Signals & Systems - Exam 2
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1. AM: Keep track of the spectrum at every step.

Let $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$: f in Hz.

Assume:
- "Bandwidth" $:= B$.
- $m : \mathbb{R} \mapsto \mathbb{R}$ and band-limited to $|f| \leq B$.
- "Carrier frequency" $:= f_0$: $f_0 >> B$.
- "Carrier phase" $:= \theta_0$: $\theta_0 = 0$.

Message Signal:

$$x_0(t) = m(t) \Leftrightarrow X_0(f) = M(f) \longrightarrow M(f) = 0 \text{ for } |f| > B$$

A useful modulation property:

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

If $x(t) \Leftrightarrow X(f)$, then

$$x(t)\, e^{j2\pi f_0 t} \Longleftrightarrow X(f - f_0), \quad x(t)\, e^{-j2\pi f_0 t} \Longleftrightarrow X(f + f_0)$$

So $x(t)\cos(2\pi f_0 t) \longleftrightarrow \frac{1}{2}\left[X(f - f_0) + X(f + f_0)\right]$

Multiplication by a cosine $\Rightarrow$ Two shifted copies of the spectrum

DSB-SC AM modulation (transmitter):

$x_1(t) = s(t) = A_1 m(t) \cos(2\pi f_0 t)$, then

$X_1(f) = S(f) = \frac{A_1}{2}[M(f-f_0) + M(f+f_0)]$

So:
- Two sidebands: One centered at $+f_0$, one at $-f_0$,
- Each sideband has shape $M(\cdot)$, scaled by $A_1/2$.

Received signal (input to the mixer):

Assuming an ideal channel, the received signal is just

$x_2(t) = r(t) = x_1(t) = A_1 m(t) \cos(2\pi f_0 t)$

So $X_2(f) = X_1(f) = \frac{A_1}{2}[M(f-f_0) + M(f+f_0)]$

Mixer output (product with local oscillator):

LO: "Carrier" $:= c(t) = \cos(2\pi f_0 t)$

Mixer output: $x_3(t) = x_2(t) \cdot c(t) = A_1 m(t) \cos^2(2\pi f_0 t)$.

Using identity $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$:

$x_3(t) = A_1 m(t) \frac{1}{2}[1 + \cos(4\pi f_0 t)] = \frac{A_1}{2} m(t) + \frac{A_1}{2} m(t) \cos(4\pi f_0 t)$

then $X_3(f) = \underbrace{\frac{A_1}{2} M(f)}_{\text{Baseband}} + \underbrace{\frac{A_1}{4}[M(f-2f_0) + M(f+2f_0)]}_{M(\cdot) \text{ shape at } \pm 2f_0 \text{ scaled } \frac{A_1}{4}}$

Ideal low-pass filter (LPF):

Frequency response $[H(f)]$ of an ideal LPF with gain 1 for $|f| \le B$ and gain 0 elsewhere:

$$H(f) = \begin{cases} 1, & |f| \le B, \\ 0, & |f| > B. \end{cases}$$

Given $f_0 \gg B$, the high-frequency copies at $\pm 2f_0$ sit completely outside the passband.

So the output of the LPF is

$$X_4(f) = H(f) \cdot X_3(f) = \begin{cases} \frac{A_1}{2} M(f), & |f| \le B, \\ 0, & |f| > B \end{cases}$$

so $x_4(t) = \frac{A_1}{2} m(t) \rightarrow$ "Almost recovered" message

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Gain/Scaling Stage:

$$x_5(t) = \frac{2}{A_1} x_4(t) = m(t)$$

so $X_5(f) = M(f)$

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2. Mass-spring-damper $\leftrightarrow$ RLC:

Take upward displacement $y(t)$ as positive, external force $F_E(t)$ upward.

Forces on the mass:

- Spring: $F_K = -ky$, • Damper: $F_C = -c\dot{y}$, • External: F_E

Then, $\sum F = ma$ (Newton): $m\ddot{y} + c\dot{y} + Ky = F_E$

so $\mathcal{L}\{m\ddot{y} + c\dot{y} + Ky\} = \mathcal{L}\{F_E\}$

$(ms^2 + cs + K)Y(s) = F_E(s)$ (Assume zero ICs)

Thus the transfer function $G(s) = \dfrac{Y(s)}{F_E(s)}$

$$G(s) = \frac{1}{ms^2 + cs + K}$$

Standard second-order parameters:

$$G(s) = \frac{1/m}{s^2 + \frac{c}{m}s + \frac{K}{m}} \longrightarrow \omega_n = \sqrt{\frac{K}{m}}$$

$$\zeta = \frac{c}{2\sqrt{Km}}, \quad K = \frac{1}{K}$$

$$G(s) = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \longrightarrow$$

- No finite zeros.
- Two Poles.

RLC circuit: equivalent model:

KCL: $i_L = i_R + i_C \longrightarrow LC\ddot{v}_o + \frac{L}{R}\dot{v}_o + v_o = v_i$

so $\mathcal{L}\{LC\ddot{v}_o + \frac{L}{R}\dot{v}_o + v_o\} = \mathcal{L}\{v_i\}$

$$\left(LCs^2 + \frac{L}{R}s + 1\right)V_o(s) = V_i(s)$$

Standard $H(s) = \dfrac{V_o(s)}{V_i(s)}$ 2^{nd}-order parameters:

$$H(s) = \frac{1/LC}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \longrightarrow \omega_n = \sqrt{\frac{1}{LC}} \quad K = 1$$

$$\zeta = \frac{1}{2RC}\sqrt{LC}$$

$$= K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \longrightarrow$$ Exactly the same canonical expression!

so, for every values of m, c, K there is a 3-tuple of R, L, C values that are equivalent in their standard 2^{nd}-order params values.

$$R = c \quad ; \quad L = m \quad ; \quad C = \frac{1}{K}$$

Generic 2^{nd}-order formulas (continuous-time, open loop):

- Poles (Zeros: None):

$$P_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

- Bode (Mag and phase):

$$s = \sigma + j\omega \; : \; \sigma = 0 \rightarrow s = j\omega$$

$$G(j\omega) = H(j\omega) = K \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + j(2\zeta\omega_n\omega)}$$

Magnitude:

$$K \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

Phase:

$$-\arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

Closed-loop system with unity feedback:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{H(s)}{1 + H(s)} = K \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2(1 + K)}$$

This is again a 2^{nd} order with modified parameters:

$$\omega_n' = \omega_n \sqrt{1 + K} \; , \; \zeta' = \frac{\zeta}{\sqrt{1 + K}}$$

So closing the loop:

- Increases ω_n (faster dynamics).
- Decreases ζ (more oscillatory for given plant).

Notes.

Basic Parameters:

"Undamped natural frequency" $:= \omega_n$

"Damping Ratio" $:= \zeta$

"Damped natural frequency" $:= \omega_d = \omega_n\sqrt{1-\zeta^2}$

"Poles" $:= p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

"DC gain" $:= K$

For $\zeta < 1$ the poles are complex conjugated, for $\zeta = 1$ the coalesce at $-\omega_n$; for $\zeta > 1$ they're dinstinct real.

Time-response measures for the step response:

"Peak time" $:= T_p = \frac{\pi}{\omega_d}$ $(0 < \zeta < 1)$

"Rise time" $:= T_r = \frac{1}{p_1 - p_2}\ln\left(\frac{p_1}{p_2}\right)$ ±2% step criterion

"Settling time" $:= T_s \approx \frac{4}{\zeta\omega_n}$ $(0{,}2 \lesssim \zeta \lesssim 0{,}8)$