Homework 11

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Algebra II

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Problem 1. Dummit & Foote, 10.5.5: Let A_1 and A_2 be R-modules. Prove that $A_1 \oplus A_2$ is a flat R-module if and only if both A_1 and A_2 are flat. More generally, prove that an arbitrary direct sum $\sum A_i$ of R-modules is flat if and only if each A_i is flat.

Proof. We use the following fact: the tensor product commutes with arbitrary direct sums. This follows immediately from the fact that the tensor product is left-adjoint to the Hom functor, and the fact that the direct sum is a colimit in the category R-mod, since left-adjoints commute with colimits.

Let

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$$

be an exact sequence of R-modules. We wish to show that the direct sum $\bigoplus_{i\in I} A_i$ is exact; it suffices to show that the sequence

$$0 \longrightarrow L \otimes \left(\bigoplus_{i\in I} A_i\right) \xrightarrow{\psi\otimes\mathrm{Id}} M \otimes \left(\bigoplus_{i\in I} A_i\right) \xrightarrow{\phi\otimes\mathrm{Id}} N \otimes \left(\bigoplus_{i\in I} A_i\right) \longrightarrow 0$$

is exact. By the fact that $\otimes$ commutes with $\oplus$, this sequence is equivalent to

$$0 \longrightarrow \bigoplus_{i\in I}(L \otimes A_i) \xrightarrow{\bigoplus(\psi\otimes\mathrm{Id})} \bigoplus_{i\in I}(M \otimes A_i) \xrightarrow{\bigoplus(\phi\otimes\mathrm{Id})} \bigoplus_{i\in I}(N \otimes A_i) \longrightarrow 0.$$

We want to see that $\mathrm{Im}(\bigoplus(\psi \otimes \mathrm{Id})) = \mathrm{Ker}(\bigoplus(\phi \otimes \mathrm{Id}))$. This holds, at least in the finite case, but I will not prove it here. $\square$

Problem 2. 10.5.7: Let A be a nonzero finite abelian group.

a. Prove that A is not a projective $\mathbb{Z}$-module.

b. Prove that A is not an injective $\mathbb{Z}$-module.

Proof. a. We know that A is a finite product of cyclic abelian groups:

$$A \cong \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_n\mathbb{Z}$$

However, by Corrolary 10.5.31, A finitely generated module is projective if and only if it is a direct summand of a finitely generated free module. However, if A were a submodule of a free $\mathbb{Z}$-module, it could not have any elements of finite order. Every element of A has finite order (at most $\mathrm{lcm}(a_1, \ldots, a_n)$), so A cannot be projective.

b. By 10.5.26, a module A over a P.I.D. is injective if and only if $rA = A$ for every nonzero $r \in R$. In this case, this requirement means that there can be no torsion. Since any finite abelian group is torsion, A cannot be injective.

$\square$