Numerical Methods in Aeronautical Engineering
HW3 - Theoretical Questions

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The differencing equation is given by:

$$\begin{aligned}\frac{u_{i,j+1}-u_{i,j}}{kG}+\mathcal{O}\left(\frac{k}{G}\right) &= \alpha\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}+\mathcal{O}\left(h^2\right) \\ u_{i,j+1}-u_{i,j} &= \frac{\alpha kG}{h^2}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)+\mathcal{O}\left(\frac{k}{G},h^2\right)\end{aligned} \tag{1}$$

Where:

- $h = \Delta x$
- $k = \Delta y$
- $C = \dfrac{\alpha k}{h^2}$
- $G = \dfrac{1}{2C}\left(1-e^{-2C}\right) = \dfrac{h^2}{2\alpha k}\left(1-e^{-2\frac{\alpha k}{h^2}}\right)$

Hence the truncation error is of the order of $\mathcal{O}\left(\frac{k}{G},h^2\right)$.

We will use von-Neumann Stability Analysis. Let's consider the error:

$$e_{r,t} = T_{r,t} - \hat{T}_{r,t} \tag{2}$$

- $T_{r,t}$ - The solution of the differencing equation.
- $\hat{T}_{r,t}$ - The solution of the differencing equation with a small noise in the initial conditions.

Let's define:

$$E_p = T_{p,0} - \hat{T}_{p,0} \tag{3}$$

We can rewrite it as:

$$E_p = \sum_{p=1}^{N} A_p e^{i\beta_n hp}, \qquad i=\sqrt{-1}, \qquad \beta_n = \frac{n\pi}{N+1} \tag{4}$$

We want to check if there is a mode that diverges. Since the equation is linear we only need one mode to diverge to consider the hole scheme as diverged:

$$E_p = A_p e^{i\beta_n hp} \tag{5}$$

We need to check how the error behaves over time and to make sure it diminishes to E_p when $t=0$. Let's assume the error is of the form of:

$$E_{p,j} = A_p e^{i\beta hp}\cdot e^{\alpha t_j} \tag{6}$$

We are considered stable when $\Re\{\alpha\} < 0$.
We can rewrite the error:

$$E_{p,j} = A_p e^{i\beta hp}\cdot e^{\alpha\Delta tj} = A_p e^{i\beta hp}\cdot \xi^j \tag{7}$$

Therefore the stability condition is $|\xi| \leq 1$ Since the differencing equation is linear we can demand the error to satisfy the differencing equation:

$$E_{p,j+1} = E_{p,j} + \frac{\alpha kG}{h^2}\left(E_{p+1,j}-2E_{p,j}+E_{p-1,j}\right) \tag{8}$$

After substituting the error we get:

$$e^{i\beta hp} \cdot \xi^{j+1} = e^{i\beta hp} \cdot \xi^{j} + \frac{\alpha k G}{h^2} \left(e^{i\beta h(p+1)} \cdot \xi^{j} - 2e^{i\beta h(p)} \cdot \xi^{j} + e^{i\beta h(p-1)} \cdot \xi^{j} \right) \tag{9}$$

Dividing by $e^{i\beta hp}\xi^{j}$ we get:

$$\xi = 1 + \frac{\alpha k G}{h^2} \left(e^{i\beta h} - 2 + e^{-i\beta h} \right) \tag{10}$$