Homework 2

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Topology II

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Problem 1. Write down all of the data producing a finite CW complex structure $X = \bigcup_n X_n$ for Σ_2.

Proof. We start with one 0-cell e^0, four 1-cells $e^1_1, e^1_2, e^1_3, e^1_4$, and one 2-cell e^2.

Our 0-Skeleton X^0 is the one-point space $e^0 = *$.

Next, the 1-skeleton X_1 is, setwise, the union

$$X_1 = \left(e^1_1 \amalg e^1_2 \amalg e^1_3 \amalg e^1_4\right) \amalg X_0,$$

Combined with gluing maps $\Phi^1_i : \partial D^1 \to X_0$ for each e^1_i. The boundary ∂D^1 can be identified with $S^0 = \{-1, 1\}$, the 0-sphere, which consists of only 2 points. The 0-skeleton has only one point, which we call $*$, so each boundary map Φ^1_i here is the trivial map which takes both -1 and 1 to $*$.

Now, the 2-skeleton X_2 is the union

$$X_2 = e^2 \amalg X_1$$

Combined with a gluing map $\Phi^2 : \partial D^2 \to X_1$. The boundary ∂D^2 of the 2-cell can be identified with the 1-sphere S^1: the set of points $\{(x, y); x^2 + y^2 = 1\}$. Reparameterizing this set as

$$\partial D^2 = \{(\cos(\theta), \sin(\theta); 0 \leq \theta < 2\pi\}$$

We can then construct a map from this set to the 1-skeleton $X_1 = \coprod_i e^1_i \amalg e^0$.

Let $\pi_i(x) : I^\circ \to e^1_i$ be the function taking $0 < x < 1$ to its image in the 1-cell e^1_i, and let $*(x)$ be the constant function $x \mapsto * \in e^0$.

$$\Phi((\cos(\theta), \sin(\theta))) = \begin{cases} \pi_1(4\theta/\pi) & 0 < \theta < \pi/4 \\ \pi_2(4(\theta - \pi/4)/\pi) & \pi/4 < \theta < \pi/2 \\ \pi_1(1 - 4(\theta - \pi/2)/\pi) & \pi/2 < \theta < 3\pi/4 \\ \pi_2(1 - 4(\theta - 3\pi/4)/\pi) & 3\pi/4 < \theta < \pi \\ \pi_3(4(\theta - \pi)/\pi) & \pi < \theta < 5\pi/4 \\ \pi_4(4(\theta - 5\pi/4)\pi) & 5\pi/4 < \theta < 3\pi/2 \\ \pi_3(1 - 4(\theta - 3\pi/2)/\pi) & 3\pi/2 < \theta < 7\pi/4 \\ \pi_4(1 - 4(\theta - 7\pi/4)/\pi) & 7\pi/4 < \theta < 2\pi \\ *(x) & x \equiv 0 \bmod (\pi/4) \end{cases}$$

This is a continuous map: it is a linear function in the parameter θ to the interval $e_i^1 \cong I$ on each domain $n\pi/4 < \theta < (n+1)\pi/4$, and its value at the boundaries of these intervals is $* \in e^0$, which has been identified with the boundaries of each e_i^1 via the gluing maps Φ_i^1. $\square$

Problem 2. Using the presentation of the Möbius band M as $I \times I/\sim$, where $\sim$ is the equivalence relation generated by

$$(x, 0) \sim (1 - x, 1), \text{ for } x \in I,$$

write down a deformation retract from M to the circle S^1.

Proof. Let $r : M \times I \to I$ be the map taking (x, y, a) to $(a(x - 1/2) + 1/2, y)$ - that is, leaving y fixed, and taking x to $a \cdot (x - 1/2) + 1/2$. For $a = 0$, this is the identity map. We also see that it is well-defined for other values of a, as it is constant on equivalence classes of $\sim$ - the only thing we need to check is that $r(x, 0, a) = r(1 - x, 1, a)$:

$$\begin{aligned} r(x, 0, a) = (a(x - 1/2) + 1/2, 0) \\ &= (ax - a/2 + 1/2, 0) \\ &= (1 - (ax - a/2 + 1/2), 1) \\ &= (a(-x + 1/2) + 1/2, 1) \\ &= (a((1 - x) - 1/2) + 1/2, 1) \\ &= r(1 - x, 1, a) \end{aligned}$$

Therefore, r is a well-defined function $M \times I \to I$. And, for $a = 1$, we see that r is a retraction of M to the subspace $\{(1/2, y); y \in I\} / \sim *$, where $\sim *$ is the restriction of $\sim$ to the space $x = 1/2$ - the only nontrivial equivalence being $(1/2, 0) \sim *(1/2, 1)$. This space is homeomorphic to S^1, and $r(\cdot, \cdot, 1)$ is constant on it, making it a retract of M to S^1.

Finally, because r is defined simply by multiplication in x, y, and a, it is continuous in all of these variables, except possibly at the glued edge of the mobius strip, $y = 0, 1$. However, it is continuous here as well - the inverse image of an open neighborhood of an element $[(x, 1)]$ of this edge is a stretched (by a factor of $1/a$) open neighborhood of $[(x/a, 1)]$.

As a continuous function $M \times I \to M$, which is equal to Id_M on $M \times \{0\}$, and to a retract to S^1 on $M \times \{1\}$, this is a deformation retract of the möbius band to the circle. $\square$