The Basics: Matrices, Solutions and Dependence.

$$\begin{array}{c} \\ \text{row} \rightarrow \end{array} \overset{\text{column}}{\left[\begin{array}{ccc|c} a & b & c & h \\ d & e & f & i \end{array}\right]} = [A \mid B] \rightarrow \text{Augmented (joined) Matrix}$$

~~dim~~ lenr. #r #c. 2×4

eg: $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right]_{2\times4} \Rightarrow \begin{array}{l} 1x_1 + 1x_3 = 2 \\ 1x_2 = 3 \end{array} \Rightarrow \begin{array}{l} x_1 = 2 - x_3 \\ x_2 = 3 \end{array} \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2-x_3 \\ 3 \\ x_3 \end{bmatrix}$

Elementary row operations. Solutions don't change with:

- row switching
- row scaling
- row addition.

Solution to the System. $= \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \rightarrow$ parametric vector form.

$\rightarrow A \rightarrow REF(A)$ does not change solutions

inconsistent $\rightarrow$ no sol. $\rightarrow$ a physical math $\rightarrow [0\,0\,0 \mid b]$ $b \neq 0$ row.

consistent $\rightarrow$ 1 or more solutions.

for REF ↙	↘ for REF
• linearly independent column vectors	• linearly dependent column vectors.
• all columns have pivot columns.	• non-pivot columns exist.
• no free variables	• free variables exist.
• rank = # of column vecs	• nullity $\neq 0$. $\rightarrow$ more later.
↓ # independent var. $\dim(\text{Row}(A))$	↓ # dependent var. $\dim(\text{Null}(A))$

rank + nullity = # of columns

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 3 \end{bmatrix} \rightarrow \underline{REF}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \underline{RREF}$$

Matrix Operations, Inverses, Transformations & Determinants.

$A = [a_1 \ a_2 \ a_3 \dots a_n]$ $\cancel{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \cancel{AB} = Ax = a_1x_1 + a_2x_2 \dots a_nx_n$

"linear combination of columns of A weighted by corresponding entries in x".

$\rightarrow$ changing basis from $[e_1 \ e_2 \ e_3 \dots e_n] \rightarrow [a_1 \ a_2 \ a_3 \dots a_n]$.

$\Rightarrow$ Transforming space. $\Rightarrow A =$ special matrix.

$T(cu + dv) = T(cu) + T(dv) = cT(u) + dT(v)$

$\cdot\ T(\vec{0}) = T(0 \cdot \vec{0}) = 0.$ since matrices have same prop.

$[T(e_1) \ T(e_2) \dots T(e_n)] = A =$ special matrix.

Matrix multiplication $\Longleftrightarrow$ Linear Transformation

$Ax = b \rightarrow$ non Homogenous
$Ax = 0 \rightarrow$ Homo Matrix-Vector eq.

Span $\rightarrow$ set of all possible linear combinations of set of vecs

Solution sets $\rightarrow$ set of all vectors satisfying a condition.

Domain $\xrightarrow{T}$ Co Domain

One-to-one : ~~many~~ $\rightarrow$ one one input $\rightarrow$ one unique output.

onto : all vectors $b : Ax = b \Rightarrow \exists x$.

$\boxed{\text{One-to-one + onto : bijective}} \Rightarrow Ax = b$ is consistent $\forall b$.

linearly independent ↓ $Ax = b$ has only 1 solution ⇓ $Ax = 0$ has 1 solution

Matrix Multiplication:

$[A][b_1 \ b_2 \ b_3] = [(A)b_1 \ [A]b_2 \ (A)b_3] \Rightarrow$ row vector rule.

$AB = 0 \not\Rightarrow A = 0$ or $B = 0$ $\begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$; $AB = AC \not\Rightarrow B = C$.

$[\ \dots\ ] = [\ \dots\ ]$

$* \ x \neq 0.$

$Q(x) > 0 \rightarrow$ positive def.
$Q(x) \geq 0 \rightarrow$ pos semi def
$Q(x)$ can be $(0, >0, <0)$ indefinite.

not quite $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\rightarrow$ in this case $Q = 0$ pseudo

$x^T(Ax) \rightarrow$

$x_1^2 + x_2^2 = [x_1 \ x_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^Tx$; $2x_1x_2 = [x_1 \ x_2]\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} =$; $[x_1 \ x_2]\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 4x_1x_2 + 3x_2^2$

$Q(x) = x^TAx$ symm. $A = \begin{bmatrix} \text{c}(x_1^2) & \text{c}(x_1x_2)/2 \\ \text{c}(x_1x_2)/2 & \text{c}(x_2^2) \end{bmatrix}$ eg.

Since A is symm. we can find orthogonal Diag. $A = PDP^T$. $P^T \! = P^{-1}$ $\Rightarrow$ defining $x = Py \Rightarrow Q(x) = x^TAx = (Py)^TA(Py) = y^TP^TAPy = y^TDy = \lambda_1y_1^2 + \lambda_2y_2^2$

Now easier to find type. P is orth $\Rightarrow x = Py$ special. y can live on rot. space

λ_1, λ_2 +ve $\rightarrow$ pos def. type depends only on λ_1, λ_2

$\lambda_1 = 0$ or $\lambda_2 = 0$ $\rightarrow$ semi; λ_1, λ_2 -ve $\rightarrow$ neg def.

$\lambda_1 + \ \lambda_2 -$ $\Rightarrow$ ind.

Transposing also doesn't change solutions since relation b/w var stay same but null(A) might change.

Elementary matrix:

when multiplied → performs elementary row operation.

$[A \mid I] \xrightarrow[\text{or}]{\text{elementary}} [RREF(A) \mid B]$. ~~if RREF(~~

$\Rightarrow AB = RREF(A) \rightarrow$ if $RREF(A) = I$

$\Rightarrow B = A^{-1}$.

$AB = I \Rightarrow B = A^{-1}$

$AA^{-1} = A^{-1}A = I$. $\Rightarrow x \rightarrow Tx$ maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Inverses:

A is invertible ⇒ has a pair partner A^{-1} : $AA^{-1} = I$ $A^{-1} = \frac{\text{adj}\, A}{|A|} = \frac{(\text{cof}(A))^T}{|A|}$

$Ax = b \Rightarrow A^{-1}A\,x = A^{-1}b \Rightarrow x = A^{-1}b$.

$A_{n\times n}$: linearly independent ⇔ bijective transformation ⇔ $Ax = 0$ has only trivial solution

rank(A) = # col. vectors ⇔ Span{col}$A = \mathbb{R}^n$ ⇔ no free variables

Determinants: How much a unit volume is stretched.

$|A| = \sum_{i=1}^{n} |A_{ij}| \cdot a_{ij}(-1)^{ij}$ → recursive definition. $|A_{1\times1}| = a_{11}$

→ cofactor c_{ij}. A_{ij} = Everything but i j $\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = ad - bc$. $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Note: you can chose to go about any i or any j.

$|A_{T\Delta}|$ = prod(diagonals) ⇒ row switching : $|B| = -|A|$ $|cA| = c^n|A|$

row scaling : $|B| = c|A|$ $|A^T| = |A|$

row addition : $|B| = |A|$. $|A||B| = |AB|$

$|A^{-1}| = \frac{1}{|A|}$ $|A^n| = |A|^n$

Vector Spaces

{vect} → 0 vect, $a+b$, ca. {sub ⊆ set}.

Bases: linearly Independent Generating set.

Change of Basis: $b_1 = \begin{bmatrix} + \\ - \end{bmatrix}$ $b_2 = \begin{bmatrix} x \\ x \end{bmatrix}$ $P_B = [b_1\ b_2] = \begin{bmatrix} + & x \\ - & x \end{bmatrix}$.

x (Std. Basis) $\underset{P_B^{-1}}{\overset{P_B}{\rightleftarrows}}$ $[x]_B$ $x = P_B[x]_B$ $[x]_B \xrightarrow{P_B[x]_B} x \xrightarrow{P_C^{-1}x} [x]_C$ $\Rightarrow P_C^{-1}\underbrace{P_B[x]_B}_{\text{to std.}} = [x]_C$

($e_1\ e_2 \ldots$)

$\text{Col}(A_{m\times n}) = \text{Span}(\text{col } A) \subseteq \mathbb{R}^m$ col(A) corresp. to REF(A)

$\text{Row}(A_{m\times n}) = \text{Span}(\text{col } A^T) \subseteq \mathbb{R}^n$ rows of REF(A)

$\text{Null}(A_{m\times n}) \Leftrightarrow Au = 0 \subseteq \mathbb{R}^n$ {max possible free var}.
(Kernel)   ↳ vectors in parametric vect form.

Diagonalize

$A = PDP^{-1}$ ($D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$) → Solve $A = \lambda x \Rightarrow |A - I\lambda| = 0$ complex eig. values: if $\lambda_1 \rightarrow v_1 \Rightarrow \bar{\lambda}_1 = \bar{v}_1$ $\bar{A} = A$ {real valued matrix}.

$A^n = PD^nP^{-1}$.   Diagonalizable ⇒ eig. vectors ~~span~~ span all space ⇒ multi alg(power of $t-\lambda$) = multi geo(# eig. vec).

$m_g \leq m_a$.   defect(λ) = $m_a - m_g$.   ∀ λ.

orthogonal: ⊥   orthonormal: ⊥ & $\|v\| = 1$.   orthogonal comp. : set of vectors ⊥ to S but in $\mathbb{R}^n$

proj(w) on space = $\sum_{v \in S} \text{proj}_v(w)$. $\text{Col}(A)^\perp = \text{Row}(A)^\perp = \text{Null}(A^T)$ $S^\perp = (\text{Col}(A))^\perp = \text{Null}(A^T)$.

~~proj(w)~~ {w_1, ~~w_2~~} if $v_1 v_2 \ldots = c_1v_1 + c_2v_2 \ldots$ → unique. $\dim(W) + \dim(W^\perp) = m$: $W \in \mathbb{R}^m$

Gram Smith:

$v_1 = w_1$

$w_2 = v_2 - \text{proj}_{\{w_1\}}(v_2) = \frac{w_2 \cdot w_1}{w_1 \cdot w_1} w_1$

$w_3 = v_3 - \text{proj}_{\{w_1, w_2\}}(v_3) = (\text{proj}_{w_1}(v_3) + \text{proj}_{w_2}(v_3)) = v - \frac{w_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{w_3 \cdot w_2}{w_2 \cdot w_2} w_2$

⇒ QR:

⇒ $v_1 = w_1$

→ $v_2 = w_2 + \text{proj}_{w_1}(v_2)$, $v_3 = w_3 + \text{proj}_{\{w_1, w_2\}}(w_3)$

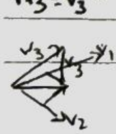
$A = [v_1\ v_2\ v_3]$.

: ~~$[w_1\ \ell_{12}w_1\ \ell_{13}w_1]$~~

: $\left[w_1 \quad w_2 + \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 \quad w_3 + \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2 + \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1\right]$

Q(orthogonal)   R(upper)

$: \left[\frac{w_1}{\|w_1\|}\ \frac{w_2}{\|w_2\|}\ \frac{w_3}{\|w_3\|}\right] \begin{bmatrix} \|w_1\| & \|w_1\| \frac{w_1 \cdot v_2}{w_1 \cdot w_1} & \|w_1\| \frac{w_1 \cdot v_3}{w_1 \cdot w_1} \\ 0 & \|w_2\| \cdot 1 & \|w_2\| \frac{w_2 \cdot v_3}{w_2 \cdot w_2} \\ 0 & 0 & \|w_3\| \end{bmatrix}$



Least Squares: $Ax^* = \text{proj}_{\text{col}(A)} b$

$|Ax^* - b| = \min$.   → find ortho basis using G.S.

$x^* \leftarrow [A^TA \mid A^Tb]$ solve $b - \hat{b} \in \text{Col}(A)^\perp$ ← $\in \text{Null}(A^T) \Rightarrow A^T(b - \hat{b}) = 0$ ⇒ $A^TAx^* = A^Tb$.

Easier to solve. $[R \mid Q^Tb]$ ← $Ax = b = QRx = b \Rightarrow Rx = Q^Tb$

Spectral Decomposition: A is symm. ⇒ $A = PDP^T$

If $A = PDP^{-1}$ & P is/can be orthogonal

⇒ eigenvectors from diff. λ are ⊥ ⇒ e_{ij}, A eig

⇒ $A = \lambda_1 v_1v_1^T + \ldots + \lambda_m v_mv_m^T$, $P_{ij} = v_jv_j^T$ ortho. proj matrix onto v_j, rank(P_j) = 1

$P_{ij} = \{\ldots\}$ if $i = j$

$u \cdot v = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = ac + bc = [a\ b]\begin{bmatrix} c \\ d \end{bmatrix} = u^Tv$.

$u \cdot v = v \cdot u$ $\|cu\| = |c|\,\|u\|$ $\|c \cdot u\| = c(u \cdot u)$.

$u \perp v \Rightarrow u \cdot v = 0$. $c \cdot u$ not $c \cdot v$ $\|v\| = 1 \Leftrightarrow$ unit vect.

Back. in P space, vectors $Ax = \lambda x$ → scaled.

$A_1 = \hat{P}\, D\, P^T$ change to P space.

$\text{proj}_{w_1}(w_2) = w_2 \cdot \hat{w}_1 \hat{w}_1 = \frac{w_2 \cdot w_1}{w_1 \cdot w_1} w_1$

orthogonal matrix ⇔ columns form orthonormal basis of $\mathbb{R}^n$ ⇒ $Q^TQ = I$, $Q^T = Q^{-1}$ ⇒ preserves inner prod (Qu)·(Qv) = $u \cdot v$, norms $\|Qu\| = \|u\|$

⇒ $|Q| = \pm 1$, Q^T & Q orthogonal

$\{w_1, w_2, w_3\}$ → orthogonal set