Homework 2

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Algebra II

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1 Problems

Problem 1. Let R be a commutative ring and let S be an R-algebra. Let M be a left S-module. We know that it follows that any left S-module is also naturally an R module.

(a) Write doesn the natural R-module structure on M

(b) Write down the natural R-module structure on S.

(c) Assume that S is finitely generated as an R-module. Prove that M is a finitely generated left S-module if and only if it is finitely generated as an R-module.

Proof. (a) We define the R-module structure on M:

This is really immediate. An S-module M is a ring homomorphism $S \to \mathrm{End}(M)$, and the structure of S as an R-module is a certain kind of ring homomorphism $R \to S$, so by composing these maps we have an R-module $R \to \mathrm{End}(M)$.

More explicitly, let $\phi : R \to S$ be the map defining S as an R-algebra, and let $\cdot_S$ be the left action of S on M. We define a left action $\cdot_R$ of R on M by $r \cdot_R m = \phi(r) \cdot_S m$. This satisfies the axioms of a left action:

- $(r_1 + r_2) \cdot_R x = r_1 \cdot_R x + r_2 \cdot_R x$: It inherits this distributivity from S through ϕ.
- $r \cdot_R (x + y) = r \cdot_R x + r \cdot_R y$: This, too, is implied by the distributivity of the S-action
- $(r_1 r_2) \cdot_R x = r_1 \cdot_R (r_2 \cdot_R x)$: Associativity of the S-action
- $1_R \cdot_R x = x$: Identity property of the S-action, along with the fact that our homomorphisms take units to units.

(b) We now define the R-module structure on S. Again, this is implied by composition of maps: any ring is a module over itself, so we have a map of rings $S \to \mathrm{End}(S)$. Composition with ϕ gives an R-module structure. In fact, the module axioms are satisfied without use of the R-algebra structure of S; R could simply be a subring - not even necessarily a commutative one - and S would still be an R-module.

Explicitly, the R-module structure on S is defined by $r \cdot s = rs$, where rs is just multiplication in the ring S. The axioms of an R-module are immediate from the axioms of a ring.

(c) We now assume that S is finitely generated as an R-module, and show that M is finitely generated as an S module is and only if it is finitely generated as an R-module.

In one direction, this is immediate. Again, let ϕ be the map taking R into the center of S. If M is finitely generated as an R-module, say by n elements $\{m_i\}_{i=1}^n$, then any element written as a sum $\sum_{i=1}^n r_i m_i$ can also be written as a finite sum $\sum_{i=1}^n \phi(r_i) m_i$ of generators with coefficients in S.

Now, assume that M is finitely generated as an S module by n elements $\{s_i\}_{i=1}^n$, and that S is finitely generated as an R module by m elements, $\{s_j\}_{j=1}^m$. We show that the set $\{s_j m_i\}_{i=1}^{n}{}_{j=1}^{m}$ of mn elements of M generates M as an R module.

Let a be an arbitrary element of M. By assumption, a may be written as $a = \sum_{i=1}^n s_i m_i$, for some coefficients s_i in S. Now, each s_i may also, by assumption, be written as a sum $s_i = \sum_{j=1}^m r_{ij} s_j$, for some coefficients r_{ij} in R. Therefore,

$$\begin{aligned} a &= \sum_{i=1}^n s_i m_i \\ &= \sum_{i=1}^n (\sum_{j=1}^m r_{ij} s_j) m_i \\ &= \sum_{i=1}^n \sum_{j=1}^m r_{ij}(s_j m_i) \end{aligned}$$

Thus our element can be written as a sum of mn generators, times mn elements of R, making M a finitely generated R-module.

□

Problem 2. Let R be a ring. Let the free object functor $\mathcal{F} : \mathbf{Sets} \to \mathbf{R-mod}$ be defined on objects by sending a set X to the R-module $R^{\oplus X}$. Let the forgetful functor $\mathcal{G} : \mathbf{R-mod} \to \mathbf{Sets}$ be defined by sending an R-module to its underlying set.

(a) Write down what the free object functor $\mathcal{F}$ does to morphisms.

(b) Write down what the forgetful functor $\mathcal{G}$ does to morphisms.

(c) Prove that $\mathcal{F}$ and $\mathcal{G}$ are a pair of adjoint functors.

Proof. (a) Let the basis elements of the free R-module $R^{\oplus X}$ be written e_x, indexed over X, where e_x is the element with a 1 in the x-th place and 0s elsewhere.

Let ϕ be a morphism $X \to Y$ in **Sets**. We define a morphism $\mathcal{F}(\phi)$ by taking an element $\sum_X r_x e_x$ of $R^{\oplus X}$ to the element $\sum_X r_x e_{\phi(x)}$ of $R^{\oplus Y}$. This is a valid morphism, since composition with another morphism $\mathcal{F}(\psi)$ gives $\sum_X r_x e_{(\psi\circ\phi)(x)}$.

(b) Let ϕ be a morphism in $\mathbf{R-mod}$. Then the forgetful functor $\mathcal{G}$ takes ϕ to the underlying set-map, which is the same function element-wise but has "forgotten" the fact that it is a homomorphism of R-modules.

(c) In order to show that $\mathcal{F} \vdash \mathcal{G}$, we form the unit and counit natural transformations $\varepsilon : \mathcal{F}\mathcal{G} \Rightarrow 1$ and $\eta : 1 \Rightarrow \mathcal{G}\mathcal{F}$.

let X be an object of $\mathbf{R-mod}$, and let $\mathcal{F}\mathcal{G}(X)$ be the free R-module generated by the set of elements of X. An element a of $\mathcal{F}\mathcal{G}(X)$ is a finite sum of basis elements:

$$a = \sum_{\mathcal{G}(X)} r_x e_x$$

Now, for each $x \in \mathcal{G}(X)$, there is an overlying $x \in X$, which is the "same" element, but as an element of the R-module X. Therefore, the sum

$$\varepsilon_X(a) = \sum_X r_x \cdot x$$

is a valid element of X, and we may define the maps ε_X by taking elements of $\mathcal{F}\mathcal{G}(X)$ to the corresponding elements of X. These maps assemble into a natural transformation $\varepsilon : \mathcal{F}\mathcal{G} \Rightarrow 1$, as they act reasonably on morphisms- the following square commutes:

$$\begin{array}{ccc} \mathcal{F}\mathcal{G}(X) & \xrightarrow{\varepsilon_X} & X \\ \big\downarrow{\scriptstyle \mathcal{F}\mathcal{G}(\phi)} & & \big\downarrow{\scriptstyle \phi} \\ \mathcal{F}\mathcal{G}(\mathcal{Y}) & \xrightarrow{\varepsilon_Y} & Y \end{array}$$

This is because the sum of elements $\sum r_x \phi(x)$ is equal to $\phi\left(\sum r_x x\right)$, by the fact that ϕ is a module morphism.

In the opposite direction, let A be an arbitrary set, and let $\mathcal{G}\mathcal{F}(A)$ be the underlying set of the free R-module on A. For any element $a \in A$, there is a unique element $1_R \cdot a$ in the free module; we let $\eta_A(a) = 1_R \cdot a$, taken as an element of the underlying set of the free R-module. These maps η_A also assemble into a natural transformation, as we see that the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathcal{G}\mathcal{F}(A) \\ \big\downarrow{\scriptstyle \psi} & & \big\downarrow{\scriptstyle \mathcal{G}\mathcal{F}(\psi)} \\ B & \xrightarrow{\eta_B} & \mathcal{G}\mathcal{F}(B) \end{array}$$

The commutativity comes from the fact that the element $1 \cdot_R a$ is taken to $1 \cdot_R \psi(a)$along the right side, and that the element $\psi(a)$ is taken to $1 \cdot_R \psi(a)$ along the bottom.

Therefore there is a unit-counit pair of natural transformations $\mathcal{F}\mathcal{G} \Rightarrow 1$ and $1 \Rightarrow \mathcal{G}\mathcal{F}$. We finally need to see that the compositions

$$\mathcal{F} \xrightarrow{\mathcal{F}\eta} \mathcal{F}\mathcal{G}\mathcal{F} \xrightarrow{\varepsilon\mathcal{F}} \mathcal{F}$$

$$\mathcal{G} \xrightarrow{\eta\mathcal{G}} \mathcal{G}\mathcal{F}\mathcal{G} \xrightarrow{\mathcal{G}\varepsilon} \mathcal{G}$$

Are both the identity functors on $\mathcal{F}$ and $\mathcal{G}$. They are, and in fact I wrote a proof of it, but at this point this solution is getting kind of hairy and categorical and I regret not just using the $\mathrm{Hom}(\mathcal{F}(X), Y) \simeq \mathrm{Hom}(X, \mathcal{G}(Y))$ definition of adjunction (although there is still a naturality proof involved).

□

2 Extra Stuff

Problem 2. Give an example of another pair of adjoint functors.

Proof. In the categories $\mathcal{L}(x_1, \dots x_n)$ of formal sentences in a language $\mathcal{L}$ with at most n variables free, we have three functors:

- $\exists : \mathcal{L}(x_1, \dots x_n, y) \to \mathcal{L}(x_1, \dots x_n)$ which takes a sentence with $n+1$ free variables and "quantifies out" the last variable by adding an $\exists y$ to the beginning of the sentence.
- $* : \mathcal{L}(x_1, \dots x_n) \to \mathcal{L}(x_1, \dots x_n, y)$, the weakening functor, which takes a sentence with at most n variables and simply considers it to have at most $n+1$ variables.
- $\forall : \mathcal{L}(x_1, \dots x_n, y) \to \mathcal{L}(x_1, \dots x_n)$, which does the same as $\exists$, except that it adds a $\forall y$ instead of an $\exists y$.

These three functors form two adjunctions, $\exists \vdash * \vdash \forall$. The first adjunction essentially says that if $\exists y.(A) \Rightarrow B$, then it does not matter what y we choose, since it does not show up in B and only its existence is needed in A. Therefore, $A \Rightarrow *(B)$, if we let y be free in both sentences.

the second adjunction says that, if $*(A) \Rightarrow B$, where A is a sentence which does not contain y, and B might contain y, then this implication holds no matter what value y has: $A \Rightarrow \forall y.(B)$. □

Problem 3. Think about how the category of $k[x, y]$-modules is the same as the category of k-vector spaces with *commuting* endomorphisms.

Proof. The reason that commutativity matters is that we want the elements xy and yx to be the same element - considered as endomorphisms, we wans $A \circ B$ to be the same as $B \circ A$, where A corresponds to x and B to y. This commutation is enough to have a $k[x, y]$-module structure, as any polynomial in A and B with scalars from k is a valid endomorphism of the vector space, and multiplying the polynomials is the same as composing transfmormations. If A and B do not commute, we can work in the sub-category of $k\langle x, y\rangle$-modules, where $k\langle x, y\rangle$ is the free k-algebra on two variables. In this category, xy and yx are separate elements, so we do not need AB and BA to be the same. □

Problem 4. Work out whether injections of R-modules are monomorphisms, and if surjections are epimorphisms.

Proof. Because the morphisms of R-mod have underlying set maps, we can use surjectivity of the maps to right-cancel, and injectivity to left-cancel without any knowledge of the structure of the category; therefore, every surjective map of R-modules is an epimorphism, and every injective map is a monomorphism.

We can also show that epimorphisms and surjective maps are exactly the same in R-mod, and that the same holds for monomorphisms and injective maps. First, let some map $f : X \to Y$ of left R-modules be an epimorphism. Then f is right-cancellable, i.e. for any g_1, g_2, if $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$. Now, because $\mathrm{Im}(f)$ is a submodule of Y, we may form the quotient module $Y/\mathrm{Im}(f)$. Let g_1 be the projection map taking an element of Y to its coset in the quotient, and let g_2 be the zero map from Y to $Y/\mathrm{Im}(f)$. Then composing both g_1 and g_2 with f gives zero; by the property of epimorphisms, g_1 and g_2 must both be the zero map. But then $\mathrm{Im}(f) = Y$, making it surjective as a set-map.

Assume now that $h : X \to Y$ is some monomorphism in R-mod. Let x_1, x_2 be elements of X such that $h(x_1) = h(x_2)$. We show that $x_1 = x_2$, and that therefore h is an injective set-map.

Let $g_1 : R \to X$ be the map from the free module on one generator, which take 1 to x_1, and $g_2 : R \to X$ be the map taking 1 to x_2. Then $h \circ g_1$ and $h \circ g_2$ are equal on the generator of R, which by the universal property of the free module means that $h \circ g_1 = h \circ g_2$. By the universal property of monomorphisms, this means that $g_1 = g_2$, in particular that $x_1 = g_1(1) = g_2(1) = x_2$. Therefore h is injective. □