13. For linear dependence, at least one of the vectors must be a linear combination of the others.

$\Rightarrow$ for say set $\{v_1, v_2, v_3\}$,

$$v_1 = c_1 v_2 + c_2 v_3.$$

or $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. in general.

Thus in the given question, for vectors to be dependent,

$[v_1 \quad v_2 \quad v_3 \mid 0]$ must have a non trivial solution.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{array}\right] \xRightarrow[r_3+3r_1]{r_2-5r_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2h-27 & 0 \\ 0 & 1 & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$\Rightarrow$ ~~$1 + 2h - 27 \neq 0$ & $1 + h - 15 = 0$. $\Rightarrow h = 14$.~~

~~$1x_1 + 2hx_3 - 27x_3 = 0$~~

~~$1x_2 + hx_3 - 15x_3 = 0$.~~

Since we have a free variable, h can take any value.

$\{$ unless $2h-27 = h-15 = 0$ which is not possible $\}$

1.

$$u = 1\begin{bmatrix}1\\0\end{bmatrix} + (-3)\begin{bmatrix}0\\1\end{bmatrix}.$$

$$A = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix} \Rightarrow T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$$

$$\& \; T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\2\end{bmatrix}$$

$$\therefore T(u) = 1\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + -3\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= \begin{bmatrix}2 & 0\\0 & -6\end{bmatrix}$$

{Basically, the matrix scales each vector by 2}.

Similarly, $v = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}.$

$$\therefore T(v) = a\begin{bmatrix}2\\0\end{bmatrix} + b\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}2a\\2b\end{bmatrix}.$$

4. $Ax = b$. $\Rightarrow$ To find x, we need to find a linear combination of column vectors of A that would give us b.

We hence solve $[A|b]$.

$$\left[\begin{array}{ccc|c}1 & -3 & 2 & 6\\0 & 1 & -4 & -7\\3 & 5 & -9 & -9\end{array}\right] \xrightarrow{r_3 - 3r_1} \left[\begin{array}{ccc|c}1 & -3 & 2 & 6\\0 & 1 & -4 & -7\\0 & 14 & -15 & -27\end{array}\right] \xrightarrow{r_3 - 14r_2} \left[\begin{array}{ccc|c}1 & -3 & 2 & 6\\0 & 1 & -4 & -7\\0 & 0 & 41 & 71\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 71/41 \end{array}\right] \xrightarrow{r_2 + 4r_1} \left[\begin{array}{cccc} 1 & -3 & 2 & 6 \\ 0 & 1 & 0 & -3/41 \\ 0 & 0 & 1 & 71/41 \end{array}\right].$$

$r_1 - 2r_3 + 3r_2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 95/41 \\ 0 & 1 & 0 & -3/41 \\ 0 & 0 & 1 & 71/41 \end{array}\right].$$

hence, x is a uniqe vector with value

$$x = \begin{bmatrix} 95/41 \\ -3/41 \\ 71/41 \end{bmatrix}$$

5. $[A \mid b] = \left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{array}\right].$

$\Rightarrow r_2 + 3r_1 \Rightarrow$ new r_2: $\left[\begin{array}{ccc|c} 1 & -5 & -7 & -2 \\ 0 & -8 & -16 & -8 \end{array}\right].$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -5 & -7 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{r_1 + 5r_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 7 \\ 0 & 1 & 2 & 1 \end{array}\right].$$

x in Not a a unique vector.

$x_1 + 3x_3 = 7, \quad x_2 + 2x_3 = 1.$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 + 7 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}.$$

where parameter x_3 can take any value.