frac{\overline{u_1^2}}{u_\tau^2} = C_1 - C_2 \log\left(\frac{y}{h}\right) - \frac{C_3}{\sqrt{y^+}} \quad (84)$$

which takes the form of a log-law for the intensity of turbulence fluctuations. This spectral similarity theory was developed by Perry

and coworkers and provides both a powerful prediction for spectral behavior in wall-bounded flows, but also insights into the physical structure of wall-bounded turbulence, as we will see in the coming chapters.

Perry, et. al. JFM 1986 and subsequent papers

Boundary Layers

We will continue the discussion of wall-bounded turbulence for the case of an external flow, the boundary layer. Like we did for the turbulent jet, first, we review the mathematical description of the laminar boundary layer before turning to the turbulent case.

Laminar Boundary Layer

For the laminar boundary layer, we adopt the same assumptions as we employed for the laminar jet, namely very high Reynolds number (albeit laminar flow) and a corresponding small ratio of the width of the layer of rotational flow compared to the relevant streamwise lengthscales of the problem, $\delta/L \ll 1$. For jet, we defined a characteristic streamwise velocity as $u_s(x)$ which we chose to represent the centerline velocity of the jet; for the boundary layer, $u_s(x)$ can be taken as the irrotational, free-stream velocity far from the wall. Thus the dimensional analysis will be exactly the same, resulting in the same leading order momentum balance.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2}$$

The y-momentum balance indicates that the pressure gradient is uniform across the boundary layer, in which case we conclude

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{3}$$

or, in other words, that the pressure gradient within the boundary layer is the same as the pressure gradient driving the irrotational flow far away from the wall. For simplicity, we assume that the irrotational flow is uniform, and that there is no free stream pressure gradient , in which case we have

This might be achieved by the case where a flat plate is moving in a quiescent medium, such that we have a uniform flow without a pressure gradient. In a wind tunnel, the tunnel geometry can be used to 'cancel' the pressure gradient driving the flow over a flat plate.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{4}$$

In order to solve this momentum balance while obeying the continuity equation, we introduce a streamfunction, $\psi(x, y)$ and seek a

similarity solution of the form

$$\psi(x,y) = u_s(x)\delta(x)f(\eta) \qquad \eta = \frac{y}{\delta(x)} \tag{5}$$

which yields, upon substitution, the same governing equation we obtained for the laminar jet

$$-\left[\frac{1}{2}(u_s\delta^2)'\right]ff'' + \left[u_s'\delta^2\right]\left(f'^2 - \frac{1}{2}ff''\right) = [\nu]f''' \tag{6}$$

Of course, if there is no pressure gradient in the irrotational flow, then $u_s(x) = u_s$ constant and we can simplify to obtain

The irrotational momentum balance for the outer flow of the boundary layer, far from the wall, is $u_s\frac{\partial u_s}{\partial x} + v_s\frac{\partial u_s}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx}$, so that if the irrotational flow is only streamwise, then $v_s = 0$, and in the absence of a pressure gradient then $\frac{du_s}{dx} = 0$. This corresponds to the earlier eigenvalue problem with $\lambda = \frac{u_s'}{u_s}x = 0$.

$$-\left[\frac{1}{2}(u_s\delta^2)'\right]ff'' = [\nu]f''' \tag{7}$$

The condition for a similarity solution is then very simply

$$\delta^2 \propto \frac{\nu x}{u_s} \tag{8}$$

or, the spreading rate of the laminar boundary layer is $\delta \propto x^{1/2}$ compared to the laminar jet, which was $\delta \propto x^{2/3}$. The governing, non-linear differential equation is then

Note that the two velocity components are defined as $u = u_s f'$ and $v = u_s\delta'(\eta f' - f)$

$$f''' + \frac{1}{2}ff'' = 0 \tag{9}$$

with boundary conditions defined as

Boundary Condition	Similarity Form
$u = 0$ at $y = 0$	$f'(0) = 0$
$v = 0$ at $y = 0$	$f(0) = 0$
$u = u_s$ as $y \to \infty$	$f'(\infty) = 1$

Table 11: Boundary conditions for the laminar boundary layer

Then the non-linear boundary value problem can be solved numerically to obtain a solution for this developing flow, known as the Blasius problem. The system can also be rewritten as an initial value problem by an appropriate change of variables. The solution cannot be written in closed form, but we can write an asymptotic approximation of the similarity function as

$$f(\eta) = \begin{cases} \frac{1}{2!}C_0\eta^2 - \frac{1}{2\cdot 5!}C_0^2\eta^5 + \cdots, & \text{if } \eta \ll 1. \\ \eta - \beta, & \text{if } \eta \gg 1. \end{cases} \tag{10}$$

where C_0 is the curvature of the function at the origin, $C_0 = f''(0)$ The numerical evaluation of $f''(0) \approx 0.33205733621519630$ and $\beta = \lim_{\eta\to\infty}[\eta - f(\eta)] \approx 1.720787657520503$.

The asymptotic solution was first developed by Blasius in 1908, the high precision calculation comes from Boyd, Experimental Mathematics, 1999. Boyd explains why this seemingly well-behaved, smooth function is so resistent to simple analytical representation.

For the laminar jet, we calculate the total momentum flux, J and also the mass flux, $\dot{m}$, and we determined that the momentum flux is constant along the axis of the jet, while the mass flux increases due to entrainment. For the laminar boundary layer, we expect a similar entrainment process at the outer edge, and thus as with the jet we expect $\frac{d\dot{m}}{dx} > 0$. Let's calculate the entrainment velocity at

the outer edge of the boundary layer. We start with the continuity equation for the irrotational flow and the boundary layer flow

$$\frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{11}$$

Subtracting the two continuity equations and integrating across the thickness of the boundary layer yields

$$v - v_s\Big|_0^\infty = \frac{d}{dx}\int_0^\infty (u_s - u)dy \tag{12}$$

Applying the boundary conditions of no penetration at the wall and one-dimensional flow in the free stream, we obtain

$$v(\infty) = \frac{d}{dx}\left[u_s \underbrace{\int_0^\infty \left(1 - \frac{u}{u_s}\right)dy}_{\delta^*}\right] \tag{13}$$

$$= \frac{d}{dx}[u_s\delta^*] \tag{14}$$

where the entrainment velocity $v(\infty)$ varies as a function of a characteristic length scale of the velocity profile, called the displacement thickness, δ^*. Because $\frac{d\delta^*}{dx} > 0$, the entrainment velocity is positive, meaning that the boundary layer is displacing the irrotational flow farther away from the wall, hence the name displacement thickness. Indeed, we can think about the displacement thickness as the perceived thickness of the wall, from the perspective of the irrotational flow. In other words, the amount of mass flux over the wall with zero thickness in the boundary layer is equivalent to the mass flux of an irrotational flow over a wall with thickness $\delta^*(x)$, which can be shown graphically by displacing the rotational mass flux near the wall to irrotational flux away from the wall a distance δ^*.

What about the momentum flux, J, across the boundary layer? In the jet, we showed that the momentum flux was constant in the axial direction: the decrease in velocity was balanced by the increase in mass, through entrainment. However, for the boundary layer, we expect a decrease in velocity due to frictional forces but no new mass, since the mass is displacement away from the wall, not entrained. Therefore, we expect a decrease in the momentum flux, as momentum is transferred from the flow to the wall, frictionally. Let's calculate this momentum flux to the wall by combining continuity and the momentum balance, like we did in the case of the jet.

Note that the laminar boundary layer 'entrainment' is opposite the case of the laminar jet, where $v(+\infty) < 0$. In both cases, the rotational flow region grows by diffusion of vorticity; in the jet, mass is entrained into the jet; in the boundary layer, mass is displaced away from the wall. Hence, for the boundary layer, the velocity $v(\infty)$ should be called the displacement velocity instead of the entrainment velocity.

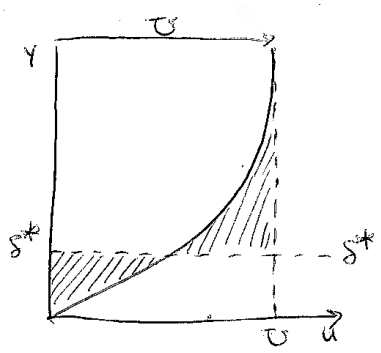


Figure 35: The displacement thickness illustrated graphically

Momentum Integral

We start with the momentum equation for the boundary layer and rewrite it into a more convenient form, by making a few substitu-

tions

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{15}$$

We write the momentum integral in its most general form, allowing for unsteadiness in the boundary layer and also for transpiration of fluid through the wall, i.e. violation of no penetration

We know that the pressure gradient is matched to the irrotational, outer flow according to

$$\frac{\partial u_s}{\partial t} + u_s\frac{\partial u_s}{\partial x} = -\frac{1}{\rho}\frac{dp_\infty}{dx} \tag{16}$$

so we can substiute to find

$$-\nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial t}(u_s - u) + u_s\frac{\partial u_s}{\partial x} - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \tag{17}$$

We can also premultiply the continuity equation by $(u_s - u)$ to find

$$0 = (u_s - u)\frac{\partial u}{\partial x} + (u_s - u)\frac{\partial v}{\partial y} \tag{18}$$

Then add momentum and continuity together to obtain

$$-\nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial t}(u_s - u) + \frac{\partial}{\partial x}(u_s u - u^2) + \frac{\partial u_s}{\partial x}(u_s - u) + \frac{\partial}{\partial y}v(u_s - u) \tag{19}$$

where we made use of the fact that $u_s = u_s(t, x)$ only and not y. Now we can integrate this more convenient form of the momentum equation over the entire flow field

$$-\nu\left.\frac{\partial u}{\partial y}\right|_0^\infty = \frac{\partial}{\partial t}\int_0^\infty (u_s - u)dy + \frac{\partial}{\partial x}\int_0^\infty (u_s u - u^2)dy + \frac{\partial u_s}{\partial x}\int_0^\infty (u_s - u)dy + v(u_s - u)|_0^\infty \tag{20}$$

and we know that $\left.\frac{\partial u}{\partial y}\right|^\infty = 0$ since the outer irrotational flow is uniform far from the wall, and $v(u_s - u)|^\infty = 0$ because $(u_s - u) = 0$ far away also. We can also give a name to the quanity $v(x, y = 0) \equiv v_0$, which is the transpiration at the wall, in the case where the wall is permeable and there is suction or blowing through the wall.

$$\overbrace{\nu\left.\frac{\partial u}{\partial y}\right|_0}^{\frac{\tau_w}{\rho}} = \frac{\partial}{\partial t}u_s\overbrace{\int_0^\infty\left(1 - \frac{u}{u_s}\right)dy}^{\delta^*} + \frac{\partial}{\partial x}u_s^2\overbrace{\int_0^\infty\frac{u}{u_s}\left(1 - \frac{u}{u_s}\right)dy}^{\theta} + \frac{\partial u_s}{\partial x}u_s\overbrace{\int_0^\infty\left(1 - \frac{u}{u_s}\right)dy}^{\delta^*} - v_0 u_s \tag{21}$$

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial t}[u_s\delta^*] + \frac{\partial}{\partial x}\left[u_s^2\theta\right] + \frac{\partial u_s}{\partial x}[u_s\delta^*] - v_0 u_s \tag{22}$$

We find that the integrated momentum equation represents the flux of momentum into the wall, τ_w, (per unit width of the boundary layer), as a function of two different characteristic lengthscales of the velocity profile, the displacement thickness, δ^*, and the so-called momentum thickness, θ. For steady flow and no transpiration at the wall, we obtain a simple relationship between the momentum flux at the wall and the momentum thickness

$$\frac{\tau_w}{\rho} = \frac{\partial}{\partial x}\left[u_s^2\theta\right] \tag{23}$$

Note that this momentum integral applies for laminar and turbulent boundary layers, because $\overline{u}\overline{v}$ is zero at both $y = 0$ (no-slip) and $y \to \infty$ (no turbulence in irrotational flow), so even if we included the additional Reynolds stress term, $\frac{\partial}{\partial y}\overline{u}\overline{v}$, it produces no net impact on the total momenum balance at the wall – although as we showed above, the presence of turbulence will affect the velocity profile and thus δ^* and θ

The interpretation of the momentum thickness, θ, is not as simple as the displacement thickness. Let's return to the displacement thickness for a moment and consider multiplying through by u_s to rewrite the thickness in terms of volumetric flowrates, Q (per unit width in the out-of-plane direction),

$$\underbrace{\int_0^\infty u dy}_{Q_{\text{rot}}} = \underbrace{\int_0^\infty u_s dy}_{Q_{\text{irrot}}} - \underbrace{u_s \delta^*}_{\text{Deficit}} \tag{24}$$

and we see that the actual amount of fluid going past the plate is equivalent to the amount of irrotational flow minus some deficit of fluid. In other words, we could represent the actual rotational flow over a plate (for instance) as if it were irrotational flow around a body of thickness δ^*, as we explained above. Now, let's expand the momentum thickness definition in this same format.

Of course, in this representation, that thicker body would need a vortex sheet at $y = \delta^*$ in order to maintain the no-slip conditions. It is easy to show that the centroid of vorticity in the actual flow is located precisely at δ^*, making this model work.

$$\theta = \int_0^\infty \frac{u}{u_s}\left(1 - \frac{u}{u_s}\right) dy \tag{25}$$

$$\underbrace{\int_0^\infty u^2 dy}_{uQ_{\text{rot}}} = \underbrace{u_s \int_0^\infty u dy}_{u_s Q_{\text{rot}}} - \underbrace{u_s (u_s \theta)}_{\text{Deficit}} \tag{26}$$

Here we see that instead of the volumetric flowrate, we have the volumetric advection rate. Advection is just the transport of some quantity – here the quantity is the volumetric flowrate, $u_s\theta$ and it is being advected at velocity u_s. When we multiply by ρ, of course we are talking about the momentum advection, just like before we were talking about the momentum. In the case of the volumetric flowrate, the presence of rotational flow reduced the overall amount of momentum in the flow. But here, we show that even when we incorporate the reduced total amount of momentum, Q_{rot}, there is still a further loss of momentum from the fact that the remaining momentum is not being advected at u_s, but rather there is a different distribution of advective velocities, u, doing the advecting in the rotational flow case. In other words, simply accounting for the loss of overall momentum (the total) does not adequately represent the momentum loss, because there is also momentum loss due to the distribution of momentum. We can see that there are, indeed, two factors reducing momentum, by incorporating δ^* inside the momentum thickness expression

$$\int_0^\infty u^2 dy = u_s \left[\int_0^\infty u_s dy - u_s \delta^* \right] - u_s^2 \theta \tag{27}$$

Regrouping, we find

$$\underbrace{\int_0^\infty u^2 dy}_{\text{Rotational}} = \underbrace{\int_0^\infty u_s^2 dy}_{\text{Irrotational}} - \underbrace{u_s^2(\delta^* + \theta)}_{\text{Deficit}} \tag{28}$$

In addition to δ^* and θ, we also encounter $\Delta = \int_0^\infty \frac{u_s - u}{u_\tau} dy$, which is called the Rotta thickness and is designed for use in the defect law formulation of the momentum balance.

In this representation, we are representing the total difference in momentum transport between the rotational and irrotational flows. That total difference includes two contributions, one from the fact that the volumetric flux is reduced by the presence of vorticity, as compared to an irrotational flow; and a second contribution from the fact that even within the reduced total amount of momentum, the momentum is distributed differently in the rotational flow, which further reduces the momentum.

Where does the rotation come from in the boundary layer? We showed earlier that, according to Kelvin's circulation theorem, any material volume of fluid that is initially irrotational will remain irrotational, unless vorticity diffuses into it. In the case of the boundary layer, before the boundary layer develops, none of the flow is rotational, hence there is no vorticity to diffuse anywhere, so how does the boundary layer flow become rotational?

Vorticity Generation

Let's begin by writing the boundary layer momentum equation in terms of vorticity.

Recall that in two dimensions, we can write the out-of-plane vorticity, $\omega_z = -\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} - \nu\frac{\partial \omega_z}{\partial y} \tag{29}$$

Then evaluate this momentum balance at the wall, assuming no transpiration

$$\frac{1}{\rho}\frac{dp_\infty}{dx} = -\nu\frac{\partial \omega_z}{\partial y}\bigg|_{y=0} \tag{30}$$

in which case we find that the diffusive flux of vorticity at the wall is driven by the presence of a pressure gradient (or, in the case of an accelerating wall, by $\frac{\partial u_w}{\partial t}$). Kelvin's theorem tells us that vorticity cannot be generated in a material volume of fluid, but vorticity can be generated at a wall or interface of fluid, due to an acceleration or net force. But, in the laminar boundary layer, we assumed the flow is steady, so there is no net acceleration at the wall, and we assumed no pressure gradient, in which case there is no vorticity flux at the wall. So where does the vorticity come from? It must be generated at the leading edge, at $x = 0$. The leading edge must have some finite thickness, and therefore must exhibit some pressure gradient. All of the vorticity in the entire laminar boundary layer is then generated at the leading edge, due to the pressure gradient there, and is then advected downstream. In the case of a turbulent

boundary layer, we have an additional contribution to the vorticity flux at the wall

$$\frac{\partial}{\partial y}\overline{uv}\bigg|_{y=0} + \frac{1}{\rho}\frac{dp_\infty}{dx} = -\nu\frac{\partial \omega_z}{\partial y}\bigg|_{y=0} \tag{31}$$

in which case we need to determine the flux of Reynolds stress at the wall. Let's write the fluctuating velocities in a Taylor series, where the coefficients are all zero-mean, stationary random variables.

$$u(x,y,t) = a_0(x,t) + a_1(x,t)y + a_2(x,t)y^2 + \cdots \tag{32}$$
$$v(x,y,t) = b_0(x,t) + b_1(x,t)y + b_2(x,t)y^2 + \cdots \tag{33}$$

At the wall, $y = 0$, no-slip requires that $a_0(x) = 0$ and similarly no penetration requires $b_0(x) = 0$. No slip also means that $\frac{\partial u}{\partial x}\Big|_{y=0} = 0$ which means, according to continuity, that $\frac{\partial v}{\partial y}\Big|_{y=0} = 0$, which demands that $b_1(x) = 0$, leaving us with

$$u(x,y) = a_1 y + a_2 y^2 + \cdots \tag{34}$$
$$v(x,y) = b_2 y^2 + \cdots \tag{35}$$

Then the Reynolds stresses are given by

$$\overline{u^2} = \overline{a_1^2} y^2 + \cdots \tag{36}$$
$$\overline{v^2} = \overline{b_2^2} y^4 + \cdots \tag{37}$$
$$\overline{uv} = \overline{a_1 b_2} y^3 + \cdots \tag{38}$$

and the flux of Reynolds stress at the wall is

$$\frac{\partial}{\partial y}\overline{uv}\bigg|_{y=0} = 0 \tag{39}$$

We see that there is no net vorticity production at the wall in turbulent flows, on average.

Of course, vorticity can be produced instantaneously, but the distribution of u and v fluctuations is often close to Gaussian (see Jimenez JFM 1998), in which case we expect the average of the vorticity $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ to be zero (instantaneous vorticity of positive and negative signs will cancel each other out).

With this review of the laminar boundary layer complete, we can now apply what we have already learned about wall-bounded turbulence to the problem of the turbulent boundary layer, in order to understand how it differs from the laminar boundary layer.

Turbulent Boundary Layer

Let's review what we already know about the turbulent boundary layer, based on the general theory of wall-bounded turbulence. We know how to describe the wall layer reasonably well, by using the law of the wall at sufficiently high Reynolds number. We obtained both the linear velocity variation very near the wall, as well as a logarithmic velocity variation at the outer edge of the constant

stress layer. We also showed that we can use inner and outer scaling to write the ensemble averaged statistics of a wall-bounded flow in universal form, both near the wall and far from the wall.

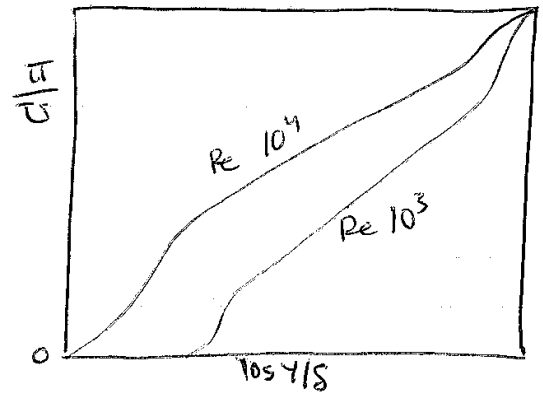


Figure 36: A sketch of the mean velocity profile for the turbulent boundary layer at two Reynolds numbers, plotted in outer scaling, using the boundary layer thickness, δ. Note that the two profiles appear entirely different, and, in particular, the slope at the wall representing the skin friction is very different at each Reynolds number.

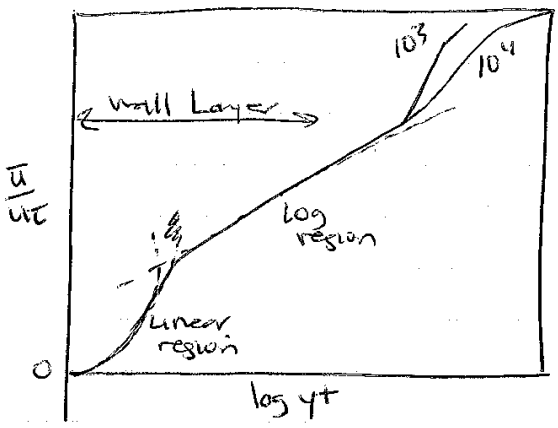


Figure 37: A sketch of the mean velocity profile for the turbulent boundary layer at two Reynolds numbers, plotted in inner scaling. Note the universal collapse of the mean behavior in the inner region, although far from the wall the two flows show different behavior still.

In the inner region, we derived a log law variation of the mean velocity profile in the form

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa} \log y^+ + B_i \tag{40}$$

and a linear sublayer very near the wall in the form,

$$\frac{\bar{u}}{u_\tau} = y^+ \tag{41}$$

Between these two regions we expect a rapid variation in the two forms of wall stress— viscous and turbulent — while the total stress is approximately constant. Therefore, the region between the linear layer and the log layer is very important, in terms of production of turbulence, and is referred to as the buffer layer. We can write a composite form of the velocity profile near the wall, combing the linear and log regions, by means of an asymptotic expansion . This expansion is valid out to approximately $y^+ \lesssim 100$

Developed by Spalding 1961

and can be written, implicitly, as

$$y^+ = u^+ + e^{-\kappa B_i}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{1}{2!}(\kappa u^+)^2 - \frac{1}{3!}(\kappa u^+)^3\right] \quad (42)$$

The problem then becomes: how can we analytically describe the outer region of the boundary layer, beyond where this approximation is valid?

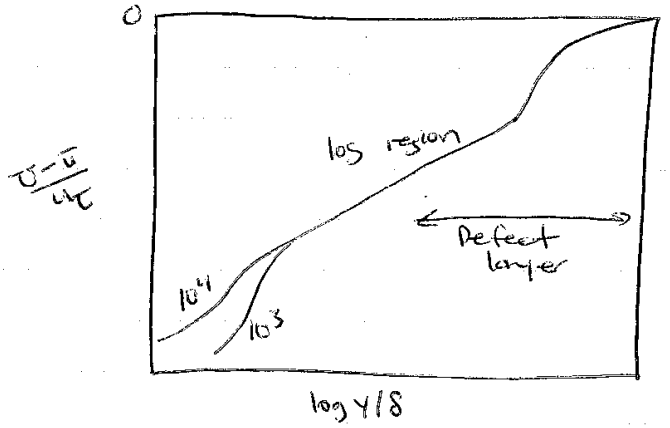


Figure 38: A sketch of the mean velocity profile for the turbulent boundary layer at two Reynolds numbers, plotted in defect scaling. Note the collapse of the mean behavior in the defect region, very far from the wall, although now the two flows show non-universal behavior in the wall layer.

We have two options: seek a similarity solution or use more empirical, fitting methods. Let's begin with the similarity solution, since we already have experience with that approach from the laminar boundary layer.

Outer Layer Similarity Solution

To begin, we need to develop the appropriate equations for the turbulent boundary layer. We can't employ the same equations from the channel flow, because there we assumed the flow was fully developed (i.e. no variation in x) whereas in the boundary layer, there is a slow variation in x. The closer analogy is therefore the problem of the turbulent jet. However, in the turbulent jet, we assumed that the Reynolds number was sufficiently high to neglect all of the viscous terms. Specifically, we assumed

$$\frac{1}{\mathrm{Re}} \ll \left(\frac{\delta}{L}\right)^2 \quad (43)$$

in order to ignore the leading order viscous term in the x-momentum equation, $\frac{\partial^2 \bar{u}}{\partial y^2}$. However, for the turbulent boundary layer, ignoring this leading order term is inconsistent with our law of the wall assumption, that the flow very near the wall depends explicitly on viscosity. In order to preserve this term (as we did in the channel flow) we must assume that

$$\frac{1}{\mathrm{Re}} \sim \left(\frac{\delta}{L}\right)^2 \quad (44)$$

Then we can write the leading order momentum equations as

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \frac{\partial}{\partial y}\overline{uv} = -\frac{1}{\rho}\frac{d\bar{p}_\infty}{dx} + \nu\frac{\partial^2 \bar{u}}{\partial y^2} + \underbrace{\left\{\frac{\partial}{\partial x}\left(\overline{v^2} - \overline{u^2}\right)\right\}}_{\mathcal{O}\left(\frac{\delta}{L}\right)} \tag{45}$$

$$\underbrace{\left\{\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + \frac{\partial}{\partial x}\overline{uv}\right\}}_{\mathcal{O}\left(\frac{\delta}{L}\right)^2} + \frac{\partial}{\partial y}\overline{v^2} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \underbrace{\left\{\nu\frac{\partial^2 \bar{v}}{\partial y^2}\right\}}_{\mathcal{O}\left(\frac{\delta}{L}\right)^2} \tag{46}$$

where we note that the second order terms are slightly different than the jet case, and we have included an explicit dependence on the pressure gradient in the outer, irrotational flow. We expect a self-similar solution in the defect scaling, far from the wall, in the limit of high Reynolds number. So we can define a similarity solution as

Technically, we could seek a solution with incomplete similarity of the form $f(\eta, x)$ and $g(\eta, x)$, allowing for an explicit x-dependence — see Schlichting's derivation in §7.2 and §18.2. However, we assume here perfect similarity.

$$\bar{u} - u_s = u_\tau(x) f(\eta) \qquad \eta = \frac{y}{\delta(x)} \tag{47}$$

$$\overline{uv} = u_\tau^2(x) g(\eta) \tag{48}$$

Now we rewrite the x-momentum equation to leading order in terms of the similarity variables, using the relation $u_s \frac{du_s}{dx} = -\frac{1}{\rho}\frac{d\bar{p}_\infty}{dx}$, to obtain

$$\overbrace{\underbrace{\left[u_\tau u_\tau'\right]}_{①} f^2 + \underbrace{\left[(u_\tau u_s)'\right]}_{②} f - \underbrace{\left[\frac{u_\tau}{\delta}(u_s\delta)'\right]}_{③} \eta f' - \underbrace{\left[\frac{u_\tau}{\delta}(u_\tau\delta)'\right]}_{④} f' \int_0^\eta f d\eta}^{Advective} + \overbrace{\underbrace{\left[\frac{u_\tau^2}{\delta}\right]}_{⑤} g'}^{Turbulent} = \overbrace{\underbrace{\left[\nu\frac{u_\tau}{\delta^2}\right]}_{⑥} f''}^{Viscous} \tag{49}$$

where we have six different dimensional quantities to analyze, unlike the case of the jet in which we had only three. Let's consider the results of the proportionality between these quantities.

Note that we have two velocity scales now, u_s and u_τ, like the case of the jet where we used u_s and $r(x)$. But due to the defect scaling the results will be different from the jet. For instance,

$$\bar{v} = u_\tau \delta' \eta f - (u_\tau\delta)' \int_0^\eta f d\eta - \delta\eta u_s'$$

differs from the jet in the final term. Although recall that we employed the self-similar TKE balance to resolve the ambiguity in the momentum equation.

Description	Balance	Scaling
Advective	③ & ④	$u_s \propto u_\tau$
Advective	① & ④	$u_\tau \propto \delta^n$
Turbulent	① & ⑤	$\frac{u_\tau'}{u_\tau} \propto \frac{1}{\delta}$
Turbulent	④ & ⑤	$\delta' \propto 1$

We find that there is only a single velocity scale in the problem, since $u_s \propto u_\tau$, which reminds us of the problem of overspecification that we encountered in the turbulent jet. We expect this single velocity scale to be valid only in the limit of infinite Reynolds number. We also see that the boundary layer thickness grows linearly, $\delta \propto x$, as in the case of the turbulent jet. Combining the two turbulent balances yields the conclusion that

$$u_\tau \propto x^n \tag{50}$$

where the exponent, n, is arbitrary. However, we haven't made use of the leading order viscous term. This seems reasonable because we are seeking a similarity solution valid far from the wall, in the defect region of the boundary layer. If instead of balancing the turbulent terms, we balance the viscous terms ① & ⑥ and the advective terms ① & ④, we obtain $u_\tau \propto x^m$ and $\delta \propto x^{\frac{1-m}{2}}$, which are the Falkner Skan similarity solutions of the laminar boundary layer; for zero pressure gradient, $m = 0$, we recover our traditional laminar solution $\delta \propto x^{1/2}$. We cannot identify a consistent similarity solution which includes both turbulent ⑤ and viscous terms ⑥.

As Reynolds number increases, we expect a steeper velocity gradient near the wall, and a blunter velocity profile. Therefore, at very high Reynolds number, we expect the velocity defect to be very small, with a uniform velocity across most of the boundary layer. In this case, we expect $f(\eta)$ to be small, such that the deficit is much smaller than the freestream velocity, $u_\tau f(\eta) \ll u_s$. In this case, based on our previous analysis, we can neglect terms that are quadratic in f, including f^2 and $f' \int f d\eta$, as well as f'', assuming that η is large in the outer region. Then we obtain a simplified version of the momentum balance

Compare to equation 5.5.31 in Tennekes & Lumley

$$\overbrace{\underbrace{\left[(u_\tau u_s)'\right]}_{②} f - \underbrace{\left[\frac{u_\tau}{\delta}(u_s\delta)'\right]}_{③} \eta f'}^{Advective} + \overbrace{\underbrace{\left[\frac{u_\tau^2}{\delta}\right]}_{⑤} g'}^{Turbulent} = 0 \tag{51}$$

This equation has the advantage of being linear in both f and g. We cannot avoid the closure problem, and thus we still need some sort of model to represent the turbulence, g, but given such a model, this equation can be solved easily. However we must define the boundary conditions, which are a bit trickier given the assumption that this equation applies only far from the wall. At the outer edge of the boundary layer, as $\eta \to 1$, we expect matched mean velocity and no turbulence, so

$$f(1) = 0 \qquad g(1) = 0 \tag{52}$$

However, the boundary conditions as $\eta \to 0$ are a bit trickier, because we have excluded the viscous term, ⑥, and thus we need to consider the asymptotic behavior of f and g as $\eta \to 0$. From the definition of our similarity function we know

$$f'(\eta) = \frac{\delta}{u_\tau}\frac{\partial \bar{u}}{\partial y} \tag{53}$$

and therefore we can write

$$\eta f'(\eta) = \underbrace{\frac{y}{u_\tau}\frac{\partial \bar{u}}{\partial y}}_{\text{as } \eta \to 0} \to \frac{1}{\kappa} \tag{54}$$

and along the outer edge of the constant stress region, very near the wall, we know that the velocity gradient is logarithmic, in which

case $\eta f'$ is a constant, which serves as the boundary conditions on f. The behavior of g is also tricky; if we would have retained the viscous term ⑥ then we could argue that $g(0) = 0$ just like we could have argued that $f(0) = 0$. But because we have excluded the viscous term, the Reynolds stress must match the wall shear stress ($-\rho\overline{uv} \approx \tau_w$ at the outer edge of the constant stress layer), never actually approaching the wall itself, in which case $g(0) = -1$. We can show this formally by rewriting our equation with the viscous term included

$$\overbrace{\underbrace{\left[(u_\tau u_s)'\right]}_{②} f - \underbrace{\left[\frac{u_\tau}{\delta}(u_s\delta)'\right]}_{③}\eta f'}^{Advective} + \overbrace{\underbrace{\left[\frac{u_\tau^2}{\delta}\right]}_{⑤} g'}^{Turbulent} = \overbrace{\underbrace{\left[\nu\frac{u_\tau}{\delta^2}\right]}_{⑥} f''}^{Viscous} \tag{55}$$

Note that $\int_0^1 \eta f' d\eta = \eta f\Big|_0^1 - \int_0^1 f d\eta$ by integration by parts. And note that $\tau_w = \mu\frac{u_\tau}{\delta} f'(0)$ and $0 = \frac{u_\tau}{\delta} f'(1)$

Then integrate across the boundary layer, from $\eta = 0$ to 1, to obtain

$$\left[(u_\tau u_s)' + \frac{u_\tau}{\delta}(u_s\delta)'\right] \underbrace{\int_0^1 f d\eta}_{-\frac{u_s\delta^*}{u_\tau\delta}} - \left[\frac{u_\tau}{\delta}(u_s\delta)'\right] \underbrace{\eta f\Big|_0^1}_{=0} = -\underbrace{\left[\frac{u_\tau^2}{\delta}\right] g\Big|_0^1}_{\left[\frac{u_\tau^2}{\delta}\right] g(0)} + \underbrace{\left[\nu\frac{u_\tau}{\delta^2}\right] f'\Big|_0^1}_{-\frac{u_\tau^2}{\delta}} \tag{56}$$

Now we can see that, in the absence of the viscous term, $g(0) = -1$ will produce the identical effect on the integrated total momentum across the layer. Let's return to the simplified version of the integrated momentum balance in order to consider the effect of the pressure gradient on the formulation of a similarity solution.

Rotta-Clauser Parameter

In order to describe the effect of the pressure gradient on similarity solutions for the turbulent boundary layer, we begin by rewriting the momentum integral with just the turbulent term, for the outer region of the boundary layer, yielding

$$\left[(u_\tau u_s)' + \frac{u_\tau}{\delta}(u_s\delta)'\right]\left(-\frac{u_s\delta^*}{u_\tau\delta}\right) = -\frac{u_\tau^2}{\delta} \tag{57}$$

$$\left[\frac{u_s}{\delta}(u_\tau\delta)' + 2u_\tau u_s'\right]\left(-\frac{u_s\delta^*}{u_\tau^3}\right) = 1 \tag{58}$$

$$\left(\frac{\delta^* u_s}{\delta u_\tau}\right)\left(\frac{u_s}{u_\tau^2}\right)(\delta u_\tau)' = 1 - 2\left(\frac{u_s}{u_\tau}\right)\left(\frac{\delta^*}{u_\tau}\right)u_s' \tag{59}$$

We have never really specified an exact definition for δ, unlie δ^* or θ. Often δ is defined as δ_{99} which represents the y-location where $\bar{u}(y = \delta_{99}) = 0.99u_s$. But here, let's use a more convenient definition for the boundary layer thickness. Let's replace δ with Δ, Rotta's weighted integral thickness, which is defined as

$$\Delta = \int_0^\infty \frac{u_s - \bar{u}}{u_\tau} dy \quad \Rightarrow \frac{\Delta}{\delta^*} = \frac{u_s}{u_\tau} \tag{60}$$

and thus is just a rescaled displacement thickness. Using the Rotta thickness Δ in place of δ (since the choice was arbitrary to begin with), we can rewrite the momentum integral as

$$\left(\frac{u_s}{u_\tau^2}\right)(\Delta u_\tau)' = 1 - 2\left(\frac{\Delta}{u_\tau}\right)u_s' \tag{61}$$

and finally we can write this momentum balance in terms of the strength of the non-dimensional pressure gradient, which we define using the Rotta-Clauser parameter, β, as

$$\beta = \frac{\delta^*}{\tau_w}\frac{d\bar{p}_\infty}{dx} = -\frac{\delta^*}{\tau_w}\rho u_s u_s' = -\frac{u_s}{u_\tau}\frac{\delta^*}{u_\tau}u_s' = -\frac{\Delta}{u_\tau}u_s' \tag{62}$$

to obtain the integrated momentum balance

$$\left(\frac{u_s}{u_\tau^2}\right)(\Delta u_\tau)' = 1 + 2\beta \tag{63}$$

This serves as an integral constraint on the non-integrated balance, in terms of f and g.

$$\left[\frac{\Delta}{u_\tau^2}(u_\tau u_s)'\right]f - \left[\frac{1}{u_\tau}(\Delta u_s)'\right]\eta f' + g' = 0 \tag{64}$$

According to our similarity analysis, there is only a single velocity scale, and thus $u_\tau \propto u_s$, in which case we can rewrite the momentum balance as

$$[-2\beta]f - [1 + 2\beta]\eta f' + g' = 0 \tag{65}$$

In other words, the existence of a similarity solution is guaranteed by β being constant. And since β is the only parameter in the similarity formulation, we expect all boundary layers with the same constant pressure gradient to be similar, independent of particular Reynolds number. We can try to solve this equation by using the integral constraint we developed above, as well as overlap boundary conditions near the wall. But this similarity solution will still involve numerical integration for each value of β, and even then it may be applicable only at very high Reynolds numbers. Therefore, we will consider another technique for describing the mean velocity profile in the outer region which may be more useful for engineering applications.

Recalling that the equation of u_s and u_τ implies a sufficiently high Reynolds number

Law of the Wake

When we plot the mean velocity profile using inner scaling, we notice that the profile 'peels away' from the log-law curve, and this peeling varies with Reynolds number. We could imagine generalizing the law of the wall to include this wake region, where the velocity profile peels away from the log law.

$$\frac{\bar{u}}{u_\tau} = f_i\left(\frac{yu_\tau}{\nu}\right) + g\left(\pi, \frac{y}{\delta}\right) \tag{66}$$

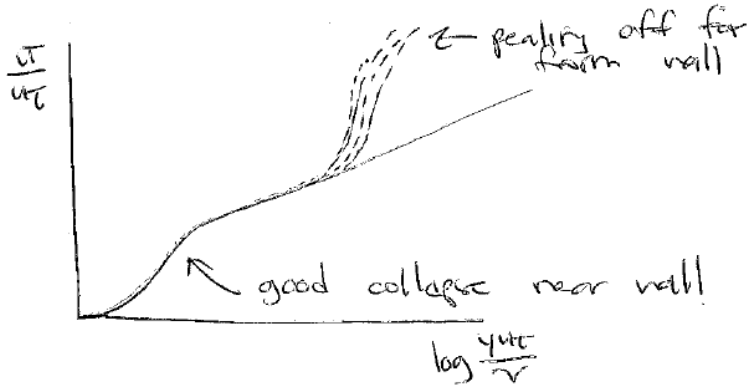


Figure 39: A sketch of the mean velocity profile for the turbulent boundary layer at multiple Reynolds numbers, plotted in inner scaling. Note that the collapse is good near the wall, but the velocity profile 'peels away' from the log law far from the wall.

This formulation implies that the correction to the velocity profile far from the wall can be written independent of the viscosity. The unknown function which describes the discrepancy could be found from similarity techniques like we discussed in the previous section. However, we could empirically model the discrepancy between the known inner behavior and the unknown outer behavior.

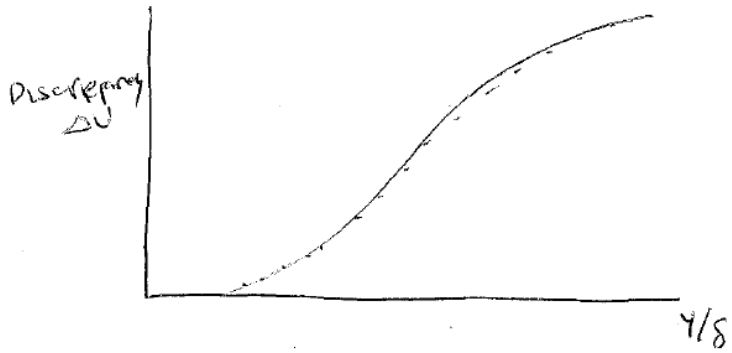


Figure 40: A sketch of the discrepancy between the actual mean velocity profile and the profile predicted by the Spalding fit of the near-wall behavior, plotted in outer scaling.

When we plot this discrepancy, it takes the form of a simple, smooth function. Coles suggested that perhaps we can empirically describe this function in order to extend our theoretical model of the mean velocity profile far from the wall. Let's write the generalized law of the wall more specifically as

$$\frac{\bar{u}}{u_\tau} = f_i\left(\frac{yu_\tau}{\nu}\right) + \frac{\pi(x)}{\kappa} w(y/\delta) \tag{67}$$

where $w(y/\delta)$ is called the wake function and $\pi(x)$ represents variation along the length of the boundary layer. Now we need to define the wake function in terms of empirically measurable quantities so that we can use it to describe the boundary layer profile. In other words, we would like to set boundary conditions and constraints on $w(y)$ and $\pi(x)$ so that we can use them universally. Let's start by calculating the displacment thickness by using the generalized formula including the law of the wall

$$\delta^* = \int_0^\delta \left(1 - \frac{u_\tau}{u_s}\frac{\bar{u}}{u_\tau}\right) dy \tag{68}$$

$$= \int_0^\delta dy - \frac{u_\tau}{u_s}\int_0^\delta f_i(y^+)dy - \frac{u_\tau}{u_s}\int_0^\delta \frac{\pi(x)}{\kappa} w(y/\delta)dy \tag{69}$$

$$= \delta - \frac{u_\tau}{u_s}\left[\left(f_i y\Big|_0^\delta - \int_0^{f_i(\delta^+)} y df_i\right) + \frac{\pi(x)}{\kappa}\left(w(y)y\Big|_0^\delta - \int_0^{w(1)} y dw\right)\right] \tag{70}$$

$$= \frac{u_\tau}{u_s}\left[\int_0^{f_i(\delta^+)} y df_i + \frac{\pi}{\kappa}\int_0^{w(1)} y dw\right] \tag{71}$$

$$= \frac{\delta u_\tau}{u_s \kappa}\left(1 + \pi(x)\int_0^{w(1)} y dw\right) \tag{72}$$

Recall that by integration by parts we have $\int_0^\delta f_i dy = f_i y\Big|_0^\delta - \int_0^{f_i} y df_i$ and in the overlap region we have $\frac{\partial f_i}{\partial y} = \frac{1}{\kappa y}$ so $\partial f_i = \frac{\partial y}{\kappa y}$ and $\frac{u_s}{u_\tau} = f_i(\delta^+) + \frac{\pi}{\kappa}w(1)$ and $w(0) = 0$.

We can arbitrarily define a normalization condition on the wake function such that

$$\int_0^{w(1)} y dw = 1 \tag{73}$$

in which case we write an expression for the function $\pi(x)$ which describes the evolution of the boundary layer in the streamwise direction

$$\frac{\delta^* u_s}{\delta u_\tau}\kappa = 1 + \pi(x) \tag{74}$$

Note that $\pi(x)$ is similar in form to $\beta(x)$ and both present the streamwise variation of the boundary layer, but $\beta \equiv -\frac{\delta^* u_s}{\delta u_\tau}\left(\frac{u_s' \delta}{u_\tau}\right)$ depends on the variation in the freestream velocity, and thus represents the pressure gradient effect, whereas π represents the thickening of the boundary layer, irrespective of pressure gradient.

In addition to the normalization condition on $w(y)$ above, and the boundary condition $w(0) = 0$, we can also set another boundary condition, without loss of generality, such that $w(1) = 2$. Now, we simply need to choose a functional form for $w(y)$ that obeys the boundary conditions and constraints. Coles chose

$$w(y/\delta) = 1 - \cos\left(\pi\frac{y}{\delta}\right) = 2\sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) \tag{75}$$

but this isn't very convenient for use in analysis that involves integrals of the mean velocity profile, so White proposed the power series expression

$$w(y/\delta) = 2\left(\frac{y}{\delta}\right)^2\left(3 - 2\frac{y}{\delta}\right) \tag{76}$$

The wake formula provides a practical engineering formula to represent the outer, turbulent mean velocity profile, based on two empirically fitted functions. In combination with Spalding's formula for the inner region, the entire boundary layer can be described in convenient form. However, even Spalding's formula depends on empirical measurement of u_τ, κ, and B_i, so we have not

obtained any theoretical understanding of the turbulence by means of this descriptive analysis.

Momentum Balance and Mean Velocity Profile

Let's recap the physical picture of the boundary layer we have been developing. Our analysis has been based on the behavior of the mean velocity profile, and we showed that the mean velocity profile exhibits different behaviors in four distinct regions.

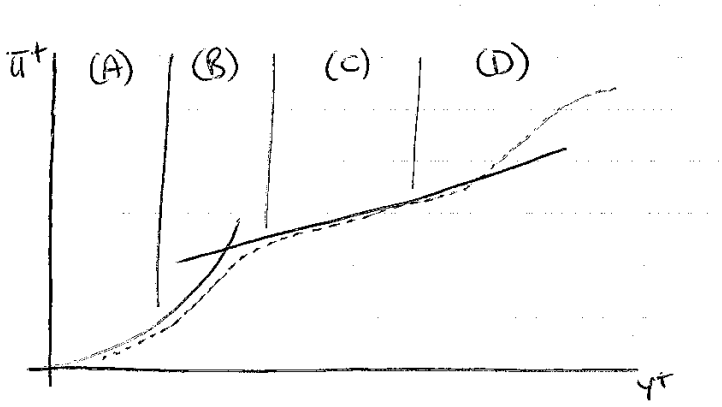


Figure 41: A sketch of the mean velocity profile for the turbulent boundary layer, identifying the four regions we have analyzed: the linear sublayer (A), the buffer layer (B), the log layer (C), and the wake (D).

(A) The linear sublayer exhibits a linear variation in velocity with wall-normal height over a region very near the wall ($0 < y^+ \lesssim 4$). Viscosity is dominant in this region, although the flow is not laminar.

There are still fluctuations in the viscous sublayer, and thus it is a mistake to refer to this as a 'laminar sublayer' as some books do — see Cantwell, Ann. Rev of Fluid Mech. 1981 and Kline, et. al. JFM 1967 who report that even at $y^+ \approx 2.7$ the trajectories of tracer bubbles in flow are irregular

(B) The buffer layer connects the linear sublayer and the log-layer, and encompasses the constant stress region of wall bounded turbulence, where both viscous and turbulent stresses are important. It extends from approximately $4 \lesssim y^+ \lesssim 30 - 50$.

(C) The log layer is a region where the magnitude of the stress is fixed by τ_w but the stresses are dominated by turbulent/inertial effects. It represents an overlap region of inner and outer scales and extends from $30 - 50\frac{\nu}{u_\tau} \lesssim y \lesssim 0.2\delta$

(D) The outer region includes the log layer and the wake, and extends from $0.2\delta \lesssim y < \delta$. It is also dominated by inertial effects.

In the above description, we have basically drawn an analogy between outer-scaling/inertial scaling and inner-scaling/viscous scaling. In the outer region, we assume inertial processes are dominant, and in the inner region we assume viscous processes are dominant. And then we posited an overlap region between the inner and outer scalings. We might think therefore that an overlap region of the two scalings (inner and outer) implies an overlap between the two physical processes (viscous and inertial). But this is not true!

The linear and buffer layers, (A) and (B), both scale on viscous units. So holding the outer scale δ fixed, as the Reynolds number increases, $\mathrm{Re}_\tau = \frac{\delta u_\tau}{\nu}$, the inner scale $\frac{\nu}{u_\tau}$ decreases, and thus the inner region of the boundary layer becomes a smaller fraction of the boundary layer.

The overlap region occurs at the outer edge of the inner-scaled region, i.e. where inertial processes are important and viscous

processes are not important. The log layer is therefore an inertial region, not a combination of viscous and inertial processes. The combination of viscous and inertial processes occurs in the buffer layer, where the maximum in production exists, i.e. where $\frac{\partial \overline{u}}{\partial y}$ and $\overline{u}\overline{v}$ are equally significant. But the buffer layer exists in the inner region of the boundary layer, where inner scaling is used, even though inertial effects are important there! We have a serious conflict between the physical interpretation of the different layers, from the point of view of energy or momentum, and the description of the different layers in terms of the lengthscales of the mean velocity profile. The characteristic lengthscales and momentum interpretations do not align. From the perspective of momentum and energy, it makes sense to describe 'overlap' in the buffer layer; from the perspective of mean-velocity scaling, it makes sense to describe the 'overlap' in the log layer. In order to resolve this contradiction, let's think more carefully about the idea of 'overlap' starting from the momentum balance, and see where it leads us.

We begin wtih the mean momentum balance for the turbulent boundary layer, written in inner units for the case of zero pressure gradient flow.

See Klewicki, et. al. PTRS 2007 and Wei, et. al. JFM 2005 for more on the momentum analysis of the layer structure of the turbulent boundary layer

$$\underbrace{\overline{u}^+\frac{\partial \overline{u}^+}{\partial x^+} + \overline{v}^+\frac{\partial \overline{u}^+}{\partial y^+}}_{\text{Advection}} = \underbrace{-\frac{\partial}{\partial y^+}\overline{u}\overline{v}^+}_{\text{Turbulence, } \frac{\partial T^+}{\partial y^+}} + \underbrace{\frac{\partial^2 \overline{u}^+}{\partial y^{+2}}}_{\text{Diffusion}} \tag{77}$$

in which case we see that there are fundamentally three processes in the momentum balance: advective transport of momentum, transport by viscous diffusion, and transport by turbulent fluctuations (Reynolds stress). In order to maintain a balance between these processes, at least 2 out of the 3 must be in balance, i.e. non-negligible. This leaves just three cases

(a) $\left|\frac{\frac{\partial^2 \overline{u}}{\partial y^{+2}}}{\frac{\partial T^+}{\partial y^+}}\right| \gg 1$ Mean advection balances the viscous diffusion (and turbulent flux are negligible)

(b) $\left|\frac{\frac{\partial^2 \overline{u}}{\partial y^{+2}}}{\frac{\partial T^+}{\partial y^+}}\right| \ll 1$ Mean advection balances the turbulent transport (and viscous diffusion is negligible)

(c) $\left|\frac{\frac{\partial^2 \overline{u}}{\partial y^{+2}}}{\frac{\partial T^+}{\partial y^+}}\right| \sim 1$ Turbulent and diffusive momentum fluxes are in balance, and both are greater than or comparable to advection.

We can plot the ratio of the viscous to turbulent momentum transport as it varies across the thickness of the boundary layer in order to see how these three cases exhibit in different layers within the boundary layer. The plot shows the existance of 4 distinct layers

Layer I: Viscous diffusion balances means advection, case (a) above.

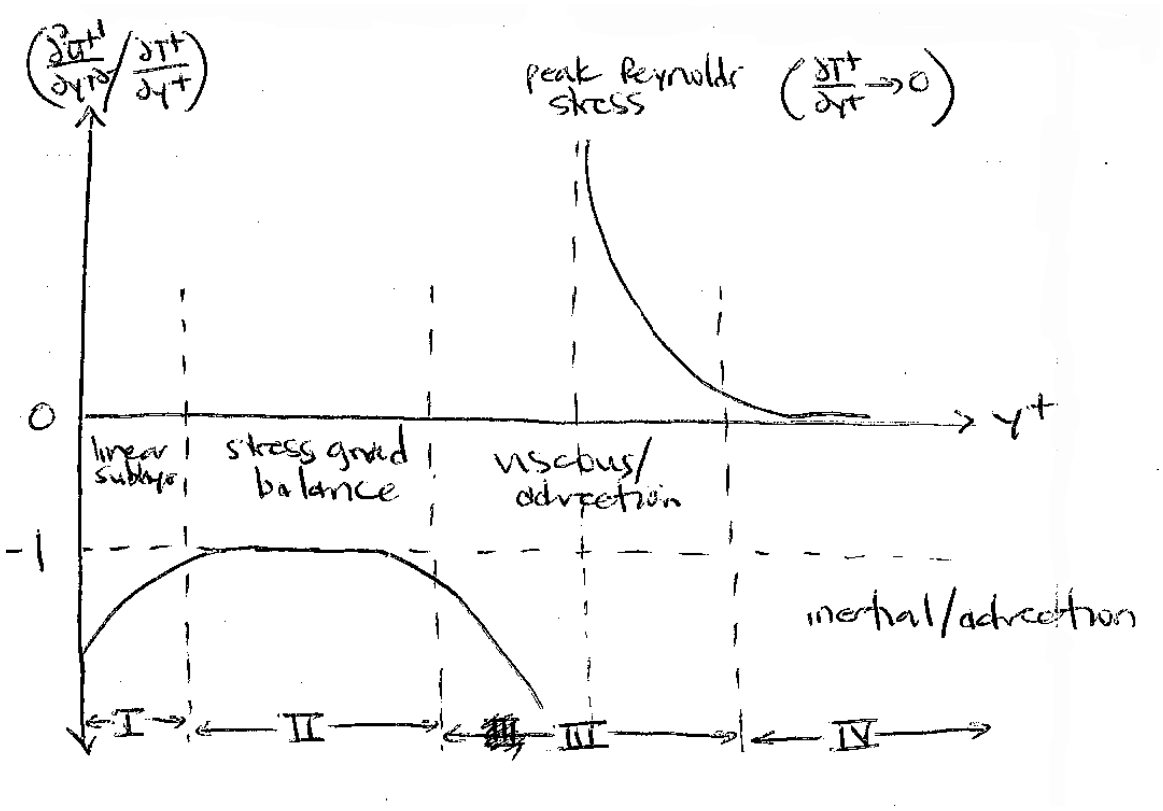


Figure 42: A sketch of the ratio of viscous to turbulent momentum transport across the thickness of the boundary layer.

Layer II: Diffusive and turbulent stresses are in balance, case (c) above.

Layer III: All three terms – diffusion, advection, and turbulent transport – are in balance. The turbulent momentum flux changes sign (the Reynolds stress itself has a maximum) and thus there is a discontinuity in the ratio in this layer. The peak Reynolds stress indicates the changing role of turbulence: from a momentum source near the wall to a momentum sink away from the wall.

Layer IV: Advection and turbulent transport are in balance, case (b) above.

We see that by analyzing the boundary layer from the physically-motivated view of the momentum equations, instead of the mathematically-motivated view of the mean velocity profile, we obtain a four-layer structure which differs from the structure we described earlier. In particular, this momentum structure suggests that the boundaries of a log-layer may not be where we expected, because the log layer is an inertial layer, and according to the momentum picture, inertially dominant layers start only at layer IV. The question of the existence and boundaries of a log layer remain an open question, but physically motivated descriptions may provide better insights into answering it.

Coherent Structures

Describing the mean dynamics of turbulence is useful for obtaining and engineering description of the net effect of turbulent flows, but of course mean dynamics ignore the fact that turbulence is composed of a variety of coherent motions (eddies) of different sizes and shapes and evolving at different timescales. We now return to the coherent structures which compose turbulence and ask: how do the individual, chaotic, coherent structures relate to the mean description of turbulence we developed in the last few sections?

Attached Eddy Hypothesis

In the inner region of the turbulent boundary layer, there is a surprisingly strong flow of TKE toward the wall. Recall the shape of the TKE distribution

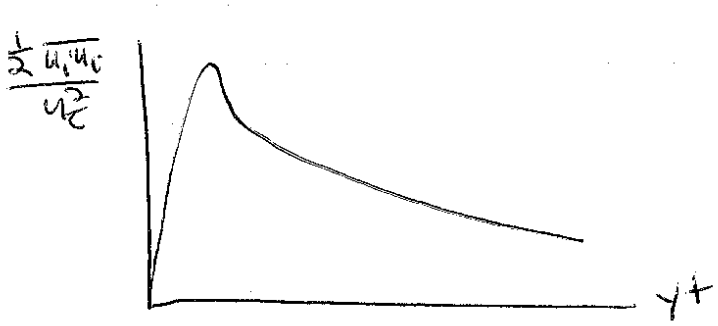


Figure 43: A sketch of the TKE distribution across the thickness of the boundary layer.

It has a peak very near the wall, which indicates that, if energy transport were purely diffusive in the classical gradient-diffusion sense, the TKE would tend to move away from the wall, away from the high energy concentration. However, when we look at the TKE budget we see that near the wall, TKE is increasing and far away TKE is decreasing, which suggests that the net effect is a movement of TKE towards the wall. As energy moves towards the wall, it is dissipated, and we can write the rate of dissipation, using the law of the wall, in the general form

$$\varepsilon = \frac{1}{\kappa}\frac{u_\tau^3}{y} h(y^+) \tag{1}$$

and in the limit that $\mathrm{Re}_\tau \to \infty$ we assume that $h(y^+)$ reaches an asymptotic limit of 1 such that

$$\varepsilon = \frac{1}{\kappa}\frac{u_\tau^3}{y} \tag{2}$$

Recall that, according to Barenblatt, this asymptotic assumption is not valid, and the dependence on Re_τ persists.

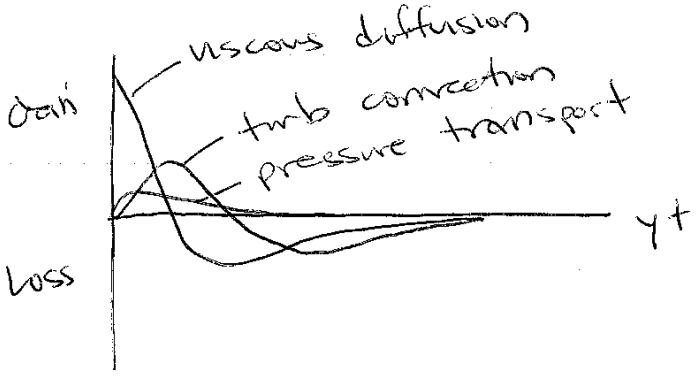


Figure 44: A sketch of the individual ingredients within the TKE budget, across the thickness of the boundary layer, showing that energy is actually transported towards the wall (decreasing away from the wall and increasing near to the wall).

But the dissipation rate can also be written, by dimensional arguments, on a per-eddy basis. In other words, for a given eddy with characteristic velocity u and characteristic length scale ℓ, we can approximate the dissipation rate from that particular eddy as

$$\varepsilon \sim \frac{u^3}{\ell} \tag{3}$$

like we did in the very first chapter. Comparing these two expressions, we can conclude that the characteristic dimension of an eddy in wall-bounded turbulence that satisfies the law of the wall in the asymptotic limit of high Reynolds number is $\ell \sim y$. And these eddies are involved in a transfer of TKE towards the wall, due to the high rate of shear. So we have a class of eddies associated with transfer of energy to the wall where it is dissipated, but the characteristic size of the eddies scales with the distance from the wall, y. How is it possible that this intense viscous dissipation is happening near the wall but the eddies involved in the dissipation do not scale on viscous units (e.g. all eddies are of size $5\nu/u_tau$) but rather the eddies scale with y? It must be that these eddies of scale y remain attached to the wall, despite their size. Even when the eddy is larger than the viscous length scales, it keeps its 'feet' attached at the wall, where the dissipation is occuring. Of course, this conclusion implies that non-local effects are important, and structure matters. The shape of the eddies is crucial to understanding how we relate their function (dissipation, energy transport) to their characteristic size (y).

This idea of an attached eddy means that as the eddy grows in size, it grows in such a way that its velocity field (or vorticity field) still extends to the wall, and thus the eddy still 'feels' the influence of the wall, even as its center moves farther away. The diameter of the eddy grows in proportion to the eddy's distance to the wall. We then formulate a hypothesis about these eddies.

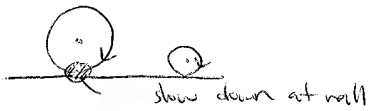


Figure 45: A cartoon of an eddy with characteristic dimension y, where the bottom of the eddy still 'feels' the effect of the wall.

Townsend's Attached Eddy Hypothesis: The main energy-containing motion of a turbulent, wall-bounded flow is described by a random superposition of attached eddies of different sizes but similar velocity distribution.

By this hypothesis, we don't mean that individual eddies actually look similar, rather we mean that the eddy structure is similar in a statistical sense. Ensemble averaging all of the different eddies located near the wall produces an 'average' eddy which this hypothesis claims is an attached eddy of universal form. But this does not imply that attached eddies exist individually, only statistically.

Consider a hierarchy of eddies in which each eddy has size λ and the largest characteristic size in the hierarchy is Λ and the smallest is $\lambda_{\min}$.

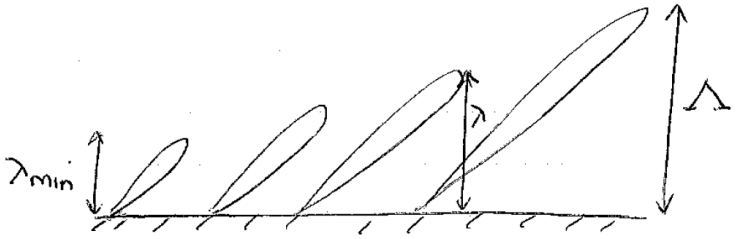


Figure 46: A hierarchy of eddies attached to the wall.

Each member of the hierarchy has a similar velocity distribution (which can be found via Biot-Savart from the vorticity distribution). Let's assume that the instantaneous velocity velocity field in space, at location $\vec{x}$, induced by eddy λ in hierarchy Λ is given by

$$\frac{u_i}{u_\tau} = f_i\left(\frac{\vec{x}}{\Lambda}, \frac{\lambda}{\Lambda}\right) \tag{4}$$

i.e. it depends on the measurement location, $\vec{x}$, the particular eddy inducing the velocity, λ, and the particular hierarchy, Λ. The contribution to the covariance functions $u_i u_j$ of that eddy λ at distance y from the wall is then

The ensemble average here is represented by integration over x and z thus retaining the variation in the wall-normal direction. Note that $\vec{x} = (x, y, z)$

$$\frac{\Delta\overline{u_i u_j}}{u_\tau^2} = \Lambda^2 \int_{-\infty}^{\infty}\!\!\int f_i\left(\frac{\vec{x}}{\Lambda}, \frac{\lambda}{\Lambda}\right) f_j\left(\frac{\vec{x}}{\Lambda}, \frac{\lambda}{\Lambda}\right) d\left(\frac{x}{\Lambda}\right) d\left(\frac{z}{\Lambda}\right) \tag{5}$$

where $\Delta\overline{u_i u_j}$ is a function of $\left(\frac{y}{\Lambda}, \frac{\lambda}{\Lambda}\right)$ since we integrate over the x and z directions. Now, if we want to integrate the contributions to the covariance of all of the individual eddies in the entire hierarchy, we need to know the probability density function (PDF) of all of the members of the hierarchy, $P_\lambda(\lambda)$. Given, the PDF, we can write the cumulative contribution over the hierarchy, for eddies ranging in size from $\lambda_{\min}$ up to the size of the hierarchy, Λ, as

$$\frac{\delta\overline{u_i u_j}}{u_\tau^2} = \int_{\lambda_{\min}}^{\Lambda} \frac{\Delta\overline{u_i u_j}}{u_\tau^2}\left(\frac{y}{\Lambda}, \frac{\lambda}{\Lambda}\right) P_\lambda(\lambda) d\lambda \tag{6}$$

where $\delta\overline{u_i u_j}$ is a function of $\frac{y}{\Lambda}$, since we are integrating over all of the individual eddies, λ. The question is, how do we define the minimum eddy size $\lambda_{\min}$ the contributes to the hierarchy? We suspect that the contribution towards $\overline{u_i u_j}$ measured at height y from a given attached eddy of size $\lambda \ll y$ is negligible, because the velocity induced by an eddy decays rapidly with distance, according to the Biot-Savart law.

Therefore, neglecting the contribution from eddies smaller than the wall-position of interest, y, we obtain the integrated contribution to the covariance from an entire hierarchy, which we call I_{ij}, which is often referred to as Townsend's eddy intensity function.

$$\frac{\delta\overline{u_i u_j}}{u_\tau^2} = \int_y^\Lambda \frac{\Delta\overline{u_i u_j}}{u_\tau^2}\left(\frac{y}{\Lambda}, \frac{\lambda}{\Lambda}\right) P_\lambda(\lambda) d\lambda \quad \equiv I_{ij}\left(\frac{y}{\Lambda}\right) \tag{7}$$

See Perry & Chong (JFM 1982) Appendix A where they show that the reciprocal distribution preserves the geometrical progression desired, in which there are stacked hierarchies, each larger than the previous, with more stacks (and thus more, smaller scales) as Reynolds number increases. This sort of stacking is visible, in side view, in the figure above.

But of course there are many different hierarchies, each defined by some characteristic scale Λ. To find the total covariance contribution from all hierarchies, ranging from the smallest contributing hierarchy size that contributes to the convariance, Λ_i, to a hierarchy that scales with the outer length scales of the flow, Λ_o, we need to know the PDF of the hierarchies, $P_\Lambda(\Lambda)$, after which we can write

$$\frac{\overline{u_i u_j}}{u_\tau^2} = \int_{\Lambda_i}^{\Lambda_o} I_{ij}\left(\frac{y}{\Lambda}\right) P_\Lambda(\Lambda) d\Lambda \tag{8}$$

In order to express the covariance, we need to know the functional form of both the PDF for the hierarchies, $P_\Lambda(\Lambda)$, and the eddy intensity function for a given hierarchy, I_{ij}. Following Townsend, we assume that the PDF takes the functional form

$$P_\Lambda(\Lambda) = \frac{M(\Lambda)}{\Lambda} \tag{9}$$

where $M(\Lambda)$ can vary with Λ but is assumed to be constant for simplicity, yielding a simple reciprocal distribution for the PDF. Then we can rewrite

$$\frac{\overline{u_i u_j}}{u_\tau^2} = \int_{\Lambda_i}^{\Lambda_o} I_{ij}\left(\frac{y}{\Lambda}\right) \frac{M}{\Lambda} d\Lambda \tag{10}$$

See Perry & Chong (JFM 1982) Appendix A where they show that the reciprocal distribution preserves the geometrical progression desired, in which there are stacked hierarchies, each larger than the previous, with more stacks (and thus more, smaller scales) as Reynolds number increases. This sort of stacking is visible, in side view, in the figure above.

If y is fixed relative to the wall, then $d\Lambda = -\frac{y}{\eta^2} d\eta$

By changing variables from Λ to $\eta = \frac{y}{\Lambda}$, we can rewrite the integral as

$$\frac{\overline{u_i u_j}}{u_\tau^2} = \int_{y/\Lambda_i}^{y/\Lambda_o} I_{ij}(\eta) \frac{M\eta}{y}\left(-\frac{y}{\eta^2}\right) d\eta \quad = \int_{y/\Lambda_o}^{y/\Lambda_i} I_{ij}(\eta) \frac{M}{\eta} d\eta \tag{11}$$

We still need to specify the eddy intensity function, I_{ij}, but this function ultimately depends on the generator functions f_i, f_j and the PDF of the individual eddies, $P_\lambda(\lambda)$. It will help to specify the generator functions in the streamwise (f_1), wall-normal (f_2), and

spanwise directions (f_3). Recall that the attached eddy hypothesis is designed to describe the population of eddies which are attached to the wall, but which inhabit a region where the characteristic scale is not, in fact, a viscous length scale, but rather y, in the limit of high Reynolds number. This is essentially a description of the region we previously described as the logarithmic or overlap layer. Within this layer, we assume that f_1 and f_3 are independent of viscosity and remain finite even in the limit as $\eta \to 0$ (approaching the wall). The wall-normal generator function must obey the impermeability constraint at the wall, and thus $f_2 = 0$ at $\eta = 0$, and in the vicinity of the wall $f_2 \sim \eta$ (by continuity).

Assuming that f_1 and f_3 tend to a finite limit at the wall, then I_{11} and I_{33} are also constant in the limit that $\eta \to 0$. Impermeability means that $I_{12} \sim \eta$ and $I_{22} \sim \eta^2$ in the same limit.

The function I_{ij} is an integral over λ so the PDF dependence is integrated out.

I_{ij}	Rationale
$I_{11}, I_{33} \sim$ const	$f_1, f_3 \to$ const as $\eta \to 0$
$I_{12} \sim \eta$	$f_2 \sim \eta$ as $\eta \to 0$
$I_{22} \sim \eta^2$	$f_2 \sim \eta$ as $\eta \to 0$

Table 12: Townsend's Eddy Intensity Function as $\eta \to 0$

Based on the asymptotic behavior of I_{ij} near the wall, we can sketch the general form of the functions.

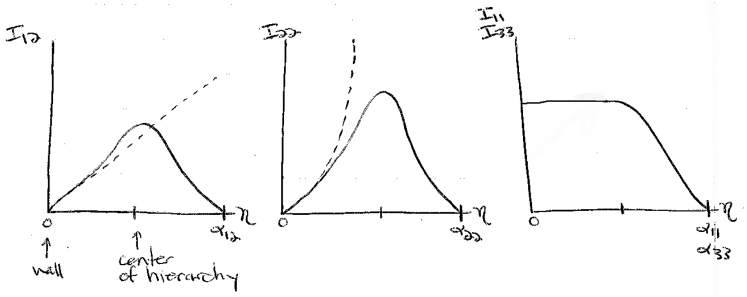


Figure 47: A sketch of the eddy intensity functions based on the asymptotic behavior near the wall, $\eta \to 0$.

The eddy intensity function must decay as $\eta \gg 1$ because this means that $y \gg \Lambda$ and, according to the Biot Savart argument above, we expect the contribution from a hierarchy of eddies of size Λ to points much farther away from the wall than Λ to be very small. We label the limit at which I_{ij} is negligible α_{ij}, which we expect to be $\mathcal{O}(1)$. Therefore, the larger hierarchy limit from the above integral, $\frac{y}{\Lambda_i}$ can be replaced with α_{ij}

$$\frac{\overline{u_i u_j}}{u_\tau^2} = \int_{y/\Lambda_o}^{\alpha_{ij}} I_{ij}(\eta) \frac{M}{\eta} d\eta \tag{12}$$

We are assuming that the contribution from hierarchies above $\eta = \alpha_{ij}$ is negligible, such that the integral $\int_{\alpha_{ij}}^{y/\Lambda_i} I_{ij}(\eta) \frac{M}{\eta} d\eta \approx 0$

Evaluating for the Reynolds stress $\overline{u_1 u_2}$ then yields

$$\frac{\overline{u_1 u_2}}{u_\tau^2} = \int_{y/\Lambda_o}^{\alpha_{12}} \underbrace{I_{12}(\eta)}_{\eta} \frac{M}{\eta} d\eta \quad = M\left(\alpha_{12} - \frac{y}{\Lambda_o}\right) \tag{13}$$

In the log layer, we can also assume that $\frac{y}{\Lambda_o} \ll 1$ since Λ_o represents the largest (outer-scale) structures in the flow. Then we find that

the Reynolds stress is just a constant in the log layer, and moreover this constant is universal if we assume that the distributions of the hierarchies of eddies are independent of Reynolds number. This result is consistent with our earlier conclusion that the Reynolds stress is constant at the outer edge of the constant total stress region of the boundary layer, i.e. the log layer.

Note that the constant Reynolds stress result is dependent on the choice of the reciprocal PDF, $P_\Lambda(\Lambda) \propto \frac{1}{\Lambda}$ and thus there is an intimate connection between the statistical/geometrical structure of eddies and the stress distribution.

We can also calculate the turbulence intensities

$$\frac{\overline{u_1^2}}{u_\tau^2} = \int_{y/\Lambda_o}^{\alpha_{11}} \underbrace{I_{11}(\eta)}_{\text{const}} \frac{M}{\eta} d\eta \quad = I_{11} M \log \eta \Big|_{y/\Lambda_o}^{\alpha_{11}} \quad = I_{11} M \log\left(\frac{\alpha_{11}\Lambda_o}{y}\right) \tag{14}$$

This result is consistent with Perry's spectrals similarity theory, where we found a logarithmic variation in the streamwise turbulence intensity, $\overline{u_1^2}$. The other velocity components also follow by the same analysis,

$$\frac{\overline{u_3^2}}{u_\tau^2} = \int_{y/\Lambda_o}^{\alpha_{33}} \underbrace{I_{33}(\eta)}_{\text{const}} \frac{M}{\eta} d\eta \quad = I_{33} M \log\left(\frac{\alpha_{33}\Lambda_o}{y}\right) \tag{15}$$

$$\frac{\overline{u_2^2}}{u_\tau^2} = \int_{y/\Lambda_o}^{\alpha_{22}} \underbrace{I_{22}(\eta)}_{\eta^2} \frac{M}{\eta} d\eta \quad = \frac{M}{2}\left[\alpha_{22}^2 - \left(\frac{y}{\Lambda_o}\right)^2\right] \tag{16}$$

where we find that $\overline{u_3^2}$ also has a logarithmic variation, but that the wall-normal intensity, $\overline{u_2^2}$, is roughly constant very near the wall.

We can use the same attached eddy analysis to consider the mean shear, $\frac{\partial \bar{u}}{\partial y}$, measured at location y. The contribution of hierarchy Λ to the mean shear is then

We skip the integration step over the individual eddies, which we wrote explicitly above, and jump directly to the contribution over the entire hierarchy

$$\frac{\delta \partial \bar{u}}{\partial y} = \frac{u_\tau}{\Lambda} f\left(\frac{y}{\Lambda}\right) \tag{17}$$

where the function f is analogous to the eddy intensity function, I_{ij} for turbulence intensity. To obtain the total contribution to the mean shear over all hierarchies, we write the integral

$$\frac{\partial \bar{u}}{\partial y} = \int_{\Lambda_i}^{\Lambda_o} \frac{\delta \partial \bar{u}}{\partial y} P_\Lambda(\Lambda) d\Lambda \quad = \int_{\Lambda_i}^{\Lambda_o} \frac{u_\tau}{\Lambda} f\left(\frac{y}{\Lambda}\right)\left(\frac{M}{\Lambda}\right) d\Lambda \quad = u_\tau \int_{\Lambda_i}^{\Lambda_o} \frac{M}{\Lambda^2} f\left(\frac{y}{\Lambda}\right) d\Lambda \tag{18}$$

and making the same change of variables as earlier, $\eta = \frac{y}{\Lambda}$, we obtain

$$\frac{\partial \bar{u}}{\partial y} = u_\tau \int_{y/\Lambda_i}^{y/\Lambda_o} \frac{M\eta^2}{y^2} f(\eta)\left(-\frac{y}{\eta^2}\right) d\eta \quad = \frac{u_\tau M}{y} \int_{y/\Lambda_o}^{y/\Lambda_i} f(\eta) d\eta \tag{19}$$

We can write this non-dimensionally as

$$\frac{\partial \bar{u}^+}{\partial y^+} = \frac{M}{y^+} \underbrace{\int_{y/\Lambda_o}^{y/\Lambda_i} f(\eta) d\eta}_{\text{const}} \quad \propto \frac{1}{\kappa y^+} \tag{20}$$

We can see that if we assume that the integral converges to a constant then we recover the logarithmic mean velocity profile. But whether the integral converges to a constant depends on the form of the function $f(\eta)$, just like our previous depended on the form of the function, $I_{ij}(\eta)$. Earlier, $I_{ij}(\eta)$ represented the induced Reynolds stresses from a given hierarchy of eddies; here $f(\eta)$ represents the induced viscous stresses from a given hierarchy of eddies. The choice of the structure of the eddies in the hierarchy fixes $f(\eta)$ and thus fixes the statistical properties of the turbulent flow. So again we have a tool which allows us to relate the structure of the eddies in turbulence (i.e. their average shape) with the statistical description of turbulence.

Hairpin Vortices

It turns out that there are a number of different choices of hierarchy shapes with corresponding functions $f(\eta)$ that all satisfy the integral constraint and produce a logarithmic velocity profile. We show just two samples below. The common feature of these two is the hairpin or Greek Λ shape of the eddy.

See Perry et. al. (JFM 1986) for details of all of the possible eddy hierarchy shapes

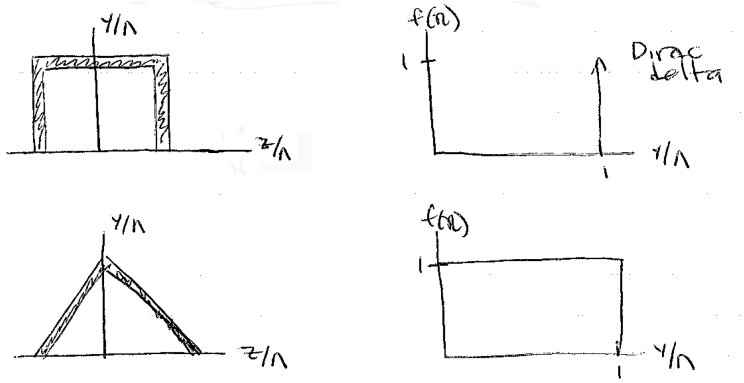


Figure 48: Two examples of attached eddy distributions consistent with the logarithmic velocity profile. (Top row) A distribution where shear is mostly confined to a spanwise vortex at height $y = \Lambda$. (Bottom row) Inclined spanwise vortices which produce an even distribution of shear at all y locations.

The idea of a hairpin-shaped eddy as being the key to wall-bounded turbulence slightly predates the attached eddy hypothesis and was first proposed by Theodorsen (1952). In his model, we observe vortex heads with legs trailing behind, inclined at 45°, with characteristic dimensions scaling on the distance from the wall. The attached eddy hypothesis makes the connection between this family of hairpin shaped eddies (or eddy hierarchies) and the statistical picture of turbulence velocity and fluctuations.

It is important to stress that the shape of the eddies is still only a statistical shape, i.e. the average shape of an ensemble of eddies, and thus no single eddy must actually exhibit this eddy shape in order to produce the statistical picture of the attached eddy. In Theodorsen's hairpin model, the region between the legs of the hairpin (behind the head) shows an upwelling of low speed fluid from near the wall. The region in front of the hairpin head shows a downsweep of high speed fluid from away from the wall.

We can denote the upwelling as $(-u, +v)$ to indicate that the fluid is moving slower than the mean flow in the streamwise direction, and positively quicker than the mean flow in the wall-normal direction. The downsweep is then denoted as $(+u, -v)$

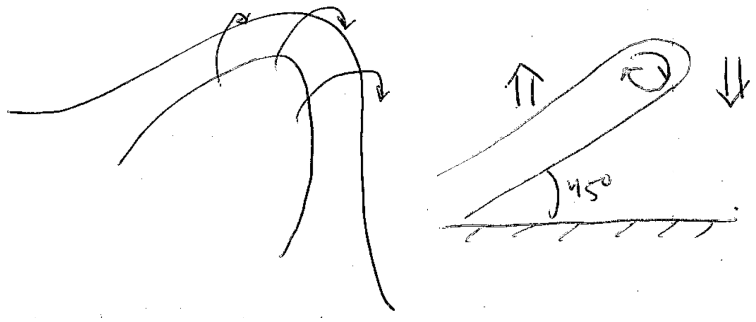


Figure 49: Theodorsen's hairpin model

Both the upwelling and downstream motions are motions in which the net velocity fluctuation in the streamwise and wall-normal directions have opposite signs, i.e. for the upwelling, there is fluid moving slowly with respect to the mean $u < 0$ but moving away from the wall $v > 0$, such that $\overline{uv} < 0$ in the vicinity of the upwelling or the downsweep. Recall from our discussion of turbulence production that, in general, τ_{ik}^R has the same sign as the mean velocity gradient. So for wall-bounded flows, we have $\tau_{ik}^R = -\rho\overline{u_i u_k} > 0$ which means that $\overline{u_i u_k} < 0$. Therefore, the two motions which surround the hairpin eddy are also the two possible motions which contribute to the Reynolds stress with the sign that is significant for the turbulent energy balance. Because of this significance, these two motions have been given names: the upwelling is called an ejection and the downsweep is referred to as a sweep. We can illustrate these motions by labeling the different possible signed fluctuations in four quadrants.

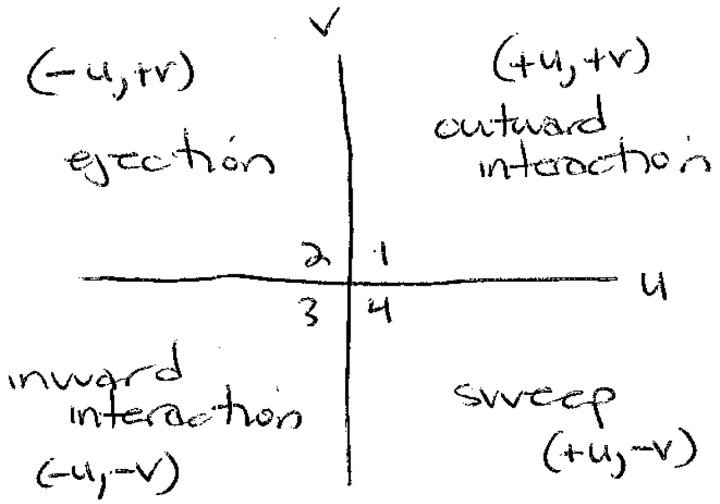


Figure 50: Quadrant analysis for two-dimensional streamwise (u) and wall-normal (v) velocity fluctuations.

Note that these motions should technically involve some sort of non-local averaging, since we are interested in identifying coherent motions which necessarily extend over more than a single point in the flow. However, for now, we will consider the quadrant criteria as defined pointwise, and we will return to the question of how to

define coherent structures non-locally. Ejections are motions that occur instantaneously in quadrant 2; sweeps are motions that occur instantaneously in quadrant 4. The two motions are, of course, related by continuity, since new fluid must replace the fluid displaced from the wall. Together, ejections and sweeps contribute 120% of the negative Reynolds stress, $\overline{uv} < 0$, meaning that motions in quadrants 1 and 3 contribute the remaining -20% of the negative stress (i.e. they contribute 20% of positive Reynolds stress), when measured at $y^+ \approx 12$.

While hairpins are crucially important to the structural description of the turbulent energy balance, due to their role in production in the region of the log layer, they are not the only significant structures in wall-bounded turbulence. There is another class of structures which inhabit the region even nearer to the wall, which are associated with the hairpin legs, and are known as low speed streaks.

Low-Speed Streaks

This viscous sublayer is composed of elongated, alternating streaks of high and low speed fluid, where low speed indicates that the speed within the streak is about half the mean streamwise velocity across the entire flow at that height, and high speed indicates that the streak velocity is 3/2 of the mean velocity. Some of these streaks lift up into the buffer layer. They show a spanwise periodicity of about 100 viscous units (ν/u_τ), independent of Reynolds number. Beyond around $y+ > 40$, the streaks are not as well defined and the spanwise spacing increases with increasing y^+. The streaks extend around 1000 viscous units in the streamwise direction, although they also meander and appear to connect to one another to form larger assemblages. The streaks are bounded by quasi-streamwise oriented vortices, which generate the streaks by means of momentum transport between the wall and the outer flow.

The streamwise vortices have their origins in the vorticity sheet created along the wall during the instantaneous startup of any fluid flow, laminar or turbulent. At sufficiently high Reynolds number, this vortex sheet becomes unstable and rolls up into streamwise vortex rolls, which then recede away from the wall. Before the vortices lift away, we can illustrate their affect on the local velocity fluctuations in the flow.

Between the pair of vortices, the distribution of the streamwise velocity component, u, is lifted away from the wall. The distribution is pushed towards the wall outside of the vortex pair. Therefore, between the vortices there is an ejection of low speed fluid (quadrant 2) and outside of the vortex pair exist sweeps of high speed fluid (quadrant 4). The ejection results in a low speed streak between the vortices; the sweep results in high speed streaks bounding the vortex pair. Both of these motions contribute towards the negative

Reynolds stress and thus contribute to the production of turbulent kinetic energy.

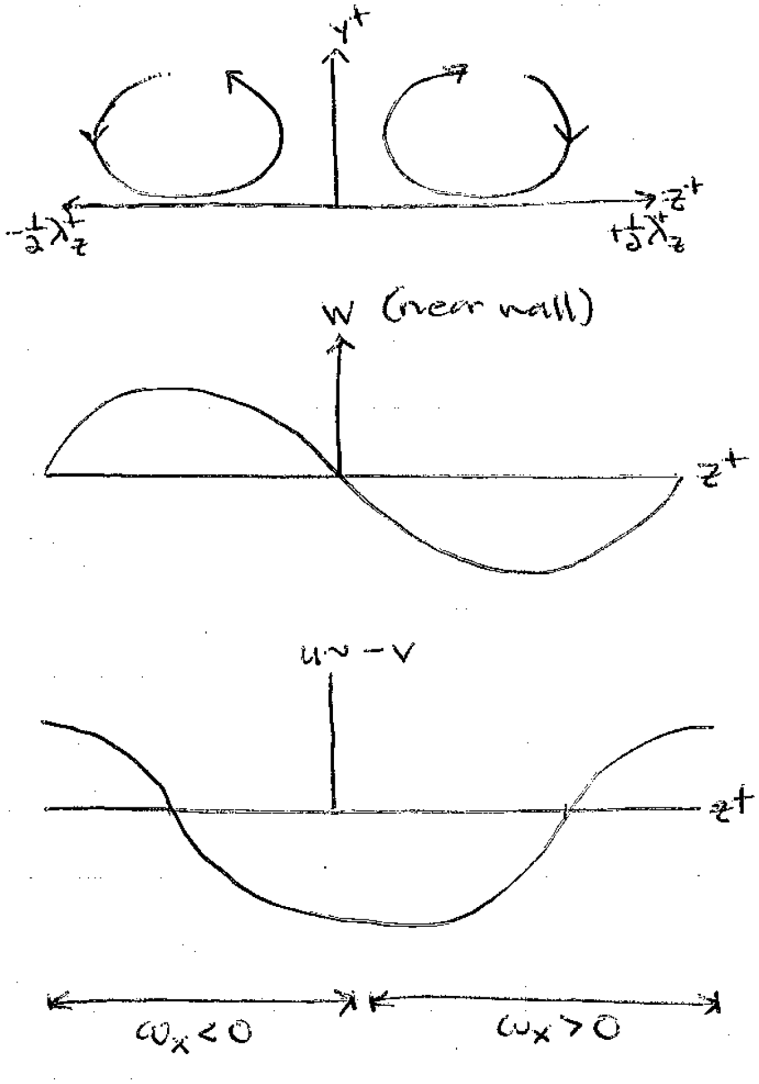
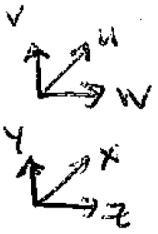


Figure 51: Quasi-streamwise vortices and the corresponding velocity components induced by the vortices.



Imagine an initially laminar flow with no velocity fluctuations; the presence of vortices results in a transport and mixing of mean flow, generating local fluctuations and thus an increase in TKE.

We can see that the average TKE over a spanwise period of vortices exhibits a net increase due to the action of the vortices. But in this simplified description, the vortices themselves remain unchanged, so the energy for the increased TKE come from the mean shear, and the vortices are just tools by which the energy is extracted from the mean shear and transported into turbulence.

Since the increased TKE leads to increased dissipation, ε, (at least for close to equilibrium conditions), and the sign of $\overline{uv}$ is uniformly negative across the high and low speed streaks, it might seem obvious that the skin friction increases due to the quasi-streamwise turbulent vortices. However, the mean velocity gradient itself will also be affected by the low speed streaks, thus reducing the gradient and reducing the skin friction where there are low speed streaks, and increasing it where there are high speed streaks. Because of these alternating streaks, we will need to be more precise

in order to show that, in fact, there is a net increase in skin friction due to the existence of the quasi-streamwise vortices.

Let's start by assuming no streamwise (x) variation in our model of streaks, and thus $\frac{\partial u}{\partial x} = 0$. This leaves us with a simplified continuity equation of

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{21}$$

We further assume no pressure gradient in the streamwise direction, and we can rewrite the streamwise momentum equation as

$$\frac{\partial u}{\partial t} + \underbrace{v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{= \frac{\partial}{\partial y}uv + \frac{\partial}{\partial z}uw} = \nu\nabla^2 u \tag{22}$$

Note that we can write: $\frac{\partial}{\partial y}uv + \frac{\partial}{\partial z}uw = u\underbrace{\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{=0} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$

Then, differentiating with respect to y yields

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial^2}{\partial y^2}uv + \frac{\partial^2}{\partial z \partial y}uw = \nu\nabla^2\left(\frac{\partial u}{\partial y}\right) \tag{23}$$

Now we assume that the vortices are periodic in the spanwise direction, z, and average over a single period such that the resulting dynamics will have no z-dependence.

$$\frac{\partial}{\partial t}\left\langle\frac{\partial u}{\partial y}\right\rangle + \frac{\partial^2}{\partial y^2}\langle uv\rangle + \underbrace{\frac{\partial}{\partial z}\left\langle\frac{\partial}{\partial y}uw\right\rangle}_{=0} = \nu\nabla^2\left\langle\frac{\partial u}{\partial y}\right\rangle \tag{24}$$

We rewrite this equation in terms of the spanwise-vorticity, $\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Applying the assumptions of spanwise averaging and full-developed flow in the x direction, we obtain $\langle\omega_z\rangle = -\left\langle\frac{\partial u}{\partial y}\right\rangle$ and thus

$$\frac{\partial}{\partial t}\langle\omega_z\rangle = \frac{\partial^2}{\partial y^2}[\nu\langle\omega_z\rangle + \langle uv\rangle] \tag{25}$$

If we neglect viscosity, then the righthand side can be rewritten as

$$\frac{\partial}{\partial t}\langle\omega_z\rangle \approx \frac{\partial^2}{\partial y^2}\langle uv\rangle \quad \equiv \left\langle 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2}\right\rangle \tag{26}$$

and then evaluating the dynamics at the wall, $y = 0$, we have $u = v = 0$ and thus

$$\left.\frac{\partial}{\partial t}\langle\omega_z\rangle\right|_{y=0} \approx 2\left\langle\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\rangle\Bigg|_{y=0} \tag{27}$$

We see that the wall vorticity increases in time if $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ are positively correlated over a spanwise period

Streak	$\frac{\partial u}{\partial y}$	$\frac{\partial v}{\partial y}$
High Speed Sweep, $v < 0$	−	−
Low Speed Ejection, $v > 0$	+	+

Table 13: Wall Vorticity in Near-Wall Streaks. Note that $u = v = 0$ at the wall. For a high speed sweep, u decreases moving away from the wall; for a low-speed ejection, u increases away from the wall.

and these two quantities are, indeed, positively correlated everywhere. So the effect of the quasi-streamwise vortices at the wall is to increase the wall vorticity everywhere, and the wall vorticity is equivalent to the skin friction, so flows with alternating high and low speed streaks have high skin friction, averaged across a full spanwise period, than flows without the vortices.

Because the increased skin friction of turbulent flows can be explained by the action of these quasi-streamwise vortices at the wall, many drag reduction strategies are designed to disrupt these vortices by means of spanwise oscillations, spanwise distributed roughness (riblets), or other types of actuation to prevent these vortices from persisting.

These quasi-streamwise vortices tend to lift up, away from the wall. So why does the process continue? In fact, new vortices develop due to an instability in the streaks themselves. In other words, there is an autonomous near-wall cycle of quasi-streamwise vortices begetting streaks begetting quasi-streamwise vortices, all driven by the presence of a mean shear (and independent of whether there is a wall or not).

Coherent Structure Indentification

We have identified a number of important vortex structures that are crucial to turbulence dynamics, including attached eddies and low speed streaks. And we introduced a quadrant criterion which is useful for describing these structures on a point-wise basis, but we know that the structures are not pointwise, but extended in space and time. Therefore, we need to develop tools to describe these structures non-locally. But we first face the problem of defining what we mean by an eddy or a coherent structure. In the very first lecture, we attempted to define coherent motions in terms of definitions by Corrsin and Robinson. We return now to Robinson's definition:

> A vortex exists when instantaneous streamlines mapped onto a plane normal to the vortex core exhibit a roughly circular or spiral pattern when viewed in a reference frame moving with the center of the vortex core

This definition is a tautology in the sense that we need to know where the core of the vortex is in order to define the vortex itself. It is also problematic because we need to know how fast the vortex core is moving in order to define the vortex, but we need to define the vortex in order to determine its speed. However, we could imagine an iterative algorithm for locating vortices by means of Galilean transformations: cycle through a range of different possible mean convection velocities for vortex cores; for each velocity substract the mean velocity from the instantaneous velocity and visualize the cores by means of streamlines; when the correct convection velocity is selected, the streamlines will appear in the circular pattern; then repeat to identify the remaining vortex cores.

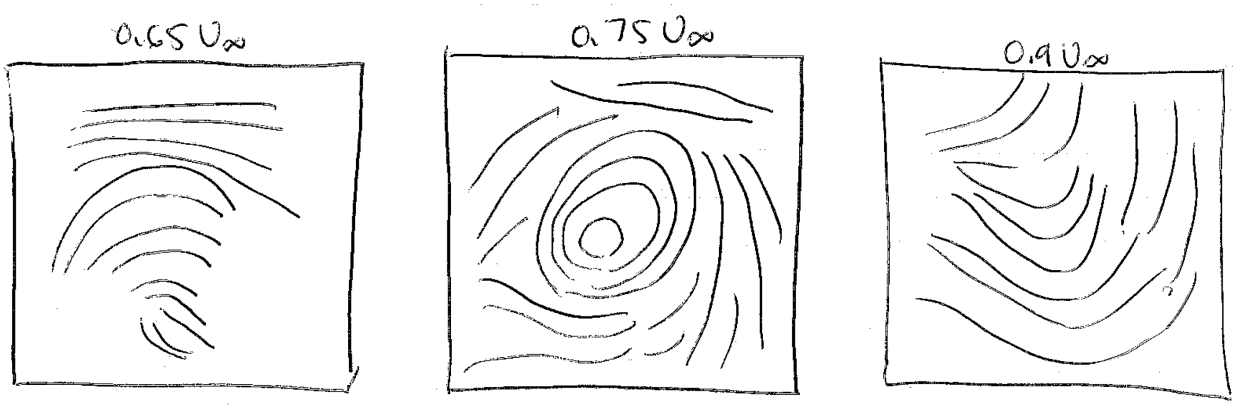


Figure 52: Galilean transformations of the instantaneous velocity field. When the correct mean velocity is substracted, the vortex core appears with closed streamlines. See Adrian, et. al. (Exp. in Fluids 2000) for more details on the Galilean decomposition.

To avoid this iteration, we might consider a different composition. Instead of the Galilean decomposition, where we substract a single mean convection velocity from the entire flow field, chosen via iteration, we could use a Reynolds decomposition, in which we substract the local, ensemble mean velocity field, in order to reveal the Reynolds fluctuations. This decomposition tends to reveal many more vortex cores than any single Galilean decomposion because the convection velocity of vortex cores tends to be very close to the local mean convection velocity in the flow. But, if the vortex cores do not convect at the local mean velocity, because the instantaneous velocity profile is very different from the mean profile, then this technique will not work. And this divergence between instantaneous and mean profiles occurs wherever the turbulence intensity is large, in particular near the wall of wall-bounded flows. In such a case, simply substracting the local mean velocity will not actually reveal vortex cores that are travelling at a different velocity. Moreover, the Reynolds decomposition will remove large-scale features of the flow, in the process of removing the local average, obscuring large-scale vortices.

The basic problem is that the definition of the vortex is not Galilean-invariant (i.e. it changes with the addition of a constant) and many vortex identification techniques are Galilean-invariant (i.e. they produce the same result independent of the addition of a constant, e.g. the Reynolds decomposition). Despite this fundamental problem that most of the non-iterative techniques are Galilean invariant, there is still some hope for successful vortex identification using the following basic approaches: 1) vorticity, 2) instantaneous dissipation, $\tilde{\varepsilon}$, 3) complex eigenvalues of $\nabla\vec{u}$, 4) second invariant of $\nabla\vec{u}$, 5) low pressure cores, 6) local pressure minimum in a plane, 7) swirling strength.

1. Vorticity: This seems to be the intuitive choice for identifying vortices, since we expect a vortex core to be a region of concentrated voticity. Let's consider how it might work. Consider the two-dimensional instantaneous vorticity field in the x-y plane,

i.e. $\tilde{\omega}_z$, defined as

$$\tilde{\omega}_z = \frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y} \tag{28}$$

Recall the usual decomposition: $\tilde{u} = \bar{u} + u$

Now consider a shear flow where $\frac{\partial \bar{u}}{\partial y}$ is very large. This mean shear will then appear as a significant contribution to vorticity even though it does not represent coherent vortex cores. Any time the background shear (the mean shear) is comparable in magnitude to the vorticity of a vortex core, the vortex core may be obscured by the shear. So the vorticity field is strongly biased by the presence of mean shear, which is worst near the wall. To get rid of the mean shear contamination, we could use the Reynolds decomposition and then calculate the vorticity field of the Reynolds fluctuations, but since the Reynolds decomposition is not Galilean, it will not identify vortices according to our Galilean definition. These problems persist in three-dimensions as well. In the x-z plane near the wall, $\tilde{\omega}_y$ is dominated by the shear between the near-wall streaks.

Vorticity can provide the sign (sense of rotation) of a vortex core that has already been identified, but is not useful for identifying the cores themselves.

2. Instantaneous Dissipation: We know the dissipation of TKE occurs at the smallest scales of the flow, and we showed that the physical mechanism of energy transfer from larger scales down to the smallest scales is vortex stretching. Therefore, we might expect that locations of intense vortex stretching will coincide with locations of intense dissipation, so we can use dissipation to find the stretching vortices. The instananeous dissipation is written as

$$\tilde{\varepsilon} = 2\nu \tilde{s}_{ij} \tilde{s}_{ij} \tag{29}$$

Recall the instantaneous rate of strain tensor $\tilde{s}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right)$

or, in more illustrative form, as

$$\tilde{\varepsilon} = \nu \left(\tilde{\omega}_i \tilde{\omega}_i + \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \tilde{u}_j}{\partial x_i} \right) \tag{30}$$

where we can see that both shear and vorticity are associated with the dissipation, and thus the dissipation will suffer the same bias towards detecting shear that the vorticity itself did.

3. Complex Eigenvalues of $\nabla \vec{u}$: Let's consider the velocity gradient tensor as part of a simple linear dynamics to describe the trajectories (pathlines) of fluid particles, $\vec{x}$

$$\frac{d\vec{x}}{dt} = (\nabla \vec{u})\vec{x} \tag{31}$$

To identifiy vortices, we seek pathlines which appear to be closed orbits in the Galilean frame moving with the center of the orbit. In the linear system, we can describe these trajectories by the eigenvalues of the matrix $\nabla \vec{u}$. Complex eigenvalues represent trajectories that are closed or in spiral patterns in a reference

frame moving at the velocity of the local point of interest. To calculate the eigenvalues for a matrix $(\nabla \vec{u})$, we need to solve

$$[(\nabla \vec{u}) - \lambda \vec{I}]\vec{v} = 0 \tag{32}$$

where λ are the eigenvalues and $\vec{v}$ are the eigenvectors. The eigenvalues are found by solving

$$\det[(\nabla \vec{u}) - \lambda \vec{I}] = 0 \tag{33}$$

which, if $(\nabla \vec{u})$ is three-dimensional, results in a characteristic equation of the form

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0 \tag{34}$$

where the matrix invariants are defined as $P = -\mathrm{tr}(\nabla \vec{u})$, $Q = \frac{1}{2}[P^2 - \mathrm{tr}\{(\nabla \vec{u})^2\}]$, and $R = -\det(\nabla \vec{u})$. If the flow is incompressible, $\nabla \cdot \vec{u} = 0$, then we can simplify the first two invariants as $P = 0$ and $Q = -\frac{1}{2}\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_j}{\partial x_i}$. Now that we have the characteristic equation for the eigenvalues, we need to determine the nature of the eigenvalues. For this we need to know the invariants $\det(\nabla \vec{u})$ and $\mathrm{tr}(\nabla \vec{u})$ as well as the discriminant, Δ. Using these three values, we can map the behavior of the eigenvalues

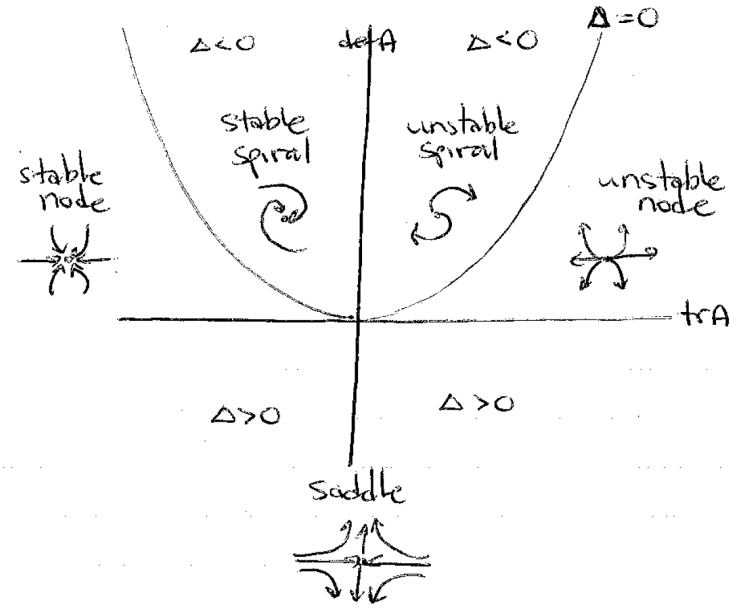


Figure 53: Map of eigenvalues for a second-order linear system, as a function of matrix invariants and the discriminant.

where we observe that the closed and spiral pathlines occur when $\Delta < 0$. This result is clear from the simplest quadratic system, $ax^2 + bx + c = 0$. The discriminant is given by $\Delta = b^2 - 4ac$, from the quadratic formula, and thus when the discriminant is negative, the result is complex eigenvalues. For a general third-degree polynomial, $ax^3 + bx^2 + cx + d = 0$ we have the discriminant $\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$. Rewriting this in terms of the characteristic equation for the velocity

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

dynamics, we obtain

$$\Delta = -(2^2 3^3)\left[\left(\frac{Q}{3}\right)^3 + \left(\frac{R}{2}\right)^2\right] \tag{35}$$

and thus the discriminant criterion $\Delta < 0$ is equivalent to the requirement that

$$\left(\frac{Q}{3}\right)^3 + \left(\frac{R}{2}\right)^2 > 0 \tag{36}$$

By constructing an indicator function for $\text{sgn}\left[\left(\frac{Q}{3}\right)^3 + \left(\frac{R}{2}\right)^2\right]$, every point in the flow can be labeled as associated with locally spiralling streamlines or not, and thus the coherent vortex cores can be found by the simple criterion of requiring complex eigenvalues.

4. Second Invariant of $\nabla\vec{u}$ (Q-Criterion): Although the discriminant criterion fixes complex eigenvalues, it is a very general requirement. In order to narrow it a bit, consider the form of the discriminant again

$$\left(\frac{Q}{3}\right)^3 + \underbrace{\left(\frac{R}{2}\right)^2}_{\geq 0} > 0 \tag{37}$$

It is clear that the criterion does not depend on the value of the invariant R, since that part of the discriminant is always positive. However, the invariant Q could be positive or negative, and thus Q alone really determines the flow trajectories. We wrote the form of the Q invariant above

$$Q = -\frac{1}{2}\frac{\partial\tilde{u}_i}{\partial x_j}\frac{\partial\tilde{u}_j}{\partial x_i} = -\underbrace{\tilde{s}_{ij}\tilde{s}_{ij}}_{\text{Straining}} + \frac{1}{2}\underbrace{\tilde{\omega}_i\tilde{\omega}_i}_{\text{Rotation}} \tag{38}$$

where we can see that Q represents a balance between straining and rotation. If $Q > 0$, this means that rotation dominates straining, and automatically satisfies the discriminant requirement for spiralling pathlines. So this requirement is slightly narrower than the general requirement that the pathlines be circular, since here the requirement is that within all the cases of circular streamlines, $Q > 0$ selects those regions where rotation is the dominant physical process.

This Q-criterion has an additional advantage over measures like vorticity itself, since Q is zero at the wall, whereas vorticity reports a maximum. Writing the velocity gradient tensor at the wall, with no-slip and no-penetration conditions yields

Note that $\frac{\partial\tilde{v}}{\partial y} = 0$ since $\frac{\partial\tilde{u}}{\partial x} = 0$ and $\frac{\partial\tilde{w}}{\partial z} = 0$, and using continuity

$$\nabla\vec{u}\Big|_{y=0} = \begin{pmatrix} 0 & \frac{\partial\tilde{u}}{\partial y} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\partial\tilde{w}}{\partial y} & 0 \end{pmatrix} \tag{39}$$

and thus it is clear that at the wall, $Q = -\frac{1}{2}\text{tr}\left[(\nabla\vec{u})^2\right] = 0$, and the Q-criterion will not falsely report the presence of coherent structures at the wall itself. .

Similarly, $R = 0$ at the wall, so the discriminant criterion is also free from the problem of false-positives at the wall

5. Low Pressure Cores: On physical grounds, we expect vortex cores to coincide with pressure minima. We can show this from consideration of the radial component of the momentum equation

$$\frac{\partial\tilde{u}_r}{\partial t} + (\vec{u}\cdot\nabla)\tilde{u}_r - \frac{1}{r}\tilde{u}_\theta^2 = -\frac{1}{\rho}\frac{\partial\tilde{p}}{\partial r} + \nu\left(\nabla^2\tilde{u}_r - \frac{\tilde{u}_r}{r^2} - \frac{2}{r^2}\frac{\partial\tilde{u}_\theta}{\partial\theta}\right) \quad (40)$$

Consider an ideal vortex with perfectly circular streamlines, i.e. $\tilde{u}_r = 0$ and no variation in θ. Then

A simple force balance between the radial presure gradient and the centripetal acceleration also yields $\frac{d\tilde{p}}{dr} = \rho\Omega^2 r$ or, rewriting, $\frac{d\tilde{p}}{dr} = \rho\left(\frac{\tilde{u}_\theta}{r}\right)^2 r$

$$-\frac{1}{r}\tilde{u}_\theta^2 = -\frac{1}{\rho}\frac{\partial\tilde{p}}{\partial r} \quad (41)$$

We can define the azimuthal velocity for a discrete vortex core by using the vorticity in a finite core

$$\tilde{\omega}(r) = \begin{cases}\omega_0 & r \le r_0 \\ 0 & r > r_0\end{cases} \quad (42)$$

Then, working backwards from the definition of vorticity, we have for $r > r_0$ the result that $\tilde{u}_\theta = \frac{D}{r}$ where D is a constant of integration. Substituting into the momentum balance then yields

$\omega = \nabla\wedge\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)$ so then $0 = \frac{\partial}{\partial r}(ru_\theta)$ and const $= ru_\theta$

$$\frac{\partial\tilde{p}}{\partial r} = \frac{\rho}{r}\left(\frac{D}{r}\right)^2 \quad (43)$$

and integrating from r to ∞ yields

$$\tilde{p} = \tilde{p}_\infty - \frac{1}{2}\rho D\frac{1}{r^2} \quad (44)$$

and thus the minimum pressure occurs at the smallest value of r, which is r_0, the outer edge of the vortex core itself. Thus we have shown that vortex cores coincide with pressure minima. But the question remains: how do we identify pressure minima, particularly if we are working with instantaneous velocity data.

6. Local Pressure Minimum (λ_2): In order to establish a algorithm for finding pressure minima and thus vortex cores, we start from the momentum equation, ignoring unsteady and viscous effects

$$\tilde{u}_j\frac{\partial\tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\tilde{p}}{\partial x_i} \quad (45)$$

Taking the gradient of both sides yields

$$\frac{\partial}{\partial x_k}\left(\tilde{u}_j\frac{\partial\tilde{u}_i}{\partial x_j}\right) = -\frac{1}{\rho}\frac{\partial^2\tilde{p}}{\partial x_i\partial x_k} \quad (46)$$

and rewriting again in terms of the rate of strain ($\vec{s}$) and rate of rotation ($\vec{r}$) tensors yields

$$\tilde{s}_{ik}\tilde{s}_{kj} + \tilde{r}_{ik}\tilde{r}_{kj} = -\frac{1}{\rho}\tilde{p}_{,ij} \quad \equiv -\frac{1}{\rho}H_{ij} \quad (47)$$

The comma subscript indicates partial differentiation with respect to the subsequent indices

A local pressure minimum can now be identified by means of the second partial derivative test. Consider just a single plane, x-y, and construct the Hessian matrix, H, of instantaneous pressure

$$H(x,y) = \begin{pmatrix} \tilde{p}_{,xx} & \tilde{p}_{,xy} \\ \tilde{p}_{,yx} & \tilde{p}_{,yy} \end{pmatrix} \tag{48}$$

where $\det H = \tilde{p}_{,xx}\tilde{p}_{,yy} - \tilde{p}^2_{,xy}$. Then, we apply the second-derivative test according to

Note that $\tilde{p}_{,xy} = \tilde{p}_{,yx}$ assuming that pressure is sufficiently continuous.

Criterion	Conclusion
$\det H > 0$ and $\tilde{p}_{,xx} > 0$	Local Minimum
$\det H > 0$ and $\tilde{p}_{,xx} < 0$	Local Maximum
$\det H < 0$	Saddle Point
$\det H = 0$	Inconclusive

Table 14: Second Partial Derivative Test

Or, more generally, we can describe the test in terms of eigenvalues, where a positive definite matrix has all positive eigenvalues, and negative definite has all negative eigenvalues

H Criterion	Conclusion
Positive Definite	Local Minimum
Negative Definite	Local Maximum
Mixed Eigenvalues	Saddle Point

Table 15: Second Partial Derivative Test Eigenvalues

If two eigenvalues of H are positive, then there is a local minimum in some plane of space, i.e. when approaching that point from any direction within the plane. If all three eigenvalues of H are positive, then that point is a local minimum when approached from any direction in space. So, for a local pressure minimum in at least a single plane in space, we need the Hessian, which is equivalent to $\vec{s}^2 + \vec{r}^2$ to have 2 negative eigenvalues. We note that the Hessian is a symmetric matrix, so all of the eigenvalues are real, and can thus be ordered

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \tag{49}$$

in which case $\lambda_2 < 0$ for a local pressure minimum to exist in a plane. Hence we have arrived at a new criterion on the eigenvalues of the symmetric and antisymmetric parts of the velocity gradient tensor which identifies a local planar pressure minimum.

How does this λ_2 criterion compare with the Q-criterion? Recall that $Q = -\frac{1}{2}\mathrm{tr}(\vec{s}^2 + \vec{r}^2)$ and thus, substituting the eigenvalues, $Q = -\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3)$.

λ_1	λ_2	λ_3	$\sum \lambda_i$	$\lambda_2 < 0$	$Q > 0$
+	−	−	−	Yes	Yes
+	−	−	+	Yes	No
+	+	−	−	No	Yes
+	+	−	+	No	No

Table 16: Q and λ_2 Criteria

We can compare the two criteria for different combinations of eigenvalues, one of which depends on the sum of the eigenvalues and the other on the sign of two out of the three eigenvalues. We see that the minimum pressure criterion of λ_2 does not always agree with the Q criterion, although one is not necessarily more general than the other. They simply measure different features of the flow.

7. Swirling Strength: We return to the eigenvalue-based criteria again, but this time we include not only the sign of the eigenvalues, but also their magnitude. In order to obtain complex eigenvalue, we required $\Delta < 0$. For the cubic characteristic equation, this criterion demands one real eigenvalue, λ_r, and two complex conjugate eigenvalues, $\lambda_{cr} \pm \lambda_{ci}$. By using the analytical solution for the cubic characteristic equation, we can write these eigenvalues as

If $\Delta = 0$ then all eigenvalues are real and there is a repeated value; if $\Delta > 0$ then the characteristic equation has three distinct real eigenvalues.

$$\lambda_r = \zeta + \eta \tag{50}$$

$$\lambda_{cr} = -\frac{1}{2}(\zeta + \eta) \tag{51}$$

$$\lambda_{ci} = \frac{\sqrt{3}}{2}(\zeta - \eta) \tag{52}$$

where

$$\zeta = \left(-\frac{R}{2} + \sqrt{\left(\frac{R}{2}\right)^2 + \left(\frac{Q}{3}\right)^3} \right)^{1/3} \tag{53}$$

$$\eta = \left(-\frac{R}{2} - \sqrt{\left(\frac{R}{2}\right)^2 + \left(\frac{Q}{3}\right)^3} \right)^{1/3} \tag{54}$$

and then we simply use λ_{ci} as a measure for the amount of actual 'swirling' because the imaginary part of the complex conjugate eigenvalue is the part which is ultimately responsible for the circular streamlines, and thus the larger it is, the more dominant the vortical motion. Of course this eigenvalue does not provide the directional sense of the rotation (just like many of the other eigenvalue criteria) and thus it must be used in conjunction with the sign of the instantaneous vorticity itself.

See Jeong & Hussain (JFM 1995) for more discussion on the differences between the different vortex detection techniques. See also Robinson's NASA Technical Memorandum 103859.

Taylor's Hypothesis

One of the key problems we highlighted in vortex eduction is that Reynolds-decomposed velocity fields can find many more vortices, but they implicitly assume that coherent structures advect at the local mean velocity of their present location within the flow field. Because this is not true, particularly near the wall, we are forced to use Galilean decompositions, which often require iteration to find vortices. This raises the important question: when can we

assume that the structural features of turbulence convect along at the local mean velocity and when not? This assumption is crucially important if we want to transform between temporal and spatial measurements, by using Taylor's Hypothesis.

To explain, let's begin with the velocity correlation function, written to include both temporal and spatial variation explicitly. For the streamwise velocity component, we can write

$$R_{11}(\Delta x, \Delta t, y) = \frac{\overline{u(x,t,y)u(x+\Delta x, t+\Delta t, y)}}{u_{\text{rms}}(y)u_{\text{rms}}(y)} \tag{55}$$

for the correlation at a given y-location in the flow. We can sketch this function for the two independent cases, for spatial variation, where $\Delta t = 0$, and for temporal variation, where $\Delta x = 0$, to obtain the typical correlation plots.

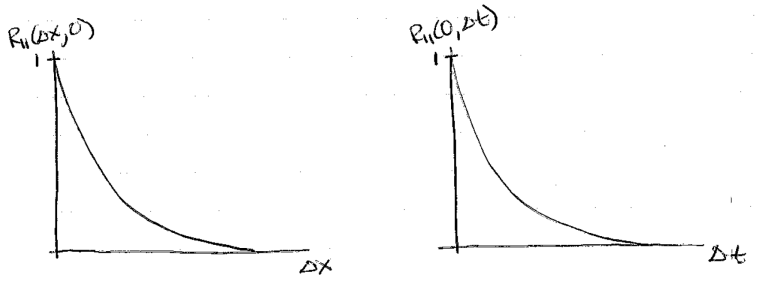


Figure 54: Streamwise autocorrelation functions in space and time.

But, what happens if we allow both time and space variation? In other words, let's compare velocity measurements at points that are separated by both space and time. Imagine considering this variation incrementally. First we calculate the spatial correlation $R_{11}(\Delta x)$ between two velocity signals with no time delay between them, $\Delta t = 0$. Then we repeat this spatial calculation for two signals that have finite time delay, $\Delta t > 0$. Each spatial correlation is then characterized by a new a different time delay, Δt_1, Δt_2, Δt_3, etc.

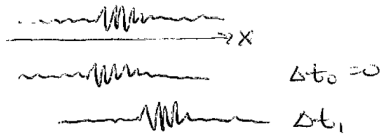


Figure 55: Two spatial signals compared with two different time delays.

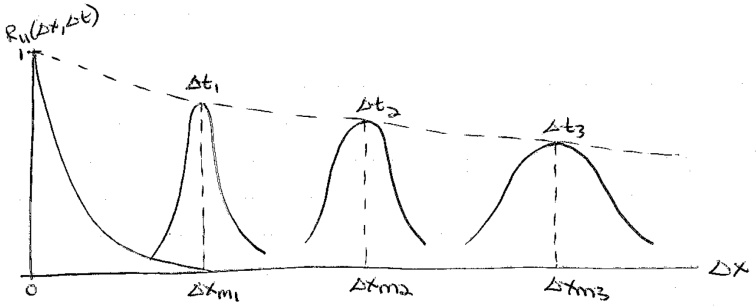


Figure 56: Spatial correlation functions, $R_{11}(\Delta x)$, between two spatial signals, evaluated with different time delays between the signals. The location of the maximum spatial correlation, Δx_m is marked for each time delay.

As the time delay, Δt increases, the location, Δx_m of the maximum spatial correlation between the signals also increases, almost as if the the spatial signals were being displaced spatially. Why does this happen?

If we imagine that each blob of fluid has a particular spatial structure (pattern/signal) and the blobs are being convected downstream by some mean convective velocity of the surrounding flow,

then it makes sense that the distance of maximal spatial correlation between two blobs increases with time, as the blobs convect farther apart in space. If the convective velocity is constant, then we should be able to define that velocity in terms of the ratio of spatial to temporal delays.

If the ratio of delays is not constant, then the equivalent convective velocity for the blob is obviously not the local constant mean velocity of the flow, but rathter some velocity that evolves in either space or time.

$$\bar{u}_{\text{conv}} = \frac{\Delta x_{m_i}}{\Delta t_i} \tag{56}$$

Similarly, we can construct the same plot for the time-correlations of two signals, $R_{11}(\Delta t)$, that are measured at different spatial delays, Δx_i, and then track the maximum time delays corresponding to each spatial delay pair, Δ_{m_i}. And we can combine these two types of plots in three-dimensions to see the overall behavor of $R_{11}(\Delta x, \Delta t)$.

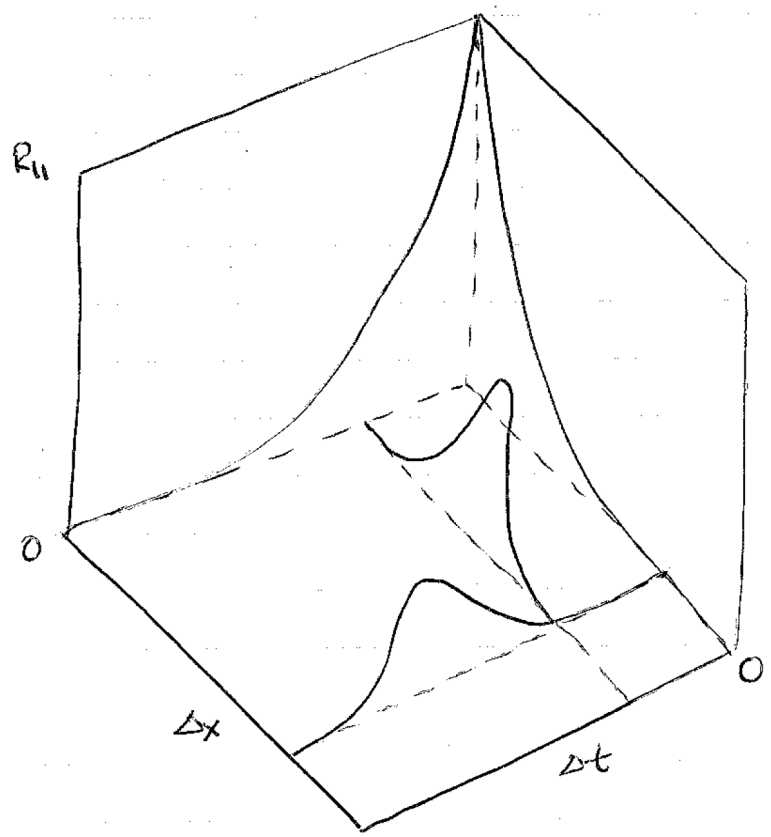


Figure 57: $R_{11}(\Delta x, \Delta t)$, between two measurement points, separated by both time and space.

To get a better view of this three-dimensional correlation function, we can flattern the plot into a two-dimensional contour plot showing levels of correlation intensity. The peak correlation appears to follow along a roughly straight line in the Δx-Δt plane, representing the mean convection velocity at which the blobs are moving. To state this mathematically, we define the convection velocity as the set of points $(\Delta x, \Delta t)$ along which the change in $R_{11}(\Delta x, \Delta t)$ is minimized, i.e. we wanto minimize the differential change, dR_{11}

$$dR_{11}(\Delta x, \Delta t) = \frac{\partial R_{11}}{\partial \Delta t} d\Delta t + \frac{\partial R_{11}}{\partial \Delta x} d\Delta x \tag{57}$$

Experimentally, if we had 2 hotwires measuring the velocity fluctuations in a turbulent flow, and the sensors are separated by a fixed spatial distance Δx, then $d\Delta x = 0$ and the convective velocity is defined by the constant Δx and the set of points Δt for which

$$\frac{\partial R_{11}}{\partial \Delta t} = 0 \tag{58}$$

which is just the peak in the $R_{11}(\Delta t)$ curve.

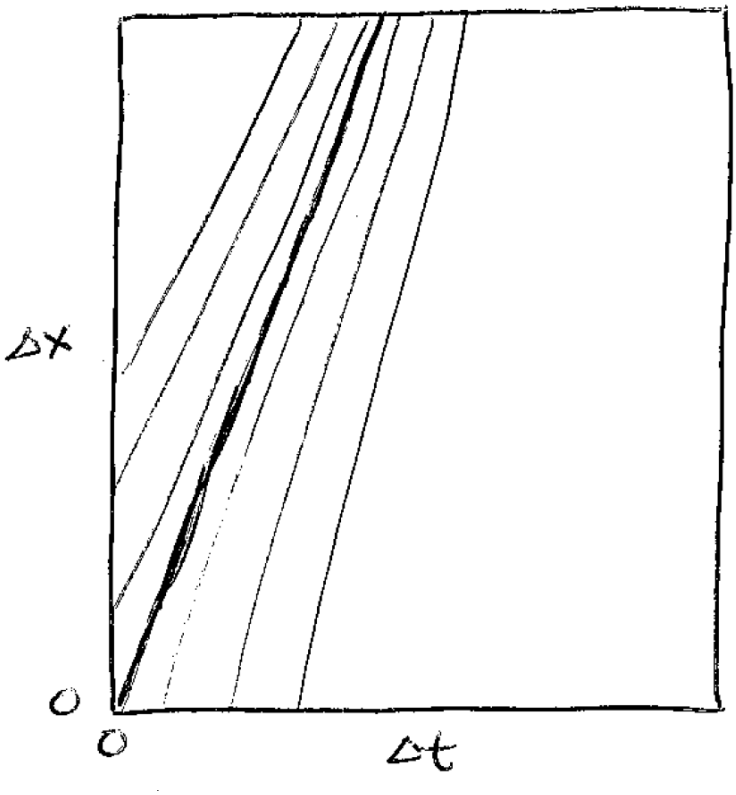


Figure 58: A contour map of $R_{11}(\Delta x, \Delta t)$. Lines represent levels of the correlation function; the dark line is the global maximum; the lighter lines represent lower levels of correlation. Note that the lighter lines are *not* parallel.

Now, experimentally, it is very easy to record flow field measurements at a single spatial location and many different time locations. It is typically much more difficult to record the flow at a single temporal moment but many different spatial locations. But the spatial correlations are what we want if we want to understand how spatially coherent structures move and behave in the turbulent flow. In order to solve this problem, we might be tempted to simply take the easier temporal measurements and then convert them to 'equivalent' spatial measurements by rescaling the Δt axis of our measurements into a Δx axis by use of the convection velocity, $\Delta x = \bar{u}_{\text{conv}} \Delta t$. In other words, we assume that the turbulent structures are 'frozen' in space and are simply being convected downstream, unchanged, with only time evolution. This is called Taylor's hypothesis of frozen turbulence.

There are immediately two problems with this hypothesis. The first is how to know the appropriate convection velocity a priori? So far, the only way to identify the correction convection velocity is to measure both spatial and temporal correlations and conduct the

minimization procedure above. But if we have only temporal measurements, the best we can do is to assume that turbulent structures convect at the local mean velocity of the flow. The second problem is that we must assume that a single convective velocity is relevant for all the turbulent structures, uniformly. But is that really true?

Consider again the spatial correlation function measured at different time delays. We already pointed out that the position of the spatial correlation peak, Δx_m increases, roughly speaking, in proportion to the time delay Δt, which is the basic idea of the frozen turbulence hypothesis.

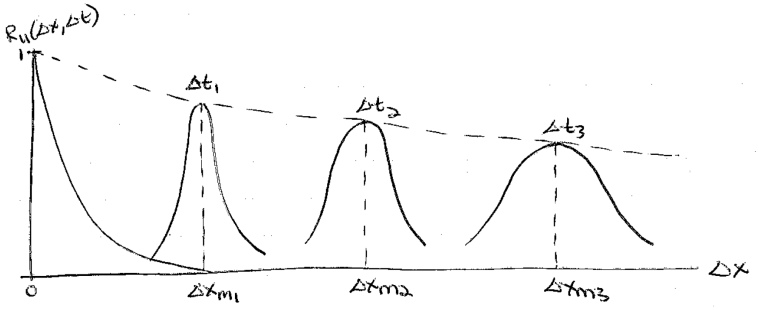


Figure 59: Spatial correlation functions, $R_{11}(\Delta x)$.

However, we also note that the magnitude of the maximum correlation decreases with increasing timelag. The radius of curvature of the correlation about its maximum also increases with timelag. And finally, the shape of the correlation function does not necessarily remain symmetric as the timelag increases.

The fact that the correlation decreases with time means that the turbulence is not actually frozen, since if it were perfectly frozen, the correlation would remain unchanged at each time delay. Obviously, the turbulence is changing spatially simultaneous with the temporal changes. The fact that the radius of curvature is also changing with timelag is significant too, because the radius of curvature is related directly to the Taylor microscale. Therefore, as the radius of curvature increases, the Taylor microscale increases, and thus the small, vicous scales of turbulence are affected differently from larger scales. If the large and small scales are evolving differently with time, then we expect that these different scales should be characterized by different convective velocities, and not, as we had assumed, a single constant velocity.

The Taylor microscale, λ, represents the largest scales at which viscosity is still important, and thus represents roughly a cutoff between dissipation and inertial regions of the TKE spectrum. We previously saw the Taylor microscale appear in the definition of the dissipation rate, ε.

Let's return to Taylor's original formulation of his frozen turbulent hypothesis (1938), which he wrote as

$$u_i(\vec{x}, t) = u_i(\vec{x} - \bar{u}_{\text{conv}}\tau, t + \tau) \tag{59}$$

such that measuring forward in time is equivalent to measuring backwards in space. At a single, fixed measurement location, the later some blobs flys past, the further upstream it was originally situated. In other words, according to Taylor's hypothesis, the velocity (or a given velocity component) can be written as a function of a

composite variable of the d'Alembert traveling wave form,

$$u_i = f(x - \bar{u}_{\text{conv}} t) \tag{60}$$

d'Alembert's variable change, $\eta = x - ut$, is usually used for solving the one-dimensional wave equation, since it is a 'traveling' wave solution, meaning the shape of the wave stays the same, and just translates in space. f' is the derivative with respect to the composite variable, $(x - \bar{u}_{\text{conv}} t)$

so that moving in space, x, is entirely equivalent to moving in time t at the convective velocity $\bar{u}_{\text{conv}}$ (where $\bar{u}_{\text{conv}}$ is in the x direction). We can then write the spatial and temporal derivaties in terms of this composite variable as $\frac{\partial u_i}{\partial t} = -\bar{u}_{\text{conv}} f'$ and $\frac{\partial u_i}{\partial x} = f'$. Then we can relate this two partial derivatives as

$$\frac{\partial u_i}{\partial t} = -\bar{u}_{\text{conv}} \frac{\partial u_i}{\partial x} \tag{61}$$

What is the implication of this relation? Let's consider homogeneous isotropic turbulence convecting downstream at some convective velocity $\bar{u}_{\text{conv}}$. The fluctuating momentum equation is given by

$$\frac{\partial u_i}{\partial t} + \bar{u}_{\text{conv}} \frac{\partial u_i}{\partial x} + \underbrace{u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}}_{\text{if frozen, then} \approx 0} \tag{62}$$

where the only non-zero mean velocity is the streamwise convective velocity, $\bar{u}_{\text{conv}}$, and we assume there are no mean velocity gradients. The derivative formulation of Taylor's hypothesis, written above, implies that all but the first two terms are neglible if the turbulence is frozen. If we scale the momentum equation, with fluctuation magnitude u, $p \sim \rho u^2$, lengthscale L, and convective timescale $t \sim L/U$ we obtain

$$\left[\frac{u}{L/U}\right] + \left[U\frac{u}{L}\right] + \underbrace{\left[\frac{u^2}{L}\right] \sim \left[\frac{u^2}{L}\right] + \left[\nu\frac{u}{L^2}\right]}_{\text{if frozen, then} \ll \frac{uU}{L}} \tag{63}$$

and the condition under which the momentum equation matches the frozen turbulence hypothesis is

$$\frac{u}{U} \ll 1 \tag{64}$$

Therefore, in the case of homogeneous isotropic turbulence, as long as the turbulence intensity level is low, we expect Taylor's hypothesis to hold. Anywhere the turbulence intensity is high would indicate that turbulence cannot be assumed frozen.

We can rationalize this another way. Assume that the magnitude of random displacements (with characteritic velocity u) of the center of any given eddy in time τ is small compared to the size of the eddy itself, ℓ; in other words

$$\tau u \ll \ell \tag{65}$$

The largest timescale that is associated with the overall eddy is defined in terms of the mean velocity of the flow, U, such that $\tau \sim$

$\frac{\ell}{U}$. Again we arrive at the conclusion that

$$\frac{u}{U} \ll 1 \tag{66}$$

in order for the internal, spatial evolution of the eddy to be insignificant in comparison to the convection of the overall eddy downstream. This is the general criterion for the applicability of Taylor's hypothesis.

Taylor's hypothesis can also be written in spectral form, where the temporal frequency of turbulent structures, ω, is related to spatial wavenumbers, k, via the mean convective velocity

The radial frequency $\omega = 2\pi f$

$$k_i(\bar{u}_i + u_i) \quad = k_1 \bar{u}_{\text{conv}} + k_i u_i \quad = \omega \tag{67}$$

And we can write

$$k_1 + k_i\left(\frac{u_i}{\bar{u}_{\text{conv}}}\right) = \left(\frac{\omega}{\bar{u}_{\text{conv}}}\right) \tag{68}$$

so we see that if $\left|\frac{u_i}{\bar{u}_{\text{conv}}}\right| \ll 1$, then we obtain the standard Taylor's hypothesis result that $\bar{u}_{\text{conv}} k_1 \approx \omega$, allowing us to convert temporal spectral measurements into spatial spectral measurements by rescaling the frequency with the mean convective velocity.

So far, we considered Taylor's hypothesis with respect to homogeneous isotropic turbulence with low turbulence intensity. What happens when we apply Taylor's hypothesis to shear flows? Let's rewrite the momentum balance for all three velocity components, for the case where $\frac{d\bar{u}}{dy} \neq 0$

$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} + v\frac{d\bar{u}}{dy} + \qquad u_j\frac{\partial u}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x_j^2} \tag{69}$$

$$\frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial x} + \qquad u_j\frac{\partial v}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial x_j^2} \tag{70}$$

$$\frac{\partial w}{\partial t} + \bar{u}\frac{\partial w}{\partial x} + \qquad u_j\frac{\partial w}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\frac{\partial^2 w}{\partial x_j^2} \tag{71}$$

If Taylor's hypothesis holds, then we can write the three velocity components as $u = f(x - \bar{u}t)$, $v = g(x - \bar{u}t)$, and $w = h(x - \bar{u}t)$. For large Reynolds number and small turbulence intensity (i.e. $u \ll U$), we can substitute the same differential relation we found above

$$\frac{\partial u_i}{\partial t} = -\bar{u}\frac{\partial u_i}{\partial x} \tag{72}$$

Previously we assumed that the pressure gradient term was small; here, however, because we are assuming that the velocity gradient term in the streamwise direction is not small, we must retain the pressure gradient term in that direction and assume that it balances the velocity gradient.

which means we can rewrite the momentum balances as

$$v\frac{d\bar{u}}{dy} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{73}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{74}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} \tag{75}$$

We then obtain, after substituting our definition of $v = g(x - \bar{u}t)$,

$$p = -\rho \frac{d\bar{u}}{dy} \int g(\eta) d\eta = -\rho \frac{d\bar{u}}{dy} G(x - \bar{u}t) \tag{76}$$

We integrate over the composite variable η because $\frac{\partial \eta}{\partial x} = 1$.

where $G' = g$. Then we can calculate the pressure gradient in the y direction by differentiating

$$\frac{\partial p}{\partial y} = -\rho \frac{d^2\bar{u}}{dy^2} G(\eta) + \rho \left(\frac{d\bar{u}}{dy} \right)^2 g(\eta) \frac{x - \eta}{\bar{u}} = 0 \tag{77}$$

This equation is satisfied (and thus Taylor's hypothesis is expected to be true) only if

$$\frac{d^2\bar{u}}{dy^2} = 0 \quad \text{and} \quad \frac{d\bar{u}}{dy} = 0 \tag{78}$$

because g and G are general, arbitrary functions. Therefore, we have a internal inconsistency when we try to apply Taylor's hypothesis to shear flows: we must assume that the pressure gradients are negligible, but this, in turn, implies that the velocity gradients are also negligible. The only way around this contradiction is to accept that the velocity gradient is sufficiently small that it is no worse than the other advective terms in the momentum equation

$$\left| v \frac{d\bar{u}}{dy} \right| \ll \left| \bar{u} \frac{\partial u}{\partial x} \right| \tag{79}$$

This implies that

$$\frac{d\bar{u}}{dy} \ll \bar{u} \underbrace{\left(\frac{1}{v} \frac{\partial u}{\partial x} \right)}_{\sim k} \tag{80}$$

where we approximate the relevant, streamwise wavenumber k in terms of the two velocity components and the streamwise length scale. Then we obtain

Recall that the mean turbulent boundary layer profile can be approximated as $u \sim y^{1/7}$ and thus $\frac{d \log \bar{u}}{d \log y} = \frac{1}{7}$

$$k\bar{u} \gg \frac{d\bar{u}}{dy} \quad \Rightarrow \quad ky \gg \frac{y}{\bar{u}} \frac{d\bar{u}}{dy} \quad \Rightarrow \quad ky \gg \frac{d \log \bar{u}}{d \log y} \tag{81}$$

as the criterion for the applicability of Taylor's hypothesis. This immediately implies that the applicability is dependent on wavenumber. For instance, consider measurements in a boundary layer at location $y \sim 0.2\delta$.

$$k \frac{y}{\delta} \gg \frac{1}{\delta} \frac{d \log \bar{u}}{d \log y} \tag{82}$$

$$k\delta \gg \frac{1}{0.2} \frac{1}{7} \tag{83}$$

$$\frac{2\pi}{\lambda} \delta \gg 0.7 \tag{84}$$

$$9\delta \gg \lambda \tag{85}$$

This means that Taylor's hypothesis is expected to fail for large structures; in the example case, for structures that are on the order

of 9δ in length. This is reasonable since small eddies are carried around by larger ones, so the convection velocity of smaller eddies is closer to the local velocity, whereas large eddies contribute to the local velocity and thus do not necessarily follow it

See Lin (Quart. Applied Math. 1953) for a detailed discussion of the fundamental assumptions inherent in Taylor's hypothesis, aside from more recent experimental efforts at validation.

Transition to Turbulence

We have spent a long time discussing the nature of turbulence itself and how it is distinct from laminar flow, but we have not explained how a flow transitions from a laminar state to a turbulent state. The first empirical observations of this transition were performed by Reynolds (1883) using a thin stream of dye in a pipe flow. He observed that the transition from laminar to turbulent flow depended on the Reynolds number of the flow; generally transition in the pipe occurred at a critical Reynolds number, $\mathrm{Re}_c \approx 2000$, but the value of this critical Reynolds number appeared to be very sensitive to disturbances at the entrance of the pipe. In fact, if the entrance was very well controlled to be smooth and undisturbed, then the critical Reynolds number could increase to as much as $13,000$.

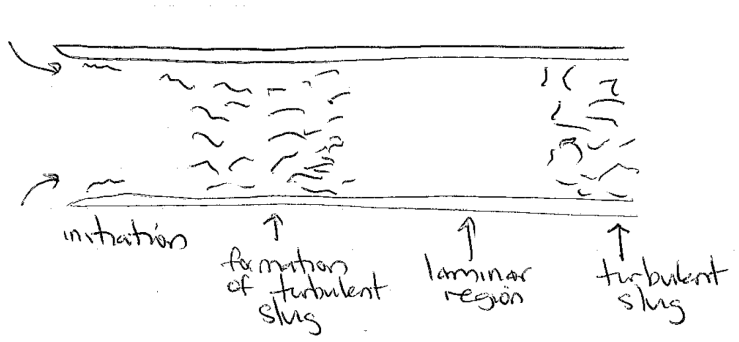


Figure 60: The general schematic of turbulence transition in a pipe. 'Slugs' of turbulent fluid form due to finite amplitude disturbances in the pipe.

These empirical observations were later justified by (linear stability) analysis which showed that, in fact, pipe flow is stable to infinitessimal disturbances at all Reynolds numbers (i.e. for very small disturbances, the pipe flow always remains laminar), and thus the observed transition must be the result of larger (finite) disturbances in the flow, including disturbances at the inlet.

More generally, we can say that there are at least two mechanisms of transition: The first is what Reynolds observed in pipe flow, that turbulent motions appear intermittently in localized patches/regions/slugs/puffs and then these patches grow and spread until eventually the entire flow field is turbulent. The second, seen in the smoke plume from a fire, occurs when a certain threshold flow condition is exceeded, after which turbulence develops uniformly everywhere in the flow field, possibility via a chain of growing instabilities.

Traditional explanations of both of these transition mechanisms

start with linear stability analysis of the governing laminar Navier Stokes equations, in which an infinitesimal disturbance is introduced to the velocity and pressure fields, the mean equations are subtracted from the perturbed equations, the resulting disturbance equations are then linearized in terms of this disturbance, the disturbance is represented as a harmonic wave in space and time, and the amplitude of this wave is then investigated to see whether it tends to grow or decay in time. Growing disturbance amplitudes indicate linear instability. But, we already noted that the assumption of an infinitesimal disturbance is inconsistent with Reynolds's observations in the pipe; if we want to explain transition, we need to deal with finite amplitude disturbances. So we will consider a toy problem exhibits many of the key features of transition in turbulence, and consider first a linearized stability analysis, followed by an analysis for finite disturbances.

Linear Stability Analysis

Our toy version of the Navier Stokes equation describes an average velocity, $\bar{u}(y,t)$, in terms of a Reynolds number-like parameter, R and an advective term, $f(\bar{u})$

$$\frac{\partial \bar{u}}{\partial t} + f(\bar{u}) = \frac{1}{R}\frac{\partial^2 \bar{u}}{\partial y^2} \tag{1}$$

The toy problem can also describe the temperature, $\bar{u}$, in a 1D bar, with heat source $-f(\bar{u})$ inside, since advective terms in a material volume can always be thought of as source terms in a stationary control volume; see Matkowsky 1970 'A simple non-linear dynamic stability problem'

with homogeneous boundary conditions, $\bar{u}(y=0) = \bar{u}(y=\pi) = 0$. The time-independent problem is obviously satisfied by the trivial solution, $\bar{u} = 0$. But let's introduce a perturbation to this trivial base flow via the initial condition: $\bar{u}(y, t=0) = \epsilon h(y)$ where the function $h(y)$ also satisfies the homogenous boundary conditions, $h(0) = h(\pi) = 0$ and the parameter ϵ is some small perturbation. Since the base flow is assumed trivial, we can expand the advective term as

$$f(\bar{u}) = f(0) + f'(0)\bar{u} + \frac{1}{2!}f''(0)\bar{u}^2 + \cdots \tag{2}$$

The advective function, $f(\bar{u})$, is assumed non-linear and can be described as the derivative of a symmetric potential, $w(\bar{u})$, with two minima of equal depth, e.g. $w(\bar{u}) = \frac{1}{2}(\bar{u}^2 - 1)^2$. This 'double-well' potential, in the form of the letter 'w', means the origin, $\bar{u} = 0$, is unstable and thus the advective forcing tends destabilize the system, whereas diffusive forcing tends to stabilize it, like in the real physics, where the advective terms are the source of turbulence.

with $f(0) = 0$, $f'(0) < 0$, $f''(0) = 0$, and $f'''(0) > 0$. If we assume that the perturbations to $\bar{u}$ are very small, then we can approximate the advective contribution by taking just the linear term in $f(\bar{u})$ (since non-linear terms involve products of very small perturbations, which are assumed negligible) yielding

$$\frac{\partial \bar{u}}{\partial t} + f'(0)\bar{u} = \frac{1}{R}\frac{\partial^2 \bar{u}}{\partial y^2} \tag{3}$$

and a solution obeying the boundary and initial conditions can be written as a Fourier sine series (due to the homogeneous boundary conditions), in the form:

$$\bar{u} = \epsilon \sum_n h_n e^{\sigma_n t} \sin(ny) + \mathcal{O}(\epsilon^2) \tag{4}$$

The coefficients h_n are found via the Fourier series representation of the initial function, $h(y)$, written in the form, $h(y) = \sum_n h_n \sin(ny)$

where σ_n are eigenvalues. These real eigenvalues can be found by substituting the solution into the linearized equation to obtain

$$\sigma_n + f'(0) = \frac{1}{R}(-n^2) \tag{5}$$

A linearly stable solution means that the solution cannot grow in time, i.e. $\sigma_n \leq 0$, yielding a stability criterion

$$R \leq \frac{n^2}{-f'(0)} \tag{6}$$

Recall that $f'(0) < 0$ by assumption, so the equivalent Reynolds number is some positive, real number

for all values of n. Thus, more generally we can define a critical Reynolds number, R_c, below which stability is guaranteed for any component n of the initial perturbation $h(y)$

$$R \leq \frac{\min(n^2)}{-f'(0)} = -\frac{1}{f'(0)} \equiv R_c \tag{7}$$

We find that, according to the linear analysis, a single critical Reynolds number determines the stability of the system, without regard to the size of the perturbations (because we assumed they were infinitesimally small). What's wrong with this linear approximation?

Imagine the Reynolds number of the flow is just at the cusp of instability, i.e.

$$0 < R - R_c \ll R_c \tag{8}$$

Recall the definition $R \leq \frac{n^2}{-f'(0)}$, so that if the first ($n = 1$) mode is stable, then $R \leq \frac{1}{-f'(0)}$ and if the second ($n = 2$) mode is stable, then $R \leq \frac{4}{-f'(0)}$, etc.

Substitute the definition for the critical Reynolds number to obtain

$$0 < R + \frac{1}{f'(0)} \ll -\frac{1}{f'(0)} \tag{9}$$

$$-\frac{1}{f'(0)} < R \ll -\frac{2}{f'(0)} \tag{10}$$

Therefore, at this Reynolds number, just at the cusp of instability, the first ($n = 1$) mode is unstable, but the second ($n = 2$) mode is stable, because $R \ll \frac{2^2}{-f'(0)}$. And of course, all modes greater than the first ($n > 1$) are also stable, because

$$\sigma_n = -f'(0) - \frac{n^2}{R}, \tag{11}$$

so as n increases, the eigenvalue decreases, and thus the perturbation decays exponentially for $n > 1$. Because all of these modes decay, we can represent the solution by just its first mode

$$\bar{u} = \epsilon h_1 e^{\sigma_1 t} \sin(y) \tag{12}$$

This perturbation grows slowly. We can approximate the timescale, t_ϵ at which the perturbation amplitude becomes $\mathcal{O}(\epsilon)$ as

$$\epsilon \sim \epsilon h_1 e^{\sigma_1 t_\epsilon} \quad \Rightarrow \quad t_\epsilon \sim -\frac{\log h_1}{\sigma_1} \tag{13}$$

Recall that $\sigma_1 = \frac{1}{R_c} - \frac{1}{R}$

Therefore, as the Reynolds number, R increase for a given value of R_c, then σ_1 increase, and the timescale t_ϵ decreases. In other words, as Reynolds number increases, the disturbance amplitude increases faster. But that means that the linear assumption (that the disturbance is infinitesimally small) is only valid for a very short time. If we want to analyze the stability past a short time, we need to take into account that the disturbance is finite in amplitude, and thus we need to include the non-linearity of the advection, $f(\bar{u})$.

Nonlinear Stability Analysis

We showed that the linear stability analysis is valid for $t < t_\epsilon$. After this time period, non-linear analysis must be applied. Thus, the non-linear analysis should apply for very large times, $t \gg t_\epsilon$. We can then imagine writing the dynamics in terms of two different timescales: t for short times $t < t_\epsilon$, in the linear regime, and τ, for very long times, $\tau \gg t_\epsilon$. We could take the timescale t_ϵ to be of order ϵ for simplicity, since both represent some small number. Then, in order to compare the two timescales, we can define the long timescale as

$$\tau = \epsilon^2 t \tag{14}$$

Using this definition, for very short times, $t < \epsilon$ and the non-linear timescale is negligible — we need consider only the linear problem and we can ignore any dependence on τ. But for long times, $t \gg \epsilon$, then τ is not negligible and we need to consider the non-linear dynamics. Similarly, let's define the extent to which the Reynolds number exceeds the (linear) critical Reynolds number using this same small quantity

We choose the power of ϵ^2 to represent a time period that is asymptotically bigger than the timescale, ϵ.

$$\epsilon^2 = R - R_c \tag{15}$$

We can write this more generally as $\epsilon^\beta = R - R_c$ and then determine the power β by asymptotic matching in the time domain, but here we just use the result that $\beta = 2$.

Now, we write the solution to the non-linear equation in the form of an asymptotic series (with power-law gauge function, ϵ^j)

$$\bar{u} = \sum_j \epsilon^j w_j(y, t, \tau) \tag{16}$$

where the functions w_j satisfy the boundary conditions for all j, $w_j(y = 0, \pi) = 0$, and the the first mode satisfies the initial condition, $w_1(y, t = \tau = 0) = h(y)$. Now, substitute into the governing equation, taking into account both timescale, approximating R for small values of ϵ, and including the first non-linear term in the series for $f(\bar{u})$, to obtain

Take into account the two timescales: $\frac{\partial}{\partial t}\bar{u}(t, \tau(t)) = \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{\partial \bar{u}}{\partial t} + \epsilon^2 \frac{\partial \bar{u}}{\partial \tau}$ and expand the Reynolds number in terms of ϵ as $\frac{1}{R} = \frac{1}{R_c + \epsilon^2} \approx \frac{1}{R_c} - \frac{\epsilon^2}{R_c^2} + \mathcal{O}(\epsilon^4)$ and recall that $f''(0) = 0$ so the first non-linear term of the series for f is third-order.

$$\frac{\partial}{\partial t}\left(w_1\epsilon + w_2\epsilon^2 + w_3\epsilon^3\right) + \epsilon^2\frac{\partial}{\partial \tau}(w_1\epsilon + \cdots) + f'(0)(w_1\epsilon + \cdots) + \frac{1}{3!}f'''(0)(w_1\epsilon + \cdots)^3 = \frac{1}{R_c}\frac{\partial^2}{\partial y^2}(w_1\epsilon + \cdots) - \frac{\epsilon^2}{R_c^2}\frac{\partial^2}{\partial y^2}(w_1\epsilon + \cdots) \tag{17}$$

Now, we can find the functions w_j by arguing that the governing equation should be satisfied independently at all orders of ϵ, thus

$$\mathcal{O}(\epsilon): \quad \mathcal{L}(w_1) \equiv \left(-\frac{\partial}{\partial t} - f'(0) + \frac{1}{R_c}\frac{\partial^2}{\partial y^2}\right) w_1 = 0 \tag{18}$$

$$\mathcal{O}(\epsilon^2): \quad \mathcal{L}(w_2) = 0 \tag{19}$$

$$\mathcal{O}(\epsilon^3): \quad \mathcal{L}(w_3) = \frac{\partial w_1}{\partial \tau} + \frac{f'''(0)}{3!} w_1^3 + \frac{1}{R_c^2}\frac{\partial^2 w_1}{\partial y^2} \tag{20}$$

where we defined an operator $\mathcal{L}(\cdot)$ which operates on each w_j function. The result of this operator can be written recursively as some function r_j of the previously known functions w

$$\mathcal{L}(w_j) = r_j(w_1, w_2, \ldots, w_{j-2}) \tag{21}$$

where the equation for w_j depends on the solutions up to w_{j-2}. Now, we apply the Fredholm alternative which states that for a linear, inhomogeneous ODE with homogeneous boundary conditions, of the form above then either: 1) $w = 0$ is the only homogeneous solution, and the non-homogeneous equation has a unique solution; or 2) there are non-trivial homogeneous solutions for which the non-homogeneous equation has zero or infinite solutions that satisfy an orthogonality condition. We know that $\mathcal{L}(w_1) = 0$ has a non-trivial solution (since that is our linear problem), thus we follow the second branch of the Fredholm alternative and apply the orthogonality condition to find the non-homogeneous solutions. The homogeneous solution to the operator $\mathcal{L}$ must be orthogonal to the non-homogeneous forcing, such that

$$\lim_{T\to\infty} \int_0^T \int_0^\pi w_1 r_k(w_1, w_2, \ldots, w_{k-2}) dy dt = 0 \quad \text{for} \quad k = 2, 3, 4, \ldots \tag{22}$$

over the entire domain in space/time.

We already know the general form of the solution for w_1 because this is just the linear solution we found above, but now we have time-dependent coefficients $A_{1n}(\tau)$ that vary on the non-linear timescale, instead of constant coefficients h_n

$$w_1 = \sum_n A_{1n}(\tau) e^{\sigma_n(R_c)t} \sin(ny) \tag{23}$$

Recall that $\sigma_n(R) = \frac{1}{R_c} - \frac{n^2}{R_c}$ and thus $\sigma_n(R_c) = \frac{1-n^2}{R_c}$ and $\sigma_1(R_c) = 0$

which we can expand as

$$w_1 = A_{11}(\tau)\sin(y) + \sum_{n=2} A_{1n}(\tau) e^{\sigma_n(R_c)t} \sin(ny) \tag{24}$$

where $\sigma_n(R_c) < 0$ for $n = 2, 3, \ldots$ and thus all of the contributions to w_1 for $n > 1$ decay exponentially in time. Thus, the very first unstable, non-linear mode has amplitude $A_{11}(\tau)$, in contrast to the first unstable linear mode we found earlier, which had amplitude

$\epsilon h_1 e^{\sigma_1 t}$. We showed above that w_2 satisfies the same equation as w_1. And finally we can obtain w_3 via the orthogonality requirement

$$\lim_{T\to\infty} \int_0^T \int_0^\pi w_1 r_3(w_1) dy dt = 0 \tag{25}$$

Substituting just the non-decaying terms of the w_1 solution yields

$$\lim_{T\to\infty} \int_0^T \int_0^\pi A_{11}(\tau)\sin(y)\left[\frac{dA_{11}}{d\tau}\sin(y) + \frac{1}{3!}f'''(0)(A_{11}\sin(y))^3 + \frac{A_{11}}{R_c^2}\frac{\partial^2}{\partial y^2}(\sin(y))\right] dy dt = 0 \tag{26}$$

and after integration over y we obtain

Note that $\int_0^\pi \sin^2(y)dy = \frac{\pi}{2}$ and $\int_0^\pi \sin^4(y)dy = \frac{3\pi}{8}$

$$\lim_{T\to\infty} \frac{\pi}{2} \int_0^T A_{11}(\tau)\left[\frac{dA_{11}}{d\tau} + \frac{f'''(0)}{8}A_{11}^3 - \frac{A_{11}}{R_c^2}\right] dt = 0 \tag{27}$$

and thus

$$\frac{dA_{11}}{d\tau} = \frac{1}{R_c^2}A_{11} - \frac{f'''(0)}{8}|A_{11}|^2 A_{11} \tag{28}$$

Recall that $A_{11}(\tau)$ represents the (signed) amplitude of the first unstable mode that varies on the non-linear timescale, thus it will determine whether the system is stable in a non-linear sense, and we see that it depends on non-linear contributions from the forcing, $f'''(0)$. We are interested in the magnitude of this amplitude, so we multiply through by $A_{11}(\tau)$ to obtain

$$\frac{d|A_{11}|^2}{d\tau} = \frac{2}{R_c^2}|A_{11}|^2 - \frac{f'''(0)}{4}|A_{11}|^4 \tag{29}$$

How can we interpret this equation? Consider the linearized solution we found earlier, $\bar{u} \approx \epsilon h_1 e^{\sigma_1 t}\sin(y)$, which has an amplitude function $A_{\text{lin}}(t) \sim e^{\sigma_1 t}$. Then we can write:

$$\frac{d|A_{\text{lin}}|^2}{dt} = 2\sigma_1|A_{\text{lin}}|^2 \tag{30}$$

which looks like the first term of the non-linear dynamics equation. But the linear solution is valid only for short times, thus we can think of it as a truncated form of the actual non-linear dynamics. Thus we identify the first parameter, σ_1 with the critical Reynolds number, $\frac{2}{R_c^2}$ and the first term in the equation measures the linear stability. The second parameter in the following term, $\frac{f'''(0)}{4}$ represents the non-linearity of the problem. For simplicity, we can rename these parameters

ℓ is known as the Landau constant and the dynamical equation is called the Landau equation; see Landau & Lifshitz §26

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 - \ell|A|^4 \tag{31}$$

If $\ell = 0$, then the non-linearity is negligible and we are back to our linear analysis. If $R > R_c$, then the regime is linearly unstable, which means that $\sigma_1 > 0$ and thus we take $\gamma > 0$, which means exponential growth in the amplitude of the perturbation. The non-linear term therefore will moderate or accelerate the exponential

growth of the perturbation, depending on the relative signs of γ and ℓ. Let's investigate the possible outcomes.

First, divide through by $|A|^4$ to obtain

Note that $\frac{d}{dt}\left(\frac{1}{|A|^2}\right) = -\frac{1}{|A|^4}\frac{d|A|^2}{dt}$

$$\frac{d}{dt}|A|^{-2} + 2\gamma|A|^{-2} = \ell \tag{32}$$

and note that this ODE is linear in $|A|^{-2}$. Using a change of variable $v = \ell - 2\gamma|A|^{-2}$, we can integrate this over time, from $t = 0$ where $A(t=0) = A_0$, to obtain

$$|A|^{-2} = \frac{\ell}{2\gamma} + \left(|A_0|^{-2} - \frac{\ell}{2\gamma}\right)e^{-2\gamma t} \tag{33}$$

or rewriting, we find

$$|A|^2 = \frac{|A_0|^2}{\frac{\ell|A_0|^2}{2\gamma} + \left(1 - \frac{\ell|A_0|^2}{2\gamma}\right)e^{-2\gamma t}} \tag{34}$$

Finally, let's multiply through by $e^{2\gamma t}$ to get

$$|A|^2 = \frac{|A_0|^2 e^{2\gamma t}}{\frac{\ell|A_0|^2}{2\gamma}e^{2\gamma t} + \left(1 - \frac{\ell|A_0|^2}{2\gamma}\right)} \tag{35}$$

Now, let's define $\pm A_e^2 = \frac{2\gamma}{\ell}$ depending on the signs of γ and ℓ.

$$|A|^2 = \frac{|A_0|^2 e^{2\gamma t}}{\frac{|A_0|^2}{\pm A_e^2}(e^{2\gamma t} - 1) + 1} \tag{36}$$

We note that the amplitude becomes singular when the denominator is zero, i.e. when

$$t^* = \frac{1}{2\gamma}\log\left(1 \mp \frac{A_e^2}{|A_0|^2}\right) > 0 \tag{37}$$

Now, let's consider the behavior of the solutions, $|A|^2$ for different scenarios.

1. If $R > R_c$, then $\gamma > 0$ (the system is linearly unstable, past the critical Reynolds number). If, in addition, $\ell > 0$, then the parameter $+A_e^2$ is taken with the positive sign. This means that the time singularity occurs for either $t^* < 0$ or t^* is imaginary, or in other words there is no singularity in time. In this regime

$$\lim_{t\to\infty}|A|^2 = +A_e^2 \tag{38}$$

   Thus it appears that, despite the fact that the regime is linearly unstable, the system reaches a steady state for long times, after some initial period of growth or decay, a super-critical stability. Therefore, independent of the initial conditions, this linearly unstable flow reaches a steady state, and we see that there must be a bifurcation in stability about the critical Reynolds number, where there are two possible paths a system could take.

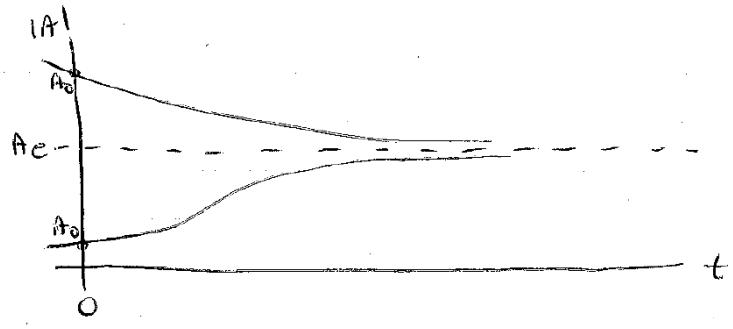


Figure 61: Supercritical stability

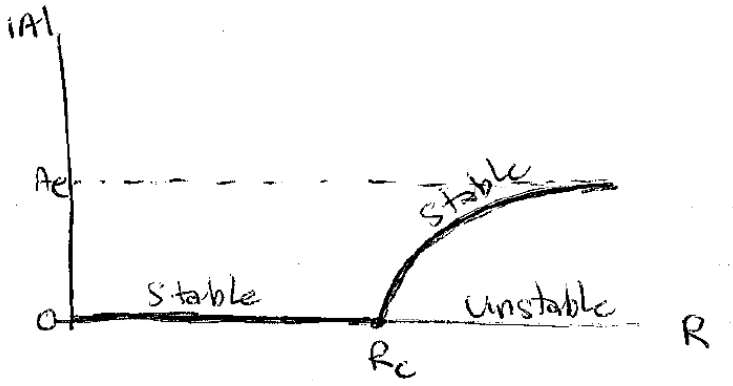


Figure 62: Bifurcation at the critical Reynolds number between linear instability and (non-linear) supercritical stability

2. Now let's consider the case of a linearly stable flow, $R < R_c$ then $\gamma < 0$, still with $\ell > 0$. The parameter $-A_e^2$ is taken as negative and therefore the time singularity occurs for $t^* < 0$, i.e. there is no singularity in time again. Then, in the limit of infinite time,

$$\lim_{t\to\infty} |A|^2 = 0 \tag{39}$$

and we have the usual form of linear stability, that the disturbance decays and because the disturbance decays, the non-linear term, $-\ell|A|^4$ remains very small for all time.

3. Now, consider the case of linear stability ($\gamma < 0$) with $\ell < 0$. In this case, we take A_e^2 as positive. Now, if $\frac{|A_e|^2}{|A_0|^2} < 1$, then $t^* > 0$ and there is a blow up in finite time. In this case,

$$\lim_{t\to t^*} |A|^2 = \infty \tag{40}$$

But if $\frac{|A_e|^2}{|A_0|^2} > 1$, then t^* is imaginary and there is no time singularity. In this case,

$$\lim_{t\to\infty} |A|^2 = 0 \tag{41}$$

and we have the traditional linear stability. So we must distinguish between these two cases of initial conditions; for some initial conditions, there is a subcritical instability.

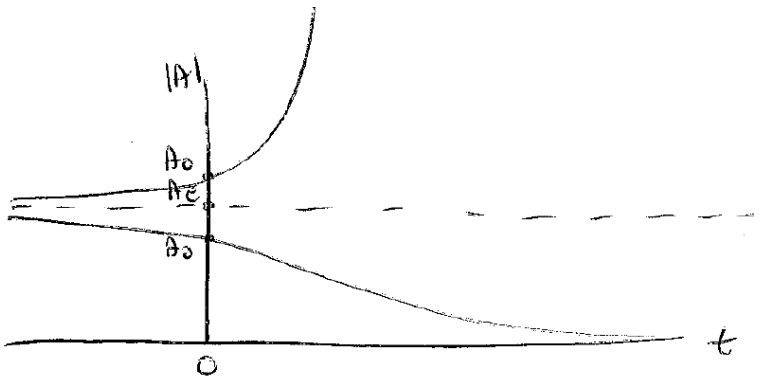


Figure 63: Subcritical instability for $|A_e|^2 < |A_0|^2$; regular linear stability for $|A_e|^2 > |A_0|^2$

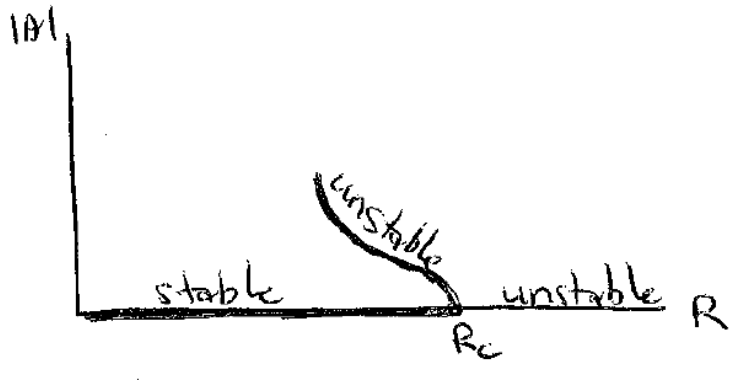


Figure 64: Bifurcation at the critical Reynolds number between linear stability and (non-linear) subcritical instability

4. If we return to linear instability, $\gamma > 0$, but this time allow for $\ell < 0$, then the parameter $-A_e^2$ is taken as negative. In this case, $t^* > 0$ and there is a finite time singularity, irrespective of initial conditions. We can interpret this blow-up from the differential equation itself

$$\frac{d|A|^2}{dt} = 2\gamma|A|^2 - \ell|A|^4, \tag{42}$$

since both the $2\gamma|A|^2$ and $-\ell|A|^4$ terms are positive, so we expect a super-exponential growth rate, or a very fast transition to turbulence.

We can summarize the results

	Linearly Stable, $R < R_c$ $\gamma < 0$	Linearly Unstable, $R > R_c$ $\gamma > 0$
$\ell > 0$	Stable	Supercritical Stability
$\ell < 0$	Subcritical Instability for $\|A_e\|^2 < \|A_0\|^2$ Stable for $\|A_e\|^2 > \|A_0\|^2$	Finite-time Blow-up

Table 17: Stability of Landau Equation

The Landau picture is the superposition of a very large number of 'modes' of a disturbance, each with different frequencies. Each new mode is introduced by a bifurcation process, like the one we modeled for the very first mode, A_{11}, which results in a new flow state due to its non-linear growth. The infinite sequence of these

bifurcations and new flow states for all of the modes eventually results in turbulence.

Now, this picture is obviously not literally correct, since we already stated at the outset that turbulence sometimes transitions all of the sudden, globally, without any long chain of steps; and other times turbulence transitions locally in a particular region of the flow, and then the turbulence spreads from that region. But this simple non-linear model does give us some sense that there are indeed many paths to instability, and even multiple states of stability and instability at a given Reynolds number, none of which was apparent from the simple linear analysis. We also see explicitly that the amplitude of the initial disturbance, $|A_0|^2$ affects the subsequent growth rate, and in fact for sufficiently large initial perturbations, a subcritical instability can be triggered even at a Reynolds number which linear theory predicts should produce a stable flow regime.

Turbulence Modeling

We recall from the discussion of two-dimensional flows (specifically wall-bounded flows) that we can write the total stress as

$$\tau_{xy} = \mu \frac{\partial \overline{u}}{\partial y} - \rho \overline{u}\overline{v} \tag{1}$$

but we do not know the form for $\overline{u}\overline{v}$, and we cannot write a closed system of equations to describe it, due to the non-linearity of the momentum equation, which we refer to as the closure problem. Also, we recall from dimensional analysis that the turbulent length and velocity scales, ℓ and u, can be used to form a turbulent 'viscosity', of the form

$$\nu_t = u\ell \tag{2}$$

such that the ratio of turbulent to molecular momentum diffusivities is

$$\frac{\nu_t}{\nu} \sim \mathrm{Re} \tag{3}$$

and thus the turbulent viscosity is dominant as Reynolds number increases. This 'viscosity' expresses the idea that the turbulent mixing can be thought of as an enhanced form of molecular mixing. We made this argument dimensionally, but we also showed that in shear flow, the sign of the molecular diffusive flux of momentum, $\mu \frac{\partial \overline{u}}{\partial y}$, and the turbulent flux of momentum, $-\rho \overline{u}\overline{v}$, have the same sign, and thus it makes sense to treat them as if they are analogous, working in the same direction.

Eddy Viscosity Models

The idea that there is some form of analogy between turbulent momentum flux and molecular diffusive flux dates back to the 1870s and the early work of Boussinesq. Boussinesq (1877) first performed a version of what we now call Reynolds averaging (1895) and made the following substitution for the Reynolds stresses

The same Boussinesq as for the buoyancy approximation in flows with natural convection.

$$-\rho \overline{u_i u_j} = 2\rho \nu_t \bar{s}_{ij} - \frac{1}{3}\rho \overline{u_k u_k}\delta_{ij} \tag{4}$$

$$\overline{\tau}_{ij} = 2\rho \nu \bar{s}_{ij} \tag{5}$$

Recall that $\bar{s}_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right)$ and for an incompressible Newtonian fluid, $\overline{\tau}_{ij} = 2\rho\nu\bar{s}_{ij}$.

where the analogy with the viscous stresses is obvious. Boussinesq observed that 'turbulent fluctuations are dissipative to the mean flow' in the same way that molecular viscosity is dissipative to the mean flow. Is this true? Recall the mean and turbulent kinetic energy balances that we wrote earlier

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{u_i u_i}\right) + \frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i u_k u_i} + \frac{1}{\rho}\overline{p u_k} - 2\nu\overline{u_i s_{ki}}\right) = \mathcal{P} - \varepsilon \quad (6)$$

$$\frac{D}{Dt}\left(\frac{1}{2}\bar{u}_i\bar{u}_i\right) + \frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i u_k}\bar{u}_i + \frac{1}{\rho}\bar{p}\bar{u}_k - 2\nu\bar{u}_i\bar{s}_{ki}\right) = -\mathcal{P} - \bar{\varepsilon} \quad (7)$$

From the perspective of the mean flow, the turbulent production, $\mathcal{P} = -\overline{u_i u_k}\bar{s}_{ik}$, acts as a dissipation pathway, moving kinetic energy from the mean flow into the turbulence. The actual dissipation, $\bar{\varepsilon} = 2\nu\bar{s}_{ik}\bar{s}_{ik}$. Therefore if these two terms are functionally equivalent then

$$(-\overline{u_i u_k})\bar{s}_{ik} \Longleftrightarrow (2\nu\bar{s}_{ik})\bar{s}_{ik} \quad (8)$$

$$-\overline{u_i u_k} \Longleftrightarrow 2\nu\bar{s}_{ik} \quad (9)$$

$$-\overline{u_i u_k} = 2\nu_t\bar{s}_{ik} - \frac{1}{3}\overline{u_j u_j}\delta_{ik} \quad (10)$$

which is the basic form of Boussinesq's model. So we see the Boussinesq was inspired by the analogy between turbulent and diffusive moment fluxes, and the analogy between turbulent and diffusive energy dissipation.

But, Boussinesq included an extra term, $-\frac{1}{3}\overline{u_j u_j}\delta_{ik}$, in his model. Consider the situation for the normal Reynolds stresses ($i = k$). The model yields

$$-\overline{u_i u_i} = 2\nu_t\bar{s}_{ii} - \frac{1}{3}\overline{u_j u_j}\delta_{ii} = -\overline{u_j u_j} \quad (11)$$

and for an incompressible fluid, $\bar{s}_{ii} = 0$. So this model cannot actually predict the Reynolds normal stresses (i.e. the turbulence 'pressure'). Without this additional term, we would implicitly assume that the the normal stresses are zero, which is not true.

However, we showed in our analysis of the turbulent boundary layer that the term $\frac{\partial}{\partial x}\overline{u^2}$ was negligible in comparison to $\frac{\partial}{\partial y}\overline{uv}$, and thus in many flows, we don't need a model for the Reynolds normal stresses in any case. But if we use such a model in three-dimensions, it is important to consider what happens to the normal stresses.

To complete the model, we need to decide on a value of ν_t. Boussinesq assumed it was a constant, which may be a valid assumption in homogeneous flows without mean gradients, or perhaps with a single dominant gradient in one direction. But in general, we would expect ν_t to vary with direction and position (near the wall, far from the wall) and also with Reynolds number, since we already showed that $\nu_t/\nu \sim \mathrm{Re}$. Thus we now need a second model, in addition to the Boussinesq model, to represent ν_t.

Prandtl suggested that ν_t is a momentum diffusivity, and thus can be defined by analogy with molecular diffusivity, ν, from the kinetic theory of gases

$$\nu = \frac{1}{3}\lambda v_{\mathrm{rms}} \quad (12)$$

where λ is the mean free path of the gas and v_{rms} is the rms speed of the particle. For a simple 2D laminar shear flow, layers of fluid particles slide past one another. Random fluctuations amongst the particle result in randomness in their momentum; diffusion then causes slower particles to shift upwards, towards the faster fluid, and faster particles to shift downward, towards the slower fluid, in order to even out the momentum gradient. . The net effect of this diffusive transport, over a thin layer of the flow with thickness λ and particles with velocity scale v_{rms}, produces a momentum flux of magnitude

Diffusive fluxes always operate against the direction of the concentration gradient of momentum, due to the second law of thermodynamics

$$\tau_{xy} = \frac{1}{3}\rho\lambda v_{\text{rms}}\frac{\partial \bar{u}}{\partial y} \tag{13}$$

Technically, this flux should be written with a minus sign, to signify a flux opposite the direction of the mean gradient; however, by the convention of this course, we typically write the flux positively

We can extend this process from the molecular level to the macro scale of turbulence, with a mixing length ℓ instead of a mean free path λ, to represent the characteristic length scale over which momentum is transferred by turbulent motions. The relative variation in velocity over this length scale can be defined by Taylor series.

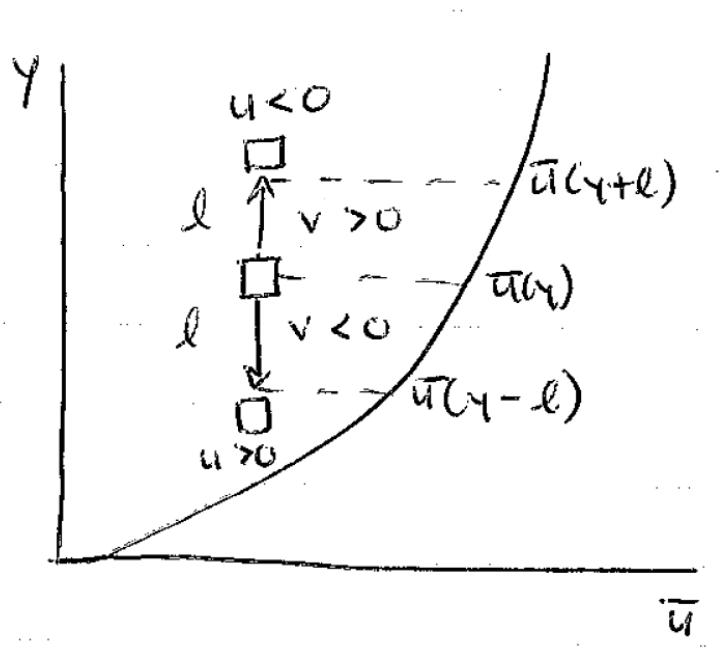


Figure 65: The momentum transport of a blob of fluid by means of turbulent fluctuations, in analogy with the molecular transport of momentum by means of random motions in a perfect gas.

$$\bar{u}(y+\ell) = \bar{u}(y) + \left.\frac{\partial \bar{u}}{\partial y}\right|_y \ell + \frac{1}{2}\left.\frac{\partial^2 \bar{u}}{\partial y^2}\right|_y \ell^2 + \cdots \tag{14}$$

We are interested in representing the turbulent transport and yet we use use the mean velocity variation because we assume that the magnitude of the fluctuations can be represented by the velocity gradient, which represents the large scale vorticity, i.e. the vorticity of the largest eddies in the flow, according to $\omega_z \sim \frac{u}{\ell} \sim \frac{\partial \bar{u}}{\partial y}$. From this it follows immediately that $u^2 \sim \ell^2\omega_z^2 \sim \ell^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2$. Thus Prandtl's theory can be developed directly from a simple dimensional argument.

and thus the magnitude of the velocity variation is approximately

$$|\bar{u}(y+\ell) - \bar{u}(y)| \approx \ell\left|\left.\frac{\partial \bar{u}}{\partial y}\right|_y\right| \tag{15}$$

which represent the mean velocity variation associated with the transport. We can call this Prandtl's 'momentum transfer theory'.

Putting the mixing length and velocity scales together yields

$$\nu_t = \ell^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \tag{16}$$

and a 2D model of the form

$$-\rho \overline{uv} = \rho \ell^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \frac{\partial \bar{u}}{\partial y} \tag{17}$$

where ℓ represents the size of the large eddies of turbulence over which the mixing occurs; this scale, could, of course, vary with Reynolds number. But how do we choose ℓ? Far from the wall, the primary length scale is the boundary layer thickness, δ, so we could set $\ell \sim \delta$.

Now, if we take $\ell \sim y$ as we would expect in the log-region, in which y is the only relevant length scale, then we can equivalently write $\ell \sim \kappa y$ and then

$$-\rho \overline{uv} = \rho \ell^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \frac{\partial \bar{u}}{\partial y} \tag{18}$$

$$\tau_{xy}^R = \rho \ell^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \tag{19}$$

$$u_\tau^2 = \ell^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \tag{20}$$

$$\frac{u_\tau}{\kappa y} = \frac{\partial \bar{u}}{\partial y} \tag{21}$$

Note that in the log layer, $\tau_{xy}^R \approx \tau_w$ since it is at the outer edge of the constant stress region, thus we can use τ_{xy}^R to define the friction velocity, $u_\tau^2 = \frac{\tau_w}{\rho}$.

Thus we see that the log-law is actually a consequence of assuming an eddy diffusion model, and a lengthscale y (i.e. that Reynolds number is asymptotically large and thus wall-normal location has no viscous dependence). But we are still left with a situation where we need to estimate ℓ in different areas of the flow field; is there a way to eliminate ℓ altogether?

In 1930, Karman proposed a hypothesis to relate the turbulence behavior everywhere in a flow field

> **Hypothesis of Local Kinematic Similarity**: The fields of turbulent fluctuations in the neighborhood of every point in a turbulent flow (except for the viscous sublayer) are similar; they differ only in length and time scales.

Let's write this hypothesis for a given point $\vec{x}_0(x_0, y_0, z_0)$, and a neighborhood about this point, measured in the $\hat{y}$-direction of width, ℓ, i.e. the neighborhood is $y_0 < y < y_0 + \ell$. In this neighborhood, we expect similarity in the mean flow of the form

$$\frac{\bar{u}(y) - \bar{u}(y_0)}{\bar{u}(y_0 + \ell) - \bar{u}(y_0)} = f(\eta) \qquad \eta = \frac{y - y_0}{\ell} \tag{22}$$

Thus, a universal function $f(\eta)$ describes the flow field in an ℓ-sized neighborhood about the point of interest, y_0 for all y in the neighborhood. Let's expand all of the velocity terms in a Taylor series about y_0 to find

$$\frac{\bar{u}(y)-\bar{u}(y_0)}{\bar{u}(y_0+\ell)-\bar{u}(y_0)} \approx \frac{\left.\frac{\partial\bar{u}}{\partial y}\right|_{y_0}(y-y_0)+\frac{1}{2}\left.\frac{\partial^2\bar{u}}{\partial y^2}\right|_{y_0}(y-y_0)^2}{\left.\frac{\partial\bar{u}}{\partial y}\right|_{y_0}(\ell)+\frac{1}{2}\left.\frac{\partial^2\bar{u}}{\partial y^2}\right|_{y_0}(\ell)^2} \tag{23}$$

$$f(\eta) \approx \frac{\eta+\frac{1}{2}\ell\eta^2\left(\left.\frac{\partial^2\bar{u}}{\partial y^2}\right|_{y_0}\Big/\left.\frac{\partial\bar{u}}{\partial y}\right|_{y_0}\right)}{1+\frac{1}{2}\ell\left(\left.\frac{\partial^2\bar{u}}{\partial y^2}\right|_{y_0}\Big/\left.\frac{\partial\bar{u}}{\partial y}\right|_{y_0}\right)} \tag{24}$$

However, by construction, $f(\eta)$ is not a function of ℓ and thus we know that

$$\ell \sim \left(\left.\frac{\partial\bar{u}}{\partial y}\right|_{y_0}\Big/\left.\frac{\partial^2\bar{u}}{\partial y^2}\right|_{y_0}\right) \tag{25}$$

and thus we obtain a definition for the mixing length defined in terms of the local velocity gradients. Substituting into Prandtl's relation, we find

Let $a = \frac{\partial\bar{u}}{\partial y}$, then we have $\frac{u_\tau}{\kappa a^2} = \left(\frac{\partial a}{\partial y}\right)^{-1}$ and $\frac{u_\tau}{\kappa}\frac{da}{a^2} = dy$ and thus $\frac{u_\tau}{\kappa}a^{-1} = y$

$$u_\tau^2 = \ell^2\left(\frac{\partial\bar{u}}{\partial y}\right)^2 \tag{26}$$

$$u_\tau^2 = \kappa^2\left(\left.\frac{\partial\bar{u}}{\partial y}\right|_{y_0}\Big/\left.\frac{\partial^2\bar{u}}{\partial y^2}\right|_{y_0}\right)^2\left(\frac{\partial\bar{u}}{\partial y}\right)^2 \tag{27}$$

$$u_\tau = \kappa\left(\left(\left.\frac{\partial\bar{u}}{\partial y}\right|_{y_0}\right)^2\Big/\left.\frac{\partial^2\bar{u}}{\partial y^2}\right|_{y_0}\right)^2 \tag{28}$$

$$\frac{u_\tau}{\kappa y} = \frac{\partial\bar{u}}{\partial y} \tag{29}$$

This is actually how Karman got his name attached to κ. Obviously, he did not come up with the log-law, but he came up with this way of enhancing Prandtl's mixing length model and defining the mixing length in terms of a universal similarity function and a supposedly universal scaling parameter, κ.

At this point, we have essentially reached the limit of algebraic models for the Reynolds stress. If these models aren't sufficient for a particular flow simulation, we need to develop more sophisticated, non-algebraic models. We can take the original eddy viscosity model of Boussinesq again, but instead of defining the viscosity in terms of momentum transfer (Prandtl), we will define it in terms of the TKE balance (also due to Prandtl, 1945). From dimensional arguments, we can write the velocity scale for the eddy viscosity in terms of the TKE, $k = \frac{1}{2}\overline{u_i u_i}$

$$\nu_t \sim u\ell \sim k^{1/2}\ell \tag{30}$$

But we still need a length scale, ℓ, also based on the energy balance.

$$\frac{Dk}{Dt} + \frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i u_k u_i} + \frac{1}{\rho}\overline{p u_k} - 2\nu\overline{u_i s_{ki}}\right) = \mathcal{P} - \varepsilon \tag{31}$$

We know that, if we integrate the TKE over all of space (or a closed volume), the transport terms (in the divergence) have no net effect

on the total energy, and thus only $\mathcal{P}$ and ε are the significant parameters. The production term is composed of the exact quantity we want to model, $\mathcal{P} = \frac{1}{\rho}\tau_{ij}^R \bar{s}_{ij}$, so the only remaining parameter is the dissipation. From dimensional arguments, we know that

$$\varepsilon \sim \frac{u^3}{\ell} \sim \frac{(k^{1/2})^3}{\ell} \tag{32}$$

and thus

$$\ell \sim \frac{k^{3/2}}{\varepsilon} \tag{33}$$

Combining the length and velocity scales yields

$$\nu_t \sim k^{1/2}\frac{k^{3/2}}{\varepsilon} \sim \frac{k^2}{\varepsilon} \tag{34}$$

And we can write this with an arbitrary constant as

$$\nu_t = c_\mu \sim \frac{k^2}{\varepsilon} \tag{35}$$

The mixing length can be defined by $\ell = c_\varepsilon \frac{k^{3/2}}{\varepsilon}$ and $c_\mu \approx 0.09$ and $c_\varepsilon \approx 0.168$ based on empirical studies

But how do we know what k and ε are? Obviously, we need more equations. We have an equation to describe the dynamics of k already, but it depends on triple product terms as well as pressure-velocity correlation terms in the divergence. (We can't simply eliminate the transport by integrating the TKE over the entire volume because we want to obtain a local value of ν_t). So we need to develop a model for the transport of TKE,

$$\frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i u_k u_i} + \frac{1}{\rho}\overline{p u_k} - 2\nu\overline{u_i s_{ki}}\right) \tag{36}$$

We know that this term serves to redistribute the TKE spatially. The pressure term is associated with spreading the TKE among different velocity components; the inertial term is associated with spreading TKE among different size eddies. Recall that Boussinesq's entire theory is based on the idea that turbulent momentum transport can be considered diffusive; let's assume that TKE transport is also diffusive. Then we can write

$$\frac{\partial}{\partial x_k}\left(\frac{1}{2}\overline{u_i u_k u_i} + \frac{1}{\rho}\overline{p u_k} - 2\nu\overline{u_i s_{ki}}\right) \approx -\frac{\partial}{\partial x_k}\left(\nu_t \frac{\partial k}{\partial x_k}\right) \tag{37}$$

The sign of the diffusive is against the mean gradient of TKE and that flux must operate in the same sense as the dominant (non-viscous) momentum flux in the original transport terms, i.e. $\frac{1}{2}\overline{u_i u_k u_i} + \frac{1}{\rho}\overline{p u_k} \approx -\nu_t \frac{\partial k}{\partial x_k}$

See the thesis of Hanjalic 1970 for the details

With this Boussinesq-type modeling of the TKE transport, we can rewrite the TKE dynamics as

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_k}\left(\nu_t \frac{\partial k}{\partial x_k}\right) + \frac{1}{\rho}\tau_{ij}^R \bar{s}_{ij} - \varepsilon \tag{38}$$

We recall that $\tau_{ij}^R = 2\rho\nu_t \bar{s}_{ij} - \frac{1}{3}\rho\overline{u_k u_k}\delta_{ij}$ and $\nu_t = \frac{k^2}{\varepsilon}$ so the TKE equation is defined entirely in terms of mean flow quantities, k, and ε.

See the thesis of Hanjalic 1970 for the details

However, we still need an equation for the evolution of ε. Let's derive it, the way we derived our original equation for the Reynolds stress itself, starting from the momentum dynamics of the velocity fluctuations.

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} + u_k \frac{\partial \bar{u}_i}{\partial x_k} + \bar{u}_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu \frac{\partial u_i}{\partial x_j} + \overline{u_j u_i}\right] \tag{39}$$

Now, take the gradient of the entire equation, $\frac{\partial}{\partial x_l}$ to obtain

$$\frac{\partial}{\partial t}\frac{\partial u_i}{\partial x_l} + \bar{u}_k \frac{\partial}{\partial x_k}\frac{\partial u_i}{\partial x_l} + \frac{\partial \bar{u}_k}{\partial x_l}\frac{\partial u_i}{\partial x_k} = -u_k \frac{\partial^2}{\partial x_k \partial x_l}(\bar{u}_i + u_i) - \frac{\partial u_k}{\partial x_l}\frac{\partial}{\partial x_k}(\bar{u}_i + u_i) + \frac{\partial^2}{\partial x_j \partial x_l}\overline{u_i u_j} + \nu \frac{\partial^2}{\partial x_j^2}\frac{\partial u_i}{\partial x_l} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x_i \partial x_l} \tag{40}$$

and now multiply through by $2\nu \frac{\partial u_i}{\partial x_l}$ and then ensemble average the entire equation, where we recall that the pseudo-dissipation is defined as

'Destruction' is the decay of the dissipation rate itself, by means of viscosity; like a dampening of the dissipation process

$$\hat{\varepsilon} = \nu \overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}} \tag{41}$$

to obtain

$$\begin{aligned}\frac{\partial \hat{\varepsilon}}{\partial t} + \bar{u}_k \frac{\partial \hat{\varepsilon}}{\partial x_k} = &\underbrace{-2\nu \frac{\partial \bar{u}_i}{\partial x_k}\left(\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} + \overline{\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}}\right) - 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} - 2\nu \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_l}\left(\overline{u_k \frac{\partial u_i}{\partial x_l}}\right)}_{\text{Generation}} \\ &+ \underbrace{\nu \frac{\partial^2}{\partial x_j^2}\hat{\varepsilon}}_{\text{Viscous Diffusion}} \underbrace{-\nu \frac{\partial}{\partial x_k}\left(\overline{u_k \frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}}\right)}_{\text{Turbulent Diffusion}} \underbrace{-2\nu^2 \overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right)^2}}_{\text{Viscous Destruction}} \underbrace{-\frac{2\nu}{\rho}\frac{\partial}{\partial x_i}\left(\overline{\frac{\partial p}{\partial x_l}\frac{\partial u_i}{\partial x_l}}\right)}_{\text{Pressure Transport}}\end{aligned} \tag{42}$$

This equation depends on many additional unknown quantities for which we have no description. Thus we will need to model these unknown quantities in order to solve for the pseudo-dissipation dynamics. We assume that the dominant generation terms for the dissipation are terms which involve a mean gradient, $\frac{\partial \bar{u}_i}{\partial x_k}$, and neglect the others. Then the dominant generation term can be modeled in simpler form.

$$-2\frac{\partial \bar{u}_i}{\partial x_k}\underbrace{\nu\left(\overline{\frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}} + \overline{\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_k}}\right)}_{\text{if } i=k \text{ then} \sim \hat{\varepsilon}} \sim -2\frac{\partial \bar{u}_i}{\partial x_k}\left(\frac{\overline{u_i u_k}}{k}\hat{\varepsilon}\right) = c_1 \bar{s}_{ik}\left(\frac{\tau_{ik}^R}{\rho k}\hat{\varepsilon}\right) \tag{43}$$

The viscous destruction term can also be modeled, dimensionally, in terms of $\hat{\varepsilon}$ of k, although there is very little theoretical justification for this choice.

By contracting $i = k$, the directional dependence of the transport is lost; this directionality can be added back by multiplying through by the non-dimensional Reynolds stress $\frac{\overline{u_i u_k}}{k}$

$$2\nu^2 \overline{\left(\frac{\partial^2 u_i}{\partial x_j \partial x_l}\right)^2} = c_2 \frac{\hat{\varepsilon}^2}{k} \tag{44}$$

The turbulent diffusion term can be modeled, following Boussinesq theory, as diffusive using a turbulent viscosity. The pressure transport is assumed negligible.

$$\nu \frac{\partial}{\partial x_k}\left(\overline{u_k \frac{\partial u_i}{\partial x_l}\frac{\partial u_i}{\partial x_l}}\right) = -\frac{1}{\sigma}\nu \frac{\partial}{\partial x_k}\left(\nu_t \frac{\partial \hat{\varepsilon}}{\partial x_k}\right) \tag{45}$$

Combining the modeled terms together and neglecting the appropriate terms yields

$$\frac{\partial \hat{\varepsilon}}{\partial t} + \bar{u}_k \frac{\partial \hat{\varepsilon}}{\partial x_k} = \underbrace{c_1 \bar{s}_{ik} \left(\frac{\tau_{ik}^R}{\rho k} \hat{\varepsilon} \right)}_{\text{Generation}} + \underbrace{\nu \frac{\partial^2}{\partial x_j^2} \hat{\varepsilon}}_{\text{Viscous Diffusion}} + \underbrace{\frac{1}{\sigma} \nu \frac{\partial}{\partial x_k} \left(\nu_t \frac{\partial \hat{\varepsilon}}{\partial x_k} \right)}_{\text{Turbulent Diffusion}} - \underbrace{c_2 \frac{\hat{\varepsilon}^2}{k}}_{\text{Viscous Destruction}} \tag{46}$$

where, empirically, we can find the fitting constants by comparison with experimental results, for a given geometry and flow type. We note that this equation depends on only mean quantities and k and $\hat{\varepsilon}$. Combined with the dynamical equation for k, the system of equations needed to describe the eddy viscosity, ν_t, is closed and the problem is solvable.

Typical values are $\sigma = 1.3$, $c_1 = 1.44$, and $c_2 = 1.92$

However, the dissipation equation was nearly made up out of thin air. We used so many different approximations and models, there is very little of the original equation left. We might as well have just invented the equation from nothing! And yet it works well in a variety of circumstances, and thus the k-ε approach has become one of the most popular in turbulence engineering.

There are many other closure models in physical space, using two or even more equations. The k-ε model is called a 'two-equation, one-point' closure scheme, since it describes the one-point statistics, but we can develop similar closure schemes for the two point dynamics, $\frac{\partial}{\partial t} Q_{ij}(\vec{r})$, where we need to model the third-order terms. But all of these physical closure schemes essentially follow the same Boussinesq approach of an eddy viscosity.

There are, however, problems with the eddy viscosity assumption.

1. The derivation is questionable, since it is ultimately based on an analogy between blobs of fluid and molecules of a perfect gas (from the kinetic theory of gases). Blobs of fluid are continuous and constantly interacting; gas molecules are discrete and only occasionally interacting. The mean free path, λ_{rms} is very small compared to the largest scales of gas motion, whereas the mixing length ℓ is comparable to the largest scales of motion.

2. The eddy viscosity, ν_t, should be a tensor quantity such that different values are produced for the different components of the Reynolds stress, based on direction. The way we have defined eddy viscosity so far, it incorporates directional dependence through the mean velocity gradient, but the mixing length scales are the same in all directions. We expect this approach to perform poorly in strongly anisotropic and inhomogeneous flows.

3. The model ultimately assumes that the Reynolds stress is a local phenomenon, that it depends only on the local strain rate tensor at a single instant in time. This is implicitly a quasi-static assumption, which treats the turbulence as if history doesn't matter and it has no memory, which we have seen is not a justified assumption.

Because of these problems, we will consider another closure scheme in spectral space, which avoids the underlying assumptions of the eddy viscosity approach.

Spectral Closure

Recall the spectral version of the Karman-Howarth equation for the decay of homogeneous, isotropic turbulence (without mean gradients)

$$\frac{\partial E(k)}{\partial t} = T(k) - 2\nu k^2 E(k) \tag{47}$$

where the transfer term $T(k,t)$ represents the energy cascade, moving energy from large scales to small scales. This interpretation is a consequence of the fact that

$$\int_0^\infty T(k')dk' = 0 \tag{48}$$

indicating that there is no net effect on the total TKE from the transfer, merely a redistribution of energy among different wavenumbers. For a given wavenumber k, we can write

$$\int_0^k T(k')dk' + \int_k^\infty T(k')dk' = 0 \tag{49}$$

or, rewriting

$$\int_k^\infty T(k')dk' = -\int_0^k T(k')dk' \equiv \Pi_E(k) \tag{50}$$

Recall that $T(k)$ represents a power density (i.e. turbulent power per wavenumber). Thus, the quantity $\Pi_E(k)$ represents the turbulent energy transfer from wavenumbers less than k toward wavenumbers greater than k (the sign was chosen to reflect the fact that $T(k)$ is negative for lower wavenumbers, as energy is lost from the larger scales). We can sketch this

$T(k)$ has units TKE per unit wavenumber per unit time per volume of fluid; $T(k)$ was originally $ik_j\Phi_{ij,i}(k)$ integrated over a spherical differential shell of area $4\pi k^2$. The spectral density $\Phi_{ij,i}(k)$ was a TKE flux density (TKE per unit wavenumber per unit time per unit area), the divergence of which (k_j) yielded a spectral power density (TKE per unit wavenumber per unit time per unit volume)

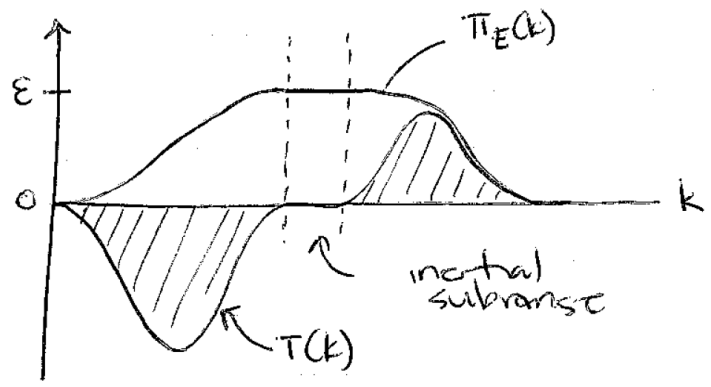


Figure 66: The transfer function $T(k)$ and its integral $\Pi_E(k)$ as a function of wavenumber; $T(k)$ is negative for low wavenumbers and positive for high wavenumbers; according to the definition $\Pi_E(k)$ is positive.

In the inertial subrange, ε is the only relevant parameter, and thus the integrated turbulent power transferred in this region is equal to Π_E (dissipation is insignificant at these wavenumbers). Above the inertial subrange, dissipation becomes more significant and there is less energy available to be transferred from scale to scale, since some of that energy is going to dissipation; thus $\Pi_E < \varepsilon$ at high wavenumbers. At low wavenumbers, the transfer is also less than the dissipation rate, because some large scale retain their original energy or transfer it among other large scales.

For higher wavenumbers, we can write the integrated energy dynamics

$$\underbrace{\frac{\partial}{\partial t}\int_k^\infty E(k')dk'}_{\frac{\partial k}{\partial t}(k\to\infty)} = \Pi_E(k) - \underbrace{2\nu\int_k^\infty k'^2 E(k')dk'}_{\varepsilon(k\to\infty)} \tag{51}$$

Above the inertial subrange, $k \gg k_e$, above the energetic scales, the transfer roughly balances the dissipation

$$0 \approx \Pi_E(k \gg k_e) - \varepsilon \tag{52}$$

For lower wavenumbers, we can write the integrated dynamics as

$$\underbrace{\frac{\partial}{\partial t}\int_0^k E(k')dk'}_{\frac{\partial k}{\partial t}(0\to k)} = -\Pi_E(k) - \underbrace{2\nu\int_0^k k'^2 E(k')dk'}_{\varepsilon(0\to k)} \tag{53}$$

and below the inertial subrange (in the large scale range, $k \ll k_d$, below the dissipation scales), dissipation is insignificant but the time evolution of the large scale energy matters, and thus

$$\frac{\partial}{\partial t}\int_0^{k\ll k_d} E(k')dk' \approx -\Pi_E(k \ll k_d) \tag{54}$$

And inside the inertial sublayer, the energy coming in from the large scales balances the energy going into the small scales, and thus

$$0 \approx -\Pi_E(k \ll k_d) + \Pi_E(k \gg k_e) \tag{55}$$

Substituting the above relations we find, as expected

$$\frac{\partial}{\partial t}\int_0^{k\ll k_d} E(k')dk' \approx -\varepsilon \tag{56}$$

that most of the energy of the large scale motions goes into the dissipation. We see that this spectral transfer function, Π_E describes everything we need to know about the energy cascade; thus we can state the closure problem as essentially a problem of how to represent that function. For simplicity, instead of writing the equation for

a fixed wavenumber bound (e.g. $(0, k)$), let's write it in differential form

$$\frac{\partial E(k)}{\partial t} = \underbrace{-\frac{\partial}{\partial k}\Pi_E(k)}_{T(k)-T(0)} - 2\nu k^2 E(k) \tag{57}$$

Note that $T(0) = 0$ based on its definition, where $h(r)$ is an odd function for isotropic turbulence

We can represent $\Pi_E(k)$ in two ways, just like we represents ν_t in two ways: 1) algebraically (mixing length) and 2) via a dynamical relationship (k-ε). Let's consider just the simpler, algebraic models. A number of different models have been proposed to describe the transfer of energy, particularly at small scales, $k \ll k_e$. We start with dimensional arguments:

$$[k] \sim \frac{1}{L}, \qquad [E(k)] \sim \frac{L^3}{T^2}, \qquad [\Pi_E] \sim \frac{L^2}{T^3} \tag{58}$$

So if we want to construct a model of Π_E from $E(k)$ and k, we can write a power law of the form

$$\Pi_E \sim E^\alpha k^\beta \quad \Rightarrow \quad \frac{L^2}{T^3} \sim \left(\frac{L^3}{T^2}\right)^\alpha \left(\frac{1}{L}\right)^\beta \tag{59}$$

and we conclude that, on purely dimensional grounds,

$$\Pi_E \sim E^{3/2} k^{5/2} \tag{60}$$

How does this model behave in the inertial subrange, where we know that $E(k) \sim \varepsilon^{2/3} k^{-5/3}$? Plugging in, we obtain

This is the Bernoulli ODE $y' + P(x)y = Q(x)y^n$ with $n = \frac{1}{2}$

$$\Pi_E \sim \left(\varepsilon^{2/3} k^{-5/3}\right)^{3/2} k^{5/2} \sim \varepsilon \tag{61}$$

exactly what we expected for the inertial subrange. What about within the dissipation range, above the inertial subrange? In this range, the dynamics are determined by the transfer and dissipation,

$$0 \approx -\frac{\partial}{\partial k}\Pi_E(k) - 2\nu k^2 E(k) \tag{62}$$

and we have already shown earlier that we expected $E(k)$ to decay exponentially quickly as $k \to \infty$. Does the model reflect this behavior?

$$\frac{\partial}{\partial k}\Pi_E(k) \approx -2\nu k^2 E(k) \tag{63}$$

$$\frac{\partial}{\partial k}\left(E^{3/2} k^{5/2}\right) \sim -2\nu k^2 E(k) \tag{64}$$

$$\frac{\partial E(k)}{\partial k} + \frac{5}{2}\left(\frac{E}{k}\right) \sim -\frac{4}{3}\nu \left(\frac{E}{k}\right)^{1/2} \tag{65}$$

This is the Bernoulli ODE $y' + P(x)y = Q(x)y^n$ with $n = \frac{1}{2}$

The solution is found in the form

$$E(k) \sim k^{-5/3} \underbrace{\left(\cdots + k^{8/3} + \cdots + k^{4/3} + \cdots\right)}_{\text{polynomial of degree } \frac{8}{3}} \tag{66}$$

and this obviously does not decay exponentially for high k, and thus our simple closure model fails in the dissipative region of the spectrum, although it works well in the inertial subrange. Can we find a model that works throughout the universal equilibrium region, from the inertial subrange out towards the dissipation range?

In the universal equilibrium range, we know that $E(k)$ depends on ε and ν. Let's assume that Π_E has the same functional dependence, based on Kolmogorov theory, and again use dimensional arguments.

$$[k] \sim \frac{1}{L}, \qquad [E(k)] \sim \frac{L^3}{T^2}, \qquad [\Pi_E] \sim \frac{L^2}{T^3}, \qquad [\nu] \sim \frac{L^2}{T}, \qquad [\varepsilon] \sim \frac{L^2}{T^3} \tag{67}$$

We introduce non-dimensional, universal forms as we did earlier, $\hat{E}(\eta k)$ and $\hat{\Pi}_E(\eta k)$ and then

Recall that $\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

$$E \sim \varepsilon^{\alpha} k^{\beta} \hat{E}, \qquad \Pi_E \sim \varepsilon^{\gamma} k^{\delta} \hat{\Pi}_E \tag{68}$$

$$\left(\frac{L^3}{T^2}\right) \sim \left(\frac{L^2}{T^3}\right)^{\alpha} \left(\frac{1}{L}\right)^{\beta}, \qquad \left(\frac{L^2}{T^3}\right) \sim \left(\frac{L^2}{T^3}\right)^{\gamma} \left(\frac{1}{L}\right)^{\delta} \tag{69}$$

$$E \sim \varepsilon^{2/3} k^{-5/3} \hat{E}, \qquad \Pi_E \sim \varepsilon \hat{\Pi}_E \tag{70}$$

With this model, let's check its behavior in the equilibrium range (i.e. dissipative range)

$$\frac{\partial}{\partial k} \Pi_E(k) \approx -2\nu k^2 E(k) \tag{71}$$

$$\frac{\partial}{\partial k} \left(\varepsilon \hat{\Pi}_E\right) \sim -2\nu k^2 \left(\varepsilon^{2/3} k^{-5/3} \hat{E}\right) \tag{72}$$

$$\frac{\partial}{\partial \eta k} \hat{\Pi}_E(\eta k) \sim -2(k\eta)^{1/3} \hat{E}(\eta k) \tag{73}$$

$$\frac{\partial}{\partial (\eta k)^{4/3}} \hat{\Pi}_E \sim -\frac{3}{2} \hat{E} \tag{74}$$

$$\frac{d}{d\zeta} \hat{\Pi}_E(\zeta) \sim -\frac{3}{2} \hat{E}(\zeta) \tag{75}$$

We use the relation $d\left(a^{4/3}\right) = \frac{4}{3} a^{1/3} da$ which yields $\frac{3}{4} d\left(a^{4/3}\right) = a^{1/3} da$ and also define $\zeta = (\eta k)^{4/3}$

Now, since $\hat{\Pi}_E$ and $\hat{E}$ are both assumed power-law functions of ζ, we can write

$$\hat{\Pi}_E \sim \hat{E}^m \tag{76}$$

and we can plug this into the above relation

$$\frac{d}{d\zeta} \left(\hat{E}^m\right) \sim -\frac{3}{2} \hat{E}(\zeta) \tag{77}$$

$$\frac{d\hat{E}^m}{\hat{E}} \sim -\frac{3}{2} d\zeta \tag{78}$$

and we conclude that $m = 1$ to obtain exponential decay in the energy spectral density, $\hat{E}$ as k (or here, equivalently, $\zeta \to \infty$). Thus we conclude that $\hat{\Pi}_E \sim \hat{E}$, and we can write the form of the model which provide the correct decay in the dissipative range as

$$\Pi_E \sim \varepsilon \hat{\Pi}_E \tag{79}$$
$$\sim \varepsilon \hat{E}(\eta k) \tag{80}$$
$$\sim \varepsilon \left(\frac{E}{\varepsilon^{2/3} k^{-5/3}} \right) \tag{81}$$
$$\sim \varepsilon^{1/3} k^{5/3} E(k) \tag{82}$$

And this closure model was produced by Pao (1965). Empirically, it tends to behave quite well in describing the equilibrium range, down to scales as small as $\eta k \approx 0.5$, and is one of the most widely used algebraic spectral closure schemes.