Numerical Methods in Aeronautical Engineering
HW2 - Theoretical Questions

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1.1 A

We are asked to prove:

$$\delta^2 = \Delta - \nabla \tag{1}$$

Where:

- $\delta f = f_{\left(x+\frac{h}{2}\right)} - f_{\left(x-\frac{h}{2}\right)}$
- $\Delta f = f_{(x+h)} - f_{(x)}$
- $\nabla f = f_{(x)} - f_{(x-h)}$

$$\begin{aligned} \delta^2 f &= \delta\left(f_{\left(x+\frac{h}{2}\right)} - f_{\left(x-\frac{h}{2}\right)}\right) \\ &= \delta f_{\left(x+\frac{h}{2}\right)} - \delta f_{\left(x-\frac{h}{2}\right)} \\ &= f_{(x+h)} - f_{(x)} - f_{(x)} + f_{(x-h)} \\ &= f_{(x+h)} - 2f_{(x)} + f_{(x-h)} \end{aligned} \qquad \begin{aligned} \Delta f - \nabla f &= f_{(x+h)} - f_{(x)} - f_{(x)} + f_{(x-h)} \\ &= f_{(x+h)} - 2f_{(x)} + f_{(x-h)} \end{aligned} \tag{2}$$

$$\Downarrow$$

$$\delta^2 = \Delta - \nabla \qquad \blacksquare$$

1.2 B

The next ODE is given:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \tag{3}$$

The following finite differencing method is suggested::

$$u_{i,j+1} = u_{i,j} + R\left(u_{i-1,j} - u_{i,j} - u_{i,j+1} + u_{i+1,j+1}\right) \tag{4}$$

- $R = \dfrac{\Delta t}{h^2}$

in order to solve the method explicitly we will isolate $u_{i,j+1}$ in the LHS:

$$\begin{aligned} (1+R)\, u_{i,j+1} &= u_{i,j} + R\left(u_{i-1,j} - u_{i,j} + u_{i+1,j+1}\right) \\ u_{i,j+1} &= \frac{1}{1+R} u_{i,j} + \frac{R}{1+R}\left(u_{i-1,j} - u_{i,j} + u_{i+1,j+1}\right) \end{aligned} \tag{5}$$

This step might look like not enough, however, if we solve in a Gauss-Sidle like method, from the end to the start, so at a specific i we would already know $u_{i+1,j+1}$.

1.3 C

Let's use forward differencing for the time derivative:

$$\frac{\partial U}{\partial t} = \frac{1}{\Delta t}\Delta_t u = \frac{1}{\Delta t}\left(u_{i,j+1} - u_{i,j}\right) \tag{6}$$

and central differencing for the spacial derivative:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2}\delta_x^2 u = \frac{1}{h^2}\left(\Delta_x - \nabla_x\right) u \tag{7}$$

To achieve the desired scheme we will define the forward differencing at $j+1$ and the backward differencing at j:

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{h^2}\left(\Delta_x|_{j+1} - \nabla_x|_j\right) u \tag{8}$$

substituting the derivative into the ODE, we get:

$$\begin{aligned}\frac{1}{\Delta t}\left(u_{i,j+1} - u_{i,j}\right) &= \frac{1}{h^2}\left(\Delta_x|_{j+1} - \nabla_x|_j\right) u \\ \left(u_{i,j+1} - u_{i,j}\right) &= \frac{\Delta t}{h^2}\left(u_{i+1,j+1} - u_{i,j+1} - u_{i,j} + u_{i-1,j}\right) \\ u_{i,j+1} &= u_{i,j} + R\left(u_{i+1,j+1} - u_{i,j+1} - u_{i,j} + u_{i-1,j}\right) \qquad \blacksquare\end{aligned} \tag{9}$$

2 Q3

2.1 A

The two dimensional heat equation is given by:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{10}$$

u is a function of x, y, t, namely $u = u_{(x,y,t)}$. We will derive the equation of $u_{(x,y,t+\Delta t)}$ by expanding it into a Taylor series:

$$\begin{aligned}u_{(x,y,t+\Delta t)} &= u_{(x,y,t)} + \Delta t \frac{\partial u_{(x,y,t)}}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u_{(x,y,t)}}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u_{(x,y,t)}}{\partial t^3} + \cdots \\ u_{(x,y,t+\Delta t)} &= \left(1 + \Delta t\frac{\partial}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3}{\partial t^3} + \cdots\right) u_{(x,y,t)}\end{aligned} \tag{11}$$

From the PDE we get the following relation:

$$\frac{\partial}{\partial t} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{12}$$

So, the Taylor expansion can be rewritten as:

$$u_{(x,y,t+\Delta t)} = \left(1 + \Delta t\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{(\Delta t)^2}{2!}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 + \frac{(\Delta t)^3}{3!}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^3 + \cdots\right) u_{(x,y,t)} \tag{13}$$

We can identify the the Taylor series of an exponential:

$$u_{(x,y,t+\Delta t)} = \exp\left(\Delta t\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right) u_{(x,y,t)} \tag{14}$$

The derivative can be substituted by using the following operators relation:

$$\begin{aligned}\frac{\partial}{\partial x} = D_x = \frac{2}{h_x}\sinh^{-1}\left(\frac{\delta_x}{2}\right) &\qquad \frac{\partial}{\partial y} = D_y = \frac{2}{h_y}\sinh^{-1}\left(\frac{\delta_y}{2}\right) \\ \frac{\partial^2}{\partial x^2} = D_x^2 = \frac{4}{h_x^2}\left(\sinh^{-1}\left(\frac{\delta_x}{2}\right)\right)^2 &\qquad \frac{\partial^2}{\partial y^2} = D_y^2 = \frac{4}{h_y^2}\left(\sinh^{-1}\left(\frac{\delta_y}{2}\right)\right)^2\end{aligned} \tag{15}$$

The following equation is reached:

$$u_{(x,y,t+\Delta t)} = \exp\left[\frac{4\Delta t}{h_x^2}\left(\sinh^{-1}\left(\frac{\delta_x}{2}\right)\right)^2 + \frac{4\Delta t}{h_y^2}\left(\sinh^{-1}\left(\frac{\delta_y}{2}\right)\right)^2\right] u_{(x,y,t)} \tag{16}$$

To further simplify, we will expand the hyperbolic sin into it's Taylor series:

$$u_{(x,y,t+\Delta t)} = \exp\left[\frac{4\Delta t}{h_x^2}\left(\frac{\delta_x}{2} - \frac{1}{3!}\left(\frac{\delta_x}{2}\right)^3 + \cdots\right)^2 + \frac{4\Delta t}{h_y^2}\left(\frac{\delta_y}{2} - \frac{1}{3!}\left(\frac{\delta_y}{2}\right)^3 + \cdots\right)^2\right] u_{(x,y,t)} \tag{17}$$

Now let's expand the exponent:

$$\begin{aligned} u_{(x,y,t+\Delta t)} &= \left[1 + \frac{4\Delta t}{h_x^2}\left(\frac{\delta_x}{2} - \frac{1}{3!}\left(\frac{\delta_x}{2}\right)^3 + \cdots\right)^2 + \frac{4\Delta t}{h_y^2}\left(\frac{\delta_y}{2} - \frac{1}{3!}\left(\frac{\delta_y}{2}\right)^3 + \cdots\right)^2\right. \\ &\left. + \frac{1}{2!}\left(\frac{4\Delta t}{h_x^2}\left(\frac{\delta_x}{2} - \frac{1}{3!}\left(\frac{\delta_x}{2}\right)^3 + \cdots\right)^2 + \frac{4\Delta t}{h_y^2}\left(\frac{\delta_y}{2} - \frac{1}{3!}\left(\frac{\delta_y}{2}\right)^3 + \cdots\right)^2\right)^2\right] u_{(x,y,t)} \quad \blacksquare \end{aligned} \tag{18}$$

2.2 B

From the this infinite Taylor series we can derive a lot of approximations. For example we could take only the elements up to the order of δ_x^2 or δ_y^2:

$$\begin{aligned} u_{(u,y,t+\Delta t)} &= \left[1 + \left(\frac{4\Delta t}{h_x^2}\frac{\delta_x^2}{4} + \frac{4\Delta t}{h_y^2}\frac{\delta_y^2}{4}\right)\right] u_{(x,y,t)} \\ u_{i,j,k+1} &= \left[1 + \frac{\Delta t}{h_x^2}\delta_x^2 + \frac{\Delta t}{h_y^2}\delta_y^2\right] u_{i,j,k} \\ u_{i,j,k+1} &= u_{i,j,k} + R_x\left(u_{i-1,j,k} - 2u_{i,j,k} + u_{i+1,j,k}\right) + R_y\left(u_{i,j-1,k} - 2u_{i,j,k} + u_{i,j+1k}\right) \quad \blacksquare \end{aligned} \tag{19}$$

- $R_x = \dfrac{\Delta t}{h_x^2}$
- $R_y = \dfrac{\Delta t}{h_y^2}$

3 Q4

The next equation is given:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \tag{20}$$

With the following boundary and initial conditions:

$$\begin{aligned} U_{(0,t)} &= 0 \\ \frac{\partial U_{(1,t)}}{\partial x} &= M \end{aligned} \qquad U_{(x,0)} = U_{0(x)} \tag{21}$$

In order to solve this equation we will use forward differencing in time and central differencing in space:

$$u_{i,j+1} = u_{i,j} + R\left(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right) \tag{22}$$

By rearranging we get:

$$\underbrace{(-R)}_{\alpha_{i,j}} u_{i-1,j+1} + \underbrace{(1+2R)}_{\beta_{i,j}} u_{i,j+1} + \underbrace{(-R)}_{\gamma_{i,j}} u_{i+1,j+1} = \underbrace{u_{i,j}}_{RHS_{i,j}} \tag{23}$$

There is no problem to solve the equations for $i = 1, 2, \cdots N$ but in the equation at $i = N+1$ we don't know $i = N+2$. We will use the boundary condition on the derivative at $i + N + 1$. We will use central differencing to write the derivative:

$$\begin{gathered} \frac{\partial u_{N+1,j}}{\partial x} = \frac{1}{2h}\left(u_{N+2,j} - u_{N,j}\right) = M \\ u_{N+2,j} = 2hM + u_{N,j} \end{gathered} \tag{24}$$

In matrix form we get:

$$\begin{pmatrix} \beta_1, j & \gamma_1, j & 0 & \cdots & \cdots & \cdots & 0 \\ \alpha_2, j & \beta_2, j & \gamma_2, j & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \alpha_i & \beta_i & \gamma_i & 0 & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \alpha_{N,j} & \beta_{N,j} & \gamma_{N,j} \\ 0 & 0 & \cdots & \cdots & 0 & \alpha_{N+1,j} + \gamma_{N+1,j} & \beta_{N+1,j} \end{pmatrix} \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \cdots \\ \cdots \\ \cdots \\ u_{N,j+1} \\ u_{N+1,j+1} \end{pmatrix} = \begin{pmatrix} RHS_{1,j} - \alpha_{1,j} \cdot u_{0,j} \\ RHS_{2,j} \\ \cdots \\ \cdots \\ \cdots \\ RHS_{N,j} \\ RHS_{N+1,j} - \gamma_{N+1,j} \cdot 2hM \end{pmatrix} \tag{25}$$

Using Thomas algorithm we would get a direct result for the u_j.