Homework 5

Andrew Tindall
Algebra II

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Problem 1. Dummit & Foote 15.1.3: Prove that the field $k(x)$ of rational functions over k in the variable x is not a finitely generated k-algebra.

Proof. The following proof is is adapted from a stackexchange comment at [1].

Say $k(x)$ were generated by finitely many rational functions $f_1, \dots f_n$. Any element of the k-algebra generated by these elements may be written as a polynomial in $f_1, \dots f_n$, with coefficients in k. Say y is such an element. Then y may also be written as a fraction $f(x)/(g(x))^n$, where f is some polynomial, g is the product of the denominators of the generators f_i, and n is the degree of y. However, there are infinitely many irreducibles in $k[x]$, and one of them, say $p(x)$, does not divide $g(x)$. Then the element $1/p(x)$ may not be written as a ratio f/g^n, and is therefore not in the algebra generated by $f_1, \dots f_n$. Thus we cannot find a finite set of generators of $k(x)$. □

Problem 2. Dummit & Foote, 15.1.6: Suppose that $0 \to M' \to M \to M'' \to 0$ is an exact sequence of R-modules. Prove that M is a Noetherian R-module if and only if M' and M'' are Noetherian R-modules.

Proof.

- M Noetherian $\Rightarrow M''$ Noetherian: let $I_1 \subset I_2 \subset I_3 \dots$ be an increasing chain of submodules in M''. Then their preimages in M are an increasing chain of submodules, and by surjectivity of the map $M \to M''$, the preimages are distinct if the submodules in M'' are distinct. The chain in M must stabilize by Noetherianness; so therefore so must the chain in M''.

- M Noetherian $\Rightarrow M'$ Noetherian: let $J_1 \subset J_2 \subset J_3 \dots$ be ain increasing chain of submodules in M'. Their images in M also form an increasing chain of submodules, which eventually stabilizes; because the map $M' \to M$ is injective, the equality of the image of two submodules means that those two submodules are equal, and thus that the chain in M' terminates as well. Therefore M' is Noetherian.

- M' and M'' Noetherian $\Rightarrow M$ Noetherian: Let $L_1 \subset L_2 \subset L_3 \dots$ be an increasing chain of submodules in M. Then the inverse images of each submodule forms a chain in M', and the images of each form a chain in M'', both of which eventually terminate by Noetherianness. The conclusion that the chain L_i itself terminates follows from the following bit of diagram chasing:

Let $\alpha : M' \to M$ be injective and $\beta : M \to M''$ be surjective, with the image of α equal to the kernel of β. Let $L_1 \subset L_2$ be two submodules of M such that $\alpha^{-1}(L_1) = \alpha^{-1}(L_2)$, and $\beta(L_1) = \beta(L_2)$. Then $L_1 = L_2$.

It suffices to show that $L_2 \subset L_1$. Let $l \in L_2$. Then $\beta(l) \in \beta(L_2) = \beta(L_1)$, so there exists some $l' \in L_1$ such that $\beta(l) = \beta(l')$. Therefore $\beta(l - l') = 0$, so $l - l'$ is in the kernel of β, which equals the image of α. Thus $l - l' = \alpha(m)$ for some element $m \in M'$. Now, because $L_1 \subset L_2$, both l and l' are in L_2, meaning that $l - l' \in L_2$, so $m \in \alpha^{-1}(L_2) = \alpha^{-1}(L_1)$. Therefore $\alpha(m) = l - l' \in L_1$. Because $l' \in L_1$, we see that l is in L_1, which was to be shown.

Thus, any chain of submodules $L_1 \subset L_2 \subset L_3 \dots$ such that $\alpha^{-1}(L_i) = \alpha^{-1}(L_{i+1})$ and $\beta(L_i) = \beta(L_{i+1})$ must stabilize, meaning M is Noetherian.

$\square$

References

[1] Prove that the field $k(x)$ of rational functions over k in the variable x is not a finitely generated k-algebra., URL (version: 2018-01-17): https://math.stackexchange.com/q/2608744