

Numerical Methods in Aeronautical Engineering  
HW2 - Theoretical Questions

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## 1 Q2

### 1.1 A

We are asked to prove:

$$\delta^2 = \Delta - \nabla \quad (1)$$

Where:

- $\delta f = f_{(x+\frac{h}{2})} - f_{(x-\frac{h}{2})}$
- $\Delta f = f_{(x+h)} - f_{(x)}$
- $\nabla f = f_{(x)} - f_{(x-h)}$

$$\begin{aligned}
 \delta^2 f &= \delta \left( f_{(x+\frac{h}{2})} - f_{(x-\frac{h}{2})} \right) & \Delta f - \nabla f &= f_{(x+h)} - f_{(x)} - f_{(x)} + f_{(x-h)} \\
 &= \delta f_{(x+\frac{h}{2})} - \delta f_{(x-\frac{h}{2})} & &= f_{(x+h)} - 2f_{(x)} + f_{(x-h)} \\
 &= f_{(x+h)} - f_{(x)} - f_{(x)} + f_{(x-h)} & & \\
 &= f_{(x+h)} - 2f_{(x)} + f_{(x-h)} & & \\
 &\Downarrow & & \\
 \delta^2 &= \Delta - \nabla \quad \blacksquare
 \end{aligned} \quad (2)$$

### 1.2 B

The next ODE is given:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (3)$$

The following finite differencing method is suggested::

$$u_{i,j+1} = u_{i,j} + R(u_{i-1,j} - u_{i,j} - u_{i,j+1} + u_{i+1,j+1}) \quad (4)$$

- $R = \frac{\Delta t}{h^2}$

in order to solve the method explicitly we will isolate  $u_{i,j+1}$  in the LHS:

$$\begin{aligned}
 (1 + R)u_{i,j+1} &= u_{i,j} + R(u_{i-1,j} - u_{i,j} + u_{i+1,j+1}) \\
 u_{i,j+1} &= \frac{1}{1+R}u_{i,j} + \frac{R}{1+R}(u_{i-1,j} - u_{i,j} + u_{i+1,j+1})
 \end{aligned} \quad (5)$$

This step might look like not enough, however, if we solve in a Gauss-Sidle like method, from the end to the start, so at a specific  $i$  we would already know  $u_{i+1,j+1}$ .

### 1.3 C

Let's use forward differencing for the time derivative:

$$\frac{\partial U}{\partial t} = \frac{1}{\Delta t} \Delta_t u = \frac{1}{\Delta t} (u_{i,j+1} - u_{i,j}) \quad (6)$$

and central differencing for the spacial derivative:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2} \delta_x^2 u = \frac{1}{h^2} (\Delta_x - \nabla_x) u \quad (7)$$



To achieve the desired scheme we will define the forward differencing at  $j + 1$  and the backward differencing at  $j$ :

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2} \left( \Delta_x|_{j+1} - \nabla_x|_j \right) u \quad (8)$$

substituting the derivative into the ODE, we get:

$$\begin{aligned} \frac{1}{\Delta t} (u_{i,j+1} - u_{i,j}) &= \frac{1}{h^2} \left( \Delta_x|_{j+1} - \nabla_x|_j \right) u \\ (u_{i,j+1} - u_{i,j}) &= \frac{\Delta t}{h^2} (u_{i+1,j+1} - u_{i,j+1} - u_{i,j} + u_{i-1,j}) \\ u_{i,j+1} &= u_{i,j} + R(u_{i+1,j+1} - u_{i,j+1} - u_{i,j} + u_{i-1,j}) \quad \blacksquare \end{aligned} \quad (9)$$

## 2 Q3

### 2.1 A

The two dimensional heat equation is given by:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (10)$$

$u$  is a function of  $x, y, t$ , namely  $u = u_{(x,y,t)}$ . We will derive the equation of  $u_{(x,y,t+\Delta t)}$  by expanding it into a Taylor series:

$$\begin{aligned} u_{(x,y,t+\Delta t)} &= u_{(x,y,t)} + \Delta t \frac{\partial u_{(x,y,t)}}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u_{(x,y,t)}}{\partial t^2} + \frac{(\Delta t)^3}{3!} \frac{\partial^3 u_{(x,y,t)}}{\partial t^3} + \dots \\ u_{(x,y,t+\Delta t)} &= \left( 1 + \Delta t \frac{\partial}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2}{\partial t^2} + \frac{(\Delta t)^3}{3!} \frac{\partial^3}{\partial t^3} + \dots \right) u_{(x,y,t)} \end{aligned} \quad (11)$$

From the PDE we get the following relation:

$$\frac{\partial}{\partial t} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (12)$$

So, the Taylor expansion can be rewritten as:

$$u_{(x,y,t+\Delta t)} = \left( 1 + \Delta t \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{(\Delta t)^2}{2!} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 + \frac{(\Delta t)^3}{3!} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^3 + \dots \right) u_{(x,y,t)} \quad (13)$$

We can identify the the Taylor series of an exponential:

$$u_{(x,y,t+\Delta t)} = \exp \left( \Delta t \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \right) u_{(x,y,t)} \quad (14)$$

The derivative can be substituted by using the following operators relation:

$$\begin{aligned} \frac{\partial}{\partial x} &= D_x = \frac{2}{h_x} \sinh^{-1} \left( \frac{\delta_x}{2} \right) & \frac{\partial}{\partial y} &= D_y = \frac{2}{h_y} \sinh^{-1} \left( \frac{\delta_y}{2} \right) \\ \frac{\partial^2}{\partial x^2} &= D_x^2 = \frac{4}{h_x^2} \left( \sinh^{-1} \left( \frac{\delta_x}{2} \right) \right)^2 & \frac{\partial^2}{\partial y^2} &= D_y^2 = \frac{4}{h_y^2} \left( \sinh^{-1} \left( \frac{\delta_y}{2} \right) \right)^2 \end{aligned} \quad (15)$$



The following equation is reached:

$$u_{(x,y,t+\Delta t)} = \exp \left[ \frac{4\Delta t}{h_x^2} \left( \sinh^{-1} \left( \frac{\delta_x}{2} \right) \right)^2 + \frac{4\Delta t}{h_y^2} \left( \sinh^{-1} \left( \frac{\delta_y}{2} \right) \right)^2 \right] u_{(x,y,t)} \quad (16)$$

To further simplify, we will expand the hyperbolic sin into it's Taylor series:

$$u_{(x,y,t+\Delta t)} = \exp \left[ \frac{4\Delta t}{h_x^2} \left( \frac{\delta_x}{2} - \frac{1}{3!} \left( \frac{\delta_x}{2} \right)^3 + \dots \right)^2 + \frac{4\Delta t}{h_y^2} \left( \frac{\delta_y}{2} - \frac{1}{3!} \left( \frac{\delta_y}{2} \right)^3 + \dots \right)^2 \right] u_{(x,y,t)} \quad (17)$$

Now let's expand the exponent:

$$\begin{aligned} u_{(x,y,t+\Delta t)} = & \left[ 1 + \frac{4\Delta t}{h_x^2} \left( \frac{\delta_x}{2} - \frac{1}{3!} \left( \frac{\delta_x}{2} \right)^3 + \dots \right)^2 + \frac{4\Delta t}{h_y^2} \left( \frac{\delta_y}{2} - \frac{1}{3!} \left( \frac{\delta_y}{2} \right)^3 + \dots \right)^2 \right. \\ & \left. + \frac{1}{2!} \left( \frac{4\Delta t}{h_x^2} \left( \frac{\delta_x}{2} - \frac{1}{3!} \left( \frac{\delta_x}{2} \right)^3 + \dots \right)^2 + \frac{4\Delta t}{h_y^2} \left( \frac{\delta_y}{2} - \frac{1}{3!} \left( \frac{\delta_y}{2} \right)^3 + \dots \right)^2 \right) \right] u_{(x,y,t)} \quad \blacksquare \end{aligned} \quad (18)$$

## 2.2 B

From the this infinite Taylor series we can derive a lot of approximations. For example we could take only the elements up to the order of  $\delta_x^2$  or  $\delta_y^2$ :

$$\begin{aligned} u_{(u,y,t+\Delta t)} &= \left[ 1 + \left( \frac{4\Delta t}{h_x^2} \frac{\delta_x^2}{4} + \frac{4\Delta t}{h_y^2} \frac{\delta_y^2}{4} \right) \right] u_{(x,y,t)} \\ u_{i,j,k+1} &= \left[ 1 + \frac{\Delta t}{h_x^2} \delta_x^2 + \frac{\Delta t}{h_y^2} \delta_y^2 \right] u_{i,j,k} \\ u_{i,j,k+1} &= u_{i,j,k} + R_x (u_{i-1,j,k} - 2u_{i,j,k} + u_{i+1,j,k}) + R_y (u_{i,j-1,k} - 2u_{i,j,k} + u_{i,j+1,k}) \quad \blacksquare \end{aligned} \quad (19)$$

- $R_x = \frac{\Delta t}{h_x^2}$
- $R_y = \frac{\Delta t}{h_y^2}$

## 3 Q4

The next equation is given:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (20)$$

With the following boundary and initial conditions:

$$\begin{aligned} U_{(0,t)} &= 0 \\ \frac{\partial U_{(1,t)}}{\partial x} &= M \end{aligned} \quad U_{(x,0)} = U_{0(x)} \quad (21)$$



In order to solve this equation we will use forward differencing in time and central differencing in space:

$$u_{i,j+1} = u_{i,j} + R(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}) \quad (22)$$

By rearranging we get:

$$\underbrace{(-R)}_{\alpha_{i,j}} u_{i-1,j+1} + \underbrace{(1+2R)}_{\beta_{i,j}} u_{i,j+1} + \underbrace{(-R)}_{\gamma_{i,j}} u_{i+1,j+1} = \underbrace{u_{i,j}}_{RHS_{i,j}} \quad (23)$$

There is no problem to solve the equations for  $i = 1, 2, \dots, N$  but in the equation at  $i = N + 1$  we don't know  $i = N + 2$ . We will use the boundary condition on the derivative at  $i = N + 1$ . We will use central differencing to write the derivative:

$$\frac{\partial u_{N+1,j}}{\partial x} = \frac{1}{2h} (u_{N+2,j} - u_{N,j}) = M \quad (24)$$

$$u_{N+2,j} = 2hM + u_{N,j}$$

In matrix form we get:

$$\begin{pmatrix} \beta_{1,j} & \gamma_{1,j} & 0 & \dots & \dots & \dots & 0 \\ \alpha_{2,j} & \beta_{2,j} & \gamma_{2,j} & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \dots & 0 \\ 0 & 0 & \alpha_i & \beta_i & \gamma_i & 0 & 0 \\ 0 & \dots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \alpha_{N,j} & \beta_{N,j} & \gamma_{N,j} \\ 0 & 0 & \dots & \dots & 0 & \alpha_{N+1,j} + \gamma_{N+1,j} & \beta_{N+1,j} \end{pmatrix} \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \dots \\ \dots \\ \dots \\ u_{N,j+1} \\ u_{N+1,j+1} \end{pmatrix} = \begin{pmatrix} RHS_{1,j} - \alpha_{0,j} \cdot u_{0,j} \\ RHS_{2,j} \\ \dots \\ \dots \\ \dots \\ RHS_{N,j} \\ RHS_{N+1,j} - \gamma_{N+1,j} \cdot 2hM \end{pmatrix} \quad (25)$$

Using Thomas algorithm we would get a direct result for the  $u_j$ .