Homework 3

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Algebraic Geometry

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Problem 1. Let X be a (quasi-projective) algenraic variety and let $p \in X$. Show that $\mathcal{O}_{p,X}$ is an integral domain if and only if p belongs to a unique irreducible component of X. (hint: first reduce the problem to the case where X is an affine variety).

Proof. First, assume that $X = \mathrm{Spec}(A)$ is an affine variety, so that $\mathcal{O}_X = A$, and $\mathcal{O}_{p,X} = A_p$ for any prime ideal $p \in \mathrm{Spec}(A)$. Irreducible components of $\mathrm{Spec}(A)$ are the sets $V(q)$, for q a minimal prime.

First, assume that p belongs to a unique irreducible component of $\mathrm{Spec}(A)$: that is, there is a unique minimal prime q such that $q \subset p$. We want to show that A_p is an integral domain. Let $x = a/b$, $y = f/g$ be two elements of A_p such that $xy = 0$. By definition, this is equivalent to the existence of some $h \in A \backslash p$ such that $h(ga - bf) = 0$. □

Problem 2. Recall that the field of rational functions $k(X)$ of an irreducible, quasi-projective, algebraic variety X, is the collection of all rational functions on X.A rational function is a regular function f on some non-empty open subset $U \subset X$ (up to the equivalence that $(f, U) \simeq (g, V)$ if $f = g$ on $U \cap V$).

(a) Verify that any two open subsets in X intersect and hence the field operations on $k(X)$ are well-defined. Then show that $k(X)$ is in fact a field.

Proof. incomplete □

(b) Show that if X is an affine variety, then its field of rational functions coincides (is naturally isomorphic to) the field of fractions of its coordinate ring $k[X]$.

Proof. incomplete □

Problem 3. Show that any two smooth quadrics in $\mathbb{P}^n$ are isomorphic. Recall that a quadric (in $\mathbb{P}^n$) is a subvariety defined by a (homogeneous) quadratic polynomial.

Proof. incomplete □

Problem 4. Consider the affine curves C in $\mathbb{A}^2$ below. Find the points at infinity on these curves (that is, points in the closure of C in $\mathbb{P}^2$ that are not in $\mathbb{A}^2$). Decide for each point at infinity if it is singular or non-singular and find its tangent space and tangent cone.

1. $y^2 = x^3 + ax + b$

   Proof. The homogenization of this polynomial is $y^2z = x^3 + axz^2 + bz^3$. Points at infinity correspond to tuples $(x, y, 0)$, where x and y are not both 0, which satisfy the equation

   $$\begin{aligned} y^2 \cdot 0 &= x^3 + ax \cdot 0 + b \cdot 0 \\ 0 &= x^3 \end{aligned}$$

   So, the point $(0, 1, 0)$ is the unique point at infinity on any elliptic curve.

   To find the tangent space at this point, we find the projective closure of the tangent space in $\mathbb{A}^2$ of the intersection of X with a chart of $\mathbb{P}^2$ containing $(0, 1, 0)$. The only such chart is $E_1 : \mathbb{A} \to \mathbb{P}^2$, defined by taking $(x, z) \mapsto (x : 1 : z)$, and the restriction of X to this chart is the variety defined by the equation

   $$z = x^3 + axz^2 + bz^3$$

   The tangent space at $(0, 0)$ is defined by the linearization of this equation, which is $z = 0$. This polynomial is homogenous, so its projective closure has no extra points: the tangent space of X at $(0, 1, 0)$ is the algebraic variety $V(z)$, the zero set of the polynomial z. □

2. $y = x^3$

   Proof. incomplete □

3. $x^3 + x^2y - y = 0$

   Proof. incomplete □