Homework 6

Andrew Tindall
Topology II

February 21, 2020

Problem 1. Let $\mathbb{H}$ be the Hamilton Quaternions, with its standard $\mathbb{R}$-basis $\{1, i, j, k\}$. Draw an isomorphism of $\mathbb{R}$-vector spaces $\mathbb{H} \simeq \mathbb{R}^{\oplus 4}$ using this basis. COnsider S^3 to be a subspace of $\mathbb{H}$ under this isomorphism. Let $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion subgroup of the group of units $\mathbb{H}^\times$ of $\mathbb{H}$.

(a) Prove that the natural left action of Q_8 on $\mathbb{H}^\times$ by left multiplication induces a left action on S^3 which is a "covering space action."

Proof. We first define the left action $Q_8 \curvearrowright S^3$. Under the identification of $\mathbb{H}$ with $\mathbb{R}^{\oplus 4}$, the definition of S^3 as unit vectors in $\mathbb{R}^{\oplus 4}$ embeds S^3 as a subset of $\mathbb{H}$. Specifically, S^3 can be identified with the set

$$\left\{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\right\}.$$

We show that any of the elements of Q_8 preserve the magnitude of any unit element in $\mathbb{H}$. Let $x = a + bi + cj + dk$ be an arbitrary element of $S^3 \subset \mathbb{H}$. Then the actions of the 8 elements of Q_8 on x are as follows:

$$\begin{aligned} 1 \cdot x &= a + bi + cj + dk & (-1) \cdot x &= -a - bi - cj - dk \\ i \cdot x &= -b + ai - dj + ck & (-i) \cdot x &= b - ai + dj - ck \\ j \cdot x &= -c + di + aj - bk & (-j) \cdot x &= c - di - aj + bk \\ k \cdot x &= -d - ci + bj + ak & (-k) \cdot x &= d + ci - bj - ak \end{aligned}$$

In each case, we see that the sum of squares of the coefficients of $g \cdot x$ are preserved, meaning that $g \cdot x \in S^3$ as well, and so the action $Q_8 \curvearrowright \mathbb{H}$ does indeed induce an action $Q_8 \curvearrowright S^3$.

Now, we want to show that this action is a covering space action. Given some $x \in S^3 \subset \mathbb{H}$, we want to find some open set U containing x such that the $g \cdot U$ are disjoint for each $g \in Q_8$.

First, we can show that a free action of a finite group on a Hausdorff space is always a covering space action. Let G be a finite group, X a Hausdorff space, and $\rho : G \hookrightarrow \mathrm{Aut}(x)$ a free group action, i.e. an action such that for each $g \in G$, $\rho(g)(x) \neq x$. We write $g \cdot x$ for $\rho(g)(x)$.

Since X is Hausdorff, there exist open sets $U_1 \ni x$, $U_2 \ni g{\cdot}x$ such that $U_1 \cap U_2 = \emptyset$. Then the set $U_g := U_1 \cap g^{-1}{\cdot}U_2$ contains x, and is a subset of U_1, while $g{\cdot}U_g = g{\cdot}(U_1 \cap g^{-1}{\cdot}U_2)$ is a subset of U_2, and so it is disjoint from U_g. Since G is finite, the intersection

$$U_x = \bigcap_{g \in G} U_g$$

is an open set, which contains x, and is disjoint from each of its translates under the action of G. Therefore, $G \curvearrowright X$ is a covering space action.

Since Q_8 is finite, S^3 is Hausdorff, and $Q_8 \curvearrowright S^3$ is free, it must be a covering space action. □

(b) Write down all of the isomorphism classes of covers of the orbit space S^3/Q_8, using the fact that S^3 is simply connected.

Proof. Since the group action of Q_8 on S^3 is a covering space action, by Hatcher 1.40 we see that

$$Q_8 \simeq \pi_1(S^3/Q_8)/p_*(\pi_1(S^3)).$$

But, since S^3 is simply connected, the group $\pi_1(S^3)$ is trivial, and so $Q_8 \simeq \pi_1(S^3/Q_8)$. Thus the set of isomorphism classes of covering spaces of S^3/Q_8 corresponds exactly to the set of subgroups of Q_8. There are 4 nontrivial subgroups of Q_8: three copies of $\mathbb{Z}_4$:

$$\{1, i, -1, -i\}$$
$$\{1, j, -1, -j\}$$
$$\{1, k, -1, -k\}$$

and one copy of $\mathbb{Z}_2$, the center of Q_8, which is $\{1, -1\}$. Thus there are 6 total isomorphism classes of covering spaces,4 of them corresponding to these 4 subgroups:

$$\begin{aligned} (S^3/\mathbb{Z}_4)_1 &\to S^3/Q_8, \\ (S^3/\mathbb{Z}_4)_2 &\to S^3/Q_8, \\ (S^3/\mathbb{Z}_4)_3 &\to S^3/Q_8, \\ S^3/\mathbb{Z}_2 &\to S^3/Q_8, \end{aligned}$$

where the three numbered covers are different, corresponding to the actions of i, j, and k on S^3 under multiplication in $\mathbb{H}$. These 4, along with the trivial cover $S^3/Q_8 \to S^3/Q_8$, which corresponds to the subgroup $\{1\} \leq Q_8$l and the universal cover $S^3 \to S^3/Q_8$, which corresponds to the whole subgroup $Q_8 \leq Q_8$, exhaust all of the isomorphism classes of covers of S^3/Q_8. □

Problem 2. Using example 1.48 as a starting point, write down all of the path-connected covering spaces of the wedge sum $\mathbb{RP}^2 \times \mathbb{R} \vee \mathbb{P}^2$, up to isomorphism.

Proof. We will show that the path-connected covering spaces of the space $\mathbb{RP}^2 \vee \mathbb{R}¶^2$ are even-numbered rings of spheres, and even-numbered chains of 2-spheres with a copy of $\mathbb{RP}^2$ wedged at either end. *incomplete* □