Parcial #1 - Señales y Sistemas 2025-2S

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① $$d^2(x_1,x_2)=\lim_{T\to\infty}\frac{1}{T}\int_T |x_1(t)-x_2(t)|^2\,dt$$

$$x_1(t)=Ae^{-jn\omega_0 t},\quad x_2(t)=Be^{jm\omega_0 t}$$

$$\omega_0=\frac{2\pi}{T},\quad T,A,B\in\mathbb{R}^+,\quad n,m\in\mathbb{Z}$$

Entonces:

$$d(x_1,x_2)=\sqrt{\lim_{T\to\infty}\frac{1}{T}\int_T |x_1(t)-x_2(t)|^2\,dt}$$

Evaluando $x_1(t)$, $x_2(t)$:

$\boxed{\text{Nota: Si } a\in\mathbb{C}\to a\cdot a^*=|a|^2}$

$$d(x_1,x_2)=\sqrt{\lim_{T\to\infty}\frac{1}{T}\int_T |Ae^{-jn\omega_0 t}-Be^{jm\omega_0 t}|^2\,dt}$$

$$=\sqrt{\lim_{T\to\infty}\frac{1}{T}\int_T (Ae^{-jn\omega_0 t}-Be^{jm\omega_0 t})(Ae^{-jn\omega_0 t}-Be^{jm\omega_0 t})^*\,dt}$$

$$=\sqrt{\lim_{T\to\infty}\frac{1}{T}\int_T (Ae^{-jn\omega_0 t}-Be^{jm\omega_0 t})([Ae^{-jn\omega_0 t}]^*-[Be^{jm\omega_0 t}]^*)\,dt}$$

$$=\sqrt{\lim_{T\to\infty}\frac{1}{T}\int_T (Ae^{-jn\omega_0 t}-B^{jm\omega_0 t})(Ae^{jn\omega_0 t}-Be^{-jm\omega_0 t})\,dt}$$

$$=\sqrt{\lim_{T\to\infty}\frac{1}{T}\int_T |Ae^{jn\omega_0 t}|^2+|-Be^{jm\omega_0 t}|^2-AB(e^{-jn\omega_0 t}\cdot e^{-jm\omega_0 t}+e^{jn\omega_0 t}\cdot e^{jm\omega_0 t})\,dt}$$

$\boxed{\text{Nota: } x \in \mathbb{R}, z \in \mathbb{C} \rightarrow |x \cdot z| = |x||z|}$

$$= \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_T |A|^2 |e^{jn\omega_0 t}|^2 + |-B|^2 |e^{jm\omega_0 t}|^2 - AB\left(e^{-j(n+m)\omega_0 t} + e^{j(n+m)\omega_0 t}\right) dt}$$

- Como $A, B \in \mathbb{R}^+ \rightarrow |A|^2 = A^2$, $|-B|^2 = B^2$
- $|e^{jn\omega_0 t}|^2 = \cos^2(n\omega_0 t) + sen^2(n\omega_0 t) = 1$
- $|e^{jm\omega_0 t}|^2 = \cos^2(m\omega_0 t) + sen^2(m\omega_0 t) = 1$
- $e^{-j(n+m)\omega_0 t} + e^{j(n+m)\omega_0 t}$

$= \cos((n+m)\omega_0 t) - j\,sen((n+m)\omega_0 t) + \cos((n+m)\omega_0 t) + j\,sen((n+m)\omega_0 t)$

$= 2\cos((n+m)\omega_0 t)$

Entonces:

$$= \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_T A^2 + B^2 - AB\left(2\cos((n+m)\omega_0 t)\right) dt}$$

$$= \sqrt{\lim_{T\to\infty} \frac{1}{T} \left[\int_T A^2 dt + \int_T B^2 dt - \int_T 2AB\cos((n+m)\omega_0 t) dt \right]}$$

$$= \sqrt{\lim_{T\to\infty} \frac{1}{T} \left[T(A^2+B^2) - 2AB \int_T \cos((n+m)\omega_0 t) dt \right]}$$

$$= \sqrt{\lim_{T\to\infty} \left[A^2 + B^2 - \frac{2AB}{T}\left(\frac{1}{(n+m)\omega_0} sen((n+m)\omega_0 T) \right) \right]}$$

$$= \sqrt{A^2 + B^2 - 2AB\left(\lim_{T\to\infty} \frac{sen((n+m)\omega_0 T)}{(n+m)\omega_0 T} \right)} \rightarrow \boxed{\frac{0}{0}}$$

$$= \sqrt{A^2 + B^2 - 2AB\left(\lim_{T\to\infty} \frac{\cos((n+m)\omega_0 T) \cdot \cancel{(n+m)\omega_0}}{\cancel{(n+m)\omega_0}} \right)}$$

$$= \sqrt{A^2 + B^2 - 2AB\left(\lim_{T\to\infty} \cos((n+m)\omega_0 T) \right)}$$

Nota: $\omega_0 = \frac{2\pi}{T}$, $n, m \in \mathbb{Z} \rightarrow n+m = k \in \mathbb{Z}$

$$= \sqrt{A^2 + B^2 - 2AB\left(\lim_{T\to\infty} \cos\left(k \frac{2\pi}{\cancel{T}} \cancel{T}\right)\right)}$$

$$= \sqrt{A^2 + B^2 - 2AB\left(\lim_{T\to\infty} \cos(k2\pi)\right)}$$

$$= \sqrt{A^2 + B^2 - 2AB(1)}$$

$$\therefore \boxed{d(x_1, x_2) = \sqrt{A^2 + B^2 - 2AB}}$$

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(2) $x(t) = 3\cos(1000\pi t) + 5\sin(3000\pi t) + 10\cos(11000\pi t)$

$x[n] = ?$: $F_s = 5\,KHz$

$\therefore T_s = \frac{1}{5\,kHz} = 2 \cdot 10^{-4}\, s$

$x(t)$ es el resultado de la superposición de tres señales:

$x(t) = x_1(t) + x_2(t) + x_3(t)$ respectivamente.

Entonces:

$x_1(t) = 3\cos(1000\pi t) \rightarrow \omega_1 = 1000\pi\ rad/s$

$x_2(t) = 5\sin(3000\pi t) \rightarrow \omega_2 = 3000\pi\ rad/s$

$x_3(t) = 10\cos(11000\pi t) \rightarrow \omega_3 = 11000\pi\ rad/s$

Nota: "Frecuencia angular digital" $:= \Omega$ [rad/sample]

Ω es periódica con periodo $2\pi \rightarrow \Omega \in (-\pi, \pi]$

Por lo tanto:

$$\Omega_1 = 2\pi \cdot \frac{f_1}{F_s} = \frac{\omega_1}{F_s} = \frac{1000\pi}{5000} = \frac{\pi}{5} \in (-\pi, \pi] \checkmark$$

$$\Omega_2 = 2\pi \cdot \frac{f_2}{F_s} = \frac{\omega_2}{F_s} = \frac{3000\pi}{5000} = \frac{3\pi}{5} \in (-\pi, \pi] \checkmark$$

$$\Omega_3 = 2\pi \cdot \frac{f_3}{F_s} = \frac{\omega_3}{F_s} = \frac{11000\pi}{5000} = \frac{11\pi}{5} \notin (-\pi, \pi] \times$$

$$\hookrightarrow \Omega_3 = \underbrace{\frac{11\pi}{5} \equiv \frac{11\pi}{5} - 2\pi}_{(\mathrm{mod}\, 2\pi)} = \frac{\pi}{5} \rightarrow \boxed{x_3(t) \text{ alias de } x_1(t)}$$

Evaluando:

$$x_1(nT_s) = 3\cos(1000\pi n T_s) = 3\cos\left(\frac{\pi}{5}n\right) = x_1[n]$$

$$x_2(nT_s) = 5\sin(3000\pi n T_s) = 5\sin\left(\frac{3\pi}{5}n\right) = x_2[n]$$

$$x_3(nT_s) = 10\cos(11000\pi n T_s) = 10\cos\left(\frac{\pi}{5}n\right) = x_3[n]$$

$$\therefore X(nT_s) = X[n] = x_1[n] + x_2[n] + x_3[n]$$

$$x[n] = 3\cos\left(\frac{\pi}{5}n\right) + 5\sin\left(\frac{3\pi}{5}n\right) + 10\cos\left(\frac{\pi}{5}n\right)$$

$$\boxed{X[n] = 13\cos\left(\frac{\pi}{5}n\right) + 5\sin\left(\frac{3\pi}{5}n\right)}$$

Periodo de $x(t)$:

$$\exists T_0 > 0 : x(t+T_0) = x(t) \; \forall t \longleftrightarrow \frac{w_i}{w_j} \in \mathbb{Q} \; \forall i,j$$

$$T_0 = \frac{2\pi}{w_0}$$

Ahora:

$$w_0 = gcd(w_1, w_2, w_3) = gcd(1000\pi, 3000\pi, 11000\pi)$$

$$w_0 = 1000\pi \longrightarrow T_0 = \frac{2\pi}{1000\pi} = \boxed{\frac{1}{500} s}$$

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③ $$x''(t) = \frac{d^2 x}{dt^2} : t \in [t_i, t_f] \rightarrow C_n = \frac{1}{(t_i - t_f) n^2 w_0^2} \int_{t_i}^{t_f} x''(t) e^{-jnw_0 t} dt$$

$n \in \mathbb{Z}$, $T = t_f - t_i$

$$x(t) = \sum_n c_n e^{jnw_0 t} \longrightarrow C_n = \frac{1}{T} \int_T x(t) \cdot e^{-jnw_0 t} dt$$

$$\frac{d^2 x}{dt^2} = \frac{d^2}{dt^2} \left(\sum_n c_n e^{jnw_0 t} \right) = \sum_n \frac{d^2}{dt^2} (C_n e^{jnw_0 t})$$

$$x''(t) = \sum_n C_n \cdot \frac{d^2}{dt^2} (e^{jnw_0 t}) = \sum_n C_n \cdot (jnw_0)^2 e^{jnw_0 t}$$

Analizando:

$$C_n \cdot (jnw_0)^2 = C_n \cdot j^2 \cdot n^2 \cdot w_0^2 = -C_n n^2 w_0^2 = K_n$$

$$\therefore x''(t) = \sum_n K_n e^{jnw_0 t} \longrightarrow \text{Otra serie de Fourier.}$$

$$\therefore K_n = \frac{1}{T}\int_T x''(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$-C_n n^2\omega_0^2 = \frac{1}{T}\int_T x''(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$C_n = \frac{1}{-T n^2\omega_0^2}\int_T x''(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$C_n = \frac{1}{-(t_f-t_i)n^2\omega_0^2}\int_{t_i}^{t_f} x''(t)e^{-jn\omega_0 t}\,dt$$

$$\boxed{C_n = \frac{1}{(t_i-t_f)n^2\omega_0^2}\int_{t_i}^{t_f} x''(t)e^{-jn\omega_0 t}\,dt\ ,\quad n\in\mathbb{Z}\setminus\{0\}}$$

Nota: Los coeficientes de la serie trigonométrica de Fourier (a_n, b_n), se relacionan con C_n de la siguiente manera:

$$a_n = C_n + C_{-n} \quad , \quad b_n = j(C_n - C_{-n})$$

Entonces:

$$a_n = \frac{1}{(t_i-t_f)n^2\omega_0^2}\left(\int_{t_i}^{t_f} x''(t)e^{-jn\omega_0 t}\,dt + \int_{t_i}^{t_f} x''(t)e^{jn\omega_0 t}\,dt\right)$$

$$a_n = \frac{1}{(t_i-t_f)n^2\omega_0^2}\left(\int_{t_i}^{t_f} x''(t)\left[e^{-jn\omega_0 t}+e^{jn\omega_0 t}\right]dt\right)$$

Nota: $\cos\theta = \dfrac{e^{j\theta}+e^{-j\theta}}{2}$

$$a_n = \frac{1}{(t_i-t_f)n^2\omega_0^2}\left(\int_{t_i}^{t_f} x''(t)\left[2\cos(n\omega_0 t)\right]dt\right)$$

$$\boxed{a_n = \frac{2}{(t_i - t_f)n^2\omega_0^2}\int_{t_i}^{t_f} x''(t)\cos(n\omega_0 t)\,dt} \quad n \in \mathbb{Z}\setminus\{0\}$$

$$b_n = j\left(\frac{1}{(t_i - t_f)n^2\omega_0^2}\left[\int_{t_i}^{t_f} x''(t)e^{-jn\omega_0 t}dt - \int_{t_i}^{t_f} x''(t)e^{jn\omega_0 t}dt\right]\right)$$

$$b_n = j\left(\frac{1}{(t_i - t_f)n^2\omega_0^2}\int_{t_i}^{t_f} x''(t)\left[e^{-jn\omega_0 t} - e^{jn\omega_0 t}\right]dt\right)$$

$$\boxed{\text{Nota: } \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$b_n = j\left(\frac{1}{(t_i - t_f)n^2\omega_0^2}\int_{t_i}^{t_f} x''(t)\left[-2j\sin(n\omega_0 t)\right]dt\right)$$

$$b_n = \frac{j^2\cdot(-2)}{(t_i - t_f)n^2\omega_0^2}\int_{t_i}^{t_f} x''(t)\sin(n\omega_0 t)\,dt$$

$$\boxed{b_n = \frac{2}{(t_i - t_f)n^2\omega_0^2}\int_{t_i}^{t_f} x''(t)\sin(n\omega_0 t)\,dt} \quad n \in \mathbb{Z}\setminus\{0\}$$

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4 Hallar: C_n, $Re\{C_n\}$, $Im\{C_n\}$, $|C_n|$, $\measuredangle C_n$, e_{rel}

$$x(t) = \begin{cases} 0 & \text{si } t \in [-\frac{T}{2}, -d_2] \cup [d_2, \frac{T}{2}) \\ m_1 t + b_1 & \text{si } t \in [-d_2, -d_1) \\ -m_2 t + b_2 & \text{si } t \in [-d_1, 0) \\ m_2 t + b_2 & \text{si } t \in [0, d_1) \\ -m_1 t + b_1 & \text{si } t \in [d_1, d_2) \end{cases}$$

Donde:

$$m_1 = \frac{A-0}{-d_1-(-d_2)} = \frac{A}{d_2-d_1} \quad , \quad b_1 = m_1 d_2$$

$$m_2 = \frac{A-0}{d_1-0} = \frac{A}{d_1} \quad , \quad b_2 = 0$$

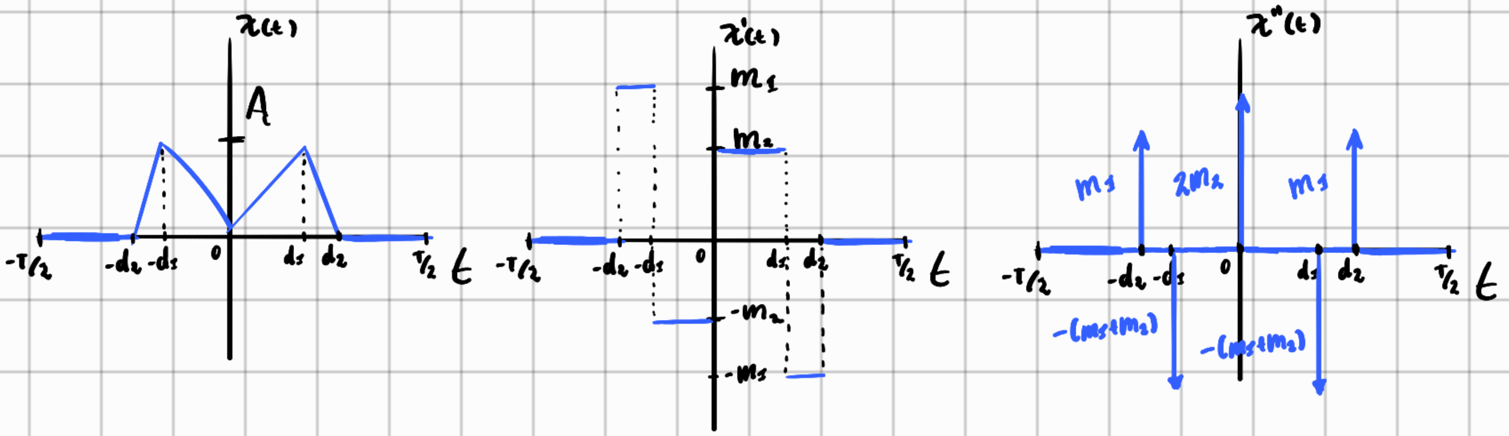
$$T, d_2, d_1, A \in \mathbb{R}^+$$

$$\frac{T}{2} > d_2 > d_1$$

$$x(t) = C_0 + \sum_n C_n e^{jn\omega_0 t} \rightarrow C_n = \frac{-1}{T n^2 \omega_0^2} \int_T x''(t) e^{-jn\omega_0 t} dt \, , n \in \mathbb{Z} \setminus \{0\}$$

Nota: Según la gráfica de $x(t)$, $m_1 > m_2$

Entonces:



De modo que:

$$x''(t) = m_1\delta(t+d_2) - (m_1+m_2)\delta(t+d_1) + 2m_2\delta(t) - (m_1+m_2)\delta(t-d_1) + m_1\delta(t-d_2)$$

$$x''(t) = m_1\left[\delta(t+d_2) + \delta(t-d_2)\right] - (m_1+m_2)\left[\delta(t+d_1) + \delta(t-d_1)\right] + 2m_2\delta(t)$$

Evaluando $x''(t)$:

$$C_n = \frac{-1}{T\,n^2\omega_0^2}\Big[\int_T m_1\left[\delta(t+d_2)+\delta(t-d_2)\right]e^{-jn\omega_0 t}dt$$

$$-\int_T (m_1+m_2)\left[\delta(t+d_1)+\delta(t-d_1)\right]e^{-jn\omega_0 t}dt$$

$$+\int_T 2m_2\,\delta(t)e^{-jn\omega_0 t}dt\Big]$$

$$\boxed{\textbf{Nota:}\ \int_T f(t)\,\delta(t\pm t_0)\,dt = f(\mp t_0)}$$

$$C_n = \frac{-1}{T\,n^2\omega_0^2}\Big[m_1\left(\int_T \delta(t+d_2)e^{-jn\omega_0 t}dt+\int_T \delta(t-d_2)e^{-jn\omega_0 t}dt\right)$$

$$-(m_1+m_2)\left(\int_T \delta(t+d_1)e^{-jn\omega_0 t}dt+\int_T \delta(t-d_1)e^{-jn\omega_0 t}dt\right)$$

$$+2m_2\left(\int_T \delta(t)e^{-jn\omega_0 t}dt\right)\Big]$$

$$C_n = \frac{-1}{T\,n^2\omega_0^2}\Big[m_1\left(e^{jn\omega_0(d_2)}+e^{-jn\omega_0(d_2)}\right)$$

$$-(m_1+m_2)\left(e^{jn\omega_0(d_1)}+e^{-jn\omega_0(d_1)}\right)$$

$$+2m_2\left(e^{-jn\omega_0(0)}\right)\Big]$$

$$C_n = \frac{-1}{T\,n^2\omega_0^2}\Big[m_1\left(\cos(d_2 n\omega_0)+\cancel{j\sin(d_2 n\omega_0)}+\cos(d_2 n\omega_0)-\cancel{j\sin(d_2 n\omega_0)}\right)$$

$$-(m_1+m_2)\left(\cos(d_1 n\omega_0)+\cancel{j\sin(d_1 n\omega_0)}+\cos(d_1 n\omega_0)-\cancel{j\sin(d_1 n\omega_0)}\right)$$

$$+2m_2\left(\overset{1}{\cancel{\cos(0)}}-j\,\overset{0}{\cancel{\operatorname{sen}(0)}}\right)\Big]$$

$$C_n = \frac{-1}{T\,n^2\omega_0^2}\left[2m_1\cos(d_2 n\omega_0) - 2(m_1+m_2)\cos(d_1 n\omega_0) + 2m_2\right] \quad n\in\mathbb{Z}\setminus\{0\}$$

$\therefore$ C_n es puramente real $\rightarrow$ $x(t)$ es par ✓

Por lo tanto:

$$\mathrm{Re}\{C_n\} = C_n \,,\ \mathrm{Im}\{C_n\} = 0 \rightarrow \measuredangle C_n = \tan^{-1}\left(\frac{\mathrm{Im}\{C_n\}}{\mathrm{Re}\{C_n\}}\right) = 0$$

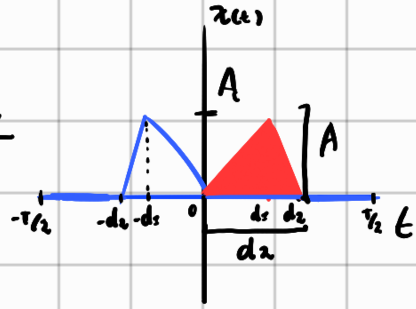
$$|C_n| = \left|\frac{-1}{T\,n^2\omega_0^2}\left[2m_1\cos(d_2 n\omega_0) - 2(m_1+m_2)\cos(d_1 n\omega_0) + 2m_2\right]\right|$$

$$|C_n| = \frac{|-1|}{|T\,n^2\omega_0^2|}\left|2m_1\cos(d_2 n\omega_0) - 2(m_1+m_2)\cos(d_1 n\omega_0) + 2m_2\right|$$

$$|C_n| = \frac{1}{T\,n^2\omega_0^2}\left|2m_1\cos(d_2 n\omega_0) - 2(m_1+m_2)\cos(d_1 n\omega_0) + 2m_2\right|$$

Coeficiente C_0: Se seguirá calculando de manera usual.

$$C_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{T}\left(2\cdot\frac{A\cdot d_2}{2}\right) = \frac{A\cdot d_2}{T}$$



Error relativo de la estimación:

$$e_r[\%] = \left(1 - \frac{\sum_{n=-N}^{N}|C_n|^2\cdot P_n}{\overline{P_x}}\right)\cdot 100$$

Recordando:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot \phi_n(t) \rightarrow \hat{x}(t) = \sum_{n=-N}^{N} C_n \cdot \phi_n(t)$$

$$\overline{P_x} = \frac{1}{T}\int_T |x(t)|^2 dt \longrightarrow \overline{P_{\hat{x}}} = \frac{1}{T}\int_T |\hat{x}(t)|^2 dt$$

$$\overline{P_{\hat{x}}} = \frac{1}{T}\int_T \left|\sum C_n \cdot \phi_n(t)\right|^2 dt = \frac{1}{T}\int_T \sum |C_n \cdot \phi_n(t)|^2 dt$$

$$= \frac{1}{T}\int_T \sum |C_n|^2 |\phi_n(t)|^2 dt = \sum |C_n|^2 \underbrace{\frac{1}{T}\int_T |\phi_n(t)|^2 dt}_{P_n}$$

$$= \sum_{n=-N}^{N} |C_n|^2 \cdot P_n$$

$$e_r = \left(1 - \frac{\overline{P_{\hat{x}}}}{\overline{P_x}}\right) = \left(1 - \frac{\sum_{n=-N}^{N} |C_n|^2 \cdot P_n}{\overline{P_x}}\right)$$

$$e_r[\%] = \left(1 - \frac{\sum_{n=-N}^{N} |C_n|^2 \cdot P_n}{\overline{P_x}}\right) \cdot 100$$

Entonces:

$$P_n = \frac{1}{T}\int_T |e^{jn\omega_0 t}|^2 dt = \frac{1}{T}\int_T \cos^2(n\omega_0 t) + \operatorname{sen}^2(n\omega_0 t)\, dt$$

$$= \frac{1}{T}\int_T 1 \cdot dt = \frac{1}{T} T = 1 \quad \forall n$$

$$\overline{P_x} = \frac{1}{T}\int_T |x(t)|^2 dt \quad // \text{como } x(t) \geq 0 \ \forall t$$

$$\overline{P_x} = \frac{1}{T}\int_T (x(t))^2 dt$$

Como $x(t)$ es par $\rightarrow (x(t))^2$ también lo es:

$$\overline{P_x} = \frac{2}{T}\int_0^{T/2} (x(t))^2 dt = \frac{2}{T}\left[\int_0^{d_1}(m_2 t)^2 dt + \int_{d_1}^{d_2}(-m_1 t + b_1)^2 dt\right]$$

$$\overline{P_x} = \frac{2}{T}\left[m_2^2 \cdot \frac{t^3}{3}\Big|_0^{d_1} + \int_{d_1}^{d_2}(b_1^2 - 2b_1 m_1 t + m_1^2 t^2)dt\right]$$

$$\overline{P_x} = \frac{2}{T}\left[\frac{m_2^2 \cdot d_1^3}{3} + b_1^2 \cdot t\Big|_{d_1}^{d_2} - 2b_1 m_1 \cdot \frac{t^2}{2}\Big|_{d_1}^{d_2} + m_1^2 \cdot \frac{t^3}{3}\Big|_{d_1}^{d_2}\right]$$

$$\overline{P_x} = \frac{2}{T}\left[\frac{m_2^2 d_1^3}{3} + b_1^2(d_2 - d_1) - 2b_1 m_1\left(\frac{d_2^2}{2} - \frac{d_1^2}{2}\right) + m_1^2\left(\frac{d_2^3}{3} - \frac{d_1^3}{3}\right)\right]$$

$$\overline{P_x} = \frac{2}{T}\left[\frac{A^2 \cdot d_1}{3} + \frac{A^2(d_2 - d_1)}{3}\right] = \frac{2}{T}\left[\frac{A^2 d_2}{3}\right]$$

$$\overline{P_x} = \frac{2A^2 d_2}{3T}$$

$$\therefore\ e_r[\%] = \left(1 - \frac{\sum_{n=-N}^{N} |C_n|^2 \cdot 1}{\frac{2A^2 d_2}{3}}\right) \cdot 100$$

$$e_r[\%] = \left(1 - \frac{3\sum_{n=-N}^{N} |C_n|^2}{2A^2 d_2}\right) \cdot 100$$

Para $n \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$

$$e_r[\%] = \left(1 - \frac{3\sum_{n=-5}^{5} |C_n|^2}{2A^2 d_2}\right) \cdot 100$$