Final Exam

Andrew Tindall
Algebraic Geometry

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Problem 1. State the following:

(a) Definitions of a ringed space and abstract variety.

Proof. A *ringed space* is a pair $(X, \mathcal{O}_X)$ of a topological space X, along with a sheaf of rings $\mathcal{O}_X$ on X. More precisely, a sheaf of rings on a space is a correspondence $U \mapsto \mathcal{O}_X$ of a ring - usually assumed commutative - for each open set U of X, along with, for each inclusion $V \subset U$ of open sets, a (commutative) ring homomorphism $\rho_{U,V} : \mathcal{O}_U \to \mathcal{O}_V$. The maps must satisfy the following property: for any chain $W \subset V \subset U$ of open sets of X,

$$\rho_{W,V} \circ \varphi_{V,U} \equiv \rho_{W,U}.$$

We usually write $\rho_{U,V}(f) = f|_V$. The above requirements make $\mathcal{O}_X$ a presheaf. Sheafness is accomplished by the following requirement:

Take any open covering $\bigcup_i U_i = U$ of an open set. If we have a family of elements $f_i \in \mathcal{O}_{U_i}$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then there exists a unique $f \in \mathcal{O}_U$ such that $f|_{U_i} = f_i$ for all i.

A morphism of ringed spaces is a continuous map $\varphi : X \to Y$ and, for each open set V of Y, a map $\varphi(V)\mathcal{O}_{Y,V} \to \mathcal{O}_{X,\varphi^{-1}(V)}$, such that, for any $U \subset V$ of Y, the following square commutes:

$$\begin{array}{ccc} \mathcal{O}_{Y,V} & \xrightarrow{\varphi^*(V)} & \mathcal{O}_{X,\varphi^{-1}(V)} \\ \Big\downarrow{\scriptstyle \rho_{V,U}} & & \Big\downarrow{\scriptstyle \rho_{\varphi^{-1}(V),\varphi^{-1}(U)}} \\ \mathcal{O}_{Y,U} & \xrightarrow{\varphi^*(U)} & \mathcal{O}_{X,\varphi^{-1}(U)} \end{array}$$

An isomorphism of ringed spaces is a morphism with a two-sided inverse.

We also need the definition of the spectrum of a ring R, as a ringed space. The topological space $X = \mathrm{Spec}(R)$ is the set of prime ideals of R, with the Zariski topology, which is generated by the distinguished opens $D(f)$; the sheaf of rings $\mathcal{O}_X$ is defined on distinguished opens by $\mathcal{O}_{X,D(f)} = R_f$, the localization of R at f. This behaves well with respect to colimits, so it extends to a definition a generic open set U of X by covering it with distinguished opens $D(f_i)$ and taking the colimit $\lim_{\to} D(f_i)$ of the cover. The upshot is that this is a ringed space, where the original ring can be recovered by $\Gamma(X, \mathcal{O}_X) = R$.

Now, we define an abstract variety. First, the definition of an abstract *prevariety*: a ringed space $(X, \mathcal{O}_X)$ is an abstract prevariety if it is quasi-compact - every open cover has a finite subcover - and if, for each $x \in X$, there is some open set U of X containing x, such that $(U, \mathcal{O}_{X,U})$ is isomorphic (as a ringed space) to the spectrum of some ring R.

An abstract variety is an abstract prevariety which satisfies the following separation axiom: For If Z is an affine variety, and φ_1, φ_2 are morphisms $Z \to X$, then the set $\{z \in Z \mid \varphi_1(z) = \varphi_2(z)\}$ (the preimage $(\varphi_1 \times \varphi_2)(\Delta)$ of the diagonal of $X \times X$) is a closed subset of Z. $\square$

(b) Hilbert's theorem on Hilbert polynomial and degree, and the BKK theorem on the number of solutions of a system of Laurent polynomial equations.

Proof. First, we state Hilbert's theorem. Let $X \subset \mathbb{P}^N$ be a projective variety with corresponding ideal I, and homogenous coordinate ring $k[X] = k[x_0, \ldots, x_N]/I$. Because I is homogeneous, it is graded, and so this is a graded ring:

$$k[X] = \bigoplus_{m \geq 0} k[X]_m$$

Where each component $k[X]_m$ is defined to be $k[x_0, \ldots, x_N]_m$ mod I: the subspace generated by the monomials in the x_i which have degree m, modulo the ideal I. The quantity $H_X(m)$ is defined as the dimension of the component $k[X]_m$ as a k-vector space:

$$H_X(m) = \dim_k(k[X]_m)$$

This is known as the Hilbert polynomial, because it is in fact polynomial in m: the first conclusion of Hilbert's theorem is that there is a unique polynomial $P_X(t)$ such that $P_X(m) = \dim_k k[V]_m$ for all m sufficiently large.

Next, Hilbert's Theorem states that the degree of P_X is equal to the dimension of X:

$$\deg(P_X(m)) = \dim(X)$$

Finally, Hilbert's theorem states that, if a_r is the leading coefficient of $P_X(m)$, then $\deg(X) = r! a_r$, where the degree $\deg(X)$ of the variety X is equal to $|X \cap L|$, for L a generic plane in $\mathbb{P}^N$ whose dimension is equal to the dimension of X.

Next, we state the BKK theorem. Let A be a finite set of elements of $\mathbb{Z}^n$, so that

$$A = \{\alpha_0, \ldots, \alpha_N\} \subset \mathbb{Z}^n$$

We recall that the definition of x^α, for $x = \langle x_1, \ldots, x_n \rangle$ and $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle$, is that

$$x^\alpha := x_1^{\alpha_1} \cdot \cdots \cdot x_n^{\alpha_n}.$$

Now, let $\mathcal{L}_A$ be the span $\{x^{\alpha_0}, \ldots, x^{\alpha_N}\}$, in the k-algebra of Laurent polynomials in the variables $x_1, \ldots, x_n$. Since each α_i is distinct, the dimension of $\mathcal{L}_A$ as a k-algebra is $|A|$, which is $N+1$.

Let $(f_1, \ldots, f_n)$ be a generic element of $\mathcal{L}_A \times \cdots \times \mathcal{L}_A$, the product of n copies of $\mathcal{L}_A$. then the number of points of $(k \backslash 0)^n$ which satisfy each polynomial $f_1, \ldots, f_n$ simultaneously is *independent* of the (generic) choice of $f_1, \ldots, f_n$, and is equal to $n!\mathrm{vol}_n(\mathrm{conv}(A))$, where $\mathrm{vol}_n(\mathrm{conv}(A))$ is Euclidean n-volume of the convex hull $\mathrm{conv}(A) \subset \mathbb{R}^n$ containing the points in A. □

Problem 2. Let $q_0, \ldots, q_N$ be positive integers with $\gcd(q_0, \ldots, q_N) = 1$. Define the *weighted projective space* $\mathbb{P}(q_0, \ldots, q_N)$ (as a set) by:

$$\mathbb{P}(q_0, \ldots, q_N) = (\mathbb{C}^{N+1} \backslash \{0\}) / \sim,$$

where $\sim$ is the equivalence relation given by:

$$(x_0, \ldots, x_N) \sim (y_0, \ldots, y_N) \iff \exists 0 \neq \lambda \in \mathbf{k}, x_i = \lambda^{q_i} y_i, \forall i = 0, \ldots, N.$$

Similarly to the projective space, we denote the equivalence class of $(x_0, \ldots, x_N)$ by $(x_0 : \cdots : x_N)$.

Consider the weighted projective space $\mathbb{P}(1,1,2)$ and the map $\Phi : \mathbb{P}(1,1,2) \to \mathbb{P}^3$ given by:

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0 x_1 : x_1^2 : x_2).$$

(a) Show that this map is one-to-one and its image $X = \mathrm{Im}(\Phi)$ is a closed subvariety. Find the defining homogeneous equation of $X \subset \mathbb{P}^3$.

Proof. We note that the usual "rescaling" that one can do in $\mathbb{P}^n$ can be done here as well, with some slight changes. If a coordinate x_i is nonzero, then

$$(x_0 : \cdots : x_i : \cdots : x_n) \sim ((x_i)^{-\alpha_0/\alpha_i} x_0 : \cdots : 1 : \cdots : (x_i)^{-\alpha_0/\alpha_i}).$$

Since we are working in the algebraically closed field $\mathbb{C}$, we do not have trouble with fractional exponents, except that there is not necessarily a unique choice for x_i^{1/α_i}. This is not any trouble in the following proofs, although we have to be careful not to assume that the coordinates of x are unique after renormalizing.

To begin with, we show that the map is one-to-one. Let $(x_0 : x_1 : x_2)$ and $(x_0' : x_1' : x_2')$ be two points which map to the same point (y_0, y_1, y_2, y_3) under Φ. First, assume that $y_3 \neq 0$, and rescale so that $y_3 = x_2 = x_2' = 1$.

We know that $x_0^2 = x_0'^2$, so $x_0 = \pm x_0'$, and similarly $x_1 = \pm x_1'$. Also, if $x_0 = -x_0$, then $x_0 x_1 = x_0' x_1'$ implies that $x_1 = -x_1'$ as well. So either $(x_0, x_1) = (x_0', x_1')$ or $(-x_0', -x_1')$.

But in fact these two cases both correspond to the same point in $\mathbb{P}(1,1,2)$:

$$\begin{aligned}(-x_0 : -x_1 : 1) &= (-1(-x_0) : -1(-x_0) : (-1)^2) \\ &= (x_0 : x_1 : 1)\end{aligned}$$

So, the two points are equal.

If $y_3 = 0$, then both x_2 and x_2' are zero as well. At least one of y_0, y_2 must be nonzero, since if they were both zero, then so would y_1. So, say $y_2 \neq 0$. Rescale so that $x_1^2 = x_1'^2 = 1$.

Again, by $x_0^2 = x_0'^2$ and $x_1^2 = x_1'^2$, along with $x_0x_1 = x_0'x_1'$, we can deduce that $(x_0, x_1) = (\pm x_0', \pm x_1')$,and in both cases we have $(x_0 : x_1 : 0) = (x_0' : x_1' : 0)$. By a symmetric argument for $y_1 \neq 0$, we have exhausted all cases. So indeed, any two points in $\mathbb{P}(1, 1, 2)$ that map to the same point under Φ must be equal, and so the map is one-to-one.

Next, we want to see that $\mathrm{Im}(\Phi)$ is a closed subvariety. We will see that the homogeneous polynomial

$$f(y_0, y_1, y_2, y_3) = y_0y_2 - y_1^2$$

is the defining equation of $\mathrm{Im}(\Phi)$. First, we see that the equation is 0 for any point $\Phi(x_0 : x_1 : x_2)$:

$$\begin{aligned} f(\Phi(x_0 : x_1 : x_2)) &= f(x_0^2 : x_0x_1 : x_1^2 : x_2) \\ &= x_0^2x_1^2 - (x_0x_1)^2 \\ &= 0 \end{aligned}$$

Now, let $y = (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^3$ be a point such that $f(y) = 0$. Pick some square root x_0 of y_0, so that $x_0^2 = y_0$. If $x_0 \neq 0$, then let $x_1 = y_1/x_0$. Otherwise, let x_1 be an arbitrary square root of y_2 and let $x_2 = y_3$. First, we see that $(x_0 : x_1 : x_2)$ is a valid point of $\mathbb{P}(1, 1, 2)$, because if all three of x_0, x_1, x_2 are zero, then $y_0 = x^2$ is zero, as is $y_2 = x_1^2$, and so is y_1, by $y_1^2 = y_0^2y_2^2$, and $y_3 = x_2 = 0$. But at least one coordinate of y must be nonzero, so x must be nonzero as well.

Finally, we see that $\Phi(x_0 : x_1 : x_2) = (y_0 : y_1 : y_2 : y_3)$, which follows quickly from construction of $(x_0 : x_1 : x_2)$:

$$\begin{aligned} x_0^2 &= y_0 \\ x_0x_1 &= y_1 \\ x_1^2 &= y_2 \\ x_2 &= y_3 \end{aligned}$$

□

(b) Is X a smooth variety? Prove or disprove your claim. Hint: look at X in different affine charts in $\mathbb{P}^3$.

Proof. incomplete □

Problem 3. Analagous to the case of projective space $\mathbb{P}^3$, write the affine charts in the weighted projective space $\mathbb{P}(1, 1, 2)$ and the gluing maps between them to show that $\mathbb{P}(1, 1, 2)$ can be realized as an abstract pre-variety.

Proof. Let U_i be the set $x_i \neq 0 \subset \mathbb{P}(1,1,2)$. We will first consider the cases $i = 0, 1$, which are similar.

We see that the set U_0 is all those $(x_0 : x_1 : x_2)$ such that $x_0 \neq 0$. As noted above, we can renormalize by $x_0^{-\alpha_i/\alpha_0}$ to obtain coordinates $(1 : x_1 : x_2)$. Let $\varphi_0 : U_0 \to \mathbb{A}^2$ be defined by

$$\varphi_0(x_1, x_2) = (1 : x_1 : x_2).$$

which has an inverse morphism taking $(1 : x_1 : x_2)$ to (x_1, x_2). Similarly, we may define a map $\varphi_1 : A^2 \to U_1$ by

$$\varphi_1((x_0, x_2)) = (x_0 : 1 : x_2),$$

which has an inverse morphism taking $(x_0 : 1 : x_2) \mapsto (x_0, x_2)$. Both of these cases are effectively the same as in the projective case. However, the same map cannot be used for U_2, since $(x_0 : x_1 : 1)$ is the same point as $(-x_0 : -x_1 : 1)$, and there is no canonical choice of sign for the coordinates. Instead, let $X \subset \mathbb{A}^3$ be the variety defined by the ideal $(y_0 y_2 - y_1^2)$ of $k[y_0, y_1, y_2]$. We have a map $\varphi_2 : X \to \mathbb{A}^2$ be defined by

$$\varphi_2(y_0 : y_1 : y_2) = (y_1 : y_2 : 1).$$

This has an inverse morphism defined by taking

$$(x_0 : x_1 : 1) \to (x_0^2 : x_0 x_1 : x_1^2)$$

The sets U_i cover $\mathbb{P}(1,1,2)$, and the charts are all isomorphisms, so it remains to define the gluing maps, for intersections $U_i \cap U_j$.

First, for $U_0 \cap U_1$, we want to map the open subset $(\varphi_0^{-1}(\varphi_1(\mathbb{A}^2)) \subset \mathbb{A}^2$ to the open subset $(\varphi_1^{-1}(\varphi_0(\mathbb{A}^2))) \subset \mathbb{A}^2$. A point is in the first set if it is the image of a point (a, b) which is first mapped to $(a : 1 : b)$, which then must be an element of U_0, meaning that $a \neq 0$. Then, φ_0^{-1} takes this point to $(1/a, b/a^2)$. Since a is arbitrary besides being nonzero, and b is totally arbitrary, we see that the points of $(\varphi_0^{-1}(\varphi_1(\mathbb{A}^2))$ are exactly those (x, y) where $x \neq 0$. Similarly, tracing elements, we see that the points of $(\varphi_1^{-1}(\varphi_0(\mathbb{A}^2))$ are also those (x, y) where $x \neq 0$. So, the map $f : (x, y) \mapsto (1/x, y/x^2)$ is a well-defined regular map in both directions: we can set our gluing map $g_{ij} = g_{ji} = f$.

We do need to check that these maps give $\varphi_0 \circ g_{1,0} = \varphi_1$, and vice versa. This can be seen by calculation:

$$\begin{aligned} \varphi_0(g_{1,0}(x, y)) &= \varphi_0(x, y) \\ &= (1 : 1/x : y/x^2) \\ &= (x : 1 : y) \qquad\qquad = \varphi_1(x, y) \end{aligned}$$

And, symmetrically, $\varphi_1 \circ g_{0,1} = \varphi_0$. Also, we see that $f \circ f = \mathrm{Id}$.

Now, we want to define the gluing maps g_{2i} and g_{i2}. We will define only the one for $i = 0$, since the other case is very similar.

The gluing maps should be isomorphisms between the open set $\varphi_2^{-1}(\varphi_0(\mathbb{A}^2))$ and $\varphi_0^{-1}(\varphi_2(X))$. An element of the first set is of the form $(1/y, x/y, x^2/y)$, where y is some nonzero value, and an element of the second set is of the form $(y^2/(xy), 1/(xy)^2)$, for some nonzero xy.

So, we can define the gluing map $g_{2,0}$ on elements as

$$(x^2, xy, y^2) \mapsto (xy/x^2, 1/x^2)$$

and the gluing map $g_{0,2}$ on elements as

$$(a, b) \mapsto (1/(ab), 1/b, a/b).$$

These maps are indeed inverses, and satisfy $\varphi_0 \circ g_{2,0} = \varphi_2$, and $\varphi_2 \circ g_{0,2} = \varphi_0$. $\square$

Problem 4. Find the Hilbert polynomial of $X = \mathbb{P}^n \times \mathbb{P}^m$ embedded in $\mathbb{P}^{(n+1)(m+1)-1}$ via the Segre embedding. Find the degree of X in $\mathbb{P}^{(n+1)(m+1)-1}$.

Proof. We recall that the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^n$ in $\mathbb{P}^{(n+1)(m+1)-1}$ is defined by sending

$$(a_0, \ldots, a_n) \times (b_0, \ldots, b_m) \mapsto (a_0 b_0, \ldots, a_i bj, \ldots a_n b_m$$

Where the $a_i b_j$ are arranged in lexographic order on (i, j). The degree-t monomials $a_i b_j$ are exactly determined by a choice of t a_is and t b_js, where order does not matter, and repetitions are allowed. The way to choose t elements fom a set of k, with repetitions, is exactly equal to $\binom{k+t-1}{t}$. We have two choices of t elements, from a set of $n+1$ a_is and $m+1$ b_js, so our total number of choices is $\binom{n+t}{t}\binom{m+t}{t}$. Any two of these monomials are unequal in the coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m \subset \mathbb{P}^{(n+1)(m+1)-1}$, so the dimension of the degree t component of the coordinate ring is exactly equal to $\binom{n+t}{t}\binom{m+t}{t}$. So, the Hilbert polynomial is

$$H(t) = \binom{n+t}{t}\binom{m+t}{t}.$$

To clean this up, we first use the identity $\binom{a}{b} = \binom{a}{a-b}$ to rewrite as

$$H(t) = \binom{n+t}{n}\binom{m+t}{m}.$$

We also want to use the identity $\frac{1}{n!m!} = \binom{n+m}{n}\frac{1}{(n+m)!}$.
Now, writing $(x)_k = x(x-1)(x-2)\cdots(x-k+1)$, we have

$$\begin{aligned} H(t) &= \binom{n+t}{n}\binom{m+t}{m} \\ &= \frac{(n+t)_n}{n!}\frac{(m+t)_m}{m!} \\ &= \binom{n+m}{n}\frac{1}{(n+m)!}(n+t)_n(m+t)_m \qquad = \binom{n+m}{n}\frac{1}{(n+m)!}((t+n)(t+n-1)\cdots(t+1)t)((t+ \\ &= \binom{n+m}{n}\frac{1}{(n+m)!}(t^n+\cdots)(t^m+\cdots) \\ &= \binom{n+m}{n}\frac{1}{(n+m)!}(t^{n+m}+\cdots) \end{aligned}$$

Since the leading coefficient of $H(t)$ is $\deg(\mathbb{P}^n \times \mathbb{P}^m)/\dim(\mathbb{P}^n \times \mathbb{P}^m)!$, we see that $\deg(\mathbb{P}^n \times \mathbb{P}^m)$, for this embedding of $\mathbb{P}^n \times \mathbb{P}^m$ in $\mathbb{P}^{(n+1)(m+1)-1}$, is equal to $\binom{n+m}{n}$. $\square$

Problem 5. Consider the affine plane curve $X = V(y^2 - x^3) \subset \mathbb{A}^2$ with coordinate ring $\mathbf{k}[X] = \mathbf{k}[x, y]/(y^2 - x^3)$.

(a) Show that $f = y/x \in k[X]$ is integral over $k[X]$.

Proof. We want to show that y/x satisfies an integral polynomial in $k[X][f]$. In fact, it satisfies the equation $f^2 - x = 0$, since

$$\begin{aligned} f^2 - x &= \frac{y^2}{x^2} - x \\ &= \frac{x^3}{x^2} - x \\ &= x - x \\ &= 0 \end{aligned}$$

Since y/x is not an element of $k[x, y]/(x^2 - y^3) = k[X]$, the ring is not normal, and neither is the variety X. $\square$

(b) Use f to construct the normalization $\tilde{X} \subset \mathbb{A}^3$ and the normalization map $\pi : \tilde{X} \to X$ (verify that the $\tilde{X}$ you constructed is actually a normal variety.)

Proof. We want to extend our variety into a third dimension, by adjoining t: $\tilde{X}$ is the subset of $\mathbb{A}^3$ consisting of points (x, y, t) which satisfy $x^3 = y^2$ and $t^2 = x$. First, we see that the variety is normal: its coordinate ring $k[\tilde{X}]$ is $k[x, y, t]/(x^3 - y^2, t^2 - x)$. This is isomorphic to $k[t]$, since we can write any element of $k[\tilde{X}]$ in terms of t alone by the identities $y = t^3$ and $x = t^2$. We know that $k[t]$ is normal, since it is a principal ideal domain, so $\tilde{X}$ is normal.

Now, we show that the map $\pi : \tilde{X} \to X$ defined by taking (x, y, t) to (x, y) is a normalization - that is, it is a morphism which is finite and birational. It is birational by the map $X \backslash \{0\} \to \tilde{X}$ defined by taking (x, y) to $(x, y, x/y)$. The set $X \backslash \{0\}$ is open and dense in X, so π is a birational equivalence.

Next, we want to show that π is finite, i.e. that the induced map $k[X] \to k[\tilde{X}]$ makes $k[\tilde{X}]$ into a finitely generated $k[X]$ module. Since $k[\tilde{X}]$ is generated by $\{t^i\}_{i \geq 0}$, it suffices to show that a finite number of these monomials generate the whole set. And in fact they do: for any $i \neq 1$, we can write i as $2a + 3b$ for some nonnegative a, b, and so

$$\begin{aligned} t^i &= (t^{2a})(t^{3b}) \\ &= x^a y^b \end{aligned}$$

By replacing each t^i, for $i \neq 1$, with a monomial in x and y, any $f \in k[\tilde{X}]$ can be written as $g(x, y) + kt$ for some $g(x, y) \in k[X]$. Thus the coordinate ring of $\tilde{X}$ is finitely genrated over the coordinate ring of X, and the map π must be finite. As a finite birational morphism from a normal variety to X, it is the normalization of X. $\square$

Problem 6. Assume we know the following fact: intersection of a projective variety $X \subset \mathbb{P}^N$ with a hyperplane H in general position has dimension $\dim(X) - 1$ (this is a corollary of the well-known Krull's principal ideal theorem of Hauptidealsatz). Prove the following:

(a) Suppose $L \subset \mathbb{P}^N$ is a plane in general position with $\operatorname{codim}(L) > \dim(X)$. Show that $X \cap L$ is empty.

Proof. We recall that an arbitrary plane $L \subset \mathbb{P}^N$ can be written as the intersection of $\operatorname{codim}(L)$-many hyperplanes H, or equivalently as $L' \cap H$, for L' a plane with $\operatorname{codim}(L') = \operatorname{codim}(L) - 1$, and H a hyperplane.

We proceed by induction on X: if X has dimension 0, then $\operatorname{codim}(L) > 1$, and $L = L' \cap H$ for some hyperplane H. But if $L \cap X$ is nonempty, we can make the following calculation:

$$\begin{aligned} \dim(L \cap X) &= \dim(L' \cap (H \cap X)) \\ &\leq \dim(H \cap X) \\ &= \dim(X) - 1 \\ &= -1 \end{aligned}$$

So in fact the set $L \cap X$ must be empty.

Now, we make the inductive hypothesis that $L \cap X = \emptyset$ for all $\dim(X) = n - 1$ and $\operatorname{codim}(L) > \dim(X)$. Let X be a space of dimension n, and let L be a plane in general postition with $\operatorname{codim}(L) > \dim(X)$. Then $L = L' \cap H$ for some plane L' with $\operatorname{codim}(L') = \operatorname{codim}(L) - 1$ and some hyperplane H in general position. Rewriting $L \cap X$ as $L' \cap (H \cap X)$, we see that $\dim(H \cap X)$ has dimension $\dim(X) - 1$, and $\operatorname{codim}(L) = \operatorname{codim}(L) - 1 > \dim(X) - 1$. So, by the inductive hypothesis, the set $L \cap X$ is empty. By induction, the theorem holds for any projective variety $X \subset \mathbb{P}^N$. ☐

(b) If $U \subset X$ is a nonempty open subset then $\deg(U) = \deg(X)$. Recall that degree is the number of intersection points with a plane of complementary dimension.

Proof. incomplete ☐