Homework 4

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Algebra II

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1 Problems

Problem 1. Dummit & Foote Problem 7.5.2: Let R be an integral domain and let D be a nonempty, multiplicatively closed subset of R. Prove that the ring of fractions $D^{-1}R$ is isomorphic to a subring of the field of fractions of R.

Proof. Let F_R be the field of fractions of R. There are two canonical maps, $j : R \to D^{-1}R$ and $i : R \to F_R$, both of which send $r \in R$ to the formal fraction $\frac{r}{1}$.

If D contains zero, then $D^{-1}R = 0$, which is trivially isomorphic to the subring 0 of F_R. So, we can assume that D does not contain zero, the only zerodivisor of R. Therefore, every element of $D \subset R$ is mapped to a unit in R_F under the map i, and we can invoke the universal property of the localization.

The universal property states that j is initial in the subcategory of R/cRing consisting of those maps taking every element of D to a unit - that is, given a map $f : R \to S$ which maps D to units, there is a unique map $g : D^{-1}R \to S$ such that $g \circ j = f$.

In particular, the map $i : R \to F_R$ can be factored through j by a unique map $g : D^{-1}R \to F_R$ such that $i = g \circ j$. We need only show that g is injective, which will show that $D^{-1}R$ is ismorphic to the image of g, which is a subring of F_R.

To see that this map is injective, let $\frac{r_1}{d_1}, \frac{r_2}{d_2}$ be two elements of $D^{-1}R$ such that $g(r_1/d_1) = g(r_2/d_2)$.

We first note that for any element $d \in D$, $g(1/d) = (g(d/1))^{-1}$, since

$$\begin{aligned} g\left(\frac{1}{d}\right) g\left(\frac{d}{1}\right) &= g\left(\frac{1}{d}\frac{d}{1}\right) \\ &= g(1) \\ &= 1 \end{aligned}$$

Using this, we make the following calculation:

$$\begin{aligned}
g\left(\frac{r_1}{d_1}\right) &= g\left(\frac{r_1}{1}\frac{1}{d_1}\right) \\
&= g\left(\frac{r_1}{1}\right) g\left(\frac{1}{d_1}\right) \\
&= g(j(r_1))g\left(\frac{1}{d_1}\right) \\
&= \frac{r_1}{1}\left(g\left(\frac{d_1}{1}\right)\right)^{-1} \\
&= \frac{r_1}{1}\frac{1}{g(j(d_1))} \\
&= \frac{r_1}{1}\frac{1}{d_1/1} \\
&= \frac{r_1}{1}\frac{1}{d_1} \\
&= \frac{r_1}{d_1}
\end{aligned}$$

Similarly, $g(r_2/d_2) = r_2/d_2$. So, if $g(r_1/d_1) = g(r_2/d_2)$, then it must be true that $\frac{r_1}{d_1} = \frac{r_2}{d_2}$ as elements of the field of fractions. By definition of equality in this field, there must be some nonzero $r \in R$ such that $rr_1d_2 = rr_2d_1$. Then, since R is an integral domain, we can cancel r to see that $r_1d_2 = r_2d_1$, and finally that $\frac{r_1}{d_1} = \frac{r_2}{d_2}$ as elements of the localization $D^{-1}R$. This is sufficient to see that the map g is injective, and thus that it is an isomorphism of $D^{-1}R$ with the subring $\mathrm{im}(g)$ of F_R. $\square$

Problem 2. Dummit & Foote, 15.4.18. Prove that R_f, the localization of R away from f, is isomorphic to the quotient ring $R[x](fx-1)$ if f is not nilpotent in R.

Proof. We define the localization R_f as $S^{-1}R$, where S is the multiplicative set formed by all powers $1, f, f^2, ...$ of f.

We first construct a surjective homomorphism φ from $R[x] \to R_f$. We extend the map taking $r \in R$ to $\frac{r}{1}$, and $x \in R$ to $\frac{1}{f}$, so that an arbitrary element $a \in R[x]$, where a is some arbitray polynomial with coefficients a_i, is mapped as follows:

$$\varphi\left(\sum a_i x^i\right) = \sum \frac{a_i}{f^i}$$

Now, we show that every element of $(fx-1)$ is mapped to zero under this map - let $b \in (fx-1)$, say $b = b' \cdot (fx-1)$. Then

$$\begin{aligned}
\varphi(b) &= \varphi(b' \cdot (fx-1)) \\
&= \varphi(b') \cdot \varphi(fx-1) \\
&= \varphi(b') \cdot (\frac{f}{f} - 1) \\
&= \varphi(b') \cdot (1-1) \\
&= 0
\end{aligned}$$

Therefore, φ can be lowered to a surjective homomorphism from $R[x]/(fx-1)$ to $S^{-1}R$.

We now show that φ admits an inverse. We can construct the inverse function by again appealing to the universal property of the localization. The canonical map $R \to R[x]$ which identifies R with the scalars of $R[x]$, when composed with the quotient map $R[x] \to R[x]/(fx-1)$, gives a function $i : R \to R[x]/(fx-1)$ which maps every element f^n of S to a unit, since $x^n f^n = 1$ in this quotient ring.

We therefore know that there is a unique map $g : S^{-1}R \to R[x]/(fx-1)$ such that $g \circ j = i$. In particular, g maps an element $\frac{r}{1}$ to r, and it maps the element $\frac{1}{f}$ to f^{-1}, which in the ring $R[x]/(fx-1)$ is equal to x.

So, calculating the value of $g \circ \varphi$ on arbitrary elements of R gives $g(\varphi(r)) = g(1/r) = r$, and on x, gives $g(\varphi(x)) = g(1/f) = x$. This determines the map $g \circ \varphi$ as the identity. Therefore, g is an inverse to φ, and the two rings are isomorphic. □

2 Extra Stuff

Problem 1. Dummit & Foote, 15.4.2: Let I be an ideal in a commutative ring R, let D be a multiplicatively closed subset of R with ring of fra ctions $S^{-1}R$, and let ${}^c({}^eI)$ be the satiration of I with respect to S.

(a) Prove that ${}^c({}^eI) = R$ if and only if ${}^eI = S^{-1}R$ if and only if $I \cap S \neq 0$.

(b) Prove that $I = {}^c({}^eI)$ is saturated with respect to S if and only if for every $s \in S$, if $sa \in I$ then $a \in I$.

(c) Prove that extension and contraction define inverse bijections between the ideals of R saturated with respect to S and the ideals of $S^{-1}R$.

(d) Let $I = (2x, 3y) \subset \mathbb{Z}[x, y]$. Show the saturation of I with respect to $\mathbb{Z} - \{0\}$ is (x, y).

Proof. Writing π for the canonical map $R \to S^{-1}R$, we note that ${}^cJ = \pi^{-1}J$ for any ideal J of $S^{-1}R$.

(a) ${}^c({}^eI) = R$ if and only if $\pi^{-1}({}^eR) = R$, if and only if the ideal ${}^eI$ contains the whole image $\pi(R)$ of the ring R. Since $\frac{1}{1} \in \pi(R)$, this occurs if and only if the ideal ${}^eI$ is the whole ring $S^{-1}R$.

In turn, ${}^eI = S^{-1}R$ if and only if ${}^eI$ contains $\frac{1}{1}$. Every element of ${}^eI$ may be written as $\frac{i}{s}$ for some elements $i \in I$, $s \in S$, so $\frac{1}{1} \in {}^eI$ if and only if $\frac{i}{s} = 1$ for some i, s, which occurs if and only if some $s \in S$ is also in I, i.e. iff they have nonempty intersection.

(b) I is saturated with respect to S iff $I = \pi^{-1}({}^eI)$. One inclusion $I \subset \pi^{-1}({}^eI)$ is immediate, since each $i \in I$ is the inverse image of $\frac{i}{1} \in {}^eI$. So we show that the reverse inclusion $\pi^{-1}({}^eI) \subset I$ holds if and only if, for every $s \in S$, if $sa \in I$ then $a \in I$.

Assume first that the condition holds, and let $a \in \pi^{-1}({}^eI)$ be arbitrary. Then $\pi(a) = \frac{a}{1}$ may be written as $\frac{i}{s}$ for some s in S, $i \in I$. This means that $s'sa = s'i$ for some $s' \in S$. But since $s'i \in I$, our condition implies that $a \in I$.

On the other hand, assume that $\pi^{-1}({}^eI) \subset I$, and let $s \in S$, $a \in R$ be arbitrary elements such that $sa \in I$. Then the element $\pi(a) = \frac{a}{1} = \frac{sa}{s} \in {}^eI$. This means that $a \in \pi^{-1}({}^eI)$, which by assumption means that $a \in I$.

(c) This follows quickly from the observation that an ideal of R being saturated means that ${}^c({}^eI) = I$, and that (as shown in D&F), for every ideal J of $S^{-1}R$, ${}^e({}^cJ) = J$. Restricted to these domains, ^{c} and ^{e} are inverses, and therefore form a bijection.

(d) We show first that any element a of (x, y) may be written as $\frac{a}{1} = \frac{i}{z}$, where $i \in (2x, 3y)$ and $z \in \mathbb{Z}$. Let $\sum_{i+j\geq 1} a_{ij}x^iy^j$ be an element of (x, y), and let I, J be the maximum values of i, j respectively, such that $a_{ij} \neq 0$. Then

$$\begin{aligned}\frac{a}{1} &= \frac{\sum_{i+j\geq 1} a_{ij}x^iy^j}{1} \\ &= \frac{\sum_{i+j\geq 1}(2^I3^J)a_{ij}x^iy^j}{2^I3^J} \\ &= \frac{\sum_{i+j\geq 1} 2^{I-i}3^{J-j}a_{ij}(2x)^i(2y)^j}{2^I3^J} \in {}^eI\end{aligned}$$

We now show that no element r of $\mathbb{Z}[x, y]\backslash(x, y)$ may be written as $\frac{r}{1} = \frac{i}{s}$ for $i \in I$, $s \in \mathbb{Z}$. Since $\mathbb{Z}[x, y]\backslash(x, y)$ is simply the scalars $\mathbb{Z}$, this is an argument by minimal degree - this can only occur if $ss'r = s'i$ for nonzero $s' \in \mathbb{Z}$. The degree of i is at least 1, s' is nonzero, and $\mathbb{Z}$ is an integral domain, so the degree of $s'r$ is at least 1. Therefore the degree of $ss'r$ is at least 1, and because s and s' are both integers, the degree of r is at least 1 and it cannot be a scalar.

□

Problem 2. Dummit & Foote, 7.5.5: If F is a field, prove that the field of fractions of $F[[x]]$ is the ring $F((x))$ of formal Laurent series. Show the field of fractions of the ring $\mathbb{Z}[[x]]$ is *properly* contained in the field of Laurent series $\mathbb{Q}((x))$.

Proof. We construct a homomorphism $\varphi : F[[x]] \to F((x))$, and show that it is both injective and surjective. Let $g, h \in F[[x]]$ be formal power series, with coefficients g_n and h_n, and let h_n be nonzero:

$$g = \sum_n g_nx^n$$
$$h = \sum_n h_nx^n$$

The element g/h is a generic element of the field of fractions of $F[[x]]$. We want $\varphi(g/h)$ to be "g/h" in some reasonable way. It is possible to divide formal power series using the formula for $1/h$, which gives a well-defined power series as long as the zero-th coefficient h_0 is nonzero (D&F, Exercise 7.2.3). This is not necessarily true for our h, but it has at least one nonzero coefficient; let $h = x^ih'$, where i is the degree of the lowest nonzero term of h. Then $h^{-1} = x^{-i}h'^{-1}$, which is a well-defined formal Laurent series.

We define φ's value on g/h as follows:

$$\varphi(g/h) = x^{-i} \cdot g \cdot h'^{-1} = gh^{-1}$$

This is a well-defined function; if $g_1/h_1 = g_2/h_2$, then $g_1h_2 = g_2h_1$, so

$$\begin{aligned}\varphi(g_1/h_1) &= g_1 \cdot h_1^{-1} \\ &= (h_2^{-1}h_2) \cdot g_1 \cdot h_1^{-1} \cdot (g_2g_2^{-1})^{-1} \\ &= h_2^{-1} \cdot (h_1g_2) \cdot (h_1g_2)^{-1}g_2 \\ &= h_2^{-1} \cdot g_2 \\ &= \varphi(g_2/h_2)\end{aligned}$$

It is also a ring homomorphism: φ takes 1 to 1, scalars factor out of the denominator, and $\varphi(g_1/h_1 + g_2/h_2) = \varphi(g_1/h_1) = \varphi(g_2/h_2)$ - we show this last one with a quick calculation:

$$\begin{aligned}\varphi\left(\frac{g_1}{h_1} + \frac{g_2}{h_2}\right) &= \varphi\left(\frac{g_1h_2 + g_2h_1}{h_1h_2}\right) \\ &= (g_1h_2 + g_2h_1) \cdot (h_1h_2)^{-1} \\ &= g_1h_2 \cdot (h_1h_2)^{-1} + g_2h_1 \cdot (h_1h_2)^{-1} \\ &= g_1h_1^{-1} + g_2h_2^{-1} \\ &= \varphi\left(\frac{g_1}{h_1}\right) + \varphi\left(\frac{g_2}{h_2}\right)\end{aligned}$$

We can also see that φ is injective, but I won't run through the proof here; we finally see that it is surjective, as if $\sum_{i=-n}^{\infty} g_i x^i$ is a formal Laurent series, then $g = x^{-n}g'$, where g' is a formal power series with no terms with negative exponents, and $g = \varphi(g'/x^i)$.

It is *not* true that, for a more general ring, the ring of fractions of its polynomial ring is equal to the ring of Laurent series over that ring's field of fractions: for example, the field of fractions of $\mathbb{Z}[[x]]$ does not contain $\mathbb{Q}((x))$: the series $\sum_{n\geq 0} \frac{x^n}{n!}$ is not equal to the formal fraction of any two power series with coefficients in $\mathbb{Z}$, as the denominators of its terms grow too quickly (?) $\square$

Problem 3. Dummit & Foote, 7.4.30: Let I be an ideal of the commutative ring R. Prove that the radical of I is an ideal containing I, and that $(\text{rad } I)/I = \mathfrak{N}(R/I)$, the nilradical of R/I.

Proof. It is clear that the radical of I contains I, since for any $x \in I$, $x^1 \in I$. On the other hand, if f and g are members of the radical of I, say $f^n \in I$ and $g^m \in I$, then $f + g$ is also a member of the radical of I, because $(f+g)^{m+n} \in I$, because every term of $(f+g)^{m+n}$ either has f^n or g^m as a factor, and if f is a member of the radical of I with $f^n \in I$, and $a \in R$ is an arbitrary element, then $(fa)^n = f^n a^n \in R$. Therefore rad I is an ideal.

We now show that $(\text{rad } I)/I = \mathfrak{N}(R/I)$. If $f + I \in (\text{rad } I)/I$, then $(f+I)^n = f^n + I = 0$, so $f + I$ is in the nilradical of R/I. On the other hand, if $f + I$ is in the nilradical, then $(f+I)^n = f^n + I = 0$, meaning that $f^n \in I$ for some n, so $f + I$ is in $\text{rad}(I) + I$. So indeed the two ideals are equal. $\square$