Numerical Methods in Aeronautical Engineering
HW3 - Theoretical Questions

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1 Q2

The differencing equation is given by:

$$\begin{aligned}
\frac{u_{i,j+1} - u_{i,j}}{kG} + \mathcal{O}\left(\frac{k}{G}\right) &= \alpha \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \mathcal{O}\left(h^2\right) \\
u_{i,j+1} - u_{i,j} &= \frac{\alpha k G}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \mathcal{O}\left(\frac{k}{G}, h^2\right)
\end{aligned} \tag{1}$$

Where:

- $h = \Delta x$
- $k = \Delta y$
- $C = \frac{\alpha k}{h^2}$
- $G = \frac{1}{2C}\left(1 - e^{-2C}\right) = \frac{h^2}{2\alpha k}\left(1 - e^{-2\frac{\alpha k}{h^2}}\right) \leq 1$

Hence the truncation error is of the order of $\mathcal{O}\left(\frac{k}{G}, h^2\right)$.

We will use von-Neumann Stability Analysis. Let's consider the error:

$$e_{r,t} = T_{r,t} - \hat{T}_{r,t} \tag{2}$$

- $T_{r,t}$ - The solution of the differencing equation.
- $\hat{T}_{r,t}$ - The solution of the differencing equation with a small noise in the initial conditions.

Let's define:

$$E_p = T_{p,0} - \hat{T}_{p,0} \tag{3}$$

We can rewrite it as:

$$E_p = \sum_{p=1}^{N} A_p e^{i\beta_n hp}, \qquad i = \sqrt{-1}, \qquad \beta_n = \frac{n\pi}{N+1} \tag{4}$$

We want to check if there is a mode that diverges. Since the equation is linear we only need one mode to diverge to consider the hole scheme as diverged:

$$E_p = A_p e^{i\beta_n hp} \tag{5}$$

We need to check how the error behaves over time and to make sure it diminishes to E_p when $t = 0$. Let's assume the error is of the form of:

$$E_{p,j} = A_p e^{i\beta hp} \cdot e^{\alpha t_j} \tag{6}$$

We are considered stable when $\Re\{\alpha\} < 0$.
We can rewrite the error:

$$E_{p,j} = A_p e^{i\beta hp} \cdot e^{\alpha \Delta t j} = A_p e^{i\beta hp} \cdot \xi^j \tag{7}$$

Therefore the stability condition is $|\xi| \leq 1$

Since the differencing equation is linear we can demand the error to satisfy the differencing equation:

$$E_{p,j+1} = E_{p,j} + \frac{\alpha k G}{h^2}\left(E_{p+1,j} - 2E_{p,j} + E_{p-1,j}\right) \tag{8}$$

After substituting the error we get:

$$e^{i\beta hp} \cdot \xi^{j+1} = e^{i\beta hp} \cdot \xi^{j} + \frac{\alpha k G}{h^2}\left(e^{i\beta h(p+1)} \cdot \xi^{j} - 2e^{i\beta h(p)} \cdot \xi^{j} + e^{i\beta h(p-1)} \cdot \xi^{j}\right) \tag{9}$$

Dividing by $e^{i\beta hp}\xi^{j}$ we get:

$$\begin{aligned} \xi &= 1 + \frac{\alpha k G}{h^2}\left(e^{i\beta h} - 2 + e^{-i\beta h}\right) \\ \xi &= 1 + \frac{2\alpha k G}{h^2}\left(\cos\left(\beta h\right) - 1\right) \\ \xi &= 1 - \frac{4\alpha k G}{h^2}\sin^2\left(\frac{\beta h}{2}\right) \end{aligned} \tag{10}$$

The stability condition is $|\xi| \leq 1$ therefore:

$$\begin{gathered} \left|1 - \frac{4\alpha k G}{h^2}\sin^2\left(\frac{\beta h}{2}\right)\right| \leq 1 \\ -1 \leq 1 - \frac{4\alpha k G}{h^2}\sin^2\left(\frac{\beta h}{2}\right) \leq 1 \end{gathered} \tag{11}$$

The right inequality is always true. Let's check the left inequality:

$$\begin{gathered} -1 \leq 1 - \frac{4\alpha k G}{h^2}\sin^2\left(\frac{\beta h}{2}\right) \\ 2 \geq \frac{4\alpha k G}{h^2}\sin^2\left(\frac{\beta h}{2}\right) \\ \frac{1}{2\sin^2\left(\frac{\beta h}{2}\right)} \geq \frac{\alpha k G}{h^2} \\ \frac{\alpha k G}{h^2} \leq \frac{1}{2} \end{gathered} \tag{12}$$

1.1 A

The stability of the method is determined by:

$$R = \frac{k}{h^2} \leq \frac{1}{2\alpha G} \tag{13}$$

Hence, the stability limitation of the normal differencing equation is:

$$R \leq \frac{1}{2\alpha} \tag{14}$$

and the stability limitation of the new differencing equation is:

$$R \leq \frac{1}{2\alpha G} \geq \frac{1}{2\alpha} \tag{15}$$

So the new differencing equation has a larger range of stable R values. Which means we can use a larger time step and remain stable.

1.2 B

The truncation error is of the order of:

$$\mathcal{O}\left(\frac{k}{G}, h^2\right) \tag{16}$$

Hence the truncation error of the normal differencing equation is:

$$\mathcal{O}\left(k, h^2\right) \tag{17}$$

and the truncation error of the new differencing equation is:

$$\mathcal{O}\left(\frac{k}{G} \geq k, h^2\right) \tag{18}$$

So the new differencing equation has a larger local precision in time than the normal differencing equation.

2 Q3

For the PDE

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \tag{19}$$

The following differencing equation is proposed:

$$u_{i,j+1} = u_{i,j} + R\left(u_{i-1,j} - u_{i,j} - u_{i,j+1} + u_{i+1,j+1}\right) \tag{20}$$

Using von-Neumann Stability Analysis we can write the error as:

$$E_{p,j} = A_p e^{i\beta hp} \cdot e^{\alpha \Delta t j} = A_p e^{i\beta hp} \cdot \xi^j \tag{21}$$

Where:

- $i = \sqrt{-1}$
- p index in space

Therefore the stability condition is $|\xi| \leq 1$

Since the differencing equation is linear we can demand the error to satisfy the differencing equation:

$$E_{p,j+1} = E_{p,j} + R\left(E_{p-1,j} - E_{p,j} - E_{p,j+1} + E_{p+1,j+1}\right) \tag{22}$$

After substituting the error we get:

$$e^{i\beta hp} \cdot \xi^{j+1} = e^{i\beta hp} \cdot \xi^j + R\left(e^{i\beta h(p-1)} \cdot \xi^j - e^{i\beta hp} \cdot \xi^j - e^{i\beta hp} \cdot \xi^{j+1} + e^{i\beta h(p+1)} \cdot \xi^{j+1}\right) \tag{23}$$

Dividing by $e^{i\beta hp}\xi^j$ we get:

$$\begin{aligned}
\xi &= 1 + R\left(e^{-i\beta h} - 1 - \xi + e^{i\beta h} \cdot \xi\right) \\
\left(1 + R - Re^{i\beta h}\right)\xi &= 1 + R\left(e^{-i\beta h} - 1\right) \\
\left(1 + R - Re^{i\beta h}\right)\xi &= 1 - R + Re^{-i\beta h} \\
\xi &= \frac{1 - R + Re^{-i\beta h}}{1 + R - Re^{i\beta h}}
\end{aligned} \tag{24}$$

The stability condition is $|\xi| \leq 1$ therefore:

$$\left|\frac{1-R+Re^{-i\beta h}}{1+R-Re^{i\beta h}}\right| \leq 1$$

$$\left|\frac{1-R+R\left(\cos\left(\beta h\right)-i\sin\left(\beta h\right)\right)}{1+R-R\left(\cos\left(\beta h\right)+i\sin\left(\beta h\right)\right)}\right| \leq 1$$

$$\left|\frac{1-R+R\cos\left(\beta h\right)-iR\sin\left(\beta h\right)}{1+R-R\cos\left(\beta h\right)+iR\sin\left(\beta h\right)}\right| \leq 1$$

$$\sqrt{\frac{\left(\frac{1}{R}-1+\cos\left(\beta h\right)\right)^2+\sin^2\left(\beta h\right)}{\left(\frac{1}{R}+1-\cos\left(\beta h\right)\right)^2+\sin^2\left(\beta h\right)}} \leq 1$$

$$\frac{\left(\frac{1}{R}-1+\cos\left(\beta h\right)\right)^2+\sin^2\left(\beta h\right)}{\left(\frac{1}{R}+1-\cos\left(\beta h\right)\right)^2+\sin^2\left(\beta h\right)} \leq 1 \tag{25}$$

$$\left(\frac{1}{R}-1+\cos\left(\beta h\right)\right)^2+\sin^2\left(\beta h\right) \leq \left(\frac{1}{R}+1-\cos\left(\beta h\right)\right)^2+\sin^2\left(\beta h\right)$$

$$\left(\frac{1}{R}-1+\cos\left(\beta h\right)\right)^2 \leq \left(\frac{1}{R}+1-\cos\left(\beta h\right)\right)^2$$

$$\frac{1}{R}-1+\cos\left(\beta h\right) \leq \frac{1}{R}+1-\cos\left(\beta h\right)$$

$$2\cos\left(\beta h\right) \leq 2$$

$$\cos\left(\beta h\right) \leq 1$$

Since $\beta h \geq 0$ the condition is always meat. So the scheme is unconditionally stable.

3 Q4

The following PDE

$$\frac{\partial U}{\partial t} = \alpha\frac{\partial^2 U}{\partial x^2} \tag{26}$$

is solve using the Crank-Nicolson method:

$$u_{i,j+1} = \frac{R}{2}\left(u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}\right)+\underbrace{u_{i,j}+\frac{R}{2}\left(u_{i-1,j}-2u_{i,j}+u_{i+1,j}\right)}_{b_{i,j}} \tag{27}$$

Where:

- $R = \dfrac{\alpha k}{h^2}$

The boundary conditions are:

$$\left.\frac{\partial U}{\partial x}\right|_{0,t} = 0 \;\Big\|\; \left.\frac{\partial U}{\partial x}\right|_{1,t} = 0 \;\Big\|\; U_{(0,x)} = U_{0(x)}$$

In order to solve the system of equations, the following iterative scheme will be in use:

$$u_{i,j+1}^{(n+1)} = \frac{R}{2}\left(u_{i-1,j+1}^{(n)} - 2u_{i,j+1}^{(n)} + u_{i+1,j+1}^{(n)}\right) + b_{i,j} \tag{28}$$

The system of equation can be rewritten as:

$$\vec{u}_{j+1}^{(n+1)} = \underline{\underline{A}}\vec{u}_{j+1}^{(n)} + \vec{b}_j \tag{29}$$

Where:

$$\underline{\underline{A}} = \begin{pmatrix} -R & \frac{R}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{R}{2} & -R & \frac{R}{2} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \frac{R}{2} & -R & \frac{R}{2} & 0 & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{R}{2} & -R & \frac{R}{2} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{R}{2} & -R \end{pmatrix} \tag{30}$$

Let's expand the scheme:

$$\begin{aligned} \vec{u}_{j+1}^{(n+1)} &= \underline{\underline{A}}\vec{u}_{j+1}^{(n)} + \vec{b}_j \\ &= \underline{\underline{A}}\left(\underline{\underline{A}}\vec{u}_{j+1}^{(n-1)} + \vec{b}_j\right) + \vec{b}_j = \underline{\underline{A}}^2\vec{u}_{j+1}^{(n-1)} + \underline{\underline{A}}\vec{b}_j + \vec{b}_j \\ &= \underline{\underline{A}}^3\vec{u}_{j+1}^{(n-2)} + \underline{\underline{A}}^2\vec{b}_j + \underline{\underline{A}}\vec{b}_j + \vec{b}_j \\ &\vdots \\ &= \underline{\underline{A}}^{n+1}\vec{u}_{j+1}^{(0)} + \underbrace{\underline{\underline{A}}^n\vec{b}_j + \cdots + \vec{b}_j}_{\vec{B}_j} \\ &= \underline{\underline{A}}^{n+1}\vec{u}_{j+1}^{(0)} + \vec{B}_j \end{aligned} \tag{31}$$

Assuming there is 'numerical noise' in the initial conditions, then we get:

$$\vec{u^*}_{j+1}^{(n+1)} = \underline{\underline{A}}^{n+1}\vec{u^*}_{j+1}^{(0)} + \vec{B}_j \tag{32}$$

The error is defined as:

$$\begin{gathered} \vec{e}_j = \vec{u}_j - \vec{u^*}_j \\ \Downarrow \\ \vec{e}_{j+1} = \vec{u}_{j+1} - \vec{u^*}_{j+1} = \cdots = \underline{\underline{A}}^{j+1}\vec{e}_0 \end{gathered} \tag{33}$$

For stability, we will demand that the error won't grow with time. We will assume that $\underline{\underline{A}}$ has all different eigenvalues and eigenvectors. Hence, each error vector can be written as a linear combination of the eigenvectors:

$$\vec{e}_j = \sum_{s=1}^{N} c_s \lambda_s^j \vec{v}_s \tag{34}$$

For stability we will demand:

$$\max_{1\leq s\leq N} |\lambda_s| \leq 1 \tag{35}$$

In our case, $\underline{\underline{A}}$ is a 'cab' matrix where:

- $c = \frac{R}{2}$
- $a = -R$
- $b = \frac{R}{2}$

In case of a 'cab' matrix, the eigenvalues are given by:

$$\begin{gathered}\lambda_s = a + 2b\sqrt{\frac{c}{b}}\cos\left(\frac{s\pi}{N+1}\right) = -R + R\cos\left(\frac{s\pi}{N+1}\right)\\ \Downarrow |\lambda_s| \leq 1\\ -1 \leq R\left(\cos\left(\frac{s\pi}{N+1}\right) - 1\right) \leq 1\\ -1 \leq R\left(-2\sin^2\left(\frac{s\pi}{2N+2}\right)\right) \leq 1\end{gathered} \tag{36}$$

The right inequality is always true. Let's check the left inequality:

$$\begin{gathered}-1 \leq -2R\sin^2\left(\frac{s\pi}{2N+2}\right)\\ \frac{1}{2\sin^2\left(\frac{s\pi}{2N+2}\right)} \geq R\\ R \leq \frac{1}{2}\\ \boxed{P = \frac{1}{2}}\end{gathered} \tag{37}$$