

Satellite Orbit Control

HW3

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1 Given

$$\begin{aligned} T_1 &= 100 [\text{min}] = 6 \cdot 10^3 [\text{sec}] & T_2 &= T_1 = 6 \cdot 10^3 [\text{sec}] \\ e_1 &= 0 & e_2 &= 0 \\ a_1 &= \sqrt[3]{\frac{\mu T_1^2}{4\pi^2}} = 7.1366 \cdot 10^3 [\text{km}] & a_2 &= a_1 = 7.1366 \cdot 10^3 [\text{km}] \\ \alpha &= \Delta i = 0.01^\circ \end{aligned}$$

In CW frame with origin at Satellite #1 and at $t = 0$:

$$\begin{pmatrix} x_2(0) = 0 \\ y_2(0) = -1 \\ z_2(0) = 1 \end{pmatrix} [\text{km}] \quad \begin{pmatrix} \dot{x}_2(0) = 0 \\ \dot{y}_2(0) = 0 \\ \dot{z}_2(0) = -0.74267 \cdot n \end{pmatrix} \left[\frac{\text{km}}{\text{sec}} \right]$$

1.1 Desired

$$\begin{pmatrix} x_2(t_f) = 0 \\ y_2(t_f) = 0 \\ z_2(t_f) = 0 \end{pmatrix} \quad \begin{pmatrix} \dot{x}_2(t_f) = 0 \\ \dot{y}_2(t_f) = 0 \\ \dot{z}_2(t_f) = 0 \end{pmatrix}$$

1.2 Limitations

$$a_{\max} = 8 \cdot 10^{-6} \left[\frac{\text{km}}{\text{sec}^2} \right]$$

2 The CW equations

$$\begin{cases} \ddot{x} - 2n\dot{y} - 3n^2x = f_x \\ \ddot{y} + 2n\dot{x} = f_y \\ \ddot{z} + n^2z = f_z \end{cases} \quad (1)$$

2.1 x-y

$$\vec{x} = \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} \quad (2)$$

$$\dot{\vec{x}} = F\vec{x} + G\vec{f} \quad (3)$$

Where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n \\ 0 & 0 & 0 & 1 \\ 0 & -2n & 0 & 0 \end{pmatrix} \quad G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \vec{f} = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \quad (4)$$

2.2 z

$$\vec{x} = \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \quad (5)$$

$$\dot{\vec{x}} = F\vec{x} + Gf \quad (6)$$

Where:

$$F = \begin{pmatrix} 0 & 1 \\ -n^2 & 0 \end{pmatrix} \quad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad f = (f_z) \quad (7)$$

2.3 x-y-z

The equations of motion in state space form are therefor:

$$\vec{x} = (x \quad \dot{x} \quad y \quad \dot{y} \quad z \quad \dot{z})^T \quad (8)$$

$$\dot{\vec{x}} = F\vec{x} + G\vec{f} \quad (9)$$

Where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -n^2 & 0 \end{pmatrix} \quad G = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \vec{f} = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \quad (10)$$

2.4 The homogeneous solution

$$\vec{x} = (x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z})^T \quad (11)$$

x-y plane:

$$\Phi_{\{t,t_0\}} = \left(\begin{array}{cc|cc} 4 - 3\cos(n\tau) & 0 & \frac{1}{n}\sin(n\tau) & \frac{2}{n}(1 - \cos(n\tau)) \\ 6(\sin(n\tau) - n\tau) & 1 & \frac{2}{n}(\cos(n\tau) - 1) & \frac{1}{n}(4\sin(n\tau) - 3n\tau) \\ \hline 3n\sin(n\tau) & 0 & \cos(n\tau) & 2\sin(n\tau) \\ 6n(\cos(n\tau) - 1) & 0 & -2\sin(n\tau) & 4\cos(n\tau) - 3 \end{array} \right) = \begin{pmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ \Phi_{21}(t,t_0) & \Phi_{22}(t,t_0) \end{pmatrix}$$

z-direction:

$$\Phi_{(t,t_0)} = \begin{pmatrix} \cos(n\tau) & \frac{1}{n}\sin(n\tau) \\ -n\sin(n\tau) & \cos(n\tau) \end{pmatrix} \quad (12)$$

Where:

$$\tau = t - t_0$$

So the full homogeneous solution in state space from:

$$\Phi_{(t,t_0)} = \left(\begin{array}{ccc|ccc} 4 - 3\cos(n\tau) & 0 & 0 & \frac{1}{n}\sin(n\tau) & \frac{2}{n}(1 - \cos(n\tau)) & 0 \\ 6(\sin(n\tau) - n\tau) & 1 & 0 & \frac{2}{n}(\cos(n\tau) - 1) & \frac{1}{n}(4\sin(n\tau) - 3n\tau) & 0 \\ 0 & 0 & \cos(n\tau) & 0 & 0 & \frac{1}{n}\sin(n\tau) \\ \hline 3n\sin(n\tau) & 0 & 0 & \cos(n\tau) & 2\sin(n\tau) & 0 \\ 6n(\cos(n\tau) - 1) & 0 & 0 & -2\sin(n\tau) & 4\cos(n\tau) - 3 & 0 \\ 0 & 0 & -n\sin(n\tau) & 0 & 0 & \cos(n\tau) \end{array} \right) = \begin{pmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ \Phi_{21}(t,t_0) & \Phi_{22}(t,t_0) \end{pmatrix} \quad (13)$$

3 case A

The desired maneuver time is 2000 [sec]. The desired approach trajectory is a straight line with constant velocity from the initial point to the origin.

From geometric considerations, the required velocity on the straight line is:

$$\dot{\vec{x}}_{req} = \begin{pmatrix} \frac{x_f - x_0}{t_f} \\ \frac{y_f - y_0}{t_f} \\ \frac{z_f - z_0}{t_f} \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ -5 \end{pmatrix} \cdot 10^{-4} \left[\frac{\text{km}}{\text{sec}^2} \right] \quad (14)$$

Which mean that the initial pulse:

$$\Delta v_1 = \dot{\vec{x}}_{req} - \dot{\vec{x}}_0 = \begin{pmatrix} 0 \\ 0.5 \\ 0.2777 \end{pmatrix} \cdot 10^{-3} \left[\frac{\text{km}}{\text{sec}^2} \right] \quad (15)$$

and the final pulse:

$$\Delta v_f = \dot{\vec{x}}_f - \dot{\vec{x}}_{req} = \begin{pmatrix} 0 \\ -5 \\ 5 \end{pmatrix} \cdot 10^{-4} \left[\frac{\text{km}}{\text{sec}^2} \right] \quad (16)$$

Since the velocity is constant:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \vec{0}$$

By substituting the constraints in the CW equations we get:

$$\begin{cases} 0 - 2n\dot{y} = f_x \\ 0 = f_y \\ 0 + n^2 z = f_z \end{cases} \quad (17)$$

$$\vec{a} = \vec{f} = \begin{pmatrix} -2n \cdot 0.5 \cdot 10^{-3} \\ 0 \\ n^2 z \end{pmatrix} \quad (18)$$

Check that the maximum acceleration hasn't been reached:

$$|\vec{a}| = \sqrt{(2n \cdot 0.5 \cdot 10^{-3})^2 + (n^2 z)^2} < \sqrt{(2n \cdot 0.5 \cdot 10^{-3})^2 + (n^2 \cdot 1)^2} \quad (19)$$

$$|\vec{a}| < 1.5163 \cdot 10^{-6} < 8 \cdot 10^{-6} \quad \checkmark$$

By solving this system numerically, we obtain the approach trajectory.

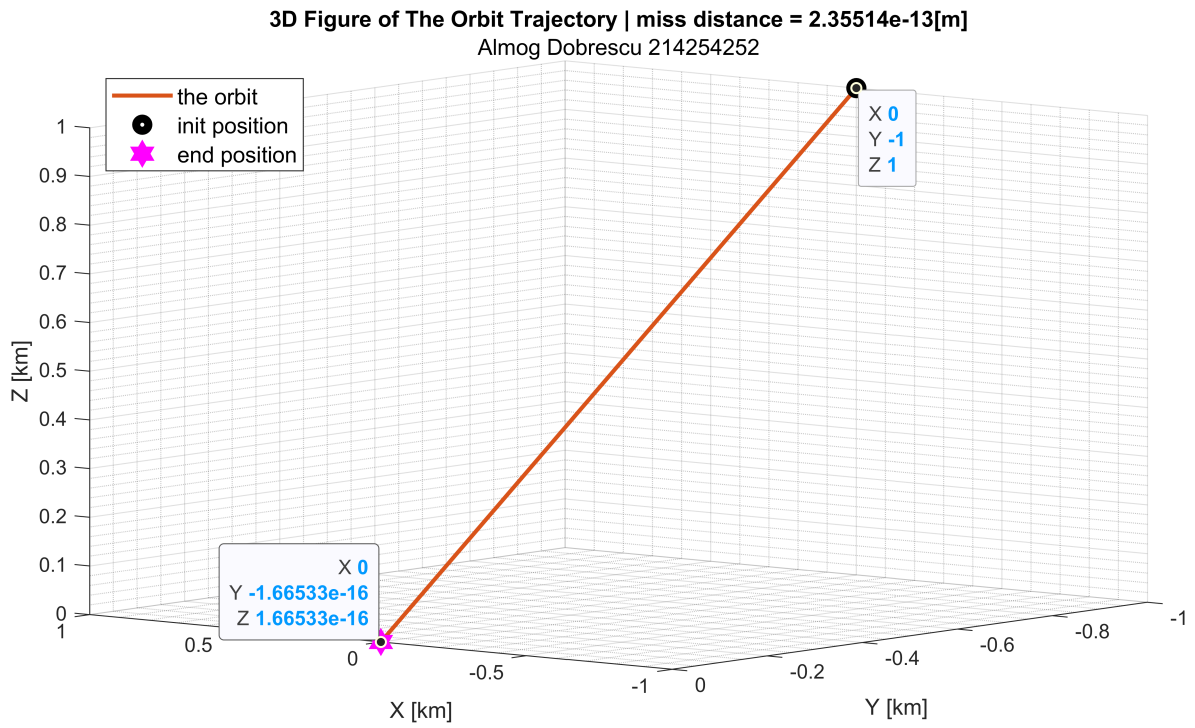


Figure 1: 3D figure of the orbit trajectory - case A

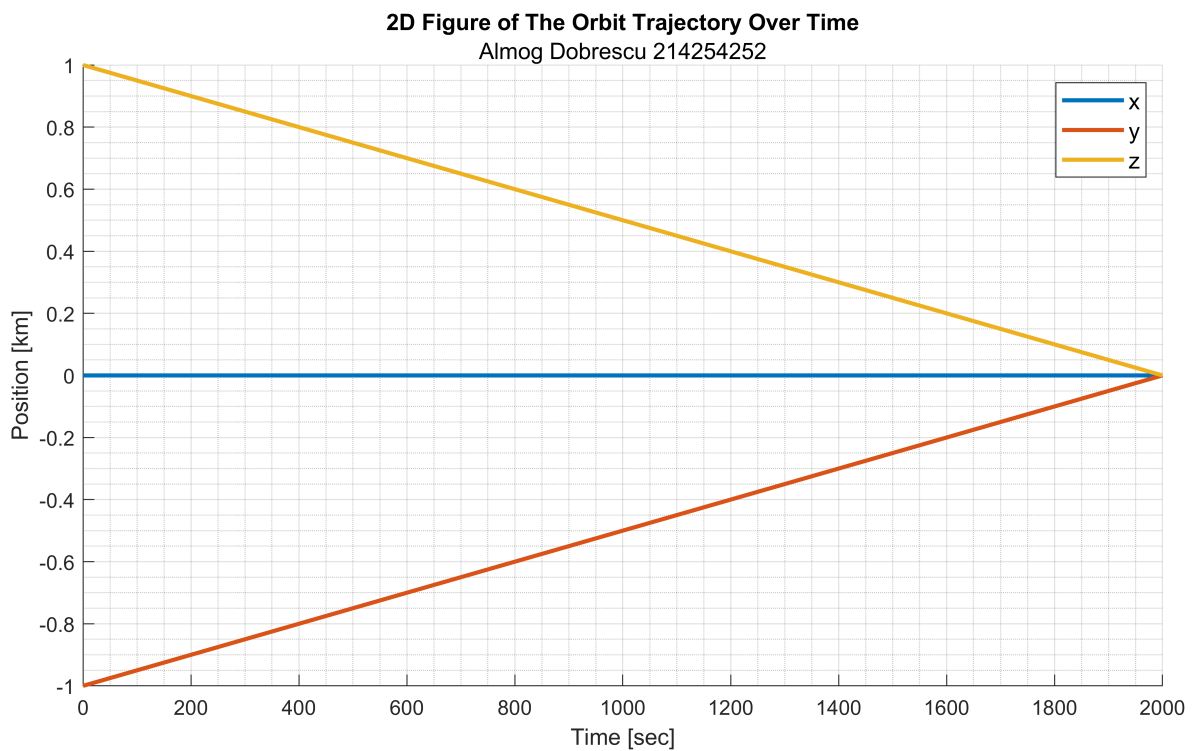


Figure 2: 2D figure of the position over time - case A

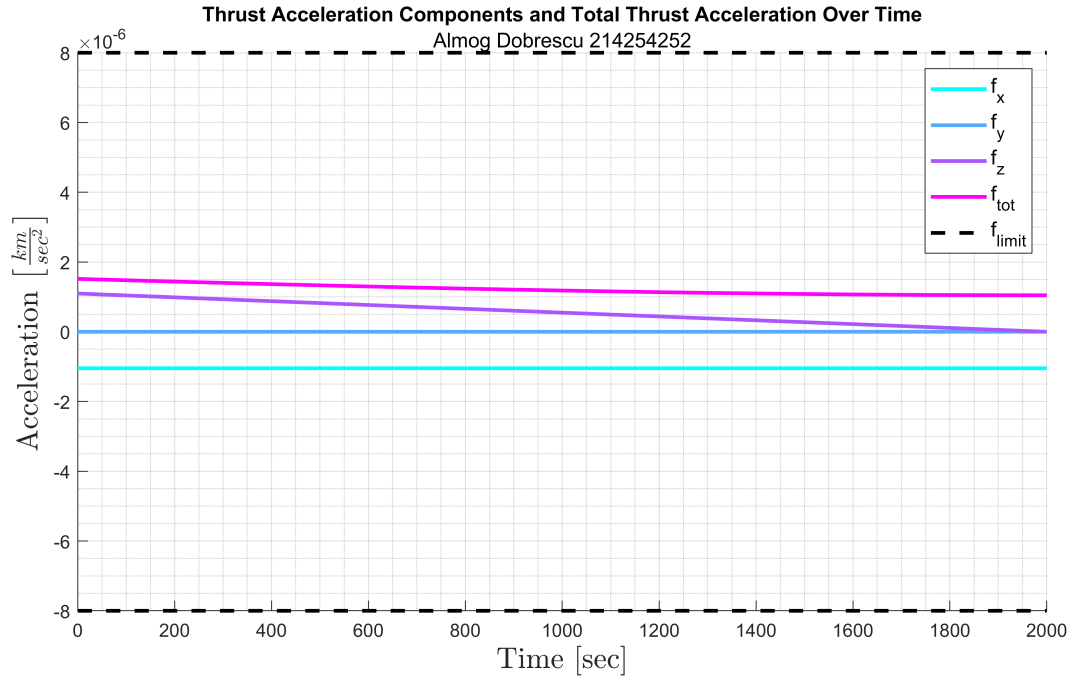
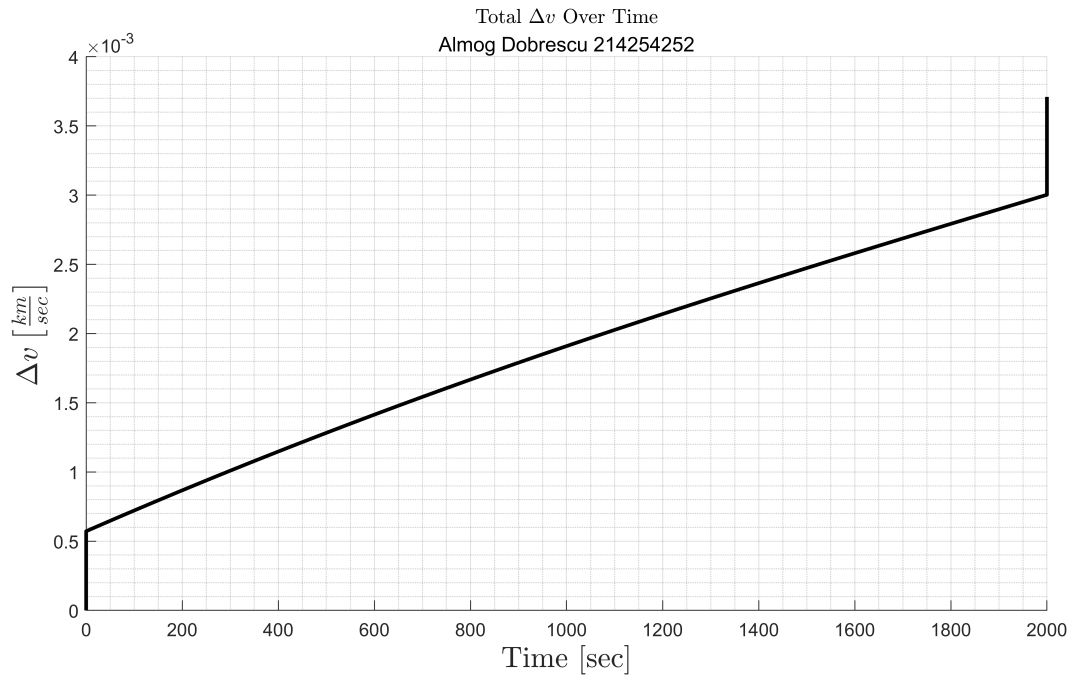


Figure 3: Thrust acceleration components and total thrust acceleration over time - case A

Figure 4: Total Δv over time - case A

We can see that we indeed accomplished the design criteria:

- The thrust doesn't exceed the maximum available thrust.
- The miss distance at the final desired time is less than 1[m] ($2.35517 \cdot 10^{-13}$ [m]).

4 case B

The velocity to be gained:

$$\vec{v}_g = \vec{v}_r - \vec{v} \quad (20)$$

The equation of motion:

$$\frac{d\vec{v}}{dt} = \vec{g}_{(\vec{r})} + \vec{a}_T \quad (21)$$

The equation of motion for the velocity to be gained:

$$\frac{d\vec{v}_g}{dt} = \frac{d\vec{v}_r}{dt} - \frac{d\vec{v}}{dt} = \frac{d\vec{v}_r}{dt} - \vec{g}_{(\vec{r})} - \vec{a}_T \quad (22)$$

\vec{v}_r is a function of time and space, so:

$$\frac{d\vec{v}_{r(t,\vec{r})}}{dt} = \frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}} \underbrace{\frac{d\vec{r}}{dt}}_{\vec{v}} = \frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}} (\vec{v}_r - \vec{v}_g)$$

The required velocity \vec{v}_r also fulfill the equation of motion with no thrust:

$$\frac{d\vec{v}_r}{dt} = \frac{\partial \vec{v}_r}{\partial t} + \frac{\partial \vec{v}_r}{\partial \vec{r}} \vec{v}_r = \vec{g}_r$$

so:

$$\frac{d\vec{v}_r}{dt} = \vec{g}_r - \frac{\partial \vec{v}_r}{\partial \vec{r}} \vec{v}_g \quad (23)$$

The equation of motion becomes:

$$\frac{d\vec{v}_g}{dt} = \vec{g}_r - \frac{\partial \vec{v}_r}{\partial \vec{r}} \vec{v}_g - \vec{g}_{(\vec{r})} - \vec{a}_T = - \underbrace{\frac{\partial \vec{v}_r}{\partial \vec{r}} \vec{v}_g}_{C^*} - \vec{a}_T \quad (24)$$

where C^* is:

$$\begin{aligned} \begin{pmatrix} x_f \\ y_f \\ z_f \end{pmatrix} &= \vec{0} = \Phi_{11(t_f,t)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \Phi_{12(t_f,t)} \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \\ \dot{z}_{req} \end{pmatrix} \\ \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \\ \dot{z}_{req} \end{pmatrix} &= -\Phi_{12(t_f,t)}^{-1} \Phi_{11(t_f,t)} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_{(t_f,t)}^* \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\Downarrow \\ C_{(t_f,t)}^* &= -\Phi_{12(t_f,t)}^{-1} \Phi_{11(t_f,t)} \end{aligned}$$

The Cross-Product fixed thrust control law:

$$\vec{v}_g \parallel \frac{d\vec{v}_g}{dt} \quad (25)$$

\Downarrow

$$\begin{aligned} \frac{d\vec{v}_g}{dt} \times \vec{v}_g &= 0 \\ (-C^* \vec{v}_g - \vec{a}_T) \times \vec{v}_g &= 0 \end{aligned} \quad (26)$$

$$\vec{a}_T \times \vec{v}_g = - \underbrace{(C^* \vec{v}_g)}_{\vec{p}} \times \vec{v}_g \quad (27)$$

Multiplying on the left by \vec{v}_g :

$$(\vec{a}_T \times \vec{v}_g) \times \vec{v}_g = (\vec{p} \times \vec{v}_g) \times \vec{v}_g \quad (28)$$

Expand according to:

$$\begin{aligned} (\vec{a} \times \vec{b}) \times \vec{c} &= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a} \\ (\vec{a}_T \cdot \vec{v}_g) \vec{v}_g - v_g^2 \vec{a}_T &= (\vec{p} \cdot \vec{v}_g) \vec{v}_g - v_g^2 \vec{p} \end{aligned} \quad (29)$$

\Downarrow

$$\vec{a}_T = \vec{p} + \frac{1}{v_g^2} ((\vec{a}_T \cdot \vec{v}_g) - (\vec{p} \cdot \vec{v}_g)) \vec{v}_g \quad (30)$$

Defining $q = \vec{a}_T \cdot \hat{v}_g$ as the projection of the acceleration vector on the velocity to be gained, we get:

$$\vec{a}_T = \vec{p} + (q - (\vec{p} \cdot \hat{v}_g)) \hat{v}_g \quad (31)$$

The terminal braking will therefor be:

$$\Delta v_f = \vec{0} - \begin{pmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{z}_f \end{pmatrix} = - \left(\Phi_{21(t_f,0)} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \Phi_{22(t_f,0)} \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \\ \dot{z}_{req} \end{pmatrix} \right) \quad (32)$$

To calculate the components of the thrust vector, we need to make sure that the solution exists and it is stable. according to Lyapunov:

$$\underbrace{U}_{\text{Lyapunov function}} = \frac{1}{2} \vec{v}_g^T \vec{v}_g \quad U_{(0)} = 0 \quad U_{\vec{v}_g} > 0 \quad (33)$$

For stability:

$$\dot{U} = \vec{v}_g^T (-C^* \vec{v}_g - \vec{a}_T) = -\vec{v}_g^T ((\vec{a}_T \cdot \hat{v}_g) - (\vec{p} \cdot \hat{v}_g)) \hat{v}_g < 0 \quad (34)$$

we need the expression in the parentheses to be positive. Let's assume that $|\vec{a}_T| = a_{max}$ and after squaring both sides we get:

$$a_T^2 = p^2 + 2(q - \vec{p} \cdot \hat{v}_g)(\vec{p} \cdot \hat{v}_g) + (q - \vec{p} \cdot \hat{v}_g)^2 \quad (35)$$

$$a_T^2 = p^2 + q^2 - (\vec{p} \cdot \hat{v}_g)^2 \quad (36)$$

$$q = \left(a_{max}^2 - p^2 + (\vec{p} \cdot \hat{v}_g)^2 \right)^{\frac{1}{2}} \quad (37)$$

To sum up:

$$\vec{a}_T = \vec{p} + (q - (\vec{p} \cdot \hat{v}_g)) \hat{v}_g$$

where:

$$\begin{aligned} \vec{p} &= -C^* \vec{v}_g \\ q &= \left(a_{max}^2 - p^2 + (\vec{p} \cdot \hat{v}_g)^2 \right)^{\frac{1}{2}} \end{aligned}$$

By solving this system numerically, we obtain the approach trajectory.

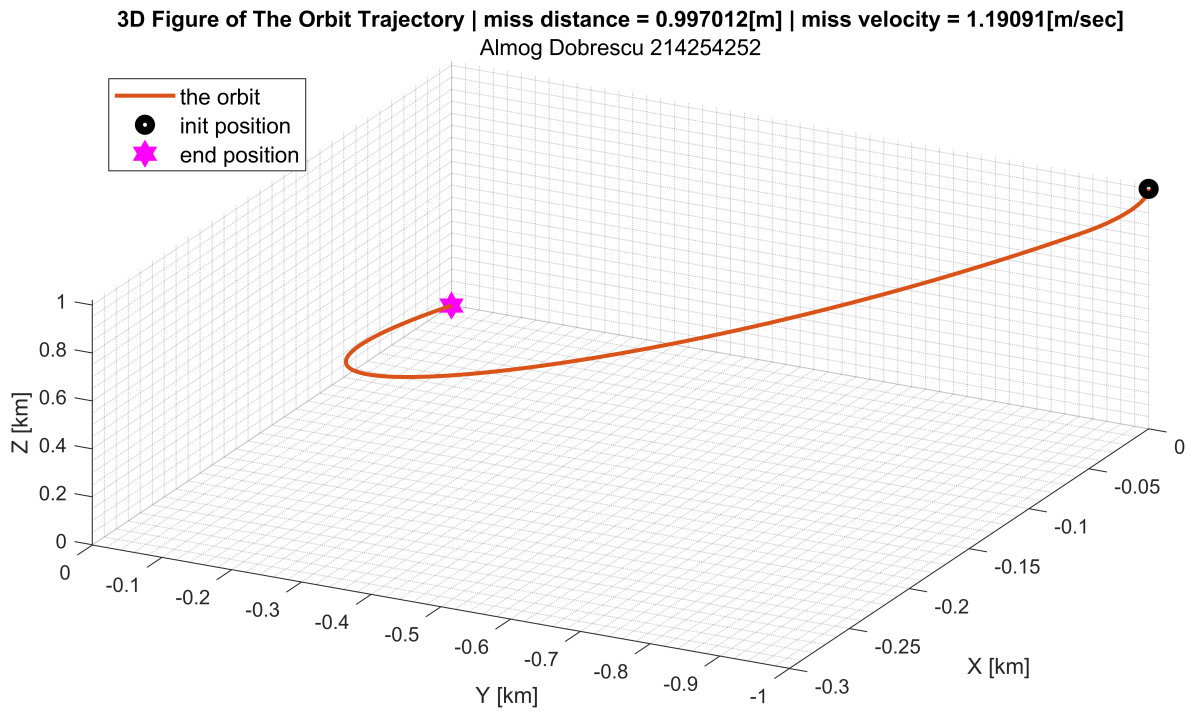


Figure 5: 3D figure of the orbit trajectory - case B

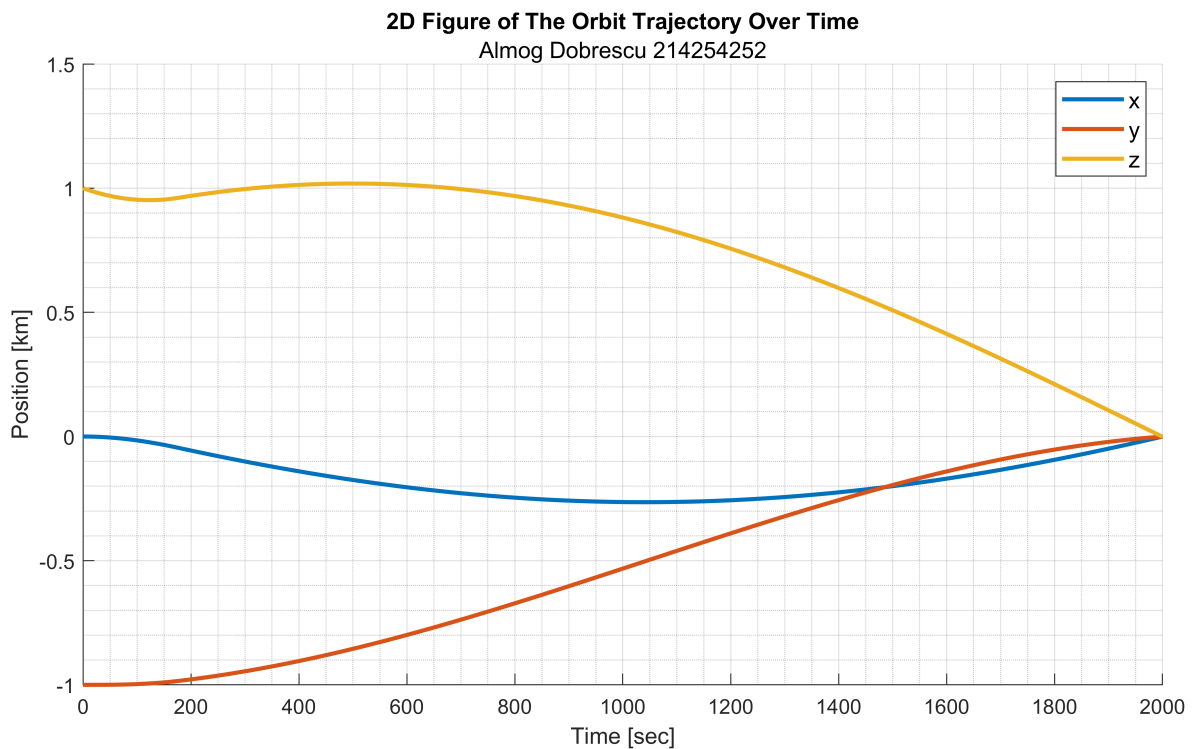


Figure 6: 2D figure of the position over time - case B

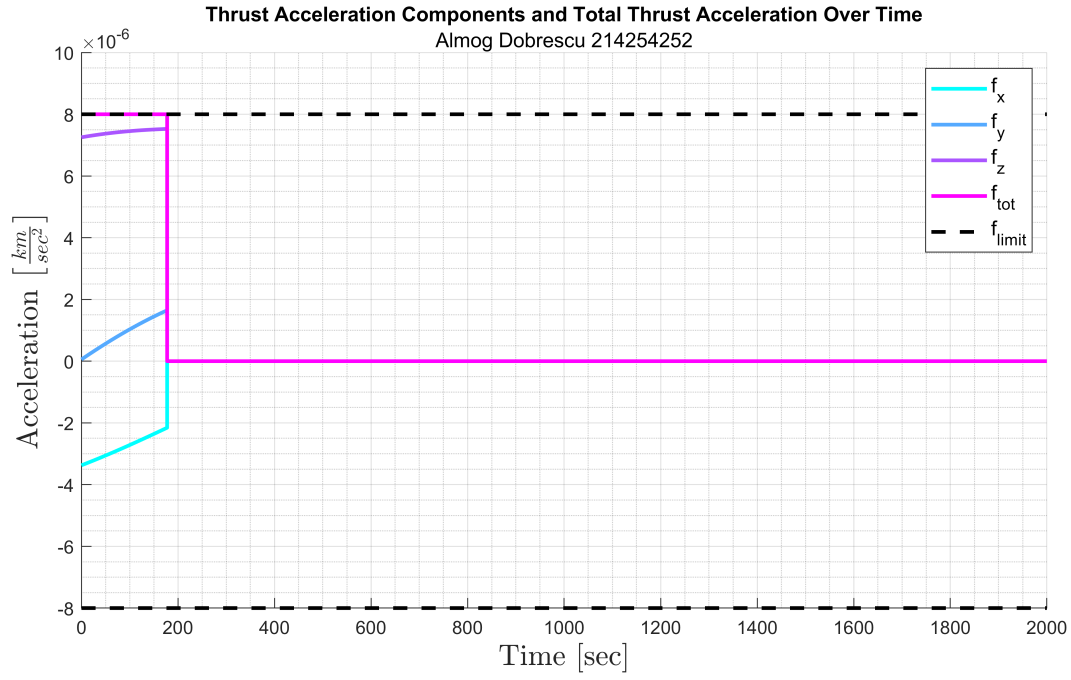
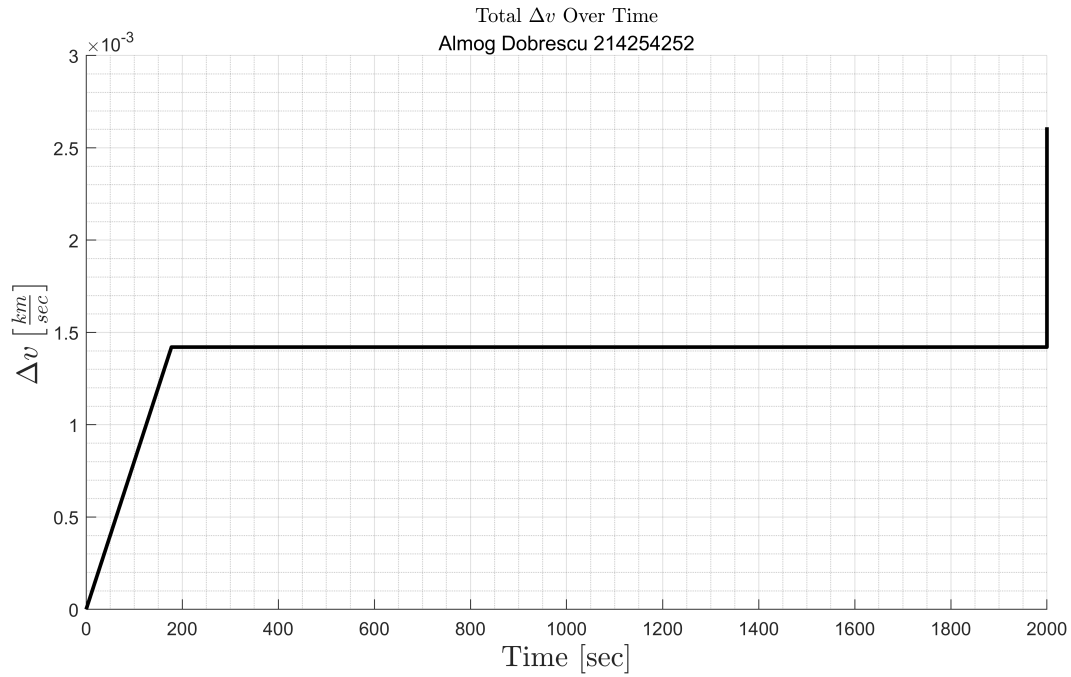


Figure 7: Thrust acceleration components and total thrust acceleration over time - case B

Figure 8: Total Δv over time - case B

We can see that we indeed accomplished the design criteria:

- The thrust doesn't exceed the maximum available thrust.
- The miss distance at the final desired time is less than 1[m] (0.997012 [m]).
- The terminal velocity is $1.19091 \left[\frac{m}{sec} \right]$