Satellite Orbit Control
HW6

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1 Given

$$T_1 = 100\,[min] = 6 \cdot 10^3\,[sec] \qquad T_2 = T_1 = 6 \cdot 10^3\,[sec]$$
$$e_1 = 0 \qquad e_2 = 0$$
$$a_1 = \sqrt[3]{\frac{\mu T_1^2}{4\pi^2}} = 7.1366 \cdot 10^3\,[km] \qquad a_2 = a_1 = 7.1366 \cdot 10^3\,[km]$$
$$\alpha = \Delta i = 0.01^\circ$$

In CW frame with origin at Satellite #1 and at $t = 0$:

$$\begin{pmatrix} x_2(0) = 0 \\ y_2(0) = -1 \\ z_2(0) = 1 \end{pmatrix} [\mathrm{km}] \qquad \begin{pmatrix} \dot{x}_2(0) = 0 \\ \dot{y}_2(0) = 0 \\ \dot{z}_2(0) = -0.74267 \cdot n \end{pmatrix} \left[\frac{\mathrm{km}}{\mathrm{sec}}\right]$$

1.1 Desired

$$\begin{pmatrix} x_2(t_f) = 0 \\ y_2(t_f) = 0 \\ z_2(t_f) = 0 \end{pmatrix} \qquad \begin{pmatrix} \dot{x}_2(t_f) = 0 \\ \dot{y}_2(t_f) = 0 \\ \dot{z}_2(t_f) = 0 \end{pmatrix}$$

1.2 Limitations

$$a_{\max} = 4 \cdot 10^{-5}\left[\tfrac{\mathrm{km}}{\mathrm{sec}^2}\right] \qquad t_f = 2000\,[sec]$$

2 The CW equations

$$\begin{cases} \ddot{x} - 2n\dot{y} - 3n^2 x = f_x \\ \ddot{y} + 2n\dot{x} = f_y \\ \ddot{z} + n^2 z = f_z \end{cases} \tag{1}$$

2.1 x-y

$$\vec{x} = \begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} \tag{2}$$

$$\dot{\vec{x}} = F\vec{x} + G\vec{f} \tag{3}$$

Where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n \\ 0 & 0 & 0 & 1 \\ 0 & -2n & 0 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad f = \begin{pmatrix} f_x \\ f_y \end{pmatrix} \tag{4}$$

2.2 z

$$\vec{x} = \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \tag{5}$$

$$\dot{\vec{x}} = F\vec{x} + Gf \tag{6}$$

Where:

$$F = \begin{pmatrix} 0 & 1 \\ -n^2 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad f = \begin{pmatrix} f_z \end{pmatrix} \tag{7}$$

2.3 x-y-z

The equations of motion in state space form are therefor:

$$\vec{x} = \begin{pmatrix} x & \dot{x} & y & \dot{y} & z & \dot{z} \end{pmatrix}^T \tag{8}$$

$$\dot{\vec{x}} = F\vec{x} + G\vec{f} \tag{9}$$

Where:

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3n^2 & 0 & 0 & 2n & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -n^2 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \tag{10}$$

3 LQR

The linear system:

$$\dot{\vec{x}} = F\vec{x} + G\vec{u} \qquad \vec{x}_{(t_0)} = \vec{x}_0 \tag{11}$$

It is needed to find a controler to bring the system to the end position while minimaizing the cost cratiria:

$$J = \frac{1}{2}\vec{x}_f^T P_f \vec{x}_f + \frac{1}{2}\int_0^{t_f} \left(\vec{x}^T Q\vec{x} + \vec{u}^T R\vec{u}\right) dt \tag{12}$$

P_f, Q, R are PD matrices that the users chose w.r.t. the requierments and limitations.
The Hamiltonian:

$$H = \vec{x}^T Q\vec{x} + \vec{u}^T R\vec{u} + \vec{\lambda}^T \left(F\vec{x} + G\vec{u}\right) \tag{13}$$

The optimum condition:

$$\frac{\partial H}{\partial \vec{u}} = 0 \qquad \rightarrow \qquad \vec{u} = -R^{-1}G^T\vec{\lambda} \tag{14}$$

The Euler Lagrange equations:

$$\dot{\vec{\lambda}}^T = -\frac{\partial H}{\partial \vec{x}} \qquad \rightarrow \qquad \dot{\vec{\lambda}} = -Q\vec{x} - F^T\vec{\lambda}, \qquad \vec{\lambda}_f^T = \frac{\partial J}{\partial \vec{x}_f} = \vec{x}_f^T P_f \tag{15}$$

By combining both equations we get:

$$\begin{pmatrix} \dot{\vec{x}} \\ \dot{\vec{\lambda}} \end{pmatrix} = \begin{pmatrix} F & -GR^{-1}G^T \\ -Q & -F^T \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{\lambda} \end{pmatrix} \tag{16}$$

We can write the system like:

$$\Phi_{(t,t_0)} = \begin{pmatrix} \Phi_{11(t,t_0)} & \Phi_{12(t,t_0)} \\ \Phi_{21(t,t_0)} & \Phi_{22(t,t_0)} \end{pmatrix} \tag{17}$$

Using the transion matrix, the instantaneous state vector w.r.t the final state:

$$\begin{aligned} \vec{x}_{(t)} &= \Phi_{11\left(t,t_f\right)}\vec{x}_f + \Phi_{12\left(t,t_f\right)}\vec{\lambda}_f \\ &\Downarrow \ \left(\vec{\lambda}_f = P_f^T\vec{x}_f\right) \\ \vec{x}_{(t)} &= \left(\Phi_{11\left(t,t_f\right)} + \Phi_{12\left(t,t_f\right)}P_f^T\right)\vec{x}_f \end{aligned} \tag{18}$$

Likewise:

$$\vec{\lambda}_{(t)} = \left(\Phi_{21\left(t,t_f\right)} + \Phi_{22\left(t,t_f\right)}P_f^T\right)\vec{x}_f \tag{19}$$

By combining the equations, we can get a linear conection between $\vec{x}$ and $\vec{\lambda}$:

$$\vec{\lambda}_{(t)} = \left(\Phi_{21\left(t,t_f\right)} + \Phi_{22\left(t,t_f\right)} P_f^T\right) \left(\Phi_{11\left(t,t_f\right)} + \Phi_{12\left(t,t_f\right)} P_f^T\right)^{-1} \vec{x}_{(t)} \equiv P_{(t)} \vec{x}_{(t)} \tag{20}$$

The porpotional controler will be writen as:

$$\vec{u}_{(t)} = -K_{(t)} \vec{x}_{(t)}, \qquad K_{(t)} = R^{-1} G^T P_{(t)} \tag{21}$$

In order to find the matrix P we will substitut $\vec{\lambda} = P\vec{x}$ inside the Euler Lagrange equations:

$$\begin{gathered} \dot{P}\vec{x} + P\dot{\vec{x}} = -Q\vec{x} - F^T P\vec{x} \\ \Downarrow \\ \left(\dot{P} + PF + F^T P - PGR^{-1} G^T P + Q\right) \vec{x} = 0 \end{gathered} \tag{22}$$

The equation is right for every x so:

$$\dot{P} + PF + F^T P - PGR^{-1} G^T P + Q = 0, \qquad P_{\left(t_f\right)} = P_f \tag{23}$$

This is the 'matrix Rikati equation'

4 The Gains Matrix

Setting Q, R to be:

$$Q = \begin{pmatrix} \frac{1}{x^2_{\text{miss}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{v^2_{\text{miss}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{x^2_{\text{miss}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{v^2_{\text{miss}}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{x^2_{\text{miss}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{v^2_{\text{miss}}} \end{pmatrix} \qquad R = \frac{1}{f^2_{max}} \tag{24}$$

By using the function *lqr* in Matlab, we get:

$$K = \begin{pmatrix} 0.0400 & 4.0100 & -0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & 0.0000 & 0.0400 & 4.0100 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0400 & 4.0100 \end{pmatrix} \tag{25}$$

5 The Results

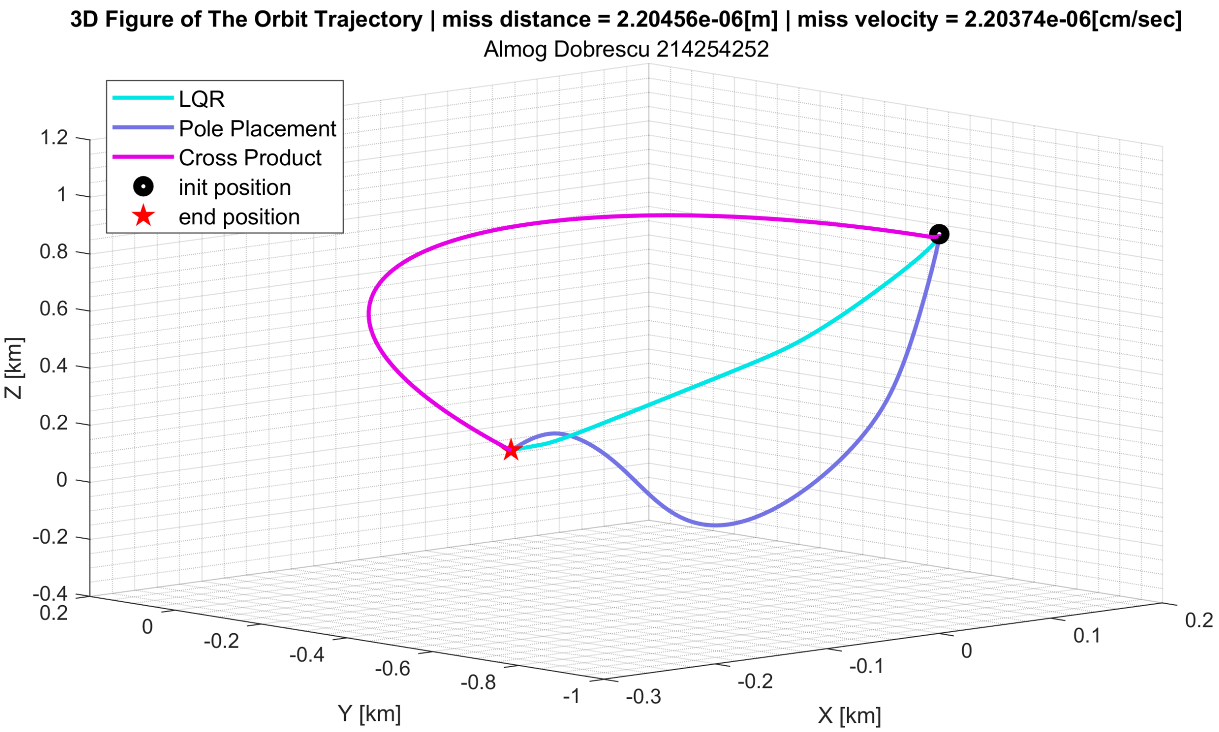

Figure 1: 3D figure of the orbit trajectory

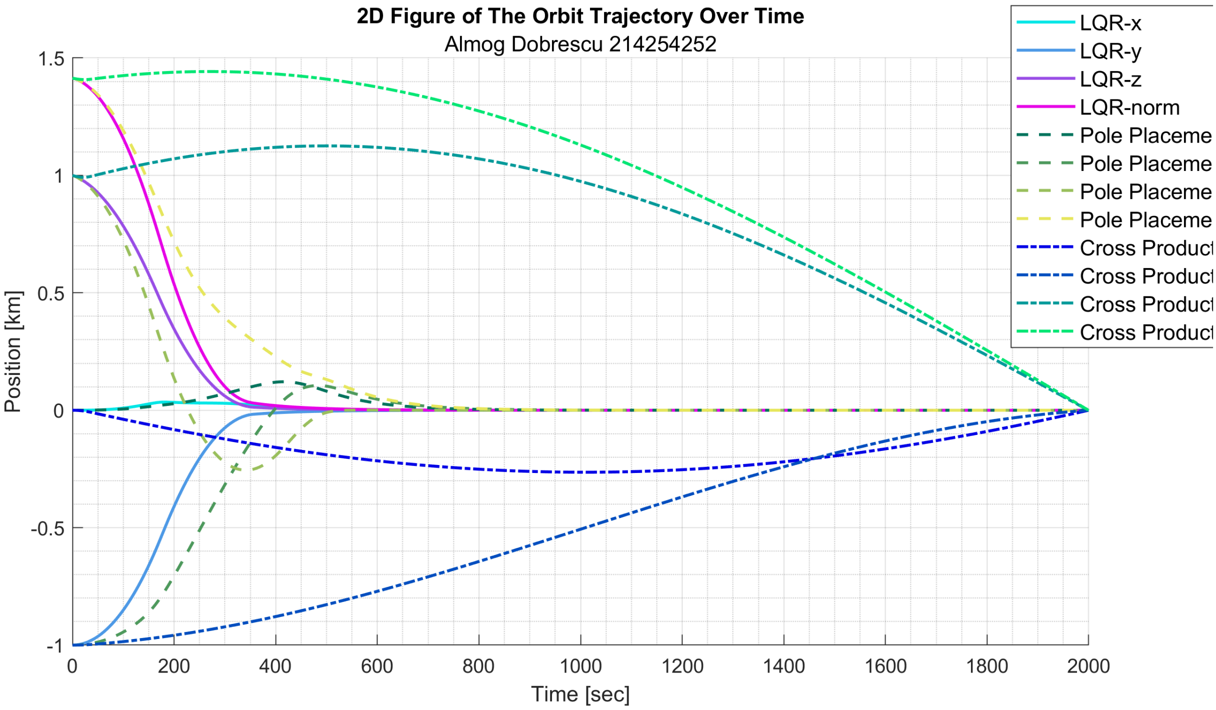

Figure 2: 2D figure of the orbit target trajectory over time

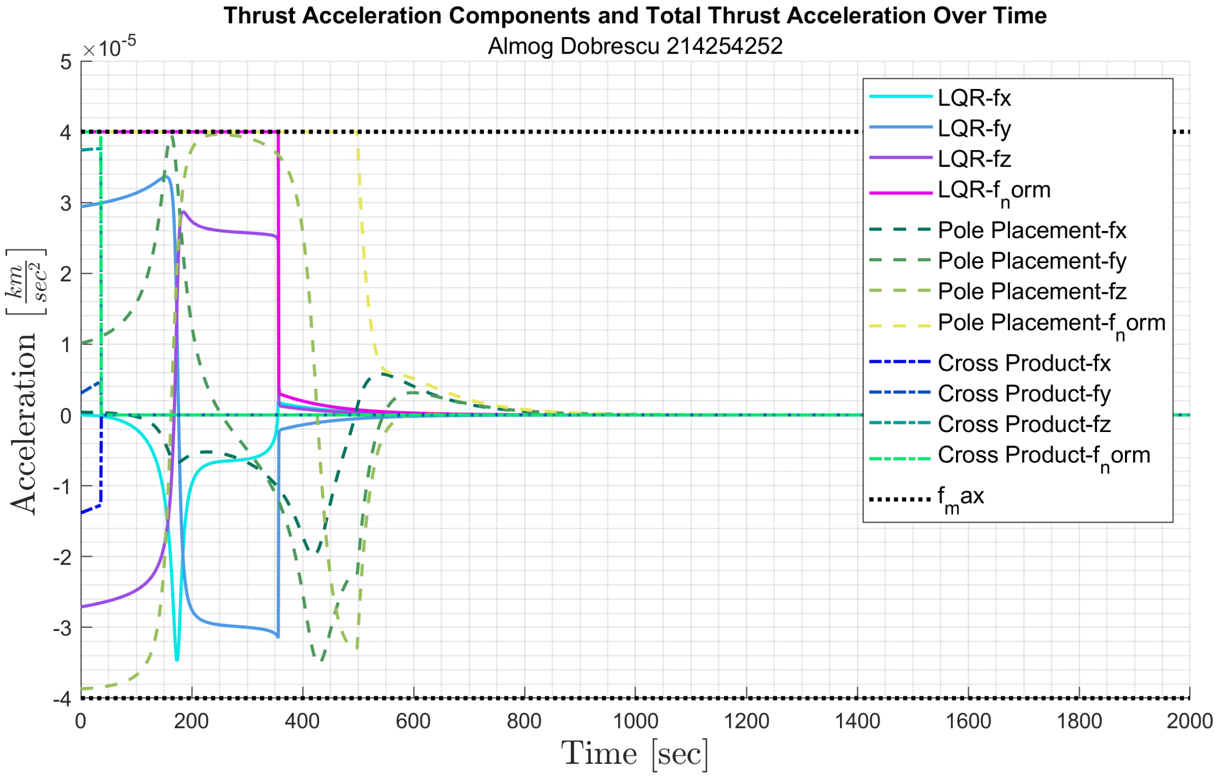

Figure 3: Thrust acceleration components and total thrust acceleration over time

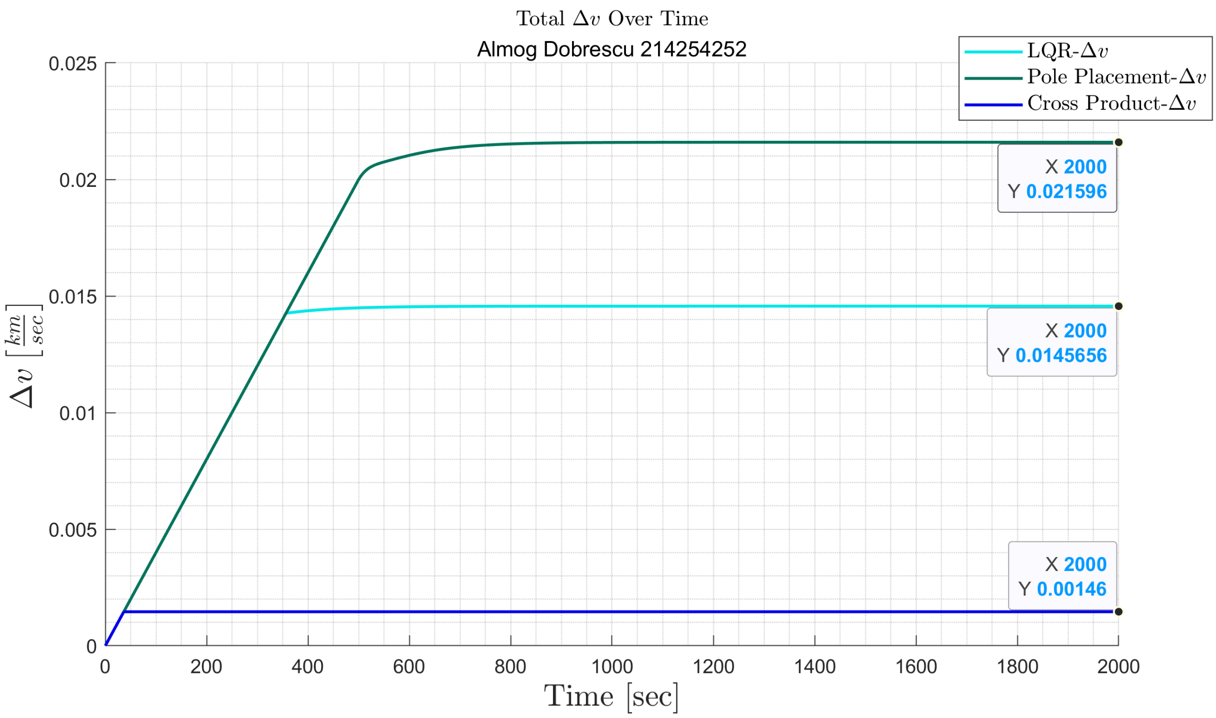

Figure 4: Total Δv over time

We can see that we indeed acomplished the design criteria:

- The thrust doesn't exceed the maximum available thrust.
- The miss distance at the final desired time is less than $1[\mathrm{m}]$ $\left(2.2047 \cdot 10^{-6}\,[\mathrm{m}]\right)$.
- The miss velocity at the final desired time is less than $1\left[\frac{\mathrm{cm}}{\mathrm{sec}}\right]$ $\left(2.2037 \cdot 10^{-6}\left[\frac{\mathrm{cm}}{\mathrm{sec}}\right]\right)$.

The total Δv is: $0.0146\left[\frac{\mathrm{km}}{\mathrm{sec}}\right]$