Numerical Methods in Aeronautical Engineering
HW1 - Theoretical Questions

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1 Q2

Given:

$$\frac{dy}{dt} = f_{(t,y)} \qquad y_{(0)} = 1 \tag{1}$$

1.1 A

Since the Euler method is based on Taylor series expansion, let's derive the expansion for y_{i+1}:

$$y_{i+1} = y_i + \frac{h^1}{1!}\frac{dy}{dt}\bigg|_i + \frac{h^2}{2!}\frac{d^2y}{dt^2}\bigg|_i + \frac{h^3}{3!}\frac{d^3y}{dt^3}\bigg|_i + \cdots \tag{2}$$

According the Euler method, the local error can be written as:

$$y_{i+1} = y_i + \frac{h^1}{1!}f_i + R_i \qquad \boxed{R_i = \frac{h^2}{2!}\frac{df}{dt}\bigg|_i \approx O\left(h^2\right)} \tag{3}$$

1.2 B

Given the function f, we can derive f and get the supremum of the local error over the hole field:

$$\boxed{\max_i R = \frac{\max_i h^2}{2!}\max_i \left|\frac{df}{dt}\right|} \tag{4}$$

1.3 C

Assuming h is constant, the supremum of the local error is:

$$\max_i R = \frac{h^2}{2!}\max_i \left|\frac{df}{dt}\right|$$

Defining the final time as T, the number of integration steps is:

$$N = \frac{T}{h} \tag{5}$$

Hence the supremum of the global error is:

$$R_{\text{global}} = \sum_{i=1}^{N}\max_i R = \sum_{i=1}^{N}\frac{h^2}{2!}\max_i\left|\frac{df}{dt}\right| = \frac{T}{h}\frac{h^2}{2!}\max_i\left|\frac{df}{dt}\right| = \frac{h}{2!}\max_i\left|\frac{df}{dt}\right| \approx \boxed{O\left(h\right)} \tag{6}$$

1.4 D

If we have the equation of f, we can derive and find the maximum value of $\frac{df}{dt}$, in the filed.

2 Q3

Given the ODE:

$$\frac{dy}{dx} = -y \qquad y_{(0)} = 1 \tag{7}$$

The analytical solution for the problem is:

$$\begin{aligned}
\frac{dy}{y} &= -dx \\
\int \frac{dy}{y} &= \int -dx \\
\ln(y) &= -x + C \\
&\Downarrow \; y_{(0)} = 1 \\
C &= 0 \\
&\Downarrow
\end{aligned} \tag{8}$$

$$\boxed{y = e^{-x}} \tag{9}$$

2.1 A - Forward Differencing

Given the differencing equation:

$$\frac{y_{n+1} - y_n}{h} = -y_n \qquad O(h) \tag{10}$$

and assuming the solution has the following form:

$$y_n = \alpha\beta^n \tag{11}$$

$$\begin{aligned}
&\Downarrow \\
\alpha\beta^{n+1} - \alpha\beta^n &= -h\alpha\beta^n \\
\beta^{n+1} &= (1-h)\,\beta^n \\
\beta &= 1 - h \\
&\Downarrow \\
y_n &= \alpha(1-h)^n \\
&\Downarrow \; y_{(0)} = 1 \\
\alpha &= 1 \\
&\Downarrow \\
y_n &= (1-h)^n
\end{aligned} \tag{12}$$

For stability we will demand y_n to approach zero, so:

$$\begin{gathered} 0 < \quad 1-h \quad < 1 \\ \boxed{0 < h < 1} \end{gathered} \tag{13}$$

2.2 B - Backward Differencing

Given the differencing equation:

$$\frac{y_n - y_{n-1}}{h} = -y_n \qquad O\left(h\right) \tag{14}$$

and assuming the solution has the following form:

$$y_n = \alpha\beta^n \tag{15}$$

$$\begin{aligned} &\Downarrow \\ \alpha\beta^n - \alpha\beta^{n-1} &= -h\alpha\beta^n \\ \beta^{n-1} &= \left(1+h\right)\beta^n \\ \beta &= \frac{1}{1+h} \\ &\Downarrow \\ y_n &= \alpha\left(\frac{1}{1+h}\right)^n \\ &\Downarrow \; y_{(0)} = 1 \\ \alpha &= 1 \\ &\Downarrow \\ y_n &= \left(\frac{1}{1+h}\right)^n \end{aligned} \tag{16}$$

For stability we will demand y_n to approach zero, so:

$$\begin{gathered} 0 < \quad \frac{1}{1+h} \quad < 1 \\ 1 \quad < \quad h+1 \\ \boxed{0 < h} \end{gathered} \tag{17}$$

2.3 C - Forward Differencing With Averaging

Given the differencing equation:

$$\frac{y_{n+1} - y_n}{h} = -\frac{1}{2}\left(y_{n+1} + y_n\right) \qquad O\left(h^2\right) \tag{18}$$

and assuming the solution has the following form:

$$y_n = \alpha\beta^n \tag{19}$$

$$\Downarrow$$

$$\begin{aligned}
\frac{\alpha\beta^{n+1} - \alpha\beta^n}{h} &= -\frac{1}{2}\left(\alpha\beta^{n+1} + \alpha\beta^n\right) \\
\left(1 + \frac{h}{2}\right)\beta^{n+1} &= \left(1 - \frac{h}{2}\right)\beta^n \\
\beta &= \frac{1 - \dfrac{h}{2}}{1 + \dfrac{h}{2}} \\
\beta &= \frac{2-h}{2+h} \\
&\Downarrow \\
y_n &= \alpha\left(\frac{2-h}{2+h}\right)^n \\
&\Downarrow \; y_{(0)} = 1 \\
\alpha &= 1 \\
&\Downarrow \\
y_n &= \left(\frac{2-h}{2+h}\right)^n
\end{aligned} \tag{20}$$

For stability we will demand y_n to approach zero, so:

$$\begin{array}{ccc}
0 < & \dfrac{2-h}{2+h} & < 1 \\
0 < & 2-h & < 2+h \\
0 < & h & < 2
\end{array} \tag{21}$$

$$\boxed{0 < h < 2}$$

2.4 D - Central Differencing

Given the differencing equation:

$$\frac{y_{n+1} - y_{n-1}}{2h} = -y_n \qquad O\left(h^2\right) \tag{22}$$

and assuming the solution has the following form:

$$y_n = \alpha\beta^n \tag{23}$$

$$\Downarrow$$

$$
\begin{aligned}
\frac{\alpha\beta^{n+1} - \alpha\beta^{n-1}}{2h} &= -\alpha\beta^n \\
\beta^{n+1} + 2h\beta^n - \beta^{n-1} &= 0 \\
\beta^2 + 2h\beta - 1 &= 0 \\
\beta &= -h \pm \sqrt{h^2+1} \\
&\Downarrow \text{ h must be positive} \\
\beta &= -h + \sqrt{h^2+1} \\
&\Downarrow \\
y_n &= \alpha\left(-h + \sqrt{h^2+1}\right)^n \\
&\Downarrow\ y_{(0)} = 1 \\
\alpha &= 1 \\
&\Downarrow \\
y_n &= \left(-h + \sqrt{h^2+1}\right)^n
\end{aligned}
\tag{24}
$$

For stability we will demand y_n to approach zero, so:

$$
\begin{array}{rcl}
0 < & -h + \sqrt{h^2+1} & < 1 \\
h < & \sqrt{h^2+1} & < 1 + h \\
h^2 < & h^2 + 1 & < 1 + 2h + h^2 \\
0 & < & 2h \\
 & \boxed{0 < h} &
\end{array}
\tag{25}
$$

2.5 Choosing The Best Method

I will choose the *Central Differencing Method* since it is the only unconditional stable method with second order local error.