Homework 4

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Algebraic Geometry

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Problem 1. Let $C \subset \mathbb{A}^2$ be the curve defined by the irreducible polynomial

$$x^6 + y^6 - xy = 0$$

1. Show that C is not a normal variety.

   Proof. A normal variety is one in which the local ring $\mathcal{O}_{p,C}$ is integrally closed in its field of fractions, for each point $p \in C$. Therefore, to show that C is not normal, we want to find one point p whose local ring is not integrally closed. We would expect this to be the nonsingular point $o = (0,0)$, and in fact we will see that this is the case.

   The local ring R at o is the localization of the coordinate ring of C at the ideal $\langle x, y\rangle$:

   $$R = (k[x,y]/\langle x^6 + y^6 - xy\rangle)_{\langle x,y\rangle}.$$

   Thus, in R, the condition $x^6 + y^6 - xy$ holds, and all elements of $k[x,y]/\langle x^6 + y^6 - xy\rangle$ which are not in $\langle x, y\rangle$, i.e. which have a nonzero constant term, are invertible.

   We want to find an element of the field of fractions of R which is integral over R, but is not an element of R itself. The element $xy/(x^3 - y^3)$ is not in R, since $x^3 - y^3$ is not invertible in R, but we see that

   $$\begin{aligned}\left(\frac{xy}{x^3 - y^3}\right)^2 &= \frac{x^2y^2}{x^6 + y^6 - x^3y^3} \\ &= \frac{x^2y^2}{-xy - x^3y^3} \\ &= -\frac{xy}{x^2y^2 + 1}\end{aligned}$$

   The element $x^2y^2 + 1$ is not in $\langle x, y\rangle$, so it is invertible in R; therefore the element $xy/(x^3 - y^3)$ satisfies the monic polynomial $t^2 + \frac{xy}{x^2y^2+1} \in R[t]$, even though it is not in R. So, R is not integrally closed, and C is not a normal variety. □

2. Show that the maximal ideal of the origin $o \in C$ is not a principal ideal.

Proof. We want to see that the ideal $\langle x, y\rangle$ is not principal in $k[x,y]/\langle x^6+y^6-xy\rangle$. For the sake of contradiction, assume that it is: then there is some $g \in k[x,y]/\langle x^6+y^6-xy\rangle$ such that $\langle x, y\rangle = \langle g\rangle$. This implies that $x = fg$ for some $f \in k[x,y]/\langle x^6+y^6-xy\rangle$.

Lifting to the ring $k[x,y]$, there must be some $h \in k[x,y]$ such that

$$x = fg + h(x^6 + y^6 - xy)$$

Because the degree of x^6+y^6-xy is greater than the degree of x, it must be true that $h = 0$. So, $x = fg$ holds in $k[x,y]$.

Up to associates, there are only two factors of x in $k[x,y]$; x itself, and 1. So, either $\langle x, y\rangle = \langle x\rangle$ or $\langle x, y\rangle = \langle 1\rangle$.

If the first were true, then $y = f'x$ must hold for some $f' \in k[x,y]/\langle x^6+y^6-xy\rangle$; lifting to $k[x,y]$, there must be some $h \in k[x,y]$ such that

$$y = f'x + h'(x^6 + y^6 - xy).$$

Again, by degree, h' must be zero. But then x would be a factor of y in $k[x,y]$, which does not hold.

On the other hand, if $\langle x, y\rangle = \langle 1\rangle$, then there would be some $a, b \in k[x,y]/\langle x^6+y^6-xy\rangle$ such that $ax + by = 1$; lifting to $k[x,y]$, there would be some $c \in k[x,y]$ such that

$$1 = ax + by + c(x^6 + y^6 - xy).$$

The right side must have zero constant term, while the left has constant term 1, so this relation cannot hold. So, $\langle x, y\rangle$ cannot be principal. ☐

3. Let $\varphi : \mathrm{Bl}_o(\mathbb{A}^2) \to \mathbb{A}^2$ be the blow-up of $\mathbb{A}^2$ at the origin o. Let $\widetilde{C}$ be the strict transform of C, i.e. $\widetilde{C} = \overline{\varphi^{-1}(C\setminus\{o\})}$. Describe the points in $\widetilde{C}$ which lie above o.

Proof. When graphing C over $\mathbb{R}^2$, we note that the singular point at the origin is a double point, with a vertical and a horizontal tangent. Therefore we expect that blowing up C at the origin will give two points, and indeed it does.

We recall that the blowup of $\mathbb{A}^2$ at the origin can be described by

$$\left\{(x,\ell) \in \mathbb{A}^2 \times \mathbb{P}^1 \;\mid\; x \in \ell\right\},$$

where $\mathbb{P}^1$ has been identified with lines through the origin in $\mathbb{A}^2$, and φ is the projection map $(x,\ell) \mapsto x$. The inverse image $\varphi^{-1}(C\setminus 0)$ is the set of points

$$\left\{((x,y),\ell) \;\mid\; (x,y) \in \ell, (x,y) \neq 0, x^6 + y^6 - xy = 0\right\}.$$

Using projective coordinates $\ell = [s:t]$,

$$\varphi^{-1}(C\setminus\{0\}) = \left\{((x,y),[s:t]) \;\mid\; xt - ys = 0, (x,y) \neq 0, x^6 + y^6 - xy = 0\right\}.$$

Now, we can take the intersection with an affine chart. Using $\{[1:t] \mid t \in k\} \simeq \mathbb{A}^1$, we have $\overline{\mathbb{A}^1} = \mathbb{P}^1$, so passing to this chart does not affect the closure:

$$\begin{aligned}\overline{\varphi^{-1}(C\setminus\{0\})} &= \overline{\varphi^{-1}(C\setminus\{0\}) \cap \mathbb{A}^2 \times \mathbb{A}^1} \\ &= \overline{\{((x,y),[1:t]) \mid xt - y = 0, (x,y) \neq 0, x^6 + y^6 - xy = 0\}}\end{aligned}$$

Now, we can use the first identity to rewrite $y = xt$:

$$\overline{\varphi^{-1}(C\setminus\{0\})} = \overline{\{(x,xt),[1:t] \mid (x,xt) \neq 0, x^6 + t^6x^6 - tx^2 = 0\}}$$

Since we have $x \neq 0$ on the domain of the variety, we can divide by x^2 in the defining polynomial:

$$\overline{\varphi^{-1}(C\setminus\{0\})} = \overline{\{(x,xtt) \mid x \neq 0, x^4 + t^6x^4 - t = 0\}}.$$

Closing this variety in $\mathbb{A} \times \mathbb{A}$ gives

$$\overline{\varphi^{-1}(C\setminus\{0\})} = \overline{\{(x,xt,t) \mid x^4 + t^6x^4 - t = 0\}}$$

Now, the right side is the closure of an affine variety on a chart of $\mathbb{A}^2 \times \mathbb{P}$. By symmetry of the polynomial in x and y, we know that the intersection of $\varphi^{-1}(C\setminus\{0\})$ with the other chart of $\mathbb{A}^2 \times \mathbb{P}$ is

$$\left\{(ys,s,s) \mid (ys,y) \neq 0, s^6y^4 + y^4 - s = 0\right\},$$

and so the closure of $\varphi^{-1}(C\setminus\{0\})$ is the union of the closures of its intersection with these two charts:

$$\overline{\{\varphi^{-1}(C\setminus\{0\})\} = \{(x,xt),[1,t] \mid x^4 + t^6x^4 - t = 0\} \cup \{(ys,y),[s,1] \mid s^{64} + y^4 - s = 0\}}.$$

Each of the two sets on the right contains a distinct point over $(x,y) = (0,0)$, one corresponding to $x = t = 0$, and the other to $y = s = 0$. These two points lie over the origin in the blowup of C, and correspond to the two tangent lines to C at $(0,0)$. □

Problem 2. Consider the circle $C = V(x^2 + y^2 - 1)$ and the line $L = V(y-1)$. Consider the point $p = (0,1)$ which lies on both C and L and let $R = \mathcal{O}_{p,C}$ and $R' = \mathcal{O}_{p,L}$ be local rings of C and L at p respectively. Verify the Cohen Structure Theorem directly by showing that the completions $\hat{R}$ and $\hat{R}'$ are both isomorphic (as k-algebras) to formal power series ring in one variable.

Proof. incomplete □

Problem 3. For simplicity let $k = \mathbb{C}$. Consider the affine curves C below (in $\mathbb{A}^2$) with given points on them.

1. If p is a non-singular point, verify directly that the maximal ideal in the corresponding local ring is principal by finding a single generator for it.

   Proof. incomplete □

2. If p is a singular point, verify directly that the corresponding local ring is not integrally closed.