Homework 10

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Algebra II

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Problem 1. Dummit & Foote 10.4.12: Let V be a vector space over the field F and let v, v' be nonzero elements of V. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_F V$ if and only if $v = av'$ for some $a \in F$.

Proof. First, assume that $v = av'$ for some $a \in F$. Then by the defining relations of the tensor product, and the fact that the left- and right-module structures for a vector space are identical;

$$\begin{aligned} v \otimes v' &= (av') \otimes v' \\ &= v' \otimes av' \\ &= v' \otimes v \end{aligned}$$

Now, assume that $v \otimes v' = v' \otimes v$ in $V \otimes_F V$. By the axiom of choice, we may assume that the space V has a basis, say $\{e_i\}_{i \in I}$, and therefore that $V \otimes_F V$ has the basis $\{e_i \otimes e_j\}_{(i,j) \in I \times I}$. We can write our two vectors uniquely in this basis:

$$v = \sum_{i \in I} v_i e_i, \qquad v' = \sum_{j \in I} v'_j e_j$$

As shown in D&F, the set $\{e_i \otimes e_j\}_{(i,j) \in I \times I}$ is a basis for $V \otimes_F V$, and we can easily calculate the value of a simple tensor in this basis in terms of its coefficients in the basis $\{e_i\}$:

$$v \otimes v' = \sum_{(i,j) \in I \times I} (v_i v'_j) e_i \otimes e_j, \qquad v' \otimes v = \sum_{(i,j) \in I \times I} v_j v'_i e_i \otimes e_j$$

We have assumed that these two tensors are equal; since equality in a vector space implies that the coefficients at each basis element are equal, this shows that $v_i v'_j = v_j v'_i$ for all $(i,j) \in I \times I$.

Since both v' and v are nonzero, there is some i for which v_i is nonzero, and some j for which v'_j is nonzero. The identity $v_i v'_j = v_j v'_i$ shows that v_j and v'_i must be nonzero as well.

Also, for any k for which v_k is nonzero, the identity $v_k v'_j = v_j v'_k$ shows that v'_k must be nonzero as well; symmetrically, if v'_k is nonzero, so is v_k. Therefore the set of $i \in I$ such that

v_i' is nonzero is the same as the set of $i \in I$ such that v_i is nonzero. Call this set I'; we can write v and v' in the basis $\{e_i\}_{i\in I'}$ such that neither have any zero coefficients:

$$v \otimes v' = \sum_{(i,j)\in I'\times I'} (v_i v_j') e_i \otimes e_j, \qquad v' \otimes v = \sum_{(i,j)\in I'\times I'} v_j v_i' e_i \otimes e_j$$

Then the relations $v_i v_j' = v_j v_i'$ give $v_i/v_i' = v_j/v_j'$ for all $(i,j) \in I' \times I$. Letting $a = v_i/v_i'$ for some $i \in I$, we see that

$$\begin{aligned} v &= \sum_{i\in I'} v_i e_i \\ &= \sum_{i\in I'} a v_i' e_i \\ &= a \sum_{i\in I'} v' e_i \\ &= a v' \end{aligned}$$

So, $v = av'$ for some $a \in F$. □

Problem 2. D&F, 10.4.27:

(a) Write down a formula for the multiplication of two elements $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ and $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4$ in the example $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ following proposition 21.

(b) Let $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$. Show that $\epsilon_e \epsilon_e = 0$, $\epsilon_1 + \epsilon_2 = 1$, and $\epsilon_j^2 = \epsilon_j$ for $j = 1, 2$. Deduce that A is isomorphic as a ring to the direct product of two principal ideals: $A \cong A\epsilon_1 \times \epsilon_2$.

(c) Prove that the map $\varphi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\varphi(z_1, z_2) = (z_1 z_2, z_1 \overline{z_2})$, where $\overline{z_2}$ denotes the complex conjugate of z_2, is an $\mathbb{R}$-bilinear map.

(d) Let Φ be the $\mathbb{R}$-module homomorphism from A to $\mathbb{C} \times \mathbb{C}$ obtained from φ in (c). Show that $\Phi(\epsilon_1) = (0, 1)$ and $\Phi(\epsilon_2) = (1, 0)$. Show also that Φ is $\mathbb{C}$-linear, where the action of $\mathbb{C}$ is on the left tensor factor in A and on both factors in $\mathbb{C} \times \mathbb{C}$. Deduce that Φ is surjective. Show that Φ is a $\mathbb{C}$-algebra isomorphism.

Proof. (a) Our basis elements are $1 = e_1 = 1 \otimes 1$, $e_2 = 1 \otimes i$, $e_3 = i \otimes 1$, and $e_4 = i \otimes i$. Using the multiplication rules defined in proposition 10.4.21, we have

$$\begin{array}{llll} e_1^2 = e_1, & e_1 e_2 = e_2, & e_1 e_3 = e_3, & e_1 e_4 = e_4 \\ *, & e_2^2 = -e_1, & e_2 e_3 = e_4, & e_2 e_4 = -e_3 \\ *, & *, & e_3^2 = -e_1, & e_3 e_4 = -e_2 \\ *, & *, & *, & e_4^2 = e_1 \end{array}$$

Using these identities, we can write the product of two elements of $\mathbb{C} \otimes \mathbb{C}$ in terms of their coefficients in this basis:

$$(a \cdot e_1 + b \cdot e_2 + c \cdot e_3 + d.e_4) \cdot (a' \cdot e_1 + b' \cdot e_2 + c' \cdot e_3 + d' \cdot e_4) =$$

$$(aa' - bb' - cc' + dd') \cdot e_1 + (ab' + ba' - cd' - dc') \cdot e_2 + \\ (ac' - bd' + ca' - db') \cdot e_3 + (ad' + bc' + cb' + da') \cdot e_4$$

(b) Letting $\epsilon_1 = \frac{1}{2}(e_1 + e_4)$ and $\epsilon_2 = \frac{1}{2}(e_1 - e_4)$, we see from the last problem that

$$\begin{aligned} \epsilon_1 \epsilon_2 &= \frac{1}{2}(e_1 + e_4) \cdot \frac{1}{2}(e_1 - e_4) \\ &= \frac{1}{4} \cdot e_1 + \frac{1}{4} \cdot e_4 - \frac{1}{4} \cdot e_4 - \frac{1}{4} \cdot e_1 \\ &= 0 \end{aligned}$$

The identity $\epsilon_1 + \epsilon_2 = e_1$ is clear. Finally, we see

$$\begin{aligned} \epsilon_1^2 &= \frac{1}{2}(e_1 + e_4) \\ &= \frac{1}{4} \cdot e_1 + \frac{1}{4} \cdot e_4 + \frac{1}{4} \cdot e_4 + \frac{1}{4} \cdot e_1 \\ &= \frac{1}{2}(e_1 + e_4) \\ &= \epsilon_1, \end{aligned}$$

and also that

$$\begin{aligned} \epsilon_2^2 &= \frac{1}{2}(e_1 - e_4) \\ &= \frac{1}{4} \cdot e_1 - \frac{1}{4} \cdot e_4 - \frac{1}{4} \cdot e_4 + \frac{1}{4} \cdot e_4 \\ &= \frac{1}{2}(e_1 - e_4) \\ &= \epsilon_2 \end{aligned}$$

Because $\epsilon_1 + \epsilon_2 = 1$, the ideals $A\epsilon_1$ and $A\epsilon_2$ are comaximal, and the intersection of these two ideals is equal to their product. Because $\epsilon_1 \cdot \epsilon_2 = 0$, the product of the ideals $A\epsilon_1 \cdot A\epsilon_2$ is (0). Therefore, the Chinese Remainder Theorem tells us that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \cdot \epsilon_1 \times (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \cdot \epsilon_2$ (where the congruence is as commutative rings).

(c) We shoe that the map φ is $\mathbb{R}$-bilinear. There are four axioms to verify:

- $\varphi(z_1 + z_1', z_2) = \varphi(z_1, z_2) + \varphi(z_1', z_2)$:

$$\begin{aligned} \varphi(z_1 + z_1', z_2) &= ((z_1 + z_1')z_2, (z_1 + z_1')\overline{z_2}) \\ &= (z_1 z_2 + z_1' z_2, z_1 \overline{z_2} + z_1' \overline{z_2}) \\ &= (z_1 z_2, z_1 \overline{z_2}) + (z_1', \overline{z_2}) \\ &= \varphi(z_1, z_2) + \varphi(z_1', z_2) \end{aligned}$$

- $\varphi(z_1, z_2 + z_2') = \varphi(z_1, z_2) + \varphi(z_1, z_2')$:

$$\begin{aligned}
\varphi(z_1, z_2 + z_2') &= (z_1(z_2 + z_2'), z_1\overline{(z_2 + z_2')}) \\
&= (z_1 z_2 + z_1 z_2', z_1\overline{z_2} + z_1\overline{z_2'}) \\
&= (z_1 z_2, z_1\overline{z_2}) + (z_1 z_2', z_1\overline{z_2'}) \\
&= \varphi(z_1, z_2) + \varphi(z_1, z_2')
\end{aligned}$$

- $\varphi(rz_1, z_2) = r \cdot \varphi(z_1, z_2)$, for any $r \in \mathbb{R}$:

$$\begin{aligned}
\varphi(rz_1, z_2) &= ((rz_1)z_2, (rz_1)\overline{z_2}) \\
&= r \cdot (z_1 z_2, z_1\overline{z_2}) \\
&= r \cdot \varphi(z_1, z_2)
\end{aligned}$$

- $\varphi(z_1, rz_2) = r \cdot \varphi(z_1, z_2)$, for any $r \in \mathbb{R}$:

$$\begin{aligned}
\varphi(z_1, rz_2) &= (z_1(rz_2), z_1\overline{(rz_2)}) \\
&= (r(z_1 z_2), r(z_1\overline{z_2})) \\
&= r(z_1 z_2, z_1\overline{z_2}) \\
&= r\varphi(z_1, z_2)
\end{aligned}$$

Therefore, the map φ is $\mathbb{R}$-bilinear.

(d) Because φ is $\mathbb{R}$-bilinear on $\mathbb{C} \times \mathbb{C}$, the universal property of the tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ gives us a unique $\mathbb{R}$-module homomorphism $\Phi : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, such that $\Phi \circ \iota = \varphi$. Since $\epsilon_1 = \frac{1}{2}(\iota(1,1) + \iota(i,i))$ and $\epsilon_2 = \frac{1}{2}(\iota(1,1) - \iota(i,i))$, we can calculate the value of Φ on ϵ_1 and ϵ_2:

$$\begin{aligned}
\Phi(\epsilon_1) &= \Phi\left(\frac{1}{2}(\iota(1,1) + \iota(i,i))\right) \\
&= \frac{1}{2}(\Phi(\iota(1,1)) + \Phi(\iota(i,i))) \\
&= \frac{1}{2}((1,1) + (-1,1)) \\
&= (0,1), \text{ and} \\
\Phi(\epsilon_2) &= \Phi\left(\frac{1}{2}(\iota(1,1) - \iota(i,i))\right) \\
&= \frac{1}{2}(\Phi(\iota(1,1)) - \Phi(\iota(i,i))) \\
&= \frac{1}{2}((1,1) - (-1,1)) \\
&= (1,0)
\end{aligned}$$

We can also show that Φ is $\mathbb{C}$-linear, i.e. that $\Phi(z \cdot x) = z \cdot \Phi(x)$, and that $\Phi(x + y) = \Phi(x) + \Phi(y)$. In fact, because Φ is an $\mathbb{R}$-module homomorphism, the second identity must hold, so we need only verify the first.

We first show that this holds for simple tensors: assume $z \in \mathbb{C}$, and $x = x_1 \otimes x_2$, for some $x_1, x_2 \in \mathbb{C}$. Then

$$\begin{aligned}
\Phi(z \cdot x) &= \Phi(z \cdot (x_1 \otimes x_2)) \\
&= \Phi((z \cdot x_1) \otimes x_2) \\
&= \Phi(\iota(z \cdot x_1, x_2)) \\
&= ((z \cdot x_1)x_2, (z \cdot x_1)\overline{x_2}) & &= z \cdot (x_1 x_2, x_1 \overline{x_2}) \\
&= z \cdot \Phi(x_1 \otimes x_1) \\
&= z \cdot \Phi(x)
\end{aligned}$$

We can extend this to every tensor in $\mathbb{C} \otimes \mathbb{C}$ by the additivity of Φ.

So, we see that Φ is a $\mathbb{C}$-linear homomorphism with the two basis vectors $(0, 1)$ and $(1, 0)$ in its image; this implies that any arbitrary element $y \in \mathbb{C} \times \mathbb{C}$, say (y_1, y_2), is in the image of Φ:

$$\begin{aligned}
y &= (y_1, y_2) \\
&= y_1 \cdot (1, 0) + y_2 \cdot (0, 1) \\
&= y_1 \cdot \Phi(\epsilon_2) + y_2 \cdot \Phi(\epsilon_1) \\
&= \Phi(y_1 \cdot \epsilon_2 + y_2 \cdot \epsilon_1)
\end{aligned}$$

So, Φ is surjective. Because $\mathbb{C} \otimes \mathbb{C}$ is a 2-dimensional $\mathbb{C}$-vector space, with basis ϵ_1 and ϵ_2, and $\mathbb{C} \times \mathbb{C}$ is also a 2 dimensional vector space over $\mathbb{C}$, Φ must be an isomorphism of $\mathbb{C}$-vector spaces.

$\square$