# cubical evaluation semantics

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I've been thinking a bit more carefully about the evaluation semantics and the semantic domain. What I think we want to do is take a term and an environment and return a family of values (indexed in restrictions); but crucially, I think it is important that the environment be a vector of *values* and not of value-families. By studying carefully the case for instantiating closures (where the rubber meets the road for variables), I think it becomes clear that this is the right way.

To interpret a variable, we just project from the environment and ignore the restriction. I think the way that we had been thinking of doing it would result in the restriction from inside the closure infecting the argument.

Colors: terms, value families, values, restrictions. I am writing $\phi^* F$ to mean the reindexing of a family by a restriction.

$$
\begin{aligned}
[\![(\lambda\{M\})]\!]^{\pi}_{\rho} &= [\phi]\ \lambda\langle M, \rho, \pi, \phi\rangle \\
[\![(\Pi\ A\ \{B\})]\!]^{\pi}_{\rho} &= [\phi]\ \mathbf{\Pi}\big(\phi^*[\![A]\!]^{\pi}_{\rho}, \langle B, \rho, \pi, \phi\rangle\big) \\
[\![(M\ N)]\!]^{\pi}_{\rho} &= \underline{\mathbf{app}}([\![M]\!]^{\pi}_{\rho}, [\![N]\!]^{\pi}_{\rho}) \\
[\![x_i]\!]^{\pi}_{\rho} &= [\_]\ \rho_i
\end{aligned}
$$

We write $\phi \vDash F \Downarrow V$ as a shorthand for $F[\phi] \equiv V$.

$$
\frac{\phi \vDash F \Downarrow \lambda\langle M, \rho, \pi, \psi\rangle}{\phi \vDash \underline{\mathbf{app}}(F, N) \Downarrow [\![M]\!]^{\pi}_{[\rho, N[\phi]]}[\psi]}
\qquad
\frac{\phi \vDash F \Downarrow \uparrow^{C} R \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}\big(A, \langle B, \rho, \pi, \psi\rangle\big)}{\phi \vDash \underline{\mathbf{app}}(F, N) \Downarrow \uparrow^{\psi^*[\![B]\!]^{\pi}_{[\rho, N[\phi]]}}\big([\chi]\ \mathbf{app}(\chi^* R, \chi \circ \phi^* N)\big)}
$$

$$
\frac{\phi \vDash F \Downarrow \mathbf{coe}^{r \rightsquigarrow r'}_{\langle\alpha\rangle C}\ M \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}\big(A, \langle B, \rho, \pi, \psi\rangle\big)}{\phi \vDash \underline{\mathbf{app}}(F, N) \Downarrow \nu\beta.\ \nu\gamma.\ \underline{\mathbf{coe}}^{r \rightsquigarrow r'}_{\langle\beta\rangle [\![B]\!]^{[\pi, (\beta\ \alpha)]}_{\left[\rho, \underline{\mathbf{coe}}^{r' \rightsquigarrow \beta}_{\langle\gamma\rangle(\gamma\ \alpha)\cdot A}\ N[\phi]\right]}}\ \underline{\mathbf{app}}(M, \underline{\mathbf{coe}}^{r' \rightsquigarrow r}_{\langle\gamma\rangle(\gamma\ \alpha)\cdot A}\ \phi^* N)}
$$

$$
\frac{\phi \vDash F \Downarrow \mathbf{hcom}^{r \rightsquigarrow r'}_{C}\ M\ \left[\overline{\xi_i \hookrightarrow \langle\alpha_i\rangle N_i}\right] \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}\big(A, \langle B, \rho, \pi, \psi\rangle\big)}{\phi \vDash \underline{\mathbf{app}}(F, O) \Downarrow \nu\beta.\ \underline{\mathbf{hcom}}^{r \rightsquigarrow r'}_{\psi^*[\![B]\!]^{\pi}_{[\rho, O[\phi]]}}\ \underline{\mathbf{app}}(M, \phi^* O)\ \left[\overline{\xi_i \hookrightarrow \langle\beta\rangle \underline{\mathbf{app}}((\alpha_i\ \beta)\cdot N_i, \phi^* O)}\right]}
$$

$$
\phi \vDash \underline{\mathbf{coe}}^{r \rightsquigarrow r'}_{\langle\alpha\rangle A}\ M \Downarrow \underline{\mathbf{coe}}^{\phi^* r \rightsquigarrow \phi^* r'}_{\nu\beta.\ \langle\beta\rangle \phi^*(\beta\ \alpha)\cdot A}\ \phi^* M
$$

$$\frac{\cdot \vDash r = r'}{\underline{\mathbf{hcom}}^{r \rightsquigarrow r'}_{C}\ M\left[\overline{\xi_i \hookrightarrow \langle\alpha_i\rangle N_i}\right] \equiv M[\cdot]} \qquad \frac{\cdot \vDash r \neq r' \qquad \exists_{min} k, \cdot \vDash \xi_k}{\underline{\mathbf{hcom}}^{r \rightsquigarrow r'}_{C}\ M\left[\overline{\xi_i \hookrightarrow \langle\alpha_i\rangle N_i}\right] \equiv N_k[\alpha_k = r'[\cdot]]} \qquad \frac{\cdot \vDash r \neq r' \qquad \overline{\cdot \nvDash \xi_i} \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}(A, B)}{\underline{\mathbf{hcom}}^{r \rightsquigarrow r'}_{C}\ M\left[\overline{\xi_i \hookrightarrow \langle\alpha_i\rangle N_i}\right] \equiv \mathbf{hcom}^{r \rightsquigarrow r'}_{C}\ M\left[\overline{\xi_i \hookrightarrow \langle\alpha_i\rangle N_i}\right]}$$

$$\frac{\cdot \vDash r \neq r' \qquad \overline{\cdot \nvDash \xi_i} \qquad \cdot \vDash C \Downarrow \mathbf{bool}}{\underline{\mathbf{hcom}}^{r \rightsquigarrow r'}_{C}\ M\left[\overline{\xi_i \hookrightarrow \langle\alpha_i\rangle N_i}\right] \equiv M[\cdot]}$$

$$\phi \vDash \underline{\mathbf{hcom}}^{r \rightsquigarrow r'}_{A}\ M\left[\overline{\xi_i \hookrightarrow \langle\alpha_i\rangle N_i}\right] \Downarrow \nu\beta.\ \underline{\mathbf{hcom}}^{\phi^* r \rightsquigarrow \phi^* r'}_{\phi^* A}\ \phi^* M\left[\overline{\phi^* \xi_i \hookrightarrow \langle\beta\rangle \phi^*(\beta\ \alpha_i) \cdot N_i}\right]$$

$$\frac{\cdot \vDash r = r'}{\underline{\mathbf{coe}}^{r \rightsquigarrow r'}_{\langle\alpha\rangle C}\ M \equiv M[\cdot]} \qquad \frac{\cdot \vDash r \neq r' \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}(A, B)}{\underline{\mathbf{coe}}^{r \rightsquigarrow r'}_{\langle\alpha\rangle C}\ M \equiv \mathbf{coe}^{r \rightsquigarrow r'}_{\langle\alpha\rangle C}\ M} \qquad \frac{\cdot \vDash r \neq r' \qquad \cdot \vDash C \Downarrow \mathbf{bool}}{\underline{\mathbf{coe}}^{r \rightsquigarrow r'}_{\langle\alpha\rangle C}\ M \equiv M[\cdot]}$$

$$\frac{\begin{array}{c} \cdot \vDash r \neq r' \qquad \cdot \vDash C \Downarrow \mathbf{fcom}^{s \rightsquigarrow s'}\ A\left[\overline{\xi_i \hookrightarrow \langle\beta\rangle B_i}\right] \\ \phi_r \triangleq (r[\cdot] = \alpha) \qquad \phi_{r'} \triangleq (r'[\cdot] = \alpha) \\ O \triangleq \boxed{?} \\ P \triangleq \underline{\mathbf{gcom}}^{r \rightsquigarrow r'}_{\langle\alpha\rangle A}\ (\beta = s[\phi_r])^* O\left[\overline{\xi_i \hookrightarrow \nu\gamma.\ \langle\gamma\rangle \underline{\mathbf{coe}}^{(\gamma\ \alpha)\cdot s' \rightsquigarrow (\gamma\ \alpha)\cdot s}_{\langle\beta\rangle(\gamma\ \alpha)\cdot B_i}\ \underline{\mathbf{coe}}^{r \rightsquigarrow \gamma}_{\langle\alpha\rangle(s'[\cdot]=\beta)^* B_i}\ M|_{\alpha\#\xi_i[\cdot]}}, s = s' \hookrightarrow \nu\gamma.\ \langle\gamma\rangle \underline{\mathbf{coe}}^{r \rightsquigarrow \gamma}_{\langle\alpha\rangle A}\ M|_{\alpha\#[s[\cdot], s'[\cdot]]}\right] \\ \overline{T} \triangleq \boxed{?} \\ \overline{U} \triangleq \boxed{?} \end{array}}{\underline{\mathbf{coe}}^{r \rightsquigarrow r'}_{\langle\alpha\rangle C}\ M \equiv \underline{\mathbf{box}}^{\phi_{r'}{}^* s \rightsquigarrow \phi_{r'}{}^* s'}\left(\underline{\mathbf{hcom}}^{\phi_{r'}{}^* s \rightsquigarrow \phi_{r'}{}^* s'}_{\phi_{r'}{}^* A}\ P\left[\overline{T}\right]\right)\left[\overline{U}\right]}$$