Satellite Orbit Control
HW2

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1 Given

$$T_1 = 100\,[min] = 6 \cdot 10^3\,[sec] \qquad T_2 = T_1 = 6 \cdot 10^3\,[sec]$$
$$e_1 = 0 \qquad e_2 = 0$$
$$a_1 = \sqrt[3]{\frac{\mu T_1^2}{4\pi^2}} = 7.1366 \cdot 10^3\,[km] \qquad a_2 = a_1 = 7.1366 \cdot 10^3\,[km]$$
$$\alpha = \Delta i = 0.01^\circ$$

In CW frame with origin at Satellite #1 and at $t = 0$:

$$\begin{pmatrix} x_2(0) = 0 \\ y_2(0) = -1 \\ z_2(0) = 1 \end{pmatrix} [km] \qquad \begin{pmatrix} \dot{x}_2(0) = ?? \\ \dot{y}_2(0) = ?? \\ \dot{z}_2(0) < 0 \end{pmatrix}$$

Desired:

$$\begin{pmatrix} x_2(t_1) = 0 \\ y_2(t_1) = 0 \\ z_2(t_1) = 0 \end{pmatrix} \qquad \begin{pmatrix} \dot{x}_2(t_1) = 0 \\ \dot{y}_2(t_1) = 0 \\ \dot{z}_2(t_1) = 0 \end{pmatrix}$$

Limitations:

$$\Delta v = t \begin{pmatrix} 0 \\ \Delta v_y \\ \Delta v_z \end{pmatrix}$$

2 A

General equations of motion in CW fram:

$$\begin{cases} \ddot{x} - 2n\dot{y} - 3n^2 x = f_x \\ \ddot{y} + 2n\dot{x} = f_y \\ \ddot{z} + n^2 z = f_z \end{cases} \tag{1}$$

The velocities:

$$\begin{cases} u_x = \dot{x} - ny(t) \\ u_y = \dot{y} + nx(t) \\ u_z = \dot{z} \end{cases} \tag{2}$$

The solution without external forces (i.e. $\vec{f} = \vec{0}$):

$$\begin{cases} x(t) = (4 - 3\cos(nt)) \cdot x_0 + \frac{\dot{x}_0}{n}\sin(nt) + \frac{2}{n}(1 - \cos(nt)) \cdot \dot{y}_0 \\ y(t) = 6\,(\sin(nt) - nt) \cdot x_0 + y_0 + \frac{2}{n}(\cos(nt) - 1) \cdot \dot{x}_0 + (4\sin(nt) - 3nt)\,\frac{\dot{y}_0}{n} \\ z(t) = z_0\cos(nt) + \frac{\dot{z}_0}{n}\sin(nt) \end{cases} \tag{3}$$

Because the two satellite have the same period, then in CW fram, it is a no-drift orbit:

$$\dot{y}_0 = -2nx_0 \tag{4}$$

From the angle between the plains we can claculate $\dot{z}_2(0)$:

$$\tan\alpha = \frac{z_{max}}{a_1} = \frac{\sqrt{z_2(0)^2 + \left(\frac{\dot{z}_2(0)}{n}\right)^2}}{a_1} \tag{5}$$

$$\frac{\dot{z}_2(0)}{n} = \underbrace{-}_{\dot{z}_2(0) < 0} \sqrt{(a_1 \cdot \tan(\alpha))^2 - z_2(0)^2} \tag{6}$$

$$\frac{\dot{z}_2(0)}{n} = -0.7426; \tag{7}$$

Because we can't apply a velocity pulse in the x direction, $\dot{x}_2(0) = 0$. The CW frame is therefor:

$$\begin{pmatrix} x_2(0) = 0 \\ y_2(0) = -1 \\ z_2(0) = 1 \end{pmatrix} [km] \qquad \begin{pmatrix} \dot{x}_2(0) = 0 \\ \dot{y}_2(0) = 0 \\ \dot{z}_2(0) = -0.74267 \cdot n \end{pmatrix} \left[\frac{km}{sec}\right] \tag{8}$$

The CW equations in state-space form:

$$\begin{array}{cc} \text{x-y plane:} & \text{z-direction:} \\ x1 = x & \\ x2 = \dot{x} & z1 = z \\ y1 = y & \\ y2 = \dot{y} & z2 = \dot{z} \end{array} \tag{9}$$

The homogeneous solution:

$$\vec{x}(t) = \Phi_{(t,t_0)}\vec{x}_0 \tag{10}$$

x-y plane:

$$\Phi_{(t,t_0)} = \left(\begin{array}{cc|cc} 4-3\cos(n\tau) & 0 & \frac{1}{n}\sin(n\tau) & \frac{2}{n}(1-\cos(n\tau)) \\ 6(\sin(n\tau)-n\tau) & 1 & \frac{2}{n}(\cos(n\tau)-1) & \frac{1}{n}(4\sin(n\tau)-3n\tau) \\ \hline 3n\sin(n\tau) & 0 & \cos(n\tau) & 2\sin(n\tau) \\ 6n(\cos(n\tau)-1) & 0 & -2\sin(n\tau) & 4\cos(n\tau)-3 \end{array}\right) = \begin{pmatrix} \Phi_{11(t,t_0)} & \Phi_{12(t,t_0)} \\ \Phi_{21(t,t_0)} & \Phi_{22(t,t_0)} \end{pmatrix}$$

z-direction:

$$\Phi_{(t,t_0)} = \begin{pmatrix} \cos(n\tau) & \frac{1}{n}\sin(n\tau) \\ -n\sin(n\tau) & \cos(n\tau) \end{pmatrix} \tag{11}$$

Where:

$$\tau = t - t_0$$

Desired:

$$\begin{pmatrix} x1 \\ y1 \\ x2 \\ y2 \end{pmatrix}(t_1) = \vec{0}$$

$$\begin{pmatrix} z1 \\ z2 \end{pmatrix}(t_1) = \vec{0} \tag{12}$$

The required velocity components are found from the first 2 lines of $\vec{x}(t_1) = \Phi_(t_1, 0)x(\vec{0})$:

$$\begin{pmatrix} x_{(t_1)} \\ y_{(t_1)} \end{pmatrix} = \vec{0} = \Phi_{11(t_1,0)} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \Phi_{12(t_1,0)} \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \end{pmatrix} \tag{13}$$

$$\Downarrow$$

$$\begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \end{pmatrix} = -\Phi^{-1}_{12(t_1,0)}\Phi_{11(t_1,0)} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \equiv \mathbf{C}^*_{(t_1,0)} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \tag{14}$$

The terminal velocity is determined from the last two rows of $\vec{x}(t_1) = \Phi_(t_1, 0)x(\vec{0})$:

$$\begin{pmatrix} \dot{x}_{(t_1)} \\ \dot{y}_{(t_1)} \end{pmatrix} = \Phi_{21(t_1,0)} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} + \Phi_{22(t_1,0)} \begin{pmatrix} \dot{x}_{req} \\ \dot{y}_{req} \end{pmatrix} \tag{15}$$

Because we can't apply a velocity pulse in the x direction, we set $\dot{x}_{t_1}$ to be zero. Using *MatLab*, the first equation after substituting the initial conditions is:

$$\frac{2\pi\left(\cos\left(\frac{\pi\tau}{3000}\right)-1\right)}{24000\cos\left(\frac{\pi\tau}{3000}\right)+3\tau\pi\sin\left(\frac{\pi\tau}{3000}\right)-24000}=0 \tag{16}$$

$$\cos\left(\frac{\pi\tau}{3000}\right)=1 \tag{17}$$

$$\frac{\pi\tau}{3000}=2\pi \tag{18}$$

$$\pi\tau=6000\pi \tag{19}$$

$$\tau=6000 \tag{20}$$

$t_1=\tau=6000\,[sec]$ (exactly on period)

3 B

3.1 x-y plane

The first pulse is the difference between the required velocity and the initial velocity:

$$\begin{pmatrix}\Delta v_{x1}\\ \Delta v_{y1}\end{pmatrix}=\begin{pmatrix}\dot{x}_{req}\\ \dot{y}_{req}\end{pmatrix}-\begin{pmatrix}\dot{x}_{(t_1)}\\ \dot{y}_{(t_1)}\end{pmatrix} \tag{21}$$

$$\begin{aligned}\Phi_{11}&=\begin{pmatrix}1&0\\-12\pi&1\end{pmatrix} & \Phi_{12}&=\begin{pmatrix}0&0\\0&-18000\end{pmatrix}\\ \Phi_{21}&=\begin{pmatrix}0&0\\0&0\end{pmatrix} & \Phi_{22}&=\begin{pmatrix}1&0\\0&1\end{pmatrix}\end{aligned} \tag{22}$$

Since $|\Phi_{11}|=0$, we can't use the state space. let's look at the CW equations of motion for no-drift orbti:

$$\begin{gathered}\begin{cases}x_{(t)}=(4-3\cos(nt))\cdot x_0+\frac{\dot{x}_0}{n}\sin(nt)+\frac{2}{n}(1-\cos(nt))\cdot\dot{y}_0\\ y_{(t)}=6(\sin(nt)-nt)\cdot x_0+y_0+\frac{2}{n}(\cos(nt)-1)\cdot\dot{x}_0+(4\sin(nt)-3nt)\frac{\dot{y}_0}{n}\\ z_{(t)}=z_0\cos(nt)+\frac{\dot{z}_0}{n}\sin(nt)\end{cases}\\ \begin{cases}\dot{x}_{(t)}=3nx_0\sin(nt)+\dot{x}_0\cos(nt)+2\dot{y}_0\sin(nt)\\ \dot{y}_{(t)}=6nx_0(\cos(nt)-1)-2\dot{x}_0\sin(nt)+\dot{y}_0(4\cos(nt)-3)\\ \dot{z}_{(t)}=-nz_0\sin(nt)+\dot{z}_0\cos(nt)\end{cases}\end{gathered} \tag{23}$$

$$x_0=0\quad y_0=-1\quad z_0=1\quad x_{(t_1)}=0\quad y_{(t_1)}=0\quad t_1=6000\quad nt=2\pi$$

From the position at the y direction:

$$y_{(t_1)} = 0 = 6\left(\sin(2\pi) - 2\pi\right)\cdot 0 + -1 + \frac{2}{n}\left(\cos(2\pi) - 1\right)\cdot 0 + \left(4\sin(2\pi) - 3\cdot 2\pi\right)\frac{\dot{y}_{req}}{n}$$
$$0 = -1 + \left(-3\cdot 2\pi\right)\frac{\dot{y}_0}{n}$$
$$\dot{y}_{req} = -\frac{n}{6\pi} = -5.5556\cdot 10^{-5}\left[\frac{km}{sec}\right]$$
$$\Downarrow$$
$$\begin{pmatrix}\Delta v_{x1}\\ \Delta v_{y1}\end{pmatrix} = \begin{pmatrix}\dot{x}_{req}\\ \dot{y}_{req}\end{pmatrix} - \begin{pmatrix}\dot{x}_{(0)}\\ \dot{y}_{(0)}\end{pmatrix} = \begin{pmatrix}0\\ -5.5556\cdot 10^{-5}\end{pmatrix} - \begin{pmatrix}0\\ 0\end{pmatrix} \tag{24}$$
$$\Downarrow$$
$$\boxed{\begin{pmatrix}\Delta v_{x1}\\ \Delta v_{y1}\end{pmatrix}_{(t=0)} = \begin{pmatrix}0\\ -5.5556\end{pmatrix}\cdot 10^{-5}\left[\tfrac{km}{sec}\right]}$$

For a rendezvous maneuver, a terminal velocity pulse is applied such that the terminal velocity is zero:

$$\begin{pmatrix}\Delta v_{x2}\\ \Delta v_{y2}\end{pmatrix} = -\begin{pmatrix}\dot{x}_{(t_1)}\\ \dot{y}_{(t_1)}\end{pmatrix} \tag{25}$$
$$= -\begin{pmatrix}3nx_0\sin(nt) + \dot{x}_0\cos(nt) + 2\dot{y}_0\sin(nt)\\ 6nx_0\left(\cos(nt) - 1\right) - 2\dot{x}_0\sin(nt) + \dot{y}_0\left(4\cos(nt) - 3\right)\end{pmatrix} \tag{26}$$
$$= -\begin{pmatrix}-2\cdot 5.5556\cdot 10^{-5}\cdot\sin(2\pi)\\ -5.5556\cdot 10^{-5}\cdot\left(4\cos(2\pi) - 3\right)\end{pmatrix} \tag{27}$$
$$= -\begin{pmatrix}0\\ -5.5556\cdot 10^{-5}\end{pmatrix} \tag{28}$$

$$\Downarrow$$
$$\boxed{\begin{pmatrix}\Delta v_{x2}\\ \Delta v_{y2}\end{pmatrix}_{(t=6000[sec])} = \begin{pmatrix}0\\ 5.5556\end{pmatrix}\cdot 10^{-5}\left[\tfrac{km}{sec}\right]}$$

3.2 z direction

Since $t_f(t_1)$ is bigger than half of a period, there is no need for an initial pulse. Let's find the time until $z = 0$:

$$z_{(t)} = 0 = z_0\cos(nt) + \frac{\dot{z}_0}{n}\sin(nt) \tag{29}$$
$$0 = 1\cdot\cos(nt) + \frac{-0.74267\cdot n}{n}\sin(nt) \tag{30}$$
$$0 = \cos(nt) - 0.74267\sin(nt) \tag{31}$$

$$a\sin(x) + b\cos(x) = \sqrt{a^2 + b^2}\sin(x + \varphi)$$
$$\varphi = \arctan\left(\frac{b}{a}\right)$$
$$1.2456\sin(nt + 2.2096) = 0 \tag{32}$$
$$nt + 2.2096 = \pi\cdot k \tag{33}$$
$$nt = 0.9320 \tag{34}$$
$$t_{(z=0)} = 889.9874[sec] \tag{35}$$

So the velocity in the z direction when $z = 0$:

$$\dot{z}_{(t)} = -nz_0 \sin(nt) + \dot{z}_0 \cos(nt) \tag{36}$$
$$\dot{z}_{(z=0)} = -n \cdot 1 \cdot \sin(0.9320) - 0.74267 \cdot n \cos(0.9320) \tag{37}$$

$$\dot{z}_{(z=0)} = -0.0013 \left[\frac{km}{sec}\right]$$

$$\Downarrow$$

$$\Delta v_{z1(t=889.9874[sec])} = -\dot{z}_{z=0} = 0.0013 \left[\frac{km}{sec}\right]$$

4 C

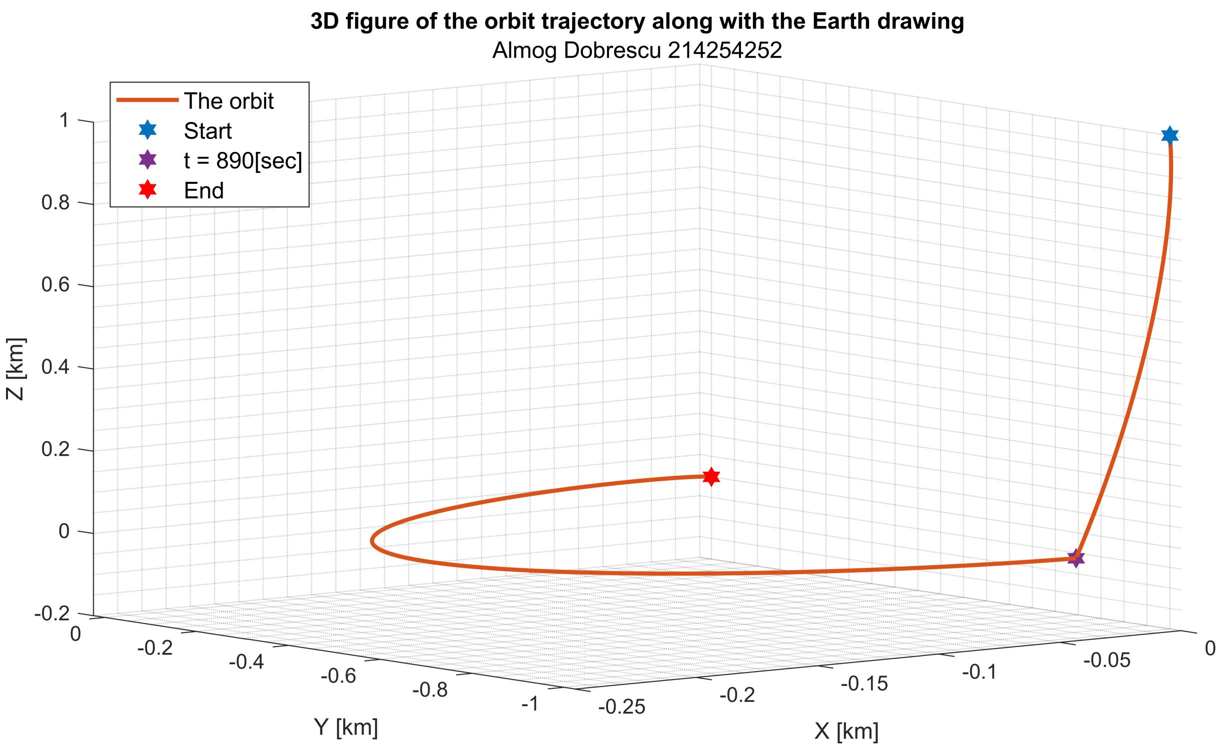

Figure 1: 3D figure of the orbit trajectory along with the Earth drawing