Homework 6

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Analysis I

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Problem 1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of C^1 maps from an open subset U of a Banach space E into a Banach space F. Assume that $\{f_n\}$ converges pointwise to a map $f : U \to F$ and that the sequence of derivatives $\{f_n'\}$ converges uniformly to a mapping $g : U \to \mathcal{L}(E, F)$. Prove that f is C^1 and that $f' = g$.

Proof. First, we see that g is continuous, from the uniform limit theorem: the uniform limit of any sequence of continuous functions, such as the sequence $\{f_n'\}$, is itself continuous. We now show that $g = f'$. By the definition of the derivative, this is equivalent to showing that, for any point $x \in U$, the following limit converges and is equal to 0:

$$\lim_{\|h\| \to 0} \frac{f(x+h) - f(x) - g(x)(h)}{\|h\|} = 0$$

By the limit definitions of f and g, this is equal to

$$\lim_{\|h\| \to 0} \frac{\lim_{m\to\infty} f_m(x+h) - \lim_{n\to\infty} f_n(x) - \lim_{k\to\infty} f_k'(x)(h)}{\|h\|}.$$

Because all three limits exist, we may take the simultaneous limit over all three of m, n, k:

$$\lim_{\|h\| \to 0} \frac{\lim_{m,n,k\to\infty} \left(f_m(x+h) - f_n(x) - f_k'(h)\right)}{\|h\|}$$

Finally, because the space is Banach, convergence to 0 is equivalent to convergence in norm to 0, so we need only for the norm of this limit to go to 0. By continuity of the norm and completeness of the space, we can take $\|\cdot\|$ inside the limits. So, we want for the following limit to exist and be equal to 0:

$$\lim_{\|h\| \to 0} \frac{\lim_{m,n,k\to\infty} \|f_m(x+h) - f_n(x) - f_k'(h)\|}{\|h\|}.$$

Let $\varepsilon > 0$. By convergence of $\{f_i\}$ at $x + h$, we can find some N_1 such that, for $n, m \geq N_1$,

$$\|f_m(x+h) - f_n(x+h)\| < \varepsilon \|h\| / 3.$$

Here h was fixed and $\|h\|$ was a constant. By convergence of f_k', there exists some N_2 such that, for $n, k \geq N_2$,

$$\|f_k'(x) - f_n'(x)\| < \varepsilon/3,$$

which implies that

$$\|f'_k(x)(h) - f'_n(x)(h)\| < \varepsilon \|h\| /3.$$

Finally, by the definition of f'_n, there exists some N_3 such that, for $n \geq N_3$,

$$\|f_n(x+h) - f_n(x) - f'_n(x)(h)\| < \varepsilon \|h\| /3.$$

Let $N = \max\{N_1, N_2, N_3\}$. Then, for $m, n, k \geq N$,

$$\begin{aligned}
&\|f_m(x+h) - f_n(x) - f'_k(x)(h)\| \\
&= \|f_m(x+h) - f_n(x+h) + f_n(x+h) - f_n(x) - f'_n(x)(h) + f'_n(x)(h) - f'_k(x)(h)\| \\
&\leq \|f_m(x+h) - f_n(x+h)\| + \|f_n(x+h) - f_n(x) + f'_n(x)(h)\| + \|f'_k(x)(h) - f'_n(x)(h)\| \\
&< \varepsilon \|h\| /3 + \varepsilon \|h\| /3 + \varepsilon \|h\| /3 \\
&= \varepsilon \|h\| .
\end{aligned}$$

Therefore, we see that

$$\begin{aligned}
\lim_{\|h\| \to 0} \frac{\lim_{m,n,k\to\infty} \|f_m(x+h) - f_n(x) - f'_k(x)(h)\|}{\|h\|} &\leq \lim_{\|h\|\to 0} \frac{\varepsilon \|h\|}{\|h\|} \\
&= \varepsilon
\end{aligned}$$

Because ε was arbitrary, we see that the original limit does indeed go to 0, showing that g is the derivative of f everywhere. Because g is continuous, this implies that f is C^1.

Now, I notice that I have not used the uniform convergence of f'_k to g, only its pointwise convergence. I figure that at one step I assumed something converged too easily, or that limits commuted when they didn't. □

Problem 2. Let $f \in \mathcal{C}^k(U, F)$, where U is an open subset of a Banach space E and F is another Banach space. Let $x_0 \in U$ and $v \in E$ be such that $x_0 + tv \in U$, for every $t \in [0, 1]$. Prove the Taylor's formula:

$$f(x_0 + v) = f(x_0) + \left(\sum_{i=1}^{k} \frac{1}{i!} D^i f(x_0)(v, \ldots, v) \right) + R_k(x_0, v)$$

where $\|R_k(x_0, v)\|/\|v\|^k \to 0$ as $v \to 0$.

Proof. incomplete □

Problem 3. Let E be a Banach space. Show that the mapping $\mathrm{Inv} : \mathcal{GL}(E, E) \to \mathcal{GL}(E, E)$ given by $\mathrm{Inv}(T) = T^{-1}$ is differentiable and find its derivative.

Proof. First, we prove the following lemma: for any continuous (i.e. bounded) map $A \in \mathcal{GL}(E, E)$, the map $A^* : L(E, E) \to L(E, E)$ defined by $B \mapsto AB$ is also continuous. Because A^* is linear (it takes 0 to 0, distributes over sums, and commutes with scalars), it suffices to prove boundedness; we show that $\|A^*\| \leq \|A\|$.

Let $\|B\| = 1$. Then

$$\begin{aligned}\|A^* B\| &= \sup\{\|ABx\|\,; x \in E\} \\ &\leq \|A\| \sup\{\|Bx\|\,; x \in E\} \\ &= \|A\|\end{aligned}$$

Therefore, the function $A^* : \mathcal{GL}(L(E,E), L(E,E))$ is continuous.

We show that the derivative of Inv at any point $T \in \mathcal{GL}(E,E)$ is the map $D\mathrm{Inv}(T) : L(E,E) \to L(E,E)$ given by sending B to $-T^{-1}BT^{-1}$:

$$\begin{aligned}
&\lim_{\|B\|\to 0} \frac{\mathrm{Inv}(T+B) - \mathrm{Inv}(T) + T^{-1}BT^{-1}}{\|B\|} \\
&= \lim_{\|B\|\to 0} \frac{(T+B)^{-1} - T^{-1} + T^{-1}BT^{-1}}{\|B\|} \\
&= \lim_{\|B\|\to 0} \frac{(T+B)^{-1}I - IT^{-1} + T^{-1}BT^{-1}}{\|B\|} \\
&= \lim_{\|B\|\to 0} \frac{(T+B)^{-1}(TT^{-1}) - ((T+B)^{-1}(T+B))T^{-1} + T^{-1}BT^{-1}}{\|B\|} \\
&= \lim_{\|B\|\to 0} \frac{(T+B)^{-1}\left[(TT^{-1}) - (T+B)T^{-1}\right] + T^{-1}BT^{-1}}{\|B\|} \\
&= \lim_{\|B\|\to 0} \frac{(T+B)^{-1}[(T - (T+B)]T^{-1} + T^{-1}BT^{-1}}{\|B\|} \\
&= \lim_{\|B\|\to 0} \frac{(T+B)^{-1}[-B]T^{-1} + T^{-1}BT^{-1}}{\|B\|} \\
&= \lim_{\|B\|\to 0} \frac{((T+B)^{-1}[-B] + T^{-1}B)T^{-1}}{\|B\|} \\
&= \lim_{\|B\|\to 0} \frac{(T^{-1} - (T+B)^{-1})BT^{-1}}{\|B\|} \\
&= \left(\lim_{\|B\|\to 0} \frac{(T^{-1} - (T+B)^{-1})B}{\|B\|}\right) T^{-1}
\end{aligned}$$

The boundedness of the function $T^{-1*} : B \mapsto T^{-1}B$ says precisely that the last limit goes to 0. Therefore the defining limit of the derivative of Inv at T goes to zero, so $D\mathrm{Inv}(T)$ is the function $B \mapsto T^{-1}BT^{-1}$. This function is continuous in both T and B, so Inv is C^1. □

Problem 4. Let U be an open subset of a Banach space E. Given a function $g \in C^1(U, \mathbb{R})$, define the mapping

$$S_g : C([0,1], U) \longrightarrow \mathbb{R}, \quad S_g(f) = \int_0^1 g(f(s))ds$$

Show that S_g is C^1 and find its derivative.

Proof. First, to find what the derivative of S_g should be, we do some infinitesimal perturbation and see what happens. Let $f, v \in C([0,1], U)$, $\delta \in \mathbb{R}$, and assume $\delta^2 \approx 0$. We want to see what the value of $S_g(f + \delta v) - S_g(f)$ is, to a first-order approximation. We write $R_2(\delta)$ for a term which includes a factor of δ^2.

$$\begin{aligned} S_g(f+\delta v) - S_g(f) &= \int_0^1 g(f(s) + \delta v(s))ds - \int_0^1 g(f(s))ds \\ &= \int_0^1 \left(g(f(s)) + \delta Dg(f(s))(v(s)) + R_2(\delta)\right) ds - \int_0^1 g(f(s))ds \\ &= \delta \int_0^1 Dg(f(s))(v(s))ds + R_2(\delta) \\ &\approx \delta \int_0^1 Dg(f(s))(v(s))ds \end{aligned}$$

So, we expect the derivative of S_g to be the map $DS_g : C([0,1], U) \to C(C([0,1], U), \mathbb{R})$, defined by

$$DS_g(f)(v) = \int_0^1 Dg(f(s))(v(s))ds$$

We first show that this function is continuous in f and linear in v.

Fixing f, $DS_g(f)(v)$ is linear in v by virtue of the facts that $Dg(f(s))$ is linear, and that the integral operator is linear. We can see that the axioms of linearity hold:

- $DS_g(f)(\alpha x) = \alpha DS_g(f)(x)$:

$$\begin{aligned} DS_g(f)(\alpha x) &= \int_0^1 Dg(f(s))(\alpha x(s))ds \\ &= \int_0^1 \alpha Dg(f(s))(x(s))ds \\ &= \alpha \int_0^1 Dg(f(s))(x(s))ds \\ &= \alpha Ds_g(f)(x) \end{aligned}$$

- $DS_g(f)(x+y) = DS_g(f)(x) + DS_g(f)(y)$:

$$\begin{aligned} DS_g(f)(x+y) &= \int_0^1 Dg(f(s))(x(s) + y(s))ds \\ &= \int_0^1 (Dg(f(s))(x(s)) + Dg(f(s))(y(s)))ds \\ &= \int_0^1 Dg(f(s))(x(s))ds + \int_0^1 Dg(f(s))(y(s))ds \\ &= DS_g(f)(x) + DS_g(f)(y) \end{aligned}$$

Therefore, the function $DS_g(f)$ is a linear map.

Now, we show that $DS_g(f)$ is continuous in f. Let $f_n \to f$ as $n \to \infty$. We want to see that $DS_g(f_n) \to DS_g(f)$. As we are working in Banach spaces, we can equivalently check that $\|DS_g(f - f_n)\| \to 0$. Let v be an arbitrary element of $C([0,1], U)$ with norm 1. Because Dg is continuous, and therefore bounded on $C([0,1], U)$, we see:

$$\begin{aligned}
\lim_{n\to\infty} \|DS_g(f - f_n)(v)\| &= \lim_{n\to\infty} \|\int_0^1 Dg(f(s) - f_n(s))(v(s))ds\| \\
&\leq \lim_{n\to\infty} \int_0^1 \|Dg(f(s) - f_n(s))(v(s))\| \, ds \\
&\leq \lim_{n\to\infty} \int_0^1 \|Dg(f(s) - f_n(s))\| \, \|v(s)\| \, ds \\
&\leq \lim_{n\to\infty} \int_0^1 \|Dg\| \, \|f(s) - f_n(s)\| \, ds \\
&\leq \lim_{n\to\infty} \|Dg\| \cdot \sup\{\|f(s) - f_n(s)\| \, ; s \in [0,1]\} \\
&= 0
\end{aligned}$$

So, DS_g is continuous. Therefore, S_g is continuously differentiable, and its derivative is DS_g. □

Problem 5. Let M be some σ-algebra of subsets of X, let N be some σ-algebra of subsets of Y. Given is a function $f : X \to Y$. Which of the following families is a σ-algebra? Give a proof or a counterexample.

a) $\{B \subset Y; f^{-1}(B) \in M\}$,

b) $\{f(A); A \in M\}$

c) $\{A \subset X; f(A) \in N\}$,

d) $\{f^{-1}(B); B \in N\}$

Proof. The three axioms of a σ-algebra Ω on a set X that we use are:

- $\emptyset \in \Omega$
- $X \in \Omega$
- Ω is closed under countable unions

a) This is a σ-algebra.

  - It must contain the empty set $\emptyset_Y \subset Y$, because $\emptyset_X \subset X$ must lie in M, as M is a σ-algebra, and $\emptyset_X = f^{-1}(\emptyset_Y)$.
  - It also contains Y, because $f^{-1}(Y) = X,$, and $X \in M$, as M is a σ-algebra.

- It is closed under countable unions, because the inverse-image map respects unions: if $f^{-1}(A_i) \in M$ for all i in some countable indexing set I, then

$$f^{-1}\left(\bigcup_i A_i\right) = \bigcup_i f^{-1}(A_i),$$

which is in M because M is closed under countable unions.

b) This is not a σ-algebra, because it might fail to contain the whole set Y. This occurs if and only if f is not surjective; for instance, the function $x \mapsto x^2$, $\mathbb{R} \to \mathbb{R}$, does not generate a σ-algebra on $\mathbb{R}$, because each set in its image is contained in $[0, \infty)$.

If f is surjective, then this is a σ-algebra, as $\emptyset = f(\emptyset), Y = f(X)$, and $\bigcup f(A_i) = f\left(\bigcup A_i\right)$.

c) This is not a σ-algebra, as it might not contain the whole set X; this occurs if and only if $f(X) \notin N$. For instance, if N is the trivial σ-algebra $\{, \mathbb{R}\}$ on R, and f is again $x \mapsto x^2$, then $f(X) = [0, \infty)$, which is not in N.

d) This is a σ-algebra.

- It contains $\emptyset$, because $\emptyset = f^{-1}(\emptyset)$, and $\emptyset \in N$.
- It contains X, because $X = f^{-1}(Y)$, and $Y \in N$.
- It is closed under countable unions, because $\bigcup_i f^{-1}(A_i) = f^{-1}\left(\bigcup_i A_i\right)$.

We conclude that the inverse image functor is more well-behaved than the direct image functor, as usual. $\square$

References

[1] Lax, P. Linear Algebra. Wiley, 1997.