Satellite Orbit Control
HW7

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1 Given

$$T_1 = 100\,[min] = 6 \cdot 10^3\,[sec] \qquad T_2 = T_1 = 6 \cdot 10^3\,[sec]$$
$$e_1 = 0 \qquad e_2 = 0$$
$$a_1 = \sqrt[3]{\frac{\mu T_1^2}{4\pi^2}} = 7.1366 \cdot 10^3\,[km] \qquad a_2 = a_1 = 7.1366 \cdot 10^3\,[km]$$
$$\alpha = \Delta i = 0.01^\circ$$

In CW frame with origin at Satellite #1 and at $t = 0$:

$$\begin{pmatrix} x_2(0) = 0 \\ y_2(0) = -1 \\ z_2(0) = 1 \end{pmatrix} [\mathrm{km}] \qquad \begin{pmatrix} \dot{x}_2(0) = 0 \\ \dot{y}_2(0) = 0 \\ \dot{z}_2(0) = -0.74267 \cdot n \end{pmatrix} \left[\frac{\mathrm{km}}{\mathrm{sec}}\right]$$

1.1 Desired

$$\begin{pmatrix} x_2(t_f) = 0 \\ y_2(t_f) = 0 \\ z_2(t_f) = 0 \end{pmatrix} \qquad \begin{pmatrix} \dot{x}_2(t_f) = 0 \\ \dot{y}_2(t_f) = 0 \\ \dot{z}_2(t_f) = 0 \end{pmatrix}$$

1.2 Limitations

$$a_{\max} = 4 \cdot 10^{-5} \left[\frac{\mathrm{km}}{\mathrm{sec}^2}\right] \qquad t_f = 2000\,[sec]$$

2 The CW equations

$$\begin{cases} \ddot{x} - 2n\dot{y} - 3n^2 x = f_x \\ \ddot{y} + 2n\dot{x} = f_y \\ \ddot{z} + n^2 z = f_z \end{cases} \tag{1}$$

2.1 x-y

$$\vec{x} = \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} \tag{2}$$

$$\dot{\vec{x}} = F\vec{x} + G\vec{f} \tag{3}$$

Where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n \\ 0 & 0 & 0 & 1 \\ 0 & -2n & 0 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \tag{4}$$

2.2 z

$$\vec{x} = \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \tag{5}$$

$$\dot{\vec{x}} = F\vec{x} + Gf \tag{6}$$

Where:

$$F = \begin{pmatrix} 0 & 1 \\ -n^2 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad f = \begin{pmatrix} f_z \end{pmatrix} \tag{7}$$

2.3 x-y-z

The equations of motion in state space form are therefor:

$$\vec{x} = \begin{pmatrix} x & \dot{x} & y & \dot{y} & z & \dot{z} \end{pmatrix}^T \tag{8}$$

$$\dot{\vec{x}} = F\vec{x} + G\vec{f} \tag{9}$$

Where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -n^2 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \tag{10}$$

3 Optimal Linear Time-Varying Gains

It is needed to fulfill the end condition:

$$\Psi\left(\vec{x}_{(t_f)}\right) = \underbrace{\Psi\vec{x}_f}_{\text{q x n}} \tag{11}$$

while minimazing the use of fuel:

$$J = \frac{1}{2}\int_0^{t_f} \vec{f}^{\,T}\vec{f}dt \tag{12}$$

We will use the Pontriagin optimization method:

$$J = \vec{\nu}^T\Psi\vec{x}_f + \frac{1}{2}\int_0^{t_f} \vec{f}^{\,T}\vec{f}dt \tag{13}$$

The Hemiltonian:

$$H = \frac{1}{2}\vec{f}^{\,T}\vec{f} + \vec{\lambda}^T\left(F\vec{x} + G\vec{f}\right) \qquad \text{where} \qquad \vec{\lambda} = \begin{pmatrix} \lambda_x \\ \lambda_{\dot{x}} \\ \lambda_y \\ \lambda_{\dot{y}} \end{pmatrix} \tag{14}$$

The optimization criteria:

$$\frac{\partial H}{\partial \vec{f}} = \vec{0} \qquad \rightarrow \qquad \vec{f} = -G^T\vec{\lambda} \tag{15}$$

and the Euler Lagrange equations:

$$\dot{\vec{\lambda}}^{\,T} = -\frac{\partial Hh}{\partial \vec{x}} = -\vec{\lambda}^{\,T}F \qquad \rightarrow \qquad \dot{\vec{\lambda}} = -F^T\vec{\lambda}, \quad \vec{\lambda}_f^{\,T} = \frac{\partial J}{\partial \vec{x}_f} = \vec{\nu}^{\,T}\Psi \tag{16}$$

Since the current problem is linear with square preformance criteria, the Euler Lagrange equation becomes linear differential equations independent of the state vector. Hence:

$$\begin{gathered} \vec{\lambda}_{(t)} = e^{-F^Tt}\vec{\lambda}_0 \\ \downarrow \ \vec{\lambda}_f \\ \vec{\lambda}_{(t)} = e^{-F^T(t-t_f)}\Psi^T\vec{\nu} \end{gathered} \tag{17}$$

The equation of motion is therefore:

$$\dot{\vec{x}} = F\vec{x} - GG^T e^{-F^T\left(t-t_f\right)}\Psi^T\vec{\nu} \tag{18}$$

The solution of the equation is:

$$\vec{x}_{(t)} = e^{Ft}\vec{x}_0 - \int_0^t \left(e^{F(t-\tau)}GG^T e^{-F^T\tau} d\tau\right) e^{F^T t_f}\Psi^T\vec{\nu} \tag{19}$$

and can be writen like:

$$\vec{x}_{(t)} = e^{Ft}\vec{x}_0 + M_{(t)}e^{F^T t_f}\Psi^T\vec{\nu} \qquad M_{(t)} = \int_0^t \left(e^{F(t-\tau)}GG^T e^{-F^T\tau} d\tau\right) \tag{20}$$

The vector $\vec{\nu}$ can be found from the known final conditions:

$$\begin{gathered} \Psi\vec{x}_{\left(t_f\right)} = 0 = \Psi\left(e^{Ft}\vec{x}_0 + M_{(t)}e^{F^T t_f}\Psi^T\vec{\nu}\right) \\ \Downarrow \\ \vec{\nu} = -\left(\Psi M_{\left(t_f\right)}e^{F^T t_f}\Psi^T\right)^{-1}\Psi e^{Ft_f}\vec{x}_0 \end{gathered} \tag{21}$$

4 The Results

In our case (rendezvous), $\Psi = I$.

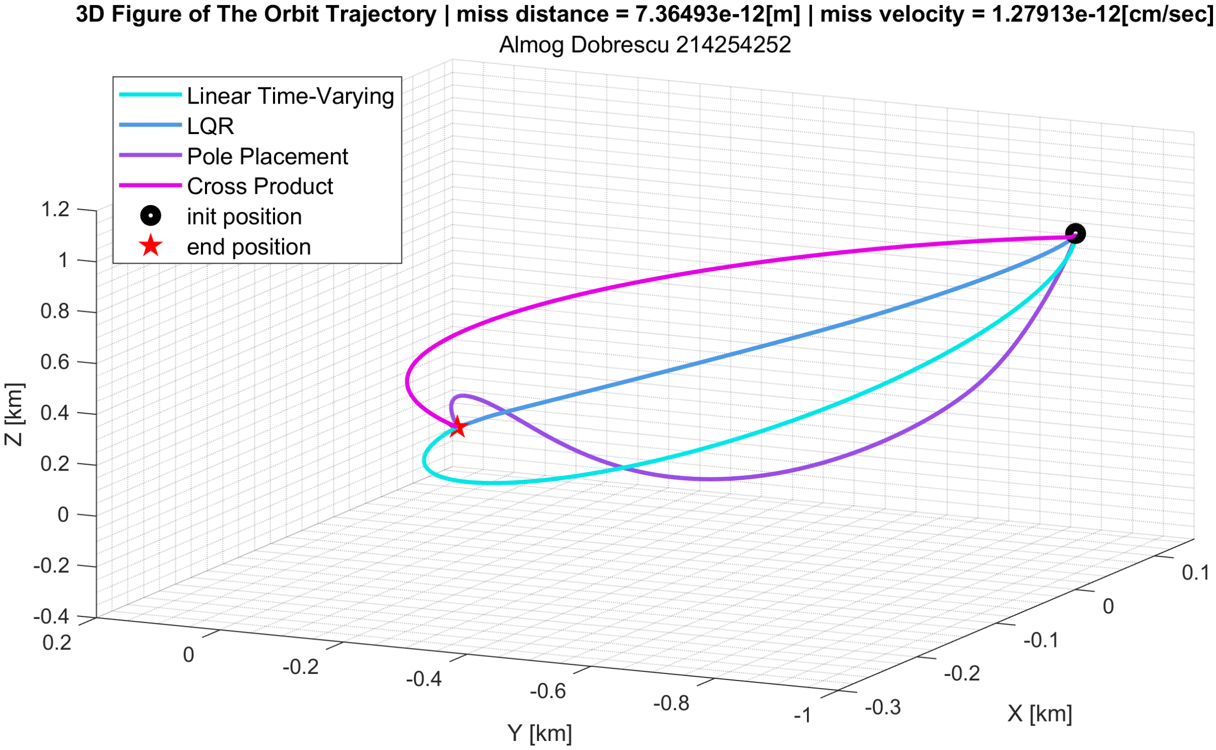

Figure 1: 3D figure of the orbit trajectory

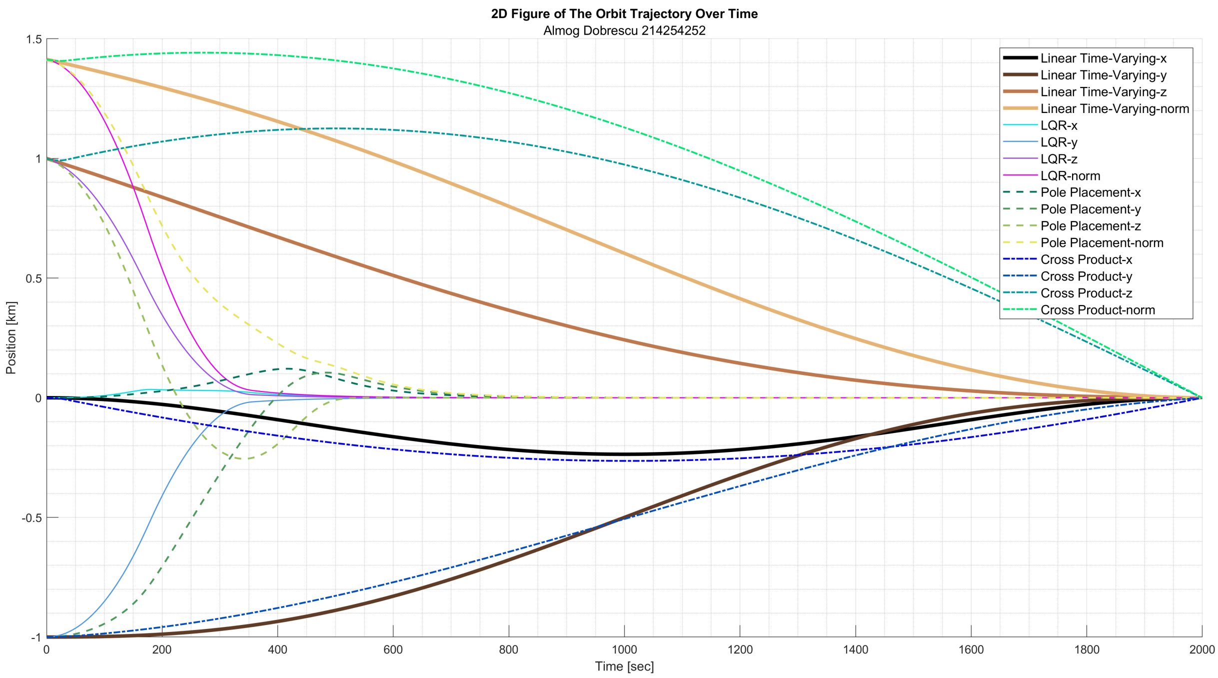

Figure 2: 2D figure of the orbit target trajectory over time

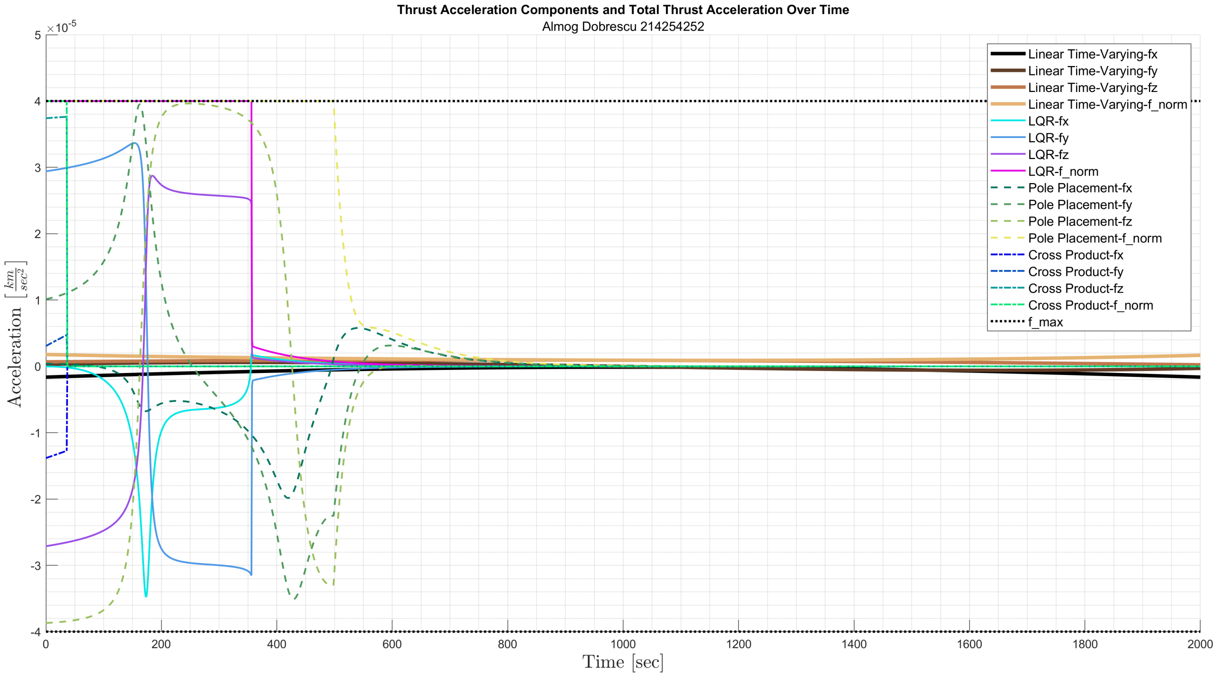

Figure 3: Thrust acceleration components and total thrust acceleration over time

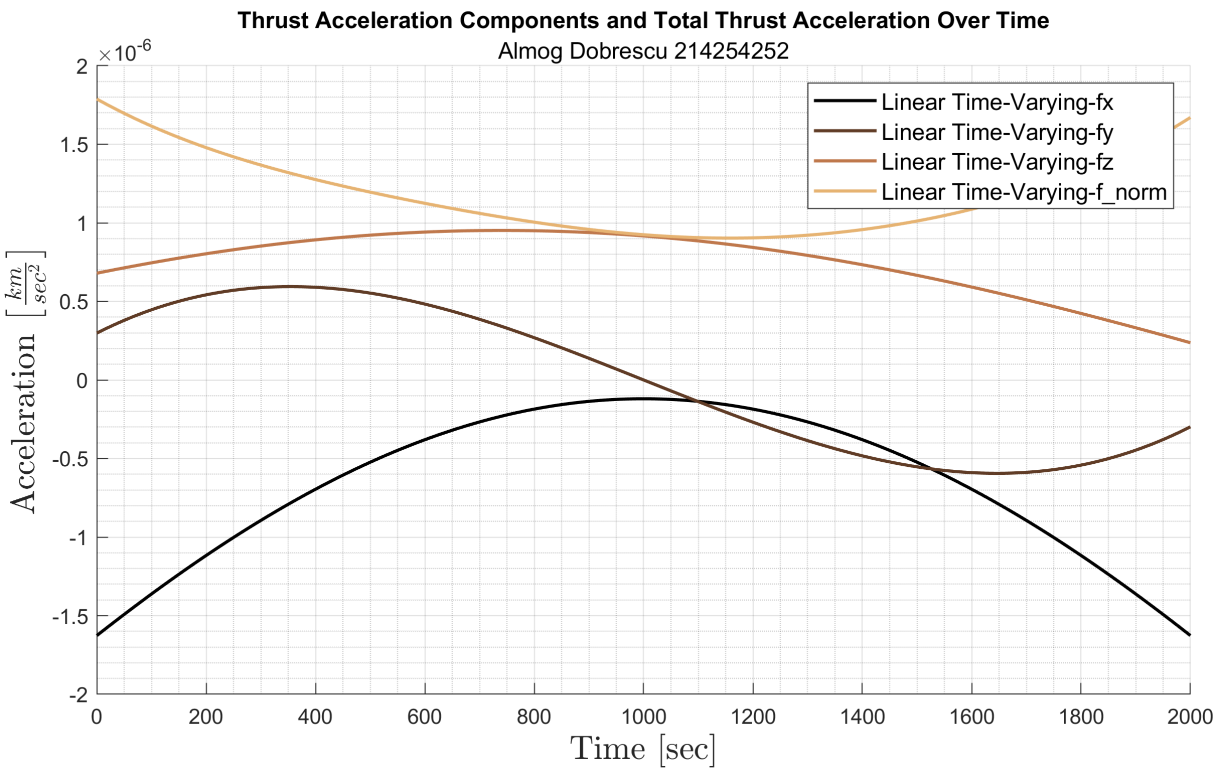

Figure 4: Thrust acceleration components and total thrust acceleration over time - only HW7

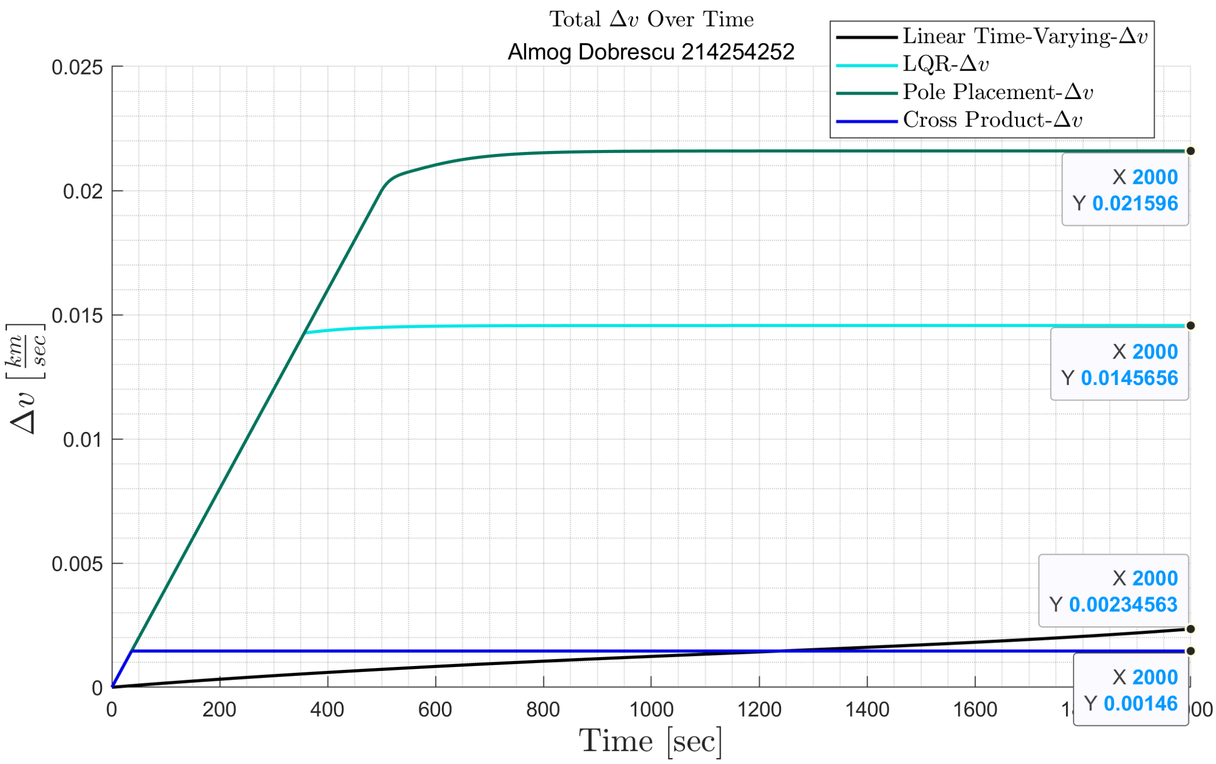

Figure 5: Total Δv over time

We can see that we indeed acomplished the design criteria:

- The thrust doesn't exceed the maximum available thrust.
- The miss distance at the final desired time is less than $1[\mathrm{m}]$ $\left(7.3649 \cdot 10^{-12}\,[\mathrm{m}]\right)$.
- The miss velocity at the final desired time is less than $1\left[\frac{\mathrm{cm}}{\mathrm{sec}}\right]$ $\left(1.2791 \cdot 10^{-12}\left[\frac{\mathrm{cm}}{\mathrm{sec}}\right]\right)$.

The total Δv is: $0.0023\left[\frac{\mathrm{km}}{\mathrm{sec}}\right]$