Homework 1

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Topology II

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1 Problems

Problem 1. Prove that S^n is homeomorphic to the quotient topological space $D^n/\sim$, where $\sim$ denotes the following equivalence relation on D^n:

$$x \sim y \iff x = y \text{ or } x \text{ and } y \in \partial D^n.$$

Let S^n be the set of unit vectors in $\mathbb{R}^{n+1}$: $S^n = \{x \in \mathbb{R}^{n+1}; |x| = 1\}$.

Proof. We first construct a map $\varphi : S^n \to D^n/\sim$. Let $x \in S^n = \langle x_1, \dots x_n, x_{n+1}\rangle$. Let $u(x_1, \dots x_n)$ be the function taking a vector to the parallel unit vector:

$$u(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}.$$

Define φ as follows:

$$\varphi(\langle x_1, \dots x_n, x_{n+1}\rangle = \begin{cases} \frac{1}{2}\langle x_1, \dots x_n\rangle & 1 \le x_{n+1} \le 0 \\ u(x_1, \dots x_n) - \frac{1}{2}\langle x_1, \dots x_n\rangle & 0 < x_{n+1} < 1 \\ [\partial D^n] & x_{n+1} = 1 \end{cases}$$

Where $[\partial D^n]$ is the equivalence class of points in ∂D^n under the relation $\sim$.

This function is clearly continuous in the regions $1 \le x_{n+1} < 0$ and $0 < x_{n+1} < 1$, as on each region it is defined by a single continuous function.

On the boundary $x_{n+1} = 0$, the two functions $x \mapsto \frac{1}{2}\langle x_1, \dots x_n\rangle$ and $x \mapsto u(x_1, \dots x_n) - \frac{1}{2}\langle x_1, \dots x_n\rangle$ agree, as we can see by computation. Letting $\mathbf{x} = \langle x_1, \dots, x_n\rangle$ and using the fact that $x_{n+1} = 0$ implies $|\langle x_1, \dots x_n\rangle| = 1$:

$$\begin{aligned} \frac{1}{2}\langle x_1, \dots x_n\rangle &= \langle \frac{x_1}{2}, \dots \frac{x_n}{2}\rangle \\ &= \langle \frac{2x_1 - x_1}{2}, \dots, \frac{2x_n - x_n}{2}\rangle \\ &= \langle \frac{x_1}{1}, \dots, \frac{x_n}{1}\rangle - \langle \frac{x_1}{2}, \dots, \frac{x_n}{2}\rangle \\ &= \langle \frac{x_1}{|\mathbf{x}|}, \dots, \frac{x_n}{|\mathbf{x}|}\rangle - \frac{1}{2}\langle x_1, \dots, x_n\rangle \\ &= u(x_1, \dots, x_n) - \frac{1}{2}\langle x_1, \dots x_n\rangle \end{aligned}$$

Since the two functions approach the same value at the boundary between the regions, $x_{n+1} = 0$, the function φ is continuous there. We finally see that it is continuous near $x_{n+1} = 1$. We show that, for every convergent sequence of points $\{\mathbf{x}_i\} \in S^n$ which approach $\langle 0, \ldots, 0, 1\rangle$, the images $\varphi(\mathbf{x}_i)$ approach $[\partial D^n]$.

Let $\{\mathbf{x}_i\}$ be a sequence of points in S^n that approach $\langle 0, \ldots, 0, 1\rangle$. We want to show that, for every open neighborhood U of $[\partial D^n]$, there is some $N \in \mathbb{N}$ such that, for all $m \geq N$, the images $\varphi(\mathbf{x}_m)$ all lie in U.

Every open neighborhood U of $[\partial D^n]$ is the image of some open set in D^n which contains the entire preimage of $[\partial D^n]$: that is, it is an open set in D^n which contains the entire boundary ∂D^n. This implies that it contains some open annulus of inner radius $1 - \varepsilon$ and outer radius 1: if it did not, it would contain points arbitrarily close to the boundary ∂D^n which did not lie in U, contradicting openness.

Because the points $\mathbf{x}_i$ approach $\langle 0, \ldots, 0, 1\rangle$, there must be some $N \in \mathbb{N}$ such that, for all $m > \mathbb{N}$, $|\langle 0, \ldots, 0, 1\rangle - \mathbf{x}_m| < 2\varepsilon$. This in turn implies that $|\langle x_{m1}, \ldots x_{mn}\rangle| < 2\varepsilon$:

$$\begin{aligned}
|\langle x_{m1}, \ldots, x_{mn}\rangle| &= |\langle x_{m1}, \ldots, x_{mn}, 0\rangle| \\
\leq \left|\langle x_{m1}, \ldots, x_{mn}, (1 - x_{m(n+1)})\rangle\right| \\
&= \left|\langle 0, \ldots, 0, 1\rangle - \langle x_{m1}, \ldots x_{m(n+1)}\rangle\right| \\
&= |\langle 0, \ldots, 0, 1\rangle - \mathbf{x_m}| \\
&< 2\varepsilon
\end{aligned}$$

Now, if $|\langle x_{m1}, \ldots, x_{mn}\rangle| < 2\varepsilon$, and $x_{m(n+1)}$ is close to 1 (and therefore greater than 0) then its image under φ must lie either in the annulus $1 - \varepsilon < |x| < 1$, or at the point $[\partial D^n]$. If $x_{m(n+1)} < 1$:

$$\begin{aligned}
|\varphi(\mathbf{x}_m)| &= \left|u(\langle x_{m1}, \ldots, x_{mn}\rangle) - \frac{1}{2}\langle x_{m1}, \ldots, x_{mn}\rangle\right| \\
&= 1 - \frac{1}{2}\left|x_{m1}, \ldots, x_{mn}\right| \\
&> 1 - \frac{1}{2}(2\varepsilon) \\
&> 1 - \varepsilon
\end{aligned}$$

And, if $x_{m+1} = 1$, the image of $\varphi(\mathbf{x}_m)$ is $[\partial D^n]$ by definition. Therefore, for $m > N$, the image of $\varphi(\mathbf{x}_m)$ lies in the union of $[\partial D^n]$ with the open annulus $1 - \varepsilon < |x| < 1$, which lies in U. So, the map φ is continuous at $x_{n+1} = 1$.

We now construct a continuous map $\psi : D^n/\sim \to S^n$. Let $\mathbf{x} \in D^n$, and define $\psi(\mathbf{x})$ as follows:

$$\psi(\mathbf{x}) = \begin{cases} \langle 2x_1, \ldots, 2x_n, -(1 - |2\mathbf{x}|^2)^{1/2}\rangle & |\mathbf{x}| \leq \frac{1}{2} \\ \langle (\frac{1}{2} - \frac{1}{2}|\mathbf{x}|)x_1, \ldots, (\frac{1}{2} - \frac{1}{2}|\mathbf{x}|)x_n, (1 - \left|(\frac{1}{2} - \frac{1}{2}|\mathbf{x}|)\mathbf{x}\right|^2)^{1/2}\rangle & \frac{1}{2} < |\mathbf{x}| < 1 \\ \langle 0, \ldots, 0, 1\rangle & \mathbf{x} = [\partial D^n] \end{cases}$$

First, we see that this is indeed a two-sided inverse to φ: on the three domains in $D^n/\sim$. If

$|\mathbf{x}| \leq \frac{1}{2}$, then $-(1-|2\mathbf{x}|^2)^{1/2} \leq 0$:

$$\begin{aligned}\varphi(\psi((x))) &= \varphi(\langle 2x_1, \ldots, 2x_n, -(1-|2\mathbf{x}|^2)^{1/2}\rangle) \\ &= \frac{1}{2}\langle 2x_1, \ldots, 2x_n\rangle \\ &= \mathbf{x}\end{aligned}$$

If $\frac{1}{2} < |\mathbf{x}| < 1$, then $(1-\left|(\frac{1}{2}-\frac{1}{2}|\mathbf{x}|)\mathbf{x}\right|^2)^{1/2} > 0$:

$$\begin{aligned}\varphi(\psi(x)) &= \varphi(\langle(\frac{1}{2}-\frac{1}{2}|\mathbf{x}|)x_1, \ldots, (\frac{1}{2}-\frac{1}{2}|\mathbf{x}|)x_n, (1-\left|(\frac{1}{2}-\frac{1}{2}|\mathbf{x}|)\mathbf{x}\right|^2)^{1/2}\rangle) \\ &= \frac{\mathbf{x}}{|\mathbf{x}|} - 2\left\langle\left(\frac{1}{2}-\frac{1}{2}\mathbf{x}\right)x_1, \ldots, \left(\frac{1}{2}-\frac{1}{2}|\mathbf{x}|\right)x_n\right\rangle \\ &= \mathbf{x}\end{aligned}$$

And, finally, ψ and φ map the singletons $\langle 0, \ldots, 0, 1\rangle$ and $[\partial D^n]$ to each other. ψ is continuous as well; a full proof of this would also need to analyze the behavior of the function at the circle $|x| = \frac{1}{2}$, and at the boundary point $[\partial D^n]$. $\square$

Problem 2. Write down an example of a topological space X and an equivalence relation $\sim$ on X such that the standard map to the quotient topological space $X \to X/\sim$ is not an open map.

Proof. The following example was found on math.stackexchange: Let $X = \mathbb{R}$, the set of real numbers with its standard topology, and let $\sim$ be the equivalence relation

$$x \sim y \iff x = y \text{ or } (x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}).$$

Then the preimage of the image of any open set containing an integer, say $(-\varepsilon, \varepsilon)$, contains all of $\mathbb{Z}$:

$$\pi^{-1}(\pi((-\varepsilon, \varepsilon))) = (-\varepsilon, \varepsilon) \cup \mathbb{Z},$$

which is not in general open in $\mathbb{R}$. Thus the set $\pi((-\varepsilon, \varepsilon))$ is not open in the quotient topology. $\square$