



The Spectral Representation Method: A framework for simulation of stochastic processes, fields, and waves

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ARTICLE INFO

Keywords:

Spectral Representation Method
Stochastic processes
Stochastic fields
Stochastic waves
Simulation
Monte Carlo simulation

ABSTRACT

The Spectral Representation Method (SRM) was developed in the 1970s to simulate Gaussian stochastic processes and fields from a Fourier series expansion according to the Spectral Representation Theorem. Since those early developments, the SRM has continuously evolved into a comprehensive framework for the simulation of stochastic processes, fields, and waves with a rigorous theoretical foundation. Its major advantages are conceptual simplicity and computational efficiency. In the 1990s, much of the theory for simulation of Gaussian stochastic processes, fields, and waves was firmly established and early methods for simulation of non-Gaussian processes, fields, and waves were introduced. In the 2000s and 2010s, methods that coupled the SRM with Translation Process Theory were improved to enable efficient and accurate simulations of stochastic processes, fields, and waves with strongly non-Gaussian marginal probability distributions. More recently, the SRM was extended for higher-order non-Gaussian processes, fields, and waves by extending the Fourier stochastic expansion to include non-linear wave interactions derived from higher-order spectra. This paper reviews the key theoretical developments related with the SRM and provides the relevant algorithms necessary for its practical implementation for the simulation of stochastic processes, fields, and waves that can be either stationary or non-stationary, homogeneous or non-homogeneous, one-dimensional or multi-dimensional, scalar or multi-variate, Gaussian or non-Gaussian, or any combination thereof. The paper concludes with some brief remarks addressing the open research challenges in SRM-based theory and simulations.

1. Introduction - objective

It is well-established today that Monte Carlo simulation is the only universal methodological framework that can provide accurate solutions for any problem in stochastic mechanics (or any other discipline involving problems described by stochastic differential equations) whose corresponding deterministic solution is known (either analytical or numerical). This includes, among many others, problems with nonlinearities, system stochasticity, stochastic stability, parametric excitations, large variations of uncertainties, system failures, etc. Monte Carlo simulation is also used as a benchmark to assess the accuracy of any approximate method in stochastic mechanics, such as perturbation, statistical linearization/quadratization, closure techniques, stochastic averaging, stochastic reduced order models, stochastic collocation methods, etc. The only well-known disadvantage of Monte Carlo simulation is that it can become very time-consuming in certain cases. However, the continuous advances in computing technology (especially in parallel computing) are steadily reducing the extent of this disadvantage.

One of the most important parts of the Monte Carlo simulation methodology is the generation of sample realizations of the stochastic processes, fields, and waves involved in the stochastic problem (these processes/fields/waves can model seismic ground motion accelerations, wind velocity fluctuations, ocean wave heights, material and soil constitutive properties, geometric imperfections of structures, road roughness, air temperature and humidity variations, ocean salinity variations, mean annual precipitation, electricity demand, and many other quantities). The generated sample functions must accurately describe the probabilistic characteristics of the corresponding stochastic processes, fields, and waves that may be either stationary or non-stationary, homogeneous or non-homogeneous, one-dimensional or multi-dimensional, scalar or multi-variate, Gaussian or non-Gaussian, or any combination thereof.

Among the various methods currently available to generate sample realizations of stochastic processes, fields, and waves, the Spectral

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<https://doi.org/10.1016/j.ress.2024.110522>

Representation Method (SRM) is one of the (i) most efficient computationally, (ii) most straightforward conceptually, (iii) easiest to implement for any type of stochastic process, field, or wave, and, consequently, (iv) most widely used. The main objective of this paper is to provide a description of the Spectral Representation Method as it pertains to its application to a wide range of different types of stochastic processes, fields, and waves.

2. The spectral representation theorem

Let $f_0(t)$ be a one-dimensional, uni-variate (1D-1V) stationary stochastic process with mean value equal to zero, autocorrelation function $R_{f_0f_0}(\tau)$ and two-sided power spectral density function $S_{f_0f_0}(\omega)$. Then, the following relations hold:

$$\mathcal{E}[f_0(t)] = 0 \quad \text{and} \quad \mathcal{E}[f_0(t + \tau)f_0(t)] = R_{f_0f_0}(\tau) \quad (1)$$

$$S_{f_0f_0}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{f_0f_0}(\tau) e^{-i\omega\tau} d\tau \quad \text{and} \quad R_{f_0f_0}(\tau) = \int_{-\infty}^{\infty} S_{f_0f_0}(\omega) e^{i\omega\tau} d\omega \quad (2)$$

where Eq. (2) constitutes the Wiener–Khinchine transform pair.

The following theorem is fundamental to the theory of 1D-1V stationary stochastic processes with mean value equal to zero (e.g., Yaglom [1], Cramér and Leadbetter [2]).

To every real-valued 1D-1V stationary stochastic process $f_0(t)$ with mean value equal to zero and two-sided power spectral density function $S_{f_0f_0}(\omega)$, two mutually orthogonal real processes $u(\omega)$ and $v(\omega)$ with orthogonal increments $du(\omega)$ and $dv(\omega)$ can be assigned such that:

$$f_0(t) = \int_0^{\infty} [\cos(\omega t) du(\omega) + \sin(\omega t) dv(\omega)] \quad (3)$$

The processes $u(\omega)$ and $v(\omega)$ and the corresponding increments $du(\omega)$ and $dv(\omega)$ are defined for $\omega \geq 0$ and satisfy a set of requirements (e.g., Yaglom [1], Cramér and Leadbetter [2]), including:

$$\mathcal{E}[u^2(\omega)] = \mathcal{E}[v^2(\omega)] = 2S_{F_0F_0}(\omega) \quad \text{for} \quad \omega \geq 0 \quad (4)$$

$$\mathcal{E}[du^2(\omega)] = \mathcal{E}[dv^2(\omega)] = 2S_{f_0f_0}(\omega)d\omega \quad \text{for} \quad \omega \geq 0 \quad (5)$$

where it is assumed that $f_0(t)$ is associated with a differentiable power spectral distribution function $S_{F_0F_0}(\omega)$ whose derivative is the power spectral density function $S_{f_0f_0}(\omega)$:

$$\frac{dS_{F_0F_0}(\omega)}{d\omega} = S_{f_0f_0}(\omega) \quad \text{for} \quad \omega \geq 0 \quad (6)$$

The power spectral distribution function $S_{F_0F_0}(\omega)$ is finite in the limit as $\omega \rightarrow \infty$ which is equivalent to:

$$\int_0^{\infty} 2S_{f_0f_0}(\omega)d\omega < \infty \quad (7)$$

Eq. (3) is now discretized and rewritten in the following form:

$$f_0(t) = \sum_{k=0}^{\infty} [\cos(\omega_k t) du(\omega_k) + \sin(\omega_k t) dv(\omega_k)], \quad \omega_k = k \Delta\omega \quad (8)$$

with sufficiently small but finite $\Delta\omega$, such that Eq. (8) can be used for Eq. (3).

If $du(\omega_k)$ and $dv(\omega_k)$ are defined as:

$$du(\omega_k) = X_k \quad \text{and} \quad dv(\omega_k) = Y_k \quad (9)$$

and if the X_k 's and the Y_k 's are independent random variables with mean value equal to zero and standard deviation $\sqrt{2S_{f_0f_0}(\omega_k)\Delta\omega}$, it is easy to show (Shinozuka and Deodatis [3]) that all requirements imposed on $du(\omega_k)$ and $dv(\omega_k)$ (e.g., Yaglom [1], Cramér and Leadbetter [2]) are satisfied. Then, substituting Eq. (9) into Eq. (8), the following series representation is obtained:

$$f_0(t) = \sum_{k=0}^{\infty} [\cos(\omega_k t) X_k + \sin(\omega_k t) Y_k] \quad (10)$$

On the other hand, if $du(\omega_k)$ and $dv(\omega_k)$ are defined as in Shinozuka [4]:

$$du(\omega_k) = \sqrt{2} A_k \cos \Phi_k \quad \text{and} \quad dv(\omega_k) = -\sqrt{2} A_k \sin \Phi_k \quad (11)$$

in which:

$$A_k = \sqrt{2S_{f_0f_0}(\omega_k)\Delta\omega}, \quad \omega_k = k \cdot \Delta\omega \quad (12)$$

and the Φ_k 's are independent random phase angles uniformly distributed in the range $[0, 2\pi]$, it is a straightforward task to show again (Shinozuka and Deodatis [3]) that all requirements imposed on $du(\omega_k)$ and $dv(\omega_k)$ (e.g., Yaglom [1], Cramér and Leadbetter [2]) are satisfied. Then, substituting Eqs. (11) and (12) into Eq. (8), the following series representation is obtained:

$$f_0(t) = \sqrt{2} \sum_{k=0}^{\infty} \sqrt{2S_{f_0f_0}(\omega_k)\Delta\omega} \cos(\omega_k t + \Phi_k) \quad (13)$$

Both series representation expressions displayed in Eqs. (10) and (13) are consistent with the spectral representation theorem stated in Eq. (3).

Shinozuka and Deodatis [3] and Grigoriu [5] showed that when Eq. (13) is used for simulation purposes (in such a case, as will be shown later, the summation in Eq. (13) goes up to a finite number N rather than infinity), it produces ergodic sample functions, in the sense that the temporally-averaged mean value and autocorrelation function of each and every generated sample function are identical with the corresponding targets when the averaging takes place over the fundamental period of the cosine series. On the other hand, generated sample functions produced by Eq. (10) are not ergodic. This is the main reason why Eq. (13) and its derivatives will be used exclusively in this paper. Grigoriu [5] showed that when Eq. (13) is used for simulation purposes (in such a case, the summation in Eq. (13) goes to a finite number N rather than infinity), it generates sample functions that are Gaussian only in the limit as $N \rightarrow \infty$. In contrast, sample functions generated by Eq. (10) (also with a finite summation going up to N instead of ∞) are always Gaussian regardless of the value of N when the X_k 's and Y_k 's are normally distributed.

The expressions given in Eqs. (10) (with X_k and Y_k normally distributed) and (13) were mentioned for the first time in a seminal paper by Rice [6] where the summations were provided as finite (N instead of ∞). As stated in Rice [6], the representation in Eq. (10) was used by Einstein and Hopf [7] for black-body radiation and by Schottky [8] for representing the shot effect current but without considering the X_k 's and Y_k 's to be normally distributed. However, Eqs. (10) and (13) were not used for simulation purposes by either Einstein and Hopf [7], Schottky [8] or Rice [6]. The first to use Eq. (13) to generate sample functions of a stochastic process was Borgman [9], but without making a connection to the Spectral Representation Theorem or Rice's 1954 work. Borgman [9] proposed a version of Eq. (13) to simulate ocean wave processes according to the concept of wave superposition. It was Shinozuka [4] and Shinozuka and Jan [10] who proposed both Eqs. (10) and (13) for simulation of stochastic processes and clearly mentioned that Rice [6] was the one who introduced the (finite) series representations in Eqs. (10) and (13). Following Shinozuka's work, Yang [11,12] showed that the Fast Fourier Transform (FFT) technique can be used on Eq. (13) to dramatically improve the computational efficiency of the algorithm.

The Spectral Representation Method (SRM) includes any simulation algorithm for stochastic processes, fields, and waves that is based on the (finite) series representations in Eqs. (10) and (13).

3. One-dimensional, uni-variate (scalar), stationary, and Gaussian stochastic processes

The simulation of one-dimensional, uni-variate (1D-1V) stationary stochastic processes using the Spectral Representation Method was first proposed in Borgman [9], Shinozuka [4] and Shinozuka and Jan [10]. The contents of this section follow mainly Shinozuka and Deodatis [3] and the reader should refer to this paper for additional detail and examples.

Let $f_0(t)$ be a 1D-1V stationary stochastic process as defined in Eqs. (1) and (2). In the following, distinction will be made between the stochastic process $f_0(t)$ and its simulation $f(t)$. From the infinite series representation displayed in Eq. (13), it follows that the stochastic process $f_0(t)$ can be simulated by the following series as $N \rightarrow \infty$:

$$f(t) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos(\omega_n t + \Phi_n) \quad (14)$$

where:

$$A_n = \sqrt{2S_{f_0 f_0}(\omega_n) \Delta\omega} \quad ; \quad \omega_n = n\Delta\omega \quad ; \quad \Delta\omega = \frac{\omega_u}{N} \quad (15)$$

and:

$$A_0 = 0 \quad \text{or} \quad S_{f_0 f_0}(\omega_0) = 0 \quad (16)$$

In Eq. (15), ω_u represents an upper cut-off frequency beyond which the power spectral density function $S_{f_0 f_0}(\omega)$ may be assumed to be zero for either mathematical or physical reasons. As such, ω_u is a fixed value and hence $\Delta\omega \rightarrow 0$ as $N \rightarrow \infty$ so that $N\Delta\omega = \omega_u$. The $\Phi_0, \Phi_1, \Phi_2, \dots, \Phi_{N-1}$ appearing in Eq. (14) are independent random phase angles distributed uniformly over the interval $[0, 2\pi]$. It is easy to show that the simulated stochastic process $f(t)$ given by Eq. (14) is periodic with period T_0 :

$$T_0 = \frac{2\pi}{\Delta\omega} \quad (17)$$

Eq. (17) indicates that the smaller $\Delta\omega$, or equivalently the larger N under a specified upper cut-off frequency value ω_u , the longer the period of the simulated stochastic process.

The simulated stochastic process $f(t)$ is asymptotically Gaussian as $N \rightarrow \infty$ because of the central limit theorem.

A sample function $f^{(i)}(t)$ of the simulated stochastic process $f(t)$ can be obtained by replacing the sequence of random phase angles $\Phi_0, \Phi_1, \dots, \Phi_{N-1}$ with their respective i th realizations $\phi_0^{(i)}, \phi_1^{(i)}, \dots, \phi_{N-1}^{(i)}$:

$$f^{(i)}(t) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos(\omega_n t + \phi_n^{(i)}) \quad (18)$$

The condition set in Eq. (16) is necessary (and must be forced if $S_{f_0 f_0}(0) \neq 0$) to guarantee that the temporal average and the temporal autocorrelation function of any sample function $f^{(i)}(t)$ are identical to the corresponding targets, $\mathcal{E}[f_0(t)] = 0$ and $R_{f_0 f_0}(\tau)$, respectively (this is the ergodicity property that will be discussed later in this section). However, to avoid having to impose the condition in Eq. (16), the alternative of using the frequency shifting theorem (e.g. Papoulis [13]) has been proposed by Zerva [14].

When generating sample functions of the simulated stochastic process according to Eq. (18), the time step Δt separating the generated values of $f^{(i)}(t)$ in the time domain has to obey the condition:

$$\Delta t \leq \frac{2\pi}{2\omega_u} \quad (19)$$

to avoid aliasing according to the sampling theorem (e.g. Bracewell [15]).

The generated values of $f^{(i)}(t)$ according to Eq. (18) are bounded as follows:

$$f^{(i)}(t) \leq \sqrt{2} \sum_{n=0}^{N-1} A_n = \sqrt{2} \sum_{n=0}^{N-1} \sqrt{2S_{f_0 f_0}(\omega_n) \Delta\omega} \quad (20)$$

The above bound is large enough for practical applications, even for relatively small values of N . For additional information, the reader is referred to Shinozuka and Deodatis [3].

The following two properties of generated sample functions using Eq. (18) are central to the Spectral Representation Method based on the series representation shown in Eq. (13). For a detailed derivation of both properties, the reader is referred to Shinozuka and Deodatis [3].

1. **Ensemble Average Property:** The ensemble expected value $\mathcal{E}[f(t)]$ and the ensemble autocorrelation function $R_{ff}(\tau)$ of the simulated stochastic process $f(t)$ are identical to the corresponding

targets, $\mathcal{E}[f_0(t)] = 0$ and $R_{f_0 f_0}(\tau)$, respectively. Specifically for the autocorrelation function:

$$R_{ff}(\tau) = R_{f_0 f_0}(\tau), \quad \text{for } 0 \leq \tau \leq T \quad (21)$$

The value of T appearing in Eq. (21) can be less than or equal to the period T_0 defined in Eq. (17). If, in addition, $T = T_0$, then:

$$S_{ff}(\omega) = S_{f_0 f_0}(\omega), \quad \text{for } 0 \leq \omega \leq \omega_u \quad (22)$$

2. **Ergodicity Property:** The temporal average $\langle f^{(i)}(t) \rangle_T$ and the temporal autocorrelation function $\langle f^{(i)}(t + \tau) f^{(i)}(t) \rangle_T$ of any sample function $f^{(i)}(t)$ are identical to the corresponding targets, $\mathcal{E}[f_0(t)] = 0$ and $R_{f_0 f_0}(\tau)$, respectively. These two identities are valid only when the length T of the sample function $f^{(i)}(t)$ is either equal to the period T_0 or when it approaches infinity. Specifically:

$$\langle f^{(i)}(t) \rangle_T = 0 = \mathcal{E}[f_0(t)], \quad \text{when either } T = T_0 \text{ or } T \rightarrow \infty \quad (23)$$

and:

$$\langle f^{(i)}(t + \tau) f^{(i)}(t) \rangle_T = R_{f_0 f_0}(\tau), \quad \text{when either } T = T_0 \text{ or } T \rightarrow \infty \quad (24)$$

Eq. (24) leads to:

$$\langle S_{ff}^{(i)}(\omega) \rangle_T = S_{f_0 f_0}(\omega), \quad \text{for } 0 \leq \omega \leq \omega_u, \quad \text{when } T = T_0 \text{ or } T \rightarrow \infty \quad (25)$$

where $\langle S_{ff}^{(i)}(\omega) \rangle_T$ is the temporal power spectral density function.

Use of Fast Fourier Transform (FFT) Technique

The computational cost of generating sample functions using Eq. (18) can be drastically reduced by using the FFT technique (e.g. 16). To take advantage of the FFT technique, Eq. (18) is rewritten as:

$$f^{(i)}(p \Delta t) = \text{Re} \left\{ \sum_{n=0}^{M-1} B_n \exp[i(n\Delta\omega)(p\Delta t)] \right\}, \quad p = 0, 1, \dots, M-1 \quad (26)$$

where Re indicates the real part and B_n stands for:

$$B_n = \sqrt{2} A_n e^{i\phi_n^{(i)}}, \quad n = 0, 1, \dots, M-1 \quad (27)$$

Sample function $f^{(i)}(p \Delta t)$ in Eq. (26) is periodic with period T_0 given in Eq. (17). Hence, Δt and $\Delta\omega$ are related in the following way:

$$M \Delta t = T_0 = \frac{2\pi}{\Delta\omega} \quad (28)$$

With the aid of Eq. (28), Eq. (26) can be written as:

$$f^{(i)}(p \Delta t) = \text{Re} \left\{ \sum_{n=0}^{M-1} B_n \exp \left[i n p \frac{2\pi}{M} \right] \right\}, \quad p = 0, 1, \dots, M-1 \quad (29)$$

It is easy to show that the condition set in Eq. (19) is equivalent to:

$$M \geq 2N \quad (30)$$

Instead of using Eq. (18) involving a straightforward summation of cosines, the FFT technique can be readily applied on Eq. (29), resulting in a drastic reduction of the computational cost.

The following conditions must be imposed when using Eq. (29):

$$B_0 = 0 \quad \text{and} \quad B_n = 0 \quad \text{for } N \leq n \leq M-1 \quad (31)$$

The first condition in Eq. (31) is equivalent to the condition set in Eq. (16), while the second condition in Eq. (31) represents the assumption that the power spectral density function $S_{f_0 f_0}(\omega)$ is assumed to be zero for values of the frequency larger than ω_u .

To avoid having to impose the first condition in Eq. (31), the alternative of using the frequency shifting theorem (e.g. Papoulis 1962) has been proposed by Zerva (1992). The basic idea is that the power spectral density function is computed at the following discrete frequencies: $\omega_n = n\Delta\omega + \frac{\Delta\omega}{2}$ ($n = 0, 1, 2, \dots, N-1$), rather than at: $\omega_n = n\Delta\omega$ ($n = 0, 1, 2, \dots, N-1$). This alternative definition of sampling frequencies

avoids having to define the power spectral density function at zero frequency (the first sampling frequency is at $\frac{\Delta\omega}{2}$). A side effect is the doubling of the period of the simulated stochastic process to $\frac{4\pi}{\Delta\omega}$.

4. Multi-dimensional, uni-variate (scalar), homogeneous, and Gaussian stochastic fields

The simulation of multi-dimensional (mD), uni-variate (1V), homogeneous, and Gaussian stochastic fields is a rather straightforward extension of the case of one-dimensional, uni-variate (scalar), stationary, and Gaussian stochastic processes presented in the previous section. The simulation of mD-1V homogeneous stochastic fields using the Spectral Representation Method was first proposed in Shinozuka [4] and Shinozuka and Jan [10] (Borgman [9] proposed a very similar algorithm, although it was not based on the series representations of Rice [6]). This section presents the simulation of 2D-1V, homogeneous, and Gaussian stochastic fields, mainly following Shinozuka and Deodatis [17]. The reader should refer to Shinozuka and Deodatis [17] for additional detail and examples, as well as for the simulation of 3D-1V and mD-1V stochastic fields.

Let $f_0(x_1, x_2)$ be a two-dimensional, uni-variate (2D-1V), homogeneous stochastic field with mean value equal to zero, autocorrelation function $R_{f_0f_0}(\xi_1, \xi_2)$ and power spectral density function $S_{f_0f_0}(\kappa_1, \kappa_2)$. Then, the following relations hold:

$$\mathcal{E}[f_0(x_1, x_2)] = 0 \quad (32)$$

$$\mathcal{E}[f_0(x_1 + \xi_1, x_2 + \xi_2)f_0(x_1, x_2)] = R_{f_0f_0}(\xi_1, \xi_2) \quad (33)$$

$$S_{f_0f_0}(\kappa_1, \kappa_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{f_0f_0}(\xi_1, \xi_2) e^{-i(\kappa_1\xi_1 + \kappa_2\xi_2)} d\xi_1 d\xi_2 \quad (34)$$

$$R_{f_0f_0}(\xi_1, \xi_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{f_0f_0}(\kappa_1, \kappa_2) e^{i(\kappa_1\xi_1 + \kappa_2\xi_2)} d\kappa_1 d\kappa_2 \quad (35)$$

where ξ_1 and ξ_2 are the separation distances along the x_1 and x_2 directions respectively, and κ_1 and κ_2 are the corresponding wave numbers. Eqs. (34) and (35) constitute the two-dimensional version of the Wiener-Khinchine transform pair. In general, $R_{f_0f_0}(\xi_1, \xi_2)$ and $S_{f_0f_0}(\kappa_1, \kappa_2)$ are symmetric around the origin. For the special case of a *quadrant* 2D-1V homogeneous stochastic field (VanMarcke [18]), the following extra symmetries are valid:

$$R_{f_0f_0}(\xi_1, \xi_2) = R_{f_0f_0}(\xi_1, -\xi_2) = R_{f_0f_0}(-\xi_1, \xi_2) = R_{f_0f_0}(-\xi_1, -\xi_2) \quad (36)$$

$$S_{f_0f_0}(\kappa_1, \kappa_2) = S_{f_0f_0}(\kappa_1, -\kappa_2) = S_{f_0f_0}(-\kappa_1, \kappa_2) = S_{f_0f_0}(-\kappa_1, -\kappa_2) \quad (37)$$

The reader should refer to Shinozuka and Deodatis [17] for simulation algorithms concerning quadrant 2D-1V homogeneous stochastic fields.

In the following, distinction will be made between the stochastic field $f_0(x_1, x_2)$ and its simulation $f(x_1, x_2)$. The following series can be used to simulate $f_0(x_1, x_2)$, as $N_1, N_2 \rightarrow \infty$ simultaneously:

$$f(x_1, x_2) = \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left[A_{n_1n_2} \cos(\kappa_{1n_1}x_1 + \kappa_{2n_2}x_2 + \Phi_{n_1n_2}^{(1)}) + \tilde{A}_{n_1n_2} \cos(\kappa_{1n_1}x_1 - \kappa_{2n_2}x_2 + \Phi_{n_1n_2}^{(2)}) \right] \quad (38)$$

where:

$$A_{n_1n_2} = \sqrt{2S_{f_0f_0}(\kappa_{1n_1}, \kappa_{2n_2}) \Delta\kappa_1 \Delta\kappa_2} \quad (39)$$

$$\tilde{A}_{n_1n_2} = \sqrt{2S_{f_0f_0}(\kappa_{1n_1}, -\kappa_{2n_2}) \Delta\kappa_1 \Delta\kappa_2} \quad (40)$$

$$\kappa_{1n_1} = n_1 \Delta\kappa_1 \quad ; \quad \kappa_{2n_2} = n_2 \Delta\kappa_2 \quad ; \quad \Delta\kappa_1 = \frac{\kappa_{1u}}{N_1} \quad ; \quad \Delta\kappa_2 = \frac{\kappa_{2u}}{N_2} \quad (41)$$

and:

$$A_{0n_2} = A_{n_10} = 0, \text{ for } n_1 = 0, 1, \dots, N_1 - 1 \text{ and } n_2 = 0, 1, \dots, N_2 - 1 \quad (42)$$

$$\tilde{A}_{0n_2} = \tilde{A}_{n_10} = 0, \text{ for } n_1 = 0, 1, \dots, N_1 - 1 \text{ and } n_2 = 0, 1, \dots, N_2 - 1 \quad (43)$$

Eqs. (42) and (43) are equivalent to:

$$S_{f_0f_0}(0, \kappa_2) = S_{f_0f_0}(\kappa_1, 0) = 0, \text{ for } -\infty < \kappa_1 < \infty \text{ and } -\infty < \kappa_2 < \infty \quad (44)$$

In Eq. (41), κ_{1u} and κ_{2u} are the upper cut-off wave numbers along κ_1 and κ_2 , respectively. This implies that the power spectral density function $S_{f_0f_0}(\kappa_1, \kappa_2)$ is assumed to be zero, for either mathematical or physical reasons, outside the region defined by:

$$-\kappa_{1u} \leq \kappa_1 \leq \kappa_{1u} \quad \text{and} \quad -\kappa_{2u} \leq \kappa_2 \leq \kappa_{2u} \quad (45)$$

As such, κ_{1u} and κ_{2u} are fixed values and therefore $\Delta\kappa_1 \rightarrow 0$ and $\Delta\kappa_2 \rightarrow 0$ as $N_1 \rightarrow \infty$ and $N_2 \rightarrow \infty$, so that $N_1\Delta\kappa_1 = \kappa_{1u}$ and $N_2\Delta\kappa_2 = \kappa_{2u}$. The $\Phi_{n_1n_2}^{(1)}$ and $\Phi_{n_1n_2}^{(2)}$; $n_1 = 0, 1, \dots, N_1 - 1$; $n_2 = 0, 1, \dots, N_2 - 1$ appearing in Eq. (38) are two sets of independent random phase angles distributed uniformly over the interval $[0, 2\pi]$. It is easy to show that the simulated stochastic field $f(x_1, x_2)$ given by Eq. (38) is periodic along the x_1 and x_2 axes with periods:

$$L_{x_10} = \frac{2\pi}{\Delta\kappa_1} \text{ along the } x_1 \text{ axis} \quad ; \quad L_{x_20} = \frac{2\pi}{\Delta\kappa_2} \text{ along the } x_2 \text{ axis} \quad (46)$$

Eq. (46) indicates that the smaller $\Delta\kappa_1$, or equivalently the larger N_1 under a specified upper cut-off wave number value κ_{1u} , the longer the period of the simulated stochastic field along the x_1 axis. A similar conclusion can be drawn from Eq. (46) concerning the period of the simulated stochastic field along the x_2 axis.

The simulated stochastic field $f(x_1, x_2)$ is asymptotically Gaussian as $N_1, N_2 \rightarrow \infty$ simultaneously, because of the central limit theorem.

A sample function $f^{(i)}(x_1, x_2)$ of the simulated stochastic field $f(x_1, x_2)$ can be obtained by replacing the sequences of random phase angles $\Phi_{n_1n_2}^{(1)}$ and $\Phi_{n_1n_2}^{(2)}$ with their respective i th realizations $\phi_{n_1n_2}^{(1)(i)}$ and $\phi_{n_1n_2}^{(2)(i)}$; $n_1 = 0, 1, \dots, N_1 - 1$; $n_2 = 0, 1, \dots, N_2 - 1$:

$$f^{(i)}(x_1, x_2) = \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left[A_{n_1n_2} \cos(\kappa_{1n_1}x_1 + \kappa_{2n_2}x_2 + \phi_{n_1n_2}^{(1)(i)}) + \tilde{A}_{n_1n_2} \cos(\kappa_{1n_1}x_1 - \kappa_{2n_2}x_2 + \phi_{n_1n_2}^{(2)(i)}) \right] \quad (47)$$

The conditions set in Eqs. (42) and (43) are necessary (and must be forced if $S_{f_0f_0}(0, \kappa_2)$ and/or $S_{f_0f_0}(\kappa_1, 0)$ are different from zero) to guarantee that the spatial average and the spatial autocorrelation function of any sample function $f^{(i)}(x_1, x_2)$ are identical to the corresponding targets, $\mathcal{E}[f_0(x_1, x_2)] = 0$ and $R_{f_0f_0}(\xi_1, \xi_2)$, respectively (this is the ergodicity property that will be discussed later in this section). However, to avoid having to impose the conditions shown in Eqs. (42) and (43), the alternative of using the frequency shifting theorem (e.g. Papoulis [13]) has been proposed by Zerva [14].

When generating sample functions of the simulated stochastic field according to Eq. (47), the space increments Δx_1 and Δx_2 along the x_1 and x_2 axes respectively, separating the generated values of $f^{(i)}(x_1, x_2)$, have to obey the conditions:

$$\Delta x_1 \leq \frac{2\pi}{2\kappa_{1u}} \quad ; \quad \Delta x_2 \leq \frac{2\pi}{2\kappa_{2u}} \quad (48)$$

to avoid aliasing according to the sampling theorem (e.g. Bracewell [15]).

The generated values of $f^{(i)}(x_1, x_2)$ according to Eq. (47) are bounded as follows:

$$f^{(i)}(x_1, x_2) \leq \sqrt{2} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} [A_{n_1n_2} + \tilde{A}_{n_1n_2}] \quad (49)$$

The above bound is large enough for practical applications, even for relatively small values of N_1 and N_2 . For additional information, the reader is referred to Shinozuka and Deodatis [17].

The following two properties of generated sample functions using Eq. (47) are central to the Spectral Representation Method based on

the series representation shown in Eq. (13). For a detailed derivation of both properties, the reader is referred to Shinozuka and Deodatis [17].

1. **Ensemble Average Property:** The ensemble expected value $\mathcal{E}[f(x_1, x_2)]$ and the ensemble autocorrelation function $R_{ff}(\xi_1, \xi_2)$ of the simulated stochastic field $f(x_1, x_2)$ are identical to the corresponding targets, $\mathcal{E}[f_0(x_1, x_2)] = 0$ and $R_{f_0f_0}(\xi_1, \xi_2)$, respectively. Specifically for the autocorrelation function:

$$R_{ff}(\xi_1, \xi_2) = R_{f_0f_0}(\xi_1, \xi_2), \text{ for } 0 \leq \xi_1 \leq L_{x_1} \text{ and } 0 \leq \xi_2 \leq L_{x_2} \quad (50)$$

The values of L_{x_1} and L_{x_2} appearing in Eq. (50) can be less or equal to the periods L_{x_10} and L_{x_20} defined in Eq. (46). If, in addition, $L_{x_1} = L_{x_10}$ and $L_{x_2} = L_{x_20}$, then:

$$S_{ff}(\kappa_1, \kappa_2) = S_{f_0f_0}(\kappa_1, \kappa_2), \text{ for } 0 \leq \kappa_1 \leq \kappa_{1u} \text{ and } -\kappa_{2u} \leq \kappa_2 \leq \kappa_{2u} \quad (51)$$

2. **Ergodicity Property:** The spatial average $\langle f^{(i)}(x_1, x_2) \rangle_{L_{x_1} \times L_{x_2}}$ and the spatial autocorrelation function $\langle f^{(i)}(x_1 + \xi_1, x_2 + \xi_2) f^{(i)}(x_1, x_2) \rangle_{L_{x_1} \times L_{x_2}}$ of any sample function $f^{(i)}(x_1, x_2)$ are identical to the corresponding targets, $\mathcal{E}[f_0(x_1, x_2)] = 0$ and $R_{f_0f_0}(\xi_1, \xi_2)$, respectively. These two identities are valid only when the rectangular area $L_{x_1} \times L_{x_2}$ over which the sample function $f^{(i)}(x_1, x_2)$ is generated is either equal to the period ($L_{x_1} = L_{x_10}$ and $L_{x_2} = L_{x_20}$) or when it approaches infinity ($L_{x_1} \rightarrow \infty$ and $L_{x_2} \rightarrow \infty$). Specifically:

$$\langle f^{(i)}(x_1, x_2) \rangle_{L_{x_1} \times L_{x_2}} = 0 = \mathcal{E}[f_0(x_1, x_2)], \text{ when either } (L_{x_1} = L_{x_10} \text{ and } L_{x_2} = L_{x_20}) \text{ or } (L_{x_1} \rightarrow \infty \text{ and } L_{x_2} \rightarrow \infty) \quad (52)$$

and:

$$\langle f^{(i)}(x_1 + \xi_1, x_2 + \xi_2) f^{(i)}(x_1, x_2) \rangle_{L_{x_1} \times L_{x_2}} = R_{f_0f_0}(\xi_1, \xi_2), \text{ when either } (L_{x_1} = L_{x_10} \text{ and } L_{x_2} = L_{x_20}) \text{ or } (L_{x_1} \rightarrow \infty \text{ and } L_{x_2} \rightarrow \infty) \quad (53)$$

Eq. (53) leads to:

$$\langle S_{ff}^{(i)}(\kappa_1, \kappa_2) \rangle_{L_{x_1} \times L_{x_2}} = S_{f_0f_0}(\kappa_1, \kappa_2), \text{ for } 0 \leq \kappa_1 \leq \kappa_{1u} \text{ and } -\kappa_{2u} \leq \kappa_2 \leq \kappa_{2u}, \text{ when either } (L_{x_1} = L_{x_10} \text{ and } L_{x_2} = L_{x_20}) \text{ or } (L_{x_1} \rightarrow \infty \text{ and } L_{x_2} \rightarrow \infty) \quad (54)$$

where $\langle S_{ff}^{(i)}(\kappa_1, \kappa_2) \rangle_{L_{x_1} \times L_{x_2}}$ is the spatial power spectral density function.

Use of Fast Fourier Transform (FFT) Technique. The computational cost of generating sample functions using Eq. (47) can be drastically reduced by using the FFT technique. The reader is referred to Shinozuka and Deodatis [17] for details concerning the implementation of FFT for 2D-1V, homogeneous, and Gaussian stochastic fields.

5. One-dimensional, multi-variate (vector), stationary, and Gaussian stochastic processes

The simulation of one-dimensional, multi-variate (vector), stationary, and Gaussian stochastic processes is *not* a straightforward extension of the case of one-dimensional, uni-variate (scalar), stationary, and Gaussian stochastic processes.¹ The simulation of stationary, Gaussian, stochastic vector processes (1D-mV) using the Spectral Representation

Method was first proposed in Shinozuka [4] and Shinozuka and Jan [10]. This section presents the simulation of 1D-3V (tri-variate), stationary, and Gaussian stochastic processes, mainly following Shinozuka and Jan [10] and Deodatis [19]. The reader should refer to these two papers for additional detail and examples. The simulation of bi-variate (1D-2V) and general multi-variate (1D-mV) stochastic vector processes are rather straightforward extensions of the tri-variate case presented in this section.

Consider a one-dimensional, tri-variate (1D-3V), stationary and Gaussian stochastic vector process with components $f_1^0(t), f_2^0(t), f_3^0(t)$, having mean value equal to zero:

$$\mathcal{E}[f_j^0(t)] = 0 \quad ; \quad j = 1, 2, 3 \quad (55)$$

cross-correlation matrix given by:

$$\mathbf{R}^0(\tau) = \begin{bmatrix} R_{11}^0(\tau) & R_{12}^0(\tau) & R_{13}^0(\tau) \\ R_{21}^0(\tau) & R_{22}^0(\tau) & R_{23}^0(\tau) \\ R_{31}^0(\tau) & R_{32}^0(\tau) & R_{33}^0(\tau) \end{bmatrix} \quad (56)$$

and cross-spectral density matrix given by:

$$\mathbf{S}^0(\omega) = \begin{bmatrix} S_{11}^0(\omega) & S_{12}^0(\omega) & S_{13}^0(\omega) \\ S_{21}^0(\omega) & S_{22}^0(\omega) & S_{23}^0(\omega) \\ S_{31}^0(\omega) & S_{32}^0(\omega) & S_{33}^0(\omega) \end{bmatrix} \quad (57)$$

In Eq. (56), $R_{jj}^0(\tau)$; $j = 1, 2, 3$ are the autocorrelation functions of the three components $f_j^0(t)$; $j = 1, 2, 3$ of the vector process and $R_{jk}^0(\tau)$; $j = 1, 2, 3$; $k = 1, 2, 3$; $j \neq k$ are the corresponding cross-correlation functions. Due to the stationarity hypothesis, the following relations are valid:

$$R_{jj}^0(\tau) = R_{jj}^0(-\tau) \quad ; \quad j = 1, 2, 3 \quad (58)$$

$$R_{jk}^0(\tau) = R_{kj}^{0*}(-\tau) \quad ; \quad j = 1, 2, 3; k = 1, 2, 3; j \neq k \quad (59)$$

The elements of the cross-correlation matrix are related to the corresponding elements of the cross-spectral density matrix through the Wiener-Khinchine transformation:

$$S_{jk}^0(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{jk}^0(\tau) e^{-i\omega\tau} d\tau \quad ; \quad j, k = 1, 2, 3 \quad (60)$$

$$R_{jk}^0(\tau) = \int_{-\infty}^{\infty} S_{jk}^0(\omega) e^{i\omega\tau} d\omega \quad ; \quad j, k = 1, 2, 3 \quad (61)$$

In Eq. (57), $S_{jj}^0(\omega)$; $j = 1, 2, 3$ are the power spectral density functions of the three components of the vector process, and $S_{jk}^0(\omega)$; $j = 1, 2, 3$; $k = 1, 2, 3$; $j \neq k$ are the corresponding cross-spectral density functions. While the power spectral density function is a real and non-negative function of ω , the cross-spectral density function is a generally complex function of ω . Because of Eqs. (58)–(61), the following relations are valid:

$$S_{jj}^0(\omega) = S_{jj}^{0*}(-\omega) \quad ; \quad j = 1, 2, 3 \quad (62)$$

$$S_{jk}^0(\omega) = S_{kj}^{0*}(-\omega) \quad ; \quad j = 1, 2, 3; k = 1, 2, 3; j \neq k \quad (63)$$

$$S_{jk}^0(\omega) = S_{kj}^{0*}(\omega) \quad ; \quad j = 1, 2, 3; k = 1, 2, 3; j \neq k \quad (64)$$

where the asterisk denotes the complex conjugate. Eq. (64) indicates that the cross-spectral density matrix $\mathbf{S}^0(\omega)$ is Hermitian. It can be shown (Cramér and Leadbetter [2]) that matrix $\mathbf{S}^0(\omega)$ is non-negative definite.

5.1. Simulation of non-ergodic, tri-variate, stochastic vector processes

In the following, distinction will be made between the stochastic vector process $f_j^0(t)$; $j = 1, 2, 3$ and its simulation $f_j(t)$; $j = 1, 2, 3$. To simulate the tri-variate stochastic vector process, its cross-spectral

¹ The concepts of a multi-dimensional stochastic field and a multi-variate (vector) stochastic process should not be confused. An m-variate stochastic process consists of m stochastic processes cross-correlated to each other. On the other hand, an n-dimensional stochastic field is a stochastic field defined over an n-dimensional space.

density matrix $\mathbf{S}^0(\omega)$ must be first decomposed into the following product:

$$\mathbf{S}^0(\omega) = \mathbf{H}(\omega) \mathbf{H}^{T*}(\omega) \quad (65)$$

To perform the decomposition in Eq. (65), Shinozuka and Jan [10] suggested using Cholesky's method, while Shinozuka et al. [20] proposed a modal decomposition. Cholesky's method will be used in the following, in which case $\mathbf{H}(\omega)$ is a lower triangular matrix:

$$\mathbf{H}(\omega) = \begin{bmatrix} H_{11}(\omega) & 0 & 0 \\ H_{21}(\omega) & H_{22}(\omega) & 0 \\ H_{31}(\omega) & H_{32}(\omega) & H_{33}(\omega) \end{bmatrix} \quad (66)$$

whose diagonal elements are real and non-negative functions of ω and whose off-diagonal elements are generally complex functions of ω . The following relations are valid for the elements of matrix $\mathbf{H}(\omega)$:

$$H_{jj}(\omega) = H_{jj}(-\omega) \quad ; \quad j = 1, 2, 3 \quad (67)$$

$$H_{jk}(\omega) = H_{jk}^*(-\omega) \quad ; \quad j = 2, 3; \quad k = 1, 2; \quad j > k \quad (68)$$

The off-diagonal elements $H_{jk}(\omega)$ are written in polar form as:

$$H_{jk}(\omega) = |H_{jk}(\omega)| e^{i\theta_{jk}(\omega)} \quad ; \quad j = 2, 3; \quad k = 1, 2; \quad j > k \quad (69)$$

where:

$$\theta_{jk}(\omega) = \tan^{-1} \left(\frac{\text{Im}[H_{jk}(\omega)]}{\text{Re}[H_{jk}(\omega)]} \right) \quad (70)$$

with $|H_{jk}(\omega)|$ and $\theta_{jk}(\omega)$ possessing the following symmetries:

$$|H_{jk}(\omega)| = |H_{jk}(-\omega)| \quad ; \quad j = 2, 3; \quad k = 1, 2; \quad j > k \quad (71)$$

$$\theta_{jk}(\omega) = -\theta_{jk}(-\omega) \quad ; \quad j = 2, 3; \quad k = 1, 2; \quad j > k \quad (72)$$

Once matrix $\mathbf{S}^0(\omega)$ is decomposed according to Eqs. (65)–(66), the stochastic vector process $f_j^0(t)$; $j = 1, 2, 3$ can be simulated by the following series as $N \rightarrow \infty$

$$f_j(t) = 2 \sum_{q=1}^j \sum_{l=0}^{N-1} |H_{jq}(\omega_l)| \sqrt{\Delta\omega} \cos[\omega_l t - \theta_{jq}(\omega_l) + \Phi_{ql}]; \quad j = 1, 2, 3 \quad (73)$$

where:

$$\omega_l = l\Delta\omega, \quad l = 0, 1, \dots, N-1 \quad ; \quad \Delta\omega = \frac{\omega_u}{N} \quad (74)$$

In Eq. (74), ω_u represents an upper cut-off frequency beyond which the elements of the cross-spectral density matrix (Eq. (57)) may be assumed to be zero for either mathematical or physical reasons. As such, ω_u is a fixed value and hence $\Delta\omega \rightarrow 0$ as $N \rightarrow \infty$ so that $N\Delta\omega = \omega_u$. The $\Phi_{1l}, \Phi_{2l}, \Phi_{3l}$; $l = 0, 1, \dots, N-1$ appearing in Eq. (73) are three sequences of independent random phase angles distributed uniformly over the interval $[0, 2\pi]$. The simulated stochastic process $f_j(t)$; $j = 1, 2, 3$ is periodic with period T_0 :

$$T_0 = \frac{2\pi}{\Delta\omega} \quad (75)$$

The simulated stochastic vector process $f_j(t)$; $j = 1, 2, 3$ is asymptotically Gaussian as $N \rightarrow \infty$ because of the central limit theorem.

A sample function $f_j^{(i)}(t)$; $j = 1, 2, 3$ of the simulated stochastic process $f_j(t)$; $j = 1, 2, 3$ can be obtained by replacing the three sequences of random phase angles $\Phi_{1l}, \Phi_{2l}, \Phi_{3l}$; $l = 0, 1, \dots, N-1$ with their respective i th realizations $\phi_{1l}^{(i)}, \phi_{2l}^{(i)}, \phi_{3l}^{(i)}$; $l = 0, 1, \dots, N-1$:

$$f_j^{(i)}(t) = 2 \sum_{q=1}^j \sum_{l=0}^{N-1} |H_{jq}(\omega_l)| \sqrt{\Delta\omega} \cos[\omega_l t - \theta_{jq}(\omega_l) + \phi_{ql}^{(i)}], \quad j = 1, 2, 3 \quad (76)$$

When generating sample functions of the simulated stochastic process according to Eq. (76), the time step Δt separating the generated values of $f_j^{(i)}(t)$ in the time domain has to obey the condition:

$$\Delta t \leq \frac{2\pi}{2\omega_u} \quad (77)$$

to avoid aliasing according to the sampling theorem (e.g. Bracewell [15]).

The generated values of $f_j^{(i)}(t)$ according to Eq. (76) are bounded as follows:

$$f_j^{(i)}(t) \leq 2 \sum_{q=1}^j \sum_{l=0}^{N-1} |H_{jq}(\omega_l)| \sqrt{\Delta\omega}, \quad j = 1, 2, 3 \quad (78)$$

The above bound is large enough for practical applications, even for relatively small values of N .

The ensemble mean value $\mathcal{E}[f_j(t)]$; $j = 1, 2, 3$ and the ensemble auto-/cross-correlation function $R_{jk}(\tau)$; $j, k = 1, 2, 3$ of the simulated tri-variate stochastic vector process $f_j(t)$ according to Eq. (73), are identical to the corresponding targets, $\mathcal{E}[f_j^0(t)] = 0$; $j = 1, 2, 3$ and $R_{jk}^0(\tau)$; $j, k = 1, 2, 3$, respectively.

The simulation formula for tri-variate stochastic vector processes shown in Eq. (73) is generating sample functions that can be ergodic in the mean value, but not in correlation. This limitation is rectified through an alternative simulation formula for tri-variate stochastic vector processes that is described in the next section.

5.2. Simulation of ergodic, tri-variate, stochastic vector processes

In this section, the algorithm given in Eq. (73) is modified to generate sample functions that are ergodic both in the mean value and in correlation. The tri-variate, stochastic vector process considered in the previous section is again used here. Specifically, this 1D-3V process has components $f_1^0(t)$, $f_2^0(t)$ and $f_3^0(t)$ that fulfill all of the properties described in Eqs. (55)–(72).

Instead of Eq. (73), the stochastic process $f_j^0(t)$; $j = 1, 2, 3$ is now simulated by the following series as $N \rightarrow \infty$

$$f_j(t) = 2 \sum_{q=1}^j \sum_{l=1}^N |H_{jq}(\omega_{ql})| \sqrt{\Delta\omega} \cos[\omega_{ql} t - \theta_{jq}(\omega_{ql}) + \Phi_{ql}]; \quad j = 1, 2, 3 \quad (79)$$

where the discretization in the frequency domain is defined as:

$$\omega_{1l} = l\Delta\omega - (2/3)\Delta\omega \quad ; \quad l = 1, 2, \dots, N \quad (80)$$

$$\omega_{2l} = l\Delta\omega - (1/3)\Delta\omega \quad ; \quad l = 1, 2, \dots, N \quad (81)$$

$$\omega_{3l} = l\Delta\omega \quad ; \quad l = 1, 2, \dots, N \quad (82)$$

with $\Delta\omega$ being defined in Eq. (74). The idea of double-indexing the frequencies was proposed by Shinozuka et al. [21] in an effort to establish a simulation formula for stochastic vector processes able to generate sample functions that are ergodic both in the mean value and in correlation. Although Shinozuka et al. [21] was not successful in this effort, the double-indexing idea was used by Deodatis [19], who proposed the first simulation formula for stochastic vector processes (Eq. (79)) capable of generating ergodic sample functions. The reader is referred to Deodatis [19] for additional detail regarding Eq. (79) and its properties.

As with Eq. (73), the $\Phi_{1l}, \Phi_{2l}, \Phi_{3l}$; $l = 1, 2, \dots, N$ appearing in Eq. (79) are three sequences of independent random phase angles distributed uniformly over the interval $[0, 2\pi]$.

The simulated stochastic process $f_j(t)$; $j = 1, 2, 3$ according to Eq. (79) is periodic with period T_0 given by:

$$T_0 = 3 \frac{2\pi}{\Delta\omega} \quad (83)$$

Note that the period T_0 has tripled compared to the period of the non-ergodic formula in Eq. (73).

The simulated stochastic process $f_j(t)$; $j = 1, 2, 3$ (Eq. (79)) is asymptotically Gaussian as $N \rightarrow \infty$ because of the central limit theorem.

A sample function $f_j^{(i)}(t)$; $j = 1, 2, 3$ of the simulated stochastic process $f_j(t)$; $j = 1, 2, 3$ (Eq. (79)) can be obtained by replacing the

three sequences of random phase angles $\Phi_{1l}, \Phi_{2l}, \Phi_{3l}; l = 1, 2, \dots, N$ with their respective l th realizations $\phi_{1l}^{(i)}, \phi_{2l}^{(i)}, \phi_{3l}^{(i)}; l = 1, 2, \dots, N$:

$$f_j^{(i)}(t) = 2 \sum_{q=1}^j \sum_{l=1}^N |H_{jq}(\omega_{ql})| \sqrt{\Delta\omega} \cos[\omega_{ql} t - \theta_{jq}(\omega_{ql}) + \phi_{ql}^{(i)}], \quad j = 1, 2, 3 \quad (84)$$

When generating sample functions of the simulated stochastic vector process according to Eq. (84), the time step Δt separating the generated values of $f_j^{(i)}(t)$ in the time domain has to obey the condition:

$$\Delta t \leq \frac{2\pi}{2\omega_u} \quad (85)$$

to avoid aliasing according to the sampling theorem (e.g. Bracewell [15]).

The generated values of $f_j^{(i)}(t)$ according to Eq. (84) are bounded as follows:

$$f_j^{(i)}(t) \leq 2 \sum_{q=1}^j \sum_{l=1}^N |H_{jq}(\omega_{ql})| \sqrt{\Delta\omega}, \quad j = 1, 2, 3 \quad (86)$$

The above bound is large enough for practical applications, even for relatively small values of N .

The ensemble mean value $\mathcal{E}[f_j(t)]; j = 1, 2, 3$ and the ensemble auto-/cross-correlation function $R_{jk}(\tau); j, k = 1, 2, 3$ of the simulated tri-variate stochastic vector process $f_j(t)$ according to Eq. (79), are identical to the corresponding targets, $\mathcal{E}[f_j^0(t)] = 0; j = 1, 2, 3$ and $R_{jk}^0(\tau); j, k = 1, 2, 3$, respectively. This property of Eq. (79) is identical to the corresponding one of Eq. (73).

It can be shown (Deodatis [19]) that the temporal average and the temporal auto-/cross-correlation function of any sample function $f_j^{(i)}(t); j = 1, 2, 3$ generated using Eq. (84) are identical to the corresponding targets, $\mathcal{E}[f_j^0(t)] = 0; j = 1, 2, 3$ and $R_{jk}^0(\tau); j, k = 1, 2, 3$, respectively, when the length of the sample function $f_j^{(i)}(t); j = 1, 2, 3$ is equal to the period T_0 given by Eq. (83). This means that the simulation formula in Eq. (79) is generating sample functions that are ergodic both in the mean value and in correlation. This property of Eq. (79) is different from the corresponding one of Eq. (73) that is not able to generate ergodic sample functions (in correlation).

5.3. Comparison of non-ergodic (Eq. (73)) and ergodic (Eq. (79)) simulation formulas for stationary stochastic vector processes

The simulation formula in Eq. (79) has a major advantage compared to the one in Eq. (73): it generates sample functions of a tri-variate stochastic vector process that are ergodic both in the mean value and in correlation. However, this ergodicity property is achieved by extending the period of the simulated vector process to $T_0 = 3 \frac{2\pi}{\Delta\omega}$, from $T_0 = \frac{2\pi}{\Delta\omega}$ in the case of non-ergodic tri-variate processes (Eq. (73)). This increase of the period T_0 will become even larger for cases of multi-variate processes with more components than three, and is considered, in general, to be a drawback. Another drawback is the complex discretization in the frequency domain using the double indexing of the frequencies shown in Eqs. (80)–(82).

The choice between the non-ergodic and ergodic simulation formulas depends on the problem under consideration. For example, if it is desired to extract exactly the complete correlation structure of the stochastic process from each and every generated sample function of sufficient length, then the ergodic formula should be used. This could be the case in stochastic response analysis problems where the response convergence is slow. On the other hand, if ergodicity of the generated sample functions is not an issue, then the non-ergodic formula should be used because of its simplicity and constant period $T_0 = \frac{2\pi}{\Delta\omega}$ for any number of components of the multi-variate stochastic process.

5.4. Use of Fast Fourier Transform (FFT) technique

The computational cost of generating sample functions of a tri-variate stochastic vector process using either Eq. (73) or Eq. (79) can be drastically reduced through the use of the FFT technique. The reader is referred to Deodatis [19] for details concerning the FFT implementation for 1D-3V processes. It is noted that the FFT technique can be used for any number of components of the multi-variate stochastic process. For generation of very-long-duration sample functions, the reader is referred to Li and Kareem [22].

5.5. Stochastic multi-variate (vector) processes with very large number of components – modeling as stochastic waves

The simulation of multi-variate (vector), stationary, and Gaussian stochastic processes $f_j^0(t); j = 1, 2, \dots, m$ requires the decomposition of their $m \times m$ cross-spectral density matrix $S^0(\omega)$ as indicated in Eq. (65). As mentioned earlier, this can be accomplished using either Cholesky's method or modal decomposition. Regardless of the approach used, such a decomposition will face increasingly more severe numerical problems as the number m of components of the vector process increases (especially when these components are strongly correlated to each other). Beyond some value of m , the decomposition in Eq. (65) will inevitably break down.

Today, there are several practical applications where the number of components m of the stochastic vector process can reach in the hundreds or even thousands (e.g. wind velocity time histories at a very large number of locations along the length of a long-span bridge). To address this issue, several approximate techniques have been suggested over the years to perform the decomposition in Eq. (65) for large values of m . However, none of these techniques will eventually work with sufficient accuracy and computational efficiency when m becomes too large.

Fortunately, there is a simple way to resolve this issue: by modeling the uncertain quantity as a stochastic wave rather than as a stochastic vector process (in the case of wind velocity time histories, the stochastic wave is defined in space and time and becomes essentially a multi-dimensional stochastic field). The stochastic wave modeling approach involves no approximations, no loss of accuracy, does not require performing the decomposition in Eq. (65), is extremely efficient computationally (as FFT can be readily used), and thousands of strongly correlated time histories can be simulated without any numerical problem whatsoever. This stochastic wave modeling approach was introduced by Benowitz and Deodatis [23], and then extended by Peng et al. [24,25], Song et al. [26], Chen et al. [27], Zhou et al. [28], Yang and Lei [29], and Zhao et al. [30].

6. Non-stationary, Gaussian stochastic processes

One of the most challenging problems associated with the theory of non-stationary stochastic processes is the definition of their spectral characteristics. Several models have been proposed in the past to define such non-stationary spectra. Three of the most notable are (1) The "instantaneous power spectrum" introduced by Page [31], (2) the "physical spectrum" defined by Mark [32], and (3) the "evolutionary power spectrum" developed by Priestley [33]. Among these three models, it is the evolutionary power concept that appears to offer the most palatable transition from the power spectra associated with stationary stochastic processes to spectra associated with non-stationary stochastic processes (within the framework of the spectral representation of stochastic processes). In this respect, Priestley's evolutionary power spectrum permits a physical interpretation of its spectral contents quite similar to that of the power spectrum associated with stationary stochastic processes. Consequently, the concept of evolutionary power spectra will be used in this chapter for the development of simulation formulas for non-stationary stochastic processes.

6.1. The concept of evolutionary power spectrum

The development of the theory in this section follows closely Priestley [33,34]. If a one-dimensional, uni-variate (1D-1V) stochastic process $f_0(t)$ (stationary or non-stationary) with mean value equal to zero can be represented in the following form:

$$f_0(t) = \int_{-\infty}^{\infty} A(t, \omega) e^{i\omega t} dz(\omega) \quad (87)$$

where $A(t, \omega)$ is a modulating function and $z(\omega)$ is an orthogonal process, then the stochastic process $f_0(t)$ is said to be “oscillatory”. Note that the physical notion of frequency is preserved by including the complex exponential in Eq. (87), and that if $A(t, \omega)$ is constant, then $f_0(t)$ is a stationary stochastic process.

The mean square of the oscillatory process $f_0(t)$ is expressed as:

$$\mathcal{E}[f_0^2(t)] = \int_{-\infty}^{\infty} A^2(t, \omega) dS_0(\omega) \quad (88)$$

where:

$$dS_0(\omega) = \mathcal{E}[dz(\omega)]^2 \quad (89)$$

The evolutionary power spectrum $dS_0(\omega, t)$ of the oscillatory stochastic process $f_0(t)$ is defined as (Priestley [33]):

$$dS_0(\omega, t) = A^2(t, \omega) dS_0(\omega) \quad (90)$$

Eq. (90) defines the non-stationary spectral contents of the oscillatory stochastic process $f_0(t)$ and can be written as:

$$S_0(t, \omega) = A^2(t, \omega) S_0(\omega) \quad (91)$$

where $dS_0(t, \omega) = S_0(t, \omega)d\omega$, if $S_0(\omega)$ exists such that $dS_0(\omega) = S_0(\omega)d\omega$. Although Priestley [34] defines $dS_0(t, \omega)$ as the evolutionary power spectrum and $S_0(t, \omega)$ as the evolutionary spectral density function, in this paper, $S_0(t, \omega)$ is called both the evolutionary spectral density function and the evolutionary power spectrum. Priestley [34] states that “the evolutionary spectrum has the same physical interpretation as the spectrum of a stationary process, namely that it describes a distribution of power over frequency, but whereas the latter is determined by the behavior of the process over all time, the former represents specifically the spectral content of the process in the neighborhood of the time instant t ”. Consequently, for any time instant t , the evolutionary spectrum is a real and non-negative function of ω satisfying:

$$S_0(t, \omega) = S_0(t, -\omega) \quad (92)$$

It can be shown (Priestley [33,34]) that an oscillatory stochastic process $f_0(t)$ of the form shown in Eq. (87) has the following autocorrelation function:

$$R_0(t, t + \tau) = \int_{-\infty}^{\infty} A(t + \tau, \omega) A(t, \omega) S_0(\omega) e^{i\omega\tau} d\omega \quad (93)$$

Because of the non-stationarity of the stochastic process, the autocorrelation function in Eq. (93) is a function of two time instants: t and $t + \tau$ (t = time and τ = time lag).

There are two broad categories of non-stationary stochastic processes with evolutionary power: uniformly modulated non-stationary stochastic processes and non-stationary stochastic processes with amplitude and frequency modulation. These two categories are defined as follows:

1. **Uniformly Modulated Non-Stationary Stochastic Processes:** In this rather trivial case, the modulating function $A(t, \omega)$ is only a function of time. This means that the frequency content of the non-stationary stochastic process remains constant in time, and it is only the amplitude of the process that changes with time. The evolutionary power spectrum of a uniformly modulated non-stationary stochastic process is expressed as:

$$S_0(t, \omega) = A^2(t) S_0(\omega) \quad (94)$$

and is related to the autocorrelation function through the following relationship:

$$R_0(t, t + \tau) = A(t + \tau)A(t) \int_{-\infty}^{\infty} S_0(\omega) \exp[i\omega\tau] d\omega \quad (95)$$

A uniformly modulated non-stationary stochastic process $f_0(t)$ can be simply considered as a stationary stochastic process $g_0(t)$ multiplied by a modulating (envelope) function $A(t)$:

$$f_0(t) = A(t) g_0(t) \quad (96)$$

where $g_0(t)$ is a stationary stochastic process with mean value equal to zero and spectral density function $S_0(\omega)$. Although uniformly modulated non-stationary stochastic processes constitute a trivial class of non-stationary stochastic processes, they are used quite often in practical applications because of their simplicity.

2. **Non-Stationary Stochastic Processes With Amplitude and Frequency Modulation:** In this most general case of non-stationary stochastic processes, the modulating function $A(t, \omega)$ is a function of both frequency and time, in such a way that it is not possible to factor out the time dependence. This means that both the frequency content and the amplitude of the non-stationary stochastic process change with time. In this case, the evolutionary power spectrum is given by Eq. (91).

6.2. Simulation formula - case with amplitude and frequency modulation

Consider a 1D-1V non-stationary stochastic process $f_0(t)$ with mean value equal to zero and evolutionary power spectrum $S_0(t, \omega)$ as defined in Eq. (91). In the following, distinction will be made between the stochastic process $f_0(t)$ and its simulation $f(t)$.

The 1D-1V non-stationary stochastic process $f_0(t)$ can be simulated using a formula very similar to the one used for 1D-1V stationary stochastic processes (Eqs. (14)–(15)). Specifically, $f_0(t)$ can be simulated by the following series as $N \rightarrow \infty$:

$$f(t) = \sqrt{2} \sum_{n=0}^{N-1} \sqrt{2A^2(t, \omega_n) S_0(\omega_n) \Delta\omega} \cos(\omega_n t + \Phi_n) \quad (97)$$

where:

$$\omega_n = n\Delta\omega \quad ; \quad n = 0, 1, 2, \dots, N-1 \quad (98)$$

$$\Delta\omega = \frac{\omega_u}{N} \quad (99)$$

In Eq. (99), ω_u represents an upper cut-off frequency beyond which the evolutionary power spectrum $S_0(t, \omega)$ may be assumed to be zero at any time instant t , for either mathematical or physical reasons. As such, ω_u is a fixed value and hence $\Delta\omega \rightarrow 0$ as $N \rightarrow \infty$ so that $N\Delta\omega = \omega_u$. An alternative definition for ω_n is the following: $\omega_n = n\Delta\omega + \Delta\omega/2$, $n = 0, 1, 2, \dots, N-1$ (refer to previous chapters for details about this definition). The $\Phi_0, \Phi_1, \Phi_2, \dots, \Phi_{N-1}$ appearing in Eq. (97) are independent random phase angles distributed uniformly over the interval $[0, 2\pi]$.

Comparing Eqs. (14)–(15) to Eq. (97), it is obvious that the only difference is the addition of term $A^2(t, \omega_n)$ under the square root in Eq. (97). This difference makes the coefficients multiplying the cosine terms in Eq. (97) functions of time (this was not the case in Eqs. (14)–(15) for stationary stochastic processes) and leads to some important differences in their properties that are discussed in the following.

Eq. (14) yields a simulated (stationary) stochastic process that is periodic with period given by Eq. (17). This is not the case with Eq. (97). With the exception of the special (and rather trivial) case where the modulating function $A(t, \omega)$ is periodic in time, the simulated non-stationary stochastic process $f(t)$ given by Eq. (97) is not periodic. It should be noted that in several practical applications in engineering (e.g. earthquake engineering), the modulating function $A(t, \omega)$ is of finite duration.

Although the simulated non-stationary stochastic process $f(t)$ given by Eq. (97) is not periodic, it is still asymptotically Gaussian as $N \rightarrow \infty$ because of the central limit theorem.

A sample function $f^{(i)}(t)$ of the simulated stochastic process $f(t)$ can be obtained by replacing the sequence of random phase angles $\Phi_0, \Phi_1, \Phi_2, \dots, \Phi_{N-1}$ with their respective i th realizations $\phi_0^{(i)}, \phi_1^{(i)}, \phi_2^{(i)}, \dots, \phi_{N-1}^{(i)}$:

$$f^{(i)}(t) = \sqrt{2} \sum_{n=0}^{N-1} \sqrt{2A^2(t, \omega_n) S_0(\omega_n) \Delta\omega} \cos(\omega_n t + \phi_n^{(i)}) \quad (100)$$

To avoid aliasing according to the sampling theorem (e.g. Bracewell [15]), the time step Δt separating the generated values of $f^{(i)}(t)$ in the time domain has to obey the condition:

$$\Delta t \leq \frac{2\pi}{2\omega_u} \quad (101)$$

The generated values of $f^{(i)}(t)$ according to Eq. (100) are bounded as follows:

$$f^{(i)}(t) \leq \sqrt{2} \sum_{n=0}^{N-1} \sqrt{2A^2(t, \omega_n) S_0(\omega_n) \Delta\omega} \quad (102)$$

The above bound varies as a function of the time instant t . It can be easily calculated for any form of the evolutionary power spectrum and can be made large enough for practical applications using a reasonable value for N .

Ensemble Average Property: Following an approach similar to that used for 1D-1V stationary stochastic processes, it is straightforward to show that the ensemble expected value $\mathcal{E}[f(t)]$ and the ensemble autocorrelation function $R(t, t + \tau)$ of the simulated non-stationary stochastic process $f(t)$ according to Eq. (97) are identical to the corresponding targets, $\mathcal{E}[f_0(t)] = 0$ and $R_0(t, t + \tau)$, respectively.

Ergodicity Property: The concept of ergodicity is associated, in general, with stationary processes. For the non-stationary case considered here, it can be shown that the temporal autocorrelation function of a sample function $f^{(i)}(t)$ generated using Eq. (100) is, in general, not identical to the corresponding target.

Comments on Computational Efficiency: In general, it is not possible to take advantage of the FFT technique when using the non-stationary simulation formula in Eq. (100) (in contrast to the corresponding formula in Eq. (18) for simulation of stationary stochastic processes). This is due to the fact that the coefficients $\sqrt{2A^2(t, \omega_n) S_0(\omega_n) \Delta\omega}$ in the summation of Eq. (100) are now functions of both frequency and time. In several cases of practical interest, this should not be of any great concern computationally since the non-stationary stochastic process $f_0(t)$ is limited to relatively short durations by the modulating function $A(t, \omega)$ (e.g. ground motion acceleration time histories). In cases where computational efficiency becomes vital, it is recommended to follow Li and Kareem [35] who proposed an asymptotic, stochastic-decomposition-based methodology to simulate general non-stationary stochastic processes with evolutionary power using the FFT technique and Li and Kareem [36] who suggested an FFT-based hybrid approach combining a discrete Fourier transform algorithm with a digital filtering scheme. A special (and rather trivial) case of non-stationary stochastic processes where the FFT technique can be readily used in the simulation formula is that of uniformly modulated processes (refer to the next section).

6.3. Simulation formula - case with uniform modulation

For the special case of a uniformly modulated non-stationary stochastic process, simulation can be performed on the basis of Eq. (96), instead of using Eq. (97). The simulation formula corresponding to Eq. (96) is:

$$f(t) = A(t) g(t) \quad (103)$$

where $g(t)$ is the simulation of the stationary stochastic process $g_0(t)$ having mean value equal to zero and spectral density function $S_0(\omega)$. It is reminded that simulation of stationary stochastic processes can be performed with great computational efficiency using the Fast Fourier Transform (FFT) technique, as described earlier in this paper.

6.4. Extensions beyond 1D-1V non-stationary and Gaussian stochastic processes

The simulation of multi-dimensional (mD), uni-variate (1V), non-homogeneous and Gaussian stochastic fields (with evolutionary power spectra) is a rather straightforward extension of the case of 1D-1V non-stationary and Gaussian stochastic processes (also with evolutionary power). The reader is referred to Li and Kareem [35], Deodatis [37] and Li and Kareem [36] for simulation of one-dimensional (1D), vector (mV), non-stationary and Gaussian stochastic processes (with evolutionary power).

7. Non-Gaussian, stationary stochastic processes and waves

As detailed earlier, simulations using the Spectral Representation Method (SRM) are Gaussian whether derived from Eq. (10) (exactly) or Eq. (13) (asymptotically). Consequently, the simulation of non-Gaussian stochastic processes is not necessarily straightforward using the SRM. As a general principle, to simulate a non-Gaussian stochastic process, it is necessary to perform a nonlinear transformation of an appropriately selected Gaussian process. Numerous such nonlinear transformations have been proposed in the literature with the transformation methods generally falling into two classes of methods. The first class of methods employs various polynomial transformations (e.g., Gurley et al. [38], Sakamoto and Ghanem [39], Puig et al. [40], Lu et al. [41]) with the general objective of matching a number of moments of the process or approximating its marginal probability distribution. The second class of methods, which will be described in more detail here, employs the non-Gaussian cumulative probability distribution function to transform an appropriately selected Gaussian process (named underlying Gaussian) into the specified non-Gaussian one. These methods rely on Translation Process Theory (Grigoriu [42, 43]) which is described in the following section.

7.1. Translation process theory

Let $f_0(t)$ be a stationary, Gaussian stochastic process having zero mean and unit standard deviation (without any loss of generality), and autocorrelation function $R_{f_0 f_0}(\tau)$. This process may then be mapped to a stationary, non-Gaussian process $h_0(t)$ with prescribed marginal cumulative probability distribution function (CDF), denoted by $F(\cdot)$, through the following non-linear transformation (Grigoriu [42, 43]):

$$h_0(t) = F^{-1}(\Phi[f_0(t)]) \quad (104)$$

where $\Phi[\cdot]$ is the standard normal CDF. The non-Gaussian process $h_0(t)$ is referred to as a translation process, with Eq. (104) referred to as the standard translation. The process $h_0(t)$ is a stationary non-Gaussian process having marginal CDF $F(\cdot)$ and autocorrelation function given by (Grigoriu [42, 43]):

$$R_{h_0 h_0}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F^{-1}(\Phi[f_1]) F^{-1}(\Phi[f_2]) \phi(f_1, f_2; R_{f_0 f_0}(\tau)) df_1 df_2 \quad (105)$$

where $f_1 = f_0(t_1)$, $f_2 = f_0(t_2)$, and $\phi(\cdot)$ is the standard joint Gaussian probability density function. The translation in Eq. (104) is referred to as the *forward translation* and the change in autocorrelation that results from Eq. (105) is referred to as *correlation distortion*. Importantly, this forward translation is always possible. That is, given the Gaussian process $f_0(t)$ with autocorrelation $R_{f_0 f_0}(\tau)$, it is always possible to

define the non-Gaussian process $h_0(t)$ with marginal CDF $F(\cdot)$ and resulting autocorrelation $R_{h_0 h_0}(\tau)$.

However, the more practical and challenging case is the *inverse translation* in which the marginal non-Gaussian CDF $F(\cdot)$ and non-Gaussian autocorrelation function $R_{h_0 h_0}(\tau)$ are specified, and it is sought to identify the corresponding underlying Gaussian process $f_0(t)$ with autocorrelation $R_{f_0 f_0}(\tau)$ that, when translated using Eq. (104), produces this non-Gaussian process $h_0(t)$. This inverse translation is not always defined/possible because Eq. (105) is not always invertible. This is referred to as *translation process incompatibility* and may arise under two conditions (Grigoriu [42,44]):

1. The specified normalized non-Gaussian autocorrelation takes values that lie outside of the admissible range $[\xi^*, 1]$ where ξ^* is determined by applying Eq. (105) with $R_{f_0 f_0}(\tau) = -1$. Note that ξ^* is not necessarily equal to -1 and, in fact, $\xi^* = -1$ only for odd memoryless transformations (Grigoriu [43]).
2. The resulting Gaussian autocorrelation function from the inversion of Eq. (105) is not positive semi-definite, and is therefore not a valid autocorrelation function (as Eq. (105) cannot be inverted analytically, its inversion typically requires solving an optimization problem). This type of incompatibility is easy to identify as the positive semi-definiteness of the autocorrelation function is equivalent to the corresponding power spectral density function being strictly non-negative (Bochner's theorem, e.g. Grigoriu [45]).

7.2. Iterative translation approximation method (ITAM)

The challenges associated with the inverse translation have long been recognized, yet simulation of non-Gaussian processes in many practical cases necessitates an inverse translation. Approaches to address inverse translation and simulate non-Gaussian processes using the Spectral Representation Method (SRM) began with the work of Yamazaki and Shinozuka [46] who proposed an iterative scheme that updates the underlying Gaussian power spectral density $S_{f_0 f_0}(\omega)$ using the statistically estimated non-Gaussian power spectral density $S_{h_0 h_0}(\omega)$ from an ensemble of non-Gaussian simulations generated using the SRM with translation process theory. Deodatis and Micaletti [47] modified this iterative algorithm to correct for a loss of Gaussianity in the translated sample functions, and while their algorithm improves practical simulation performance in certain ways (e.g., generating realizations that better match the prescribed non-Gaussian probability distribution, $F(\cdot)$), the resulting non-Gaussian process is no longer strictly a translation process and certain (desirable) theoretical characteristics of translation processes are lost. Motivated by the need to retain the theoretical properties of the translation process, Shi and Deodatis [48] and later Bocchini and Deodatis [49] proposed a different iterative scheme based on acceptance or rejection of random perturbations applied to the Gaussian power spectral density. This approach, while effective for generating sample functions that satisfy translation process theory, is computationally very intensive requiring a very large number of iterations. In fact, all of the methods described thus far come at high computational costs due to the need to generate sample functions at each iteration.

Two significant methodological advances occurred with the works of Grigoriu [44] and Shields et al. [50]. Both works recognized that sample function generation is not necessary to solve the inverse translation problem. Grigoriu [44] first posed the inverse translation problem as an optimization problem, in which criteria were identified to determine the existence of a translation model for compatible cases and algorithms were devised to minimize the error in translation for incompatible cases. Shields et al. [50] then took motivation from the iterative approaches originally proposed by Yamazaki and Shinozuka [46] and Deodatis and Micaletti [47] and showed that the inverse

translation can be solved (in compatible cases) and approximated (in incompatible cases) using a similar iterative scheme without requiring sample function generation during the iterations. Due to its computational efficiency and conceptual simplicity, this method – referred to as the Iterative Translation Approximation Method (ITAM) – is the focus of this work and is described in the following.

Consider a non-Gaussian random process $h_0(t)$ having marginal CDF $F(\cdot)$ and power spectral density $S_{h_0 h_0}(\omega)$, where $F(\cdot)$ and $S_{h_0 h_0}(\omega)$ need not be compatible according to translation process theory. The goal of the ITAM is to determine an underlying Gaussian stochastic process $\hat{f}_0(t)$, having power spectral density $S_{\hat{f}_0 \hat{f}_0}(\omega)$ that, when translated using Eq. (104)–(105), yields stochastic process $\hat{h}_0(t)$ having power spectral density $S_{\hat{h}_0 \hat{h}_0}(\omega)$ that is as close as possible to $S_{h_0 h_0}(\omega)$. This is achieved using a sequence of iterations of the form (again motivated by Yamazaki and Shinozuka [46]):

$$S_{\hat{f}_0 \hat{f}_0}^{(i+1)}(\omega) = \left[\frac{S_{h_0 h_0}(\omega)}{S_{\hat{h}_0 \hat{h}_0}^{(i)}(\omega)} \right]^\beta S_{\hat{f}_0 \hat{f}_0}^{(i)}(\omega) \quad (106)$$

where $S_{\hat{f}_0 \hat{f}_0}^{(i)}(\omega)$, $S_{\hat{h}_0 \hat{h}_0}^{(i)}(\omega)$ are the Gaussian and non-Gaussian power spectral densities at iteration i , respectively. $S_{\hat{h}_0 \hat{h}_0}^{(i)}(\omega)$ is computed directly using Eq. (105) and the Wiener–Khinchine transform (i.e. without requiring sample function generation). The power β is selected to optimize convergence and is typically in the range $1.3 \leq \beta \leq 1.5$. The iterative scheme in Eq. (106) has been shown to converge rapidly (in less than 100 iterations), yielding an $S_{\hat{h}_0 \hat{h}_0}(\omega)$ that is very close to the prescribed $S_{h_0 h_0}(\omega)$ at little computational cost even for strongly non-Gaussian processes with a high degree of translation process incompatibility (Shields et al. [50]).

The ITAM for multi-dimensional non-Gaussian random fields follows from a straightforward extension of Eq. (106), which replaces the one-dimensional power spectral density with the multi wave-number power spectral density. However, as shown by Shields and Deodatis [51], the extension of the ITAM to multi-variate non-Gaussian stochastic processes is not trivial and involves iterations (similar to Eq. (106)) on the Cholesky-decomposed cross-spectral density matrix along with a shuffling at each iteration to ensure adequate convergence in all vector components. To mitigate the complexity of this decomposition for vector processes with a large number of components (as discussed in Section 5.5), the ITAM has been extended for the simulation of non-Gaussian stochastic waves by Zhou et al. [28].

7.3. Simulation formula: Non-Gaussian translation processes

Once the underlying Gaussian power spectral density function $S_{f_0 f_0}(\omega)$, or more specifically $S_{\hat{f}_0 \hat{f}_0}(\omega)$, has been determined, simulation of non-Gaussian stochastic translation processes is relatively straightforward. First, the Gaussian process is simulated using the method described earlier (Eq. (14)). Given a realization of the underlying Gaussian process, its non-linear translation according to Eq. (104) is then applied to generate a realization of the non-Gaussian process as:

$$h(t) = F^{-1}(\Phi[f(t)]) \quad (107)$$

Importantly, the corresponding collection of realizations possesses the desired non-Gaussian marginal distribution, $F(\cdot)$, and spectral density function $S_{h_0 h_0}(\omega)$ in ensemble average sense. Extension of this concept for the simulation of non-Gaussian multi-dimensional random fields, multi-variate random processes, and stochastic waves follows directly with a straightforward extension of Eq. (107).

8. Higher-order spectral representation method

The classical SRM for Gaussian stochastic processes derives from second-order properties of the process – i.e., its power spectral density function – which completely defines the process. Section 7 presented

an SRM-based methodology to simulate non-Gaussian processes according to their spectral density function (SDF) and marginal probability distribution function (PDF) following Translation Process Theory (Grigoriu [42,43]) that is the most commonly used approach to model non-Gaussian processes in engineering and applied science because of its mathematical rigor, physical meaning, and conceptual simplicity.

An alternative approach to modeling non-Gaussian processes is through higher-order spectra, or polyspectra, that describe higher-order correlations (equivalently non-linear interactions) of various orders between frequency components in the Spectral Representation Theorem (Eq. (3)). Shields and Kim [52] showed that the Spectral Representation Method can be generalized to develop a higher-order expansion of a general non-Gaussian stochastic process based on polyspectra. This approach, referred to as the Higher-Order Spectral Representation Method (HO-SRM), differs from Translation Process Theory based SRM approaches, which match the marginal PDF and second-order correlations/spectra, by simulating non-Gaussian stochastic processes directly from polyspectra. However, it should be mentioned that in contrast to the ease of estimating SDFs from data and the associated general availability of analytic expressions for SDFs describing various uncertain physical quantities, there are currently no corresponding analytic expressions for polyspectra and their estimation is generally challenging and data-intensive.

8.1. Higher-order spectra

The Spectral Representation Theorem in Eq. (3) defines mutually orthogonal real processes $u(\omega)$ and $v(\omega)$ with orthogonal increments $du(\omega)$ and $dv(\omega)$, respectively satisfying the orthogonality properties given in Eqs. (4)–(5). For Gaussian stochastic processes, these second-order orthogonality properties are sufficient to express the process according to Eq. (3). However, for non-Gaussian processes, a more general set of higher-order orthogonality properties must be satisfied. These orthogonality properties were established by Rosenblatt and Van Ness [53] and applied to the SRM by Shields and Kim [52]. Much like the second-order orthogonality conditions are expressed in terms of the SDF, the higher-order orthogonality ones are expressed in terms of higher-order spectra (polyspectra), which are briefly introduced next.

In general, for a non-Gaussian stochastic process $f_0(t)$, the polyspectrum of order n , $C_{f_0 f_0}^{(n)}(\cdot)$, can be expressed through a generalized Wiener–Khinchine transform (Eq. (2)) as the Fourier transform of the n th-order cumulant function, $c_{f_0 f_0}^{(n)}(\cdot)$ [54]:

$$C_{f_0 f_0}^{(n)}(\omega_1, \omega_2, \dots, \omega_{n-1}) = \frac{1}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} c_{f_0 f_0}^{(n)}(\tau_1, \tau_2, \dots, \tau_{n-1}) \prod_{i=1}^{n-1} e^{-i\omega_i \tau_i} d\tau_1 d\tau_2 \dots d\tau_{n-1} \quad (108)$$

where the cumulant functions can be expressed in terms of the moment (correlation) functions of the process. In fact, for zero-mean processes, the cumulant and moment (correlation) functions are identical up to the third-order and Eq. (108) reduces to exactly Eq. (2) for the second-order case. In general, the polyspectrum is a complex function, having both a real and an imaginary component.

Of particular interest here is the bispectrum, which is given by the Fourier transform of the three-point correlation function $R_{f_0 f_0}(\tau_1, \tau_2)$ (equivalently, the third-order cumulant function) as:

$$B_{f_0 f_0}(\omega_1, \omega_2) = C_{f_0 f_0}^{(3)}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{f_0 f_0}(\tau_1, \tau_2) e^{-i(\omega_1 \tau_1 + \omega_2 \tau_2)} d\tau_1 d\tau_2 \quad (109)$$

The bispectrum has a convenient physical interpretation as it expresses the power in quadratic wave interactions in the process that lead to asymmetry, such that integrating the real part of the bispectrum

yields the skewness of the process while integrating the imaginary part of the bispectrum gives the skewness of the derivative of the process [55,56]. For convenience, the bispectrum can be normalized to define the bicoherence [57]:

$$b_{f_0 f_0}^2(\omega_1, \omega_2) = \frac{|B_{f_0 f_0}(\omega_1, \omega_2)|^2}{E[|F_{f_0 f_0}(\omega_1)F_{f_0 f_0}(\omega_2)|^2]S_{f_0 f_0}(\omega_1 + \omega_2)} \quad (110)$$

where $F_{f_0 f_0}(\omega)$ are the Fourier coefficients of the non-Gaussian process $f_0(t)$. By Schwartz' inequality, the bicoherence is bounded on $[0, 1]$ which provides a convenient interpretation as the fraction of total energy associated with phase coupling.

To enable SRM-based simulation, Shields and Kim [52] defined a partial bicoherence (analogous to the partial coherence of Bendat and Piersol [58]) as:

$$b_p^2(\omega_i, \omega_j) = \frac{|B_{f_0 f_0}(\omega_i, \omega_j)|^2}{S_{f_0 f_0}^P(\omega_i)S_{f_0 f_0}^P(\omega_j)S_{f_0 f_0}(\omega_i + \omega_j)} \quad (111)$$

where $S_{f_0 f_0}^P(\omega_k)$ is the corresponding pure power spectrum:

$$S_{f_0 f_0}^P(\omega_k) = S_{f_0 f_0}(\omega_k) \left[1 - \sum_{\substack{i+j=k \\ i \geq j \geq 0}} b_p^2(\omega_i, \omega_j) \right] \quad (112)$$

The pure power spectrum in Eq. (112) can be interpreted as the remaining power spectrum after all power associated with quadratic coupling has been removed. Like the complete power spectrum, it cannot be negative, which is attributed to the property: $0 \leq \sum_{\substack{i+j=k \\ i \geq j \geq 0}} b_p^2(\omega_i, \omega_j) \leq 1$.

8.2. Higher-order spectral representation theorem

The Spectral Representation Theorem presented in Section 2 can be generalized for higher-order stochastic processes by defining the orthogonal processes $u(\omega)$ and $v(\omega)$, and their corresponding increments $du(\omega)$ and $dv(\omega)$, in a way that they can satisfy higher-order orthogonality properties. However, these orthogonal processes may not remain real-valued, especially as the process grows to fourth-order or higher [59].

Shields and Kim [52] showed that the spectral representation for third-order processes can be established by dividing the orthogonal increments into components derived from independent (pure) waves ($du_p(\omega_k)$, $dv_p(\omega_k)$) and those from wave interactions ($du_I(\omega_k)$, $dv_I(\omega_k)$) as:

$$\begin{aligned} du(\omega_k) &= du_p(\omega_k) + du_I(\omega_k) \\ dv(\omega_k) &= dv_p(\omega_k) + dv_I(\omega_k) \end{aligned} \quad (113)$$

where:

$$\begin{aligned} du_p(\omega_k) &= \sqrt{2 \cdot 2S_{f_0 f_0}^P(\omega_k)\Delta\omega_k} \cos \Phi_k \\ dv_p(\omega_k) &= \sqrt{2 \cdot 2S_{f_0 f_0}^P(\omega_k)\Delta\omega_k} \sin \Phi_k \end{aligned} \quad (114)$$

and:

$$\begin{aligned} du_I(\omega_k) &= \sqrt{2 \cdot 2S(\omega_k)\Delta\omega_k} \left\{ \sum_{\substack{i+j=k \\ i \geq j \geq 0}} b_p(\omega_i, \omega_j) \cos [\Phi_i + \Phi_j + \beta(\omega_i, \omega_j)] \right\} \\ dv_I(\omega_k) &= \sqrt{2 \cdot 2S(\omega_k)\Delta\omega_k} \left\{ \sum_{\substack{i+j=k \\ i \geq j \geq 0}} b_p(\omega_i, \omega_j) \sin [\Phi_i + \Phi_j + \beta(\omega_i, \omega_j)] \right\} \end{aligned} \quad (115)$$

where $S_{f_0 f_0}^P(\omega_k)$ is the pure power spectrum from Eq. (112), the Φ_i are independent, uniformly distributed, random phase angles on $[0, 2\pi]$, and:

$$\beta(\omega_1, \omega_2) = \arctan \left[\frac{\text{Im } B_{f_0 f_0}(\omega_1, \omega_2)}{\text{Re } B_{f_0 f_0}(\omega_1, \omega_2)} \right] \quad (116)$$

is the biphas (Re and Im denote the real and imaginary parts).

Applying these orthogonal increments to the discretized spectral representation theorem (Eq. (8)) gives the following series representation for third-order stochastic processes:

$$f_0(t) = \sqrt{2} \sum_{k=0}^{\infty} \sqrt{2S_{f_0 f_0}^P(\omega_k) \Delta \omega_k} \cos(\omega_k t - \Phi_k) + \sqrt{2} \sum_{k=0}^{\infty} \sum_{i+j=k}^{\infty} \sqrt{2S_{f_0 f_0}(\omega_i + \omega_j) \Delta(\omega_i + \omega_j)} |b_p(\omega_i, \omega_j)| \cos[(\omega_i + \omega_j)t - (\Phi_i + \Phi_j + \beta(\omega_i, \omega_j))] \quad (117)$$

The Higher-Order Spectral Representation Method (HO-SRM) includes any simulation algorithm for stochastic processes, fields, and waves that is based on the (finite) series representations in Eq. (117), or any extension thereof to higher-orders (see e.g., Vandanapu [59]). Simulation algorithms based on the series representation in Eq. (117) are specifically referred to as the Bispectral Representation Method (BSRM).

8.3. Simulation formula: One-dimensional, uni-variate, stationary, third-order stochastic processes

From the infinite series representation displayed in Eq. (117), it follows that the third-order stochastic process $f_0(t)$ can be simulated by the following series as $N \rightarrow \infty$:

$$f(t) = \sqrt{2} \sum_{k=0}^{N-1} \sqrt{2S_{f_0 f_0}^P(\omega_k) \Delta \omega_k} \cos(\omega_k t - \Phi_k) + \sqrt{2} \sum_{k=0}^{N-1} \sum_{i+j=k}^{\infty} \sqrt{2S_{f_0 f_0}(\omega_i + \omega_j) \Delta(\omega_i + \omega_j)} |b_p(\omega_i, \omega_j)| \cos[(\omega_i + \omega_j)t - (\Phi_i + \Phi_j + \beta(\omega_i, \omega_j))] \quad (118)$$

where all definitions and conventions (e.g. frequency discretization) from Section 3 are adopted. Likewise, a sample function $f^{(i)}(t)$ of the simulated stochastic process $f(t)$ can be obtained by replacing the sequence of random phase angles $\Phi_0, \Phi_1, \dots, \Phi_{N-1}$ with their respective i th realizations $\phi_0^{(i)}, \phi_1^{(i)}, \dots, \phi_{N-1}^{(i)}$.

1. **Ensemble Average Property:** The ensemble expected value $\mathcal{E}[f(t)]$, the ensemble autocorrelation function $R_{ff}(\tau)$, and the ensemble three-point correlation function $R_{fff}(\tau_1, \tau_2)$ of the simulated stochastic process $f(t)$ are identical to the corresponding targets, $\mathcal{E}[f_0(t)] = 0$, $R_{f_0 f_0}(\tau)$, $R_{f_0 f_0}(\tau_1, \tau_2)$, respectively.

The value of T appearing in Eq. (21) can be less than or equal to the period T_0 defined in Eq. (17). If, in addition, $T = T_0$, then:

$$S_{ff}(\omega) = S_{f_0 f_0}(\omega), \quad \text{for } 0 \leq \omega \leq \omega_u \quad (119)$$

and:

$$B_{ff}(\omega_1, \omega_2) = B_{f_0 f_0}(\omega_1, \omega_2), \quad \text{for } 0 \leq \omega_1, \omega_2 \leq \omega_u \quad (120)$$

Detailed proofs can be found in Shields and Kim [52], Vandanapu and Shields [60].

2. **Ergodicity Property:** The temporal average $\langle f^{(i)}(t) \rangle_T$, the temporal autocorrelation function $\langle f^{(i)}(t + \tau) f^{(i)}(t) \rangle_T$, and the temporal three-point correlation function of any sample function $f^{(i)}(t)$ are identical to the corresponding targets, $\mathcal{E}[f_0(t)] = 0$, $R_{f_0 f_0}(\tau)$, and $R_{f_0 f_0}(\tau_1, \tau_2)$ respectively. Correspondingly, the temporal power spectral density function $\langle S_{ff}^{(i)}(\omega) \rangle_T$ and bispectrum $\langle B_{ff}^{(i)}(\omega_1, \omega_2) \rangle_T$ are equal to their respective targets $S_{f_0 f_0}(\omega)$ and $B_{f_0 f_0}(\omega_1, \omega_2)$. These identities are valid only when the length T of the sample function $f^{(i)}(t)$ is either equal to the period T_0 or when it approaches infinity. Detailed proofs can be found in Vandanapu and Shields [60].

Use of Fast Fourier Transform (FFT) Technique

The computational cost of generating sample functions using Eq. (118) can be drastically reduced by using the FFT technique. To

take advantage of the FFT technique, Eq. (118) is rewritten as Vandanapu and Shields [60,61]:

$$f(p\Delta t) = \text{Re} \left[\sum_{n=0}^{M-1} B_n e^{i((n\Delta\omega)(p\Delta t))} \right], \quad p = 0, 1, \dots, M-1 \quad (121)$$

where:

$$B_n = \sqrt{2} C_n \left[\sqrt{1 - \sum_{i+j=n}^{\infty} b_p^2(\omega_i, \omega_j)} e^{i\phi_n} + \sum_{i+j=n}^{\infty} |b_p(\omega_i, \omega_j)| e^{i(\phi_i + \phi_j + \beta(\omega_i, \omega_j))} \right] \quad (122)$$

and:

$$C_n = \sqrt{2S_{f_0 f_0}(n\Delta\omega) \Delta\omega} \quad (123)$$

Applying the same definitions and conventions adopted in Section 3, the FFT technique may be applied to Eq. (121), resulting in a drastic reduction of the computational cost.

8.4. Extensions of the higher-order spectral representation method

The HO-SRM has been extended to multi-dimensional, uni-variate, homogeneous, third-order stochastic fields and to one-dimensional, multi-variate, stationary, third-order stochastic processes by Vandanapu and Shields [61]. Vandanapu and Shields [62] further extended the HO-SRM formulation to third-order stochastic waves and applied it for the simulation of stochastic wind fields on long-span bridges. The theoretical properties of the HO-SRM are detailed in a pair of manuscripts that elaborate on ergodicity, orthogonality, and other important properties of third-order stochastic processes and fields [60,63].

Vandanapu [59] further provides the foundations for a fourth-order SRM, arguing that representations of order higher than three are easier to formulate for complex-valued random processes. Meanwhile, Li et al. [64] provide a fourth-order SRM that simulates directly from the trispectrum, but provides no explicit control over third-order properties of the process.

9. Open problems

As detailed herein, the theoretical development of the Spectral Representation Method (SRM) has reached a high level of maturity. In particular, the theory underpinning simulations of stationary and non-stationary (or equivalently, homogeneous and non-homogeneous), scalar and vector, one-dimensional and multi-dimensional, Gaussian and non-Gaussian stochastic processes, fields, and waves is thoroughly developed. Nonetheless, some noteworthy gaps remain.

Perhaps the most important gap in the development of the SRM relates to the simulation of non-stationary and non-Gaussian stochastic processes. The challenge of simulating stochastic processes with an evolving, non-Gaussian probability structure is large. One of the major challenges in simulating non-Gaussian and non-stationary processes is adequately quantifying the evolving probability structure, and specifically the evolutionary power spectral density for non-Gaussian processes. Numerous approaches have been developed to estimate the evolutionary power spectrum for non-Gaussian processes, including estimates based on approximate locally stationary Fourier transforms [65] and optimization schemes [66] to invert the relationship between the non-stationary autocorrelation and the evolutionary spectrum (Eq. (93)) which is not uniquely invertible in general. Other approaches, such as the Iterative Power and Amplitude Correction (IPAC) are built from the S-transform or the Wavelet Transform [67].

A closely related challenge is to estimate the evolutionary spectral properties of the stochastic process from data. Evolutionary spectra are incredibly data-intensive to estimate, and this challenge increases when the process deviates from being Gaussian. For example, using the

non-stationary extension of the BSRM requires estimation of both the evolutionary spectrum and evolutionary bispectrum [68]. Such estimation requires enormous data sets. This has motivated direct data-driven approaches such as the approach by Huang et al. [69] that employs empirical mode decomposition and the SRM. But all such approaches are still in their infancy. In general, methods for simulating stochastic processes that are simultaneously non-stationary and non-Gaussian lag far behind all other cases.

10. Availability and software implementation

The SRM and BSRM are implemented in the open-source Python package UQpy [70,71] and most of the simulation algorithms presented herein are readily available for application.

CRediT authorship contribution statement

George Deodatis: Writing – review & editing, Writing – original draft. **Michael Shields:** Writing – review & editing, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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