Homework 3

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Analysis I

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Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x_0 \in \mathbb{R}^n$.

(i) Prove that $f'(x_0)$ is the linear, automatically continuous, operator from $\mathbb{R}^n$ to $\mathbb{R}^m$ that is given by the matrix $A \in \mathbb{R}^{n\times m}$ whose components are partial derivatives: $A_{ij} = \frac{\partial f^i}{\partial x_j}(x_0)$. This is the Jacobi matrix.

(ii) Prove that if all partial derivatives of f exist and are continuous, then f is strongly differentiable at each $x_0 \in \mathbb{R}^n$.

Proof. (i) *incomplete*

(ii) *incomplete*

□

Problem L. et E be a Banach space and let F be a finite-dimensional subspace of E. Show that there exists a closed subspace $G \subset E$ such that:

$$F + G = E \text{ and } F \cap G = \{0\}.$$

Proof. This proof is adapted from a Stackexchange answer given at [1] Assuming that F is n-dimensional, let $e_1, ...e_n$ be a basis of F. Define the linear functionals f_i on the basis e_i by $f_i(e_i) = 1$, and $f_i(e_j) = 0$ if $i \neq j$. Then by Hahn-Banach, each of these functionals may be extended to functionals g_i defined on the whole of E. Let the space G be defined as

$$G = \bigcap_{i=1}^{n} \ker(g_i).$$

We now show that this space is closed, that $F + G = E$, and that $F \cap G = \{0\}$. First, G is closed because the kernel of any functional is closed, and therefore so is their intersection.

Now, let $x \in E$ be arbitrary; we decompose it into elements of F and G. Let $y = g_1(x)e_1 + \cdots + g_n(x)e_n$, and $z = x - (g_1(x)e_1 + \cdots + g_n(x)e_n)$. It is clear that $y \in F$, and we see that $z \in G$ because, for any g_i,

$$\begin{aligned} g_i(z) &= g_i(x - (g_1(x))e_1 + \cdots + g_n(x)e_n) \\ &= g_i(x) - g_i(x)g_i(e_i) && = 0 \end{aligned}$$

Therefore, g is in the intersection of the kernels of all g_i, meaning it is in G, and so any arbitrary element of E may be decomposed as $y+z$, with $y \in F$ and $z \in G$. So $E = F + G$.

Now, we show that $F \cap G = \{0\}$. Let x be an arbitary element of $F \cap G$. Because $x \in F$, we may write it as $x = x_1 e_1 + \cdots + x_n e_n$; therefore for any g_i,

$$g_i(x) = x_n g_i(e_i) = x_i.$$

However, $x \in \ker(g_i)$, so each component x_i is 0. □

Problem 3. (i) Let $(E, \|\cdot\|)$ be the Banach space defined in problem 5 of Homework 3 (the space of Lipchitz functions $f : X \to \mathbb{R}$, where $f(x_0) = 0$, and $\|f\|$ is the lowest Lipschitz constant of f). For each $x \in X$, let $T_x(f) = f(x)$. Prove that each $T_x \in E^*$, and that $\|T_x - T_y\| = d(x, y)$. Deduce that X is therefore isometric to a subset of E^*.

(ii) Let E be a linear normed space. Prove that it is isometric with a linear subspace of the Banach space $\mathcal{B}(B_{E^*}(1))$ of bounded real functions on the closed unit ball in the dual space E^*: $B_{E^*}(1) = \{T \in E^*; \|T\| \leq 1\}$.

Proof. (i) Let x in X be arbitrary. We show first that $T_x \in E^*$; i.e. that it is a bounded real functional on E.

- $T_x(f+g) = T_x(f) + T_x(g)$: This follows from direct calculation, as

$$\begin{aligned} T_x(f+g) &= (f+g)(x) \\ &= f(x) + g(x) \\ &= T_x(f) + T_x(g) \end{aligned}$$

- $T_x(\alpha f) = \alpha T_x(f)$: This too, follows from calculation:

$$\begin{aligned} T_x(\alpha f) &= (\alpha f)(x) \\ &= \alpha(f(x)) \\ &= \alpha T_x(f) \end{aligned}$$

- $\|T_x\| < \infty$: We show that the norm of T_x is bounded by $d(x, x_0)$. For if f is an arbitrary element of E with lowest Lipschitz constant 1, then in particular

$$\frac{|f(x) - f(x_0)|}{d(x, x_0)} \leq 1,$$

  And so $|T_x(f)| \leq d(x, x_0)$.

Therefore T_x is an element of the continuous dual. Now, let $x, y \in E$ be arbitrary. We may bound the value of $\|T_x - T_y\|$ by $d(x, y)$: if $f \in E$ is an arbitrary Lipschitz function with $f(x_0) = 0$ and Lipschitz constant 1, then $\frac{|f(x)-f(y)|}{d(x,y)} \leq 1$, so

$$\begin{aligned} |(T_x - T_y)(f)| &= |(f(x) - f(y))| \\ &\leq d(x, y) \end{aligned}$$

Now, we can also show that this bound is attained: let f be the function:

$$z \mapsto \frac{d(z,y)d(x_0,x)}{d(x_0,x)+d(x_0,y)} - \frac{d(z,x)d(x_0,y)}{d(x_0,x)+d(x_0,y)}.$$

Then f is Lipschitz, because the distance function has Lipschitz constant 1, and it is an element of E, because $f(x_0) = 0$:

$$f(x_0) = \frac{d(x_0,y)d(x_0,x)}{d(x_0,x)+d(x_0,y)} - \frac{d(x_0,x)d(x_0,y)}{d(x_0,x)+d(x_0,y)} = 0$$

Also, $\|f\| \leq 1$, because for any two $w, z \in X$,

$$\begin{aligned}
|f(z) - f(w)| &= \left| \frac{d(z,y)d(x_0,x) - d(w,y)d(x_0,x)}{d(x_0,x)+d(x_0,y)} - \frac{d(z,x)d(x_0,y) - d(w,x)d(x_0,y)}{d(x_0,x)+d(x_0,y)} \right| \\
&\leq \left| \frac{((d(z,x) - d(w,x))d(x_0,y)}{d(x_0,x)+d(x_0,y)} \right| + \left| \frac{(d(z,y) - d(w,y))d(x_0,x)}{d(x_0,x)+d(x_0,y)} \right| \\
&\leq \frac{d(z,w)d(x_0,y)}{d(x_0,x)+d(x_0,y)} + \frac{d(z,w)d(x_0,x)}{d(x_0,x)+d(x_0,y)} \\
&= d(w,z)
\end{aligned}$$

In fact, $\|f\| = 1$, because $|f(x) - f(y)| = d(x,y)$. So, f is an element of the unit ball in E.

Now, we can show that $|T_x(f) - T_y(f)| = d(x,y)$. In fact, this follows from the earlier observation that $|f(x) - f(y)| = d(x,y)$, which we can see by calculation:

$$\begin{aligned}
|f(x) - f(y)| &= \left| \frac{(d(x,y) - d(x,x))d(x_0,x)}{d(x_0,x)+d(x_0,y)} + \frac{(d(x,y) - d(y,y)d(x_0,x))}{d(x_0,x)+d(x_0,y)} \right| \\
&= d(x,y)
\end{aligned}$$

Therefore, $\|T_x - T_y\| = d(x,y)$. Therefore the subspace of elements $\{T_x; x \in X\}$ is isometric to the space X itself.

(ii) *incomplete*

□

Problem 4. Let (Y,d) be a complete metric space and let $f : B \to Y$ be a contractive mapping with Lipschitz constant $\alpha < 1$, where B is an open ball centered at some y_0 with radius $r > 0$. Prove that if $d(f(y_0), y_0) < (1-\alpha)r$ then f has a fixed point.

Proof. Given the above assumptions, let $k = d(f(y_0), y_0)/r$, which we assume to be less than $(1-\alpha)$. We now show that f restricts to a contraction mapping on the closed ball B' with radius $\frac{kr}{1-\alpha}$: Let $x \in B'$. Then we want to show that $f(x) \in B'$, i.e. that $d(f(x), y_0) \leq \frac{kr}{1-\alpha}$. This in fact holds:

$$\begin{aligned}
d(f(x), y_0) \leq d(f(x), f(y)) + d(f(y), y) & \\
&\leq \alpha d(x,y) + kr \\
&\leq \alpha \frac{kr}{1-\alpha} + kr \\
&= \frac{kr}{1-\alpha}
\end{aligned}$$

Therefore f restricts to a function $B' \to B'$, and the restriction of a contraction mapping is a contraction mapping. Because Y is a complete space, the closed subspace B' is complete, and by the contraction mapping theorem f admits a fixed point $x_0 \in B' \subset B$. □

Problem L. et (X, d) be a complete metric space and $f : X \to X$ a map such that for some $n > 1$ the composition of the function f with itself n times: $f^{(n)} : X \to X$ is a contraction.

(i) Does f have to be continuous?

(ii) Prove that f has a unique fixed point in X.

Proof. (i) No, f does not have to be continuous - for example, let f be the function on the unit ball in $\mathbb{R}^2$:

$$f(x,y) = \begin{cases} (\frac{x}{2}, \frac{y}{2}) & y \neq 0 \\ (-\frac{x}{2}, 0) & y = 0 \end{cases}$$

Then f is discontinuous on the line $y = 0$, but $f^{(2)}$ is the continuous contraction mapping $(x, y) \mapsto (\frac{x}{4}, \frac{y}{4})$.

(ii) *incomplete*

□

References

[1] Tsemo Aristide, Complement a finite dimensional subspace in a Banach space, URL: https://math.stackexchange.com/q/3224629