Homework 3

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Analysis I

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Problem 1. Prove that every metric space is paracompact; that is, every open covering admits a locally finite refinement.

Proof. This was proven first by A.H. Stone in 1948, in [3]. The following proof follows a 2010 proof by Akhil Mathew, from [1], which in turn adapts a 1968 proof by M. H. Rudin in [2].

Let (E, d) be a metric space, and let $\{U_i\}_{i \in I}$ be an open covering of E in the metric topology. We show that there is a covering $\{V_j\}_{j \in J}$, such that

- $\bigcup_{j \in J} V_j = E$
- For each $j \in J$, there is some $i \in I$ such that $V_j \subset U_i$
- For each $x \in E$, the collection of all V_j such that $x \in V_j$ is finite.

Our open cover $\{U_i\}_{i \in I}$ is not necessarily countable, as our metric space is not necessarily second countable (on a metric space, this is equivalent with being separable). However, by taking the axiom of choice, we may assume that the set I is well-ordered.

Now, for each U_i we define a sequence of sets $V_i^n = \{x \in U_i \mid d(x, E \backslash U_i) \geq 2^- n\}$. A point in V_i^n is a point of U_i which is not too close to the boundary of U_i. Taking the union of V_i^n over all $n \in \mathbb{N}$ gives us U_i again.

Now, for each $i \in I$, define the set

$$W_i^n = V_i^n - \bigcup_{j<i} V_j^{n+1}$$

This gets rid of redundancies while still covering E: For each point x, there is some U_i which contains x. Because U_i is open, the distance from x to the exterior of U_i must be positive, so it must be greater than some 2^{-n}. Therefore, x is contained in V_i^m for all $m \geq n$, and not in any V_j^{m+1} for any $j < i$. Therefore, $x \in W_i^m$ for all $m \geq n$. It is also not in any W_k^m for $k > i$, $m \geq n$.

However, the W_i^m are not necessarily open. We can take a small neighborhood of each;

$$Z_i^n = \left\{x \in E \mid d(x, W_i^n) < 2^{-n-3}\right\}$$

These are open sets, and like the W_i^ns, each x is contained in only one Z_i^n for large enough n. Further, because the radius 2^{-n-3} around W_i^n is strictly smaller than 2^{-n}, each Z_i^n is contained in U_i, so the collection $\{\{Z_i^n\}_{i\in I}\}_{n=1}^{\infty}$ is a refinement of $\{U_i\}$.

Now, it is in fact true that this cover is locally finite - this construction follows closely the construction done by M. E. Rudin in [2]. The proof, which I do not understand well enough to reproduce here, shows that for any x, after choosing some $Z_i^n \ni x$, and j such that the open ball $B_{2^{-j}}(x) \subset Z_i^n$, there are *no* $Z_i^k \supset B_{2^{-j}}(x)$ for $k \geq n+j$, and that for $k < n+j$, there is only one such Z_i^k; therefore any open ball around x is contaned in finitely many of the open sets Z_i^n, and it is a locally finite cover. □

Problem 2. Recall that a metric space Y is an extensor, if for every continuous function $f : A \to Y$ defined on a close dsubset A of a metric space Z, there exists a continuous function $\tilde{f} : X \to Y$ such that $\tilde{f}(x) = f(x)$ for every $x \in A$. Prove that:

(i) A space homeomorphic to an extensor is also an extensor.

(ii) A retract of an extensor is an extensor.

(iii) if Y_1 and Y_2 are extensors, then $Y_1 \times Y_2$ is an extensor.

Proof. (i) Let Y be an extensor, and let Z be homeomophic to Y along the homeomorphism $\varphi : Y \to Z$. Now, take some closed subspace A of a metric space X, and let $f : A \to Z$ be a continuous function.

We see then that $\varphi^{-1} \circ f$ is a continuous function $A \to Y$, and thus that there is a continuous function $\widetilde{\varphi^{-1} \circ f} : X \to Y$ which agrees with $\varphi^{-1} \circ f$ on A. Composing with φ, we see that $\varphi \circ \widetilde{\varphi^{-1} \circ f}$ is a continuous function $X \to Z$, such that for any $x \in A$,

$$\begin{aligned} \varphi \circ \widetilde{\varphi^{-1} \circ f}(x) &= \varphi \circ \varphi^{-1} \circ f(x) \\ &= f(x), \end{aligned}$$

as desired.

(ii) Let Y be an extensor, and let $Z \subset Y$ be a retract of Y, with $\iota : Z \to Y$ be the inclusion map, and $\pi : Y \to Z$ a continuous projection such that $\pi \circ \iota = \mathrm{id}_Z$. Let A be a closed subset of a metric space X, and let $f : A \to Z$ be a continuous map.

Then $\iota \circ f$ is a continuous map $A \to Y$, and there exists some map $\widetilde{\iota \circ f} : X \to Y$ which agrees with $\iota \circ f$ on A. Then, composing with π, we see that for any $x \in A$,

$$\begin{aligned} \pi \circ \widetilde{\iota \circ f}(x) &= \pi \circ \iota \circ f(x) \\ &= f(x) \end{aligned}$$

So Z is an extensor.

(iii) Let Y_1, Y_2 be extensors, and let A be a closed subset of a metric space X, with map $f : A \to Y_1 \times Y_2$. By the universal property of the product, f is determined by two continuous maps $f_1 : A \to Y_1$, and $f_2 : A \to Y_2$. Because these two target spaces are extensors, we have two maps $\tilde{f}_1 : X \to Y_1$, and $\tilde{f}_2 : X \to Y_2$, which agree with f_1 and f_2, respectively, on A. Again, by the mapping property of the product, together these give us a map $\tilde{f}_1 \times \tilde{f}_2$, such that for any $x \in A$,

$$\begin{aligned}\tilde{f}_1 \times \tilde{f}_2(x) &= (\tilde{f}_1(x), \tilde{f}_2(x)) \\ &= (f_1(x), f_2(x)) \\ &= f(x)\end{aligned}$$

So the product space is also an extensor.

□

Problem 3. Find the norms of the following linear functionals on $\mathcal{C}[-1,1]$:

(i) $T(f) := \int_0^1 f(x)dx$,

(ii) $T(f) := \int_{-1}^1 (\text{sgn}(x))f(x)dx$,

(iii) $T(f) := \int_{-1}^1 f(x)dx - f(0)$,

(iv) $T(f) := \frac{f(\varepsilon)+f(-\varepsilon)-2f(x)}{\varepsilon^2}$,

(v) $T(f) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} f(1/n)$

Proof. (i) We show that $\|T\| = 1$. It is clear that it is at most one, as for any f with $\sup\{|f(x)|; x \in [-1,1]\} = 1$, the value of $|T(f)|$ is at most 1:

$$\begin{aligned}|T(f)| &= \left| \int_0^1 f(x)dx \right| \\ &\leq \int_0^1 |f(x)|dx \\ &\leq \int_0^1 dx \\ &= 1\end{aligned}$$

This bound is attained, at for example the function

$$f(x) = \begin{cases} -\text{erf}(1/x) & 1 \leq x < 0 \\ 1 & 0 \leq x \leq 1. \end{cases}$$

Where erf is the real error function, $\text{erf}(x) = \frac{1}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. [1]

[1] I'm kidding - just take $f(x) = 1$.

(ii) The value of $\|T\|$ is bounded above in this case by 2, as for any f such that $|f| = 1$, the value of $|\text{sgn}(f(x))|$ is bounded by 1, giving us

$$\begin{aligned}
|T(f)| &= \left|\int_{-1}^{1} \text{sgn}(f(x))dx\right| \\
&\leq \int_{-1}^{1} |\text{sgn}(f(x))|dx \\
&\leq \int_{-1}^{1} 2dx \\
&= 2
\end{aligned}$$

This bound is not attained, but we may approach it with a sequence such as $\{f_n\}_{n\geq 1}$, where f_n is defined as:

$$f_n(x) = \begin{cases} -1 & x < -\frac{1}{n} \\ nx & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & \frac{1}{n} < x \end{cases}$$

The integral on the region $[-\frac{1}{n}, \frac{1}{n}]$ is 0, so the value of $|T(f_n)|$ is $2 - \frac{2}{n}$, which approaches 2 as $n \to \infty$. Therefore the value of $\|T\|$ is 2.

(iii) The value of $\|T\|$ in this case is 3. It is bounded above by this value; for f such that $|f| = 1$, it is true that $|f(x)| \leq 1$, and that $|f(0)| \leq 1$, so

$$\begin{aligned}
|T(f)| &= \left|\int_{-1}^{1} f(x)dx - f(0)\right| \\
&\leq \int_{-1}^{1} |f(x)|dx + |f(x)| \\
&\leq 2 + 1
\end{aligned}$$

This value is not attained at any function in $\mathcal{C}[-1, 1]$, but we may approach it with a sequence of functions $\{f_n\}$ like the following:

$$f_n(x) = \begin{cases} 1 & -1 \leq x < -\frac{1}{n} \\ -2nx - 1 & -\frac{1}{n} \leq x \leq 0 \\ 2nx - 1 & 0 < x \leq \frac{1}{n} \\ 1 & \frac{1}{n} < x \end{cases}$$

Again, the intregral of such a function is 0 on $[-\frac{1}{n}, \frac{1}{n}]$. The value of $|T(f_n)|$ is $2 - \frac{2}{n} + 1$, which approaches 3 as $n \to \infty$.

(iv) The value of $\|T\|$ can be bounded above by $4/\varepsilon^2$, since $|f(\varepsilon)| \leq 1$, and the same for $|f(-\varepsilon)|$ and $|f(0)|$. In fact, the value $|T(f)| = 4/\varepsilon^2$ is attained, at any function where

$f(\varepsilon) = f(-\varepsilon) = -f(0) = 1$, or at the negative of such a function. A continuous, norm-1 example is

$$f(x) = \begin{cases} 1 & -1 \le x \le -\varepsilon \\ -2x/\varepsilon - 1 & -\varepsilon < x \le 0 \\ 2x/\varepsilon - 1 & 0 < x \le \varepsilon \\ 1 & \varepsilon < x \le 1 \end{cases}$$

The value of $f(\varepsilon)$ and $f(-\varepsilon)$ is 1, and the value of $f(0)$ is -1, so $|T(f)| = (1 + 1 - (-2))/\varepsilon^2 = 4/\varepsilon^2$.

(v) The value of $\|T\|$ is $\pi^2/6$. We can bound by this value, because, given some f such that $|f| = 1$,

$$\begin{aligned} |T(f)| &= \left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} f(1/n) \right| \\ &\le \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} f(1/n) \right| \\ &\le \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{\pi^2}{6} \end{aligned}$$

This value is not attained, as f would need to oscillate infinitely often between -1 and 1 as $1/n \to 0$. However, we can approximate such a function with the following sequence $\{f_m\}$:

$$f_m(x) = \begin{cases} 0 & -1 \le x < \frac{2}{2m+1} \\ \cos(\pi/x) & \frac{2}{2m+1} \le x \le 1 \end{cases}$$

This function is continuous, since $f(2/(2m+1)) = \cos((2m+1)\pi/2) = 0$, and also $\cos(1/x)$ is continuous away from 0. For any $n \le m$,

$$\begin{aligned} f_m(1/n) &= \cos(n\pi) \\ &= (-1)^n \end{aligned}$$

And, for any $n \ge m+1$, $f_m(1/n) = 0$. Therefore, the value of $|T(f_m)|$ is $\sum_{n=1}^{m} \frac{1}{n^2}$, which approaches $\frac{\pi^2}{6}$ as $m \to \infty$.

$\square$

Problem 4. Prove that the space $\mathcal{C}_b(\mathbb{R}^N)$ of bounded continuous functions on $\mathbb{R}^N$, with the supremum norm $\|\cdot\|_\infty$, is not separable.

Proof. We exhibit an uncountable set Ω of elements of $\mathcal{C}_b(\mathbb{R}^N)$, such that each element of Ω is distance 1 from each other element. Let $\Omega' = 2^{\mathbb{N}}$, the uncountable power set of the natural numbers, and let $\Omega = \{f_U\}_{U \in \Omega'}$ be the set of functions f_U indexed by subsets U of the naturals, defined as follows.

Let B_n, for $n \in \mathbb{N}$, be the following bump function on $\mathbb{R}$:

$$B_n(x) = \begin{cases} 0 & x \notin [n, n+1] \\ 1 - 2|(x - n - 1/2)| & x \in [n, n+1] \end{cases}$$

The maximum value of $B_n(x)$ on $[n, n+1]$ is 1, it is 0 elsewhere, and it is continuous. Then, for any subset U of $\mathbb{N}$, let $f'_U : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f'_U(x) = \begin{cases} 0 & \lfloor x \rfloor \notin U \\ B_{\lfloor x \rfloor}(x) & \lfloor x \rfloor \in U \end{cases}$$

So, if the nearest integer below x is in U, we take x to a bump function; otherwise, to zero. This is like the sum of the bump functions B_n over $n \in U$, but U might be infinite.

This defines an uncountable set of functions f'_U on $\mathbb{R}$; we now define $\{f_U\}$ as the function sending $\{x_1, ...x_N\}$ to $f'_U(x_1)$. This set $\{f_U\}_{U \in 2^{\mathbb{N}}}$ is an uncountable set of functions which are all distance 1 from each other.

The fact that two f_U, $f_{U'}$ are distance 1 apart for distinct U, U' follows from the fact that there must be some $n \in U \backslash U' \cup U' \backslash U$. Assume without loss of generality that $n \in U \backslash U'$; then

$$\begin{aligned} |f_U(n + 1/2, 0, \ldots 0) - f_{U'}(n + 1/2, 0, \ldots 0)| &= |1 - 0| \\ &= 1 \end{aligned}$$

This contradicts separability, because any countable subset could only intersect a countable number of balls of radius $\frac{1}{2}$ around these elements, meaning it could not be dense. □

Problem L. et (X, d) be a metric space. Fix a reference point $x_0 \in X$ and let E be the vector space of all the Lipschitz continuous functions $f : X \to \mathbb{R}$ such that $f(x_0) = 0$. Define $\|f\|$ to be the smallest Lipschitz constant of f, that is:

$$\|f\| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

Prove that $(E, \|\cdot\|)$ is a Banach space.

Proof. We first show that $\|\cdot\|$ is a norm.

- $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$:

  It is clear that the norm of x is nonnegative, because it is the supremum of a set of ratios of nonnegative numbers $|f(x) - f(y)|$ with positive numbers $d(x, y)$. Now, if $\|x\| = 0$, then the value of
  $frac|f(x) - f(y)|d(x, y)$ must be zero for each pair $y \neq x$, meaning $|f(x) - f(y)|$ is always zero, and that f is constant. Because $f(x_0)$ is required to be zero, this implies that f can only be the zero function.

- $\|\alpha f\| = |\alpha| \|f\|$: This follows from the fact that $|\alpha f(x) - \alpha f(y)| = |\alpha| |f(x) - f(y)|$, and that a nonnegative constant can be factored out of a supremum.

- $\|f+g\| \leq \|f\| + \|g\|$: Pointwise, we see that

$$\begin{aligned}\frac{|(f+g)(x)-(f+g)(y)|}{d(x,y)} &= \frac{|f(x)-f(y)+g(x)-g(y)|}{d(x,y)} \\ &\leq \frac{|f(x)-f(y)|}{d(x,y)} + \frac{|g(x)-g(y)|}{d(x,y)}\end{aligned}$$

  Taking the supremum over all $x \neq y$, we see that indeed $\|f+g\| \leq \|f\| + \|g\|$.

We now need to show that the space is Banach with respect to the norm $\|\cdot\|$. Let f^k be a Cauchy sequence in this space; we need to construct a function f and then show that it lies in E and that it is the limit of the f^ks with respect to $\|\cdot\|$.

We first see that the Lipschitz constants of f^k must converge, as in any normed space the convergence of a sequence in $\|\cdot\|$ implies that the norms of the elements $\left\|f^k\right\|$ themselves converge, say to a value $L \in R$.

Fix $x \neq x_0 \in X$; we wish to calculate $f(x)$. We will see that the values $f^k(x)$ themselves converge as $k \to \infty$: Let $\varepsilon > 0$. By convergence of $\left\|f^k\right\|$ to L, we can find $n \in \mathbb{N}$ such that, for $k, l \geq n$, $\sup_{z \neq y} \frac{\left|(f^k - f^j)(z) - (f^k - f^j)(y)\right|}{d(z,y)} < \varepsilon / d(x, x_0)$. In particular,

$$\frac{\left|(f^k - f^j)(x) - (f^k - f^j)(x_0)\right|}{d(x,x_0)} < \frac{\varepsilon}{d(x,x_0)}$$

Since $f^k(x_0) = f^j(x_0) = 0$, this shows that $f^j(x) - f^k(x)$ converges to 0 as j and k go to ∞, so this is a Cauchy sequence of real numbers and has a limit in $\mathbb{R}$, which we call $f(x)$. If $x = x_0$, define $f(x) = 0$.

We now show that the function $x \mapsto f(x)$ is Lipschitz. Since the Lipschitz constants of the f^k converge to L, and the value of $f^k(x)$ converges pointwise to $f(x)$, this means that we may bound the Lipschitz constant of f as follows:

$$\begin{aligned}\sup_{x \neq y} \frac{|f(x)-f(y)|}{d(x,y)} &= \sup_{x \neq y} \frac{|\lim_{n\to\infty}(f^n(x)) - \lim_{m\to\infty}(f^m(y))|}{d(x,y)} \\ &= \sup_{x \neq y} \frac{|\lim_{n\to\infty}(f^n(x) - f^n(y))|}{d(x,y)} \\ &= \sup_{x \neq y} \lim_{n\to\infty} \frac{|f^n(x) - f^n(y)|}{d(x,y)} \\ &\leq \lim_{n\to\infty} \sup_{x \neq y} \frac{|f^n(x) - f^n(y)|}{d(x,y)} \\ &= L.\end{aligned}$$

Thus f is Lipschitz, with Lipschitz constant no greater than L.

Finally, we show that the functions f^n converge in Lipschitz norm to f - we wish to show that, as n goes to ∞, the quantity

$$\sup_{x \neq y} \frac{|(f^n(x) - f(x)) - (f^n(y) - f(y))|}{d(x,y)}$$

goes to 0. Calculating:

$$\begin{aligned}\lim_{n\to\infty}\sup_{x\neq y}\frac{|(f^n(x)-f(x))-(f^n(y)-f(y))|}{d(x,y)} &= \lim_{n\to\infty}\sup_{x\neq y}\lim_{m\to\infty}\frac{|(f^n(x)-f^m(x))-(f^n(y)-f^m(y))|}{d(x,y)} \\ &\leq \lim_{n\to\infty}\lim_{m\to\infty}\sup_{x\neq y}\frac{|(f^n(x)-f^m(x))-(f^n(y)-f^m(y))|}{d(x,y)}\end{aligned}$$

By Cauchy-ness of the sequence f^n in this norm, the last limit is equal to 0; therefore, the sequence converges to the limit f.

□

References

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[3] Stone, A.H. Paracompactness and Product Spaces. Bulletin of the AMS, Vol 54, Number 10, 1948.