Homework 5

Andrew Tindall
Analysis I

October 11, 2019

Problem 1. Let Λ be a metric space, E a Banach space, and let $F : \Lambda \times E \to E$ be a function such that

$$\exists \kappa < 1 \forall \lambda \in \Lambda \forall x, y \in E \; \|F(\lambda, x) - F(\lambda, y)\| \leq \kappa \|x - y\| .$$

Prove that:

(i) For every $\lambda \in \Lambda$ there exists a unique $x(\lambda) \in E$ such that $x(\lambda) = F(\lambda, x(\lambda))$,

(ii) For every $\lambda \in \Lambda, y \in E$, one has:

$$\|x(\lambda) - F(\lambda, y)\| \leq \frac{\kappa}{1-\kappa} \|y - F(\lambda, y)\|$$

$$\|y - x(\lambda)\| \leq \frac{1}{1-\kappa} \|y - F(\lambda, y)\|$$

Proof. (i) We show that for each λ, the function $F_\lambda : x \mapsto F(\lambda, x)$ has a unique fixed point $x(\lambda)$. Fix an arbitrary λ; then for any $x, y \in E$,

$$\|F_\lambda(x) - F_\lambda(y)\| \leq k \|x - y\| < \frac{k+1}{2} \|x - y\| .$$

Because $\frac{k+1}{2} < 1$, this makes F_λ a contraction mapping. E is Banach, so it is in particular complete, so the Brouwer fixed-point theorem gives a unique $x(\lambda)$ such that $F_\lambda(x(\lambda)) = x(\lambda)$. Because λ was arbitrary, this holds for every $\lambda \; in \Lambda$.

(ii) Using the fixed-point equality $x(\lambda) = F_\lambda(x(\lambda))$, and the existence of the almost-contraction-constant κ, we see:

$$\begin{aligned} \|x(\lambda) - F_\lambda(y)\| &= \|F_\lambda(x(\lambda)) - F_\lambda(y)\| \\ &\leq \kappa \|x(\lambda) - y\| \\ &\leq \kappa(\|y - F_\lambda(y)\| + \|x(\lambda) - F_\lambda(y)\|) \\ &= \kappa(\|y - F_\lambda(y)\| + \|F_\lambda(x(\lambda)) - F_\lambda(y)\|) \end{aligned}$$

Therefore,

$$\|F_\lambda(x(\lambda)) - F_\lambda(y)\| \le \kappa(\|y - F_\lambda(y)\| + \|F_\lambda(x(\lambda)) - F_\lambda(y)\|),$$

Or, rearranging,

$$\|F_\lambda(x(\lambda)) - F_\lambda(y)\| \le \frac{\kappa}{1-\kappa} \|y - F_\lambda(y)\| ,$$

which was to be shown.

Now, we show the second inequality to be true. Again using the constant κ and the fixed-point $x(\lambda)$,

$$\begin{aligned} \|y - x(\lambda)\| &\le \|y - F_\lambda(y)\| + \|F_\lambda(y) - x(\lambda)\| \\ &= \|y - F_\lambda(y)\| + \|F_\lambda(y) - F_\lambda(x(\lambda))\| \\ &\le \|y - F_\lambda(y)\| + \kappa \|y - x(\lambda)\| \end{aligned}$$

Which, rearranging, gives us

$$\|y - x(\lambda)\| \le \frac{1}{1+\kappa} \|y - F_\lambda(y)\| ,$$

which was to be shown.

□

Problem 2. Let $f_1 : E \to F_1$ and $f_2 : E \to F_2$ be two differentiable mappings between Banach spaces E, F_1, F_2. Define $f : E \to F_1 \times F_2$ by $f(x) = (f_1(x), f_2(x))$. Prove that f is differentiable and find its derivative. (Here $F_1 \times F_2$ is the Banach space equipped with the norm $\|(y_1, y_2)\| = \|y_1\|_{F_1} + \|y_2\|_{F_2}$).

Proof. We show that, for any $x \in E$, the map $Df : x \mapsto (Df_1(x), Df_2(x))$ is linear, and that it satisfies the definition of the derivative,

$$\lim_{\|h\|\to 0} \frac{f(x+h) - f(x) - Df(x)h}{\|h\|} = 0.$$

Expanding the functions f and Df by their definitions, the above limit is the same as

$$\lim_{\|h\|\to 0} \frac{(f_1(x+h), f_2(x+h)) - (f_1(x), f_2(x)) + (Df_1(x)h, Df_2(x)h)}{\|h\|}.$$

Now, using the componentwise definition of addition in the vector space $F_1 \times F_2$, this limit is equal to the following:

$$\lim_{\|h\|\to 0} \left(\frac{f_1(x+h) - f_1(x) + Df_1(x)h}{\|h\|}, \frac{f_2(x+h) - f_2(x) + Df_2(x)h}{\|h\|} \right);$$

The inner two limits exist and are equal to zero by the fact that f_1 and f_2 are differentiable. We finally need only see that convergence to 0 in $F_1 \times F_2$ under the given norm is equivalent to componentwise convergence to 0 in F_1 and F_2, and that the product of two linear functions into F_1 and F_2 is itself linear.

First, we show that componentwise convergence implies convergence. Let x^i be a sequence in $F_1 \times F_2$ such that the sequences of components x_1^i and x_2^i independently converge to 0. Let $\varepsilon > 0$. Then there exist N_1, N_2 such that for all $i > N_1$, $\|x_1^i\|_{F_1} < \varepsilon/2$, and for all $j > N_2$, $\left\|x_2^j\right\|_{F_2} < \varepsilon/2$. Then, for all $i > \max(N_1, N_2)$,

$$\begin{aligned} \left\|x^i\right\| &= \left\|x_1^i\right\|_{F_1} + \left\|x_2^i\right\|_{F_2} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

So the sequence x^i converges in $F_1 \times F_2$.

Now, we show that $Df(x)$ is a linear map for all x - more generally, that given any two linear maps $L_1 : E \to F_1$, and $L_2 : E \to F_2$, the product map $L_1 \times L_2 : E \to F_1 \times F_2$ is also linear.

- $(L_1 \times L_2)(\alpha x) = \alpha(L_1 \times L_2)(x)$:

$$\begin{aligned} (L_1 \times L_2)(\alpha x) &= (L_1(\alpha x), L_2(\alpha x)) \\ &= (\alpha L_1(x), \alpha L_2(x)) \\ &= \alpha(L_1(x), L_2(x)) \\ &= \alpha(L_1 \times L_2)(x) \end{aligned}$$

- $(L_1 \times L_2)(x + y) = (L_1 \times L_2)(x) + (L_1 \times L_2)(x)$:

$$\begin{aligned} (L_1 \times L_2)(x + y) &= (L_1(x + y), L_2(x + y)) \\ &= (L_1(x) + L_1(y), L_2(x) + L_2(y)) \\ &= (L_1(x), L_2(x)) + (L_1(y), L_2(y)) \\ &= (L_1 \times L_2)(x) + (L_1 \times L_2)(y) \end{aligned}$$

So the product of any two linear maps is linear; in particular, the map $Df(x) = Df_1(x) \times Df_2(x)$ is linear. Since it satisfies the limit definition of the derivative, it is the derivative of f at x.

Because f is differentiable at every point x of E, it is a differentiable map. Therefore, the product $f_1 \times f_2$ of any two differentiable maps into any two spaces is itself differentiable. □

Problem 3. Let E, F, G be normed spaces and let $\varphi : E \times F \to G$ be a bilinear map (i.e. such that both maps $\varphi(\cdot, y) : E \to G$ and $\varphi(x, \cdot) : E \to F$ are linear, for every $x \in E$ and $y \in F$).

(i) Prove that φ is continuous if and only if it is bounded; that is:

$$\exists C > 0 \forall x \in E \forall y \in F \;\; \|\varphi(x, y)\| \leq C \cdot \|x\| \cdot \|y\|,$$

(ii) Let $\mathcal{L}(E, F; G)$ be the linear space of all continuous bilinear mappings φ, as above. Prove that it is a normed space, with the norm defined as:

$$\|\varphi\| : \sup \{\|\varphi(x, y)\| ; \|x\| \leq 1, \|y\| \leq 1\}.$$

(iii) Prove that if G is Banach then $\mathcal{L}(E, F; G)$ is also Banach.

Proof. Some of the following ideas were found in the lecture notes [1].

(i) First, assume that φ is continuous. If it were not bounded, there would exist some sequence (x_n, y_n) of points in $E \times F$ whose norm under φ was unbounded - that, say, $\|\varphi(x_n, y_n)\| > n^2 \|x_n\| \cdot \|y_n\|$.

Now, we can use these points to construct a convergent sequence in $E \times F$ whose values under φ diverge, contradicting continuity. Define the following sequences:

$$\bar{x}_n = \frac{x_n}{n \|x_n\|} \text{ and } \bar{y}_n = \frac{y_n}{n \|y_n\|}$$

Then $\bar{x}_n \to 0$ and $\bar{y}_n \to 0$, but the value of $\|\varphi(\bar{x}_n, \bar{y}_n)\| > n^2 \frac{\|x_n\|}{n\|x_n\|} \cdot$ $frac\|y_n\|n \|y_n\|$, meaning the value of $\varphi(\bar{x}_n, \bar{y}_n)$ cannot converge to the value $0 = \varphi(0, 0)$. Thus φ cannot be continuous.

Now, We show the reverse implication. Let φ be a bounded bilinear mapping $E \times F \to G$ with bounding constant C, and let $x_n \to x$ and $y_n \to y$ be convergent sequences in E and F. We show that $\varphi(x_n, y_n) \to \varphi(x, y)$.

By convergence of the sequences x_n, y_n, there is some upper bound to the norms $\|x_n\|$ and $\|y_n\|$; say M. Then

$$\begin{aligned}
\|\varphi(x_n, y_n) - \varphi(x, y)\| &= \|\varphi(x_n, y_n) - \varphi(x_n, y) + \varphi(x_n, y) - \varphi(x, y)\| \\
&\leq \|\varphi(x_n, y_n) - \varphi(x_n, y)\| + \|\varphi(x_n, y) - \varphi(x, y)\| \\
&= \|\varphi(x_n, y_n - y)\| + \|\varphi(x_n - x, y)\| \\
&\leq C \cdot \|x_n\| \cdot \|y_n - y\| + C \cdot \|x_n - x\| \cdot \|y\| \\
&\leq CM \cdot \|x_n - x\| + CM \cdot \|y_n - y\|,
\end{aligned}$$

which converges to 0 because $\|x_n - x\|$ and $\|y_n - y\|$ do. Therefore φ is continuous.

(ii) We show that the function

$$\|\varphi\| = \sup \{\|\varphi(x, y)\|; \|x\| \leq 1, \|y\| \leq 1\},$$

defined on bounded bilinear maps $E \times F \to G$, is a norm. We first note that this function is equivalent to a supremum defined on the whole of $E \times F$:

$$\|\varphi\| = \sup \left\{ \frac{\|\varphi(x, y)\|}{\|x\| \cdot \|y\|}; x \in E, y \in F \right\}.$$

- $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$: This follows from the pointwise inequality

  $$\|\varphi(x, y) + \psi(x, y)\| \leq \|\varphi(x, y)\| + \|\psi(x, y)\|,$$

  which holds at all points (x, y) of $E \times F$ by the triangle inequality in G.

- $\|\alpha\varphi\| = |\alpha| \cdot \|\varphi\|$: This also follows pointwise, from the norm properties of G:

$$\begin{aligned}\|(\alpha\varphi)(x,y)\| &= \|\alpha(\varphi(x,y))\| \\ &= |\alpha| \cdot \|\varphi(x,y)\|\end{aligned}$$

- If $\|\varphi\| = 0$, then φ is the 0 function: if $\|\varphi\| = 0$, then $\|\varphi(x,y)\| = 0$ at each point (x,y), which implies that $\varphi(x,y) = 0$ for all (x,y).

(iii) Assume that G is Banach, and let $\{\varphi_n\}$ be a Cauchy sequence of functions in $\mathcal{L}(E,F;G)$, i.e. the norms $\|\varphi_n - \varphi_m\|$ go to zero. Because the norm in this space is the supremum of pointwise norms, then for any point (x,y), the norms $\|\varphi_n(x,y) - \varphi_m(x,y)\|$ must go to 0 as n,m go to ∞. Because G is Banach, the values $\varphi_n(x,y)$ must converge to a unique point, which we define as $\varphi(x,y)$. This gives a well-defined set-function $E \times F \to G$, and it is the only possible function that φ_n may converge to. We now show that it is a bounded bilinear function.

First, bilinearity. By symmetry in x and y, it suffices to show that $\varphi(\cdot,y)$ is linear on E for any fixed $y \in F$.

- $\varphi(\alpha x,y) = \alpha\varphi(x,y)$: by the definition of $\varphi(x,y)$, we see

$$\begin{aligned}\varphi(\alpha x,y) &= \lim_{n\to\infty} \varphi_n(\alpha x,y) \\ &= \lim_{n\to\infty} \alpha\varphi(x,y) \\ &= \alpha \lim_{n\to\infty} \varphi(x,y) \\ &= \alpha\varphi(x,y)\end{aligned}$$

- $\varphi(x_1+x_2,y) = \varphi(x_1,y) + \varphi(x_2,y)$:

$$\begin{aligned}\varphi(x_1+x_2,y) &= \lim_{n\to\infty} \varphi_n(x_1+x_2,y) \\ &= \lim_{n\to\infty} (\varphi_n(x_1,y) + \varphi_n(x_2,y)) \\ &= \left(\lim_{n\to\infty} \varphi_n(x_1,y)\right) + \left(\lim_{n\to\infty} (x_2,y)\right) \\ &= \varphi(x_1,y) + \varphi(x_2,y)\end{aligned}$$

Now, we show that φ is bounded: that there exists some global bound C such that, for any $(x,y) \in E \times F$,

$$\|\varphi(x,y)\| \leq C \cdot \|x\| \cdot \|y\|.$$

Because the functions φ_n converge in the norm on $\mathcal{L}(E,F;G)$, the bounds of each φ_n converge to some number C:

$$\lim_{n\to\infty} (\sup\{\|\varphi_n(x,y); \|x\| \leq 1, \|y\| \leq 1\|\}) = C < \infty.$$

We show that C is a bound for φ. Let $(x,y) \in E \times F$. Then

$$\|\varphi(x,y)\| = \left\|\lim_{n\to\infty} \varphi_n(x,y)\right\|$$

By continuity of the norm, we have

$$\begin{aligned}
\left\| \lim_{n\to\infty} \varphi_n(x,y) \right\| &= \lim_{n\to\infty} \|\varphi_n(x,y)\| \\
&= \frac{\lim_{n\to\infty} \|\varphi_n(x,y)\|}{\|x\|\cdot\|y\|} \cdot \|x\| \cdot \|y\| \\
&\le \lim_{n\to\infty} \sup \left\{ \frac{\|\varphi_n(x,y)\|}{\|x\|\cdot\|y\|} x \in E, y \in F \right\} \cdot \|x\| \cdot \|y\| \\
&= C \cdot \|x\| \cdot \|y\|
\end{aligned}$$

Thus, φ is an element of $\mathcal{L}(E,F;G)$. We have shown that any Cauchy sequence converges to an element of this normed vector space; thus it is Banach.

□

Problem 4. For $p \in (0,1)$ define l_p and $\|\cdot\|_{l_p}$ by the standard formula. Show that $\|\cdot\|_{l_p}$ is not a norm.

Proof. The definition we use is that l_p is the space of all sequences of real numbers $\{x_i\}$ such that

$$\sum_{i=1}^{\infty} |x_i|^p$$

is less than ∞, with the norm $\|x\|_{l_p}$ defined as

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$$

This function works well with scalars, and takes 0 to 0, but it does not satisfy the triangle inequality, because the function $x \mapsto |x|^p$ is not convex for $p \in (0,1)$. For example, let $p \in (0,1)$, and let x be the sequence $\{2^{1/p}, 0, 0, \dots\}$, and y the sequence $\{0, 2^{1/p}, 0, \dots\}$. Then $\|x\| = \|y\| = 2$, but $\|x+y\| = 4^{1/p}$, which is greater than $2+2$ for any $p \in (0,1)$. □

Problem 5. Give an example of a discontinuous linear map between normed spaces, so that:

(i) Its graph is closed and its target space is Banach,

(ii) Its graph is closed and its domain space is Banach.

Proof. (i) Let A be the space of real-valued smooth functions on $[0,1]$ with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$, let B be the Banach space $\mathbb{R}$, and let $T : A \to B$ be the map $f \mapsto f'(0)$. The derivative map is well-known to be linear, and its target space $\mathbb{R}$ is Banach. Further, the graph of T is closed: if $\{f_i\}$ is a sequence of smooth functions on $[0,1]$ which is Cauchy, and the sequence $\{f_i'(0)\}$ is also Cauchy, then the sequence $\{(f_i, f_i'(0))\}$ in $A \times B$ converges to the value $(f, f'(0))$, where f is the limit of the f_i in the space A.

However, the derivative-at-0 operator is not continuous. Let $\{f_n\}$ be the sequence of functions $f_n(x) = \frac{\sin(n^2 x)}{n}$. This sequence converges in the supremum norm to 0, but the derivative at 0 grows without bound - $f_n'(0) = n$.

(ii) Incomplete.

$\square$

¡++¿

References

[1] Vesely, Libor. Continuity of Bilinear Mappings. Università degli Studi di Milano, 2017.