cubical evaluation semantics

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I've been thinking a bit more carefully about the evaluation semantics and the semantic domain. What I think we want to do is take a term and an environment and return a family of values (indexed in restrictions); but crucially, I think it is important that the environment be a vector of *values* and not of value-families. By studying carefully the case for instantiating closures (where the rubber meets the road for variables), I think it becomes clear that this is the right way.

To interpret a variable, we just project from the environment and ignore the restriction. I think the way that we had been thinking of doing it would result in the restriction from inside the closure infecting the argument.

Colors: terms, value families, values, restrictions. I am writing ϕ^*F to mean the reindexing of a family by a restriction.

$$\begin{aligned}
[\![(\lambda\{M\})]\!]^{\pi}_{\rho} &= [\phi]\ \lambda\langle M, \rho, \pi, \phi\rangle \\
[\![(\Pi\ A\ \{B\})]\!]^{\pi}_{\rho} &= [\phi]\ \mathbf{\Pi}\big(\phi^*[\![A]\!]^{\pi}_{\rho}, \langle B, \rho, \pi, \phi\rangle\big) \\
[\![(M\ N)]\!]^{\pi}_{\rho} &= \underline{\mathbf{app}}([\![M]\!]^{\pi}_{\rho}, [\![N]\!]^{\pi}_{\rho}) \\
[\![x_i]\!]^{\pi}_{\rho} &= [_]\ \rho_i
\end{aligned}$$

We write $\phi \vDash F \Downarrow V$ as a shorthand for $F[\phi] \equiv V$.

$$\frac{\phi \vDash F \Downarrow \lambda\langle M, \rho, \pi, \psi\rangle}{\phi \vDash \underline{\mathbf{app}}(F, N) \Downarrow [\![M]\!]^{\pi}_{[\rho, N[\phi]]}[\psi]}
\qquad
\frac{\phi \vDash F \Downarrow\ \uparrow^{C} R \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}\big(A, \langle B, \rho, \pi, \psi\rangle\big)}{\phi \vDash \underline{\mathbf{app}}(F, N) \Downarrow\ \uparrow^{\psi^*[\![B]\!]^{\pi}_{[\rho, N[\phi]]}}([\chi]\ \mathbf{app}(\chi^*R, \chi \circ \phi^*N))}$$

$$\frac{\phi \vDash F \Downarrow \mathbf{coe}^{r \rightsquigarrow r'}_{\langle\alpha\rangle C}\ M \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}\big(A, \langle B, \rho, \pi, \psi\rangle\big)}{\phi \vDash \underline{\mathbf{app}}(F, N) \Downarrow \nu\beta.\ \nu\gamma.\ \underline{\mathbf{coe}}^{r \rightsquigarrow r'}_{\langle\beta\rangle[\![B]\!]^{[\pi, (\beta\ \alpha)]}_{\left[\rho, \underline{\mathbf{coe}}^{r' \rightsquigarrow \beta}_{\langle\gamma\rangle(\gamma\ \alpha)\cdot A}\ N[\phi]\right]}}\ \underline{\mathbf{app}}(M, \underline{\mathbf{coe}}^{r' \rightsquigarrow r}_{\langle\gamma\rangle(\gamma\ \alpha)\cdot A}\ \phi^*N)}$$

$$\frac{\phi \vDash F \Downarrow \mathbf{hcom}^{r \rightsquigarrow r'}_{C}\ M\ \left[\overline{\xi_i \hookrightarrow \langle\alpha_i\rangle N_i}\right] \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}\big(A, \langle B, \rho, \pi, \psi\rangle\big)}{\phi \vDash \underline{\mathbf{app}}(F, O) \Downarrow \nu\beta.\ \underline{\mathbf{hcom}}^{r \rightsquigarrow r'}_{\psi^*[\![B]\!]^{\pi}_{[\rho, O[\phi]]}}\ \underline{\mathbf{app}}(M, \phi^*O)\ \left[\overline{\xi_i \hookrightarrow \langle\beta\rangle\underline{\mathbf{app}}((\alpha_i\ \beta)\cdot N_i, \phi^*O)}\right]}$$

$$\phi \vDash \underline{\mathbf{coe}}^{r \rightsquigarrow r'}_{\langle\alpha\rangle A}\ M \Downarrow \underline{\mathbf{coe}}^{\phi^*r \rightsquigarrow \phi^*r'}_{\nu\beta.\ \langle\beta\rangle\phi^*((\beta\ \alpha)\cdot A)}\ \phi^*M$$

$$\frac{\cdot \vDash r = r'}{\underline{\mathbf{hcom}}_{C}^{r \rightsquigarrow r'}\ M\left[\overline{\xi_i \hookrightarrow \langle \alpha_i \rangle N_i}\right] \equiv M[\cdot]} \qquad \frac{\cdot \vDash r \neq r' \qquad \exists_{min} k, \cdot \vDash \xi_k}{\underline{\mathbf{hcom}}_{C}^{r \rightsquigarrow r'}\ M\left[\overline{\xi_i \hookrightarrow \langle \alpha_i \rangle N_i}\right] \equiv N_k[\alpha_k = r'[\cdot]]} \qquad \frac{\cdot \vDash r \neq r' \qquad \overline{\cdot \nvDash \xi_i} \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}(A, B)}{\underline{\mathbf{hcom}}_{C}^{r \rightsquigarrow r'}\ M\left[\overline{\xi_i \hookrightarrow \langle \alpha_i \rangle N_i}\right] \equiv \mathbf{hcom}_{C}^{r \rightsquigarrow r'}\ M\left[\overline{\xi_i \hookrightarrow \langle \alpha_i \rangle N_i}\right]}$$

$$\frac{\cdot \vDash r \neq r' \qquad \overline{\cdot \nvDash \xi_i} \qquad \cdot \vDash C \Downarrow \mathbf{bool}}{\underline{\mathbf{hcom}}_{C}^{r \rightsquigarrow r'}\ M\left[\overline{\xi_i \hookrightarrow \langle \alpha_i \rangle N_i}\right] \equiv M[\cdot]}$$

$$\phi \vDash \underline{\mathbf{hcom}}_{A}^{r \rightsquigarrow r'}\ M\left[\overline{\xi_i \hookrightarrow \langle \alpha_i \rangle N_i}\right] \Downarrow \nu\beta.\ \underline{\mathbf{hcom}}_{\phi^* A}^{\phi^* r \rightsquigarrow \phi^* r'}\ \phi^* M\left[\overline{\phi^* \xi_i \hookrightarrow \langle \beta \rangle \phi^*((\beta\ \alpha_i) \cdot N_i)}\right]$$

$$\frac{\cdot \vDash r = r'}{\underline{\mathbf{coe}}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'}\ M \equiv M[\cdot]} \qquad \frac{\cdot \vDash r \neq r' \qquad \cdot \vDash C \Downarrow \mathbf{\Pi}(A, B)}{\underline{\mathbf{coe}}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'}\ M \equiv \mathbf{coe}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'}\ M} \qquad \frac{\cdot \vDash r \neq r' \qquad \cdot \vDash C \Downarrow \mathbf{bool}}{\underline{\mathbf{coe}}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'}\ M \equiv M[\cdot]}$$

$$\frac{\begin{array}{c} \cdot \vDash r \neq r' \qquad \cdot \vDash C \Downarrow \mathbf{fcom}^{s \rightsquigarrow s'}\ A\left[\overline{\xi_i \hookrightarrow \langle \beta \rangle B_i}\right] \qquad \phi_r \triangleq (r[\cdot] = \alpha) \qquad \phi_{r'} \triangleq (r'[\cdot] = \alpha) \\ O_t \triangleq \underline{\mathbf{hcom}}_{\phi_r{}^* A}^{\phi_r{}^* s' \rightsquigarrow t}\left(\underline{\mathbf{cap}}^{\phi_r{}^* s \leftsquigarrow \phi_r{}^* s'}\ M\left[\overline{\phi_r{}^* \xi_i \hookrightarrow \langle \beta \rangle (\phi_r \setminus \beta)^* B_i}\right]\right)\left[\overline{\phi_r{}^* \xi_i \hookrightarrow \nu\delta.\ \langle \delta \rangle \underline{\mathbf{coe}}_{\langle \beta \rangle (\phi_r \setminus \beta)^* B_i}^{\delta \rightsquigarrow \phi_r{}^* s}\ \underline{\mathbf{coe}}_{\langle \beta \rangle (\phi_r \setminus \beta)^* B_i}^{\phi_r{}^* s' \rightsquigarrow \delta}\ M}\right] \\ P \triangleq \underline{\mathbf{gcom}}_{\langle \alpha \rangle A}^{r \rightsquigarrow r'}\ (\gamma = s[\phi_r])^* O_\gamma\left[\overline{\xi_i \hookrightarrow \nu\gamma.\ \langle \gamma \rangle \underline{\mathbf{coe}}_{\langle \beta \rangle (\gamma\ \alpha) \cdot B_i}^{(\gamma\ \alpha) \cdot s' \rightsquigarrow (\gamma\ \alpha) \cdot s}\ \underline{\mathbf{coe}}_{\langle \alpha \rangle (s'[\cdot] = \beta)^* B_i}^{r \rightsquigarrow \gamma}\ M|_{\alpha \# \xi_i[\cdot]}}, s = s' \hookrightarrow \nu\gamma.\ \langle \gamma \rangle \underline{\mathbf{coe}}_{\langle \alpha \rangle A}^{r \rightsquigarrow \gamma}\ M|_{\alpha \# [s[\cdot], s'[\cdot]]}\right] \\ Q_i^t \triangleq \underline{\mathbf{gcom}}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\phi_{r'}{}^* s \rightsquigarrow t}\ P\left[\overline{\xi_i \hookrightarrow \nu\eta.\ \langle \eta \rangle \underline{\mathbf{coe}}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\phi_{r'}{}^* s' \rightsquigarrow \eta}\ \underline{\mathbf{coe}}_{\langle \alpha \rangle (s'[\cdot] = \beta)^* B_i}^{r \rightsquigarrow r'}\ M|_{\alpha \# \xi_i[\cdot]}}, r = r' \hookrightarrow \nu\eta.\ \langle \eta \rangle \underline{\mathbf{coe}}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\phi_{r'}{}^* s' \rightsquigarrow \eta}\ M\right] \\ \overline{\overline{T} \triangleq \xi_i \hookrightarrow \nu\delta.\ \langle \delta \rangle \underline{\mathbf{coe}}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\delta \rightsquigarrow \phi_{r'}{}^* s}\ \underline{\mathbf{coe}}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\phi_{r'}{}^* s' \rightsquigarrow \delta}\ \underline{\mathbf{coe}}_{\nu\eta.\ \langle \eta \rangle ((s'[\phi_{r'}] = \delta) \circ \phi_{r'})^* (\eta\ \beta) \cdot B_i}^{r \rightsquigarrow r'}\ M|_{\alpha \# \xi_i[\cdot]}, \phi_{r'}{}^* \xi_i \hookrightarrow \nu\delta.\ \langle \delta \rangle \underline{\mathbf{coe}}_{\langle \beta \rangle (\phi_{r'} \setminus \beta)^* B_i}^{\delta \rightsquigarrow \phi_{r'}{}^* s}\ Q_i^\delta|_{\alpha \in \xi_i[\cdot]}}, r = r' \hookrightarrow \nu\gamma.\ \langle \gamma \rangle O_\gamma \\ \overline{U} \triangleq \overline{\xi_i \hookrightarrow \underline{\mathbf{coe}}_{\nu\gamma.\ \langle \gamma \rangle ((\gamma\ \alpha) \cdot s'[\cdot] = \beta)^* (\gamma\ \alpha) \cdot B_i}^{r \rightsquigarrow r'}\ M|_{\alpha \# \xi_i[\cdot]}, \phi_{r'}{}^* \xi_i \hookrightarrow Q_i^{\phi_{r'}{}^* s'}|_{\alpha \in \xi_i[\cdot]}} \end{array}}{\underline{\mathbf{coe}}_{\langle \alpha \rangle C}^{r \rightsquigarrow r'}\ M \equiv \underline{\mathbf{box}}^{\phi_{r'}{}^* s \rightsquigarrow \phi_{r'}{}^* s'}\left(\underline{\mathbf{hcom}}_{\phi_{r'}{}^* A}^{\phi_{r'}{}^* s \rightsquigarrow \phi_{r'}{}^* s'}\ P\left[\overline{T}\right]\right)\left[\overline{U}\right]}$$