Homework 9

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Analysis I

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Problem 1. Given an open box $A = (a_1, b_1) \times \cdots \times (a_n, b_n) \subset \mathbb{R}^n$ of finite, positive volume, construct a sequence φ_n of functions in $\mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\varphi_n \nearrow \mathbf{1}_A$.

Proof. We use the following step function:

$$\sigma(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & -1 < x < 0 \\ 0, & x \leq -1 \\ 1, & x \geq 0 \end{cases}$$

It is smooth, and it is 0 for $x \leq -1$ and 1 for $x \geq 0$. Then, using the two linear transformations

$$x \mapsto \frac{x-a}{\varepsilon} - 1, \text{ and}$$
$$y \mapsto -\frac{y-b+\varepsilon}{\varepsilon},$$

which map $[a, a+\varepsilon]$ and $[b, b-\varepsilon]$ to $[-1, 0]$, respectively, we get the two step functions

$$f_{\varepsilon,a}(x) = \begin{cases} \exp\left(-\frac{1}{1-((x-a)/\varepsilon-1)^2}\right) & , x \in (a, a+\varepsilon) \\ 0, & x \leq a \\ 1, & x \geq a+\varepsilon \end{cases}$$

$$g_{\varepsilon,b}(x) = \begin{cases} \exp\left(-\frac{1}{1-(-(y-b+\varepsilon)/\varepsilon)^2}\right), & b-\varepsilon < x < b \\ 0, & x \geq b \\ 1, & x \leq b-\varepsilon \end{cases}$$

Then, using these two step functions, and assuming that $a+\varepsilon < b-\varepsilon$, we have the following bump function:

$$h_{a,b,\varepsilon}(x) = \begin{cases} f_{\varepsilon,a}(x), & x \leq a+\varepsilon \\ 1, & a+\varepsilon < x < b-\varepsilon \\ g_{\varepsilon,b}(x), & x \geq b-\varepsilon \end{cases}$$

This is a smooth function with compact support that is 0 outside of (a, b), and is 1 inside of $(a + \varepsilon, b - \varepsilon)$.

Now, let m be the minimum value of $b_i - a_i$ for $1 \leq i \leq n$, and let $\varepsilon < m/2$. Define the function $H_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by

$$H_\varepsilon(x_1, \ldots x_n) = (h_{a_1,b_1,\varepsilon}(x_1), \ldots h_{a_n,b_n,\varepsilon}(x_n)).$$

Because each component function is smooth, and has compact support, H is $\mathcal{C}_c^\infty$, and is 0 outside of A and 1 inside the box $(a_1 + \varepsilon, b_1 - \varepsilon) \times \cdots \times (a_n + \epsilon, b_n - \epsilon)$.

Finally, let $\varphi_n = H_{m/2^{n+1}}$. This is a monotone increasing sequence of functions which, for any point in the interior of A, is eventually 1, and is always 0 for any point in the complement of A. Therefore, $\varphi_n \nearrow \mathbf{1}_A$. □

Problem 2. Prove that if $A \in \mathcal{L}_{n+m}$ has measure 0, then for almost every $x \in \mathbb{R}^n$ the set:

$$A_y := \{x \in \mathbb{R}^n; (x, y) \in A\}$$

is in $\mathcal{L}_n$ and has measure 0. (I did this problem in x instead of y before noticing my mistake; the argument is symmetric)

Proof. We first prove that the set A_x is in $\mathcal{L}_m$, for all x. The following proof follows Rudin in [1]:

Let Ω be the class of all $A \in \mathcal{L}_{n+m}$ such that $A_x \in \mathcal{L}_m$for every $x \in \mathbb{R}^n$. If $A = (a, b) \times (c, d)$ then $A_x = (c, d)$ if $x\ in(a, b)$, and $\emptyset$ otherwise. Therefore, every measurable rectangle $(a, b) \times (c, d)$ belongs to Ω. The following statements show that Ω is a σ-algebra, and therefore that it is equal to $\mathcal{L}_{n+m}$:

(a) $\mathbb{R}^{n+m} \in \Omega$

(b) If $A \in \Omega$, then $(A^c)_x = (A_x)^c$, so $(A^c)_x$ is measurable and $A^c \in \Omega$

(c) If $A_i \in \Omega$ for $i \in (1, 2, 3, \ldots)$ and $A = \bigcup A_i$, then $A_x = \bigcup (A_i)_x$, so it A_x is measurable, and $A \in \Omega$

So, $\Omega = \mathcal{L}_{n+m}$, and every set A in $\mathcal{L}_{n+m}$ is such that A_x is measurable for all x.

Now, assume that A has measure 0; we show that A_x has measure 0 for almost every $x \in \mathbb{R}^n$. Let X_ε be the set of all $x \in \mathbb{R}^n$ such that $m(A_x) \geq \varepsilon$ for all $x \in X_\varepsilon$; by the above argument, $\left(\bigcup_{\varepsilon>0} X_\varepsilon\right)^c$ is the set of all $x \in \mathbb{R}^n$ such that A_x has measure 0.

Fix some $\varepsilon > 0$; we show that X_ε has measure 0. To do this, fix $\delta > 0$; we show that $m(X_\varepsilon) \leq \delta$.

Let A_ε be the set $\{(x, y) \in A; x \in X_\varepsilon\}$. Since A has measure 0, we can cover it with a union of boxes with total size $\leq \delta\varepsilon$. By definition of A_ε, the width of each box (its measure when projected onto its y-component) must be $\geq \varepsilon$; therefore the sum of the heights of the boxes must be $\leq \delta$. Because δ was arbitrary, the projection of A_ε onto its x-components must be measure 0, meaning that X_ε has measure 0.

Let X be the set of all x such that A_x has measure > 0. Because $X = \bigcup X_\varepsilon$, a countable union of measure 0 sets, it must also have measure 0, which was to be shown. □

Problem 3. *incomplete*

Problem 4. Let $f_n : [0, \pi] \to \mathbb{R}$ be given by:

$$f_n(x) = n \frac{\sin x}{1 + n^2 \sin^2 x}.$$

For a given $\varepsilon > 0$, find explicitly the Egoroff set E_ε on which the sequence f_n converges uniformly, and such that $\mu(E_\varepsilon) > \pi - \varepsilon$.

Proof. We show that $[\varepsilon/3, \pi - \varepsilon/3]$ satisfies the requirements. The measure of this set is $\pi - 2\varepsilon/3$, so it satisfies the measure requirement; we can also show that $f_n \to 0$ uniformly on E_ε as $n \to \infty$.

Each f_n is the composition of the rational function $g_n : [0, 1] \to \mathbb{R}$ given by

$$g_n(x) = n \frac{x}{1 + n^2 x^2}$$

with the function $\sin : [0, \pi] \to [0, 1]$. Because sin is smooth and monotone on $[0, \pi/2]$ and on $[\pi/2, \pi]$, with bounded first derivative, it suffices to show that g_n converges smoothly to 0 on $\sin([\varepsilon/3, \pi - \varepsilon/3])$, which is $[\sin(\varepsilon/3), 1]$.

We wish to show that, for $\delta > 0$, there exists some $N \in \mathbb{N}$ such that, for all $n \geq N$, $|g_n(x)| < \delta$ for all $x \in [\sin(\varepsilon/3), 1]$.

First, we note that for large enough n, g_n is monotone decreasing on $[\sin(\varepsilon/3), 1]$. In particular, this holds for all $n > \sin^{-1}(1/\varepsilon)$, as g_n' has only one 0 in $[0, 1]$, which is at $1/n$. For $n > \sin^{-1}(3/\varepsilon)$, this maximum falls outside of the range $[\sin(\varepsilon/3), 1]$, so g_n is positive and monotone decreasing on this interval, and we need only check that $g_n(\sin(\varepsilon/3)) < \delta$ for large enough n:

$$\begin{aligned} n \frac{\sin(\varepsilon/3)}{1 + n^2 \sin^2(\varepsilon/3)} &< \delta \\ n \sin(\varepsilon/3) &< \delta + \delta n^2 \sin^2(\varepsilon/3) \\ 0 &< \delta n^2 \sin^2(\varepsilon/3) - n \sin(\varepsilon/3) + \delta \end{aligned}$$

Because this is a quadratic polynomial in the sin terms with positive first coefficient, it suffices to show that, for sufficiently large n, both zeroes of the polynomial

$$\delta n^2 x^2 - nx + \delta$$

are less than $\sin(\varepsilon/3)$. By the quadratic formula, the zeroes are

$$\frac{n \pm \sqrt{n^2 - 4\delta n^2}}{2\delta^2 n^4} = \frac{1 \pm \sqrt{1 - 4\delta}}{2\delta^2 n^3}.$$

Assuming δ is small enough, $\sqrt{1 - 4\delta}$ is close to 1, so the zeroes are at approximately 0 and $1/\delta^2 n^3$, with neither one greater than $1/\delta^2 n^3$. For $n > (\delta^2 \sin(\varepsilon/3))^{-1/3}$, both zeroes are less than $[\sin(\varepsilon/3)]$, and so the function g_n is bounded as desired. Therefore, f_n converge uniformly to 0 on E_ε. $\square$

Problem 5. Prove that for every set $A \subset \mathbb{R}^n$ which is not of Lebesgue measure 0, there holds:

$$\forall c \in (0,1) \quad \exists P \subset \mathbb{R}^n \qquad \mu^*(A \cap P) > c\mu^*(P).$$

Proof. We show the contrapositive: if A is a subset of $\mathbb{R}^n$ such that there exists a c for which, for all closed boxes $P \subset \mathbb{R}^n$, $\mu^*(A \cap P) \leq c\mu^*(P)$, then A is of Lebesgue measure 0.

First, we note that if this property holds for all closed boxes P, it holds for all countable unions of these sets as well, by subadditivity of the outer measure.

Now, assume A has the given property, for some $c \in (0,1)$, and let $\varepsilon = 1 - c$. Let A_m be the set $A \cap ([-m,m] \times \cdots \times [-m,m])$, which has finite outer measure. The same property, with $c\mu^*(P)$ bounding the measure of $\mu^*(P \cap A_m)$, assuming that $P \subset [-m,m] \times \cdots \times [-m,m]$. Take $P = [-m,m] \times [-m,m]$; then $\mu^*(P) = (2m)^n$, which is less than ∞. Write $M = (2m)^n$, then we see that

$$\mu^*(A_m \cap P) \leq (1-\varepsilon)M$$

By definition of the Lesbesgue outer measure, there must be some countable set of boxes whose union contains A_m, and such that the outer measure of their union is $(1-\varepsilon/2)M$. Call the union of these boxes P_1; then

$$\mu^*(A_m \cap P_1) \leq (1-\varepsilon)(1-\varepsilon/2)M$$

Inductively, we can construct a sequence of measurable sets P_i, such that for each one,

$$\mu^*(A_m \cap P_i) \leq (1-\varepsilon)(1-\varepsilon/2)^i M$$

Thus A_m can be covered by a measurable set of arbitrarily small measure, and it must have Lebesgue measure 0. Because $A = \bigcup A_m$, a countable union of measure 0 sets, it too must have Lebesgue measure 0.

$\square$

References

[1] Rudin, W. Real and Complex Analysis, 3rd edition.