7.
$$\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\downarrow \; r_4 \longleftrightarrow r_3$$

$$\begin{bmatrix} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{REF!}$$

$$\downarrow \; r_1 + 4r_4, \quad r_2 - 3r_4, \quad r_3 + 2r_4$$

$$\begin{bmatrix} 1 & 7 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow \; r_1 - 3r_3, \quad r_2 + r_3$$

$$\begin{bmatrix} 1 & 7 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow \; r_1 - 7r_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{RREF!}$$

Assuming it's an Augmented matrix, the last row (r_4) suggests $0x_1 + 0x_2 + 0x_3 = 1 \rightarrow \boxed{\text{NO SOLUTION}}$

8. $\left[\begin{array}{ccc|c} 1 & 1 & 5 & 0 \\ 0 & 1 & 9 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right] \rightarrow$ REF

$\downarrow \frac{1}{7} r_3 \rightarrow$ new r_3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 5 & 0 \\ 0 & 1 & 9 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\downarrow r_1 - 5r_3, \quad r_2 - 9r_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$r_1 - r_2 \rightarrow r_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \text{RREF} \Rightarrow \begin{cases} x = -4 \\ y = 9 \\ z = -1 \end{cases}$$

$\boxed{\text{Unique solution}}$

35. $\left[\begin{array}{ccc|c} 1 & -3 & 5 & g \\ 0 & 2 & -3 & h \\ -3 & 5 & -4 & k \end{array}\right]$

$\downarrow$ $\cancel{r_3 \Rightarrow}\ r_3 + 3r_1 \to$ new r_3.

$\left[\begin{array}{ccc|c} 1 & -3 & 5 & g \\ 0 & 2 & -3 & h \\ 0 & -4 & 6 & k+3g \end{array}\right]$

$\downarrow$ $r_3 + 2r_2 \to$ new r_3.

$\left[\begin{array}{ccc|c} 1 & -3 & 5 & g \\ 0 & 2 & -3 & h \\ 0 & 0 & 0 & k+3g+2h \end{array}\right] \to$ REF.

the last row suggests,

$$0 + 0 + 0 = \boxed{0 = k + 3g + 2h.}$$

hence, an equation involving g, k, h that makes this augmented matrix consistent is:

$$\boxed{k + 3g + 2h = 0}$$

2. a) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \boxed{\text{RREF}}$

All leading nonzero terms are 1's & have no other non-zero terms above them.

b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow \boxed{\text{RREF}}$

All leading nonzero terms are 1's & have no other non-zero terms above them.

c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \rightarrow$ NOT even REF

All leading nonzero terms are NOT strictly to the right of previous nonzero leading terms.

d) $\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$ REF.

NOT RREF.

23. $\begin{cases} x_1 + hx_2 = 2 \\ 4x_1 + 8x_2 = k. \end{cases}$

$$\therefore A = \begin{array}{c} \left[\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right] \end{array} \begin{array}{l} r_1 \\ r_2. \end{array}$$

A is an equivalent form for the system as an augmented matrix.

$$r_2 - 4r_1 \rightarrow \text{new } r_2.$$

$$\therefore R = \left[\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right].$$

b) $k = 10$, $h = 0$ {1 of many situations}

$$R = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 8 & 2 \end{array}\right]$$

gives the <u>unique solution</u> $\begin{cases} x_1 = 2 \\ x_2 = \frac{1}{4} \end{cases}$

a) $k = 10$, $h = 2$.

$$\therefore R = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 2 \end{array}\right]$$ suggesting $0 + 0 = 2$.

$\therefore$ <u>NO SOLUTION</u>

c) $k=8$, $h=2$.

$$\therefore R = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right].$$

$\Rightarrow$ $x_1 = 2.$ & $0=0.$

Hence, x_2 is a free variable

We thus have INFINITE SOLUTIONS.

41. If each column is a variable / unknown, and each row corresponds to an equation,

we can have the system of equations represented by the Augmented matrix

$$A = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & & & & \\ a_{m1} & & & a_{mn} & b_m \end{array}\right]$$

it has n co-efficients in each equation. & m equations.

for a so called "undetermined system"

$m < n$.

If given that A is consistent, the system can be either
- having unique solutions.
- having infintely-many Solutions

By definition, if the system has a unique solution, then there must be a RREF where all columns of R are pivot columns.

*→ i.e. R must only have leading non-zero entries to the right of other non-zero leading entries above it.

*→ those leading entries must be the only non-zero Entry in thier column.

for eg:

pivot positions.

$$\left[\begin{array}{cccc|c} a_1 & 0 & 0 & 0 & b_1 \\ 0 & 0 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & a_3 & b_3 \\ 0 & 0 & 0 & 0 & b_4 \end{array}\right]$$

pivot columns.

[Not a matrix for conditions in the question]

$$a_1, a_2, a_3 \neq 0$$

The best shot ~~ath~~ at having such a matrix is to have elements on the immidiate down-left to the previous nonzero elements,

i.e. to have elements along the matrix's diagnal:

$$\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix}$$

This makes sure all columns are pivotal & hence there are <u>maximun number of pivotal columns.</u>

Each pivotal column would correspond to 1 variable being defined / independent.

But for an untetermined system,

\# columns < \# rows.

a_6 ?

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \left[\begin{array}{cccccc|c} a_1 & 0 & 0 & 0 & 0 & 0 & b_1 \\ 0 & a_2 & 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & a_3 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & a_4 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & a_5 & 0 & b_5 \end{array}\right] \end{array}$$

There is hence 1 non-pivotal column where the last unknown would have to fit in, making atleast 1 variable ~~indepenc~~ dependent & causing the system to have INFINITE SOLUTIONS.

$$\left[\begin{array}{cccccc|c} a_1 & 0 & 0 & 0 & 0 & 0 & b_1 \\ 0 & a_2 & 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & a_3 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & a_4 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 & a_5 & a_6 & b_5 \end{array}\right]$$

atleast ~~of~~ one of the two have to be dependent!

Hence, an undetermined system, i.e. a system with more equations than unknowns that is consistent will have

INFINITELY MANY SOLUTIONS.