

cubical evaluation semantics

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I've been thinking a bit more carefully about the evaluation semantics and the semantic domain. What I think we want to do is take a term and an environment and return a family of values (indexed in restrictions); but crucially, I think it is important that the environment be a vector of *values* and not of value-families. By studying carefully the case for instantiating closures (where the rubber meets the road for variables), I think it becomes clear that this is the right way.

To interpret a variable, we just project from the environment and ignore the restriction. I think the way that we had been thinking of doing it would result in the restriction from inside the closure infecting the argument.

Colors: **terms**, **value families**, **values**, **restrictions**. I am writing ϕ^*F to mean the reindexing of a family by a restriction.

$$\begin{aligned} \llbracket (\lambda \{M\}) \rrbracket_\rho^\pi &= [\phi] \lambda \langle M, \rho, \pi, \phi \rangle \\ \llbracket (\Pi A \{B\}) \rrbracket_\rho^\pi &= [\phi] \Pi(\phi^* \llbracket A \rrbracket_\rho^\pi, \langle B, \rho, \pi, \phi \rangle) \\ \llbracket (M N) \rrbracket_\rho^\pi &= \text{app}(\llbracket M \rrbracket_\rho^\pi, \llbracket N \rrbracket_\rho^\pi) \\ \llbracket x_i \rrbracket_\rho^\pi &= [_] \rho_i \end{aligned}$$

We write $\phi \models F \Downarrow V$ as a shorthand for $F[\phi] \equiv V$.

$$\begin{array}{c} \frac{\phi \models F \Downarrow \lambda \langle M, \rho, \pi, \psi \rangle}{\phi \models \text{app}(F, N) \Downarrow \llbracket M \rrbracket_{[\rho, N[\phi]]}^\pi [\psi]} \qquad \frac{\phi \models F \Downarrow \uparrow^C R \quad \cdot \models C \Downarrow \Pi(A, \langle B, \rho, \pi, \psi \rangle)}{\phi \models \text{app}(F, N) \Downarrow \uparrow^{\psi^* \llbracket B \rrbracket_{[\rho, N[\phi]]}^\pi} ([\chi] \text{app}(\chi^* R, \chi \circ \phi^* N))} \\[10pt] \frac{\phi \models F \Downarrow \text{coe}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'} M \quad \cdot \models C \Downarrow \Pi(A, \langle B, \rho, \pi, \psi \rangle)}{\phi \models \text{app}(F, N) \Downarrow \nu \beta. \nu \gamma. \text{coe}_{\langle \beta \rangle [B]}^{r \rightsquigarrow r'} \left[\frac{\pi, (\beta \alpha)}{\rho, \text{coe}_{\langle \gamma \rangle (\gamma \alpha) \cdot A}^{r' \rightsquigarrow \beta}} N[\phi] \right] \text{app}(M, \text{coe}_{\langle \gamma \rangle (\gamma \alpha) \cdot A}^{r' \rightsquigarrow r} \phi^* N)} \qquad \frac{\phi \models F \Downarrow \text{hcom}_C^{r \rightsquigarrow r'} M \left[\xi_i \mapsto \langle \alpha_i \rangle N_i \right] \quad \cdot \models C \Downarrow \Pi(A, \langle B, \rho, \pi, \psi \rangle)}{\phi \models \text{app}(F, O) \Downarrow \nu \beta. \text{hcom}_{\psi^* [B]}^{r \rightsquigarrow r'} \left[\xi_i \mapsto \langle \beta \rangle \text{app}(\langle \alpha_i \rangle \beta \cdot N_i, \phi^* O) \right]} \\[10pt] \phi \models \text{coe}_{\langle \alpha \rangle A}^{r \rightsquigarrow r'} M \Downarrow \text{coe}_{\nu \beta. \langle \beta \rangle \phi^* ((\beta \alpha) \cdot A)}^{\phi^* r \rightsquigarrow \phi^* r'} \phi^* M \end{array}$$

$$\begin{array}{c}
\frac{\cdot \models r = r'}{\text{hcom}_C^{r \rightsquigarrow r'} M [\xi_i \hookrightarrow \langle \alpha_i \rangle N_i] \equiv M[\cdot]} \quad \frac{\cdot \models r \neq r' \quad \exists_{\min} k, \cdot \models \xi_k}{\text{hcom}_C^{r \rightsquigarrow r'} M [\xi_i \hookrightarrow \langle \alpha_i \rangle N_i] \equiv N_k[\alpha_k = r'[\cdot]]} \quad \frac{\cdot \models r \neq r' \quad \overline{\cdot \not\models \xi_i} \quad \cdot \models C \Downarrow \Pi(A, B)}{\text{hcom}_C^{r \rightsquigarrow r'} M [\xi_i \hookrightarrow \langle \alpha_i \rangle N_i] \equiv \text{hcom}_C^{r \rightsquigarrow r'} M [\xi_i \hookrightarrow \langle \alpha_i \rangle N_i]} \\
\\
\frac{\cdot \models r \neq r' \quad \overline{\cdot \not\models \xi_i} \quad \cdot \models C \Downarrow \text{bool}}{\text{hcom}_C^{r \rightsquigarrow r'} M [\xi_i \hookrightarrow \langle \alpha_i \rangle N_i] \equiv M[\cdot]} \\
\\
\phi \models \text{hcom}_A^{r \rightsquigarrow r'} M [\xi_i \hookrightarrow \langle \alpha_i \rangle N_i] \Downarrow \nu \beta. \text{hcom}_{\phi^* A}^{\phi^* r \rightsquigarrow \phi^* r'} \phi^* M [\phi^* \xi_i \hookrightarrow \langle \beta \rangle \phi^* ((\beta \alpha_i) \cdot N_i)] \\
\\
\frac{\cdot \models r = r'}{\text{coe}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'} M \equiv M[\cdot]} \quad \frac{\cdot \models r \neq r' \quad \cdot \models C \Downarrow \Pi(A, B)}{\text{coe}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'} M \equiv \text{coe}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'} M} \quad \frac{\cdot \models r \neq r' \quad \cdot \models C \Downarrow \text{bool}}{\text{coe}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'} M \equiv M[\cdot]} \\
\\
\begin{array}{c}
\cdot \models r \neq r' \quad \cdot \models C \Downarrow \text{fcom}^{s \rightsquigarrow s'} A [\xi_i \hookrightarrow \langle \beta \rangle B_i] \quad \phi_r \triangleq (r[\cdot] = \alpha) \quad \phi_{r'} \triangleq (r'[\cdot] = \alpha) \\
N_i \triangleq \text{coe}_{\langle \beta \rangle B_i}^{s' \rightsquigarrow \beta} \text{coe}_{\langle \alpha \rangle (s' = \beta)^* B_i}^{r \rightsquigarrow \alpha} M \\
O_i \triangleq \text{hcom}_{\phi_r^* A}^{\phi_r^* s' \rightsquigarrow t} \left(\text{cap}_{\phi_r^* s \rightsquigarrow \phi_r^* s'}^{\phi_r} M [\xi_i \hookrightarrow \langle \beta \rangle (\phi_r \setminus \beta)^* B_i] \right) \left[\phi_r^* \xi_i \hookrightarrow \nu \delta. \langle \delta \rangle \text{coe}_{\langle \beta \rangle (\phi_r \setminus \beta)^* B_i}^{\delta \rightsquigarrow \phi_r^* s} \text{coe}_{\langle \beta \rangle (\phi_r \setminus \beta)^* B_i}^{\phi_r^* s' \rightsquigarrow \delta} M \right] \\
P \triangleq \text{gcom}_{\langle \alpha \rangle A}^{r \rightsquigarrow r'} (\gamma = s[\phi_r])^* O_\gamma \left[\xi_i \hookrightarrow \nu \gamma. \langle \gamma \rangle ((\gamma \alpha) \cdot s = \beta)^* ((\gamma \alpha) \cdot N_i)|_{\alpha \# \xi_i[\cdot]}, s = s' \hookrightarrow \nu \gamma. \langle \gamma \rangle \text{coe}_{\langle \alpha \rangle A}^{r \rightsquigarrow \gamma} M|_{\alpha \# [s[\cdot], s'[\cdot]]} \right] \\
Q_i^\dagger \triangleq \text{gcom}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\phi_{r'}^* s \rightsquigarrow t} P \left[\xi_i \hookrightarrow \nu \eta. \langle \eta \rangle \phi_{r'}^* ((\eta \beta) \cdot N_i)|_{\alpha \# \xi_i[\cdot]}, r = r' \hookrightarrow \nu \eta. \langle \eta \rangle \text{coe}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\phi_{r'}^* s' \rightsquigarrow \eta} M \right] \\
\overline{T} \triangleq \xi_i \hookrightarrow \nu \delta. \langle \delta \rangle \text{coe}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\delta \rightsquigarrow \phi_{r'}^* s} \phi_{r'}^* ((\delta \beta) \cdot N_i)|_{\alpha \# \xi_i[\cdot]}, \phi_{r'}^* \xi_i \hookrightarrow \nu \delta. \langle \delta \rangle \text{coe}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\delta \rightsquigarrow \phi_{r'}^* s} Q_i^\delta|_{\alpha \in \xi_i[\cdot]}, r = r' \hookrightarrow \nu \gamma. \langle \gamma \rangle O_\gamma \\
\overline{U} \triangleq \xi_i \hookrightarrow \text{coe}_{\nu \gamma. \langle \gamma \rangle ((\gamma \alpha) \cdot s'[\cdot] = \beta)^* ((\gamma \alpha) \cdot B_i)}^{r \rightsquigarrow r'} M|_{\alpha \# \xi_i[\cdot]}, \phi_{r'}^* \xi_i \hookrightarrow Q_i^{\phi_{r'}^* s' \rightsquigarrow t}|_{\alpha \in \xi_i[\cdot]} \\
\text{coe}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'} M \equiv \text{box}_{\phi_{r'}^* s \rightsquigarrow \phi_{r'}^* s'}^{\phi_{r'}^* s \rightsquigarrow \phi_{r'}^* s'} \left(\text{hcom}_{\phi_{r'}^* A}^{\phi_{r'}^* s \rightsquigarrow \phi_{r'}^* s'} P [\overline{T}] \right) [\overline{U}]
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