Helicopter Dynamics and Aerodynamics

Bell UH-1N Twin Huey (Bell 212)

Hw3

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Theoretical Background

Blade Kinematics

A point on the blade has the position vector:

$$\vec{r} = R\left(e\vec{e}_1^h + l\vec{e}_1^b\right) \tag{1}$$

where e is the offset, and $l \in (0,1-e)$. Its velocity and acceleration are:

$$\vec{v} = \frac{d\vec{r}}{dt} = R\left(e\frac{d\vec{e}_1^h}{dt} + l\frac{d\vec{e}_1^b}{dt}\right) \tag{2}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = R\left(e\frac{d^2\vec{e}_1^h}{dt^2} + l\frac{d^2\vec{e}_1^b}{dt^2}\right) \tag{3}$$

The linear and angular momentum of the blade relative to the hub are:

$$\vec{P} = R\int_0^{1-e}\bar{m}\vec{v}dl = R^2\int_0^{1-e}\bar{m}\left(e\frac{d\vec{e}_1^h}{dt} + l\frac{d\vec{e}_1^b}{dt}\right)dl = m_0Re\frac{d\vec{e}_1^h}{dt} + m_1\frac{d\vec{e}_1^b}{dt} \tag{4}$$

$$\begin{aligned}\vec{H} &= R\int_0^{1-e}\bar{m}\vec{r}\times\vec{v}dl = R^2\int_0^{1-e}\bar{m}\left(e\vec{e}_1^h + l\vec{e}_1^b\right)\times\left(e\frac{d\vec{e}_1^h}{dt} + l\frac{d\vec{e}_1^b}{dt}\right)dl = \\ &= m_0R^2e^2\vec{e}_1^h\times\frac{d\vec{e}_1^h}{dt} + m_1eR\left(\vec{e}_1^h\times\frac{d\vec{e}_1^b}{dt} + \vec{e}_1^b\times\frac{d\vec{e}_1^h}{dt}\right) + m_2\vec{e}_1^b\times\frac{d\vec{e}_1^b}{dt}\end{aligned} \tag{5}$$

where, having $\bar{m}$ as the mass of the blade per unit of its length and

$$m_n = R^{n+1}\int_0^{1-e}\bar{m}l^n dl \tag{6}$$

is its n^{th} moment. The force acting on the blade:

$$\vec{F} = R\int_0^{1-e}\vec{f}dl = R\int_0^{1-e}\bar{m}\vec{a}dl = R\int_0^{1-e}\bar{m}\frac{d\vec{v}}{dt}dl = R\frac{d}{dt}\int_0^{1-e}\bar{m}\vec{v}dl = \frac{d\vec{P}}{dt} \tag{7}$$

The moment about the origin of the reference frame at the hub:

$$\begin{aligned}\vec{M}_0 &= R\int_0^{1-e}\vec{r}\times\vec{f}dl = R\int_0^{1-e}\bar{m}\vec{r}\times\vec{a}dl = R\int_0^{1-e}\bar{m}\vec{r}\times\frac{d\vec{v}}{dt}dl \\ &= R\int_0^{1-e}\bar{m}\left(\frac{d}{dt}(\vec{r}\times\vec{v}) - \left(\frac{d\vec{r}}{dt}\times\vec{v}\right)\right)dl \\ &= R\frac{d}{dt}\int_0^{1-e}\bar{m}\vec{r}\times\vec{v}dl = \frac{d\vec{H}}{dt}\end{aligned} \tag{8}$$

The moment about the hinge:

$$\vec{M} = \vec{M}_0 - eR\vec{e}_1^h\times\vec{F} \tag{9}$$

After a lot of algebra, we get:

$$F_1^h = \vec{F} \cdot \vec{e}_1^h = -(eRm_0 + m_1)\Omega^2 - 2m_1\Omega\dot{\zeta} + \cdots \tag{10}$$

$$F_2^h = \vec{F} \cdot \vec{e}_2^h = m_1(\ddot{\zeta} - \Omega^2\zeta) + \cdots \tag{11}$$

$$F_3^h = \vec{F} \cdot \vec{e}_3^h = m_1\ddot{\beta} - \cdots \tag{12}$$

$$M_\beta = -\vec{M} \cdot \vec{e}_2^b = m_2\ddot{\beta} + \Omega^2\beta(eRm_1 + m_2) + \cdots \tag{13}$$

$$M_\zeta = \vec{M} \cdot \vec{e}_3^h = m_2\ddot{\zeta} + eRm_1\Omega^2\zeta - 2m_2\Omega\beta\dot{\beta} + \cdots \tag{14}$$

When pitch and roll rates are added, M_β becomes:

$$M_\beta = m_2\ddot{\beta} + (eRm_1 + m_2)\left(\Omega^2\beta + \left(2q\Omega - \dot{P}\right)\sin(\psi) - (2P\Omega + \dot{q})\cos(\psi)\right) + \cdots \tag{15}$$

Angular accelerations $\dot{P}$ *and* $\dot{q}$ are neglected hereafter.

Blade Dynamics in Forward Flight

Instantaneous aerodynamic forces acting on a blade in axial flight

The apparent velocity experienced by the blade:

$$\bar{v}_a = r + \mu\sin(\psi) \tag{16}$$

The local lift coefficient of a blade is:

$$C_l = a\alpha_{eff} \tag{17}$$

and its local angle of attack is:

$$\alpha_{eff} = \theta - \theta_1 r - \frac{\lambda_s}{\bar{v}_a} + \frac{Pr}{\Omega\bar{v}_a}\sin(\psi) + \frac{qr}{\Omega\bar{v}_a}\cos(\psi) - \frac{\dot{\beta}r}{\Omega\bar{v}_a} - \frac{\beta\mu\cos(\psi)}{\bar{v}_a} \tag{18}$$

λ_s is the axial velocity in the shaft axes:

$$\lambda_s = \lambda + \lambda_i = \lambda_c - \mu\tau \tag{19}$$

$$\lambda_i = \lambda_{i,nom}\sum_{n=0}^{N}\bar{\lambda}_{i,n}(r)\cos(n\psi) = \lambda_{i,nom}\sum_{n=0}^{N}\sum_{m=0}^{M}\bar{\lambda}_{nm}r^m\cos(n\psi), \qquad \lambda_{i,nom} = \frac{C_T}{2\mu} \tag{20}$$

The aerodynamic flapping moment acting on the blade,

$$M_\beta^\alpha = \frac{1}{2}\rho\Omega^2R^4c\int_0^1 C_l\bar{v}_a^2 r\,dr \tag{21}$$

Takes on the form:

$$\begin{aligned} M_\beta^\alpha = \frac{1}{2}\rho\Omega^2R^4ca\Big(&\bar{M}_0 + \bar{M}_{1s}\sin(\psi) + \bar{M}_{1c}\cos(\psi) + \bar{M}_{2c}\cos(2\psi) \\ &-\frac{1}{4}\left(1 + \frac{4}{3}\mu\sin(\psi)\right)\frac{\dot{\beta}}{\Omega} - \frac{1}{3}\mu\left(\cos(\psi) + \frac{3}{4}\mu\sin(2\psi)\right)\beta + \cdots\Big) \end{aligned} \tag{22}$$

where:

$$\bar{M}_0 = \frac{1}{4}\theta_0(1+\mu^2) + \frac{1}{3}\theta_{1s}\mu - \frac{1}{5}\theta_1\left(1+\frac{5}{6}\mu^2\right) - \frac{1}{3}(\lambda + \lambda_{i,nom}\bar{\Lambda}_0^{(3)} + \frac{1}{6}\mu\frac{P}{\Omega} \quad (23)$$

$$\bar{M}_{1s} = \frac{1}{4}\theta_{1s}\left(1+\frac{3}{2}\mu^2\right) + \frac{2}{3}\theta_0\mu - \frac{1}{2}\theta_1\mu - \mu\frac{1}{2}\left(\lambda + \lambda_{i,nom}\left(\bar{\Lambda}_0^{(2)} - \frac{1}{2}\bar{\Lambda}_2^{(2)}\right)\right) + \frac{1}{4}\frac{p}{\Omega} \quad (24)$$

$$\bar{M}_{1c} = \frac{1}{4}\theta_{1c}\left(1+\frac{1}{2}\mu^2\right) - \frac{1}{3}\lambda_{i,nom}\mu\bar{\Lambda}_1^{(3)} + \frac{1}{4}\frac{q}{\Omega} \quad (25)$$

With any nonnegative integer n and positive integer k, $\bar{\Lambda}_n^{(k)}$ stands for the sum:

$$\bar{\Lambda}_n^{(k)} = \sum_{m=0}^{M}\frac{k\bar{\lambda}_{nm}}{k+m}$$

$$\bar{\lambda}_{nm} = \begin{bmatrix} 0.09 & 0 & 1.64 & 6.64 & -7.58 \\ 0 & -2.70 & 0 & 6.28 & 0 \\ 0 & 0 & -1.20 & -9.09 & 10.8 \\ 0 & 0 & 0 & -2.15 & 0 \end{bmatrix} \quad (26)$$

Flapping

The flapping motion for a blade is governed by:

$$M_\beta = M_\beta^s + M_\beta^\alpha \quad (27)$$

$$\ddot{\beta} + \frac{1}{8}\Omega L\dot{\beta}\left(1+\frac{4}{3}\mu\sin(\psi)\right) + \Omega^2\beta\left(1 + E_{eff} + \frac{L}{6}\left(\mu\cos(\psi) + \frac{3}{4}\mu^2\sin(2\psi)\right)\right) = \frac{1}{2}\Omega^2 L(\bar{M}_0 + \bar{M}'_{1s}\sin(\psi) + \bar{M}'_{1c}\cos(\psi) + \cdots) \quad (28)$$

where:

$$\bar{M}'_{1s} = \bar{M}_{1s} - \frac{4}{L}(1+E)\frac{q}{\Omega} \quad (29)$$

$$\bar{M}'_{1c} = \bar{M}_{1c} + \frac{4}{L}(1+E)\frac{P}{\Omega} \quad (30)$$

$$E = \frac{Rm_1}{m_2}e \approx \frac{3}{2}e \quad (31)$$

$$E_{eff} = \frac{Rm_1}{m_2}e + \frac{K_\beta}{m_2\Omega^2} = \frac{Rm_1}{m_2}e_{eff} \approx \frac{3}{2}e_{eff} \quad (32)$$

By substituting $\beta = \beta_0 + \beta_{1s}\sin(\psi) + \beta_{1c}\cos(\psi) + \cdots$, the equation becomes:

$$\begin{aligned} &\beta_0(1+E_{eff}) + \left(\beta_{1s}E_{eff} - \frac{L}{8}\beta_{1c}\left(1-\frac{1}{2}\mu^2\right)\right)\sin(\psi) \\ &+ \left(\beta_{1c}E_{eff} + \beta_0\frac{L}{6}\mu + \frac{L}{8}\beta_{1s}\left(1+\frac{1}{2}\mu^2\right)\right)\cos(\psi) + \cdots = \\ &= \frac{L}{2}(\bar{M}_0 + \bar{M}'_{1s}\sin(\psi) + \bar{M}'_{1c}\cos(\psi) + \cdots) \end{aligned} \quad (33)$$

Matching the factors with the trigonometric functions leads to the system of algebraic equations for the flapping coefficients:

$$\begin{pmatrix} 1+E_{eff} & 0 & 0 \\ 0 & E_{eff} & -\frac{L}{8}\left(1-\frac{\mu^2}{2}\right) \\ \frac{L\mu}{6} & \frac{L}{8}\left(1+\frac{\mu^2}{2}\right) & E_{eff} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_{1s} \\ \beta_{1c} \end{pmatrix} = \frac{L}{2} \begin{pmatrix} \bar{M}_0 \\ \bar{M}'_{1s} \\ \bar{M}'_{1c} \end{pmatrix} \tag{34}$$

Since $E, E_{eff} \approx 10^{-3}$, it is reasonable to assume $E, E_{eff} \to 0$. It yields:

$$\beta_0 = \frac{L}{2}\bar{M}_0 \tag{35}$$

$$\beta_{1s} = \frac{4\bar{M}'_{1c} - \left(\frac{2}{3}\mu L\right)\bar{M}_0}{1+\frac{1}{2}\mu^2} = \frac{4}{1+\frac{1}{2}\mu^2}\left(\bar{M}'_{1c} - \frac{\mu\beta_0}{3}\right) \tag{36}$$

$$\beta_{1c} = -\frac{4\bar{M}'_{1s}}{1-\frac{1}{2}\mu^2} \tag{37}$$

After substitution:

$$\beta_0 = \frac{L}{8}\left(\theta_0(1+\mu^2) - \frac{4}{5}\theta_1\left(1+\frac{5}{6}\mu^2\right) + \frac{4}{3}\theta_{1s}\mu - \frac{4}{3}\left(\lambda + \lambda_{i,nom}\bar{\Lambda}_0^{(3)}\right) + \frac{\frac{2}{3}\mu P}{\Omega}\right) \tag{38}$$

$$\beta_{1s} = \theta_{1c} + \frac{1}{1+\frac{1}{2}\mu^2}\left(-\frac{4}{3}\left(\lambda_{i,nom}\bar{\Lambda}_1^{(3)} + \mu\beta_0\right) + \left(\frac{q}{\Omega} + \frac{16}{L}\frac{P}{\Omega}\right)\right) \tag{39}$$

$$\begin{aligned} \beta_{1c} = -\theta_{1s}\frac{1+\frac{3}{2}\mu^2}{1-\frac{1}{2}\mu^2} - \frac{\mu}{1-\frac{1}{2}\mu^2}\frac{8}{3}\left(\theta_{0.75} - \frac{3}{4}\left(\lambda + \lambda_{i,nom}\left(\bar{\Lambda}_0^{(2)} - \frac{1}{2}\bar{\Lambda}_2^{(2)}\right)\right)\right) \\ - \frac{1}{1-\frac{1}{2}\mu^2}\left(\frac{P}{\Omega} - \frac{16}{L}\frac{q}{\Omega}\right) \end{aligned} \tag{40}$$

Using the disk axes:

$$\lambda_d = \lambda_s + \mu\beta_{1c} = \lambda_\perp + \lambda_i = \lambda_c + \mu\beta_{1c} + \lambda_i \tag{41}$$

where:

- λ_d is the flow velocity normal to the rotor disk (imaginary shaft)
- λ_s is the velocity along the shaft (real shaft)
- $\lambda_\perp$ is the normal to the disk plane component of the unperturbed flow.
- Along a straight path, $\lambda_\perp = \lambda_c + \mu(\beta_{1c} - \tau)$

Changing the inflow, leaves equation for β_{1s} unaltered

$$\beta_{1s} = \theta_{1c} + \frac{1}{1+\frac{1}{2}\mu^2}\left(-\frac{4}{3}\left(\lambda_{i,nom}\bar{\Lambda}_1^{(3)} + \mu\beta_0\right) + \left(\frac{q}{\Omega} + \frac{16}{L}\frac{P}{\Omega}\right)\right) \tag{42}$$

The equation for β_{1c} becomes:

$$\beta_{1c} = -\theta_{1s} - \frac{\mu}{1+\frac{3}{2}\mu^2}\frac{8}{3}\left(\theta_{0.75} - \frac{3}{4}\left(\lambda_\perp + \lambda_{i,nom}\left(\overline{\Lambda}_0^{(2)} - \frac{1}{2}\overline{\Lambda}_2^{(2)}\right)\right)\right) - \frac{1}{1+\frac{3}{2}\mu^2}\left(\frac{P}{\Omega} - \frac{16}{L}\frac{q}{\Omega}\right) \tag{43}$$

And finally, the equation for β_0 becomes:

$$\beta_0 = \frac{L}{8}\left(\theta_{0.75}\frac{1-\frac{19}{18}\mu^2+\frac{3}{2}\mu^4}{1+\frac{3}{2}\mu^2} - \frac{4}{3}\left(\lambda_\perp + \lambda_{i,nom}\overline{\Lambda}_0^{(3)}\right)\frac{1-\frac{1}{2}\mu^2}{1+\frac{3}{2}\mu^2}\right) + \frac{L}{12}\mu\left(\frac{32}{L}\frac{q}{\Omega}\frac{1}{1+\frac{3}{2}\mu^2} - \frac{P}{\Omega}\frac{1-\frac{3}{2}\mu^2}{1+\frac{3}{2}\mu^2}\right) + \frac{L}{12}\left(\frac{4\mu^2}{1+\frac{3}{2}\mu^2}\lambda_{i,nom}\left(\overline{\Lambda}_0^{(2)} - \overline{\Lambda}_0^{(3)} - \frac{1}{2}\overline{\Lambda}_2^{(2)}\right) - \frac{3}{40}\theta_1\left(1-\frac{5}{3}\mu^2\right)\right) \tag{44}$$

The effective cyclic commands in non-flapping (disk) axes:

$$\theta'_{1s} = \theta_{1s} + \beta_{1c} = -\frac{\mu}{1+\frac{3}{2}\mu^2}\frac{8}{3}\left(\theta_{0.75} - \frac{3}{4}\left(\lambda_\perp + \lambda_{i,nom}\left(\overline{\Lambda}_0^{(2)} - \frac{1}{2}\overline{\Lambda}_2^{(2)}\right)\right)\right) - \frac{1}{1+\frac{3}{2}\mu^2}\left(\frac{P}{\Omega} - \frac{16}{L}\frac{q}{\Omega}\right) \tag{45}$$

$$\theta'_{1c} = \theta_{1c} - \beta_{1s} = -\frac{1}{1+\frac{1}{2}\mu^2}\left(-\frac{4}{3}\left(\lambda_{i,nom}\overline{\Lambda}_1^{(3)} + \mu\beta_0\right) + \left(\frac{q}{\Omega} + \frac{16}{L}\frac{P}{\Omega}\right)\right) \tag{46}$$

Period-Averaged Forces and Moments in Disk Axes

The complication of using non-uniform inflow cannot be justified. The expressions become incomprehensible. We will ignore variations of the inflow across the disk, which means:

$$\lambda_d = \lambda_\perp + \lambda_i \tag{47}$$

with $\lambda_i = \lambda_{i,nom}$, zero angular rates of the shaft, and ignorance of the conicity-associated cyclic angle of attack $-\frac{\beta_0 \mu \cos(\psi)}{\bar{v}_a}$.

In disk axes, the effective angle of attack of the blade is:

$$\alpha_{eff} = (\theta' - \theta_1 r) + \frac{1}{\bar{v}_a}(-\lambda_\perp - \lambda_i) \tag{48}$$

where:

$$\theta' = \theta_0 + \theta'_{1s}\sin(\psi) + \theta'_{1c}\cos(\psi) \tag{49}$$

The local lift coefficient of a blade is:

$$C_l = a\alpha_{eff} \tag{50}$$

The period-averaged thrust of the rotor in disk axes will be obtained by summing up contributions of all blades:

$$C_T = \frac{NcR}{2A_d}\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^1 C_l \bar{v}_a^2 dr = \frac{\sigma}{2}\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^1 C_l \bar{v}_a^2 dr =$$
$$= \frac{\sigma a}{2}\int_0^1 \left(\theta_0\left(r^2 + \frac{1}{2}\mu^2\right) - \theta_1\left(r^3 + \frac{1}{2}\mu^2 r\right) + \theta'_{1s}\mu r - (\lambda_\perp + \lambda_i)r + \frac{1}{2}\frac{P}{\Omega}\mu r\right) dr =$$
$$= \frac{\sigma a}{2}\left(\frac{1}{3}\theta_0\left(1 + \frac{3}{2}\mu^2\right) - \frac{1}{4}\theta_1(1 + \mu^2) + \theta'_{1s}\mu\frac{1}{2} - \frac{1}{2}\lambda_d + \frac{1}{4}\mu\frac{P}{\Omega}\right) = \quad (51)$$
$$= \frac{\sigma a}{2}\left(\frac{1}{3}\theta_{0.75}\left(1 + \frac{3}{2}\mu^2\right) - \frac{1}{8}\theta_1\mu^2 + \theta'_{1s}\mu\frac{1}{2} - \frac{1}{2}\lambda_d + \frac{1}{4}\mu\frac{P}{\Omega}\right)$$

The in-plane drag force acting on the entire rotor, C_H:

$$C_H = \frac{\sigma}{2}\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^1 \bar{v}_a^2\left(\frac{\lambda_d}{\bar{v}_a}C_l + C_{d0}\right)\sin(\psi)\, dr \quad (52)$$

where $C_l = a\alpha_{eff}$ but $\alpha_{eff} = (\theta' - \theta_1 r) - \frac{\lambda_d}{\bar{v}_a}$.

After substituting θ' we get:

$$C_H = \frac{\sigma a}{4}\int_0^1 (\theta'_{1s} r(\theta_0 - \theta_1 r)\mu)\lambda_d dr + \frac{\sigma C_{d0}}{2}\mu\int_0^1 r dr =$$
$$\frac{\sigma a}{4}\left(\theta'_{1s}\frac{1}{2} + \left(\theta_0 - \theta_1\frac{1}{2}\right)\mu\right)\lambda_d + \frac{\sigma C_{d0}}{4}\mu \approx \frac{\sigma C_{d0}}{4}\mu \quad (53)$$

Trim

Considering a helicopter flying along a straight path with constant speed, zero angular rates, small side velocity ϖ and climb rate λ_c. Assuming that the induced velocity is uniform across the rotor disk, we get:

Force balance at (-x) direction:

$$C_T\beta_{1c} - C_H - C_W\tau - \frac{1}{2}\bar{S}_b\mu^2 + \cdots = 0 \tag{54}$$

Force balance at (y) direction:

$$-C_T\beta_{1s} + C_{T,t} + C_W\phi - C_Y - \mathrm{sgn}(\varpi)\frac{1}{2}\bar{S}_D\varpi^2 + \cdots = 0 \tag{55}$$

Force balance at (z) direction:

$$C_T + \frac{1}{2}\mu^2 C_{L,ht}\bar{S}_{ht} - C_W + \cdots = 0 \tag{56}$$

Moment balance at (-x) direction:

$$C_T\frac{\Delta y}{R} - \left(C_T\frac{\Delta z}{R} + \bar{K}_\beta\right)\beta_{1s} + C_{T,t}\frac{\Delta z - h_t}{R} - C_Y\frac{\Delta z}{R} + \cdots = 0 \tag{57}$$

Moment balance at (y) direction:

$$C_T\frac{\Delta x}{R} - \left(C_T\frac{\Delta z}{R} + \bar{K}_\beta\right)\beta_{1c} + C_H\frac{\Delta z}{R} - \frac{1}{2}\mu^2 C_{L,ht}\bar{S}_{ht}\frac{l_{ht} - \Delta x}{R} + \cdots = 0 \tag{58}$$

Moment balance at (-z) direction:

$$C_Q - C_T\frac{\Delta x}{R}\beta_{1s} + C_T\frac{\Delta y}{R}\beta_{1c} - C_{T,t}\frac{l_t - \Delta x}{R} + \cdots = 0 \tag{59}$$

Where:

- τ and ϕ are the pitch and roll angles of the shaft axes
- The effective spring at the flapping axis $\bar{K}_\beta = \frac{N}{2}\frac{K_\beta}{\rho v_t^2 A_d R} = \frac{1}{2}\frac{K_\beta}{\Omega^2}\frac{\sigma a}{L m_2} = \frac{3\sigma a e_{eff}}{4L}$
- h_t is the location of the tail rotor axis below the main rotor
- $\bar{S}_{ht}$ is the area of the horizontal tail rotor reduced by the area of the rotor disk
- l_{ht} is the distance between the aerodynamic center of the horizontal tail and the main rotor shaft
- The horizontal tail lift coefficient $C_{L,ht} = \frac{1}{2}a_{ht}\bar{S}_{ht}\mu(\mu\tau - \lambda_c - \lambda_i + \mu i_0)$. This will be neglected
- Like C_H, $C_Y \approx \frac{\sigma C_{d0}}{4}\varpi$

$$\lambda_i = \begin{cases} \dfrac{C_T}{2\sqrt{\mu^2 + \varpi^2}} & \text{in forward flight} \\ -\dfrac{\lambda_c}{2} + \sqrt{\dfrac{\lambda_c^2}{4} + \dfrac{C_T}{2}} & \text{in axial flight} \end{cases} \tag{60}$$

Using an empirical interpolation:

$$\lambda_i = \frac{C_T}{2\sqrt{\sqrt{\mu^2+\varpi^2}^2 + k_i^2(\lambda_i+\lambda_c)^2}} \tag{61}$$

When λ_c is sufficiently small, it yields:

$$\lambda_i = \sqrt{\frac{1}{2k_i^2}\left(-\sqrt{\mu^2+\varpi^2}^2 + \sqrt{\sqrt{\mu^2+\varpi^2}^4 + k_i^2C_T^2}\right)} - \frac{-\sqrt{\mu^2+\varpi^2}^2 + \sqrt{\sqrt{\mu^2+\varpi^2}^4 + k_i^2C_T^2}}{2\sqrt{\sqrt{\mu^2+\varpi^2}^4 + k_i^2C_T^2}}\lambda_\perp + \cdots = \tag{62}$$

Notably, $\lambda_\perp = \lambda_c + \cdots$ at low forward speeds, whereas at high forward speeds, the entire term involving $\lambda_\perp$ vanishes. Then, λ_i becomes:

$$\lambda_i = \sqrt{\frac{1}{2k_i^2}\left(-(\mu^2+\varpi^2) + \sqrt{(\mu^2+\varpi^2)^2 + k_i^2C_T^2}\right)} - \frac{-(\mu^2+\varpi^2) + \sqrt{(\mu^2+\varpi^2)^2 + k_i^2C_T^2}}{2\sqrt{(\mu^2+\varpi^2)^2 + k_i^2C_T^2}}\lambda_c + \cdots \tag{63}$$

Solving The Trim Equations

From force balance at z direction, one can plausibly assume that:

$$C_T \approx C_W \tag{64}$$

Under this assumption:

$$C_Q = C_P = \frac{\sigma C_{d0}}{8}\big(1 + 3(\mu^2+\varpi^2)\big) + C_W(\lambda_i+\lambda_c) + \frac{1}{2}\bar{S}_b\mu^3 + \frac{1}{2}\bar{S}_D\varpi^3 \tag{65}$$

Whereas the moment balance at (y) direction furnishes its orientation relative to the body (β_{1c}):

$$\beta_{1c} = \frac{C_W\frac{\Delta x}{R} + C_H\frac{\Delta z}{R}}{\left(C_W\frac{\Delta z}{R} + \bar{K}_\beta\right)} \tag{66}$$

With that, the moment balance at (-z) direction becomes:

$$C_Q - C_W\frac{\Delta x}{R}\beta_{1s} + C_W\frac{\Delta y}{R}\beta_{1c} - C_{T,t}\frac{l_t - \Delta x}{R} + \cdots = 0 \tag{67}$$

Consequently, the force balance at (y) direction furnishes the sidewise tilt of the disk plane relative to Earth:

$$\beta_{1s} - \phi = \frac{C_{T,t} - C_Y - \mathrm{sgn}(\varpi)\frac{1}{2}\bar{S}_D\varpi^2}{C_W} \tag{68}$$

Whereas the moment balance at (-x) direction furnishes its orientation relative to the body (β_{1s}):

$$\left(C_W \frac{\Delta z}{R} + \bar{K}_\beta\right)\beta_{1s} = C_{T,t}\frac{\Delta z - h_t}{R} + C_W \frac{\Delta y}{R} - C_Y \frac{\Delta z}{R} \tag{69}$$

By writing both equations in matrix form we get the following system:

$$\begin{pmatrix} C_W \frac{\Delta x}{R} & \frac{l_t - \Delta x}{R} \\ C_W \frac{\Delta z}{R} + \bar{K}_\beta & -\frac{\Delta z - h_t}{R} \end{pmatrix} \begin{pmatrix} \beta_{1s} \\ C_{T,t} \end{pmatrix} = \begin{pmatrix} C_W \frac{\Delta y}{R}\beta_{1c} + C_Q \\ C_W \frac{\Delta y}{R} - C_Y \frac{\Delta z}{R} \end{pmatrix} \tag{70}$$

Using Cramer's rule, we compute $\beta_{1s}\ and\ C_{T,t}$.

The bank angle ϕ is therefore:

$$\phi = \beta_{1s} - \frac{C_{T,t} - C_Y - \mathrm{sgn}(\varpi)\frac{1}{2}\bar{S}_D \varpi^2}{C_W} \tag{71}$$

Force balance at (-x) direction furnishes the forward-backward tilt of the disk plane relative to Earth:

$$\beta_{1c} - \tau = \frac{\left(\frac{1}{2}\bar{S}_b \mu^2 + C_H\right)}{C_W} \tag{72}$$

The pitch angle τ is therefore:

$$\tau = \beta_{1c} - \frac{\left(\frac{1}{2}\bar{S}_b \mu^2 + C_H\right)}{C_W} = \frac{C_W \frac{\Delta x}{R} + C_H \frac{\Delta z}{R}}{\left(C_W \frac{\Delta z}{R} + \bar{K}_\beta\right)} - \frac{\left(\frac{1}{2}\bar{S}_b \mu^2 + C_H\right)}{C_W} \tag{73}$$

Solving For the Cyclic Inputs

Assuming that $\mu \ll 1\ (\mu^2 \to 0\ and\ \lambda_d = \lambda_c + \lambda_i)\ and\ P = q = 0$, the thrust coefficient of the main rotor:

$$\begin{gathered} C_T = \frac{\sigma a}{2}\left(\frac{1}{3}\theta_{0.75} + \frac{1}{2}(\theta'_{1s}\mu + \theta'_{1c}\varpi) - \frac{1}{2}(\lambda_c + \lambda_i + \mu\beta_{1c} - \varpi\beta_{1s})\right) = \\ = \frac{\sigma a}{6}\theta_{0.75} + \frac{\sigma a}{4}\mu(\theta_{1s} + \beta_{1c}) + \frac{\sigma a}{4}\varpi(\theta_{1c} - \beta_{1s}) - \frac{\sigma a}{4}(\lambda_c + \lambda_i + \mu\beta_{1c} - \varpi\beta_{1s}) \\ \Downarrow \\ \frac{4C_T}{\sigma a} = \frac{2}{3}\theta_{0.75} + \mu\theta_{1s} + \varpi\theta_{1c} - (\lambda_c + \lambda_i) \end{gathered} \tag{74}$$

The flapping angles:

$$\beta_{1s} = \theta_{1c} + \frac{8}{3}\left(\varpi - \mu\frac{L}{16}\right)\theta_{0.75} - 2\left(\varpi - \mu\frac{L}{9}\right)(\lambda_i + \lambda_c) \tag{75}$$

$$\beta_{1c} = -\theta_{1s} - \frac{8}{3}\left(\mu + \varpi\frac{L}{16}\right)\theta_{0.75} + 2\left(\mu + \varpi\frac{L}{9}\right)(\lambda_i + \lambda_c) \tag{76}$$

where for each direction, the velocity normal to that direction was added as in the other direction.

By writing all three equations in matrix form we get the following system:

$$\begin{pmatrix} \frac{2}{3} & \mu & \varpi \\ \frac{8}{3}\left(\varpi - \mu\frac{L}{16}\right) & 0 & 1 \\ -\frac{8}{3}\left(\mu + \varpi\frac{L}{16}\right) & -1 & 0 \end{pmatrix} \begin{pmatrix} \theta_{0.75} \\ \theta_{1s} \\ \theta_{1c} \end{pmatrix} = \begin{pmatrix} \frac{4C_T}{\sigma a} + (\lambda_c + \lambda_i) \\ \beta_{1s} + 2\left(\varpi - \mu\frac{L}{9}\right)(\lambda_i + \lambda_c) \\ \beta_{1c} - 2\left(\mu + \varpi\frac{L}{9}\right)(\lambda_i + \lambda_c) \end{pmatrix} \quad (77)$$

Using Cramer's rule, we compute $\theta_{0.75}$ *and* θ_{1s} *and* θ_{1c}.

Tail Rotor Angle

Crabbing to the left

The thrust coefficient of the tail:

$$C'_{T,t} = \frac{\sigma_t a_t}{2}\left(\frac{1}{3}\theta_{0.75,t} - \frac{1}{2}(\lambda'_{d,t})\right) \quad (78)$$

where:

- $C'_{T,t} = \frac{T_t}{\rho v_{t,t}^2 A_{d,t}} = \frac{T_t}{\rho v_t^2 A_d}\frac{v_t^2 R^2}{v_{t,t}^2 R_t^2} = C_{T,t}\frac{v_t^2 R^2}{v_{t,t}^2 R_t^2}$
- $\lambda'_{d,t} = \frac{v_d}{v_{t,t}} = \frac{v_d}{v_t}\frac{v_t}{v_{t,t}} = \lambda_{d,t}\frac{v_t}{v_{t,t}}$

$$\lambda_{d,t} = \begin{cases} \frac{\varpi}{2} + \sqrt{\frac{\varpi^2}{4} + \lambda_h^2} & |\overline{\omega}| < \lambda_h \\ \varpi & |\varpi| > 2\lambda_h \end{cases} \quad (79)$$

$$\lambda_h = \sqrt{\frac{C_{T,t}}{2}} \quad (80)$$

The angle of the tail rotor is therefore:

$$\theta_{0.75,t} = 3 \cdot \left(\frac{2C'_{T,t}}{\sigma_t a_t} + \frac{1}{2}\lambda'_{d,t}\right) \quad (81)$$

Crabbing to the right

The thrust coefficient of the tail:

$$C'_{T,t} = \frac{\sigma_t a_t}{2}\left(\frac{1}{3}\theta_{0.75,t} - \frac{1}{2}(\varpi' + \lambda'_{i,t})\right) \quad (82)$$

where:

- $C'_{T,t} = \frac{T_t}{\rho v_{t,t}^2 A_{d,t}} = \frac{T_t}{\rho v_t^2 A_d}\frac{v_t^2 R^2}{v_{t,t}^2 R_t^2} = C_{T,t}\frac{v_t^2 R^2}{v_{t,t}^2 R_t^2}$
- $\varpi' = \frac{v_y}{v_{t,t}} = \frac{v_y}{v_t}\frac{v_t}{v_{t,t}} = \varpi\frac{v_t}{v_{t,t}}$
- $\lambda'_c = \frac{v_c}{v_{t,t}} = \frac{v_c}{v_t}\frac{v_t}{v_{t,t}} = \lambda_c\frac{v_t}{v_{t,t}}$
- $\mu' = \frac{v}{v_{t,t}} = \frac{v}{v_t}\frac{v_t}{v_{t,t}} = \mu\frac{v_t}{v_{t,t}}$

and since ϖ and λ_c are the "normal" to the plane and in-plane velocities respectively,

$$\lambda'_{i,t} = \begin{cases} \dfrac{C'_{T,t}}{2\sqrt{{\mu'}^2 + {\lambda'_c}^2}} & \text{in forward flight} \\ -\dfrac{\varpi'}{2} + \sqrt{\dfrac{{\varpi'}^2}{4} + \dfrac{C'_{T,t}}{2}} & \text{in axial flight} \end{cases}$$

Using an empirical interpolation:

$$\lambda'_{i,t} = \frac{C'_{T,t}}{2\sqrt{\overline{\sqrt{{\mu'}^2 + {\lambda'_c}^2}}^2 + k_{i,t}^2\left(\lambda'_{i,t} + \varpi\right)^2}} \tag{83}$$

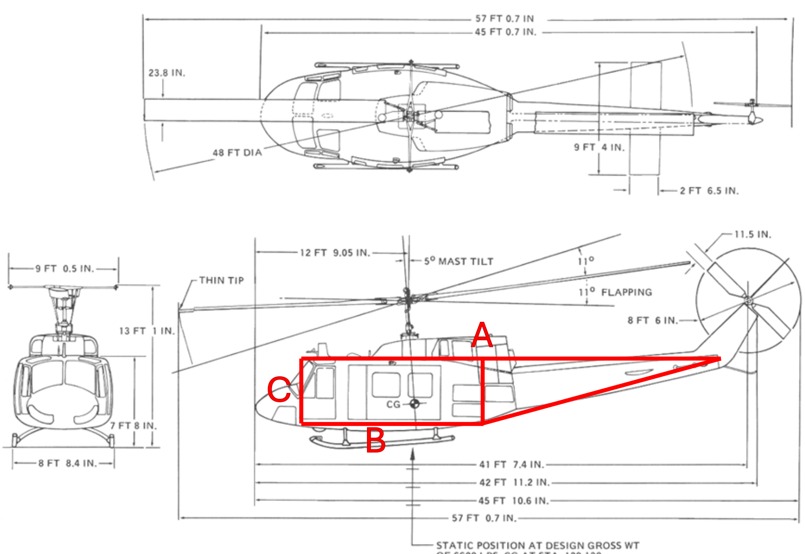
When λ_c is sufficiently small, it yields:

$$\lambda'_{i,t} = \sqrt{\frac{1}{2k_{i,t}^2}\left(-({\mu'}^2 + {\lambda'_c}^2) + \sqrt{\left({\mu'}^2 + {\lambda'_c}^2\right)^2 + \left(k_{i,t}C'_{T,t}\right)^2}\right)} - \frac{-({\mu'}^2 + {\lambda'_c}^2) + \sqrt{\left({\mu'}^2 + {\lambda'_c}^2\right)^2 + \left(k_{i,t}C'_{T,t}\right)^2}}{2\sqrt{\left({\mu'}^2 + {\lambda'_c}^2\right)^2 + \left(k_{i,t}C'_{T,t}\right)^2}}(\varpi' + \mu'(\beta_{1s} - \phi)) + \cdots \tag{84}$$

So, the equation for the tail rotor angle is:

$$\theta_{0.75,t} = 3 \cdot \left(\frac{2C'_{T,t}}{\sigma_t a_t} + \frac{1}{2}(\varpi' + \lambda'_{i,t})\right) \tag{85}$$

Calculating $\bar{S}_D$



$$
\begin{aligned}
&A = 41'7.4'' = 12.68476[m] \\
&B = 12'9.05'' \cdot 1.5 = 3.88747 \cdot 1.5[m] = 5.831205[m] \\
&C = 7'8'' = 2.3368[m]
\end{aligned}
\tag{86}
$$

$$\Downarrow$$

$$S_D = \frac{C}{2}(A+B) = \frac{2.3368}{2}(12.68476 + 5.831205) = 21.63[m^2] \tag{87}$$

$$\Downarrow$$

$$\bar{S}_D = \frac{S_D}{A_d} \approx \frac{21.63}{168} = 0.1285 \tag{88}$$

Addendum

A few additional parameters are noteworthy. By assuming standard sea-level atmospheric conditions and that the weight is half-way between the maximal and empty weights.

$$
\begin{gathered}
W = \frac{W_{max} + W_{mim}}{2} \\
\Downarrow \\
C_W = \frac{W}{\rho v_t^2 A_d}, \qquad \rho = 1.225\left[\frac{kg}{m^3}\right]
\end{gathered}
\tag{89}
$$

The effective offset (e_{eff}) is assumed to be 0.04.

The Lock number (L) is assumed to be 7.

The lift-slope coefficient of the tail and main rotor (a, a_t) are assumed to be 2π.

Note: we created one MATLAB function that takes in count all the trim conditions. Therefore, in each of the following questions we will state the given conditions and the resulting trim angles.

Remark: in all the questions $\lambda_c = 0$, hence in the implementation we neglected it.

Q1

It is given that CG is exactly below the mast, which mean $\Delta x, \Delta y = 0$. At hover $\mu = \lambda_c = \overline{\omega} = 0$.

The arguments we gave the function:

$$(4.037 \cdot 10^3 [kg], 0,0,0,0,2[m])$$

The angle	Value $[deg]$
τ	0
ϕ	-2.1767
β_{1c}	0
β_{1s}	1.3402
θ_{1c}	1.3402
θ_{1s}	0
$\theta_{0.75}$	7.1112
$\theta_{0.75,t}$	10.1372

Q2

At hover $\mu = \lambda_c = \varpi = 0$. It is given that CG is 4 inches on the left, which mean $\Delta x = 0$ and $\Delta y = -4'' = -4 \cdot 0.0254\ [m] = -0.1016\ [m]$

The arguments we gave the function:

$$(4.037 \cdot 10^3 [kg], 0,0,0, -0.1016[m], 2[m])$$

The angle	Value $[deg]$ Q2	Value $[deg]$ Q1
τ	0	0
ϕ	-3.2859	-2.1767
β_{1c}	0	0
β_{1s}	0.2310	1.3402
θ_{1c}	0.2310	1.3402
θ_{1s}	0	0
$\theta_{0.75}$	7.1112	7.1112
$\theta_{0.75,t}$	10.1372	10.1372

- The angle sidewise of the disk with respect to the horizon ($\beta - \phi = 3.5169[deg]$) doesn't change, as accepted because the disk orients itself in order to maintain force balance and since the forces don't change the angles of the disk doesn't change.
- The body tilts more to the right in order to cancel out the fact that gravity works at a different point.

Q3

It is given that CG is exactly below the mast, which mean $\Delta x, \Delta y = 0$. The helicopter is crabbing to the right at $40[knots]$, which mean $\bar{\omega} = 0.0829$.

The arguments we gave the function:

$$(4.037 \cdot 10^3[kg], 0,0.0829,0,0,2[m])$$

The angle	Value $[deg]$ Q3	Value $[deg]$ Q1
τ	0	0
ϕ	6.5067	-2.1767
β_{1c}	0	0
β_{1s}	0.9918	1.3402
θ_{1c}	0.0850	1.3402
θ_{1s}	-0.3370	0
$\theta_{0.75}$	4.8990	7.1112
$\theta_{0.75,t}$	12.8346	10.1372

- We can see that the pitch angle of the body relative to the horizon didn't change, just like we accepted because the wind is from right to left which doesn't affect the front-back direction.
- The body tilts now to the left, which is responsible since the wind is from the right.
- The disk tilts to the right ($\beta - \phi = -5.5149[deg]$) to cancel out C_Y and the cyclic command changes in order to maintain this angle.

Q4

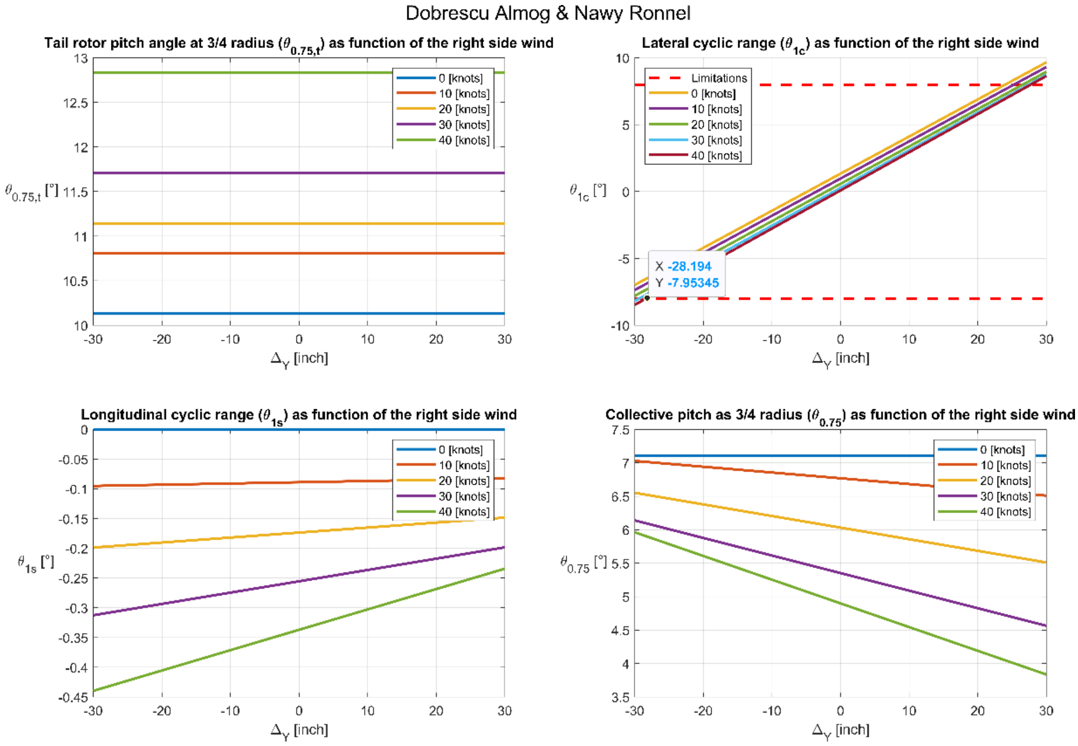


Figure 2: plot of $\theta_{0.75,t}, \theta_{1c}, \theta_{1s}, \theta_{0.75}$ as a function of Δy – Q4

We can see that the limiting angle is θ_{1c} and the maximal value for Δy (with 10% margin) is $25.56[inch]$ to the left. Moreover all the other angles didn't reach their limitation.

Q5

Redoing Q2

The angle	Value [deg]	Value $[deg]$ Q1
τ	0	0
ϕ	-1.0676	-2.1767
β_{1c}	0	0
β_{1s}	2.4493	1.3402
θ_{1c}	2.4493	1.3402
θ_{1s}	0	0
$\theta_{0.75}$	7.1112	7.1112
$\theta_{0.75,t}$	10.1372	10.1372

- The angle sidewise of the disk with respect to the horizon ($\beta - \phi = 3.5169[deg]$) doesn't change, as accepted because the disk orients itself in order to maintain force balance and since the forces don't change the angles of the disk doesn't change.
- The body tilts more to the right in order to cancel out the fact that gravity works at a different point.

Redoing Q3

The angle	Value [deg]	Value $[deg]$ Q1
τ	0	0
ϕ	-9.3670	-2.1767
β_{1c}	0	0
β_{1s}	0.7692	1.3402
θ_{1c}	1.7257	1.3402
θ_{1s}	0.3587	0
$\theta_{0.75}$	5.1241	7.1112
$\theta_{0.75,t}$	-6.4305	10.1372

- We can see that the pitch angle of the body relative to the horizon didn't change, just like we accepted because the wind is from right to left which doesn't affect the front-back direction.
- The body tilts more to the right, which is responsible since the wind is from the left.
- The disk tilts more to the left ($\beta - \phi = 10.1362[deg]$) to cancel out C_Y and the cyclic command changes in order to maintain this angle.

Redoing Q4

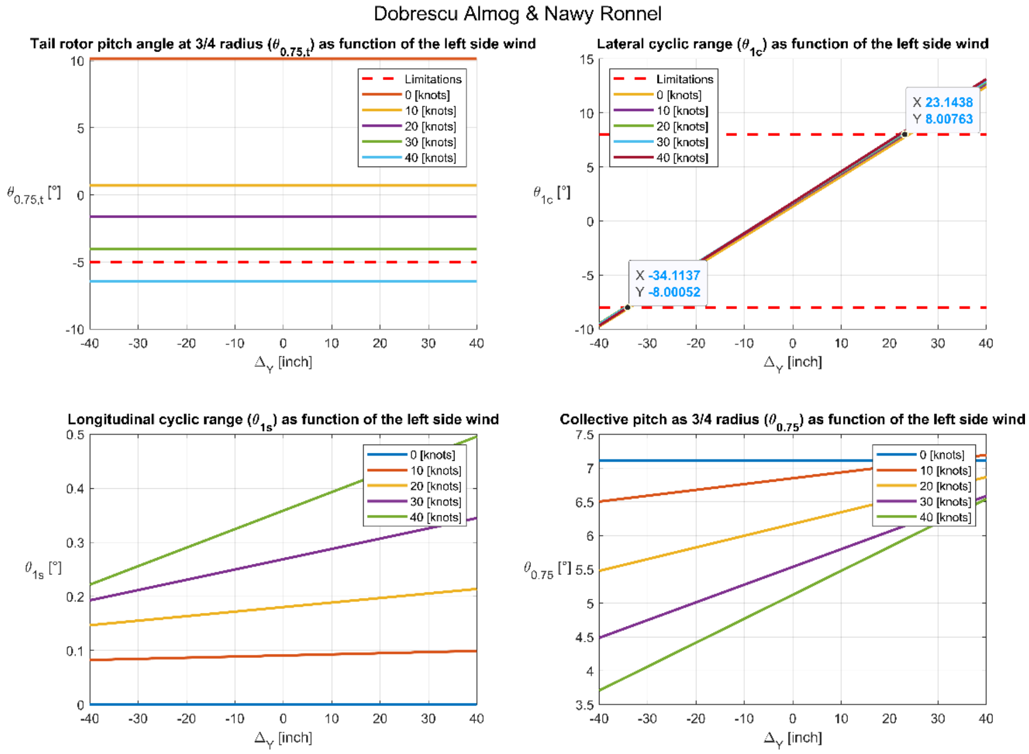


Figure 3: plot of $\theta_{0.75,t}$, θ_{1c}, θ_{1s}, $\theta_{0.75}$ as a function of Δy – Q5

As we can see $\theta_{0.75,t}$ reaches the limitation at approximately $33[knots]$. Since on the graph of θ_{1c} all the plots are basically the same, we estimate that the limit of center of gravity are determined by θ_{1c} and the maximal value for Δy (with 10% margin) is $20.83[inch]$ to the right and $30.70[inch]$ to the left.