Intro to Turbulent Flow
HW2

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1 Stationarity and Ergodicity

For a random process to be stationary, it needs to fulfill the following conditions:

1. $\mathbb{E}\{x_{k(t)}\} = \text{constant}$
2. $Q_{(t,t+s)} = \mathbb{E}\left\{x_{k(t)}x_{k(t+s)}\right\} = Q_{(s)}$

To determine if a random process is ergodic the following condition needs to be met:

1. $\lim_{T\to\infty} \frac{1}{T}\int_0^T Q_{(\hat{s})}d\hat{s} = 0$

1.1 Random Process #1

$$x_k = a_k \sin(2\pi ft + \theta) \tag{1}$$

Where:

$$a_k \sim U[0,1] \tag{2}$$

1.1.1 Statinarity

The expectation operator for x_k is given by:

$$\begin{aligned}
\mathbb{E}\{x_{k(t)}\} &= \int_0^1 a\sin(2\pi ft+\theta)da \\
&= \left[\frac{1}{2}a^2\sin(2\pi ft+\theta)\right]_0^1 \\
&= \frac{1}{2}\sin(2\pi tf+\theta) \neq \text{const}
\end{aligned} \tag{3}$$

$$\Downarrow$$

$$\boxed{\text{Not stationary}}$$

1.1.2 Ergodicity

The covariance function is given by:

$$\begin{aligned}
Q_{(t,t+s)} &= \mathbb{E}\left\{x_{k(t)}x_{k(t+s)}\right\} \\
&= \mathbb{E}\left\{a_k\sin(2\pi ft+\theta)a_k\sin(2\pi f(t+s)+\theta)\right\} \\
&\Downarrow \ \sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right] \\
&= \mathbb{E}\left\{a_k^2\frac{1}{2}\left[\cos(2\pi ft+\theta-2\pi f(t+s)-\theta)-\cos(2\pi ft+\theta+2\pi f(t+s)+\theta)\right]\right\} \\
&= \int_0^1 a^2\frac{1}{2}\left[\cos(2\pi fs)-\cos(4\pi ft+2\pi fs+2\theta)\right]da \\
&= \int_0^1 a^3\frac{1}{6}\left[\cos(2\pi fs)-\cos(4\pi ft+2\pi fs+2\theta)\right]da \neq Q_{(s)}
\end{aligned} \tag{4}$$

$$\Downarrow$$

$$\boxed{\text{Not ergodic}}$$

1.2 Random Process #2

$$x_k = a\sin(2\pi ft + \theta_k) \tag{5}$$

Where:

$$\theta_k \sim U[0, 2\pi] \tag{6}$$

1.2.1 Statinarity

The expectation operator for x_k is given by:

$$\begin{aligned}
\mathbb{E}\{x_{k(t)}\} &= \int_0^{2\pi} \frac{1}{2\pi}\alpha\sin(2\pi ft + \theta)d\theta \\
&= \left[-\frac{1}{2\pi}\alpha\cos(2\pi ft + \theta)\right]_0^{2\pi} \\
&= -\frac{1}{2\pi}\alpha\cos(2\pi ft + 2\pi) + \frac{1}{2\pi}\alpha\cos(2\pi ft + 0) \\
&= 0 = \text{const}
\end{aligned} \tag{7}$$

The covariance function is given by:

$$\begin{aligned}
Q_{(t,t+s)} &= \mathbb{E}\left\{x_{k(t)}x_{k(t+s)}\right\} \\
Q_{(t,t+s)} &= \mathbb{E}\left\{a\sin(2\pi ft + \theta_k)a\sin(2\pi f(t+s) + \theta_k)\right\} \\
&\Downarrow \ \sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right] \\
&= \mathbb{E}\left\{a^2\frac{1}{2}\left[\cos(2\pi ft + \theta_k - 2\pi f(t+s) - \theta_k) - \cos(2\pi ft + \theta_k + 2\pi f(t+s) + \theta_k)\right]\right\} \\
&= \int_0^{2\pi} a^2\frac{1}{4\pi}\left[\cos(2\pi ft + \theta - 2\pi f(t+s) - \theta) - \cos(2\pi ft + \theta + 2\pi f(t+s) + \theta)\right]d\theta \\
&= \int_0^{2\pi}\left[a^2\frac{1}{4\pi}\cos(2\pi fs) - a^2\frac{1}{4\pi}\cos(4\pi ft + 2\theta + 2\pi fs)\right]d\theta \\
&= \left[a^2\frac{1}{4\pi}\cos(2\pi fs)\,\theta - a^2\frac{1}{8\pi}\sin(4\pi ft + 2\theta + 2\pi fs)\right]_0^{2\pi} \\
&= a^2\frac{1}{2}\cos(2\pi fs) - a^2\frac{1}{8\pi}\sin(4\pi ft + 4\pi + 2\pi fs) + a^2\frac{1}{8\pi}\sin(4\pi ft + 2\pi fs) \\
&= \frac{a^2}{2}\cos(2\pi fs) = Q_{(s)} \qquad \blacksquare
\end{aligned} \tag{8}$$

$$\Downarrow$$
$$\boxed{\text{Stationary}}$$

1.2.2 Ergodicity

To prove an ergodic process we need to check the next limit:

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T Q_{(\hat{s})}d\hat{s} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\frac{a^2}{2}\cos\left(2\pi f\hat{s}\right)d\hat{s}\\ &= \lim_{T\to\infty}\frac{1}{T}\left[\frac{a^2}{4\pi f}\sin\left(2\pi f\hat{s}\right)\right]_0^T\\ &= \lim_{T\to\infty}\frac{1}{T}\frac{a^2}{4\pi f}\sin\left(2\pi fT\right)=0 \qquad \blacksquare\end{aligned} \tag{9}$$

$$\Downarrow$$

$$\boxed{\text{Ergodic}}$$

1.3 Random Process #3

$$x_k = a_k\sin\left(2\pi ft+\theta_k\right) \tag{10}$$

Where:

$$a_k \sim U\left[0,1\right] \qquad \theta_k\sim U\left[0,2\pi\right] \tag{11}$$

1.3.1 Statinarity

The expectation operator for x_k is given by:

$$\begin{aligned}\mathbb{E}\{x_{k(t)}\} &= \int_0^{2\pi}\frac{1}{2\pi}\int_0^1 a\sin\left(2\pi ft+\theta\right)dad\theta\\ &= \int_0^{2\pi}\frac{1}{2\pi}\left[\frac{a^2}{2}\sin\left(2\pi ft+\theta\right)\right]_0^1 d\theta\\ &= \int_0^{2\pi}\frac{1}{4\pi}\sin\left(2\pi ft+\theta\right)d\theta\\ &= \left[-\frac{1}{4\pi}\cos\left(2\pi ft+\theta\right)\right]_0^{2\pi} = 0 = \text{const}\end{aligned} \tag{12}$$

The covariance function is given by:

$$
\begin{aligned}
Q_{(t,t+s)} &= \mathbb{E}\left\{x_{k(t)}x_{k(t+s)}\right\} \\
Q_{(t,t+s)} &= \mathbb{E}\left\{a_k \sin\left(2\pi f t + \theta_k\right) a_k \sin\left(2\pi f\left(t+s\right) + \theta_k\right)\right\} \\
&\Downarrow \ \sin\left(\alpha\right)\sin\left(\beta\right) = \frac{1}{2}\left[\cos\left(\alpha-\beta\right) + \cos\left(\alpha+\beta\right)\right] \\
&= \mathbb{E}\left\{a_k^2 \frac{1}{2}\left[\cos\left(2\pi f t + \theta_k - 2\pi f\left(t+s\right) - \theta_k\right) - \cos\left(2\pi f t + \theta_k + 2\pi f\left(t+s\right) + \theta_k\right)\right]\right\} \\
&= \mathbb{E}\left\{a_k^2 \frac{1}{2}\left[\cos\left(2\pi f s\right) - \cos\left(4\pi f t + 2\theta_k + 2\pi f s\right)\right]\right\} \\
&= \int_0^{2\pi} \frac{1}{2\pi} \int_0^1 a^2 \frac{1}{2}\left[\cos\left(2\pi f s\right) - \cos\left(4\pi f t + 2\theta + 2\pi f s\right)\right] da d\theta \\
&= \int_0^{2\pi} \frac{1}{2\pi}\left[a^3 \frac{1}{6}\left[\cos\left(2\pi f s\right) - \cos\left(4\pi f t + 2\theta + 2\pi f s\right)\right]\right]_0^1 d\theta \\
&= \int_0^{2\pi}\left[\frac{1}{12\pi}\cos\left(2\pi f s\right) - \frac{1}{12\pi}\cos\left(4\pi f t + 2\theta + 2\pi f s\right)\right] d\theta \\
&= \left[\frac{1}{12\pi}\cos\left(2\pi f s\right)\theta - \frac{1}{24\pi}\sin\left(4\pi f t + 2\theta + 2\pi f s\right)\right]_0^{2\pi} \\
&= \frac{1}{6}\cos\left(2\pi f s\right) = Q_{(s)} \qquad \blacksquare
\end{aligned} \tag{13}
$$

$$\Downarrow$$

$$\boxed{\text{Stationary}}$$

1.3.2 Ergodicity

To prove an ergodic process we need to check the next limit:

$$
\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\int_0^T Q_{(\hat{s})} d\hat{s} &= \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{6}\cos\left(2\pi f\hat{s}\right) d\hat{s} \\
&= \lim_{T\to\infty}\frac{1}{T}\left[\frac{1}{12\pi f}\sin\left(2\pi f\hat{s}\right)\right]_0^T \\
&= \lim_{T\to\infty}\frac{1}{T}\frac{1}{12\pi f}\sin\left(2\pi f T\right) = 0 \qquad \blacksquare
\end{aligned} \tag{14}
$$

$$\Downarrow$$

$$\boxed{\text{Ergodic}}$$

2 Getting Closure

Considering the differencing equation:

$$u_{n+1} = ru_n\left(1 - u_n\right) \tag{15}$$

2.1 Ensemble Average $\langle u \rangle$

In order to calculate the ensemble average $\langle u \rangle$ we will assume that the limit $\lim_{n\to\infty} u_n$ exists. This means there is n at which $u_{n+1} = u_n = u$. Hence:

$$\begin{aligned} \langle u \rangle &= \langle u_{n+1} \rangle = \langle ru - ru^2 \rangle = r\langle u \rangle - r\langle u^2 \rangle \\ (1-r)\langle u \rangle &= -r\langle u^2 \rangle \\ \langle u \rangle &= \frac{-r}{1-r}\langle u^2 \rangle \qquad \blacksquare \end{aligned} \tag{16}$$

Where:

- $\langle u \rangle$ is the first moment
- $\langle u^2 \rangle$ is the second moment

2.2 Reynolds Stress

The Reynolds stresses can be represented as the second moment and form the first subsection, it can be written as:

$$\langle u \cdot u \rangle = \langle u^2 \rangle = \frac{1-r}{-r}\langle u \rangle \tag{17}$$

The Reynolds stress depends on the ensemble average and the variable r.

2.3 Solving For $\langle u \rangle$

The first moment depends on the second moment so we can't solve it directly. Let's try to solve for the second moment:

$$\begin{aligned} \langle u \cdot u \rangle &= \langle \left(ru - ru^2\right)\left(ru - ru^2\right) \rangle \\ &= \langle r^2u^2 - r^2u^3 - r^2u^3 + r^2u^4 \rangle \\ &= \langle r^2u^2 \rangle - \langle 2r^2u^3 \rangle + \langle r^2u^4 \rangle \\ &= r^2\left(\langle u^2 \rangle - \langle 2u^3 \rangle + \langle u^4 \rangle\right) \end{aligned} \tag{18}$$

We see that the second moment depends on the third and fourth moments. We can conclude that each moment depends on a higher-order moment and this 'series' continues to infinity. Hence we can not solve for the first moment directly.
By assuming that u is normaly distributed, $N[\mu, \sigma]$, we get the following central moments:

$$\langle u \rangle = \mu \;\Big\|\; \sigma^2 = \mathbb{E}\left\{(u-\mu)^2\right\} \;\Big\|\; Q_{(u,v)} = \mathbb{E}\left\{(u-\mu_u)(v-\mu_v)\right\} \;\Big\|\; \mu_3 = \mathbb{E}\left\{(u-\mu)^3\right\} \;\Big\|\; \mu_4 = \mathbb{E}\left\{(u-\mu)^4\right\}$$

$$\Downarrow$$

$$\sigma^2 = Q_{(u,u)} = \langle u^2 \rangle - \langle u \rangle^2$$
$$\langle u^2 \rangle = \langle u \rangle^2 + \sigma^2 \tag{19}$$
$$\langle u^2 \rangle = \mu^2 + \sigma^2$$

We also know that:

$$\begin{aligned} \langle u^2 \rangle &= r^2 \left(\langle u^2 \rangle - \langle 2u^3 \rangle + \langle u^4 \rangle \right) \\ \langle u^2 \rangle &= \frac{r^2}{1-r^2} \left(-2\langle u^3 \rangle + \langle u^4 \rangle \right) \end{aligned} \tag{20}$$

What we have here are non central moments. For normal distribution we know that:

$$\langle u^3 \rangle = \mu^3 + 3\mu\sigma^2 \quad \text{and} \quad \langle u^4 \rangle = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \tag{21}$$

Substituting Eq.21 into Eq.20:

$$\mu^2 + \sigma^2 = \frac{r^2}{1-r^2} \left(-2\left(\mu^3 + 3\mu\sigma^2 \right) + \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \right)$$
$$\text{and} \tag{22}$$
$$\mu = \frac{r}{r-1} \left(\mu^2 + \sigma^2 \right)$$

The ensemble average can be calculated using a numerical system of equations solver.

2.4 Numerical Verification

Let's compare the analytical solution to a numerical one:

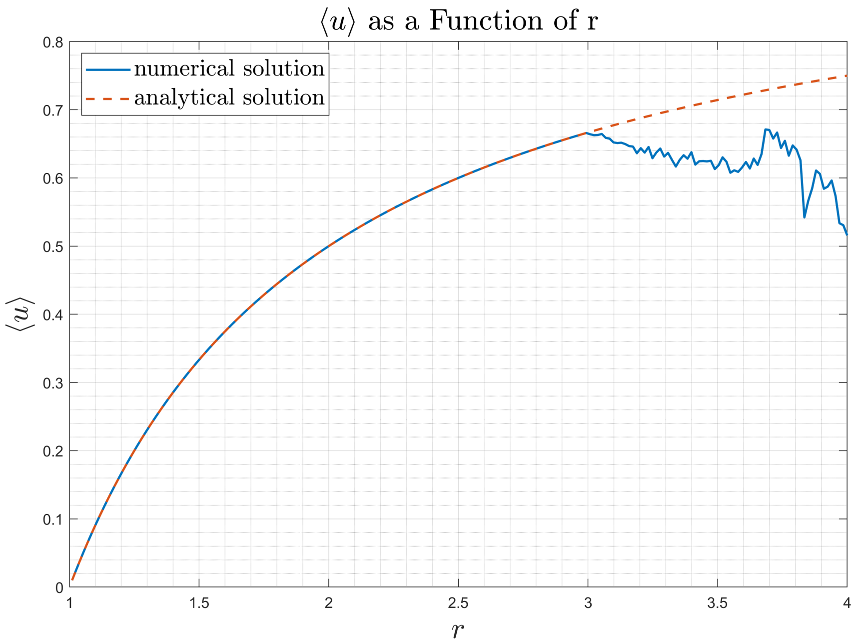

Figure 1: Calculating PDF

The figure shows that indeed the analytically calculated solution feet the numerical one for all $1 < r < 3$. This means that the ensemble average can be consider as normally distributed for $r < 3$.

3 Scalar TKE Equation

Considering the instantaneous transport equation for a passive scalar $\tilde{T}$:

$$\frac{\partial \tilde{T}}{\partial t} + \frac{\partial}{\partial x_j}\left(\tilde{T}\tilde{u}_j\right) = \alpha \frac{\partial^2 \tilde{T}}{\partial x_j^2} \tag{23}$$

Where:

- $\tilde{T} = \bar{T} + T$
- $\tilde{u} = \bar{u} + u$
- $\tilde{(\cdot)}$ - instantaneous quantity
- $\bar{(\cdot)}$ - ensemble average
- $(\cdot)$ - fluctuating component representing the turbulence

In order to describe the magnitude of the fluctuations we will derive an equivalent to the TKE for the temperature fluctuations, $\frac{1}{2}\bar{T}^2$:

$$\begin{aligned}
\frac{\partial \bar{T} + T}{\partial t} + \frac{\partial}{\partial x_j}\left(\left(\bar{T} + T\right)\left(\bar{u}_j + u_j\right)\right) &= \alpha \frac{\partial^2}{\partial x_j^2}\left(\bar{T} + T\right) \\
\frac{\partial \bar{T}}{\partial t} + \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{T}\bar{u}_j + \bar{T}u_j + T\bar{u}_j + Tu_j\right) &= \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2} + \alpha \frac{\partial^2 T}{\partial x_j^2}
\end{aligned} \tag{24}$$

Let's ensemble average:

$$\begin{gathered}
\overline{\frac{\partial \bar{T}}{\partial t} + \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{T}\bar{u}_j + \bar{T}u_j + T\bar{u}_j + Tu_j\right) = \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2} + \alpha \frac{\partial^2 T}{\partial x_j^2}} \\
\overline{\frac{\partial \bar{T}}{\partial t}} + \overline{\frac{\partial T}{\partial t}} + \overline{\frac{\partial}{\partial x_j}\left(\bar{T}\bar{u}_j + \bar{T}u_j + T\bar{u}_j + Tu_j\right)} = \alpha \overline{\frac{\partial^2 \bar{T}}{\partial x_j^2}} + \alpha \overline{\frac{\partial^2 T}{\partial x_j^2}} \\
\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\bar{T}\bar{u}_j + \bar{T}u_j + T\bar{u}_j + Tu_j}\right) = \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2} \\
\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{T}\bar{u}_j + \overline{Tu_j}\right) = \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2} \\
\frac{\partial \bar{T}}{\partial t} + \frac{\partial \bar{T}\bar{u}_j}{\partial x_j} + \frac{\partial \overline{Tu_j}}{\partial x_j} = \alpha \frac{\partial^2 \bar{T}}{\partial x_j^2}
\end{gathered} \tag{25}$$

By subtracting Eq.25 from Eq.24 we get:

$$\begin{gathered}
\cancel{\frac{\partial \bar{T}}{\partial t}} + \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}\left(\cancel{\bar{T}\bar{u}_j} + \bar{T}u_j + T\bar{u}_j + Tu_j\right) - \cancel{\frac{\partial \bar{T}}{\partial t}} - \cancel{\frac{\partial \bar{T}\bar{u}_j}{\partial x_j}} - \frac{\partial \overline{Tu_j}}{\partial x_j} = \alpha \cancel{\frac{\partial^2 \bar{T}}{\partial x_j^2}} + \alpha \frac{\partial^2 T}{\partial x_j^2} - \alpha \cancel{\frac{\partial^2 \bar{T}}{\partial x_j^2}} \\
\frac{\partial T}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{T}u_j + T\bar{u}_j + Tu_j\right) - \frac{\partial \overline{Tu_j}}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j^2}
\end{gathered} \tag{26}$$

By assuming incompressible flow and applying the ensemble average and substraction from the overall continuity equation, $\dfrac{\partial \tilde{u}}{\partial x_j} = 0$ yields:

$$\frac{\partial}{\partial x_j}\bar{u}_j = 0 \quad \text{and} \quad \frac{\partial}{\partial x_j}u_j = 0 \quad \text{and} \quad \frac{\partial \overline{T u_j}}{\partial x_j} = 0 \tag{27}$$

Hence we can rewrite Eq.26 as:

$$\frac{\partial T}{\partial t} + \frac{\partial \bar{T} u_j}{\partial x_j} + \frac{\partial T \bar{u}_j}{\partial x_j} + \frac{\partial T u_j}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j^2}$$

$$\frac{\partial T}{\partial t} + \bar{T}\cancel{\frac{\partial u_j}{\partial x_j}} + u_j \frac{\partial \bar{T}}{\partial x_j} + T\cancel{\frac{\partial \bar{u}_j}{\partial x_j}} + \bar{u}_j \frac{\partial T}{\partial x_j} + T\cancel{\frac{\partial u_j}{\partial x_j}} + u_j \frac{\partial T}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j^2} \tag{28}$$

$$\frac{\partial T}{\partial t} + u_j \frac{\partial \bar{T}}{\partial x_j} + \bar{u}_j \frac{\partial T}{\partial x_j} + u_j \frac{\partial T}{\partial x_j} = \alpha \frac{\partial^2 T}{\partial x_j^2}$$

By multiplying by T we get:

$$T\frac{\partial T}{\partial t} + T u_j \frac{\partial \bar{T}}{\partial x_j} + T\bar{u}_j \frac{\partial T}{\partial x_j} + T u_j \frac{\partial T}{\partial x_j} = T\alpha \frac{\partial^2 T}{\partial x_j^2} \tag{29}$$

and ensemble averaging the equation:

$$\overline{\frac{1}{2}\frac{\partial T^2}{\partial t} + T u_j \frac{\partial \bar{T}}{\partial x_j} + T\bar{u}_j \frac{\partial T}{\partial x_j} + T u_j \frac{\partial T}{\partial x_j} = T\alpha \frac{\partial^2 T}{\partial x_j^2}}$$

$$\frac{1}{2}\frac{\partial \overline{T^2}}{\partial t} + \overline{T u_j} \frac{\partial \bar{T}}{\partial x_j} + \frac{1}{2}\bar{u}_j \frac{\partial \overline{T^2}}{\partial x_j} + \overline{\frac{1}{2} u_j \frac{\partial T^2}{\partial x_j}} = \overline{T\alpha \frac{\partial^2 T}{\partial x_j^2}} \tag{30}$$

$$\frac{1}{2}\frac{D\overline{T^2}}{Dt} + \overline{T u_j} \frac{\partial \bar{T}}{\partial x_j} + \overline{\frac{1}{2} u_j \frac{\partial T^2}{\partial x_j}} = \overline{T\alpha \frac{\partial^2 T}{\partial x_j^2}}$$

Notice that:

$$\overline{\frac{\partial}{\partial x_j}\left(T\frac{\partial T}{\partial x_j}\right)} = \overline{\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_j} + T\frac{\partial^2 T}{\partial x_j^2}} = \overline{\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_j}} + \overline{T\frac{\partial^2 T}{\partial x_j^2}} \tag{31}$$

$$\frac{1}{2}\frac{D\overline{T^2}}{Dt} + \overline{T u_j} \frac{\partial \bar{T}}{\partial x_j} + \overline{\frac{1}{2} u_j \frac{\partial T^2}{\partial x_j}} = \alpha \overline{\frac{\partial}{\partial x_j}\left(T\frac{\partial T}{\partial x_j}\right)} - \alpha \underbrace{\overline{\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_j}}}_{\hat{\varepsilon} \sim \varepsilon} \tag{32}$$

This term is considered to be the temperature pseudo-dissipation as seen in the lecture (Pg.45) and from here on will be considered as the dissipation. Hence we can declare the rate at which these temperature fluctuations dissipate as:

$$\boxed{\varepsilon \propto \alpha \overline{\frac{\partial T}{\partial x_j}\frac{\partial T}{\partial x_j}}} \tag{33}$$

A Listing of The Computer Program

```
clc; clear; close all;

%% NUMERICAL SOLUTION =====================================================

n_max = 1e4;
N = 1e3;
r_min = 1+1e-2;
r_max = 4;
num_of_r = 2e2;
rs = linspace(r_min, r_max, num_of_r);
a = 0;
b = 1;

res.r = [];
res.mu_numerical = [];
res.mu_anal = [];
for r_index = 1:length(rs)
    r = rs(r_index);
    presenteg = r_index / length(rs) * 100;
    fprintf('r: %0.4f | n_max: %d | N: %d | done: %d%%\n', r, n_max, N, floor(presenteg))
    % numerical solution
    sum = 0;
    for realization = 1:N
        % get final u for specific r and unsamble
        u = [];
        u(1) = rand() * (b - a) + a;
        for i = 1:N
            u(i+1) = r * u(i) * (1 - u(i));
        end
        sum = sum + u(end);
    end
    mu_numerical = sum / N;
    res.r(end+1) = r;
    res.mu_numerical(end+1) = mu_numerical;

    syms mu sig_squ
    eq1 = mu^2 + sig_squ == r^2 / (1 - r^2) * (-2 * (mu^3 + 3 * mu * sig_squ) + mu^4 + 6 * mu^2 * sig_squ + 3 * sig_squ);
    eq2 = mu == r / (r - 1) * (mu^2 + sig_squ);

    anal_solution = solve([eq1, eq2], [mu, sig_squ]);
    real_mu = [];
    for i = 1:length(anal_solution.mu)
        if imag(anal_solution.mu(i)) ~= 0
            continue
        end
        real_mu(end+1) = double(anal_solution.mu(i));
    end
    % double(anal_solution.mu)
    % real_mu
    % mu_numerical
    res.mu_anal(end+1) = max(real_mu);
end

fig1 = figure('Name','1', 'Position', [0, 250, 900, 600]);
hold all
```

```
size = 20;

plot(res.r, res.mu_numerical,'LineStyle','-','LineWidth',1.5)
plot(res.r, res.mu_anal,'LineStyle','--','LineWidth',1.5)

title('$\langle u\rangle$ as a Function of r', 'FontSize', size,'Interpreter','latex')
ylabel('$\langle u\rangle$', 'FontSize', size,'Interpreter','latex')
xlabel('$r$','FontSize', size,'Interpreter', 'latex')
legend({'numerical solution', 'analytical solution'}, 'Location', 'northwest', 'FontSize', size-4, 'Interpreter','latex')
grid on
grid minor
box on
% exportgraphics(fig1, 'images/Q2.4.png','Resolution',400);
```

Listing 1: Code for Q1