Numerical Methods in Aeronautical Engineering
HW2 - Theoretical Questions

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1 Q2

1.1 A

We are asked to prove:

$$\delta^2 = \Delta - \nabla \tag{1}$$

Where:

- $\delta f = f_{\left(x+\frac{h}{2}\right)} - f_{\left(x-\frac{h}{2}\right)}$
- $\Delta f = f_{(x+h)} - f_{(x)}$
- $\nabla f = f_{(x)} - f_{(x-h)}$

$$\begin{aligned} \delta^2 f &= \delta\left(f_{\left(x+\frac{h}{2}\right)} - f_{\left(x-\frac{h}{2}\right)}\right) \\ &= \delta f_{\left(x+\frac{h}{2}\right)} - \delta f_{\left(x-\frac{h}{2}\right)} \\ &= f_{(x+h)} - f_{(x)} - f_{(x)} + f_{(x-h)} \\ &= f_{(x+h)} - 2f_{(x)} + f_{(x-h)} \end{aligned} \qquad \begin{aligned} \Delta f - \nabla f &= f_{(x+h)} - f_{(x)} - f_{(x)} + f_{(x-h)} \\ &= f_{(x+h)} - 2f_{(x)} + f_{(x-h)} \end{aligned} \tag{2}$$

$$\Downarrow$$

$$\delta^2 = \Delta - \nabla \qquad \blacksquare$$

1.2 B

The next ODE is given:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \tag{3}$$

The following finite differencing method is suggested::

$$u_{i,j+1} = u_{i,j} + R\left(u_{i-1,j} - u_{i,j} - u_{i,j+1} + u_{i+1,j+1}\right) \tag{4}$$

- $R = \dfrac{\Delta t}{h^2}$

in order to solve the method explicitly we will isolate $u_{i,j+1}$ in the LHS:

$$\begin{aligned} (1+R)\,u_{i,j+1} &= u_{i,j} + R\left(u_{i-1,j} - u_{i,j} + u_{i+1,j+1}\right) \\ u_{i,j+1} &= \frac{1}{1+R}u_{i,j} + \frac{R}{1+R}\left(u_{i-1,j} - u_{i,j} + u_{i+1,j+1}\right) \end{aligned} \tag{5}$$

This step might look like not enough, however, if we solve in a Gauss-Sidle like method, from the end to the start, so at a specific i we would already know $u_{i+1,j+1}$.

1.3 C

Let's use forward differencing for the time derivative:

$$\frac{\partial U}{\partial t} = \frac{1}{\Delta t}\Delta_t u = \frac{1}{\Delta t}\left(u_{i,j+1} - u_{i,j}\right) \tag{6}$$

and central differencing for the spacial derivative:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2}\delta_x^2 u = \frac{1}{h^2}\left(\Delta_x - \nabla_x\right)u \tag{7}$$

To achieve the desired scheme we will define the forward differencing at $j+1$ and the backward differencing at j:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2}\left(\Delta_x\big|_{j+1} - \nabla_x\big|_j\right) u \tag{8}$$

substituting the derivative into the ODE, we get:

$$\begin{aligned}
\frac{1}{\Delta t}\left(u_{i,j+1} - u_{i,j}\right) &= \frac{1}{h^2}\left(\Delta_x\big|_{j+1} - \nabla_x\big|_j\right) u \\
\left(u_{i,j+1} - u_{i,j}\right) &= \frac{\Delta t}{h^2}\left(u_{i+1,j+1} - u_{i,j+1} - u_{i,j} + u_{i-1,j}\right) \\
u_{i,j+1} &= u_{i,j} + R\left(u_{i+1,j+1} - u_{i,j+1} - u_{i,j} + u_{i-1,j}\right) \qquad \blacksquare
\end{aligned} \tag{9}$$

2 Q3

2.1 A

The two dimensional heat equation is given by:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{10}$$

u is a function of x, y, t, namely $u = u_{(x,y,t)}$. We will derive the equation of $u_{(x,y,t+\Delta t)}$ by expanding it into a Taylor series:

$$\begin{aligned}
u_{(x,y,t+\Delta t)} &= u_{(x,y,t)} + \Delta t \frac{\partial u_{(x,y,t)}}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u_{(x,y,t)}}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u_{(x,y,t)}}{\partial t^3} + \cdots \\
u_{(x,y,t+\Delta t)} &= \left(1 + \Delta t \frac{\partial}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3}{\partial t^3} + \cdots\right) u_{(x,y,t)}
\end{aligned} \tag{11}$$

From the PDE we get the following relation:

$$\frac{\partial}{\partial t} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{12}$$

So, the Taylor expansion can be rewritten as:

$$u_{(x,y,t+\Delta t)} = \left(1 + \Delta t\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{(\Delta t)^2}{2!}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 + \frac{(\Delta t)^3}{3!}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^3 + \cdots\right) u_{(x,y,t)} \tag{13}$$

We can identify the the Taylor series of an exponential:

$$u_{(x,y,t+\Delta t)} = \exp\left(\Delta t\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right) u_{(x,y,t)} \tag{14}$$

The derivative can be substituted by using the following operators relation:

$$\begin{aligned}
\frac{\partial}{\partial x} = D_x = \frac{2}{h}\sinh^{-1}\left(\frac{\delta_x}{2}\right) \qquad & \frac{\partial}{\partial y} = D_y = \frac{2}{h}\sinh^{-1}\left(\frac{\delta_y}{2}\right) \\
\frac{\partial^2}{\partial x^2} = D_x^2 = \frac{4}{h^2}\left(\sinh^{-1}\left(\frac{\delta_x}{2}\right)\right)^2 \qquad & \frac{\partial^2}{\partial y^2} = D_y^2 = \frac{4}{h^2}\left(\sinh^{-1}\left(\frac{\delta_y}{2}\right)\right)^2
\end{aligned} \tag{15}$$

The following equation is reached:

$$u_{(x,y,t+\Delta t)} = \exp\left[\frac{4\Delta t}{h^2}\left(\left(\sinh^{-1}\left(\frac{\delta_x}{2}\right)\right)^2 + \left(\sinh^{-1}\left(\frac{\delta_y}{2}\right)\right)^2\right)\right]u_{(x,y,t)} \tag{16}$$

To further simplify, we will expand the hyperbolic sin into it's Taylor series:

$$u_{(x,y,t+\Delta t)} = \exp\left[\frac{4\Delta t}{h^2}\left(\left(\frac{\delta_x}{2} - \frac{1}{3!}\left(\frac{\delta_x}{2}\right)^3 + \cdots\right)^2 + \left(\frac{\delta_y}{2} - \frac{1}{3!}\left(\frac{\delta_y}{2}\right)^3 + \cdots\right)^2\right)\right]u_{(x,y,t)} \tag{17}$$

Now let's expand the exponent:

$$\begin{aligned} u_{(x,y,t+\Delta t)} \quad = \quad & \left[1 + \frac{4\Delta t}{h^2}\left(\left(\frac{\delta_x}{2} - \frac{1}{3!}\left(\frac{\delta_x}{2}\right)^3 + \cdots\right)^2 + \left(\frac{\delta_y}{2} - \frac{1}{3!}\left(\frac{\delta_y}{2}\right)^3 + \cdots\right)^2\right)\right. \\ & \left. + \frac{1}{2!}\frac{16\Delta t^2}{h^4}\left(\left(\frac{\delta_x}{2} - \frac{1}{3!}\left(\frac{\delta_x}{2}\right)^3 + \cdots\right)^2 + \left(\frac{\delta_y}{2} - \frac{1}{3!}\left(\frac{\delta_y}{2}\right)^3 + \cdots\right)^2\right)^2\right]u_{(x,y,t)} \quad \blacksquare \end{aligned} \tag{18}$$

2.2 B

From the this infinite Taylor series we can derive a lot of approximations. For example we could take only the elements up to the order of δ_x^2 or δ_y^2:

$$\begin{aligned} u_{(u,y,t+\Delta t)} &= \left[1 + \frac{4\Delta t}{h^2}\left(\frac{\delta_x^2}{4} + \frac{\delta_y^2}{4}\right)\right]u_{(x,y,t)} \\ u_{i,j,k+1} &= \left[1 + \frac{\Delta t}{h^2}\left(\delta_x^2 + \delta_y^2\right)\right]u_{i,j,k} \\ u_{i,j,k+1} &= a \end{aligned} \tag{19}$$