Homework 10

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Analysis I

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Problem 1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. For each $c \in \mathbb{R}$ denote the number of solution of the equation $f(x) = c$. Prove that the function $\mathbb{R} \to \mathbb{R}_+$ is Lebesgue measurable.

Proof. We show that the inverse $n^{-1}(a, b)$ of any open interval in $\mathbb{R}_+$ is Lebesgue measurable. Because the image of n is contained in $\mathbb{N} \cup \{\infty\}$, we need only look at the inverse images of finite sets of natural numbers $\{n_1, \ldots n_m\}$, infinite sets of the form $\{n \in \mathbb{N}; n_1 \le n < \infty\}$, and infinite sets of the form $\{n \in \mathbb{N}; n_1 \le n \le \infty\}$.

- In the case of a finite set, $n^{-1}\{n_1, \ldots n_2\}$ is the finite union
$$\bigcup_{n_1 \le i \le n_2} n^{-1}(\{i\}),$$
- In the case of an infinite set, not including ∞, $n^{-1}(n_1, \ldots)$ is equal to the countable union
$$\bigcup_{n_1 \le i} n^{-1}(\{i\}),$$
- In the case of an infinite set including ∞, $n^{-1}(\{n_1, \ldots\} \cup \{\infty\})$ is the countable union
$$\left(\bigcup_{n_1 \le i} n^{-1}(\{i\})\right) \cup n^{-1}(\{\infty\})$$

So, it suffices to show that the set $n^{-1}(i)$ is measurable, for any $i \in \mathbb{N} \cup \{\infty\}$.

First, let $i = 0$. The set n^{-1} is the set of all numbers y in $\mathbb{R}$ which have no solutions $f(x) = y$, for $x \in [0, 1]$. Because f is continuous with compact domain, it attains a maximum value M on $[0, 1]$, as well as a minimum value m. Because $m \le f(x) \le M$ for all $x \in [0, 1]$, $n^{-1}(0)$ contains the intervals $(-\infty, m)$ and (M, ∞). Further, by the intermediate value theorem, f attains all values in the interval $[m, M]$, so $n^{-1}(0)$ is dis joint from $[m, M]$. This means that
$$n^{-1}(0) = (-\infty, m) \cup (M, \infty),$$
which is measurable.

The rest of this proof is incomplete. ☐

Problem 2. Prove that every Lebesgue measurable function $f : [0, 1] \to \mathbb{R}$ is a limit almost everywhere of a sequence $\{f_n\}$ of continuous functions. Is it always possible to choose this sequence to be monotone?

Proof. We take as given that every Lebesgue measurable function $f : [0, 1] \to \mathbb{R}$ is a limit almost everywhere of a sequence s_i of simple functions. Let $\varepsilon > 0$; we wish to construct a sequence of continuous functions f_i such that $f_i \to f$ on a set $X \subset [0, 1]$ of measure $\mu(X) \geq 1 - \varepsilon$.

There must be a set X' of measure $\mu(X) \geq 1 - \varepsilon/2$, on which $s_i \to f$. For each s_i, because s_i is a simple function on a compact set, there must be a finite number n_i of intervals $I \subset [0, 1]$, on each of which s_i is constant.

Between any two successive intervals there is a jump discontinuity; by covering the point of discontinuity with an interval $[a, b]$ of length $b - a = \varepsilon/(2n_i \cdot 2^{-i})^{-1}$, and connecting the two constant functions with a line segment from $(a, s_i(a))$ to $(b, s_i(b))$, we can construct a continuous function f_i which is equal to s_i outside of the intervals covering the jump discontinuities.

The total lengths of these intervals is at most

$$\sum_{j=1}^{n_i} \frac{\varepsilon}{2n_1 \cdot 2^i} = \frac{\varepsilon}{2^{i+1}}.$$

The f_i converge pointwise to f on the relative complement $X' \setminus \cup I_{i,j}$ of the domain of convergence of the s_is with the union of all of these intervals. The total area of the union of all the intervals is at most

$$\sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{2}.$$

Therefore, the f_i converge to f on a set of measure

$$\mu(X' \setminus \cup I_{i,j}) \geq \mu(X') - \mu(\cup I_{i,j}) \geq 1 - \varepsilon$$

. Because ε was arbitrary, the f_i converge to f almost everywhere.

It is not necessarily true that we can find a monotone sequence of continuous functions which converge to f. For example, let f be the measurable function

$$f(x) = \begin{cases} \frac{1}{x} & 0 < x < 1/2 \\ \frac{-1}{1-x} & 1/2 < x < 0 \\ 0 & x = 0, 1/2, 1 \end{cases}$$

The function f is measurable, goes to ∞ as $x \to 0$, and goes to $-\infty$ as $x \to 1$. If there were a sequence of continuous functions which converged monotonically almost everywhere to f; say an increasing sequence f_i, then for each i, the set of $x \in [0, 1]$ such that $f_i(x) > f(x)$ would need to have measure 0.

In particular, for any ε, for almost every $x \in [1-\varepsilon, 1]$, the value of $f_i(x)$ would need to be less than or equal to $-\frac{1}{\varepsilon}$. This means that we can find a sequence x_j such that $f_i(x_j) < -\frac{1}{2^j}$. Thus $f_i(x_j) \to -\infty$ as $j \to \infty$, which is impossible because f_i is a continuous function on a compact set, and cannot be unbounded. $\square$

Problem 3. Let U be a bounded open subset of $\mathbb{R}^n$ and let $f : (a,b) \times U \to \mathbb{R}$ be a continuous function such that for each $(t,x) \in (a,b) \times U$ the partial derivative $\partial f/\partial t(t,x)$ exists and satisfies $\|\partial f/\partial t(t,x)\| \leq g(x)$ for some integrable function $g : U \to \mathbb{R}$. Define the function: $F(t) := \int_U f(t,\cdot)d\mu_n$. Prove that F is differentiable and that:

$$F'(t) = \int_U \frac{\partial f}{\partial t}(t,\cdot)d\mu_n.$$

Proof. We investigate how $F(t)$ acts when we perturb t by an infinitesimal amount δt:

$$\begin{aligned} F(t+\delta t) &= \int_U f(t+\delta t,\cdot)d\mu_n \\ &= \int_U \left(f(t,\cdot) + \delta \frac{\partial f}{\partial t}(t,\cdot) + R(\delta^2)) \right) d\mu_n \\ &= \int_U f(t,\cdot)d\mu_n + \delta \int_U \frac{\partial f}{\partial t}(t,\cdot)d\mu_n + R(\delta^2) \\ &= F(t) + \delta \int_U \frac{\partial f}{\partial t}(t,\cdot)d\mu_n + R(\delta^2) \end{aligned}$$

Note that the integral $\int_U R(\delta^2)$ is again proportional to δ^2, because the size of U is bounded.

Therefore, if the function $\int_U \frac{\partial f}{\partial t}(t,\cdot)d\mu_n$ is continuous, it is the derivative of F. It is, by virtue of the fact that $\frac{\partial f}{\partial t}(t,\cdot)$ is bounded by the integrable function $|(g(x))|$, and therefore the norm

$$\left\| \int_U \right\| \frac{\partial f}{\partial t}(t,\cdot)d\mu_n$$

is bounded by the value

$$\int_U \|g(x)\| \, d\mu_n.$$

$\square$

Problem 4. Prove that $\mathcal{L}_{n+m}$ is the smallest σ-algebra of subsets of $\mathbb{R}^{n+m}$, containing the product σ-algebra $\mathcal{L}_n \otimes \mathcal{L}_m$, and all sets of zero outer measure.

Proof. incomplete $\square$

Problem 5. Let (X, M, μ) be a finite measure space, so that $\mu(X) < \infty$. We say that a sequence of real-valued, integrable functions f_n on X is *uniformly integrable*, if $\sup_n \left\{ \int_X |f_n| \, d\mu \right\} < \infty$ and:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \qquad \mu(A) < \delta \Rightarrow \forall n \qquad \int_A |f_n| \, d\mu < \varepsilon.$$

Prove that a sequence f_n satisfies $\int_X |f_n - f| \, d\mu \to 0$ if and only if both f_n converges to f in measure and the f_n are uniformly integrable.

Proof. First, assume that f_n converges to f in measure and that f_n are uniformly integrable. This implies that the set $\{f_i\}_{i\geq 1} \cup \{f\}$ is also uniformly integrable. Let $\varepsilon > 0$; we want to show that there exists $N \in \mathbb{N}$ such that, for all $n > N$,

$$\int_X |f_n - f| \, d\mu < \varepsilon.$$

By uniform integrability, there exists some $\delta > 0$ such that, for all A such that $\mu(A) < \delta$, $\int_A |f_n| < \frac{\varepsilon}{4}$, and the same is true for f.

By convergence in measure, there exists some $N \in \mathbb{N}$ such that, for all $n \geq N$, the subset X_ε of X on which $|f_n - f| \geq \frac{\varepsilon}{2\mu(X)}$ has measure $\mu(X_\varepsilon) < \delta$. Therefore, for all $n \geq N$,

$$\begin{aligned}
\int_X |f_n - f| \, d\mu &= \int_{X_\varepsilon} |f_n - f| + \int_{X_\varepsilon^c} |f_n - f| \, d\mu \\
&\leq \int_{X_\varepsilon} |f_n| \, d\mu + \int_{X_\varepsilon} |f| \, d\mu + \int_{X_\varepsilon^c} |f_n - f| \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \mu(X_\varepsilon^c)\frac{\varepsilon}{2\mu(X)} \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \mu(X)\frac{\varepsilon}{2\mu(X)} \\
&= \varepsilon
\end{aligned}$$

Therefore, $\int_X |f_n - f| \, d\mu$ goes to 0 as $n \to \infty$.

Now, assume that $\int_X |f_n - f| \to 0$ as $n \to \infty$. We frist want to show that $f_n \to f$ in measure. Let $\varepsilon > 0$; we want to find that

$$\mu(\{x \in X; |f_n(x) - f(x)| > \varepsilon\}) \to 0$$

as $n \to \infty$. If this were not true, then there would be some $\delta > 0$ such that

$$\mu(\{x \in X; |f_n(x) - f(x)| > \varepsilon\}) \geq \delta$$

for all $n \in \mathbb{N}$; this would imply that

$$\begin{aligned}
\int_X |f_n - f| \, d\mu &\geq \int_{x \in X; |f_n(x) - f(x) > \varepsilon|} |f_n - f| \, d\mu \\
&\geq \varepsilon\delta
\end{aligned}$$

For all $n \in \mathbb{N}$. However, this integral goes to 0 as $n \to \infty$ by assumption. Therefore, f_n must converge to f in measure.

Now, we want to show that the f_n are uniformly integrable. Let $\varepsilon > 0$; we want to find a $\delta > 0$ such that for all $\mu(A) < \delta$, the integral $\int_A |f_i| \, d\mu$ is less than ε.

The rest of this proof is incomplete. $\square$