Satellite Orbit Control
HW3

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1 Given

$$T_1 = 100\,[min] = 6 \cdot 10^3\,[sec] \qquad T_2 = T_1 = 6 \cdot 10^3\,[sec]$$
$$e_1 = 0 \qquad e_2 = 0$$
$$a_1 = \sqrt[3]{\frac{\mu T_1^2}{4\pi^2}} = 7.1366 \cdot 10^3\,[km] \qquad a_2 = a_1 = 7.1366 \cdot 10^3\,[km]$$
$$\alpha = \Delta i = 0.01^\circ$$

In CW frame with origin at Satellite #1 and at $t = 0$:

$$\begin{pmatrix} x_2(0) = 0 \\ y_2(0) = -1 \\ z_2(0) = 1 \end{pmatrix} [\mathrm{km}] \qquad \begin{pmatrix} \dot{x}_2(0) = 0 \\ \dot{y}_2(0) = 0 \\ \dot{z}_2(0) = -0.74267 \cdot n \end{pmatrix} \left[\frac{\mathrm{km}}{\mathrm{sec}}\right]$$

1.1 Desired

$$\begin{pmatrix} x_2(t_f) = 0 \\ y_2(t_f) = 0 \\ z_2(t_f) = 0 \end{pmatrix} \qquad \begin{pmatrix} \dot{x}_2(t_f) = 0 \\ \dot{y}_2(t_f) = 0 \\ \dot{z}_2(t_f) = 0 \end{pmatrix}$$

1.2 Limitations

$$a_{\max} = 8 \cdot 10^{-6} \left[\frac{\mathrm{km}}{\mathrm{sec}^2}\right]$$

2 The CW equations

$$\begin{cases} \ddot{x} - 2n\dot{y} - 3n^2 x = f_x \\ \ddot{y} + 2n\dot{x} = f_y \\ \ddot{z} + n^2 z = f_z \end{cases} \tag{1}$$

2.1 x-y

$$\vec{x} = \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} \tag{2}$$

$$\dot{\vec{x}} = F\vec{x} + G\vec{f} \tag{3}$$

Where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n \\ 0 & 0 & 0 & 1 \\ 0 & -2n & 0 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \tag{4}$$

2.2 z

$$\vec{x} = \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \tag{5}$$

$$\dot{\vec{x}} = F\vec{x} + Gf \tag{6}$$

Where:

$$F = \begin{pmatrix} 0 & 1 \\ -n^2 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad f = (f_z) \tag{7}$$

2.3 x-y-z

The equations of motion in state space form are therefor:

$$\vec{x} = \begin{pmatrix} x & \dot{x} & y & \dot{y} & z & \dot{z} \end{pmatrix}^T \tag{8}$$

$$\dot{\vec{x}} = F\vec{x} + G\vec{f} \tag{9}$$

Where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -n^2 & 0 \end{pmatrix} \quad G = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \tag{10}$$

2.4 The homogeneous solution

$$\vec{x} = \begin{pmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{pmatrix}^T \tag{11}$$

x-y plane:

$$\Phi_{\{t,t_0\}} = \left(\begin{array}{cc|cc} 4-3\cos(n\tau) & 0 & \frac{1}{n}\sin(n\tau) & \frac{2}{n}\left(1-\cos(n\tau)\right) \\ 6\left(\sin(n\tau)-n\tau\right) & 1 & \frac{2}{n}\left(\cos(n\tau)-1\right) & \frac{1}{n}\left(4\sin(n\tau)-3n\tau\right) \\ \hline 3n\sin(n\tau) & 0 & \cos(n\tau) & 2\sin(n\tau) \\ 6n\left(\cos(n\tau)-1\right) & 0 & -2\sin(n\tau) & 4\cos(n\tau)-3 \end{array}\right) = \begin{pmatrix} \Phi_{11(t,t_0)} & \Phi_{12(t,t_0)} \\ \Phi_{21(t,t_0)} & \Phi_{22(t,t_0)} \end{pmatrix}$$

z-direction:

$$\Phi_{(t,t_0)} = \begin{pmatrix} \cos(n\tau) & \frac{1}{n}\sin(n\tau) \\ -n\sin(n\tau) & \cos(n\tau) \end{pmatrix} \tag{12}$$

Where:

$$\tau = t - t_0$$

So the full homogeneous solution in state space from:

$$\Phi_{(t,t_0)} = \left(\begin{array}{ccc|ccc} 4-3\cos(n\tau) & 0 & 0 & \frac{1}{n}\sin(n\tau) & \frac{2}{n}\left(1-\cos(n\tau)\right) & 0 \\ 6\left(\sin(n\tau)-n\tau\right) & 1 & 0 & \frac{2}{n}\left(\cos(n\tau)-1\right) & \frac{1}{n}\left(4\sin(n\tau)-3n\tau\right) & 0 \\ 0 & 0 & \cos(n\tau) & 0 & 0 & \frac{1}{n}\sin(n\tau) \\ \hline 3n\sin(n\tau) & 0 & 0 & \cos(n\tau) & 2\sin(n\tau) & 0 \\ 6n\left(\cos(n\tau)-1\right) & 0 & 0 & -2\sin(n\tau) & 4\cos(n\tau)-3 & 0 \\ 0 & 0 & -n\sin(n\tau) & 0 & 0 & \cos(n\tau) \end{array}\right) =$$

$$\begin{pmatrix} \Phi_{11(t,t_0)} & \Phi_{12(t,t_0)} \\ \Phi_{21(t,t_0)} & \Phi_{22(t,t_0)} \end{pmatrix} \tag{13}$$

3 case A

The desired maneuver time is 2000 [sec]. The desired approach trajectory is a straight line with constant velocity from the initial point to the origin.
From geometric considerations, the required velocity on the straight line is:

$$\dot{\vec{x}}_{req} = \begin{pmatrix} \dfrac{x_f - x_0}{t_f} \\ \\ \dfrac{y_f - y_0}{t_f} \\ \\ \dfrac{z_f - z_0}{t_f} \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix} \cdot 10^{-4} \left[\frac{\text{km}}{\text{sec}^2}\right] \tag{14}$$

Which mean that the initial pulse:

$$\Delta v_1 = \dot{\vec{x}}_{req} - \dot{\vec{x}}_0 = \begin{pmatrix} 0 \\ 0.5 \\ 0.2777 \end{pmatrix} \cdot 10^{-3} \left[\frac{\text{km}}{\text{sec}^2}\right] \tag{15}$$

and the final pulse:

$$\Delta v_f = \dot{\vec{x}}_f - \dot{\vec{x}}_{req} = \begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix} \cdot 10^{-4} \left[\frac{\text{km}}{\text{sec}^2}\right] \tag{16}$$

Since the velocity is constant:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \vec{0}$$

By substituting the constraints in the CW equations we get:

$$\begin{cases} 0 - 2n\dot{y} = f_x \\ 0 = f_y \\ 0 + n^2 z = f_z \end{cases} \tag{17}$$

$$\vec{a} = \vec{f} = \begin{pmatrix} -2n \cdot 0.5 \cdot 10^{-3} \\ 0 \\ n^2 z \end{pmatrix} \tag{18}$$

Check that the maximum acceleration hasn't been reached:

$$\begin{gathered} |\vec{a}| = \sqrt{(2n \cdot 0.5 \cdot 10^{-3}) + (n^2 z)} < \sqrt{(2n \cdot 0.5 \cdot 10^{-3}) + (n^2 \cdot 1)} \\ |\vec{a}| < 1.5163 \cdot 10^{-6} < 8 \cdot 10^{-6} \qquad \surd \end{gathered} \tag{19}$$

By solving this system numerically, we obtain the approach trajectory.

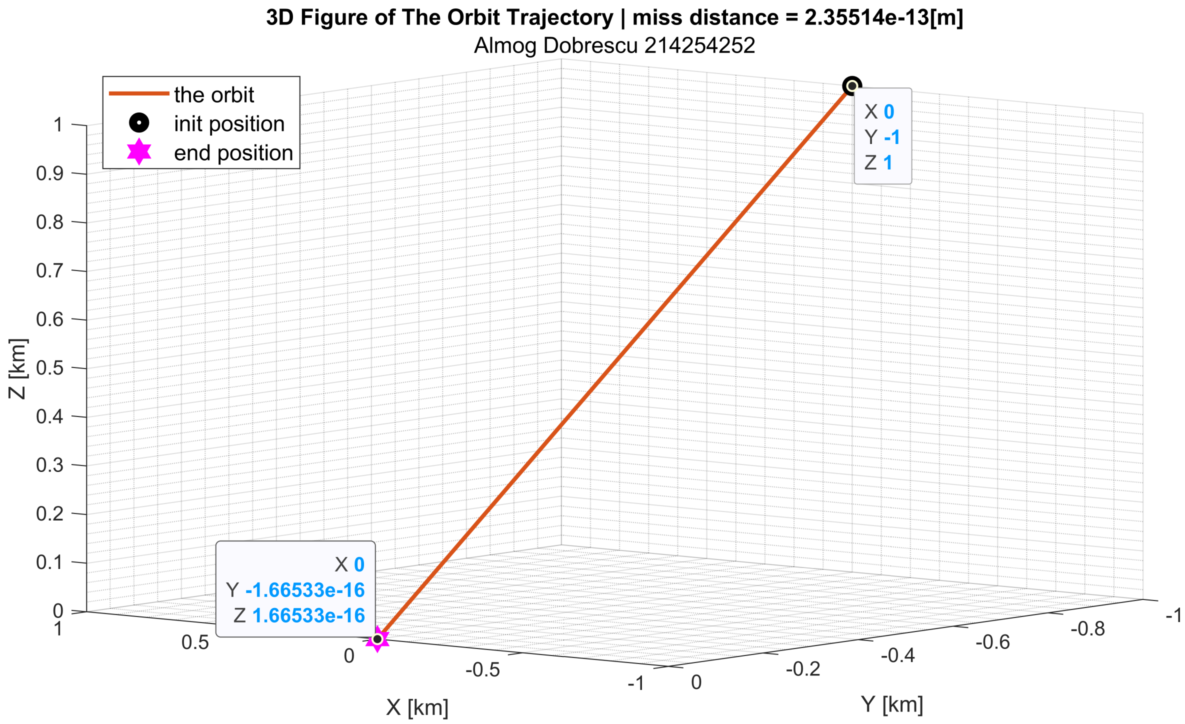

Figure 1: 3D figure of the orbit trajectory - case A

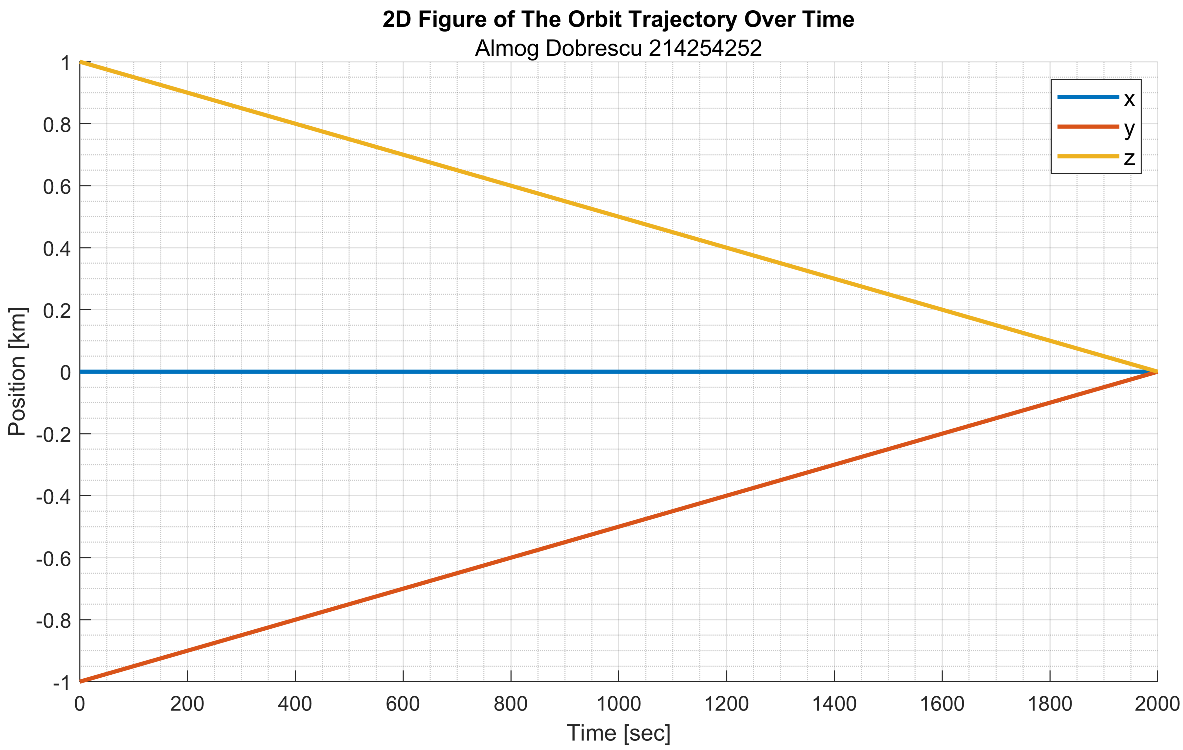

Figure 2: 2D figure of the position over time - case A

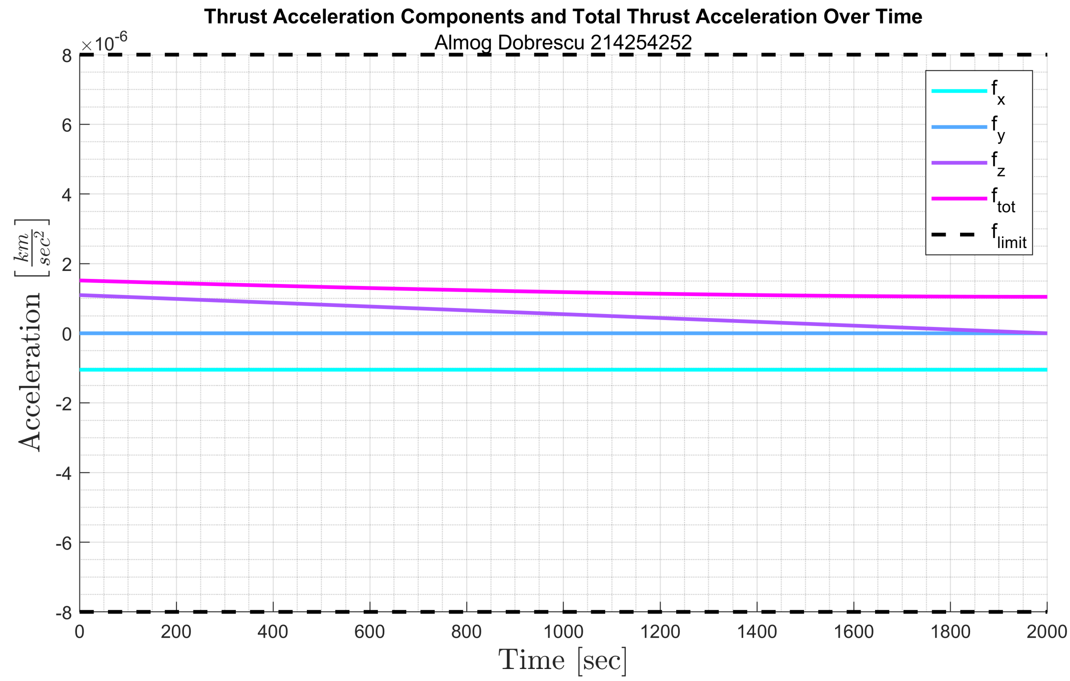

Figure 3: Thrust acceleration components and total thrust acceleration over time - case A

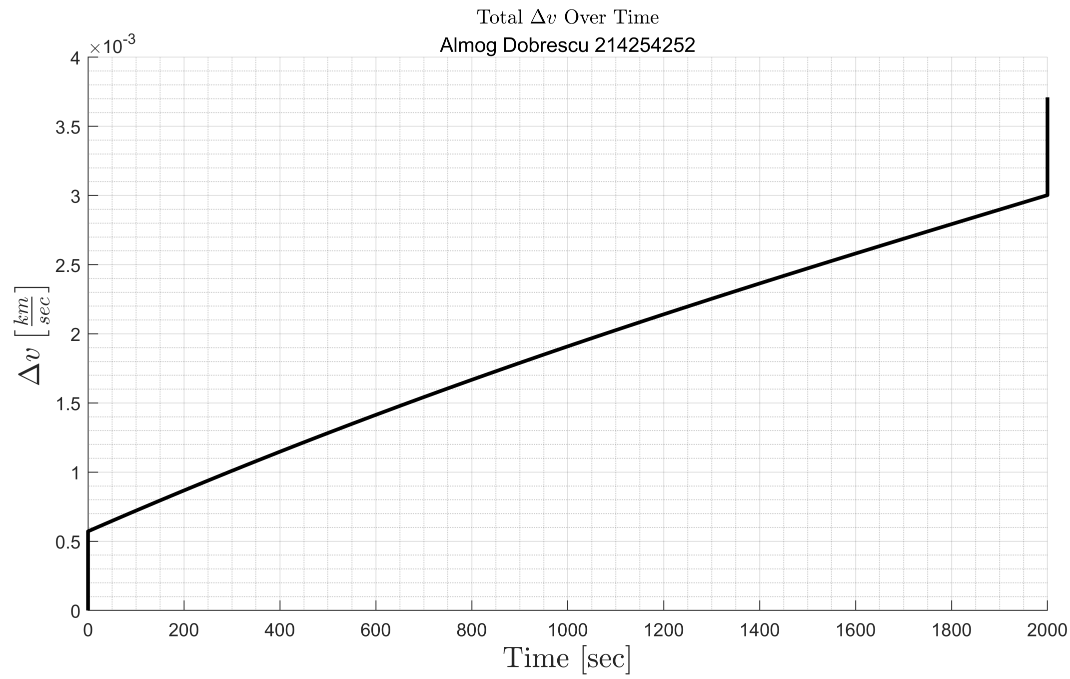

Figure 4: Total Δv over time - case A

We can see that we indeed acomplished the design criteria:

- The thrust doesn't exceed the maximum available thrust.
- The miss distance at the final desired time is less than 1[m] $\left(2.35517 \cdot 10^{-13}\,[\text{m}]\right)$.

4 case B

The velocity to be gained:

$$\vec{v}_g = \vec{v}_r - \vec{v} \tag{20}$$

The equation of motion:

$$\frac{d\vec{v}}{dt} = \vec{g}_{(\vec{r})} + \vec{a}_T \tag{21}$$

The equation of motion for the velocity to be gained:

$$\frac{d\vec{v}_g}{dt} = \frac{d\vec{v}_r}{dt} - \frac{d\vec{v}}{dt} = \frac{d\vec{v}_r}{dt} - \vec{g}_{(\vec{r})} - \vec{a}_T \tag{22}$$

$\vec{v}_r$ is a function of time and space, so:

$$\frac{d\vec{v}_{r(t,\vec{r})}}{dt} = \frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}} \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}} = \frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}} (\vec{v}_r - \vec{v}_g)$$

The required velocity $\vec{v}_r$ also fulfill the equation of motion with no thrust:

$$\frac{d\vec{v}_r}{dt} = \frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}} \vec{v}_r = \vec{g}_r$$

so:

$$\frac{d\vec{v}_r}{dt} = \vec{g}_r - \frac{\partial \vec{v}_r}{\partial \vec{r}} \vec{v}_g \tag{23}$$

The equation of motion becomes:

$$\frac{d\vec{v}_g}{dt} = \vec{g}_r - \frac{\partial \vec{v}_r}{\partial \vec{r}} \vec{v}_g - \vec{g}_{(\vec{r})} - \vec{a}_T = -\underbrace{\frac{\partial \vec{v}_r}{\partial \vec{r}}}_{C^*} \vec{v}_g - \vec{a}_T \tag{24}$$

where C^* is:

$$\begin{pmatrix} x_f \\ y_f \\ z_f \end{pmatrix} = \vec{0} = \Phi_{11(t_f,t)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \Phi_{12(t_f,t)} \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \\ \dot{z}_{req} \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \\ \dot{z}_{req} \end{pmatrix} = -\Phi^{-1}_{12(t_f,t)} \Phi_{11(t_f,t)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = C^*_{(t_f,t)} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Downarrow$$

$$C^*_{(t_f,t)} = -\Phi^{-1}_{12(t_f,t)} \Phi_{11(t_f,t)}$$

The Cross-Product fixed thrust control law:

$$\vec{v}_g \parallel \frac{d\vec{v}_g}{dt} \tag{25}$$

$$\Downarrow$$

$$\frac{d\vec{v}_g}{dt} \times \vec{v}_g = 0$$

$$(-C^* \vec{v}_g - \vec{a}_T) \times \vec{v}_g = 0 \tag{26}$$

$$\vec{a}_T \times \vec{v}_g = \underbrace{-\left(C^* \vec{v}_g\right)}_{\vec{p}} \times \vec{v}_g \tag{27}$$

Multypling on the left by $\vec{v}_g$:

$$\left(\vec{a}_T \times \vec{v}_g\right) \times \vec{v}_g = \left(\vec{p} \times \vec{v}_g\right) \times \vec{v}_g \tag{28}$$

Expand according to:

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = \left(\vec{a} \cdot \vec{c}\right) \vec{b} - \left(\vec{b} \cdot \vec{c}\right) \vec{a}$$

$$\left(\vec{a}_T \cdot \vec{v}_g\right) \vec{v}_g - v_g^2 \vec{a}_T = \left(\vec{p} \cdot \vec{v}_g\right) \vec{v}_g - v_g^2 \vec{p} \tag{29}$$

$$\Downarrow$$

$$\vec{a}_T = \vec{p} + \frac{1}{v_g^2} \left(\left(\vec{a}_T \cdot \vec{v}_g\right) - \left(\vec{p} \cdot \vec{v}_g\right)\right) \vec{v}_g \tag{30}$$

Defining $q = \vec{a}_T \cdot \hat{v}_g$ as the projection of the acceleration vector on the velocity to be gained, we get:

$$\vec{a}_T = \vec{p} + \left(q - \left(\vec{p} \cdot \hat{v}_g\right)\right) \hat{v}_g \tag{31}$$

The terminal braking will therefor be:

$$\Delta v_f = \vec{0} - \begin{pmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{z}_f \end{pmatrix} = -\left(\Phi_{21(t_f,0)} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \Phi_{22(t_f,0)} \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \\ \dot{z}_{req} \end{pmatrix}\right) \tag{32}$$

To calculate the components of the thrust vector, we need to make sure that the solution exists and it is stable. according to Lyapunov:

$$\underbrace{U}_{\text{Lyapunov function}} = \tfrac{1}{2} \vec{v}_g^T \vec{v}_g \qquad U_{(0)} = 0 \quad U_{\vec{v}_g} > 0 \tag{33}$$

For stabllity:

$$\dot{U} = \vec{v}_g^T \left(-C^* \vec{v}_g - \vec{a}_T\right) = -\vec{v}_g^T \left(\left(\vec{a}_T \cdot \hat{v}_g\right) - \left(\vec{p} \cdot \hat{v}_g\right)\right) \hat{v}_g < 0 \tag{34}$$

we need the expression in the parentheses to be positive. Let's assume that $|\vec{a}_T| = a_{max}$ and after squaring both sides we get:

$$a_T^2 = p^2 + 2\left(q - \vec{p} \cdot \hat{v}_g\right)\left(\vec{p} \cdot \hat{v}_g\right) + \left(q - \vec{p} \cdot \hat{v}_g\right)^2 \tag{35}$$

$$a_T^2 = p^2 + q^2 - \left(\vec{p} \cdot \hat{v}_g\right)^2 \tag{36}$$

$$q = \left(a_{max}^2 - p^2 + \left(\vec{p} \cdot \hat{v}_g\right)^2\right)^{\frac{1}{2}} \tag{37}$$

To sum up:

$$\vec{a}_T = \vec{p} + \left(q - \left(\vec{p} \cdot \hat{v}_g\right)\right) \hat{v}_g$$

where:

$$\vec{p} = -C^* \vec{v}_g$$
$$q = \left(a_{max}^2 - p^2 + \left(\vec{p} \cdot \hat{v}_g\right)^2\right)^{\frac{1}{2}}$$

By solving this system numerically, we obtain the approach trajectory.

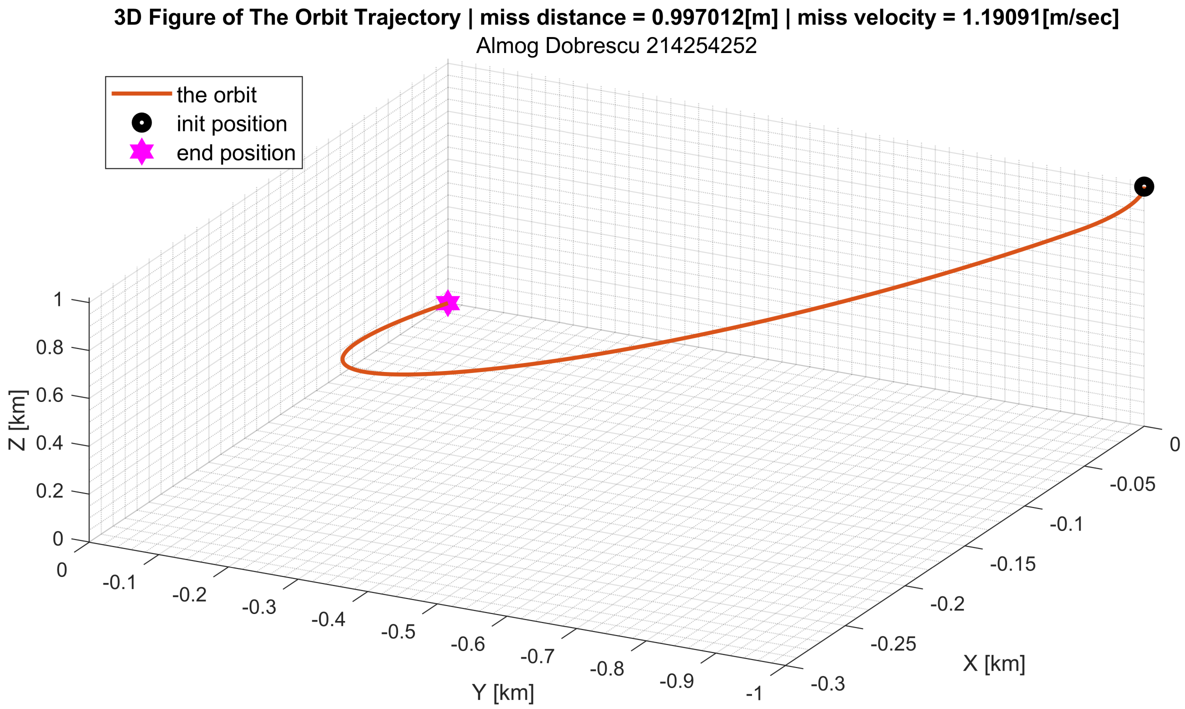

Figure 5: 3D figure of the orbit trajectory - case B

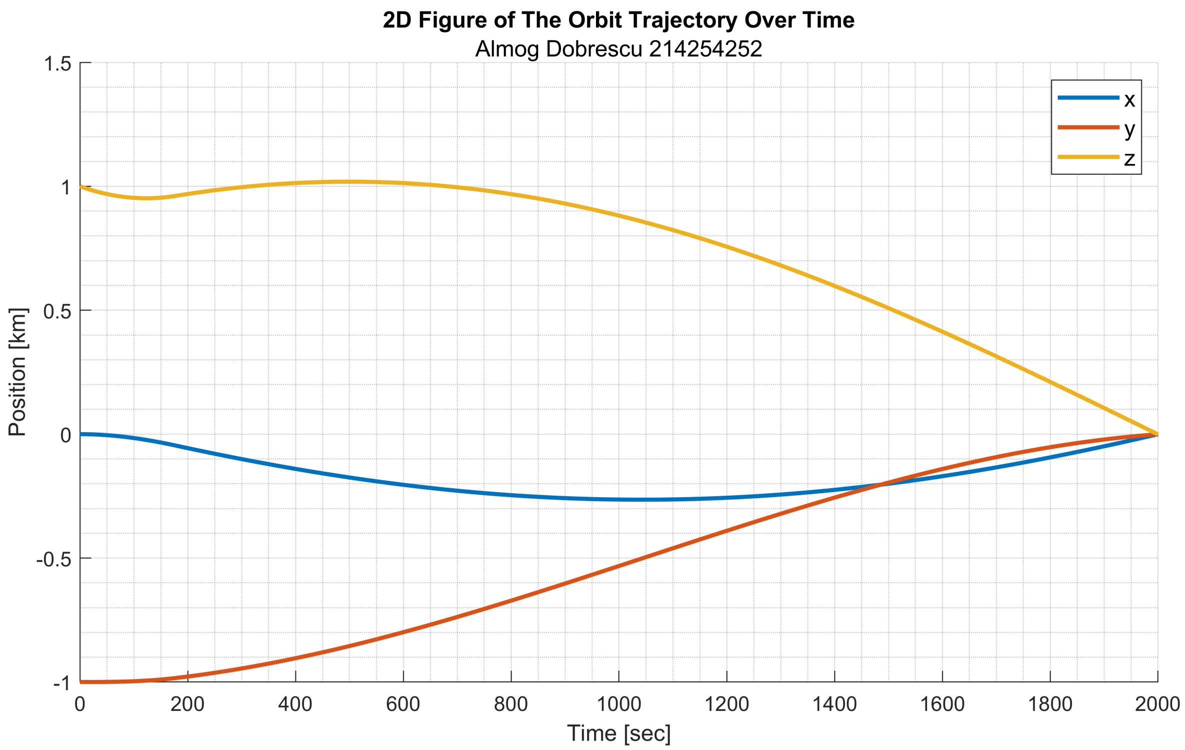

Figure 6: 2D figure of the position over time - case B

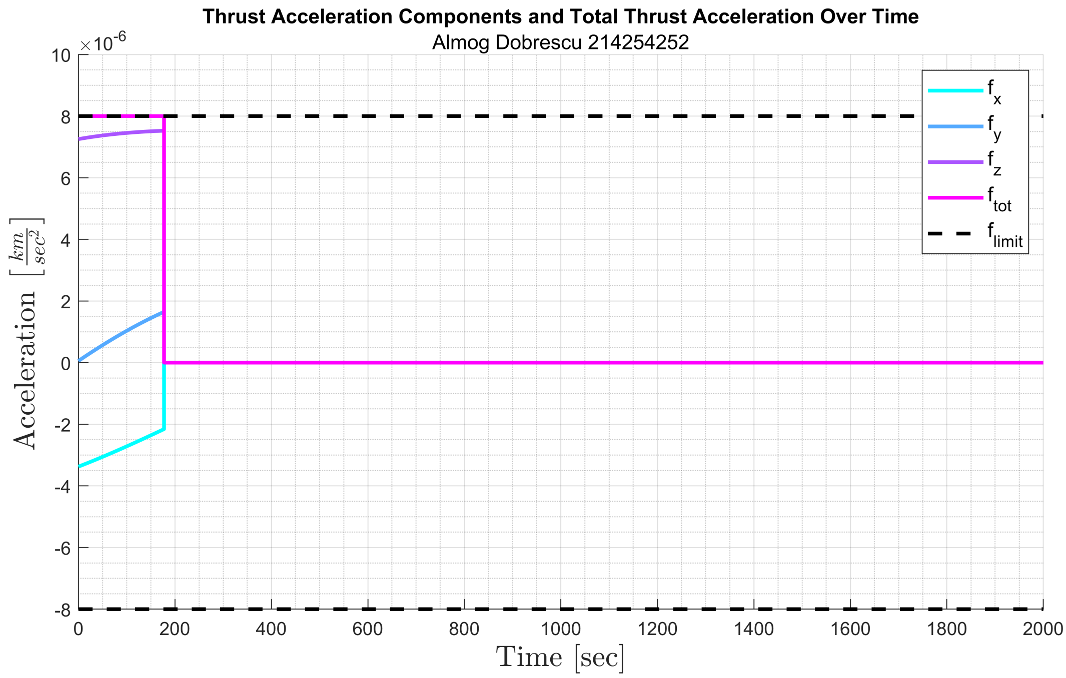

Figure 7: Thrust acceleration components and total thrust acceleration over time - case B

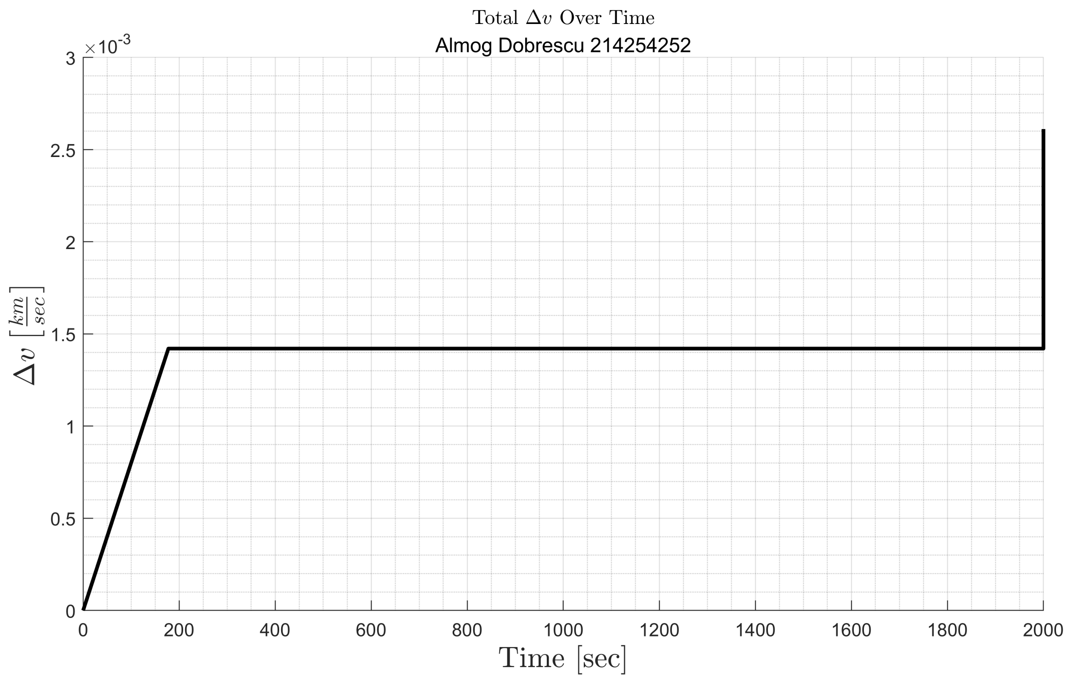

Figure 8: Total Δv over time - case B

We can see that we indeed acomplished the design criteria:

- The thrust doesn't exceed the maximum available thrust.
- The miss distance at the final desired time is less than 1[m] (0.997012 [m]).
- The terminal velocity is 1.19091 $\left[\frac{\text{m}}{\text{sec}}\right]$