

# Satellite Orbit Control

## HW2

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December 19, 2024

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## 1 Given

$$\begin{aligned}
T_1 &= 100 [\text{min}] = 6 \cdot 10^3 [\text{sec}] & T_2 &= T_1 = 6 \cdot 10^3 [\text{sec}] \\
e_1 &= 0 & e_2 &= 0 \\
a_1 &= \sqrt[3]{\frac{\mu T_1^2}{4\pi^2}} = 7.1366 \cdot 10^3 [\text{km}] & a_2 &= a_1 = 7.1366 \cdot 10^3 [\text{km}] \\
\alpha &= \Delta i = 0.01^\circ
\end{aligned}$$

In CW frame with origin at Satellite #1 and at  $t = 0$ :

$$\begin{pmatrix} x_2(0) = 0 \\ y_2(0) = -1 \\ z_2(0) = 1 \end{pmatrix} [\text{km}] \quad \begin{pmatrix} \dot{x}_2(0) = ?? \\ \dot{y}_2(0) = ?? \\ \dot{z}_2(0) < 0 \end{pmatrix}$$

Desired:

$$\begin{pmatrix} x_2(t_1) = 0 \\ y_2(t_1) = 0 \\ z_2(t_1) = 0 \end{pmatrix} \quad \begin{pmatrix} \dot{x}_2(t_1) = 0 \\ \dot{y}_2(t_1) = 0 \\ \dot{z}_2(t_1) = 0 \end{pmatrix}$$

Limitations:

$$\Delta v = t \begin{pmatrix} 0 \\ \Delta v_y \\ \Delta v_z \end{pmatrix}$$

## 2 A

General equations of motion in CW fram:

$$\begin{cases} \ddot{x} - 2n\dot{y} - 3n^2x = f_x \\ \ddot{y} + 2n\dot{x} = f_y \\ \ddot{z} + n^2z = f_z \end{cases} \quad (1)$$

The velocities:

$$\begin{cases} u_x = \dot{x} - ny(t) \\ u_y = \dot{y} + nx(t) \\ u_z = \dot{z} \end{cases} \quad (2)$$

The solution without external forces (i.e.  $\vec{f} = \vec{0}$ ):

$$\begin{cases} x(t) = (4 - 3\cos(nt)) \cdot x_0 + \frac{\dot{x}_0}{n} \sin(nt) + \frac{2}{n} (1 - \cos(nt)) \cdot \dot{y}_0 \\ y(t) = 6(\sin(nt) - nt) \cdot x_0 + y_0 + \frac{2}{n} (\cos(nt) - 1) \cdot \dot{x}_0 + (4\sin(nt) - 3nt) \frac{\dot{y}_0}{n} \\ z(t) = z_0 \cos(nt) + \frac{\dot{z}_0}{n} \sin(nt) \end{cases} \quad (3)$$

Because the two satellite have the same period, then in CW fram, it is a no-drift orbit:

$$\dot{y}_0 = -2nx_0 \quad (4)$$

From the angle between the plains we can claculate  $\dot{z}_2(0)$ :

$$\tan \alpha = \frac{z_{max}}{a_1} = \frac{\sqrt{z_2(0)^2 + \left(\frac{\dot{z}_2(0)}{n}\right)^2}}{a_1} \quad (5)$$

$$\frac{\dot{z}_2(0)}{n} = \underbrace{-}_{\dot{z}_2(0) < 0} \sqrt{(a_1 \cdot \tan(\alpha))^2 - z_2(0)^2} \quad (6)$$

$$\frac{\dot{z}_2(0)}{n} = -0.7426; \quad (7)$$

Because we can't apply a velocity pulse in the x direction,  $\dot{x}_2(0) = 0$ . The CW frame is therefor:

$$\begin{pmatrix} x_2(0) = 0 \\ y_2(0) = -1 \\ z_2(0) = 1 \end{pmatrix} [km] \quad \begin{pmatrix} \dot{x}_2(0) = 0 \\ \dot{y}_2(0) = 0 \\ \dot{z}_2(0) = -0.74267 \cdot n \end{pmatrix} \left[ \frac{km}{sec} \right] \quad (8)$$

The CW equations in state-space form:

$$\begin{array}{ll} \text{x-y plane:} & \text{z-direction:} \\ x1 = x & z1 = z \\ x2 = \dot{x} & \\ y1 = y & z2 = \dot{z} \\ y2 = \dot{y} & \end{array} \quad (9)$$

The homogeneous solution:

$$\vec{x}(t) = \Phi_{(t,t_0)} \vec{x}_0 \quad (10)$$

x-y plane:

$$\Phi_{(t,t_0)} = \left( \begin{array}{cc|cc} 4 - 3\cos(n\tau) & 0 & \frac{1}{n}\sin(n\tau) & \frac{2}{n}(1 - \cos(n\tau)) \\ 6(\sin(n\tau) - n\tau) & 1 & \frac{2}{n}(\cos(n\tau) - 1) & \frac{1}{n}(4\sin(n\tau) - 3n\tau) \\ \hline 3n\sin(n\tau) & 0 & \cos(n\tau) & 2\sin(n\tau) \\ 6n(\cos(n\tau) - 1) & 0 & -2\sin(n\tau) & 4\cos(n\tau) - 3 \end{array} \right) = \begin{pmatrix} \Phi_{11(t,t_0)} & \Phi_{12(t,t_0)} \\ \Phi_{21(t,t_0)} & \Phi_{22(t,t_0)} \end{pmatrix}$$

z-direction:

$$\Phi_{(t,t_0)} = \begin{pmatrix} \cos(n\tau) & \frac{1}{n}\sin(n\tau) \\ -n\sin(n\tau) & \cos(n\tau) \end{pmatrix} \quad (11)$$

Where:

$$\tau = t - t_0$$

Desired:

$$\begin{pmatrix} x1 \\ y1 \\ x2 \\ y2 \end{pmatrix} (t_1) = \vec{0} \quad (12)$$

$$\begin{pmatrix} z1 \\ z2 \end{pmatrix} (t_1) = \vec{0}$$

The required velocity components are found from the first 2 lines of  $\vec{x}(t_1) = \Phi_{(t_1,0)} \vec{x}(0)$ :

$$\begin{pmatrix} x(t_1) \\ y(t_1) \end{pmatrix} = \vec{0} = \Phi_{11(t_1,0)} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \Phi_{12(t_1,0)} \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \end{pmatrix} \quad (13)$$

$\Downarrow$

$$\begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \end{pmatrix} = -\Phi_{12(t_1,0)}^{-1} \Phi_{11(t_1,0)} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \equiv \mathbf{C}_{(t_1,0)}^* \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \quad (14)$$

The terminal velocity is determined from the last two rows of  $\vec{x}(t_1) = \Phi_{(t_1,0)} \vec{x}(0)$ :

$$\begin{pmatrix} \dot{x}(t_1) \\ \dot{y}(t_1) \end{pmatrix} = \Phi_{21(t_1,0)} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \Phi_{22(t_1,0)} \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \end{pmatrix} \quad (15)$$

Because we can't apply a velocity pulse in the x direction, we set  $\dot{x}_{t_1}$  to be zero. Using *MatLab*, the first equation after substituting the initial conditions is:

$$\frac{2\pi \left( \cos\left(\frac{\pi\tau}{3000}\right) - 1 \right)}{24000 \cos\left(\frac{\pi\tau}{3000}\right) + 3\tau\pi \sin\left(\frac{\pi\tau}{3000}\right) - 24000} = 0 \quad (16)$$

$$\cos\left(\frac{\pi\tau}{3000}\right) = 1 \quad (17)$$

$$\frac{\pi\tau}{3000} = 2\pi \quad (18)$$

$$\pi\tau = 6000\pi \quad (19)$$

$$\tau = 6000 \quad (20)$$

$$t_1 = \tau = 6000 \text{ [sec]} \quad (\text{exactly on period})$$

## 3 B

### 3.1 x-y plane

The first pulse is the difference between the required velocity and the initial velocity:

$$\begin{pmatrix} \Delta v_{x1} \\ \Delta v_{y1} \end{pmatrix} = \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \end{pmatrix} - \begin{pmatrix} \dot{x}_{(t_1)} \\ \dot{y}_{(t_1)} \end{pmatrix} \quad (21)$$

$$\begin{aligned} \Phi_{11} &= \begin{pmatrix} 1 & 0 \\ -12\pi & 1 \end{pmatrix} & \Phi_{12} &= \begin{pmatrix} 0 & 0 \\ 0 & -18000 \end{pmatrix} \\ \Phi_{21} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \Phi_{22} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \quad (22)$$

Since  $|\Phi_{12}| = 0$ , we can't use the state space. let's look at the CW equations of motion for no-drift orbt:

$$\begin{cases} x(t) = (4 - 3\cos(nt)) \cdot x_0 + \frac{\dot{x}_0}{n} \sin(nt) + \frac{2}{n} (1 - \cos(nt)) \cdot \dot{y}_0 \\ y(t) = 6(\sin(nt) - nt) \cdot x_0 + y_0 + \frac{2}{n} (\cos(nt) - 1) \cdot \dot{x}_0 + (4\sin(nt) - 3nt) \frac{\dot{y}_0}{n} \\ z(t) = z_0 \cos(nt) + \frac{\dot{z}_0}{n} \sin(nt) \end{cases} \quad (23)$$

$$\begin{cases} \dot{x}(t) = 3nx_0 \sin(nt) + \dot{x}_0 \cos(nt) + 2\dot{y}_0 \sin(nt) \\ \dot{y}(t) = 6nx_0 (\cos(nt) - 1) - 2\dot{x}_0 \sin(nt) + \dot{y}_0 (4\cos(nt) - 3) \\ \dot{z}(t) = -nz_0 \sin(nt) + \dot{z}_0 \cos(nt) \end{cases}$$

$$x_0 = 0 \quad y_0 = -1 \quad z_0 = 1 \quad x_{(t_1)} = 0 \quad y_{(t_1)} = 0 \quad t_1 = 6000 \quad nt = 2\pi$$

From the position at the  $y$  direction:

$$\begin{aligned}
y_{(t_1)} = 0 &= 6 (\sin(2\pi) - 2\pi) \cdot 0 + -1 + \frac{2}{n} (\cos(2\pi) - 1) \cdot 0 + (4 \sin(2\pi) - 3 \cdot 2\pi) \frac{\dot{y}_{req}}{n} \\
0 &= -1 + (-3 \cdot 2\pi) \frac{\dot{y}_0}{n} \\
\dot{y}_{req} &= -\frac{n}{6\pi} = -5.5556 \cdot 10^{-5} \left[ \frac{km}{sec} \right] \\
&\Downarrow \\
\begin{pmatrix} \Delta v_{x1} \\ \Delta v_{y1} \end{pmatrix} &= \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \end{pmatrix} - \begin{pmatrix} \dot{x}_{(0)} \\ \dot{y}_{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ -5.5556 \cdot 10^{-5} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&\Downarrow \\
\begin{pmatrix} \Delta v_{x1} \\ \Delta v_{y1} \end{pmatrix}_{(t=0)} &= \begin{pmatrix} 0 \\ -5.5556 \end{pmatrix} \cdot 10^{-5} \left[ \frac{km}{sec} \right]
\end{aligned} \tag{24}$$

For a rendezvous maneuver, a terminal velocity pulse is applied such that the terminal velocity is zero:

$$\begin{pmatrix} \Delta v_{x2} \\ \Delta v_{y2} \end{pmatrix} = - \begin{pmatrix} \dot{x}_{(t_1)} \\ \dot{y}_{(t_1)} \end{pmatrix} \tag{25}$$

$$= - \begin{pmatrix} 3nx_0 \sin(nt) + \dot{x}_0 \cos(nt) + 2\dot{y}_0 \sin(nt) \\ 6nx_0 (\cos(nt) - 1) - 2\dot{x}_0 \sin(nt) + \dot{y}_0 (4 \cos(nt) - 3) \end{pmatrix} \tag{26}$$

$$= - \begin{pmatrix} -2 \cdot 5.5556 \cdot 10^{-5} \cdot \sin(2\pi) \\ -5.5556 \cdot 10^{-5} \cdot (4 \cos(2\pi) - 3) \end{pmatrix} \tag{27}$$

$$= - \begin{pmatrix} 0 \\ -5.5556 \cdot 10^{-5} \end{pmatrix} \tag{28}$$

$$\Downarrow \\
\begin{pmatrix} \Delta v_{x2} \\ \Delta v_{y2} \end{pmatrix}_{(t=6000[sec])} = \begin{pmatrix} 0 \\ 5.5556 \end{pmatrix} \cdot 10^{-5} \left[ \frac{km}{sec} \right]$$

### 3.2 z direction

Since  $t_f(t_1)$  is bigger than half of a period, there is no need for an initial pulse. Let's find the time until  $z = 0$ :

$$z_{(t)} = 0 = z_0 \cos(nt) + \frac{\dot{z}_0}{n} \sin(nt) \tag{29}$$

$$0 = 1 \cdot \cos(nt) + \frac{-0.74267 \cdot n}{n} \sin(nt) \tag{30}$$

$$0 = \cos(nt) - 0.74267 \sin(nt) \tag{31}$$

$$a \sin(x) + b \cos(x) = \sqrt{a^2 + b^2} \sin(x + \varphi)$$

$$\varphi = \arctan \left( \frac{b}{a} \right)$$

$$1.2456 \sin(nt + 2.2096) = 0 \tag{32}$$

$$nt + 2.2096 = \pi \cdot k \tag{33}$$

$$nt = 0.9320 \tag{34}$$

$$t_{(z=0)} = 889.9874[sec] \tag{35}$$

So the velocity in the  $z$  direction when  $z = 0$ :

$$\dot{z}(t) = -nz_0 \sin(nt) + \dot{z}_0 \cos(nt) \quad (36)$$

$$\dot{z}_{(z=0)} = -n \cdot 1 \cdot \sin(0.9320) - 0.74267 \cdot n \cos(0.9320) \quad (37)$$

$$\dot{z}_{(z=0)} = -0.0013 \left[ \frac{km}{sec} \right]$$

$\Downarrow$

$$\Delta v_{z1}(t=889.9874[sec]) = -\dot{z}_{z=0} = 0.0013 \left[ \frac{km}{sec} \right]$$

## 4 C

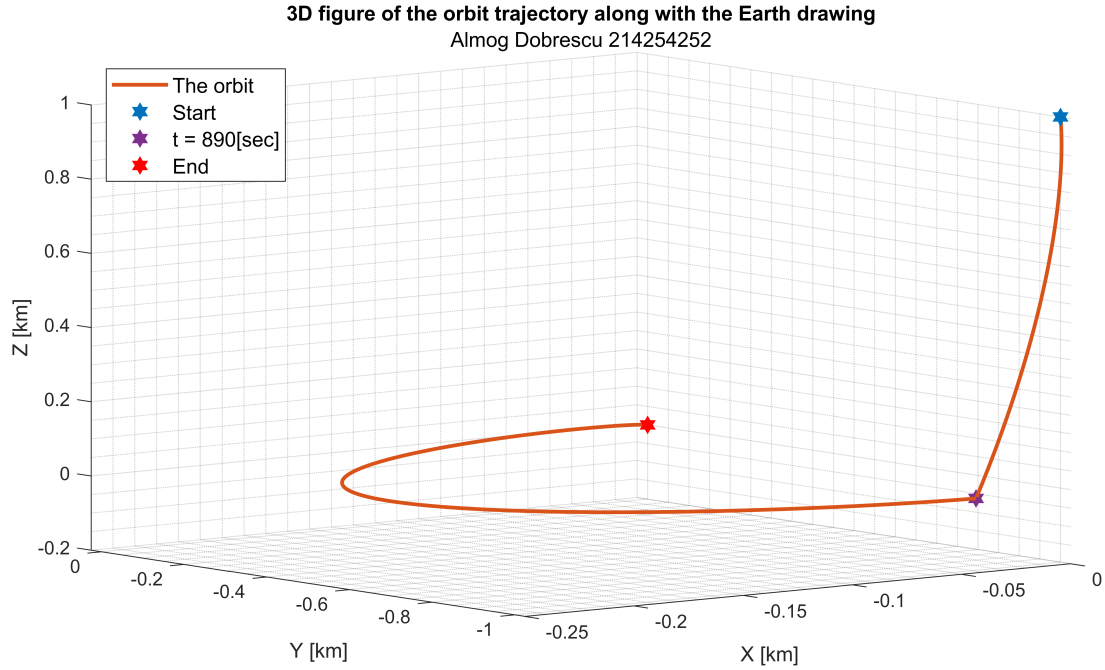


Figure 1: 3D figure of the orbit trajectory along with the Earth drawing

## 5 D

$$\begin{pmatrix} \Delta x_m \\ \Delta y_m \\ \Delta z_m \end{pmatrix} = \begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix} \cdot 10^{-3} [km] \quad \begin{pmatrix} \Delta \dot{x}_m \\ \Delta \dot{y}_m \\ \Delta \dot{z}_m \end{pmatrix} = \begin{pmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \end{pmatrix} \cdot 10^{-5} \left[ \frac{km}{sec} \right]$$

$\tau = 6000$

The worst case miss distance is the sum of the absolute values:

$$\begin{cases} \Delta x_f|_{\text{worst case}} = |\Delta x_m (4 - 3 \cos(n\tau))| + \left| \frac{\Delta \dot{x}_m}{n} \sin(n\tau) \right| + \left| \frac{2\Delta \dot{y}_m}{n} (1 - \cos(n\tau)) \right| \\ \Delta y_f|_{\text{worst case}} = |6\Delta x_m (\sin(n\tau) - n\tau)| + |\Delta y_m| + \left| \frac{2\Delta \dot{x}_0}{n} (\cos(n\tau) - 1) \right| + \left| \frac{\Delta \dot{y}_m}{n} (4 \sin(n\tau) - 3n\tau) \right| \\ \Delta z_f|_{\text{worst case}} = |\Delta z_m \cos(n\tau)| + \left| \frac{\Delta \dot{z}_m}{n} \sin(n\tau) \right| \end{cases} \quad (38)$$

$\Downarrow$

$$\left\{ \begin{array}{l} \Delta x_f|_{\text{worst case}} = 0.001 \\ \Delta y_f|_{\text{worst case}} = 0.2187 \\ \Delta z_f|_{\text{worst case}} = 0.001 \end{array} \right. \quad (39)$$