# Homework 7

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**Problem 1.** Hatcher, 1.2.8: compute the fundamental group of the space obatined from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

*Proof.* Let $X$ be the space defined above. Then $X$ is the union of two path-connected spaces with path-connected intersection, and we may apply Van Kampen's theorem: the fundamental group of $X$ is defined as the following pushout:

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_1"] \arrow[d, "i_2"] & \pi_1(S^1 \times S^1) \arrow[d] \\ \pi_1(S^1 \times S^1) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \pi_1(X) \end{tikzcd}$$

Where the homomorphisms $i_1$ and $i_2$ are both induced by the injection

$$\begin{aligned} \iota : S^1 &\to S^1 \times S^1 \\ x &\mapsto (x, x_0) \end{aligned}$$

This takes the generating loop of $\pi_1(S^1)$ to one of the two generating loops of $S^1 \times S^1$ - say, the first, and so $i_1$ and $i_2$ can both be defined as the injection

$$\begin{aligned} i : \mathbb{Z} &\hookrightarrow \mathbb{Z} \times \mathbb{Z} \\ 1 &\mapsto (1, 0) \end{aligned}$$

So, our pushout diagram is the following:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "i_1"] \arrow[d, "i_2"] & \mathbb{Z} \times \mathbb{Z} \arrow[d] \\ \mathbb{Z} \times \mathbb{Z} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \pi_1(X) \end{tikzcd}$$

Thereforem, $\pi_1(X)$ is the free product of $\mathbb{Z} \times \mathbb{Z}$ with itself, quotiented by the identification of one copy of $\mathbb{Z} \times \{0\}$ with the other:

$$\pi_1(X) \simeq (\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z} \times 0} (\mathbb{Z} \times \mathbb{Z})$$

Despite the fact that **Grp** is not in general distributive, the product and coproduct in this case commute, and so

$$\pi_1(X) \simeq \mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})$$

Explicitly, this follows from the fact that, for any element in $(\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}\times 0} (\mathbb{Z} \times \mathbb{Z})$, defined as a word formed from elements of $(\mathbb{Z} times \mathbb{Z})_0$ and $(\mathbb{Z} \times \mathbb{Z})_1$, the first factors can be "brought out" of the word:

$$\begin{aligned} x &= (a,b)_1 \cdot (c,d)_2 \cdot \dots \cdot (y,z)_i \\ &= (a,0)(0,b)_1 \cdot (c,0)(0,d)_2 \cdot \dots \cdot (y,0)(0,z)_i \\ &= (a+c+\dots+y,0) \cdot ((0,b)_1 \cdot (0,d)_2 \cdot \dots \cdot (0,z)_i) \end{aligned}$$

So, every elment of $(\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}\times 0} (\mathbb{Z} \times \mathbb{Z})$ is determined uniquely by an element of $(\mathbb{Z} \times 0)$ and an element of $(0 \times \mathbb{Z}) * (0 \times \mathbb{Z})$. After checking that operations behave well and all elements commute that should, and using the identification $\mathbb{Z} \simeq 0 \times \mathbb{Z} \simeq \mathbb{Z} \times 0$, we have

$$\pi_1(X) \simeq \mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})$$

□

**Problem 2.** Hatcher, 1.2.11: The **mapping torus** $T_f$ of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x,0)$ with $(f(x),1)$. In the case $X = S^1 \vee S^1$ with $f$ basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.

*Proof.* One way we may view the mapping torus $T_f$ is as a CW-complex obtained by affixing an $n+1$-cell for each $n$-cell $C$ in $X$, which has as boundary the cell $C$, the product of the boundary of $C$ with $I$, and its image $f(C)$.

In the case of $X = S^1 \vee S^1$, we have one 0-cell $x_0$, which is mapped to $x_0$ and thus creates a 1-cell $x_1^0$ with both of its endpoints at $x_0$, and two 1-cells $x_1^1$ and $x_1^2$, which are mapped to some loops $f(x_1^1)$ and $f(x_1^2)$ in $S^1 \vee S^1$, and which give two 2-cells in $X$: one, $x_2^1$, has boundary formed from $x_1^1$, $x_1^0$, and a reversed $f(x_1^1)$, and the other, $x_2^2$, has boundary formed from $x_1^2, x_1^0$, and a reversed $f(x_1^2)$.

Thus there are 3 one-cells in $X$, all of which are loops, which give us 3 generators for the fundamental group, and 2 two-cells, which give us relations:

$$\pi_1(X) = \left\langle [x_1^0], [x_1^1], [x_1^2] \mid \langle [x_1^1 \centerdot x_1^0 \centerdot \overline{f(x_1^1)}], [x_1^2 \centerdot x_1^0 \centerdot \overline{f(x_1^2)}] \rangle \right\rangle$$

Next, we look at the case $X = S^1 \times S^1$. In this case, $X$ has one 0-cell $x_0$, two 1-cells $x_1^1$ and $x_1^2$, and one 2-cell $x_2^0$, which has as its boundary the path $x_1^1 \centerdot x_1^2 \centerdot \overline{(x_1^1)^{-1}} \centerdot \overline{(x_1^2)^{-1}}$. The mapping torus $T_f$ adds one 1-cell $x_1^0$, with both endpoints at $x_0$, two 2-cells $x_2^1$ and $x_2^2$, with boundaries $x_1^1 \centerdot x_1^0 \centerdot \overline{f(x_1^1)}$ and $x_1^2 \centerdot x_1^0 \centerdot \overline{f(x_1^2)}$, respectively. There is also a 3-cell, which does not influence the fundamental group. These three 1-cells and three 2-cells give us a presentation of $\pi_1(X)$:

$$\pi_1(X) = \langle [x_1^0], [x_1^1], [x_1^2] \mid \langle [x_1^1 \centerdot x_1^2 \centerdot \overline{(x_1^1)^{-1}} \centerdot \overline{(x_1^2)^{-1}}], [x_1^1 \centerdot x_1^0 \centerdot \overline{f(x_1^1)}], [x_1^2 \centerdot x_1^0 \centerdot \overline{f(x_1^2)}] \rangle \rangle$$

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