Homework 3

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Topology II

February 14, 2020

Problem 1. Let $n \in \mathbb{Z}_{\geqslant 1}$. Prove that the natural map $\rho : S^n \to \mathbb{RP}^n$ discussed in class is a covering map.

Proof. Recall that the map is defined by taking $(x_0, \ldots, x_n)$ to $[x_0 : \cdots : x_n]$. This map is $2 - to - 1$; for example the preimage of the point

$$x = [x_0 : x_1 : \cdots : x_n]$$

is the pair of points

$$\left\{ \pm \langle \frac{x_0}{m_x}, \ldots, \frac{x_n}{m_x} \rangle \right\},$$

where $m_x = \sqrt{x_0^2 + \cdots + x_n^2}$ is the magnitude of the vector $\langle x_0, \ldots, x_n \rangle$ chosen to represent x.

We show that this map is a covering map. First of all, it is a local homeomorphism: it is locally 1-to-1, as the preimage of any point is a pair of separated points; and it has a continuous local inverse near any point. This can be seen by taking a point $\langle y_0, \ldots, y_n \rangle \in S^n$: y must have some nonzero component, say y_i. Let $\sigma(y_i) = \frac{y_i}{|y_i|}$ be the sign of y_i. The function

$$\varphi : [x_0 : \ldots x_n] \mapsto (\sigma(x_i)\sigma(y_i))\langle x_0, \ldots, x_n \rangle$$

is a continuous function on the open neighborhood $U_i = \{x \mid x_i \neq 0\}$ of $\rho(y)$, and is a right-inverse to ρ, as $\rho \circ \varphi \equiv \mathrm{Id}|_{U_i}$.

Finally, we see that each point has an open neighborhood which is evenly covered by ρ. Let $x = [x_0 : \ldots x_n]$ be an arbitrary point of $\mathbb{P}^n$. x must have some nonzero component, say x_j. Then the open set $U_j = \{y \mid y_j \neq 0\}$ contains x, and the preimage of U_j is the disjoint union

$$\{y \in S^n \mid y_j > 0\} \coprod \{z \in S^n \mid z_j < 0\}$$

And we have seen that the map ρ is a homeomorphism when restricted to either of these domains. $\square$

Problem U. sing $(1, 0) \in S^1$ as a base point,

(a) Write down all of the path-connected covers of S^1, up to isomorphism.

Proof. There is of course the universal cover $\mathbb{R} \to S^1$. Since $\mathbb{R}$ is simply connected, the induced map $\pi_1(\mathbb{R}, 0) \hookrightarrow \pi_1(S^1, (1,0))$ corresponds to the inclusion $0 \hookrightarrow \mathbb{Z}$.

Since $\mathbb{R}$ is the maximal connected cover, any other cover $X \to S^1$ must also be covered by $\mathbb{R}$, so that the composition $\mathbb{R} \to X \to S^1$ is the identity. And, because $\mathbb{R}$ is the unique simply connected cover, any other cover X must have a nontrivial 1st homotopy group. The inclusion p_* of this group in $\pi_1(S^1, (1,0))$ gives the sequence

$$\pi_1(\mathbb{R}, 0) \hookrightarrow \pi_1(X, x_0) \hookrightarrow \pi_1(S^1, (1,0))$$
$$0 \hookrightarrow \pi_1(X, x_0) \hookrightarrow \mathbb{Z},$$

which shows that $\pi_1(X, x_0)$ must be a nontrivial subgroup of $\mathbb{Z}$. The only nontrivial subgroups of a cyclic group are cyclic groups, so this shows that $\pi_1(X, x_0) \simeq \mathbb{Z}$ itself, with the image of $\pi_1(X, x_0)$ in $\mathbb{Z}$ being the subgroup $n\mathbb{Z}$ for some n.

Given some n, one way to construct a cover $S^1 \to S^1$ with $\rho_*(\pi_1(S^1, (1,0))) = n\mathbb{Z}$ is by the map

$$\rho_n : e^{2\pi i\theta} \mapsto e^{2\pi n i\theta},$$

which does indeed give $(\rho_n)_*(\pi_1(S^1, (1,0))) = n\mathbb{Z} \leqslant \mathbb{Z}$, which we can see by the fact that the generator $[f]$ goes to $n[f]$ - i.e. a path which winds around the source S^1 once is mapped to apath which winds around the destination S^1 n times.

By Theorem 1.37 of Hatcher, any two path-connected covering spaces with $\rho_*(X_1, x_1) = \rho_*(X_2, x_2)$ - not just isomorphism, but equality - must be homeomorphic. Since the groups 0 and $\{n\mathbb{Z}\}_{n \in \mathbb{N}_+}$ exhast all of the subgroups of $\mathbb{Z}$, we see that the only path-connected covering spaces of S^1 are $\mathbb{R}$ and S^1 itself, with the latter giving a distinct cover (S^1, ρ_n) for every $n \in \mathbb{N}_+$. $\square$

(b) For each cover $\rho : \bar{X} \to S^1$, write down formulas for every deck transformation of $\bar{X}$.

Proof. We have seen that the only covers are $(\mathbb{R}, \rho_0)$, where ρ maps $\theta \in \mathbb{R}$ to $e^{2\pi i\theta}$, and (S^1, ρ_n), where ρ_n maps $e^{2\pi i\theta}$ to $e^{2\pi i n\theta}$.

A deck transformation $\mathcal{T}$ of a path-connected cover is determined by how it acts on a fiber $\rho^{-1}(x)$, by uniqueness of path lifting - let $x_1, x_2 \in \rho^{-1}(\{x\})$ and $y_1, y_2 \in \rho^{-1}(\{y\})$. If $\mathcal{T}$ maps x_1 to x_2, then it will map a path-lift $x \sim y$ from some path $x_1 \sim y_1$ to some path $x_2 \sim y_2$, and so it must map y_1 to y_2. So, any deck transformation is determined entirely by how it acts on the preimage of, say, the base point.

In the case of $\mathbb{R}$, the preimage $\rho^{-1}(\{(1,0)\})$ is exactly $\mathbb{Z}$, and the only transformations of $\mathbb{R}$ which preserve ρ are orientation-preserving euclidean transformations which take $\mathbb{Z}$ to $\mathbb{Z}$; these are exactly the translations of $\mathbb{R}$ by integer lengths. Each of these is determined by the image of 0, which can go to any other element $z \in \mathbb{Z}$, and the composition of two deck transformations $\mathcal{T}_1 : 0 \mapsto z_1$ and $\mathcal{T}_2 : 0 \mapsto z_2$ is a transformation $\mathcal{T}_1 \circ \mathcal{T}_2 : 0 \mapsto z_1 + z_2$; thus the group of deck transformations is exactly the integers $\mathbb{Z}$, with each transformation $\mathcal{T}_z$ being given by

$$r \mapsto r + z.$$

In the case of (S, ρ_n), a deck transformation of S is an orientation-preserving isometry of S which preserves the points in $\rho_n^{-1}(\{(1,0)\})$, which is the set of nth roots of unity; therefore, each deck transformation is a rotation by some integer multiple of $2\pi/n$, and the composition of two transformations is equal to rotation by the sum of their corresponding angles; therefore the set of deck transformations is equal to some subgroup $G \leqslant \mathrm{SO}_1(\mathbb{R})$. Since one of these transformations is determined by where it takes $(1,0)$, and there are n possible destinations for this point, the group of deck transformations must be a subgroup of order n in $\mathrm{SO}_1(\mathbb{R})$; i.e. it must be $\mathbb{Z}_n$. Using the identification of the nth roots of unity with $\mathbb{Z}_n$, the deck transformation $\mathcal{T}_\xi$ corresponding to each root is given by

$$s \mapsto \xi \cdot s.$$

More simply, we could use the fact that the deck transformations $\square$

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