Intro to Turbulent Flow
HW1

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1 Computing a Turbulent Flow

Considering a commercial airliner with a chord $L = 5\,[\mathrm{m}]$ cruising at $U = 250\left[\frac{\mathrm{m}}{\mathrm{sec}}\right]$.

1.1 Estimating boundary layer thickness δ

According to Prandtl's one-seventh power law:

$$\frac{\delta}{x} \approx \frac{0.16}{Re_x^{\frac{1}{7}}} \tag{1}$$

At the trailing edge, the thickness of the boundary layer is:

$$\frac{\delta}{L} \approx \frac{0.16}{Re_L^{\frac{1}{7}}} \quad , \quad Re_L = \frac{UL}{\nu} \tag{2}$$

To calculate the kinematic viscosity of air, let's assume the airliner is cruising at 30[kft]. At this hight, the temperature is about $-44.3°[C]$ [1]. Hence, the kinematic viscosity is about $1 \cdot 10^{-5}$ [2]

$$Re_L = \frac{250 \cdot 5}{1 \cdot 10^{-5}} = 1.25 \cdot 10^8 \tag{3}$$

$$\Downarrow$$

$$\delta \approx \frac{0.16 \cdot 5}{(1 \cdot 10^{-5})^{\frac{1}{7}}} = 0.0558[\mathrm{m}] \tag{4}$$

1.2 Smallest Dynamically Important Scales

At the largest scales, the Reynolds number is very high, which indicates a turbulent regime. In a turbulent regime, there is no dissipation, which means:

$$Re_\ell = \frac{u\ell}{\nu} \gg 1 \tag{5}$$

However, we know that turbulent flows are dissipative, so there must be dissipation. For dissipation to accrue, the Reynolds number for the scales at which dissipation happens, the smallest scales, must be:

$$Re_\eta = \frac{v\eta}{\nu} \sim 1 \tag{6}$$

We can see that by fulfilling both demands for turbulent flow, there must be an energy transfer between scales, namely, an energy cascade.

1.3 Number of Grid Points

In order to accurately calculating the flow at the boundary layer, the size of one cell needs to be smaller then smallest eddy. From the energy cascade concept, we can conclude that the rate of change of the kinetic energy at the large scales:

$$\frac{du^2}{dt} \sim \frac{u^2}{\frac{\ell}{u}} \sim \varepsilon \tag{7}$$

is balanced by the energy dissipated at the small scales. By dimensional analysis, we find that:

$$\varepsilon \sim \nu \left(\frac{v}{\eta}\right)^2 \tag{8}$$

By combining Eq.7 and Eq.8 we get:

$$\frac{\ell}{\eta} \sim Re_\ell^{\frac{3}{4}} \quad \text{and} \quad \frac{v}{u} \sim Re_\ell^{-\frac{1}{4}} \quad \text{and} \quad \frac{\frac{\ell}{u}}{\frac{\eta}{v}} \sim Re_\ell^{\frac{1}{2}} \tag{9}$$

From Eq.9 we can calculate the size of one cell at the boundary layer:

$$\eta_x \sim \frac{L}{Re_L^{\frac{3}{4}}} = \frac{5}{(1.25 \cdot 10^8)^{\frac{3}{4}}} = 4.2295 \cdot 10^{-6}\,[\mathrm{m}] \tag{10}$$

So, the number of grid point along the chord is:

$$N_x \sim \frac{L}{\eta} = \frac{5}{4.2295 \cdot 10^{-6}} = 1.1822 \cdot 10^6 \tag{11}$$

To calculate the number of grid point normal to the airfoil, we need to asstemate the small scale of the boundary layer:

$$\eta_y \sim \frac{\delta}{Re_\delta \frac{3}{4}} = \frac{0.0558}{\left(\frac{U\delta}{\nu}\right)^{\frac{3}{4}}} = 1.3745 \cdot 10^{-6}\,[\mathrm{m}] \tag{12}$$

so the number of grid point normal to the airfoil:

$$N_y \sim \frac{\delta}{\eta} = \frac{0.0558}{4.2295 \cdot 10^{-6}} = 4.0574 \cdot 10^4 \tag{13}$$

The total number of grid point is therefore:

$$N = N_x N_y = \boxed{4.7965 \cdot 10^{10}} \tag{14}$$

1.4 Instantaneous Flow Over One Eddy

The turnover time of one eddy is defined as:

$$t_{large} = \frac{\delta}{U} \tag{15}$$

The time step needs to be smaller then the smallest time step. Therefore:

$$t_{small} = \frac{\eta_y}{v} \tag{16}$$

From Eq.9 we get:

$$\frac{t_{large}}{t_{small}} \sim Re_\delta^{\frac{1}{2}} \tag{17}$$

$$\Downarrow$$

$$N_t = \frac{t_{large}}{t_{small}} \sim Re_\delta^{\frac{1}{2}} \sim 1223 \text{ steps} \tag{18}$$

I finished this quistion in about 3 hours

2 Heating a Room

2.1 Characteristic Time Scale

The one dimensional heat equation for a non moving field is:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{19}$$

We will use dimensional analysis to determined the characteristic time scale for heating the room.

$$\begin{aligned} \frac{T_\infty}{t} &= \alpha \frac{T_\infty}{L^2} \\ t &= \frac{L^2}{\alpha} \end{aligned} \tag{20}$$

Assuming typical bedroom conditions [3]:

$$\alpha|_{T\,=\,25^\circ[C]} = 22.39 \cdot 10^{-6} \left[\frac{\mathrm{m}^2}{\mathrm{sec}}\right] \quad , \quad L = 3\,[\mathrm{m}]$$

$$\Downarrow$$

$$t = 4.0197 \cdot 10^5\,[\mathrm{sec}] = 4.6524\,[\mathrm{day}] \tag{21}$$

2.2 Induced Velocity Estimation

The momentum equation under Boussinesq approximation can be written as:

$$\frac{D}{Dt}\vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} + g\frac{\Delta T}{T} \tag{22}$$

Where:

- u is velocity induced by a space heater via buoyancy effects
- T is the local temperature
- ΔT is the difference between T and the newly heated air

Assuming the buoyancy-driven flow is turbulent:

$$Re \gg 1$$

The momentum equation can be rewritten as:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u} + g\frac{\Delta T}{T}$$

and by substituting the operators we get:

$$\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial^2 x} + g\frac{T - T_h}{T} \tag{23}$$

In order to estimate the magnitude of the induced velocity we will use dimensional analysis:

$$u = u_\infty \tilde{u} \;\Big|\; t = T\tilde{t} \;\Big|\; x = h\tilde{x} \;\Big|\; p = \Lambda \tilde{p} \;\Big|\; T = T_\infty \tilde{T}$$

$$\frac{u_\infty}{T}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{u_\infty^2}{h}\tilde{u}\cdot\frac{\partial \tilde{u}}{\partial \tilde{x}} = -\frac{\Lambda}{\rho h}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\nu u_\infty}{h^2}\frac{\partial^2 \tilde{u}}{\partial^2 \tilde{x}} + g\frac{\tilde{T} - \frac{T_h}{T_\infty}}{\tilde{T}} \tag{24}$$

Multiplying by $\dfrac{h^2}{\nu u_\infty}$:

$$\frac{h^2}{\nu u_\infty}\frac{u_\infty}{T}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{h^2}{\nu u_\infty}\frac{u_\infty^2}{h}\tilde{u}\cdot\frac{\partial \tilde{u}}{\partial \tilde{x}} = -\frac{h^2}{\nu u_\infty}\frac{\Lambda}{\rho h}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{h^2}{\nu u_\infty}\frac{\nu u_\infty}{h^2}\frac{\partial^2 \tilde{u}}{\partial^2 \tilde{x}} + \frac{h^2}{\nu u_\infty}g\frac{\tilde{T} - \frac{T_h}{T_\infty}}{\tilde{T}} \tag{25}$$

Multiplying by $\dfrac{h^2}{\nu u_\infty}$:

$$\underbrace{\frac{u_\infty h}{\nu}}_{Re_h}\underbrace{\frac{h}{u_\infty T}}_{St_h}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \underbrace{\frac{h u_\infty}{\nu}}_{Re_h}\tilde{u}\cdot\frac{\partial \tilde{u}}{\partial \tilde{x}} = -\frac{h\Lambda}{\rho\nu u_\infty}\frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{u}}{\partial^2 \tilde{x}} + \frac{h^2}{\nu u_\infty}g\frac{\tilde{T} - \frac{T_h}{T_\infty}}{\tilde{T}} \tag{26}$$

3 Help Karl Pearson

References

[1] T. E. ToolBox, "U.s. standard atmosphere: Temperature, pressure, and air properties vs. altitude." `https://www.engineeringtoolbox.com/standard-atmosphere-d_604.html`, 2003.

[2] E. Edge, "Viscosity of air, dynamic and kinematic." `https://www.engineersedge.com/physics/viscosity_of_air_dynamic_and_kinematic_14483.htm`, 2025.

[3] T. E. ToolBox, "Air - thermal diffusivity vs. temperature and pressure." `https://www.engineeringtoolbox.com/air-thermal-diffusivity-d_2011.html`, 2018.

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