Homework 2

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Algebraic Geometry

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Problem 1. Let $X = V(f) \subset \mathbb{A}^n$ be an affine variety defined by a single polynomial f. Prove that $\dim(X) = n - 1$.

Proof. incomplete. □

Problem (. 2) Let $X \subset \mathbb{A}^n$ be an algebraic set with $\overline{X}$ its closure in $\mathbb{P}^n$. Show that X is irreducible if and only if $\overline{X}$ is irreducible.

Proof. incomplete □

Problem (. 3) Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring. Recall that an ideal $I \subset A$ is homogeneous if it is generated by homogeneous elements.

(a) Show that if I is homogeneous then the following holds: let $f = \sum_i f_i \in A$, where f_i denotes the homogenous degree i component of f. Then $f \in I$ if and only if $f_i \in I$, for all i.

Proof. One direction is clear; for if every component of f is in I, then so is f, because I is closed under addition.

In the opposite direction, assume $f \in I$. Since I is homogeneous, it is generated by some set $\{g_j\}$ of homogenous polynomials, and f may be written as a finite sum of terms in the g_j:

$$f = \sum_{j \in J} a_j g_j$$

Denote by d_j the degree of the homogenous term g_j. Now, we can write each term a_j as a sum of homogeneous terms; say

$$a_j = \sum_{k \geq 0} a_{jk}$$

Using these decompositions, we may write each homogenous component of f as

$$f_i = \sum_{j \in J} a_{j(i-d_k)} g_j,$$

So indeed we may write each homogenous component of f in terms of the homogenous generators of I, and so each homogenous component of f is in I. □

(b) Show that if f is homogenous then its radical $\sqrt{(I)}$ is also homogenous.

Proof. incomplete □

Problem 4. Consider the n-tuple morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ given by $(t : s) \mapsto (t^n : st^{n-1} : \cdots : s^n)$ Prove that the image of φ is a projective subvariety C_n of $\mathbb{P}^n$.

Proof. Let $C_n = V(\{f_{ij}\}_{0<=i<j<=n})$, where $f_{ij} = x^i x^j - x^{i+1}x^{j-1}$. We first see that the image of φ lies within C_n, since for any point $(t^n : st^{n-1} : \cdots : s^n)$, and any i, j,

$$\begin{aligned}
f_{ij}(t^n : st^{n-1} : \cdots : s^n) &= (s^i t^{n-i})(s^j t^t n - j) - (s^{i+1}t^{n-(i+1)})(s^{j-1}t^{n-(j-1)}) \\
&= s^{i+j}t^{2n-(i+j)} - s^{i+j}t^{2n-(i+j)} \\
&= 0
\end{aligned}$$

It remains to show that, for any point $x \in C_n$, there exists some $(s : t)$ such that $\varphi(s : t) = x$. Let x be an arbitrary point of $\mathbb{P}^n$ such that each $f_{ij}(x) = 0$.

Assume first that $x_0 \neq 0$, and rescale so that

$$x = (1 : x_1 : \cdots : x_n).$$

We will show that the point $(1 : x_1)$ maps to x under φ. We see first that

$$\varphi(1 : x_1) = (1 : x_1 : x_1^2 : \cdots : x_1^n)$$

Now, using the relation $f_{03}(x) = 0$, we see that $x_2 = x_1^2$; using $f_{04}(x) = 0$, we see that $x_3 = x_1^3$, and so on. So indeed

$$\begin{aligned}
x &= (1 : x_1 : x_1^2 : \cdots : x_1^n) \\
&= \varphi(1 : x_1)
\end{aligned}$$

Next, assume that $x_0 = 0$; we will see that this forces $x_1, \ldots, x_{n-1} = 0$, and so that $x_n \neq 0$. If $x_0 = 0$, then the relation $f_{02}(x) = 0$ forces $x_1^2 = 0$, so $x_1 = 0$; that $f_{03}(x) = 0$ forces $x_2 = 0$, and so on through x_{n-1}. Since x is a point of projective space, we know that not all of its components are 0; therefore $x_n = 1$ (up to scaling), and

$$\begin{aligned}
x &= (0 : 0 : \cdots : 1) \\
&= \varphi(0 : 1)
\end{aligned}$$

So in each of the cases $x_0 = 0$ and $x_0 \neq 0$, we have $x \in \mathrm{Im}(\varphi)$, and so $\mathrm{Im}(\varphi)$ is indeed the algebraic set C_n. □

Problem 5. Consider the hyperbola $X = V(xy - 1)$ and the parabola $Y = V(x^2 - y)$ as affine varieties in $\mathbb{A}^2$. Let $\mathbb{P}^2$ be the projective plane with coordinates $(x : y : z)$, and $\mathbb{A}^2 = \{(x : y : z) \mid z \neq 0\}$. Let $\overline{X}$ and $\overline{Y}$ denote the closure of X and Y in $\mathbb{P}^2$, respectively.

(a) Find homogeneous ideals defining $\overline{X}$ and $\overline{Y}$ in $\mathbb{P}^2$.

Proof. The ideals are simply generated by the homogenization of the polynomials generating the ideals for X and Y:

$$\overline{X} = V(xy - z^2)$$
$$\overline{Y} = V(x^2 - yz)$$

□

(b) How many points of intersection do $\overline{X}$ and $\overline{Y}$ have?

Proof. One way to calculate this is to first solve the equations over $\mathbb{A}^2$, i.e. assuming that $z = 1$ and solving the original equations, and then checking for extra points "at infinity", where $z = 0$.

First, we have the thre points of intersection between the two curves in $\mathbb{A}^2$: setting $x = 1/y$ and solving $x^2 - y = 0$, we see that

$$y^3 = 1,$$

which has 3 solutions over an algebraically closed field k, which we denote $1, \zeta_3, \zeta_3^2$. Solving for $x = 1/y$, we get the three points

$$(1 : 1 : 1), (\zeta_3, \zeta_3^2, 1), (\zeta_3^2, \zeta_3, 1)$$

Finally, we check for interesting solutions when $z = 0$:

$$xy - 0 = x^2 - 0 = 0$$

Of course we have $x = y = z = 0$, but this does not correspond to a point in projective space. The second equation forces $x = z = 0$, which is a single point in projective space that does indeed satisfy both equations:

$$(0 : 1 : 0)$$

Therefore there are 4 points of intersection between the two curves. □

(c) Find an automorphism φ of the projective plane $\mathbb{P}^2$ which maps $\overline{X}$ to $\overline{Y}$ and vice versa.

Proof. incomplete □

Problem 6. Let $X = V(y - x^2, z - x^3) \subset \mathbb{A}^3$. Prove the following:

(a) $I = I(X) = \langle y - x^2, z - x^3 \rangle$.

(b) *incomplete*

Problem 7. *incomplete*