Homework 8

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Analysis I

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Problem 1. Let $F : A \to \mathbb{R}$ be a continuous function on a compact set $A \subset \mathbb{R}^n$. Verify that, setting:

$$F(x) := \min_{y \in A} \left\{ F(y) + \frac{|x-y|}{\text{dist}(x, A)} - 1 \right\} \quad \text{for all } x \in \mathbb{R}^n \backslash A,$$

defines a continuous extension of F on $\mathbb{R}^n$.

Proof. incomplete. □

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space. For every $A \subset X$, define:

$$\mu^*(A) := \inf \{\mu(B); A \subset B, B \in M\}$$

(i) Show that μ^* is a measure generator, coinciding with μ on $\mathcal{M}$ and such that it is 0 on every subset of a zero μ-measure set.

(ii) Let $\mathcal{M}_c$ be the σ-algebra generated by μ^*. Show that $\mathcal{M} \subset \mathcal{M}_c$.

(iii) Is the following characterization true?:

$$\mathcal{M}_c = \left\{ A \in 2^X; \exists B \in \mathcal{M} \ A \subset B \text{ and } \mu(B) = \mu^*(A) \text{ and } \mu^*(B \backslash A) = 0 \right\}$$

Proof. (i) *incomplete*

(ii) *incomplete*

(iii) *incomplete*

□

Problem 3. Let $f : [a, b] \to \mathbb{R}$ be a given function.

(i) If f is continuous, show that its graph is a set of (Lesbesgue) measure 0 in $\mathbb{R}^2$.

(ii) What if f is just a (possibly discontinuous) monotone function?

Proof. (i) Let $\varepsilon > 0$. We show that the graph of f may be covered with a measurable subset of $\mathbb{R}^2$ with measure $\leq \varepsilon$. Because f is a continuous function on the compact set $[a, b]$, it is uniformly continuous. Therefore, there must be some δ such that, for any $x \in [a, b]$, for all $y \in [a, b]$ such that $|x - y| < \delta$, $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$. This implies that, for any $x \in [a, b]$, the graph of f restricted to $[x, x + \delta]$ can be covered by the rectangle $[x, x+\delta] \times [f(x) - \frac{\varepsilon}{2(b-a)}, f(x) + \frac{\varepsilon}{2(b-a)}]$. Thus, dividing $[a, b]$ into the intervals

$$[a, a + \delta] \cup [a + \delta, a + 2\delta] \cup \cdots \cup [a + n\delta, b],$$

we can cover the graph of f with the rectangles

$$[a, a + \delta] \times \left[f(a) - \frac{\varepsilon}{2(b-a)}, f(a) + \frac{\varepsilon}{2(b-a)}\right] \cup$$
$$[a + \delta, a + 2\delta] \times \left[f(a + \delta) - \frac{\varepsilon}{2(b-a)}, f(a + \delta) + \frac{\varepsilon}{2(b-a)}\right] \cup \ldots$$

Each of these rectangles has a height of $\frac{\varepsilon}{(b-a)}$, and their widths add up to $b - a$, so the total area covered is $\frac{\varepsilon}{b-a} \cdot (b - a)$, or ε. Since ε was arbitrary, the graph of the function must have measure 0.

(ii) The graph of any monotone function also has measure 0. Assume f is some monotone increasing function on $[a, b]$, and let $h = f(b) - f(a)$. If f is constant, its graph clearly has measure 0, so assume that $h > 0$. Let $\varepsilon > 0$ be arbitrary. As f is a monotone function on a compact set, it can only have countably many points of discontinuity. If the set of points of discontinuity is empty, then f is continuous, and its graph has measure 0 by the last problem. Assuming that the set is nonempty and infinite, let $\{x_n\}_{n=1}^{\infty}$ be an enumeration of the points of discontinuity. Around point x_n, we put the box $\left[x_n - \frac{\varepsilon}{2^{n+2}h}, x_1 + \frac{\varepsilon}{2^{n+2}h}\right] \times [f(a), f(b)]$. Each box has height h, and the sum of their widths is

$$\sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}h} = \frac{\varepsilon}{2}$$

Further, f is continuous on the remaining parts of $[a, b]$, and so its graph can be covered by a union of boxes with measure $\varepsilon/2$ as well (this should be proven, but it is true). Thus the whole graph of f can be covered with a set of measure ε, where ε is arbitrary, meaning that its graph must have measure 0.

If the points of discontinuity are finite in number, say there are m of them, the proof can be done similarly; the width of each box can be chosen to be $\frac{\varepsilon}{2mh}$, again covering the discontinuities with a set of measure $\frac{\varepsilon}{2}$. In this case the remaining part of $[a, b]$ can be covered with finitely many intervals on which f is continuous, and so its graph on these intervals has measure 0 by the last problem.

$\square$

Problem 4. Prove that the following subsets of $[0, 1]$ are compact, of Lesbesgue measure 0, and uncountable:

(i) The set A containing all numbers which admit a binary representation $0.c_1c_2c_3\dots$ such that $c_n = 0$ for all n odd,

(ii) The set B of all numbers which admit a binary representation $0.c_1c_2c_3\dots$ such that for every n there is: $c_n = 0$ or $c_{n+1} = 0$.

Proof. (i) First, we show that the set is compact. Because it is a subset of a compact set, it is enough to show that it is closed. Let $\{x^n\}$ be a Cauchy sequence of elements of A. We show that the limit x of the sequence is also an element of A.

We note first that the set A can equivalently be defined as the set of numbers in $[0, 1]$ whose base-4 expansion can be written with only 0s and 2. Because a sequence of such numbers contains no 3s in any of their quaternary expansions, the limit x must have a unique expression in base-4 (there can be no "trailing 3s"), and, as with decimal numbers, a sequence of such numbers must "settle" at each digit: the mth quaternary digit of each x^n, for all n greater than some N, must be constant, and equal to the mth digit of the limit x. Therefore, the quaternary decimal expansion of x must be composed of only 0s and 2s, so it must also be an element of A.

Now, we show that the measure of A is 0. Because A is compact, it is measurable, so it suffices to show that the complement of A in $[0, 1]$ has measure 1.

The complement of A consists of all those numbers which have at least one 1 or one 3 in their quaternary expansion. This can be broken up into a countable union: let A_1^c be the set

$$[0.1_4, 0.2_4) \cup (0.3_4, 1],$$

Let A_2^c be the set

$$[0.01_4, 0.02_4) \cup (0.03_4, 0.01_4] \cup [0.21_4, 0.22_4) \cup (0.23_4, 0.3_4],$$

And so on. Each set A_n^c is dis joint from any other A_m^c, and each A_n^c has measure $2^n \cdot \frac{1}{2^{n+1}}$. Finally, every element of A_n^c has at least one 1 or one 3 in its quaternary expansion, meaning that $\bigcup_n A_n^c \subset A^c$ (in fact, they are equal). Therefore the measure of A^c is greater than or equal to the sum of the measures of the A_n^c, which is

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Because the complement of A^c is A, which is measurable, A must have measure $1-1 = 0$.

(ii) The set B can also be described in base 4, a little less succintly: B consists of all those numbers which have only 0, 1, and 2 in their base 4 expansion, and where a 1 is never followed by a 2 (because 12_4 in base 2 is 110, and there cannot be any 11s in the base 2 expansions).

B is compact by the same argument as for A: because there are no 3s in the base 4 expansion of any of its elements, any Cauchy sequence of elements of B will have eventually constant digits in base 4, meaning that the digits of its limit follow the same rules, and its limit is in B.

It is uncountable because it contains an uncountable subset: let B' be the set of numbers in $[0, 1]$ whose base 4 expansions have only 0 and 1, and do not have trailing 1s. This can be put in an obvious correspondence with the set of numbers in $[0, 1]$ whose base 2 expansions have only 0 and 1 and do not have trailing 1s, which is in fact all numbers in $[0, 1]$ - an uncountable set. Because $B' \subset B$, we see that B is also uncountable.

Measure 0: *incomplete*

□

Problem 5. Show that the derivative of a differentiable function $f : (a, b) \to \mathbb{R}$ is a (Lesbesgue) measurable function.

Proof. This follows quickly from the fact that a pointwise limit of measurable functions is measurable ([2]), and the fact that the sequence

$$f_n(x) = \frac{f(x + 1/n) - f(x)}{1/n}$$

Converges pointwise to the derivative of f, for any differentiable function. Because f_n is formed by adding measurable functions and multiplying by scalars, each f_n is measurable, and therefore their pontwise limit, which is the derivative of f, is also measurable.

Source: [1] □

References

[1] Henning Makholm, Is the derivative of differentiable function $f : \mathbb{R} \to \mathbb{R}$ measurable on $\mathbb{R}$?, URL: https://math.stackexchange.com/q/1803668

[2] Rudin, W, Real and Complex Analysis, Third Edition. McGraw Hill, 1987

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