Connected Complex Abelian Lie Groups: Tori and Number Theory

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In this paper we will first show that all compact connected complex Lie groups are complex tori, then show that every complex abelian variety has the structure of a complex torus, and finally discuss the issue of whether a given complex torus is an abelian variety, solved through the Riemann conditions.

1   Tori

The following treatment of compact complex connected Lie groups follows section 1.1 in [1], expanded where necessary.

Definition 1.1. A **Complex Lie Group** X of dimension n is a complex manifold of dimension n with a gorup structure on the underlying set such that the multiplication maps $X \times X \to X$ and inverse map $X \to X$ are both holomorphic.

Lemma 1.2. For any homomorphism $T : X_1 \to X_2$ of complex Lie groups,

$$T(\exp_{X_1} y) = \exp_{X_2}((dT)_e y).$$

As shown in Böcker & tom Dieck, the following diagram commutes:

$$
\begin{array}{ccc}
LX_1 & \xrightarrow{(dT)_e} & LX_2 \\
\downarrow{\scriptstyle \exp_{X_1}} & & \downarrow{\scriptstyle \exp_{X_2}} \\
X_1 & \xrightarrow{\quad T \quad} & X_2
\end{array}
$$

Theorem 1.3. Every compact connected complex Lie group is abelian.

Proof. Let X be such a group, with $x \in X$ arbitrary, and let C_x be the conjugation map $X \to X$, taking y to xyx^{-1}. We wish to show that C_x is the identity map for all x.

We have seen that the differential of C_x at the origin, $(dC_x)_e$, is always an automorphism of the Lie algebra LX. In this case, the association $x \to (dC_x)_e$ is a holomorphic map of X into $\text{Aut}(LX)$, where we are viewing $\text{Aut}(LX)$ as a submanifold of the complex vector space $\text{End}(LX)$.

Because $x \to (dC_x)_e$ is a holomorphic map from a compact connected complex manifold, and the only holomorphic maps on a compact complex manifold are locally constant (By Liouville's Theorem), the value of $(dC_x)_e$ must be independent of x. The map $(dC_e)_e$ is the identity, so $(dC_x)_x$ must also be the identity for all x.

By the lemma above, we have

$$C_x(\exp y) = \exp((dC_x)_e y).$$

Since $(dC_x)_e = 1$, we see that $C_x(\exp y) = \exp y$, so $\exp LX$ is in the center of X. Because X is connected, it is generated by a neighborhood of the identity, and $\exp$ is a local diffeomorphism near the identity, X is generated by an abelian neighborhood of the identity, and so it is itself abelian. $\qquad\square$

As shown in Bröcker & tom Dieck, every compact connected abelian Lie group is a torus: the exponential map $LX \to X$ is a surjective homomorphism, with kernel U a lattice in the vector space LX.

2　Abelian Varieties

We now discuss abelian varieties, and show their relation to complex Lie groups. Because full proofs of some of these theorems would take us far afield into algebraic geometry, we offer some theorems as unproved statements.

Definition 2.1. A **Projective Variety** X over a field k is a subset of projective n-space over k, which can be defined as the set of zeroes of a set of homogeneous polynomials with coefficients in k.

Definition 2.2. A **Meromorphic function** f on an open subset D of $\mathbb{C}^n$ is a function $D\backslash I \to \mathbb{C}$ which is holomorphic on all of D except for a set I of codimension n, which is the set of **poles** of the function. Equivalently, a meromorphic function is a function which is locally equal to a quotient of holomorphic functions.

The **Polar Divisor** of a meromorphic function $f : D\backslash I \to \mathbb{C}$ is a function $D \to \mathbb{N}$ which takes the value 0 everywhere except the zeros and poles of f, and which takes the value k at every zaero of order k, and $-j$ at every pole of order $-j$. For instance, the divisor of the function $\frac{z-\alpha}{(z-\beta)^2}$ takes value 1 at α, -2 at β, and 0 everywhere else.

Definition 2.3. An **Abelian Variety** over $\mathbb{C}$ is a projective variety X over $\mathbb{C}$, which is also a group, and for which the group law can be written as a homogeneous polynomial in the coordinates of X. Further, X must be nonsingular - the Jacobian of the polynomials defining the variety has constant rank, and X must be irreducible - it cannot be written as the disjoint union of two strictly smaller projective varieties.

The definition does not explicitly state that the group must be abelian, but in fact it always is:

Theorem 2.4. As an abstract group, an abelian variety X over $\mathbb{C}$ is commutative and divisible ($mX = X$, for all $m \in \mathbb{N}_+$).

Proof. By definition, X is a subset of the compact complex manifold $\mathbb{P}^n$ defined as the zero-set of a set of functions whose Jacobian has constant rank, and it is therefore a compact complex manifold itself. The coordinate functions are complex-analytic on X, and the group law is defined by polynomials in the coordinates, so it is itself complex-analytic. Further, X is irreducible, so it is a connected manifold. As a compact connected complex Lie group, we have seen that X must be abelian and divisible - in fact, it must be a torus. $\square$

So we see that every complex abelian varity is a torus. One might expect that every complex torus is an abelian variety, but this is not true in general - although it is true in the simplest case.

Theorem 2.5. Every complex torus of complex dimension 1 is an abelian variety.

Proof. Let T be a complex torus of complex dimension 1: $T = \mathbb{C}/U$, where U is the lattice in $\mathbb{C}$ generated by two complex numbers $\omega_1 = a + bi$ and $\omega_2 = c + di$, where $ad - bc \neq 0$.
Define the Weierstrass function $\wp(z; U)$ by

$$\wp = \frac{1}{z^2} + \sum_{n^2 + m^2 \neq 0} \left(\frac{1}{(z + n\omega_1 + m\omega_2)^2} - \frac{1}{(n\omega_1 + m\omega_2)^2} \right).$$

This is a meromorphic function with a pole of order 2 at every point of U. Further, $\wp(z; U)$ is equal to $\wp(z + u; U)$ for any $u \in U$, so it descends to a meromorphic function on the torus $\mathbb{C}/U$. We use the following statement without proof; its proof can be found in a book on elliptic functions, such as [2].
The function $\wp(z)$ satisfies the differential equation

$$(\wp')^2 - 4\wp^3(z) + g_2\wp(z) + g_3,$$

Where g_2 and g_3 are constants depending on ω_1 and ω_2:

$$g_2 = 30 \sum_{(m,n) \neq (0,0)} (n\omega_1 + m\omega_2)^{-4}$$

$$g_3 = 5 \sum_{(m,n) \neq (0,0)} (n\omega_1 + m\omega_2)^{-6}$$

Let E be the projective variety in three variables $[w : x : y]$, given by

$$wy^2 = 4x^3 + g_2w^2x + g_3w^3.$$

By the differential equation which $\wp$ satisfies, we see that the function

$$z \mapsto [1 : \wp(z) : \wp'(z)]$$
$$0 \mapsto [1 : 0 : 0]$$

maps the torus $\mathbb{C}/U$ to the projective variety E.

Therefore, the torus is homeomorphic to the set of complex points of a projective variety. This homeomorphism is also a group isomorphism: as shown in [2], for any points z_1, z_2 on the torus $\mathbb{C}/U$, where $z_1 \neq z_2$, we have

$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2 - \wp(z_1) - \wp(z_2)$$

$$\wp'(z_1 + z_2) = \frac{1}{2}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)(\wp(z_2) - \wp(z_1 + z_2)) - \wp'(z_2)$$

When $z_1 = z_2$, we have

$$\wp(2z_1) = \left(\frac{3\wp(z_1)^2 + g_3}{\wp'(z_1)}\right)^2 - 2\wp(z_1)$$

$$\wp'(2z_1) = -\wp'(z_1)$$

This group law defines an abelian variety structure on the projective variety E; so we have an isomorphism between $\mathbb{C}/U$ and an abelian variety E over $\mathbb{C}$. $\qquad\square$

Every torus $\mathbb{C}/U$ is therefore an abelian variety. However, in higher dimensions this is no longer true: most tori $\mathbb{C}^n/U$ are not complex abelian varieties, even though we have seen that every complex abelian variety is a torus. However, there is a well-defined way to determine whether or not a given torus is an abelian variety.

Statement 2.6. Let $X = \mathbb{C}^n/U$ be an n-dimensional complex torus. Then X is an abelian variety if and only if there exists a positive definite hermitian form H on $\mathbb{C}^n/U$ such that $\mathrm{Im}(H)$, the imaginary part of H, takes integral values on $U \times U$.

This theorem is proved in [1], and relies on the identification of a line bundle on X with a unique choice of a hermitian form H, and a map $\alpha : U \to \{z \in \mathbb{C}^*; |z| = 1\}$ such that

$$\alpha(u_1 + u_2) = e^{i\pi \mathrm{Im}(H)(z,u)} \cdot \alpha(u_1), \alpha(u_2), u_i \in U.$$

This project is much shorter than I wanted/expected it to be - the topic was too far afield of the course topics, and I was far too unfamiliar with the subject to really understand the meat of it.

A full proof of the determination of the algebraicity of tori, which is what I wanted most of this paper to consist of, takes some fairly advanced homological algebra, which I was just unable to synthesize into a good project. I left the work until too late to change topics, so I'm handing in what I have done.

References

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[5] Hartshorne, R. Algebraic Geometry. Springer-Verlag, 1977