Numerical Methods in Aeronautical Engineering
HW2 - Theoretical Questions

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1   Q2

1.1   A

We are asked to prove:

$$\delta^2 = \Delta - \nabla \tag{1}$$

Where:

- $\delta f = f_{\left(x+\frac{h}{2}\right)} - f_{\left(x-\frac{h}{2}\right)}$

- $\Delta f = f_{(x+h)} - f_{(x)}$

- $\nabla f = f_{(x)} - f_{(x-h)}$

$$
\begin{aligned}
\delta^2 f &= \delta\left(f_{\left(x+\frac{h}{2}\right)} - f_{\left(x-\frac{h}{2}\right)}\right) \\
&= \delta f_{\left(x+\frac{h}{2}\right)} - \delta f_{\left(x-\frac{h}{2}\right)} \\
&= f_{(x+h)} - f_{(x)} - f_{(x)} + f_{(x-h)} \\
&= f_{(x+h)} - 2f_{(x)} + f_{(x-h)}
\end{aligned}
\qquad
\begin{aligned}
\Delta f - \nabla f &= f_{(x+h)} - f_{(x)} - f_{(x)} + f_{(x-h)} \\
&= f_{(x+h)} - 2f_{(x)} + f_{(x-h)}
\end{aligned}
\tag{2}
$$

$$\Downarrow$$

$$\delta^2 = \Delta - \nabla \qquad \blacksquare$$

1.2   B

The next ODE is given:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \tag{3}$$

The following finite differencing method is suggested::

$$u_{i,j+1} = u_{i,j} + R\left(u_{i-1,j} - u_{i,j} - u_{i,j+1} + u_{i+1,j+1}\right) \tag{4}$$

- $R = \dfrac{\Delta t}{h^2}$

in order to solve the method explicitly we will isolate $u_{i,j+1}$ in the LHS:

$$
\begin{aligned}
(1+R)\,u_{i,j+1} &= u_{i,j} + R\left(u_{i-1,j} - u_{i,j} + u_{i+1,j+1}\right) \\
u_{i,j+1} &= \frac{1}{1+R}u_{i,j} + \frac{R}{1+R}\left(u_{i-1,j} - u_{i,j} + u_{i+1,j+1}\right)
\end{aligned}
\tag{5}
$$

This step might look like not enough, however, if we solve in a Gauss-Sidle like method, from the end to the start, so at a specific i we would already know $u_{i+1,j+1}$.

1.3   C

Let's use forward differencing for the time derivative:

$$\frac{\partial U}{\partial t} = \frac{1}{\Delta t}\Delta_t u = \frac{1}{\Delta t}\left(u_{i,j+1} - u_{i,j}\right) \tag{6}$$

and central differencing for the spacial derivative:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2}\delta_x^2 u = \frac{1}{h^2}\left(\Delta_x - \nabla_x\right)u \tag{7}$$

To achieve the desired scheme we will define the forward differencing at $j+1$ and the backward differencing at j:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2}\left(\Delta_x|_{j+1} - \nabla_x|_j\right)u \tag{8}$$

substituting the derivative into the ODE, we get:

$$
\begin{aligned}
\frac{1}{\Delta t}\left(u_{i,j+1} - u_{i,j}\right) &= \frac{1}{h^2}\left(\Delta_x|_{j+1} - \nabla_x|_j\right)u \\[2mm]
\left(u_{i,j+1} - u_{i,j}\right) &= \frac{\Delta t}{h^2}\left(u_{i+1,j+1} - u_{i,j+1} - u_{i,j} + u_{i-1,j}\right) \\[2mm]
u_{i,j+1} &= u_{i,j} + R\left(u_{i+1,j+1} - u_{i,j+1} - u_{i,j} + u_{i-1,j}\right) \qquad \blacksquare
\end{aligned}
\tag{9}
$$

2   Q3

2.1   A

The two dimensional heat equation is given by:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{10}$$

u is a function of x, y, t, namely $u = u_{(x,y,t)}$. We will derive the equation of $u_{(x,y,t+\Delta t)}$ by expanding it into a Taylor series:

$$u_{(x,y,t+\Delta t)} = \tag{11}$$