Homework 3

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Algebra II

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1 Problems

Problem 1. Let R be a commutative ring, and let I and J be ideals of R. Prove the following:

(a) $I + J$ is the smallest ideal of R containing both I and J.

(b) IJ is an ideal and it is contained in $I \cap J$.

(c) Give an example withere $IJ \neq I \cap J$.

(d) If $I + J = R$ then $IJ = I \cap J$.

Proof. (a) It is immediately clear that $I + J \supset I$ and $I + J \supset J$, as any element $i \in I$ can also be written as $i + 0$, where $i \in I$ and $0 \in J$, and equivalently $j = j + 0$ for any element $j \in J$.

It also follows quickly that any ideal containing I and J must also contain $I + J$: if $i + j$ is some element of $I + J$, then both i and j must be in K, as it contains both ideals. Therefore $i + j$ must be in K as well, as ideals are closed under addition. So any such K must contain $I + J$, and $I + J$ is the minimal element in the set of ideals containing both I and J.

(b) We show first that IJ is an ideal. Certainly $0 \in IJ$, since $0 = 0 \cdot 0$. IJ is also closed under addition, as it is defined as the set of all finite sums of monomials ij, where $i \in I$ and $j \in J$ - a sum of two such finite sums is also a finite sum. Finally, it is closed under multiplication by any element of R: let $a = \sum_K i_k j_k$ be an element of IJ, where $k \in K$ is a finite index set. Then, for any $r \in R$,

$$\begin{aligned} ra &= r\left(\sum_{k\in K} i_k j_k\right) \\ &= \sum_{k\in K} r\left(i_k j_k\right) \\ &= \sum_{k\in K} \left(r i_k\right) j_k \end{aligned}$$

Since I is an ideal, $ri_k \in I$, so this is also a finite sum of elements from I and J multiplied together, and is therefore an element of IJ. Therefore the set is closed under multiplication by arbitrary elements of R, and it is an ideal. (If R were not commutative, and we were checking *left*-ideal axioms, we would need to check that IJ was closed under left-multiplication by arbitrary elements of R, and under right-multiplication by other elements of IJ, because ideals also need to be subrings. Since R is commutative, the two cases collapse into one, and we only need to check multiplication by generic elements of R.)

We now show that $IJ \subset I \cap J$. Let $\sum_K i_k j_k$ be an element of IJ. Each term $i_k j_k$ is an element of both I and J, and since I and J are closed under finite sums, the element $\sum_K i_k j_k$ must also be in both I and J. Therefore any element of IJ is also in $I \cap J$.

(c) We can also see that the inclusion $IJ \subset I \cap J$ need not be an equality. Let $R = k[x, y]$, for k some field, and let $I = \langle x^2 y \rangle$ and $J = \langle xy^2 \rangle$. Then an element of IJ is a sum of monomials, each of which has an exponent greater than or equal to 3 for both x and y - for example, $k_1 x^7 y^5 + k_2 x^5 y^3 + k_3 x^3 y^3$. However, the element $x^2 y^2$ is in both $\langle x^2 y \rangle$ and $\langle xy^2 \rangle$, so it is in $I \cap J$, but it is not in IJ.

(d) We now show that if $I + J =$ R, then $IJ = I \cap J$. Since the inclusion has been proved in one direction, we show it in the other: let $a \in I \cap J$ be arbitrary, and we show that it is in IJ.

We know that $a \in I$ and $a \in J$. We also assume that $I + J = R$, which means in particular that there exist $i \in I$ and $j \in J$ such that $i + j = 1$. So, we make the following calculation:

$$\begin{aligned} a &= 1a \\ &= (i + j)a \\ &= ia + ja \\ &= ia + aj \end{aligned}$$

We see that ia and aj are both in IJ, since $a \in J$ and $a \in I$, so the sum $ia + aj \in IJ$ as well. Therefore $IJ \supset I \cap J$, and we can conclude that the two ideals IJ and $I \cap J$ are equal.

□

Problem 2. Show that if R is an integral domain and M is any nonprincipal ideal of R then M is torsion free of rank 1 but is not a free R-module.

Proof. We first show that M is torsion free and its rank is at least 1, and then that it is at most 1, and then show that it is not free.

It is immediate that it is torsion free, since it is a subring of R, and R is an integral domain - if $rm = 0$ for some nonzero $r \in R$ and nonzero $m \in M$, then this relation would also hold in R, and it would not be integral.

Being torsion free implies that the rank of M is at least 1: if the rank were 0, then every element m would be linearly dependent over R - there would be some r such that $rm = 0$, but there is no such r, since M is torsion free.

Now, let m_1 and m_2 be any elements of M. from $m_2 \cdot m_1 + (-m_1) \cdot m_2 = 0$, we see that there is no linearly independent set of more than 1 element, so the rank must be exactly 1.

Finally, we show that M is not a free R module. For if it were a free R module, it would necessarily have rank 1, making it isomorphic (as an R-module) to R itself under some isomorphism ϕ. But then M would be generated as an ideal by the image of 1 under ϕ - to see this, let $m \in M$ be arbitrary; then $m = \phi(r)$ for some r, and

$$\begin{aligned} m &= \phi(r) \\ &= \phi(r \cdot 1) \\ &= r \cdot \phi(1) \end{aligned}$$

Then any m could be written as $r \cdot \phi(1)$ for some r, and M would be equal to the principal ideal $\langle \phi(1) \rangle$, contradicting our assumption that it was nonprincipal. So no nonprincipal ideal of an integral domain can be free of rank 1, despite being torsion free of rank 1.

We show the specific case for $\{2, x\}$ directly in the following problem, but it is also implied by this problem, since $\mathbb{Z}[x]$ is an integral domain and $\{2, x\}$ is a nonprincipal ideal. □

2 Extra Stuff

Problem 1. Let $R = \mathbb{Z}[x]$ and let $M = (2, x)$ be the ideal generated by 2 and x, considered as a submodule of R. Show that $\{2, x\}$ is not a basis of M. Then show that the rank of M is 1 but that M is not free of rank 1.

Proof. If $\{2, x\}$ were a basis of M, then we could write any element a of M as a *unique* sum $f \cdot 2 + g \cdot x$. Of course such a sum always exists, by definition of the generators of an ideal. However, we can show that it is never unique.

Let $a = f \cdot 2 + g \cdot x$. Then we can also write a as

$$\begin{aligned} a &= f \cdot 2 + g \cdot x + g \cdot x - g \cdot x \\ &= f \cdot 2 + (g \cdot x) \cdot 2 - g \cdot x \\ &= (f + g \cdot x) \cdot 2 + (-g) \cdot x \end{aligned}$$

Therefore $\{2, x\}$ is not a basis, as no element of $\langle 2, x \rangle$ has a unique representation in terms of 2 and x. □

Problem 2. If M is a finitely generated module over the P.I.D. R, describe the structure of $M/\mathrm{Tor}(M)$.

Proof. We show that $M/\mathrm{Tor}(M)$ is a free R module. This follows from the structure theorem for modules over P.I.D.s, as we may decompose M as

$$M = R^{\oplus n} \oplus \frac{R}{(p_1)} \oplus \cdots \oplus \frac{R}{(p_n)}$$

A theorem in D&F shows that $\mathrm{Tor}(M) \cong R/(p_1) \oplus \cdots \oplus R/(p_n)$, and quotienting the decomposition of M by this submodule gives $M/\mathrm{Tor}(M) \cong R^{\oplus n}$, showing that the torsion-free part of M is a free R-module. □

Problem 3. Prove that the intersection of all prime ideals of a commutative ring R is equal to its nilradical.

Proof. We recall that $\text{nilrad}(R) = \text{rad}(0)$, the set of all elements $f \in R$ such that $f^n = 0$ for some $n \geq 1$. We first show that every such f belongs to every prime ideal $p \in \text{Spec}(R)$, and then that if f is not nilpotent, then there exists some $p \in \text{Spec}(A)$ such that $f \notin p$.

It is true in general that, if $f^n \in p$ for some $f \in R$ and $p \in \text{Spec}(R)$, then $f \in p$. In particular, if $f^n = 0$, then $f \in p$. So the nilradical of R is contained in the intersection of all prime ideals of R.

On the other hand, if f is not a zerodivisor, then the multiplicative set $S = \{1, f, f^2, ...\}$ does not contain 0. By taking Zorn's lemma[1], there exists some maximal ideal p' which does not intersect S, and this ideal p' is prime.[2] Thus there is a prime ideal not containing f, and we can conclude that the intersection of all prime ideals is contained in the nilradical of R, and thus that the two ideals are equal. □

References

[1] PersonX, Nilradicals without Zorn's lemma, URL: https://mathoverflow.net/q/27172

[2] Reid, Miles. Undergraduate Commutative Algebra. LMS Student Texts 29, London, 1995.

[1] If using the Axiom of Choice to prove this simple statement about prime ideals seems exessive, we could also use the Ultrafilter principle,[1] which is strictly weaker than Choice, but still stronger than ZF.