

Parcial #1 - Señales y Sistemas 2025-2S

Martín Ramírez Espinosa

$$\textcircled{1} \quad d^2(x_1, x_2) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_T |x_1(t) - x_2(t)|^2 dt$$

$$x_1(t) = A e^{-jn\omega_0 t}, \quad x_2(t) = B e^{jm\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T}, \quad T, A, B \in \mathbb{R}^+, \quad n, m \in \mathbb{Z}$$

Entonces:

$$d(x_1, x_2) = \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \int_T |x_1(t) - x_2(t)|^2 dt}$$

Evaluando $x_1(t)$, $x_2(t)$:

Nota: Si $a \in \mathbb{C} \rightarrow a \cdot a^* = |a|^2$

$$d(x_1, x_2) = \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \int_T |A e^{-jn\omega_0 t} - B e^{jm\omega_0 t}|^2 dt}$$

$$= \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \int_T (A e^{-jn\omega_0 t} - B e^{jm\omega_0 t})(A e^{-jn\omega_0 t} - B e^{jm\omega_0 t})^* dt}$$

$$= \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \int_T (A e^{-jn\omega_0 t} - B e^{jm\omega_0 t})([A e^{-jn\omega_0 t}]^* - [B e^{jm\omega_0 t}]^*) dt}$$

$$= \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \int_T (A e^{-jn\omega_0 t} - B e^{jm\omega_0 t})(A e^{jn\omega_0 t} - B e^{-jm\omega_0 t}) dt}$$

$$= \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \int_T |A e^{jn\omega_0 t}|^2 + |-B e^{jm\omega_0 t}|^2 - AB(e^{-jn\omega_0 t} \cdot e^{-jm\omega_0 t} + e^{jn\omega_0 t} \cdot e^{jm\omega_0 t}) dt}$$

Nota: $x \in \mathbb{R}, z \in \mathbb{C} \rightarrow |x \cdot z| = |x| |z|$

$$= \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \int_T |A|^2 |e^{jn\omega_0 t}|^2 + |-B|^2 |e^{jm\omega_0 t}|^2 - AB(e^{-j(n+m)\omega_0 t} + e^{j(n+m)\omega_0 t}) dt}$$

- Como $A, B \in \mathbb{R}^+$ $\rightarrow |A|^2 = A^2, |-B|^2 = B^2$

- $|e^{jn\omega_0 t}|^2 = \cos^2(n\omega_0 t) + \sin^2(n\omega_0 t) = 1$

- $|e^{jm\omega_0 t}|^2 = \cos^2(m\omega_0 t) + \sin^2(m\omega_0 t) = 1$

- $e^{-j(n+m)\omega_0 t} + e^{j(n+m)\omega_0 t}$

$$= \cos((n+m)\omega_0 t) - j \sin((n+m)\omega_0 t) + \cos((n+m)\omega_0 t) + j \sin((n+m)\omega_0 t)$$

$$= 2 \cos((n+m)\omega_0 t)$$

Entonces:

$$= \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \int_T A^2 + B^2 - AB(2 \cos((n+m)\omega_0 t)) dt}$$

$$= \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \left[\int_T A^2 dt + \int_T B^2 dt - \int_T 2AB \cos((n+m)\omega_0 t) dt \right]}$$

$$= \sqrt{\lim_{T \rightarrow \infty} \frac{1}{T} \left[T(A^2 + B^2) - 2AB \int_T \cos((n+m)\omega_0 t) dt \right]}$$

$$= \sqrt{\lim_{T \rightarrow \infty} \left[A^2 + B^2 - \frac{2AB}{T} \left(\frac{1}{(n+m)\omega_0} \sin((n+m)\omega_0 T) \right) \right]}$$

$$= \sqrt{A^2 + B^2 - 2AB \left(\lim_{T \rightarrow \infty} \frac{\sin((n+m)\omega_0 T)}{(n+m)\omega_0 T} \right)} \rightarrow \boxed{\frac{0}{0}}$$

$$= \sqrt{A^2 + B^2 - 2AB \left(\lim_{T \rightarrow \infty} \frac{\cos((n+m)\omega_0 T) \cdot (n+m)\omega_0}{(n+m)\omega_0} \right)}$$

$$= \sqrt{A^2 + B^2 - 2AB \left(\lim_{T \rightarrow \infty} \cos((n+m)\omega_0 T) \right)}$$

Nota: $W_0 = \frac{2\pi}{T}$, $n, m \in \mathbb{Z} \rightarrow n+m=k \in \mathbb{Z}$

$$= \sqrt{A^2 + B^2 - 2AB \left(\lim_{T \rightarrow \infty} \cos(K \frac{2\pi}{T} x) \right)}$$

$$= \sqrt{A^2 + B^2 - 2AB \left(\lim_{T \rightarrow \infty} \cos(K2\pi) \right)}$$

$$= \sqrt{A^2 + B^2 - 2AB(1)}$$

$$\therefore d(x_1, x_2) = \sqrt{A^2 + B^2 - 2AB}$$

(2) $X(t) = 3 \cos(1000\pi t) + 5 \sin(3000\pi t) + 10 \cos(11000\pi t)$

$$X[n] = ? \quad : F_s = 5 \text{ kHz}$$

$$\therefore T_s = \frac{1}{5 \text{ kHz}} = 2 \cdot 10^{-4} \text{ s}$$

$X(t)$ es el resultado de la superposición de tres señales:

$$X(t) = X_1(t) + X_2(t) + X_3(t) \text{ respectivamente.}$$

Entonces:

$$X_1(t) = 3 \cos(1000\pi t) \rightarrow \omega_1 = 1000\pi \text{ rad/s}$$

$$X_2(t) = 5 \sin(3000\pi t) \rightarrow \omega_2 = 3000\pi \text{ rad/s}$$

$$X_3(t) = 10 \cos(11000\pi t) \rightarrow \omega_3 = 11000\pi \text{ rad/s}$$

Nota: "Frecuencia angular digital" := Ω [rad/sample]

Ω es periódica con periodo $2\pi \rightarrow \Omega \in (-\pi, \pi]$

$P_0 = 0$ falso:

$$\Omega_1 = 2\pi \cdot \frac{f_1}{F_s} = \frac{\omega_1}{F_s} = \frac{1000\pi}{5000} = \frac{\pi}{5} \in (-\pi, \pi] \quad \checkmark$$

$$\Omega_2 = 2\pi \cdot \frac{f_2}{F_s} = \frac{\omega_2}{F_s} = \frac{3000\pi}{5000} = \frac{3\pi}{5} \in (-\pi, \pi] \quad \checkmark$$

$$\Omega_3 = 2\pi \cdot \frac{f_3}{F_s} = \frac{\omega_3}{F_s} = \frac{11000\pi}{5000} = \frac{11\pi}{5} \notin (-\pi, \pi] \quad \times$$

$$\left(\Omega_3 = \underbrace{\frac{11\pi}{5}}_{(\text{mod } 2\pi)} = \frac{11\pi}{5} - 2\pi = \frac{\pi}{5} \right) \xrightarrow{\text{red}} \boxed{x_3(t) \text{ alias de } x_1(t)}$$

Evaluando:

$$x_1(nT_s) = 3 \cos(1000\pi nT_s) = 3 \cos\left(\frac{\pi}{5}n\right) = x_1[n]$$

$$x_2(nT_s) = 5 \sin(3000\pi nT_s) = 5 \sin\left(\frac{3\pi}{5}n\right) = x_2[n]$$

$$x_3(nT_s) = 10 \cos(11000\pi nT_s) = 10 \cos\left(\frac{11\pi}{5}n\right) = x_3[n]$$

$$\therefore x(nT_s) = x[n] = x_1[n] + x_2[n] + x_3[n]$$

$$x[n] = 3 \cos\left(\frac{\pi}{5}n\right) + 5 \sin\left(\frac{3\pi}{5}n\right) + 10 \cos\left(\frac{11\pi}{5}n\right)$$

$$\boxed{x[n] = 13 \cos\left(\frac{\pi}{5}n\right) + 5 \sin\left(\frac{3\pi}{5}n\right)}$$

Período de $\chi(t)$:

$$\exists T_0 > 0 : \chi(t + T_0) = \chi(t) \quad \forall t \iff \frac{\omega_i}{\omega_j} \in \mathbb{Q} \quad \forall i, j$$

$$T_0 = \frac{2\pi}{\omega_0}$$

Ahora:

$$\omega_0 = \gcd(\omega_1, \omega_2, \omega_3) = \gcd(1000\pi, 3000\pi, 11000\pi)$$

$$\omega_0 = 1000\pi \rightarrow T_0 = \frac{2\pi}{1000\pi} = \boxed{\frac{1}{500}s}$$

③ $\chi''(t) = \frac{d^2\chi}{dt^2} : t \in [t_i, t_f] \rightarrow C_n = \frac{1}{(t_f - t_i)n^2\omega_0^2} \int_{t_i}^{t_f} \chi''(t) e^{-jnw_0 t} dt$

$$n \in \mathbb{Z}, \quad T = t_f - t_i$$

$$\chi(t) = \sum_n C_n e^{jnw_0 t} \rightarrow C_n = \frac{1}{T} \int_T \chi(t) \cdot e^{-jnw_0 t} dt$$

$$\frac{d^2\chi}{dt^2} = \frac{d^2}{dt^2} \left(\sum_n C_n e^{jnw_0 t} \right) = \sum_n \frac{d^2}{dt^2} (C_n e^{jnw_0 t})$$

$$\chi''(t) = \sum_n C_n \cdot \frac{d^2}{dt^2} (e^{jnw_0 t}) = \sum_n C_n \cdot (jn\omega_0)^2 e^{jnw_0 t}$$

Analizando:

$$C_n \cdot (jn\omega_0)^2 = C_n \cdot j^2 \cdot n^2 \cdot \omega_0^2 = -C_n n^2 \omega_0^2 = K_n$$

$$\therefore \chi''(t) = \sum_n K_n e^{jnw_0 t} \rightarrow \text{Otra serie de Fourier.}$$

$$\therefore K_n = \frac{1}{T} \int_T x''(t) \cdot e^{-jn\omega_0 t} dt$$

$$-C_n n^2 \omega_0^2 = \frac{1}{T} \int_T x''(t) \cdot e^{-jn\omega_0 t} dt$$

$$C_n = \frac{1}{-T n^2 \omega_0^2} \int_T x''(t) \cdot e^{-jn\omega_0 t} dt$$

$$C_n = \frac{1}{-(t_f - t_i) n^2 \omega_0^2} \int_{t_i}^{t_f} x''(t) e^{-jn\omega_0 t} dt$$

$$C_n = \frac{1}{(t_i - t_f) n^2 \omega_0^2} \int_{t_i}^{t_f} x''(t) e^{-jn\omega_0 t} dt, \quad n \in \mathbb{Z} \setminus \{0\}$$

Nota: los coeficientes de la serie trigonométrica de Fourier (a_n, b_n) , se relacionan con C_n de la siguiente manera :

$$a_n = C_n + C_{-n}, \quad b_n = j(C_n - C_{-n})$$

Entonces :

$$a_n = \frac{1}{(t_i - t_f) n^2 \omega_0^2} \left(\int_{t_i}^{t_f} x''(t) e^{-jn\omega_0 t} dt + \int_{t_i}^{t_f} x''(t) e^{jn\omega_0 t} dt \right)$$

$$a_n = \frac{1}{(t_i - t_f) n^2 \omega_0^2} \left(\int_{t_i}^{t_f} x''(t) [e^{-jn\omega_0 t} + e^{jn\omega_0 t}] dt \right)$$

$$\text{Nota: } \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$a_n = \frac{1}{(t_i - t_f) n^2 \omega_0^2} \left(\int_{t_i}^{t_f} x''(t) [2 \cos(n\omega_0 t)] dt \right)$$

$$a_n = \frac{2}{(t_i - t_f) n^2 \omega_0^2} \int_{t_i}^{t_f} x''(t) \cos(n\omega_0 t) dt \quad n \in \mathbb{Z} \setminus \{0\}$$

$$b_n = j \left(\frac{1}{(t_i - t_f) n^2 \omega_0^2} \left[\int_{t_i}^{t_f} x''(t) e^{-jn\omega_0 t} dt - \int_{t_i}^{t_f} x''(t) e^{jn\omega_0 t} dt \right] \right)$$

$$b_n = j \left(\frac{1}{(t_i - t_f) n^2 \omega_0^2} \int_{t_i}^{t_f} x''(t) [e^{-jn\omega_0 t} - e^{jn\omega_0 t}] dt \right)$$

Nota: $\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$$b_n = j \left(\frac{1}{(t_i - t_f) n^2 \omega_0^2} \int_{t_i}^{t_f} x''(t) [-2j \sin(n\omega_0 t)] dt \right)$$

$$b_n = \frac{j^2 \cdot (-2)}{(t_i - t_f) n^2 \omega_0^2} \int_{t_i}^{t_f} x''(t) \sin(n\omega_0 t) dt$$

$$b_n = \frac{-2}{(t_i - t_f) n^2 \omega_0^2} \int_{t_i}^{t_f} x''(t) \sin(n\omega_0 t) dt \quad n \in \mathbb{Z} \setminus \{0\}$$

④ Hallar: $C_n, \operatorname{Re}\{C_n\}, \operatorname{Im}\{C_n\}, |C_n|, \chi(t), \ell_{\text{rel}}$

$$\chi(t) = \begin{cases} 0 & \text{si } t \in [-\frac{T}{2}, -d_2] \cup [d_2, \frac{T}{2}] \\ m_1 t + b_1 & \text{si } t \in [-d_2, -d_1] \\ -m_2 t + b_2 & \text{si } t \in [-d_1, 0) \\ m_2 t + b_2 & \text{si } t \in [0, d_1) \\ -m_1 t + b_1 & \text{si } t \in [d_1, d_2] \end{cases}$$

Donde:

$$M_1 = \frac{A - 0}{-d_2 - (-d_2)} = \frac{A}{d_2 - d_1}, \quad b_1 = M_1 d_2$$

$$M_2 = \frac{A - 0}{d_1 - 0} = \frac{A}{d_1} \quad , \quad b_2 = 0$$

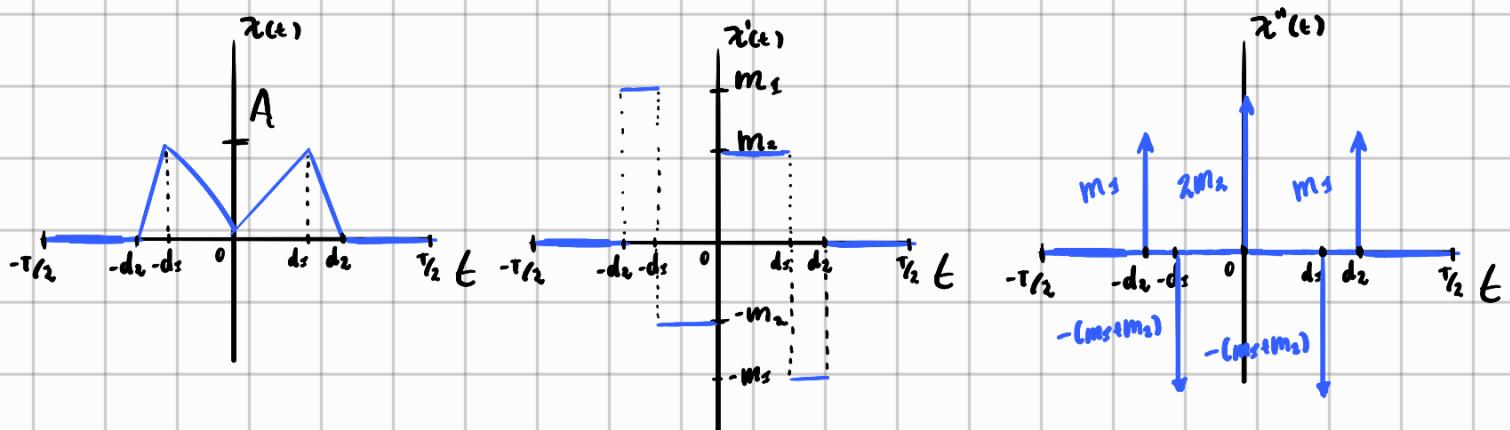
$$T, d_2, d_1, A \in \mathbb{R}^+$$

$$\frac{T}{2} > d_2 > d_1$$

$$x(t) = C_0 + \sum_n C_n e^{jn\omega_0 t} \rightarrow C_n = \frac{-1}{T n^2 \omega_0^2} \int_T x''(t) e^{-jn\omega_0 t} dt, \quad n \in \mathbb{Z} \setminus \{0\}$$

Nota: Según la gráfica de $x(t)$, $M_1 > M_2$

Entonces:



De modo que:

$$x''(t) = m_1 \delta(t+d_2) - (m_1+m_2) \delta(t+d_1) + 2m_2 \delta(t) - (m_1+m_2) \delta(t-d_1) + m_1 \delta(t-d_2)$$

$$x''(t) = M_1 [\delta(t+d_2) + \delta(t-d_2)] - (M_1+M_2) [\delta(t+d_1) + \delta(t-d_1)] + 2m_2 \delta(t)$$

Evaluando $\mathcal{X}''(t)$:

$$C_n = \frac{-1}{T n^2 w_0^2} \left[\int_T [m_1 [\delta(t+d_2) + \delta(t-d_2)] e^{-jn\omega_0 t} dt - \int_T (m_1+m_2) [\delta(t+d_1) + \delta(t-d_1)] e^{-jn\omega_0 t} dt + \int_T 2m_2 \delta(t) e^{-jn\omega_0 t} dt \right]$$

Nota: $\int_T f(t) \delta(t \pm t_0) dt = f(\pm t_0)$

$$C_n = \frac{-1}{T n^2 w_0^2} \left[m_1 \left(\int_T \delta(t+d_2) e^{-jn\omega_0 t} dt + \int_T \delta(t-d_2) e^{-jn\omega_0 t} dt \right) - (m_1+m_2) \left(\int_T \delta(t+d_1) e^{-jn\omega_0 t} dt + \int_T \delta(t-d_1) e^{-jn\omega_0 t} dt \right) + 2m_2 \left(\int_T \delta(t) e^{-jn\omega_0 t} dt \right) \right]$$

$$C_n = \frac{-1}{T n^2 w_0^2} \left[m_1 \left(e^{jn\omega_0(d_2)} + e^{-jn\omega_0(d_2)} \right) - (m_1+m_2) \left(e^{jn\omega_0(d_1)} + e^{-jn\omega_0(d_1)} \right) + 2m_2 \left(e^{-jn\omega_0(0)} \right) \right]$$

$$C_n = \frac{-1}{T n^2 w_0^2} \left[m_1 \left(\cos(d_2 n \omega_0) + j \sin(d_2 n \omega_0) + \cos(d_2 n \omega_0) - j \sin(d_2 n \omega_0) \right) - (m_1+m_2) \left(\cos(d_1 n \omega_0) + j \sin(d_1 n \omega_0) + \cos(d_1 n \omega_0) - j \sin(d_1 n \omega_0) \right) + 2m_2 \left(\cos(0) - j \sin(0) \right) \right]$$

$$C_n = \frac{-1}{T n^2 \omega_0^2} \left[2m_1 \cos(d_2 n \omega_0) - 2(m_1 + m_2) \cos(d_1 n \omega_0) + 2m_2 \right] \quad n \in \mathbb{Z} \setminus \{0\}$$

∴ C_n es puramente real $\rightarrow X(t)$ es par ✓

Por lo tanto:

$$\operatorname{Re}\{C_n\} = C_n, \operatorname{Im}\{C_n\} = 0 \rightarrow \arg C_n = \tan^{-1}\left(\frac{\operatorname{Im}\{C_n\}}{\operatorname{Re}\{C_n\}}\right) = 0$$

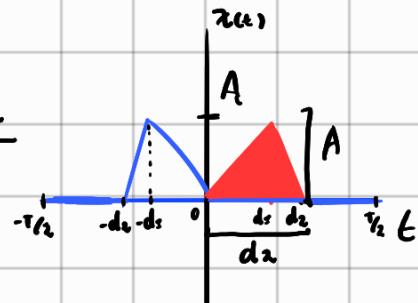
$$|C_n| = \left| \frac{-1}{T n^2 \omega_0^2} \left[2m_1 \cos(d_2 n \omega_0) - 2(m_1 + m_2) \cos(d_1 n \omega_0) + 2m_2 \right] \right|$$

$$|C_n| = \frac{1}{|T n^2 \omega_0^2|} \left| 2m_1 \cos(d_2 n \omega_0) - 2(m_1 + m_2) \cos(d_1 n \omega_0) + 2m_2 \right|$$

$$|C_n| = \frac{1}{T n^2 \omega_0^2} \left| 2m_1 \cos(d_2 n \omega_0) - 2(m_1 + m_2) \cos(d_1 n \omega_0) + 2m_2 \right|$$

Coeficiente C_0 : Se seguirá calculando de manera usual.

$$C_0 = \frac{1}{T} \int_T x(t) dt = \frac{1}{T} \left(2 \cdot \frac{A \cdot d_2}{2} \right) = \frac{A \cdot d_2}{T}$$



Error relativo de la estimación:

$$C_R [\%] = \left(1 - \frac{\sum_{n=-N}^N |C_n|^2 P_n}{\bar{P}_x} \right) \cdot 100$$

Recordando:

$$X(t) = \sum_{n=-\infty}^{\infty} C_n \cdot \phi_n(t) \rightarrow \tilde{X}(t) = \sum_{n=-N}^N C_n \cdot \phi_n(t)$$

$$\bar{P}_x = \frac{1}{T} \int_T |X(t)|^2 dt \rightarrow \bar{P}_{\tilde{x}} = \frac{1}{T} \int_T |\tilde{X}(t)|^2 dt$$

$$\begin{aligned}\bar{P}_{\tilde{x}} &= \frac{1}{T} \int_T \left| \sum C_n \cdot \phi_n(t) \right|^2 dt = \frac{1}{T} \int_T \sum |C_n \cdot \phi_n(t)|^2 dt \\ &= \frac{1}{T} \int_T \sum |C_n|^2 |\phi_n(t)|^2 dt = \sum |C_n|^2 \underbrace{\frac{1}{T} \int_T |\phi_n(t)|^2 dt}_{P_n} \\ &= \sum_{n=-N}^N |C_n|^2 \cdot P_n\end{aligned}$$

$$C_F = \left(1 - \frac{\bar{P}_{\tilde{x}}}{\bar{P}_x} \right) = \left(1 - \frac{\sum_{n=-N}^N |C_n|^2 \cdot P_n}{\bar{P}_x} \right)$$

$$C_F \% = \left(1 - \frac{\sum_{n=-N}^N |C_n|^2 \cdot P_n}{\bar{P}_x} \right) \cdot 100$$

Entonces:

$$\begin{aligned}P_n &= \frac{1}{T} \int_T |e^{jn\omega_0 t}|^2 dt = \frac{1}{T} \int_T \cos^2(n\omega_0 t) + \sin^2(n\omega_0 t) dt \\ &= \frac{1}{T} \int_T 1 \cdot dt = \frac{1}{T} T = 1 \quad \forall n\end{aligned}$$

$$\bar{P}_x = \frac{1}{T} \int_T |X(t)|^2 dt \quad // \text{como } X(t) \geq 0 \quad \forall t$$

$$\bar{P}_x = \frac{1}{T} \int_T (X(t))^2 dt$$

Como $X(t)$ es par $\rightarrow (X(t))^2$ también lo es:

$$\bar{P}_X = \frac{2}{T} \int_0^{T/2} (X(t))^2 dt = \frac{2}{T} \left[\int_0^{d_3} (m_2 t)^2 dt + \int_{d_3}^{d_2} (-m_3 t + b_1)^2 dt \right]$$

$$\bar{P}_X = \frac{2}{T} \left[m_2^2 \cdot \frac{t^3}{3} \Big|_0^{d_3} + \int_{d_3}^{d_2} (b_1^2 - 2b_1 m_3 t + m_3^2 t^2) dt \right]$$

$$\bar{P}_X = \frac{2}{T} \left[\frac{m_2^2 \cdot d_3^3}{3} + b_1^2 \cdot t \Big|_{d_3}^{d_2} - 2b_1 m_3 \cdot \frac{t^2}{2} \Big|_{d_3}^{d_2} + m_3^2 \cdot \frac{t^3}{3} \Big|_{d_3}^{d_2} \right]$$

$$\bar{P}_X = \frac{2}{T} \left[\frac{m_2^2 d_3^3}{3} + b_1^2 (d_2 - d_3) - 2b_1 m_3 \left(\frac{d_2^2}{2} - \frac{d_3^2}{2} \right) + m_3^2 \left(\frac{d_2^3}{3} - \frac{d_3^3}{3} \right) \right]$$

$$\bar{P}_X = \frac{2}{T} \left[\frac{A^2 d_1}{3} + \frac{A^2 (d_2 - d_1)}{3} \right] = \frac{2}{T} \left[\frac{A^2 d_2}{3} \right]$$

$$\bar{P}_X = \frac{2 A^2 d_2}{3 T}$$

$$\therefore Cr[Y] = \left(1 - \frac{\sum_{n=-N}^N |C_n|^2 \cdot 1}{\frac{2 A^2 d_2}{3 T}} \right) \cdot 100$$

$$Cr[Y] = \left(1 - \frac{3T \sum_{n=-N}^N |C_n|^2}{2 A^2 d_2} \right) \cdot 100$$

Para $n \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$

$$Cr[Y] = \left(1 - \frac{3T \sum_{n=-5}^5 |C_n|^2}{2 A^2 d_2} \right) \cdot 100$$

