LAB 3: Linear Equations, Rank, Nullity and Matrix multiplication

(Math 250: Sections C3)

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Learning Outcomes

- Solving a system of linear equations by using the reduced row echelon form of the augmented matrix of the system and determining the rank and nullity of the coefficent matrix.
- Forming linear combinations of a set of vectors and the fundamental concepts of linear dependence and linear independence.
- Matrix multiplication and its properties.
- Applications of the powers of (0; 1) Matrices

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Random Seed

Initialize the random number generator by typing the following code, where abcd are the last four digits of your RUID

```
rng('default');
rng(abcd, 'twister');
```

```
% Enter your code here
rng('default');
rng(8256, 'twister');
```

This will ensure that you generate your own particular random vectors and matrices.

The lab report that you hand in must be your own work. The following problems use randomly generated matrices and vectors, so the matrices and vectors in your lab report will not be the same as those of other students doing the lab. Sharing of lab report files is not allowed in this course.

Question 1. Solving Ax = b (Total Points - 10)

In this question you will find the general solution $\mathbf{x} \in \mathbb{R}^3$ to a linear system Ax = b of 3 equations in 5 variables x_1, x_2, x_3, x_4, x_5. Here, A is the 3 X 5 coefficient matrix and $\mathbf{b} \in \mathbb{R}^3$ is the given right-hand side of the system.

(a) Generate a random 3 X 5 integer matrix ***A*** and check that columns 1, 2, 3 of ***A*** are the pivot columns. (Documentation: rank)

```
A = rmat(3, 5), rank(A(:,1:3))
```

```
% Enter your code here
A = rmat(3, 5), rank(A(:,1:3))
```

```
A = 3×5
     1     3     7     6     5
     4     5     2     8     1
     9     2     5     9     1
ans =
3
```

Note the use of the colon operator to select columns 1, 2, 3 of ***A***. If the rank is less than 3, generate a new ***A*** (this is unlikely, but it can happen). Include all the matrices that you generate this way in your lab report.

When you have an A for which the rank of the first three columns is 3, then generate a random vector $\mathbf{b} \in \mathbb{R}^3$ and the reduced row echelon form **R** of the augmented matrix [A b]:

```
b = rvect(3), R = rref([A b])
```

```
% Enter your code here
b = rvect(3), R = rref([A b])
```

```
b = 3×1
     2
     8
     2
R = 3×6
    1.0000         0         0    0.5861   -0.3074    0.0984
         0    1.0000         0    0.9918    0.1721    1.7049
         0         0    1.0000    0.3484    0.6844   -0.4590
```

To get the reduced row echelon form *S = rref(A)* just remove the last column from **R**:

```
S = R(:,1:5)
```

```
% Enter your code here
S = R(:,1:5)
```

```
S = 3×5
    1.0000         0         0    0.5861   -0.3074
         0    1.0000         0    0.9918    0.1721
         0         0    1.0000    0.3484    0.6844
```

(Note the use of the colon operator to select columns 1 to 5 of **R**). Check with MATLAB that:

```
S == rref(A)
```

```
% Enter your code here
S == rref(A)
```

```
ans = 3x5 logical array
   1   1   1   1   1
   1   1   1   1   1
   1   1   1   1   1
```

Now, write answers to the following:

(i) ***Exercise (Points - 1):*** *Which columns of S are the pivot columns? (Points - 1)*

Answer: The 1st, 2nd and 3rd Columns of S are pivot columns

(ii) ***Exercise (Points - 1):*** *What is the rank of R and the rank of A? (Points - 1)*

Answer: Rank(R) = Rank(A) = 3

(iii) ***Exercise (Points - 1):*** *What is the nullity of A and which variables xi are be the free variables?*

Answer: *Nullity(A) = 2; x4 and x5 are the free variables.*

(iv) ***Exercise (Points - 1):*** *Why does the equation* ***Ax = b*** *have a solution?*

Answer: [A | b] has no row equivalent with a [0 0 0 | bi] row hence the system is consistent. Also, Col(A) = R^3. Since b is also an element in R^3, b is within the Col(A), i.e. there exists a solution x for which Ax = b.

(b)

(i) ***Exercise (Points - 1)***: Obtain c = R(:,6) (the last column of **R**), and set x = [c; 0; 0]. Note carefully that semicolons appear in this formula before the zeros, so that $\mathbf{x} \in \mathbb{R}^5$ and the last two components of x are zero.

```
% Enter your code here
c = R(:,6)
```

```
c = 3×1
    0.0984
    1.7049
   -0.4590
```

```
x = [c; 0; 0]
```

```
x = 5×1
    0.0984
```

```
    1.7049
   -0.4590
         0
         0
```

Solve S*x = c by typing ***x = S\c*** and then calculate A*x - b

```
% Enter your code here
x = S\c
```

```
x = 5×1
   -0.9091
         0
   -1.0579
    1.7190
         0
```

```
A*x - b
```

```
ans = 3×1
10^-14 ×
         0
   -0.1776
   -0.3553
```

IMPORTANT: Because of the finite precision of computer arithmetic and roundoff error, vectors or matrices that are zero (theoretically) may appear in MATLAB in exponential form such as 1.0e-014 * M (where M is a vector or matrix with entries between -1 and 1). An example of MATLAB output in this form is

```
ans =
1.0e-14 *
0.0222
0.0888
0.1776
```

which represents the vector 10^{-14} . $[0.0222;\ 0.0888;\ 0.1776]^T$. This means that each component of the answer is less than 10^{-14} in absolute value, so the vector or matrix can be treated as zero (numerically) in comparison to vectors or matrices with entries that are on the order of 1 in size. Whenever you are asked to verify by MATLAB that two matrices or vectors are equal, calculate their difference and use this meaning of "zero". (Here, answers such as 1.0 e-013 * M or 1.0 e-012 * M would also be considered as "zero".)

Learn more here

(ii) *Exercise (Points - 1):* Use properties of row reduction to explain why the ***ans*** in (i) is "zero".

Answer: Ax-b here happens to be "zero". Let's consider what each of these matrices/vectors are.

A is our randomly generated original 3x5 matrix

b is a randomly generated vector in the R^3 space.

x is the solution for Sx = c

S and c are a matrix vector pair such that [S|c] is the row equivalent form of [A|b] that we get while transforming A into RREF(A)

Then, x is the solution we get for Sx=c (by definition) and since they are row equivalent, x is also a solution for Ax=b

Therefore, Ax-b = 0

(c) Calculate u = [-S(: , 4) ; 1 ; 0], v = [-S(: , 5); 0 ; 1].

```
% Enter your code here
u = [ -S( : , 4) ; 1 ; 0], v = [ -S( : , 5); 0 ; 1]
```

```
u = 5×1
   -0.5861
   -0.9918
   -0.3484
    1.0000
         0
v = 5×1
    0.3074
   -0.1721
   -0.6844
         0
    1.0000
```

(i) ***Exercise (Points - 1):*** Give an explanation, using symbols and linear algebra, rather than numbers, to show why **u** and **v** are the vectors that appear in the parametric vector form of the general solution to **Ax = 0**.

Answer: x = [x1 x2 x3 x4 x5]

solving for x1, x2 and x3 in Sx=0, in terms of x4 and x5 by considering terms of each row and solving for x gives us that u is the vector corresponding to the free variable 4 and and v is the free variable corresponding to the vector 5.

I.e. u and v are the parametric vectors for that we can scale by any vector to span the null space.

(Essentially, for our above example [S|c] would have given us a type P + x4u + x5v. since we want the null space, we omit the constant vector that corresponds to b.)

Confirm by calculating S*u, A*u, S*v, A*v. You should get vectors that are (approximately) zero.

```
% Enter your code here
S*u, A*u, S*v, A*v
```

```
ans = 3×1
     0
     0
     0
ans = 3×1
     0
     0
     0
ans = 3×1
     0
     0
```

```
     0
ans = 3×1
     0
     0
     0
```

Now, generate a random linear combination of u and v by the commands. (Note that each occurrence of rand(1) generates a different random coefficient - Documentation: rand)

```
s = rand(1), t = rand(1), y = s*u + t*v
```

```
% Enter your code here
s = rand(1), t = rand(1), y = s*u + t*v
```

```
s = 
0.1966
t = 
0.2511
y = 5×1
   -0.0380
   -0.2382
   -0.2403
    0.1966
    0.2511
```

(ii) ***Exercise (Points - 1)*** What properties of matrix and vector algebra ensure that **Ay = 0**?

Answer: Since u and v are the null basis, any linear combination of these vectors is in the null(A). Hence, any such vector transformed by matrix A would transform to 0. Therefore, Ay=0 is ensured.

Ay = A(su + tv) = s(Au) + t(Av) = 0 (by distribution and scalar multiplication)

Confirm by a MATLAB calculation that **Ay** is approximately zero.

```
% Enter your code here
A*y
```

```
ans = 3×1
```
10^{-15} ×
```
    0.2220
         0
   -0.4441
```

(d) Calculate z = x + y .

```
% Enter your code here
 z = x + y
```

```
z = 5×1
   -0.9471
   -0.2382
   -1.2982
    1.9156
```

```
0.2511
```

(i) ***Exercise (Points - 1):*** What properties of matrices (and vectors) imply that **Az = b**?

Answer: Az = A(x+y) = Ax + Ay (by distribution) = Ax + 0 = b

Simply, adding a null vector to a constant vector does not change the output vector upon transformation.

(ii) ***Exercise (Points - 1):*** Confirm using MATLAB that **Az - b** is approximately zero.

```
% Enter your code here
A*z - b
```

```
ans = 3×1
10^-14 ×
   -0.3553
   -0.1776
   -0.7105
```

Question 2. Spanning Sets and Linear Independence (Total Points - 6)

Generate four random vectors in $\mathbb{R}^3$ by the command

```
u1 = rvect(3), u2 = rvect(3), u3 = rvect(3), u4 = rvect(3)
```

```
% Enter your code here
u1 = rvect(3), u2 = rvect(3), u3 = rvect(3), u4 = rvect(3)
```

```
u1 = 3×1
     6
     4
     3
u2 = 3×1
     8
     5
     5
u3 = 3×1
     9
     2
     7
u4 = 3×1
     7
     3
     5
```

Use these vectors in the following:

(a) Consider the set of vectors S = $\{u_1, u_2, u_3\}$. To determine whether S is **linearly independent**, form the matrix **A** with the vectors from **S** as columns and calculate its reduced row echelon form:

```
A = [u1 u2 u3], rref(A)
```

```
% Enter your code here
A = [u1 u2 u3], rref(A)
```

```
A = 3×3
     6     8     9
     4     5     2
     3     5     7
ans = 3×3
     1     0     0
     0     1     0
     0     0     1
```

Use these calculations to answer the following questions:

(i) ***Exercise (Points - 1):*** How many free variables does the equation Ax = 0 have?

Answer: Zero

(ii) ***Exercise (Points - 1):*** Is the set S linearly independent or linearly dependent? Why?

Answer: Linearly Independent. Since the column vectors all correspond to pivot columns, they are linearly independent (The only solution to [A|0] would then be that all elements x for Ax=0 must be 0 and hence you have no free variables, no dependency)

(b) Consider the set of vectors T = $\{u_1, u_2, u_3, u_4\}$. To determine whether T is linearly independent, form the matrix B with the vectors from T as columns and calculate its reduced row echelon form:

```
B = [u1 u2 u3 u4], rref(B)
```

```
% Enter your code here
B = [u1 u2 u3 u4], rref(B)
```

```
B = 3×4
     6     8     9     7
     4     5     2     3
     3     5     7     5
ans = 3×4
    1.0000         0         0   -0.1579
         0    1.0000         0    0.5789
         0         0    1.0000    0.3684
```

Use these calculations to answer the following questions:

(i) ***Exercise (Points - 1):*** How many free variables does the equation Bx = 0 have?

Answer: It has 1 free variable.

(ii) ***Exercise (Points - 1):*** Is the set T linearly independent or linearly dependent?

Answer: It is linearly dependent.

(c) Let v be a random linear combination of u_1 and u_2:

```
v = rand(1)*u1 + rand(1)*u2
```

```
% Enter your code here
v = rand(1)*u1 + rand(1)*u2
```

```
v = 3×1
    9.4181
    6.0190
    5.4882
```

Thus, v is of the form $c_1u_1 + c_2u_2$ for some scalars c_1 and c_2. Consider the set of vectors U = $\{u_1, u_2, v\}$.

(i) ***Exercise (Points - 1):*** Is the set U linearly independent or linearly dependent? Answer first without calculation using the definition of linear independent sets.

Answer: It is linearly dependent. By definition, a linearly dependent set is a set in which atleast 1 vector in the set can be written as the linear combination of the other elements. Since v is defined as a linear combination of u1 and u2, the set is linearly dependent.

(ii) ***Exercise (Points - 1):*** Check your answer using the method of part (a). As we can see, there is a not pivot column.

```
% Enter your code here
rref([u1,u2,v])
```

```
ans = 3×3
    1.0000         0    0.5308
         0    1.0000    0.7792
         0         0         0
```

Question 3. Matrix Multiplication (Total Points - 4)

For this question generate random matrices and a random vector:

```
A = rmat(2,3), B = rmat(3, 4), C = rmat(4,3), v = rvect(4)
```

```
% Enter your code here
A = rmat(2,3), B = rmat(3, 4), C = rmat(4,3), v = rvect(4)
```

```
A = 2×3
     2     4     1
     8     9     2
B = 3×4
     1     5     8     5
     1     5     6     4
     8     1     3     0
C = 4×3
     2     4     4
```

```
     1     0     4
     1     9     3
     2     9     9
v = 4×1
     3
     1
     7
     3
```

To obtain the product ***AB*** of the matrices ***A*** and ***B*** using MATLAB, you must type **A*B**. This is only defined when the number of columns of ***A*** is the same as the number of rows of ***B*** (you will get an error message when the matrix sizes are not compatible - Learn more here).

```
% Enter your code here
A*B
```

```
ans = 2×4
    14    31    43    26
    33    87   124    76
```

(a) ***Exercise (Points - 1):*** **Associativity -** The product ***AB*** is defined uniquely by the property that ***A*** applied to a vector **u** = B**v** is the same as the matrix ***AB*** applied to the vector **v**, for every vector **v** of the correct size. Verify this by calculating:

```
u = B*v, A*u, D = A*B, D*v
```

```
% Enter your code here
u = B*v, A*u, D = A*B, D*v
```

```
u = 3×1
    79
    62
    46
ans = 2×1
         452
        1282
D = 2×4
    14    31    43    26
    33    87   124    76
ans = 2×1
         452
        1282
```

This property implies the associativity of matrix multiplication: A(BC) = (AB)C. Verify this for the matrices A, B, C that you have generated.

```
% Enter your code here
A*(B*C),(A*B)*C
```

```
ans = 2×3
         154         677         543
         429        1932        1536
ans = 2×3
         154         677         543
```

```
        429         1932         1536
```

(b) Matrices Are Not Numbers: Matrix algebra has many similarities to (ordinary) algebra of numbers,

but there are important differences (which are the mathematical source of the differences between classical mechanics and quantum mechanics in physics). Here are some examples:

Generate matrices

```
A = [0 1; 0 0], B = [0 0; 1 0], C = [0 1; 1 0]
```

```
% Enter your code here
A = [0 1; 0 0], B = [0 0; 1 0], C = [0 1; 1 0]
```

```
A = 2×2
     0     1
     0     0
B = 2×2
     0     0
     1     0
C = 2×2
     0     1
     1     0
```

Use these matrices and MATLAB calculations to answer the following:

(i) ***Exercise (Points - 1):*** Is AB = BA? Is $(A+B)^2 = A^2 + 2\text{AB} + B^2$? (Note that both of these equations would be true if A and B were numbers instead of matrices)

Answer: No! AB is NOT equal to BA for all matrices. Also, (A+B)^2 = A^2 + AB + BA + B^2 and we can not simply club the middle terms since matrix mutiplication is non commutative.

```
% Enter your code here
A*B, B*A
```

```
ans = 2×2
     1     0
     0     0
ans = 2×2
     0     0
     0     1
```

(ii) ***Exercise (Points - 1):*** Calculate A^2. If A were a number instead of a matrix, what would the value of A^2 tell you about the value of A? Is this conclusion valid for matrices?

Answer: Assuming A to hold a value 1 (which is the only non zero entry in our defined Matrix), A^2 would have been 1 as well. But A^2 is not the same as A in this case, infact, A^2 is a zero matrix! Square of a non zero number would never be zero.

```
% Enter your code here
A^2
```

```
ans = 2×2
```

```
     0     0
     0     0
```

(iii) ***Exercise (Points - 1):*** Calculate AC and compare it with AB. If A, B, C were numbers with A $\neq$ 0, what would you conclude about B and C from this calculation? Is this conclusion valid for matrices?

Answer: AC would only be equal to AB (if they were numbers) if and only if C = B. But for matrices,

AC = AB implies AC-AB = 0 and hence A(C-B)=0. But unlike numbers, as we saw, the product of two non zero matrices can indeed be zero. Hence, C need not be B.

```
% Enter your code here
A*C, A*B
```

```
ans = 2×2
     1     0
     0     0
ans = 2×2
     1     0
     0     0
```

Question 4. (0,1) Matrices (Total Points - 3)

A 6×6 matrix A is given to you below, containing only entries of 0 or 1. This matrix contains information about a group of six people, each equipped with a communication device: the (i, j) entry of A is 1 if person i can send a message to person j directly, and 0 otherwise.

The (i, j) entry of A^2 will then represent the number of ways to send a message from person i to person j in exactly two steps: indeed, this entry is given by summing the entries of row i of A (which are 1 if that's a destination person i can send to) times the entries of column j (which are 1 if that's a destination that can send a message to person j). Similar logic can be used to see that higher powers A^n contain entries counting the number of ways to communicate between one person and another in exactly n steps. Powers of A are tedious to compute by hand, but we can use MATLAB to help us.

Enter the 6 x 6 matrix A into your MATLAB workspace.

```
A = [0 0 0 1 0 1; 1 0 1 1 0 0; 0 1 0 1 0 0; 1 0 1 0 0 0; 1 1 1 0 0 1; 0 0 1 1 0 0]
```

```
A = 6×6
     0     0     0     1     0     1
     1     0     1     1     0     0
     0     1     0     1     0     0
     1     0     1     0     0     0
     1     1     1     0     0     1
     0     0     1     1     0     0
```

On a plain sheet of paper draw by hand six vertices labeled 1-6 (one for each person). Make this into a directed graph by putting an arrow pointing from vertex **i** to vertex **j** for all ordered pairs (i, j) for which the entry a_{ij} = 1 in A.

Give detailed solutions in your lab report for the following:

(i) ***Exercise (Points - 1):*** Is there any person who cannot receive a message from anyone else in a single step? Obtain the answer in two ways: by the graph and by the matrix A. Give details.

Answer: Yes, the 5th person cannot recieve a message from anyone directly. This can be concluded since no link points towards the node on a graph, and as for the Matrix, all elements in the 5th column (j=5 for all), are zero.

(ii) ***Exercise (Points - 1):*** How many ways can person 1 send a message to person 4 in exactly 1, 2, 3, and 4 steps? Compute the matrices A, A^2, A^3, and A^4 to obtain your answer and consider the four cases (1, 2, 3, 4 steps) separately. For the case of two steps, check your answer by comparing to the directed graph you made. Give details.

Answer: Person 1 can, obviously reach person 4 in exactly one way (A_14 = 1)

- ***Since node 1 can message node 6 and then node 4, and no other ways, there is 1 way for person 1 to message person 4 in exactly 2 ways. (Corresponding to A^2_14 = 1)***
- ***Node 1 can message node 6 and node 4, thereafter node 3 to node 2 or node 2 to node 1 to back to node 3. Hence, 3 ways, Corresponding to A^3_14=3***

Node 1 -> Node 6 -> Node 3 -> Node 2 -> Node 4,

Node 1 -> Node 6 -> Node 4 -> Node 3 -> Node 4,

Node 1 -> Node 6 -> Node 4 -> Node 1 -> Node 4,

Node 1 -> Node 4 -> Node 3 -> Node 2 -> Node 4,

Node 1 -> Node 4 -> Node 1 -> Node 6 -> Node 4,

Hence 5 ways, corresponding to A^4_14 = 5

```
% Enter your code here
A, A^2, A^3, A^4
```

```
A = 6×6
     0     0     0     1     0     1
     1     0     1     1     0     0
     0     1     0     1     0     0
     1     0     1     0     0     0
     1     1     1     0     0     1
     0     0     1     1     0     0
ans = 6×6
```

```
     1     0     2     1     0     0
     1     1     1     2     0     1
     2     0     2     1     0     0
     0     1     0     2     0     1
     1     1     2     4     0     1
     1     1     1     1     0     0
ans = 6×6
     1     2     1     3     0     1
     3     1     4     4     0     1
     1     2     1     4     0     2
     3     0     4     2     0     0
     5     2     6     5     0     1
     2     1     2     3     0     1
ans = 6×6
     5     1     6     5     0     1
     5     4     6     9     0     3
     6     1     8     6     0     1
     2     4     2     7     0     3
     7     6     8    14     0     5
     4     2     5     6     0     2
```

(iii) ***Exercise (Points - 1):*** How many ways are there for person 3 to send a messsage to person 4 using four or fewer steps? Use the matrix B = $A + A^2 + A^3 + A^4$ to obtain your answer. Give details.

Answer: B(3,4) = 12 Looking at the previous correlation, I can say that B(3,4) would correspond to the total ways of person 3 contacting person 4 in 1,2,3 or 4 steps

```
% Enter your code here
B = A + A^2 + A^3 + A^4
```

```
B = 6×6
     7     3     9    10     0     3
    10     6    12    16     0     5
     9     4    11    12     0     3
     6     5     7    11     0     4
    14    10    17    23     0     8
     7     4     9    11     0     3
```

Helper Function (Do Not Edit)

```
function v = rvect(m)
v = fix(10*rand(m,1));
end
```

```
function A = rmat(m,n)
A = fix(10*rand(m,n));
end
```