Homework 4

Andrew Tindall
Algebra II

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1 Problems

Problem 1. For each of the following, give specific rings $R \subset S$ and explicit ideals in these rings that exhibit the specified relation:

(a) An ideal I of R such that $I \neq SI \cap R$ - so the contraction of the extension of an ideal I need not equal I.

(b) A prime ideal P of R such that there is no prime ideal Q of S with $P = Q \cap R$

(c) A maximal ideal M of S such that $M \cap R$ is not maximal in R

(d) A prime ideal P of R whose extension PS to S is not a prime ideal in S

(e) An ideal J of S such that $J \neq (J \cap R)S$ - so the extension of the contraction of an ideal J need not equal J.

Proof. (a) Say R is any integral domain, and S is its field of fractions. Then $R \subset S$, and for any nontrivial ideal $I \subset R$ (that is, $I \neq \{0\}$ and $I \neq R$), the extension IS is equal to S; so $IS \cap R = S \cap R = R$. Specific examples abound: take $R = \mathbb{Z}$, $S = \mathbb{Q}$, $I = 2\mathbb{Z}$. Then $IS \cap R = \mathbb{Z}$.

(b) By the above, say I is any nontrivial prime ideal in an integral domain R. Then I cannot be the contraction of any ideal in the field of fractions of R, because there are very few ideals in this field to begin with.

In fact, if $R \subset S$, and $I \subset R$ is prime, then I is the contraction of a prime ideal J if and only if $IS \cap R = J$. (The proof I found of this theorem relies on some subtler facts about how localizations behave under homomorphisms).

For a specific example, again let $R = \mathbb{Z}$, $S = \mathbb{Q}$, and $I = 2\mathbb{Z}$. Then $I \neq J \cap R$ for any ideal $J \subset S$. In particular, no prime lies over I.

(c) Once again, the inclusion of an integral domain into its field of fractions provides an example. If S is the field of fractions of a domain R, then $0 \subset S$ is maximal, but $0 \subset R$ is not necessarily so. (It is maximal if and only if R is itself a field). In particular, 0 is maximal in $\mathbb{Q}$ but not in $\mathbb{Z}$.

(d) Once again! Let R be any integral domain, S its field of fractions, and I a nonzero prime ideal of R. Then $IS = S$, which is not prime. In particular, $2\mathbb{Z}\mathbb{Q} = \mathbb{Q}$.

(e) Let $R = k[x]$, $S = k[x, y]$, and $J = (x, y)$. Then the contraction of J is $(x) \subset k[x]$, and the extension of this ideal is $(x) \subset k[x, y]$, which is not equal to the original ideal (x, y). Any polynomial in y alone, for instance, is in (x, y) but not in (x).

$\square$

Problem 2. Prove that if $s_1, \ldots s_n \in S$ are integral over R, then the ring $R[s_1, \ldots s_n]$ is a finitely generated R-module.

Proof. We induct on n - looking at the chain of inclusions

$$R \hookrightarrow R[s_1] \hookrightarrow R[s_1, s_2] \hookrightarrow \cdots \hookrightarrow R[s_1, \ldots s_n] \hookrightarrow S,$$

and the fact that if s_i is integral over R then it is integral over $R[s_1, \ldots s_{i-1}]$, we will see inductively that each $R[s_1, \ldots s_i]$ is a finitely generated $R[s_1, \ldots s_{i-1}]$ module. Then, applying a result from a previous homework, we see that $R[s_1, \ldots s_n]$ must be a finitely generated R-module.

Now, the meat of this solution is that, if $s_i \in S$ is integral over $R' \subset S$, then $R'[s_i]$ is a finitely generated R'-module. Let s_i be a zero of the monic polynomial

$$f(x) = x^n + \sum_{j=0}^{n-1} r_j x^j,$$

With coefficients $r_i \in R'$. Then s_i satisfies the relation

$$s_i^n = -\sum_{j=0}^{n-1} r_j s_i^j.$$

We show that the elements $1, s_i, s_i^2, \ldots s_i^{n-1}$ generate $R'[s_i]$ as an R'-module. Any element x of $R'[s_i]$ may be written as a (possibly nonunique) finite sum $x = \sum_{j=0}^{m} r_j s_i^j$. If the highest term is s_i^m, with $m \geq n$, then we may rewrite this term as $r_m(s_i^n)s_i^{m-n}$, which, in the ring S, satisfies the relation

$$r_m(s_i^n)s_i^{m-n} = r_m \left(\sum_{j=0}^{n-1} r_j s^j \right) s^{m-n}.$$

The highest degree of s_i in this term is $m-1$, which is strictly less than m, so we have found a new representation of x as

$$x = \sum_{j=0}^{m-1} r_j s_i^j.$$

Repeating this process, we may continue until we have written x as a sum of terms $r_j s_i^j$, with j running from 0 to $n-1$. Thus, $R'[s_i]$ is generated as an R'-module by $\{1, s_i, \ldots s_i^{n-1}\}$.

Now, each subring $R[s_1, \ldots s_{i-1}, s_i]$ of S can be identified with $(R[s_1, \ldots s_{i-1}])[s_i]$. Since s_i is finitely generated over R, it is also finitely generated over $R[s_1, \ldots s_{i-1}]$, and by the above argument, it is finitely generated as an $R[s_1, \ldots s_{i-1}]$-algebra. Therefore we have a chain of subrings of S, each of which is finitely generated as a module over the last:

$$R \hookrightarrow R[s_i] \hookrightarrow R[s_1, s_2] \hookrightarrow \cdots \hookrightarrow R[s_1, \ldots s_n].$$

By an argument in a previous homework assignment, this implies that $R[s_1, \ldots s_n]$ is finitely generated as an R-module. $\square$

References