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Supplementary material for " Nonparametric tests of independence for high-dimensional survival data"

BY JINHONG LI
Department of Statistics, East China Normal University, 3663 North Zhongshan Road, Shanghai 200062, China
jinhongli0106@gmail.com

JICAI LIU
School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, 995 Shangchuan Road, Shanghai 201209, China
liujicai1234@126.com

SHUJIE MA
Department of Statistics, University of California, Riverside, USA
shujie.ma@ucr.edu

AND RIQUAN ZHANG
School of Statistics and Information, Shanghai University of International Business and Economics, 1900 Wenxiang Road, Shanghai 201620, China
zhangriquan@163.com

APPENDIX A: PROOFS

Proof ***Proof of Lemma*** 1. We first show that

$$T \perp\!\!\!\perp \mathbf{Z} \Longleftrightarrow \mathbb{E}\{dN(t)|Y(t), \mathbf{Z}\} = \mathbb{E}\{dN(t)|Y(t)\}. \tag{S1}$$

Prove the " $\Leftarrow$ " part. If $\mathbb{E}\{dN(t)|Y(t), \mathbf{Z}\} = \mathbb{E}\{dN(t)|Y(t)\}$ holds, we obtain that

$$P\{dN(t) = 1 \mid \mathbf{Z}, Y(t) = 1\} = P\{dN(t) = 1 \mid Y(t) = 1\}. \tag{S2}$$

Under the assumption that C is independent of T conditional on $\mathbf{Z}$, we have that

$$\begin{aligned} P\{dN(t) = 1 \mid \mathbf{Z}, Y(t) = 1\} &= \frac{P\{dN(t) = 1, Y(t) = 1 \mid \mathbf{Z}\}}{P\{Y(t) = 1 \mid \mathbf{Z}\}} \\ &= \frac{P\{t + dt > T \geq t, T \leq C \mid \mathbf{Z}\}}{P\{T \wedge C \geq t \mid \mathbf{Z}\}} \\ &= \frac{\int_t^{t+dt} P\{C \geq u|\mathbf{Z}\} dP\{T \leq u \mid \mathbf{Z}\}}{P\{T > t \mid \mathbf{Z}\} P\{C > t \mid \mathbf{Z}\}} \\ &= \frac{P\{C \geq t \mid \mathbf{Z}\} f_{T|\mathbf{Z}}(t) dt}{P\{T \geq t \mid \mathbf{Z}\} P\{C \geq t \mid \mathbf{Z}\}} \\ &= \lambda(t \mid \mathbf{Z}) dt, \end{aligned} \tag{S3}$$

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as $dt \to 0^+$, for $t \in [0, \tau]$. Then, (S2) and (S3) suggest that we have

$$\lambda(t \mid \mathbf{Z})dt = P\{dN(t) = 1 \mid Y(t) = 1\}, \text{ as } dt \to 0^+, \tag{S4}$$

for any $\mathbf{Z} \in \mathcal{Z}$ and $t \in [0, \tau]$. Thus, $\mathbf{Z}$ and T are independent due to the one-to-one relation between the distribution function and the hazard function.

Prove the " $\Rightarrow$ " part. When $\mathbf{Z}$ and T are independent, we have that $\lambda(t \mid \mathbf{Z}) = \lambda(t)$. By the same argument of (S3), we can prove that $P\{dN(t) = 1 \mid Y(t) = 1\} = \lambda(t)dt$ as $dt \downarrow 0$, and thus (S4) holds. Then, by (S3) and (S4), we obtain that (S2) holds. This yields that $\mathbb{E}\{dN(t)|Y(t), \mathbf{Z}\} = \mathbb{E}\{dN(t)|Y(t)\}$.

Let $d\widetilde{N}(t) = dN(t) - \mathbb{E}\{dN(t)|Y(t)\}$. Note that

$$0 = \mathbb{E}\{dN(t)|Y(t), \mathbf{Z}\} - \mathbb{E}\{dN(t)|Y(t)\} = \mathbb{E}\{d\widetilde{N}(t)|Y(t), \mathbf{Z}\}. \tag{S5}$$

Note that

$$\mathbb{E}\{Y(t)d\widetilde{N}(t)|\mathbf{Z}\} = \mathbb{E}\{d\widetilde{N}(t)|Y(t), \mathbf{Z}\}\mathbb{E}\{Y(t)|\mathbf{Z}\}.$$

This, together with (S1) and (S5), implies that

$$T \perp\!\!\!\perp \mathbf{Z} \Longleftrightarrow \mathbb{E}\{Y(t)d\widetilde{N}(t)|\mathbf{Z}\} = 0. \qquad \square$$

Proof of Theorem 1. By the definition of the SICM and the Fubini's theorem, we have

$$\begin{aligned} \mathrm{SICM}(t; a, \nu) = \mathbb{E}\Big\{ &\int_0^t \int_0^t a(s_1)a(s_2)Y_1(s_1)Y_2(s_2)d\widetilde{N}_1(s_1)d\widetilde{N}_2(s_2) \\ &\times \int_{\mathbb{R}^p} \exp(i\mathbf{u}^T(\mathbf{Z}_1 - \mathbf{Z}_2))d\nu(\mathbf{u})\Big\}. \end{aligned} \tag{S6}$$

Additionally, we have the following fact that

$$\mathbb{E}\{dN(t)|Y(t)\} = Y(t)\frac{\mathbb{E}\{dN(t)\}}{\mathbb{E}\{Y(t)\}}. \tag{S7}$$

(i) By (S6), (S7) and Lemma 2 (i), some similar calculations in the proof of Theorem 1 of Székely et al. (2007) yields that

$$\begin{aligned}
\mathrm{SICM}_{L_2}(t;a) &= -\mathbb{E}\Big\{\int_0^t\int_0^t \|\mathbf{Z}_1-\mathbf{Z}_2\| a(s_1)a(s_2)Y_1(s_1)Y_2(s_2)d\widetilde{N}_1(s_1)d\widetilde{N}_2(s_2)\Big\}\\
&= -\mathbb{E}\Big\{\int_0^t\int_0^t \|\mathbf{Z}_1-\mathbf{Z}_2\| a(s_1)a(s_2)Y_1(s_1)Y_2(s_2)\Big[dN_1(s_1)\\
&\quad -Y_1(s_1)\frac{\mathbb{E}\{dN(s_1)\}}{\mathbb{E}\{Y(s_1)\}}\Big]\Big[dN_2(s_2)-Y_2(s_2)\frac{\mathbb{E}\{dN(s_2)\}}{\mathbb{E}\{Y(s_2)\}}\Big]\Big\}\\
&= -\mathbb{E}\Big\{\int_0^t\int_0^t a(s_1)a(s_2)\|\mathbf{Z}_1-\mathbf{Z}_2\| dN_1(s_1)dN_2(s_2)\Big\}\\
&\quad +2\mathbb{E}\Big\{\int_0^t\int_0^t a(s_1)a(s_2)\frac{\|\mathbf{Z}_1-\mathbf{Z}_3\|Y_3(s_2)}{\mathbb{E}\{Y(s_2)\}}dN_1(s_1)dN_2(s_2)\Big\}\\
&\quad -\mathbb{E}\Big\{\int_0^t\int_0^t a(s_1)a(s_2)\frac{\|\mathbf{Z}_1-\mathbf{Z}_2\|Y_1(s_2)Y_2(s_2)}{\mathbb{E}\{Y(s_1)\}\mathbb{E}\{Y(s_2)\}}dN_3(s_1)dN_4(s_2)\Big\}\\
&= -\mathbb{E}\Big\{\int_0^t\int_0^t \frac{a(s_1)a(s_2)\|\mathbf{Z}_1-\mathbf{Z}_2\|}{\mathbb{E}\{Y(s_1)\}\mathbb{E}\{Y(s_2)\}}[Y_3(s_1)dN_1(s_1)-Y_1(s_1)dN_3(s_1)]\\
&\quad \times[Y_4(s_2)dN_2(s_2)-Y_2(s_2)dN_4(s_2)]\Big\}\\
&= -\mathrm{E}\Big\{\|\mathbf{Z}_1-\mathbf{Z}_2\|\psi_t(\widetilde{T}_1,\widetilde{T}_3)\psi_t(\widetilde{T}_2,\widetilde{T}_4)\Big\},
\end{aligned}$$

where $\widetilde{T}_i=(X_i,\Delta_i)$ and

$$\psi_t(\widetilde{T}_i,\widetilde{T}_k)=\int_0^t \frac{a(s)}{\mathbb{E}\{Y(s)\}}\Big[I(X_k\ge s)dN_i(s)-I(X_i\ge s)dN_k(s)\Big].$$

(ii) By (S6), (S7) and Lemma 2(ii), calculations similar to those above yield that

$$\begin{aligned}
\mathrm{SICM}_K(t;a) &= \mathbb{E}\Big\{\int_0^t\int_0^t K(\mathbf{Z}_1,\mathbf{Z}_2)a(s_1)a(s_2)Y_1(s_1)Y_2(s_2)d\widetilde{N}_1(s_1)d\widetilde{N}_2(s_2)\Big\}\\
&= \mathrm{E}\Big\{K(\mathbf{Z}_1,\mathbf{Z}_2)\psi_t(\widetilde{T}_1,\widetilde{T}_3)\psi_t(\widetilde{T}_2,\widetilde{T}_4)\Big\}.
\end{aligned}$$

(ii) $\mathrm{SICM}(t;a,\nu)\ge 0$ is straightforward. By the definition of $\mathrm{SICM}(t;a,\nu)$, we have that $\mathrm{SICM}(t;a,\nu)=0$, for any $t\in[0,\tau]$, if and only if

$$\mathbb{E}\Big\{\int_0^t a(s)\exp(i\mathbf{u}^T\mathbf{Z})Y(s)d\widetilde{N}(s)\Big\}=0,\forall t\in[0,\tau] \text{ and } \mathbf{u}\in\mathbb{R}^p,$$

which is equivalent to

$$\mathbb{E}\{\exp(i\mathbf{u}^T\mathbf{Z})Y(t)d\widetilde{N}(t)\}=0,\forall t\in[0,\tau] \text{ and } \mathbf{u}\in\mathbb{R}^p.$$

This, together with (2) and (3) shows that

$$T\perp\!\!\!\perp\mathbf{Z}\Longleftrightarrow \mathrm{SICM}(t;a,\nu)=0,\forall t\in[0,\tau]. \qquad \square$$

Proof of Theorem 2. By the definition of $\mathrm{SICM}_{L_2}(t;a)$ and (4), we have

$$\begin{aligned}
&\frac{1}{\tau_z}\mathrm{SICM}_{L_2}(t;a)\\
&= -\int_0^t\int_0^t \mathbb{E}\Big\{\frac{\|\mathbf{Z}_1-\mathbf{Z}_2\|}{\tau_z}a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\
&= -\int_0^t\int_0^t \mathbb{E}\Big\{\Big[1+\frac{1}{2}\Big(\frac{\|\mathbf{Z}_1-\mathbf{Z}_2\|^2}{\tau_z^2}-1\Big)+O_p(p^{-1})\Big]a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\
&= -\frac{1}{2\tau_z^2}\int_0^t\int_0^t \mathbb{E}\Big\{(\|\mathbf{Z}_1-\mathbf{Z}_2\|^2-\tau_z^2)a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\
&\quad -O_p\Big(\frac{1}{p}\Big)\int_0^t\int_0^t \mathbb{E}\Big\{a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\
&= I + O_p(p^{-1}),
\end{aligned}\tag{S8}$$

where the third equality holds due to $\mathbb{E}\{a(s_1)Y_i(s_1)d\widetilde{N}_i(s_1)\}=0$.

Consider the term I. Note that

$$\begin{aligned}
I &= -\frac{1}{2\tau_z^2}\int_0^t\int_0^t\sum_{j=1}^p \mathbb{E}\Big\{(Z_{1j}-Z_{2j})^2 a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\
&= -\frac{1}{2\tau_z^2}\int_0^t\int_0^t\sum_{j=1}^p \mathbb{E}\Big\{\big[Z_{1j}^2+Z_{2j}^2-2Z_{1j}Z_{2j}\big]a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\
&= \frac{1}{\tau_z^2}\int_0^t\int_0^t\sum_{j=1}^p \mathbb{E}\Big\{Z_{1j}Z_{2j}a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\
&= \frac{1}{\tau_z^2}\sum_{j=1}^p \mathrm{Cov}^2\Big\{Z_j,\int_0^t a(s)Y(s)d\widetilde{N}(s)\Big\}.
\end{aligned}$$

This, together with $\tau_z = O(p^{1/2})$, implies that

$$\mathrm{SICM}_{L_2}(t;a) = \frac{1}{\tau_z}\sum_{j=1}^p \mathrm{Cov}^2\Big\{Z_j,\int_0^t a(t)Y(t)d\widetilde{N}(t)\Big\} + O_P(p^{-1/2}).$$

We next consider $\mathrm{SICM}_K(t;a)$. By a Taylor expansion and (4), we have that

$$\begin{aligned}
K(\mathbf{z}_1,\mathbf{z}_2) &= \Psi\Big(\frac{\|\mathbf{Z}_1-\mathbf{Z}_2\|}{\tau_z}\frac{\tau_z}{\gamma_z}\Big)\\
&= \Psi\Big(\Big[1+\frac{1}{2}\Big(\frac{\|\mathbf{Z}_1-\mathbf{Z}_2\|^2}{\tau_z^2}-1\Big)+O_p(p^{-1})\Big]\frac{\tau_z}{\gamma_z}\Big)\\
&= \Psi\Big(\frac{\tau_z}{\gamma_z}\Big)+\Psi'\Big(\frac{\tau_z}{\gamma_z}\Big)\Big\{\frac{L(\mathbf{Z}_1,\mathbf{Z}_2)}{2}+O_p(p^{-1})\Big\}\frac{\tau_z}{\gamma_z}+R_K(\mathbf{Z}_1,\mathbf{Z}_2),
\end{aligned}$$

where $L(\mathbf{Z}_1,\mathbf{Z}_2)=\|\mathbf{Z}_1-\mathbf{Z}_2\|^2/\tau_z^2-1$ and $R_K(\mathbf{Z}_1,\mathbf{Z}_2)$ is the remainder term. By the definition of $\mathrm{SICM}_K(t;a)$, we have

$$\begin{aligned}\mathrm{SICM}_K(t;a)&=\int_0^t\int_0^t\mathbb{E}\Big\{\Psi\Big(\frac{\|\mathbf{Z}_1-\mathbf{Z}_2\|}{\gamma_z}\Big)a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\&=\int_0^t\int_0^t\mathbb{E}\Big\{\Big(\Psi'\Big(\frac{\tau_z}{\gamma_z}\Big)\frac{L(\mathbf{Z}_1,\mathbf{Z}_2)}{2}\frac{\tau_z}{\gamma_z}+\Big[O_p(p^{-1})\Psi'\Big(\frac{\tau_z}{\gamma_z}\Big)\frac{\tau_z}{\gamma_z}+R_K(\mathbf{Z}_1,\mathbf{Z}_2)\Big]\Big)\\&\quad\times a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\&=I+II.\end{aligned}$$

By the definition of $L(\mathbf{Z}_1,\mathbf{Z}_2)=|\mathbf{Z}_1-\mathbf{Z}_2|^2/\tau_z^2-1$, we have that

$$\begin{aligned}I&=\Psi'\Big(\frac{\tau_z}{\gamma_z}\Big)\frac{1}{2\tau_z\gamma_z}\int_0^t\int_0^t\mathbb{E}\Big\{\|\mathbf{Z}_1-\mathbf{Z}_2\|^2a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\&=\Psi'\Big(\frac{\tau_z}{\gamma_z}\Big)\frac{1}{2\tau_z\gamma_z}\int_0^t\int_0^t\sum_{j=1}^p\mathbb{E}\Big\{(Z_{1j}-Z_{2j})^2a(s_1)Y_1(s_1)d\widetilde{N}_1(s_1)a(s_2)Y_2(s_2)d\widetilde{N}_2(s_2)\Big\}\\&=\Psi'\Big(\frac{\tau_z}{\gamma_z}\Big)\frac{1}{\tau_z\gamma_z}\sum_{j=1}^p\mathrm{Cov}^2\Big\{Z_j,\int_0^t a(s)Y(s)d\widetilde{N}(s)\Big\}.\end{aligned}$$

Denote $\mathcal{R}_{II}=O_p(p^{-1})\Psi'\Big(\frac{\tau_z}{\gamma_z}\Big)\frac{\tau_z}{\gamma_z}+R_K(\mathbf{Z}_1,\mathbf{Z}_2)$. By Lemma 2 in the supplemetary material of Zhu et al. (2020), we have that $\mathcal{R}_{II}=O_P(L^2(\mathbf{Z}_1,\mathbf{Z}_2))=o_P(1)$. Then, we have

$$\mathrm{SICM}_K(t;a)=\Psi'\Big(\frac{\tau_z}{\gamma_z}\Big)\frac{1}{\tau_z\gamma_z}\sum_{j=1}^p\mathrm{Cov}^2\Big\{Z_j,\int_0^t a(s)Y(s)d\widetilde{N}(s)\Big\}+o_P(1).\qquad\square$$

Proof of Theorem 3. By the definition of the conditional hazard function, we have that $\mathbb{E}\{dN(t)|\mathbf{Z},Y(t)\}=Y(t)\lambda(t|\mathbf{Z})dt$. For any $j\in\{1,\dots,p\}$, we obtain that

$$\begin{aligned}\mathbb{E}\{Z_jY(t)d\widetilde{N}(t)\}&=\mathbb{E}\{Z_jY(t)[dN(t)-\mathbb{E}\{dN(t)|Y(t)\}]\}\\&=\mathbb{E}\{Z_jY(t)\mathbb{E}\{dN(t)|\mathbf{Z},Y(t)\}\}-\mathbb{E}\{Z_jY(t)\mathbb{E}\{dN(t)|Y(t)\}\}\\&=\mathbb{E}\{Z_jY(t)\lambda(t|\mathbf{Z})\}-\mathbb{E}\{Z_jY(t)\mathbb{E}\{\lambda(t|\mathbf{Z})|Y(t)\}\}.\end{aligned}\qquad\text{(S9)}$$

We next calculate the two terms: $\mathbb{E}\{Z_jY(t)\lambda(t|\mathbf{Z})\}$ and $\mathbb{E}\{Z_jY(t)\mathbb{E}\{\lambda(t|\mathbf{Z})|Y(t)\}\}$.

Consider the term $\mathbb{E}\{Z_jY(t)\lambda(t|\mathbf{Z})\}$. Let $f_{T,\mathbf{Z}}(s,\mathbf{z})$ be the joint density function of $(T,\mathbf{Z})$. Note that

$$f_{T,\mathbf{Z}}(t,\mathbf{z})=f_{T|\mathbf{Z}}(t|\mathbf{z})f_{\mathbf{Z}}(\mathbf{z})=\lambda(t|\mathbf{z})S(t|\mathbf{z})f_{\mathbf{Z}}(\mathbf{z})=\lambda(t|\mathbf{z})\exp\Big\{-\int_0^t\lambda(s|\mathbf{z})ds\Big\}f_{\mathbf{Z}}(\mathbf{z}).$$

Since $f_{\mathbf{Z}}(\mathbf{z})$ and $\lambda(t|\mathbf{z})$ are even with respect to each component of $(z_1,\dots,z_p)$, i.e.,

$$f_{\mathbf{Z}}(z_1,\dots,z_k,\dots,z_p)=f_{\mathbf{Z}}(z_1,\dots,-z_k,\dots,z_p),$$

$$\lambda(t|z_1,\dots,z_k,\dots,z_p))=\lambda(t|(z_1,\dots,-z_k,\dots,z_p)),$$

for any $k\in\{1,\dots,p\}$, we obtain that

$$f_{T,\mathbf{Z}}(t,(z_1,\dots,z_k\dots,z_p))=f_{T,Z}(t,(z_1,\dots,-z_k,\dots,z_p)).\qquad\text{(S10)}$$

This implies that

$$\mathbb{E}\{Z_j Y(t)\lambda(t|\mathbf{Z})\} = \int_0^t \int_{\mathbb{R}^p} z_j \lambda(s|\mathbf{z}) f_{T,\mathbf{Z}}(s,\mathbf{z}) ds d\mathbf{z} = 0. \tag{S11}$$

Consider the term $\mathbb{E}\{Z_j Y(t)\mathbb{E}\{\lambda(t|\mathbf{Z})|Y(t)\}\}$. By (S10), we have that

$$\begin{aligned}\mathbb{E}\{Z_j Y(t)\mathbb{E}\{\lambda(t|\mathbf{Z})|Y(t)\}\} &= \mathbb{E}\{Z_j Y(t)\}\mathbb{E}\{\lambda(t|\mathbf{Z})|Y(t)=1\}\\ &= \mathbb{E}\{\lambda(t|\mathbf{Z})|Y(t)=1\}\int_0^t \int_{\mathbb{R}^p} z_j f_{T,\mathbf{Z}}(s,\mathbf{z}) ds\\ &= 0. \end{aligned} \tag{S12}$$

By (S9), (S11) and (S12), we have that $\mathbb{E}\{\mathbf{Z}Y(t)d\widetilde{N}(t)\} = 0$.

Proof of Theorem 4. Let $A_{ijkl} = K(Z_{is}, Z_{js})\psi(\widetilde{T}_i, \widetilde{T}_k)\psi(\widetilde{T}_j, \widetilde{T}_l)$. Then, we obtain

$$\begin{aligned}&\sum_{i,j,k,l=1}^{n} A_{ijkl}\\ &= \Big(\sum_{i\neq j\neq k\neq l}^{n} + \sum_{i=j,k,l}^{n} + \sum_{i=k,j,l}^{n} + \sum_{i=l,j,k}^{n} + \sum_{i,j=k,l}^{n} + \sum_{i,j=l,k}^{n} + \sum_{i,j,k=l}^{n} -2\sum_{i=j=k,l}^{n} -2\sum_{i=j=l,k}^{n}\\ &\quad -2\sum_{i=k=l,j}^{n} -2\sum_{i,j=k=l}^{n} - \sum_{i=j,k=l}^{n} - \sum_{i=k,j=l}^{n} - \sum_{i=l,j=k}^{n} +6\sum_{i=j=k=l}^{n}\Big)A_{ijkl}.\end{aligned}$$

Define

$$\begin{aligned}
J_1 &= \sum_{s=1}^{p}\sum_{i=j,k,l}^{n} A_{ijkl} = \sum_{s=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} K(Z_{is}, Z_{is})\psi(\widetilde{T}_i, \widetilde{T}_k)\psi(\widetilde{T}_i, \widetilde{T}_j),\\
J_2 &= \sum_{s=1}^{p}\sum_{i=l,j,k}^{n} A_{ijkl} = \sum_{s=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} K(Z_{is}, Z_{js})\psi(\widetilde{T}_i, \widetilde{T}_k)\psi(\widetilde{T}_j, \widetilde{T}_i),\\
J_3 &= \sum_{s=1}^{p}\sum_{i,j=k,l}^{n} A_{ijkl} = \sum_{s=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} K(Z_{is}, Z_{js})\psi(\widetilde{T}_i, \widetilde{T}_j)\psi(\widetilde{T}_j, \widetilde{T}_k),\\
J_4 &= \sum_{s=1}^{p}\sum_{i,j,k=l}^{n} A_{ijkl} = \sum_{s=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} K(Z_{is}, Z_{js})\psi(\widetilde{T}_i, \widetilde{T}_k)\psi(\widetilde{T}_j, \widetilde{T}_k),\\
J_5 &= \sum_{s=1}^{p}\sum_{i=j,k=l}^{n} A_{ijkl} = \sum_{s=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n} K(Z_{is}, Z_{is})\psi(\widetilde{T}_i, \widetilde{T}_j)\psi(\widetilde{T}_i, \widetilde{T}_j),\\
J_6 &= \sum_{s=1}^{p}\sum_{i=l,j=k}^{n} A_{ijkl} = \sum_{s=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n} K(Z_{is}, Z_{js})\psi(\widetilde{T}_i, \widetilde{T}_j)\psi(\widetilde{T}_j, \widetilde{T}_i),\\
J_7 &= \sum_{s=1}^{p}\sum_{i\neq j\neq k\neq l}^{n} A_{ijkl} = \sum_{s=1}^{p}\sum_{(i,j,k,l)}^{n} K(Z_{is}, Z_{js})\psi(\widetilde{T}_i, \widetilde{T}_k)\psi(\widetilde{T}_j, \widetilde{T}_l).
\end{aligned}$$

By some straightforward computations, we have that

$$\mathcal{W}_n = \frac{1}{n^4}\{J_7 + J_1 + J_2 + J_3 + J_4 - J_5 - J_6\}, \quad \mathcal{T}_n = \frac{1}{n_4} J_7.$$

Then, we can obtain that

$$\begin{aligned}\mathcal{T}_{n,p} = (n)_4^{-1} \sum_{s=1}^{p}\sum_{i=1}^{n}\sum_{j=1}^{n} \Big\{ & K(Z_{is}, Z_{js})[n^2\bar{\psi}(\widetilde{T}_i)\bar{\psi}(\widetilde{T}_j) + 2n\bar{\psi}(\widetilde{T}_i)\psi(\widetilde{T}_i, \widetilde{T}_j) \\ & - n\bar{\psi}(\widetilde{T}_i, \widetilde{T}_j) - \psi^2(\widetilde{T}_i, \widetilde{T}_j)] + K(Z_{is}, Z_{is})[\psi^2(\widetilde{T}_i, \widetilde{T}_j) - n^2\bar{\psi}^2(\widetilde{T}_i)] \Big\}, \qquad \square\end{aligned}$$

where

$$\bar{\psi}(\widetilde{T}_i) = \frac{1}{n}\sum_{k=1}^{n}\Delta_i I(X_k \geq X_i) - \frac{1}{n}\sum_{k=1}^{n}\Delta_k I(X_i \geq X_k), \quad \bar{\psi}(\widetilde{T}_i, \widetilde{T}_j) = \frac{1}{n}\sum_{k=1}^{n}\psi(\widetilde{T}_i, \widetilde{T}_k)\psi(\widetilde{T}_j, \widetilde{T}_k).$$

Proof of Theorem 5. By the Hoeffding decomposition of U-statistic (Lee, 1990) and Lemma S1 in Appendix B, we have that

$$\mathcal{T}_{n,p} = \frac{12}{n(n-1)} \sum_{1 \leq i < j \leq n} \bar{h}^{(2)}(\mathbf{W}_i, \mathbf{W}_j) + R_n^{(2)} = R_n^{(1)} + R_n^{(2)}, \tag{S13}$$

where $R_n^{(2)}$ is the remainder term, and

$$R_n^{(1)} = \frac{12}{n(n-1)} \sum_{1 \leq i < j \leq n} \bar{h}^{(2)}(\mathbf{W}_i, \mathbf{W}_j).$$

Then, we prove the theorem in the following two steps:

Step 1: We show that

$$\sqrt{\frac{n(n-1)}{2}}\frac{R_n^{(1)}}{\mathcal{S}} \xrightarrow{D} N(0,1), \text{ as } n, p \to \infty;$$

Step 2: We show that

$$\sqrt{\frac{n(n-1)}{2}}\frac{R_n^{(2)}}{\mathcal{S}} \xrightarrow{P} 0, \text{ as } n, p \to \infty.$$

We first consider Step 1. By Lemma S1, we obtain that

$$\bar{h}^{(2)}(\mathbf{W}_1, \mathbf{W}_2) = \frac{1}{6}\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s}) L_2(\tilde{T}_1, \tilde{T}_2).$$

Let $H(\mathbf{W}_1, \mathbf{W}_2) = \sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s}) L_2(\tilde{T}_1, \tilde{T}_2)$, with $\mathbf{W}_i = (X_i, \Delta_i, \mathbf{Z}_i)$. Plugging it into $\sqrt{\frac{n(n-1)}{2}} R_n^{(1)}/\mathcal{S}$, we have

$$\sqrt{\frac{n(n-1)}{2}}\frac{R_n^{(1)}}{\mathcal{S}} = \sqrt{\frac{2}{n(n-1)}} \sum_{1 \leq i < j \leq n} \frac{H(\mathbf{W}_i, \mathbf{W}_j)}{\mathcal{S}}.$$

We next prove that the above result converges to the standard normal distribution.

For the sake of convenience, we denote

$$\widetilde{R}^{(1)}_{k,p} = \sum_{i=2}^{k} B_{ni}, \;\; B_{ni} = \{n(n-1)/2\}^{-1/2} \sum_{j=1}^{i-1} H(\mathbf{W}_i, \mathbf{W}_j).$$

Let $\mathcal{F}_i = \sigma\{\mathbf{W}_1, ..., \mathbf{W}_i\}$ be the σ- filtration generated by $\{\mathbf{W}_j, j \le i\}$. It can be seen that $\mathbb{E}\{B_{ni}|\mathcal{F}_{i-1}\} = 0$, which means that $\{(\widetilde{R}^{(1)}_{k,p},\ \mathcal{F}_k)\}_{k=2}^{n}$ is a zero mean martingale. Subsequently, if we can verify two conditions in Corollary 3.1 presented by Hall & Heyde (2014), we will complete the proof for Step 1.

Specifically, define $q_{ni} = \mathbb{E}(B_{ni}^2|\mathcal{F}_{i-1})$, $2 \le i \le n$, and $Q_n = \sum_{i=2}^{n} q_{ni}$. We will prove that for any $\varepsilon > 0$, the following two conditions hold

$$\frac{Q_n}{\mathrm{Var}(\widetilde{R}^{(1)}_{n,p})} \xrightarrow{P} 1, \tag{S14}$$

$$\sum_{i=1}^{n} \mathcal{S}^{-2}\mathbb{E}\{B_{ni}^2 I(B_{ni} > \varepsilon\mathcal{S})|\mathcal{F}_{i-1}\} \xrightarrow{P} 0. \tag{S15}$$

To show (S14), we just need to verify that

$$\mathbb{E}\{Q_n\} = \mathrm{Var}\{\widetilde{R}^{(1)}_{n,p}\}, \quad \text{and} \quad \mathrm{Var}\{Q_n\} = o(\mathcal{S}^4).$$

Note that

$$q_{ni} = \mathbb{E}\{B_{ni}^2|\mathcal{F}_{i-1}\} = \mathbb{E}\left\{\sum_{j,k=1}^{i-1} H(\mathbf{W}_j, \mathbf{W}_i)H(\mathbf{W}_k, \mathbf{W}_i)|\mathcal{F}_{i-1}\right\}.$$

We can decompose it into $q_{ni} = q_{ni}^{(1)} + q_{ni}^{(2)}$, where

$$q_{ni}^{(1)} = \{n(n-1)/2\}^{-1} \sum_{j=1}^{i-1} \mathrm{Var}\{H(\mathbf{W}_i, \mathbf{W}_j)|\mathbf{W}_j\},$$
$$q_{ni}^{(2)} = \{n(n-1)\}^{-1} \sum_{1\le j<k\le i-1} \mathbb{E}\{H(\mathbf{W}_i, \mathbf{W}_j)H(\mathbf{W}_i, \mathbf{W}_k)|\mathbf{W}_j, \mathbf{W}_k\}.$$

It can be showed that

$$\mathbb{E}\{q_{ni}^{(1)}\} = \{n(n-1)/2\}^{-1}(i-1)\mathrm{Var}\{H(\mathbf{W}_i, \mathbf{W}_j)|\mathbf{W}_j\}.$$

Then, it follows immediately that $\mathbb{E}\{\sum_{i=2}^{n} q_{ni}^{(1)}\} = \mathrm{Var}\{\widetilde{R}^{(1)}_{n,p}\}$. By Lemma S1 and Assumption 2, as $p \to \infty$, we have that

$$\mathbb{E}\{V(\mathbf{Z}_1, \mathbf{Z}_2)V(\mathbf{Z}_2, \mathbf{Z}_3)V(\mathbf{Z}_3, \mathbf{Z}_4)V(\mathbf{Z}_4, \mathbf{Z}_1)\} = o(\mathcal{S}^4), \;\; n^{-1}\mathbb{E}\{V^4(\mathbf{Z}_1, \mathbf{Z}_2)\} = o(\mathcal{S}^4),$$

where $V(\mathbf{Z}_1, \mathbf{Z}_2) = \sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s})$. We next calculate the variance of $\sum_{i=2}^{n} q_{ni}^{(1)}$. Note that

$$
\begin{aligned}
&\left(\frac{n(n-1)}{2}\right)^2 \mathrm{Var}\left\{\sum_{i=2}^{n} q_{ni}^{(1)}\right\} \\
&= \mathrm{Var}\left\{\sum_{j=1}^{n}\sum_{s=1}^{p}\sum_{t=1}^{p}(n-j)\mathbb{E}\left\{L_1(Z_s, Z_{js})L_1(Z_t, Z_{jt})L_2^2(\tilde{T}, \tilde{T}_j)|\mathbf{W}_j\right\}\right\} \\
&= \sum_{j=1}^{n}(n-j)^2\mathrm{Var}\left\{\mathbb{E}\left\{V^2(\mathbf{Z}, \mathbf{Z}_j)\mathbb{E}\big[L_2^2(\tilde{T}, \tilde{T}_j|\mathbf{Z})\big]|\mathbf{W}_j\right\}\right\} \\
&\leq \sum_{j=1}^{n}(n-j)^2\mathrm{Var}\left\{\sqrt{\mathbb{E}\Big\{V^4(\mathbf{Z}, \mathbf{Z}_j)|\mathbf{W}_j\Big\}\mathbb{E}\left\{\mathbb{E}^2\left(L_2^2(\tilde{T}, \tilde{T}_j)|\mathbf{Z}\right)|\mathbf{W}_j\right\}}\right\} \\
&\leq \sum_{j=1}^{n}(n-j)^2\mathrm{Var}\left\{\sqrt{\mathbb{E}\Big\{V^4(\mathbf{Z}, \mathbf{Z}_j)|\mathbf{W}_j\Big\}\mathbb{E}\Big\{\mathbb{E}\big(L_2^4(\tilde{T}, \tilde{T}_j)|\mathbf{Z}\big)|\mathbf{W}_j\Big\}}\right\} \\
&\leq \sum_{j=1}^{n}(n-j)^2\mathrm{Var}\Big(\sqrt{\mathbb{E}\Big\{V^4(\mathbf{Z}, \mathbf{Z}_j)|\mathbf{W}_j\Big\}}\Big) = o(n\mathcal{S}^4)\sum_{j=1}^{n}(n-j)^2,
\end{aligned}
$$

where the first inequality holds from the Hölder's inequality, the second inequality holds due to the Jensen's inequality, and the third inequality holds due to the fact $\sup_{t\in[0,\tau]}|L_2^4(\tilde{t}, \tilde{t}_j)| \leq 1$. Together with the fact $\sum_{-1}^{n}(n-j)^2 = O(n^3)$, we obtain that

$$
\mathrm{Var}\left\{\sum_{i=2}^{n} q_{ni}^{(1)}\right\} = o(\mathcal{S}^4).
$$

By the Markov's inequality, we have that

$$
\sum_{i=2}^{n} q_{ni}^{(1)}/\mathrm{Var}(\widetilde{R}_{n,p}^{(1)}) \xrightarrow{P} 1.
$$

The second term can be derived by analogous calculation that $q_{ni}^{(2)}/\mathrm{Var}\{\tilde{R}_{n,p}^{(1)}\} = o_P(1)$. We first show that $\mathbb{E}\left\{\sum_{i=2}^n q_{ni}^{(2)}/\mathrm{Var}(\widetilde{R}_{n,p}^{(1)})\right\} = 0$ and

$$
\begin{aligned}
&\{n(n-1)\mathcal{S}^2\}^2\mathrm{Var}\left\{\sum_{i=2}^n q_{ni}^{(2)}\right\}\\
&= \mathrm{Var}\left\{\sum_{1\le j<k\le n}^n (n-k)\mathbb{E}\big\{[V(\mathbf{Z},\mathbf{Z}_j)]L_2(\tilde{T},\tilde{T}_j)[V(\mathbf{Z},\mathbf{Z}_k)]L_2(\tilde{T},\tilde{T}_k)|\mathbf{W}_j,\mathbf{W}_k\big\}\right\}\\
&\le \mathrm{Var}\Big\{\sum_{1\le j<k\le n}^n (n-k)\sqrt{\mathbb{E}\big[\mathbb{E}^2\{L_2(\tilde{T},\tilde{T}_j)|\mathbf{Z}\}\mathbb{E}^2\{L_2(\tilde{T},\tilde{T}_k)|\mathbf{Z}\}|\tilde{T}_j,\tilde{T}_k\big]}\\
&\quad\times\sqrt{\mathbb{E}\big[V^2(\mathbf{Z},\mathbf{Z}_j)V^2(\mathbf{Z},\mathbf{Z}_k)|\mathbf{Z}_j,\mathbf{Z}_k\big]}\Big\}\\
&\le \mathrm{Var}\Big\{\sum_{1\le j<k\le n}^n (n-k)\sqrt{\mathbb{E}\big[\mathbb{E}\{L_2^2(\tilde{T},\tilde{T}_j)|\mathbf{Z}\}\mathbb{E}\{L_2^2(\tilde{T},\tilde{T}_k)|\mathbf{Z}\}|\tilde{T}_j,\tilde{T}_k\big]}\\
&\quad\times\sqrt{\mathbb{E}\big[V^2(\mathbf{Z},\mathbf{Z}_j)V^2(\mathbf{Z},\mathbf{Z}_k)|\mathbf{Z}_j,\mathbf{Z}_k\big]}\Big\}\\
&\le \sum_{j=1}^n (j-1)(n-j)^2\mathbb{E}\left\{V(\mathbf{Z}_1,\mathbf{Z}_2)V(\mathbf{Z}_2,\mathbf{Z}_3)V(\mathbf{Z}_3,\mathbf{Z}_4)V(\mathbf{Z}_4,\mathbf{Z}_1)\right\}\\
&= o(\mathcal{S}^4)\sum_{j=1}^n (j-1)(n-j)^2,
\end{aligned}
$$

where the first and second inequalities hold due to the Hölder's inequality and the last inequality holds by the fact $\sup_{t\in[0,\tau]}|L_2^2(\tilde{t},\tilde{t}_j)|$ is bounded. Combining with the fact that $\sum_{j=1}^n(j-1)(n-j)^2 = O(n^4)$, we have

$$\mathrm{Var}\left\{\sum_{i=2}^n q_{ni}^{(2)}\right\} = o(\mathcal{S}^4).$$

Using the Markov's inequality again, we have

$$\sum_{i=2}^n q_{ni}^{(2)}/\mathrm{Var}(\widetilde{R}_{n,p}^{(1)}) = o_P(1).$$

Thus, we have

$$\mathbb{E}\{Q_n\} = \mathbb{E}\left\{\sum_{i=2}^n q_{ni}^{(1)}\right\} = \mathrm{Var}\{\tilde{R}_{n,p}^{(1)}\},\ \text{and}\ \mathrm{Var}\{Q_n\} = o(\mathcal{S}^4(1+o(1))).$$

Based on all the above results, (S14) holds.

Next we verify that (S15) holds. By the Markov's inequality, we obtain that

$$0 \le \mathcal{S}^{-2}\mathbb{E}\{B_{ni}^2 I(B_{ni} > \varepsilon\mathcal{S})|\mathcal{F}_{i-1}\} \le \frac{\mathbb{E}\{B_{ni}^4|\mathcal{F}_{i-1}\}}{\varepsilon^2\mathcal{S}^4}.$$

To this end, it suffices to show that

$$\sum_{i=2}^{n} \mathbb{E}\{B_{ni}^4\} = o(\mathcal{S}^4). \tag{S16}$$

Under the assumption $\mathbb{E}\{H^4(\mathbf{W}_1, \mathbf{W}_2)\} = o(n\mathcal{S}^4)$, (S16) can be proved by the result

$$\begin{aligned}
&\left[\frac{n(n-1)}{2}\right]^2 \sum_{2=1}^{n} \mathbb{E}\{B_{ni}^4\} \\
&= \sum_{i=2}^{n} \sum_{j_1,j_2,j,j_4=1}^{i-1} \mathbb{E}\left\{H(\mathbf{W}_{j_1}, \mathbf{W}_i)H(\mathbf{W}_{j_2}, \mathbf{W}_i)H(\mathbf{W}_{j_3}, \mathbf{W}_i)H(\mathbf{W}_{j_4}, \mathbf{W}_i)\right\} \\
&= \frac{n(n-1)}{2}\mathbb{E}\left\{H^4(\mathbf{W}_i, \mathbf{W}_j)\right\} + 3\sum_{i=2}^{n}(i-1)(i-2)\mathbb{E}\left\{H^2(\mathbf{W}_1, \mathbf{W}_3)H^2(\mathbf{W}_2, \mathbf{W}_3))\right\} \\
&\le \left[\frac{n(n-1)}{2} + 3\sum_{i=2}^{n}(i-1)(i-2)\right] o(n\mathcal{S}^4) = \frac{n^3 - 5n^2 + 3n}{2} o(n\mathcal{S}^4).
\end{aligned}$$

Based on the results in (S14) and (S15), $\sqrt{\frac{n(n-1)}{2}} R_n^{(1)}/\mathcal{S}$ converges to a standard normal distribution. Thus, we have completed the proof for Step 1.

Step 2 can be shown as follow. By the Hoeffding decomposition of U-statistic (Lee, 1990), the remainder term in (S13) can be obtained by

$$R_n^{(2)} = 4H_n^{(3)} + H_n^{(4)},$$

where $H_n^{(j)} = \binom{n}{j}^{-1} \sum_{<n,j>} \bar{l}^{(j)}(\mathbf{W}_{i_1}, ..., \mathbf{W}_{i_j})$ for $j = 2, 3, 4$ and

$$\bar{l}^{(j)}(\mathbf{w}_1, ..., \mathbf{w}_j) = \bar{h}^{(j)}(\mathbf{w}_1, ..., \mathbf{w}_j) - \sum_{s=1}^{j-1} \sum_{<j,s>} \bar{l}^{(s)}(\mathbf{w}_{i_1}, ..., \mathbf{w}_{i_s}).$$

Then, we can calculate that

$$\begin{aligned}
&\mathrm{Var}\{R_n^{(2)}\} \\
&= 16\mathrm{Var}\{H_n^{(3)}\} + \mathrm{Var}\{H_n^{(4)}\} \\
&= 16\binom{n}{3}^{-1} \mathrm{Var}\left\{\bar{l}^{(3)}(\mathbf{W}_1, ..., \mathbf{W}_3)\right\} + \binom{n}{4}^{-1} \mathrm{Var}\left\{\bar{l}^{(4)}(\mathbf{W}_1, ..., \mathbf{W}_4)\right\}. \qquad \square
\end{aligned}$$

By the Hoeffding's variance formula in Lemma S1,

$$\mathrm{Var}\{R_n^{(1)}\} = O(n^{-2}\mathcal{S}^2), \quad \mathrm{Var}\{R_n^{(2)}\} = o\left(n^{-2}\mathcal{S}^2\right),$$

which follows from the fact that $\mathrm{Var}\{\bar{h}^{(2)}\}$, $\mathrm{Var}\{\bar{h}^{(3)}\}$ and $\mathrm{Var}\{\bar{h}^{(4)}\}$ are of the same order. From Serfling (1980), we have

$$\sqrt{\frac{n(n-1)}{2}} R_n^{(2)}/\mathcal{S} = o_P(1).$$

This completes the proof.

Proof of Theorem 6. By the Slutsky's theorem and the Markov's inequality, we only need to prove that

$$\frac{\left|\mathbb{E}\Big\{L^2_{2,n}(\tilde{T}_i,\tilde{T}_j)[\sum_{s=1}^{p}L_{1,n}(Z_{is},Z_{js})]^2\Big\}-\mathcal{S}^2\right|}{\mathcal{S}^2}\to 0, \tag{S17}$$

$$\frac{\mathrm{Var}\Big\{\sum_{(i,j)}L^2_{2}(\tilde{T}_i,\tilde{T}_j)[\sum_{s=1}^{p}L_{1}(Z_{is},Z_{js})]^2\Big\}}{n^4\mathcal{S}^4}\to 0. \tag{S18}$$

We can divide these into the following Steps 1-2.

Step 1: We first prove (S17). Denote

$$\bar{\psi}(\tilde{t})=\delta\mathbb{E}(I(X\geq x))-\mathbb{E}(\Delta I(x\geq X)).$$

By the law of large numbers and the Slutsky's theorem, it is easy to see that

$$\frac{1}{n}\sum_{k=1}^{n}\big[\Delta_i(I(X_k\geq X_i))-(\Delta_k I(X_i\geq X_k))\big]\xrightarrow{P}\bar{\psi}(\tilde{T}_i).$$

Then, we have

$$L^2_{2,n}(\tilde{T}_i,\tilde{T}_j)=\bar{\psi}(\tilde{T}_i)\bar{\psi}(\tilde{T}_j)+o_P(1).$$

Based on the double U-centered property, it is easy to verify

$$\begin{aligned}
&L_{1,n}(Z_{is},Z_{js})\\
=&\frac{n-3}{n-1}L_1(Z_{is},Z_{js})-\frac{n-3}{(n-1)(n-2)}\sum_{l\notin\{i,j\}}^{n}L_1(Z_{is},Z_{ls})\\
&-\frac{n-3}{(n-1)(n-2)}\sum_{k\notin\{i,j\}}^{n}L_1(Z_{ks},Z_{js})+\frac{1}{(n-1)(n-2)}\sum_{k,l\notin\{i,j\}}^{n}L_1(Z_{ks},Z_{ls}).
\end{aligned}\tag{S19}$$

By the independence of $\mathbf{Z}_i$ and $\mathbf{Z}_j$, these four terms on the right-hand sides of (S19) are uncorrelated with each other. This provides an alternative way of justifying the unbiasedness of the U-centered estimator. The variance of $L_{1,n}(Z_{is},Z_{js})$ is equal to

$$\mathbb{E}\left\{L^2_{1,n}(Z_{is},Z_{js})\right\}=\frac{n-1}{n-3}\mathbb{E}\left\{L^2_1(Z_{is},Z_{js})\right\}.$$

Therefore, the unbiased estimator of $\mathbb{E}\left\{L^2_1(Z_{is},Z_{js})\right\}$ is

$$\sum_{(i,j)}L^2_{1,n}(Z_{is},Z_{js})/\{n(n-3)\}.$$

This is convenient to verify the condition (S17). Then, we have that

$$\begin{aligned}
\mathbb{E}\Big\{L^2_{2,n}(\tilde{T}_i,\tilde{T}_j)[\sum_{s=1}^{p}L_{1,n}(Z_{is},Z_{js})]^2\Big\}&=\mathbb{E}\Big\{\big(o_p(1)+L^2_2(\tilde{T}_i,\tilde{T}_j)\big)[\sum_{s=1}^{p}L_{1,n}(Z_{is},Z_{js})]^2\Big\}\\
&=\mathbb{E}\Big\{\big(L^2_2(\tilde{T}_i,\tilde{T}_j)+o_p(1)\big)[\sum_{s=1}^{p}L_{1}(Z_{is},Z_{js})]^2\Big\}\\
&=(1+o(1))\mathcal{S}^2+cn^{-1}\mathbb{E}\Big\{\Big(\sum_{s=1}^{p}L_{1,n}(Z_{is},Z_{js})\Big)^4\Big\}^{1/2},
\end{aligned}$$

where the second equality follows from (S19). By the Hölder's inequality, we obtain that

$$\frac{\mathbb{E}\Big\{L_{2,n}^2(\tilde{T}_i,\tilde{T}_j)[\sum\limits_{s=1}^{p}L_{1,n}(Z_{is},Z_{js})]^2-\mathcal{S}^2\Big\}}{\mathcal{S}^2}\le\frac{cn^{-1}\mathbb{E}\Big\{\Big(\sum\limits_{s=1}^{p}L_{1,n}(Z_{is},Z_{js})\Big)^4\Big\}^{1/2}}{\mathcal{S}^2}\to 0,$$

which implies that condition (S17) holds.

Step 2: We turn to prove (S18) in this step. Denote

$$d_{kl}=\sum_{s=1}^{p}L_{1,n}(Z_{ks},Z_{ls})L_{2,n}(\tilde{T}_k,\tilde{T}_l).$$

To prove (S18), it suffices to show that

$$\operatorname{Var}\left\{\sum_{k<l}d_{kl}^2\right\}=o(n^4\mathcal{S}^4). \tag{S20}$$

Note that

$$\begin{aligned}
&\operatorname{Var}\left\{\sum_{k<l}d_{kl}^2\right\}\\
&=\sum_{k<l}\sum_{k'<l'}\operatorname{Cov}(d_{kl}^2,d_{k'l'}^2)\\
&=2\sum_{k<l<l'}\operatorname{Cov}\left\{d_{kl}^2,d_{l'l}^2\right\}+\sum_{k<l}\operatorname{Var}\left\{d_{kl}^2\right\}+\sum_{k<l,k'<l',\{k,j\}\cap\{k',l'\}=\emptyset}\operatorname{Cov}\left\{d_{kl}^2,d_{k'l'}^2\right\}\\
&\le 2\sum_{k<l<l'}\mathbb{E}\left\{d_{kl}^2d_{l'l}^2\right\}+\sum_{k<l}\mathbb{E}\left\{d_{kl}^4\right\}+\sum_{k<l,k'<l',\{k,j\}\cap\{k',l'\}=\emptyset}\mathbb{E}\left\{d_{kl}^2,d_{k'l'}^2\right\}\\
&=I_1+I_2+I_3.
\end{aligned}$$

For the first term I_1, it follows from (S19) and the Hölder's inequality that

$$\begin{aligned}
2\sum_{k<l<l'}\mathbb{E}\left\{d_{kl}^2d_{l'l}^2\right\}&=O(n^3)\mathbb{E}\left\{d_{kl}^2d_{l'l}^2\right\}\\
&\le O(n^3)\left(\mathbb{E}\left\{V(\mathbf{Z}_k,\mathbf{Z}_l)^2V(\mathbf{Z}_k,\mathbf{Z}_{l'})^2\right\}+n^{-1}\mathbb{E}\left\{V(\mathbf{Z}_k,\mathbf{Z}_l)^4\right\}\right)\\
&\le O(n^3)\left(\mathbb{E}\left\{V(\mathbf{Z}_k,\mathbf{Z}_l)^4\right\}+n^{-1}\mathbb{E}\left\{V(\mathbf{Z}_k,\mathbf{Z}_l)^4\right\}\right).
\end{aligned}\tag{S21}$$

By similar calculations, the second terms I_2 and I_3 follow that

$$\sum_{k<l}\mathbb{E}\left\{d_{kl,t}^4\right\}\le O(n^2)\mathbb{E}\left\{V(\mathbf{Z}_k,\mathbf{Z}_l)^4\right\}, \tag{S22}$$

and

$$\begin{aligned}
&\sum_{k<l,k'<l',\{k,j\}\cap\{k',l'\}}\mathbb{E}\left\{d_{kl}^2,d_{k'l'}^2\right\}\\
&\le O(n^4)\Big(\mathbb{E}\left\{V(\mathbf{Z}_k,\mathbf{Z}_l)V(\mathbf{Z}_l,\mathbf{Z}_{k'})V(\mathbf{Z}_{k'},\mathbf{Z}_{l'})V(\mathbf{Z}_{l'},\mathbf{Z}_k)\right\}+n^{-1}\mathbb{E}\left\{V(\mathbf{Z}_k,\mathbf{Z}_l)^4\right\}\Big).
\end{aligned}\tag{S23}$$

These results, together with Assumption 2, lead to

$$\operatorname{var}\left\{\sum_{k<l}d_{kl}^2\right\}\le o(n^4\mathcal{S}^4).$$

Then, the result in (S18) follows from Assumption 2, (S21), (S22), (S23) and the Slutsky's theorem. Combining the results in Steps 1 and 2, we complete the proof. $\square$

Proof of Theorem 7. Since $\mathcal{T}_{n,p}$ is a U-statistic, it follows from the proof of Lemma S2 that under the local alternatives H_1', we have that

$$\begin{aligned}
&\mathcal{T}_{n,p} - \mathrm{SICM}_K(T|\mathbf{Z}) \\
&= 6\sum_{i<j}^{n} \bar{l}^{(2)}(\mathbf{W}_i, \mathbf{W}_j) + o_P(1) \\
&= 6\sum_{i<j}^{n} \Big\{ \bar{h}^{(2)}(\mathbf{W}_i, \mathbf{W}_j) - \bar{h}^{(1)}(\mathbf{W}_i) - \bar{h}^{(1)}(\mathbf{W}_j) + \mathrm{SICM}_p(X|\mathbf{Z}) \Big\} + o_P(1) \\
&= \sum_{i<j}^{n} \Big(\sum_{s=1}^{p} \Big\{ L_1(Z_{is}, Z_{js}) L_2(\tilde{T}_i, \tilde{T}_j) - \mathbb{E}(L_1(Z_{is}, Z_{js}) L_2(\tilde{T}_i, \tilde{T}_j)|\mathbf{W}_i) \\
&\quad - \mathbb{E}(L_1(Z_{is}, Z_{js}) L_2(\tilde{T}_i, \tilde{T}_j)|\mathbf{W}_j) \Big\} + \mathrm{SICM}_K(T|\mathbf{Z}) \Big) + o_p(1).
\end{aligned}$$

Denote

$$\begin{aligned}
\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2) = \Big(\sum_{s=1}^{p} \Big\{ L_1(Z_{1s}, Z_{2s}) L_2(\tilde{T}_1, \tilde{T}_2) - \mathbb{E}\Big\{ L_1(Z_{1s}, Z_{2s}) L_2(\tilde{T}_1, \tilde{T}_2)|\mathbf{W}_1 \Big\} \\
- \mathbb{E}\Big\{ L_1(Z_{1s}, Z_{2s}) L_2(\tilde{T}_1, \tilde{T}_2)|\mathbf{W}_2 \Big\} + \mathrm{SICM}_K(T|\mathbf{Z}) \Big) / (\sqrt{n(n-1)/2}\mathcal{S}).
\end{aligned}$$

It can be shown that

$$\mathbb{E}\{\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)|\mathbf{W}_1\} = 0.$$

Under the local alternatives H_1', we further have

$$\begin{aligned}
\frac{n^2}{2}\mathbb{E}\{\tilde{H}_n^2(\mathbf{W}_1, \mathbf{W}_2)\} &= 1 + O\left(\mathrm{Var}\left\{ \mathbb{E}\left\{ \sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s}) L_2(\tilde{T}_1, \tilde{T}_2|\mathbf{W}_1 \right\} \mathcal{S}^{-2} \right\} \right) \\
&= 1 + o(1).
\end{aligned}$$

To establish the asymptotic normality of $\mathcal{T}_{n,p}$, it suffices to verify the condition (2.1) in Theorem 1 of Hall (1984), namely,

$$\frac{\mathbb{E}\{G_n(\mathbf{W}_1, \mathbf{W}_2)^2\} + n^{-1}\mathbb{E}\{\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)\}}{\mathbb{E}^2\{\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)^2\}} \to 0,$$

as $n, p \to \infty$, where $G_n(\mathbf{W}_1, \mathbf{W}_2) = \mathbb{E}(\tilde{H}_n(\mathbf{W}_3, \mathbf{W}_1)\tilde{H}_n(\mathbf{W}_3, \mathbf{W}_2)|\mathbf{W}_1, \mathbf{W}_2)$. Following the proof of Lemma S2, under Assumption 2, we can show that

$$\begin{aligned}
\mathbb{E}\{G_n(\mathbf{W}_1, \mathbf{W}_2)^2\} &\le Cn^{-4}\mathbb{E}\{V(\mathbf{Z}_1, \mathbf{Z}_2)V(\mathbf{Z}_2, \mathbf{Z}_3)V(\mathbf{Z}_3, \mathbf{Z}_4)V(\mathbf{Z}_4, \mathbf{Z}_1)\}/\mathcal{S}^4 \\
&\quad + Cn^{-4}\mathrm{Var}^2\{\mathbb{E}(\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s}) L_2(\tilde{T}_1, \tilde{T}_2)|\mathbf{W}_1)\}/\mathcal{S}^4, \\
\mathbb{E}^2\{\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)^2\} &= 4n^{-4}\{1 + o(1)\}, \\
n^{-1}\mathbb{E}\{\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)^4\} &\le Cn^{-5}\mathbb{E}\{V(\mathbf{Z}_1, \mathbf{Z}_2)^4\}/\mathcal{S}^4.
\end{aligned}$$

By Assumption 2 and $|\psi| \leq 2$, it follows that

$$\mathbb{E}\{G_n(\mathbf{W}_1, \mathbf{W}_2)^2\}/\mathbb{E}^2\{\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)^2\} \to 0,$$
$$n^{-1}\mathbb{E}\{\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)^4\}/\mathbb{E}^2\{\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)^2\} \to 0,$$

as $n, p \to \infty$. Therefore, all assumptions in Theorem 1 of Hall (1984) are satisfied with the kernel $\tilde{H}_n(\mathbf{W}_1, \mathbf{W}_2)$. This completes the proof. □

APPENDIX B: LEMMAS FOR THEOREMS 5-7

To prove Theorems 5-7, we need to use the asymptotic properties of the variance of $\mathcal{T}_{n,p}$ as $p \to \infty$. We can establish this result by the Hoeffding decomposition of U-statistic (Lee, 1990).

We first obtain the symmetric kernel of $\mathcal{T}_{n,p}$. Define

$$\phi_s(i,j,k,l) = \frac{1}{4}\psi(\widetilde{T}_i, \widetilde{T}_k)\psi(\widetilde{T}_j, \widetilde{T}_l)K_{s,ijkl},$$

where $K_{s,ijkl} = K(Z_{is}, Z_{js}) - K(Z_{is}, Z_{ls}) - K(Z_{js}, Z_{ks}) + K(Z_{ks}, Z_{ls})$. Then, the symmetrized version of $\sum_{s=1}^{p} \phi_s(i,j,k,l)$ can be obtained by

$$h(i,j,k,l) = \frac{1}{3}\sum_{s=1}^{p}\left\{\phi_s(i,j,k,l) + \phi_s(i,j,l,k) + \phi_s(i,k,j,l)\right\}. \tag{S24}$$

Then, $\mathcal{T}_{n,p}$ has the following expression

$$\mathcal{T}_{n,p} = \frac{1}{n_4}\sum_{(i,j,k,l)}^{n} h(i,j,k,l).$$

Thus, $\mathcal{T}_{n,p}$ is a U-statistic of order four with the symmetric kernel $h(i,j,k,l)$. Let

$$\bar{h}^{(c)}(\mathbf{W}_1, ..., \mathbf{W}_c) = \mathbb{E}\{h(1,2,3,4)|\mathbf{W}_1, ..., \mathbf{W}_c\}, c = 1,2,3,4,$$

be the c-order projection of h, where $\mathbf{W}_i = (X_i, \Delta_i, \mathbf{Z}_i)$ is the i-th observation. Denote $\xi_c = \mathrm{Var}\{\bar{h}^{(c)}(\mathbf{W}_1, ..., \mathbf{W}_c)\}$.

Let $\mathbf{w} = (x, \delta, \mathbf{z})$ and $\tilde{t} = (x, \delta)$, where $x \in \mathbb{R}^+$, $\delta \in \{0,1\}$ and $\mathbf{z} \in \mathbb{R}^p$. Direct calculation shows that

$$\bar{h}^{(1)}(\mathbf{w}_1) = \sum_{s=1}^{p}\mathbb{E}\{L_1(z_{1s}, Z_{2s})L_2(\tilde{t}_1, \widetilde{T}_2) + L_1(Z_{3s}, Z_{4s})\psi(\widetilde{T}_3, \tilde{t}_1)\psi(\widetilde{T}_4, \widetilde{T}_2)\}/2,$$

$$\begin{aligned}\bar{h}^{(2)}(\mathbf{w}_1, \mathbf{w}_2) = \sum_{s=1}^{p}\mathbb{E}\{&L_1(z_{1s}, z_{2s})L_2(\tilde{t}_1, \tilde{t}_2) + L_1(z_{1s}, Z_{3s})\psi(\tilde{t}_1, \tilde{T}_4)\psi(\tilde{T}_3, \tilde{t}_2)\\ &+L_1(z_{2s}, Z_{3s})\psi(\tilde{t}_2, \tilde{T}_4)\psi(\tilde{T}_3, \tilde{t}_1) + L_1(Z_{3s}, Z_{4s})\psi(\tilde{T}_3, \tilde{t}_1)\psi(\tilde{T}_4, \tilde{t}_2)\\ &+L_1(z_{1s}, Z_{3s})\psi(\tilde{t}_1, \tilde{t}_2)\psi(\tilde{T}_3, \tilde{T}_4) + L_1(z_{2s}, Z_{3s})\psi(\tilde{t}_2, \tilde{t}_1)\psi(\tilde{T}_3, \tilde{T}_4)\}/6,\end{aligned}$$

$$\begin{aligned}&\bar{h}^{(3)}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)\\ =& \sum_{s=1}^{p}\mathbb{E}\Big\{[L_1(z_{1s}, z_{2s}) - L_1(z_{1s}, Z_{4s}) - L_1(z_{2s}, z_{3s}) + L_1(z_{3s}, Z_{4s})]\psi(\tilde{t}_1, \tilde{t}_3)\psi(\tilde{t}_2, \tilde{T}_4)\\ &+[L_1(z_{1s}, z_{2s}) - L_1(z_{1s}, z_{3s}) - L_1(z_{2s}, Z_{4s}) + L_1(z_{3s}, Z_{4s})]\psi(\tilde{t}_1, \tilde{T}_4)\psi(\tilde{t}_2, \tilde{t}_3)\\ &+[L_1(z_{1s}, z_{3s}) - L_1(z_{1s}, Z_{4s}) - L_1(z_{2s}, z_{3s}) + L_1(z_{2s}, Z_{4s})]\psi(\tilde{t}_1, \tilde{t}_2)\psi(\tilde{t}_3, \tilde{T}_4)\Big\}/12\end{aligned}$$

and

$$
\begin{aligned}
&\bar{h}^{(4)}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) \\
&= \sum_{s=1}^{p} \Big\{ [L_1(z_{1s}, z_{2s}) - L_1(z_{1s}, z_{4s}) - L_1(z_{2s}, z_{3s}) + L_1(z_{3s}, z_{4s})]\psi(\tilde{t}_1, \tilde{t}_3)\psi(\tilde{t}_2, \tilde{t}_4) \\
&\quad + [L_1(z_{1s}, z_{2s}) - L_1(z_{1s}, z_{3s}) - L_1(z_{2s}, z_{4s}) + L_1(z_{3s}, Z_{4s})]\psi(\tilde{t}_1, \tilde{t}_4)\psi(\tilde{t}_2, \tilde{t}_3) \\
&\quad + [L_1(z_{1s}, z_{3s}) - L_1(z_{1s}, z_{4s})\ L_1(z_{2s}, z_{3s}) + L_1(z_{2s}, z_{4s})]\psi(\tilde{t}_1, \tilde{t}_2)\psi(\tilde{t}_3, \tilde{t}_4) \Big\} / 12.
\end{aligned}
$$

LEMMA S1. *Under H_0', we have that*

(1) $\bar{h}^{(1)}(\mathbf{w}_1) = 0$, *and* $\bar{h}^{(2)}(\mathbf{w}_1, \mathbf{w}_2) = \sum_{s=1}^{p} L_1(\mathbf{z}_{1s}, \mathbf{z}_{2s}) L_2(\tilde{t}_1, \tilde{t}_2)/6$;
(2) $\mathrm{Var}\{\mathcal{T}_{n,p}\} = \frac{2}{n(n-1)} \mathcal{S}^2\{1 + o(1)\}$.

Proof of Lemma S1. Under H_0', it is easy to see that $\bar{h}^{(1)}(\mathbf{w}_1) = 0$. Moreover, we have that

$$
\bar{h}^{(2)}(\mathbf{w}_1, \mathbf{w}_2) = \sum_{s=1}^{p} L_1(z_{1s}, z_{2s}) L_2(\tilde{t}_1, \tilde{t}_2)/6.
$$

Then, we obtain that

$$
\begin{aligned}
\mathrm{Var}\{\bar{h}^{(2)}(\mathbf{W}_1, \mathbf{W}_2)\} &= \frac{1}{36} \mathbb{E}\Big\{ \Big[\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s}) \Big]^2 L_2^2(\tilde{T}_1, \tilde{T}_2) \Big\} \\
&= \frac{1}{36} \mathbb{E}\Big\{ \Big[\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s}) \Big]^2 \mathbb{E}\{L_2^2(\tilde{T}_1, \tilde{T}_2) | Z_{1s}, Z_{2s}\} \Big\}. \qquad \text{(S25)}
\end{aligned}
$$

By the definition of $\psi(\cdot, \cdot)$, we have that

$$
\mathbb{E}\{\psi(\tilde{T}_1, \tilde{T}_3) | \tilde{T}_1\} = \int_0^{\tau} \mathbb{E}\{Y(s)\} \Big[dN_1(s) - \mathbb{E}\{dN(s) | Y_1(s)\} \Big].
$$

By Theorem 2.4.2 in Fleming & Harrington (1991), under H_0', we have that

$$
\begin{aligned}
\mathbb{E}\{L_2^2(\tilde{T}_1, \tilde{T}_2) | Z_{1s}, Z_{2s}\} &= \mathrm{Var}^2\Big\{ \int_0^{\tau} \mathbb{E}\{Y(s)\} \Big[dN_1(s) - \mathbb{E}\{dN(s) | Y_1(s), Z_{1s}\} \Big\} \\
&= \mathbb{E}^2\Big\{ \int_0^{\tau} \mathbb{E}^2\{Y(s)\} d < M_1, M_1 > (s) \Big\} \\
&= \mathbb{E}^2\Big\{ \int_0^{\tau} \mathbb{E}^2\{Y(s)\} Y_1(s) \lambda(s | \mathbf{Z}_{1s}) ds \Big\}, \qquad \text{(S26)}
\end{aligned}
$$

where $< M, M > (s)$ is the predictable variation process of the martingale $M(s)$. Let

$$
\eta^2 = \mathbb{E}\Big\{ \Big[\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s}) \Big]^2 \Big\}.
$$

Then, we have that

$$\begin{aligned}
\eta^2 &= \sum_{s=1}^{p}\sum_{s'=1}^{p} \mathrm{Cov}\{L_1(Z_{1s}, Z_{2s}), L_1(Z_{1s'}, Z_{2s'})\} \\
&= \sum_{s=1}^{p}\sum_{s'=1}^{p} \mathbb{E}\{K(Z_{1s}, Z_{2s})K(Z_{1s'}, Z_{2s'})\} + \mathbb{E}\{K(Z_{1s}, Z_{2s})\}\mathbb{E}\{K(Z_{1s'}, Z_{2s'})\} \\
&\quad -2\mathbb{E}\{K(Z_{1s}, Z_{2s})K(Z_{1s'}, Z_{3s'})\} \\
&= \sum_{s=1}^{p}\sum_{s'=1}^{p} \mathrm{HSIC}(Z_s, Z_{s'}) \\
&\geq \sum_{s=1}^{p} \mathrm{HSIC}(Z_s, Z_s) = cp \to \infty, \qquad \text{(S27)}
\end{aligned}$$

as $p \to \infty$, where c is a constant and $\mathrm{HSIC}(X, Y)$ is the Hilbert-Schmidt independence criterion (Gretton et al., 2005). By (S25)-(S27), we have that

$$\begin{aligned}
\mathrm{Var}\{\bar{h}^{(2)}(\mathbf{W}_1, \mathbf{W}_2)\} &= \frac{1}{36}\mathrm{Var}\left\{\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s})L_2(\tilde{T}_1, \tilde{T}_2)\right\} \\
&= \frac{1}{36}\mathcal{S}^2 \asymp \eta^2 \to \infty, \text{ as } p \to \infty.
\end{aligned}$$

By the the boundedness of $|L_2(\tilde{t}_1, \tilde{t}_2))| \leq 4$ for any $t \in [0, \tau]$, we have that

$$\begin{aligned}
&\mathrm{Var}\{\bar{h}^{(3)}(\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3)\} \\
&= \frac{1}{144}\mathbb{E}\Big\{\Big[\sum_{s=1}^{p}[L_1(Z_{1s}, Z_{2s}) - L_1(Z_{2s}, Z_{3s})]\psi(\tilde{T}_1, \tilde{T}_3)\bar{\psi}(\tilde{T}_2) \\
&\quad +[L_1(Z_{1s}, Z_{2s}) - L_1(Z_{1s}, Z_{3s})]\bar{\psi}(\tilde{T}_1)\psi(\tilde{T}_2, \tilde{T}_3) \\
&\quad +[L_1(Z_{1s}, Z_{3s}) - L_1(Z_{2s}, Z_{3s})]\psi(\tilde{T}_1, \tilde{T}_2)\bar{\psi}(\tilde{T}_3)\Big]^2\Big\} \\
&\leq \frac{16}{144}\mathbb{E}\Big\{\big[\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s})\big]^2 + \big[\sum_{s=1}^{p} L_1(Z_{2s}, Z_{3s})\big]^2 + \big[\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s})\big]^2 \\
&\quad +\big[\sum_{s=1}^{p} L_1(Z_{1s}, Z_{3s})\big]^2 + \big[\sum_{s=1}^{p} L_1(Z_{1s}, Z_{3s})\big]^2 + \big[\sum_{s=1}^{p} L_1(Z_{2s}, Z_{3s})\big]^2\Big\} \\
&= \frac{16*6}{144}\mathbb{E}\Big\{\big[\sum_{s=1}^{p} L_1(Z_{1s}, Z_{2s})\big]^2\Big\} = \frac{2}{3}\eta^2
\end{aligned}$$

and

$$
\begin{aligned}
&\mathrm{Var}\{\bar{h}^{(4)}(\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4)\}\\
=&\frac{1}{144}\mathbb{E}\Big\{\Big[\sum_{s=1}^{p}[L_1(Z_{1s},Z_{2s})-L_1(Z_{1s},Z_{4s})-L_1(Z_{2s},Z_{3s})\\
&+L_1(Z_{3s},Z_{4s})]\psi(\tilde{T}_1,\tilde{T}_3)\psi(\tilde{T}_2,\tilde{T}_4)+[L_1(Z_{1s},Z_{2s})-L_1(Z_{1s},Z_{3s})\\
&-L_1(Z_{2s},Z_{4s})+L_1(Z_{3s},Z_{4s})]\psi(\tilde{T}_1,\tilde{T}_4)\psi(\tilde{T}_2,\tilde{T}_3)+[L_1(Z_{1s},Z_{3s})-L_1(Z_{1s},Z_{4s})\\
&-L_1(Z_{2s},Z_{3s})+L_1(Z_{2s},Z_{4s})]\psi(\tilde{T}_1,\tilde{T}_2)\psi(\tilde{T}_3,\tilde{T}_4)\Big]^2\Big\}\\
\leq&\frac{16*12}{144}\mathbb{E}\Big\{\big[\sum_{s=1}^{p}L_1(Z_{1s},Z_{2s})\big]^2\Big\}=\frac{4}{3}\eta^2.
\end{aligned}
$$

The above results imply that $\xi_3=O(\eta^2)$ and $\xi_4=O(\eta^2)$, as $p\to\infty$. Then, by Theorem 3 in
155 Lee (1990), we have that

$$
\begin{aligned}
\mathrm{Var}\{\mathcal{T}_{n,p}\}&=\binom{n}{4}^{-1}\sum_{c=1}^{4}\binom{4}{c}\binom{n-4}{k-c}\xi_c\\
&=\sum_{c=1}^{4}\binom{4}{c}\frac{4!}{(4-c)!}[n^{-c}+O(n^{-c-1})]\xi_c\\
&=\frac{2}{n(n-1)}\mathrm{Var}\Big\{\sum_{s=1}^{p}L_1(Z_{1s},Z_{2s})L_2(\tilde{T}_1,\tilde{T}_2)\Big\}\{1+o(1)\}. \qquad \square
\end{aligned}
$$

LEMMA S2. *Under the local H_1', we have that*

$$\mathrm{Var}\{\mathcal{T}_{n,p}\}=\frac{2}{n(n-1)}\mathcal{S}^2\{1+o(1)\}.$$

Proof of Lemma S2. We first consider the variance of $\bar{h}^{(1)}$. Assume that $\mathrm{Var}\{\mathbb{E}\{H(\mathbf{W}_1,\mathbf{W}_2)|\mathbf{W}_1\}\}=o(n^{-1}\mathcal{S}^2)$ and $\mathrm{Var}\{\mathbb{E}\{G_1(\mathbf{W}_1,\mathbf{W}_2)|\mathbf{W}_1]\}\}=o(n^{-1}\mathcal{S}^2)$. Using the Hölder's inequality, we have that

$$\mathrm{Var}\{\bar{h}^{(1)}(\mathbf{W}_1)\}\leq o(n^{-1}\mathcal{S}^2).$$

The variance of $\bar{h}^{(2)}$ includes the terms showed as follows:

$$\mathrm{Var}\{H(\mathbf{W}_1,\mathbf{W}_2)\},\quad \mathrm{Var}\left\{G_1(\mathbf{W}_1,\mathbf{W}_2)\right\},\quad \mathrm{Var}\left\{G_2(\mathbf{W}_1,\mathbf{W}_2)\right\}, \qquad \text{(S28)}$$

160 and the remainder cross terms can be control by these terms. The first term is the leading term, $\mathrm{Var}\{H(\mathbf{W}_1,\mathbf{W}_2)\}=\mathcal{S}^2$. Thus, we need to assume that the another two terms in (S28) are all $o(\mathcal{S}^2)$. Then, we have

$$\mathrm{Var}\{\bar{h}^{(2)}\}=(1+o(1))\mathcal{S}^2.$$

Similar to the null hypothesis, we have

$$\begin{aligned}
&\mathbb{E}\Big\{\mathbb{E}^2\big([\sum_{s=1}^{p} L_1(Z_{1s}, Z_{3s})]\psi(\tilde{T}_1,\tilde{T}_3)\psi(\tilde{T}_2,\tilde{T}_4)|\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3\big)\Big\}\\
&\le \mathbb{E}\Big\{\mathbb{E}\big([\sum_{s=1}^{p} L_1(Z_{1s}, Z_{3s})]^2|\mathbf{W}_1,\mathbf{W}_3\big)\mathbb{E}\big(\psi^2(\tilde{T}_1,\tilde{T}_3)\psi^2(\tilde{T}_2,\tilde{T}_4)|\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3\big)\Big\}\\
&\le \mathbb{E}\Big\{\mathbb{E}\big([\sum_{s=1}^{p} L_1(Z_{1s}, Z_{3s})]^2|\mathbf{W}_1,\mathbf{W}_3\big)\Big\}\\
&= O(\eta^2), \qquad\qquad (S29)
\end{aligned}$$

where the first inequality holds from the Hölder's inequality and the second inequality is due to $\sup_{t\in[0,\tau]}|\psi(\tilde{t}_1,\tilde{t}_3)\psi(\tilde{t}_2,\tilde{t}_4)|\le 1$. Similarly,

$$\begin{aligned}
&\mathbb{E}\Big\{\mathbb{E}^2\big([\sum_{s=1}^{p} L_1(Z_{1s}, Z_{4s})]\psi(\tilde{T}_1,\tilde{T}_3)\psi(\tilde{T}_2,\tilde{T}_4)|\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3\big)\Big\}\\
&\le \mathbb{E}\Big\{\mathbb{E}\big([\sum_{s=1}^{p} L_1(Z_{1s}, Z_{4s})]^2|\mathbf{W}_1\big)\mathbb{E}\big(\psi^2(\tilde{T}_1,\tilde{T}_3)\psi^2(\tilde{T}_2,\tilde{T}_4)|\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3\big)\Big\}\\
&\le \mathbb{E}\Big\{\mathbb{E}\big([\sum_{s=1}^{p} L_1(Z_{1s}, Z_{4s})]^2|\mathbf{W}_1\big)\Big\}\\
&= O(\eta^2). \qquad\qquad (S30)
\end{aligned}$$

The rest terms in $\text{Var}\{\bar{h}^{(3)}\}$ are same to (S29) and (S30). So, we can derive that $\text{Var}\{\bar{h}^{(3)}\}\le O(\mathcal{S}^2)$ as $p\to\infty$. For the variance of $\bar{h}^{(4)}$, we can calculate that

$$\begin{aligned}
&\text{Var}\{\bar{h}^{(4)}(\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4)\}\\
&= \frac{1}{144}\mathbb{E}\Big\{\Big(\sum_{s=1}^{p}[L_1(Z_{1s},Z_{2s}) - L_1(Z_{1s},Z_{4s} - L_1(Z_{2s},Z_{3s}) + L_1(Z_{3s},Z_{4s})]\\
&\quad\times\psi(\tilde{T}_1,\tilde{T}_3)\psi(\tilde{T}_2,\tilde{T}_4) + [L_1(Z_{1s},Z_{2s}) - L_1(Z_{1s},Z_{3s}) - L_1(Z_{2s},Z_{4s})\\
&\quad+L_1(\mathbf{Z}_{3s},Z_{4s})]\psi(\tilde{T}_1,\tilde{T}_4)\psi(\tilde{T}_2,\tilde{T}_3) + [L_1(Z_{1s},Z_{3s}) - L_1(Z_{1s},Z_{4s})\\
&\quad-L_1(Z_{2s},Z_{3s}) + L_1(Z_{2s},Z_{4s})]\psi(\tilde{T}_1,\tilde{T}_2)\psi(\tilde{T}_3,\tilde{T}_4)\Big)^2\Big\}\\
&\le 12\mathbb{E}\Big\{\big[\sum_{s=1}^{p} L_1(Z_{1s},Z_{2s})\big]^2\Big\} = 12\eta^2,
\end{aligned}$$

where the inequality holds from the Hölder's inequality and the boundedness of ψ. As $p\to\infty$, the remainder term in $\text{Var}\{\mathcal{T}_{n,p}\}-\mathcal{S}^2$ is asymptotically negligible if

$$\begin{aligned}
&\text{Var}\Big\{\mathbb{E}\Big\{G_1(\mathbf{W}_1,\mathbf{W}_2)\Big\} = o(\mathcal{S}^2), \quad \text{Var}\Big\{\mathbb{E}\Big\{H(\mathbf{W}_1,\mathbf{W}_2)|\mathbf{W}_1\Big\}\Big\} = o(n^{-1}\mathcal{S}^2),\\
&\text{Var}\Big\{\mathbb{E}\Big\{G_2(\mathbf{W}_1,\mathbf{W}_2)\Big\} = o(\mathcal{S}^2), \quad \text{Var}\Big\{\mathbb{E}\Big\{G_1(\mathbf{W}_1,\mathbf{W}_2)|\mathbf{W}_1\Big\}\Big\} = o(n^{-1}\mathcal{S}^2). \qquad \square
\end{aligned}$$

These results, together with Assumptions above, lead to that ξ_2, ξ_3 and ξ_4 are the same order which is $O(\eta^2)$ as $p\to\infty$ under the local alternative H_1'. The variance of $\text{Var}\{\mathcal{T}_{n,p}\}$ is same showed in the null alternative, the third and forth terms in the Hoeffding decomposition are all of smaller order.

APPENDIX C: DISCUSSIONS ON ASSUMPTION 2

In the section, we discuss Assumption 2 under the banded dependence structure for $\mathbf{Z}$. Under such structure, Z_i and Z_j are independent if $|i-j|>m$ for some given m.

By $|\psi|\leq 2$, a sufficient condition on Assumption 2 is

$$\mathbb{E}\{V(\mathbf{Z}_1,\mathbf{Z}_2)V(\mathbf{Z}_2,\mathbf{Z}_3)V(\mathbf{Z}_3,\mathbf{Z}_4)V(\mathbf{Z}_4,\mathbf{Z}_1)\}=o(\mathcal{S}^4),\quad n^{-1}\mathbb{E}\{V^4(\mathbf{Z}_1,\mathbf{Z}_2)\}=o(\mathcal{S}^4),$$

where $V(\mathbf{Z}_1,\mathbf{Z}_2)=\sum_{s=1}^{p}L_1(Z_{1s},Z_{2s})$. Under the banded dependence, we have that

$$\begin{aligned}
&\mathbb{E}\{V(\mathbf{Z}_1,\mathbf{Z}_2)V(\mathbf{Z}_2,\mathbf{Z}_3)V(\mathbf{Z}_3,\mathbf{Z}_4)V(\mathbf{Z}_4,\mathbf{Z}_1)\}\\
&=\sum_{i=1}^{p}\sum_{j=i}^{i+m}\sum_{k=j-m}^{j+m}\sum_{l=k-m}^{k+m}\mathbb{E}\{L_1(Z_{1i},Z_{2i})L_1(Z_{2j},Z_{3j})L_1(Z_{3k},Z_{4k})L_1(Z_{4l},Z_{1l})\}\\
&=O(pm^3)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\{V^4(\mathbf{Z}_1,\mathbf{Z}_2)\}=&\sum_{i=1}^{p}\mathbb{E}\{L_1^4(Z_{1i},Z_{2i})\}+4\sum_{1\leq i\neq j\leq p}\mathbb{E}\{L_1^3(Z_{1i},Z_{2i})L_1(Z_{1j},Z_{2j})\}\\
&+3\sum_{1\leq i\neq j\neq\leq p}\mathbb{E}\{L_1^2(Z_{1i},Z_{2i})L_1^2(Z_{1j},Z_{2j})\}\\
&+6\sum_{1\leq i\neq j\neq k\leq p}\mathbb{E}\{L_1^2(Z_{1i},Z_{2i})L_1(Z_{1j},Z_{2j})L_1(Z_{1k},Z_{2k})\}\\
&+\sum_{1\leq i\neq j\neq k\neq l\leq p}\mathbb{E}\{L_1(Z_{1i},Z_{2i})L_1(Z_{1j},Z_{2j})L_1(Z_{1k},Z_{2k})L_1(Z_{1l},Z_{2l})\}\\
=&\,O\left(pm^3+\mathcal{S}^4\right).
\end{aligned}$$

As $n\to\infty$ and $p\to\infty$, we have the following results

$$\begin{aligned}
&\mathcal{S}^2\geq cp\to\infty,\\
&0\leq\mathbb{E}\{H(\mathbf{W}_1,\mathbf{W}_2)H(\mathbf{W}_2,\mathbf{W}_3)H(\mathbf{W}_3,\mathbf{W}_4)H(\mathbf{W}_4,\mathbf{W}_1)\}=O(pm^3),\\
&\mathbb{E}\{H^2(\mathbf{W}_1,\mathbf{W}_3)H^2(\mathbf{W}_2,\mathbf{W}_3)\}\leq\mathbb{E}\{H(\mathbf{W}_1,\mathbf{W}_2)^4\}=O\left(pm^3+\mathcal{S}^4\right),
\end{aligned}$$

where c is a nonzero constant determined by the kernel function K. Then, we have

$$\begin{aligned}
&\mathbb{E}\{H(\mathbf{W}_1,\mathbf{W}_2)H(\mathbf{W}_2,\mathbf{W}_3)H(\mathbf{W}_3,\mathbf{W}_4)H(\mathbf{W}_4,\mathbf{W}_1)\}/\mathcal{S}^4\leq O(m^3/p),\\
&\mathbb{E}\{H(\mathbf{W}_1,\mathbf{W}_2)^4\}/n\mathcal{S}^4\leq O(m^3/np+1/n).
\end{aligned}$$

Assumption 2 holds if $m=o(p^{1/3})$ as $p\to\infty$. In particular, when m is fixed and p is divergent, the above conditions are mild. Moreover, there is no explicit relationship between p and n. If the coordinates of $\mathbf{Z}$ are independent but not necessarily identically distributed, p can grow to infinity freely as $n\to\infty$.

Table S1. *The empirical powers for the additive hazards models in Case 2 for Example 2 at the significance level 5% and 10%.*

							Gaussian Kernel		Laplace Kernel	
	n	p	α	SNR	IPCW	SICM_{L^2}	KLR	SICM_K	KLR	SICM_K
(i)	40	20	0.050	2.344	0.035	0.285	0.115	0.665	0.130	0.665
		40	0.050	1.450	0.040	0.180	0.050	0.425	0.055	0.425
		80	0.050	1.087	0.030	0.150	0.055	0.260	0.050	0.265
		20	0.100	2.344	0.075	0.390	0.240	0.790	0.270	0.780
		40	0.100	1.450	0.100	0.265	0.140	0.530	0.135	0.535
		80	0.100	1.087	0.095	0.230	0.100	0.380	0.115	0.395
	60	30	0.050	2.843	0.060	0.385	0.135	0.810	0.155	0.820
		60	0.050	1.966	0.051	0.240	0.080	0.555	0.085	0.580
		90	0.050	1.654	0.065	0.190	0.055	0.505	0.060	0.530
		30	0.100	2.843	0.110	0.505	0.250	0.885	0.275	0.915
		60	0.100	1.966	0.065	0.355	0.135	0.695	0.145	0.695
		90	0.100	1.654	0.105	0.305	0.120	0.640	0.135	0.660
	80	40	0.050	3.209	0.030	0.410	0.060	0.905	0.075	0.915
		80	0.050	2.369	0.040	0.315	0.060	0.720	0.060	0.715
		120	0.050	1.976	0.050	0.215	0.035	0.605	0.045	0.595
		40	0.100	3.209	0.090	0.590	0.195	0.950	0.210	0.950
		80	0.100	2.369	0.125	0.420	0.125	0.835	0.140	0.820
		120	0.100	1.976	0.105	0.340	0.090	0.750	0.095	0.755
(ii)	40	20	0.050	1.182	0.105	0.115	0.075	0.350	0.075	0.355
		40	0.050	0.897	0.110	0.155	0.075	0.255	0.085	0.275
		80	0.050	0.468	0.080	0.110	0.065	0.125	0.070	0.120
		20	0.100	1.182	0.160	0.195	0.165	0.460	0.170	0.475
		40	0.100	0.897	0.145	0.215	0.155	0.360	0.180	0.365
		80	0.100	0.468	0.095	0.150	0.145	0.160	0.135	0.180
	60	30	0.050	0.743	0.020	0.080	0.035	0.165	0.035	0.170
		60	0.050	0.541	0.030	0.095	0.055	0.135	0.055	0.135
		90	0.050	0.368	0.050	0.095	0 045	0.125	0.050	0.140
		30	0.100	0.743	0.075	0.135	0.065	0.255	0.080	0.265
		60	0.100	0.541	0.080	0.185	0.120	0.210	0.105	0.210
		90	0.100	0.368	0.110	0.135	0.080	0.185	0.095	0.185
	80	40	0.050	1.091	0.045	0.170	0.060	0.310	0.075	0.310
		80	0.050	0.812	0.050	0.185	0.050	0.245	0.065	0.225
		120	0.050	0.572	0.050	0.105	0.025	0.145	0.030	0.145
		40	0.100	1.091	0.095	0.255	0.130	0.395	0.155	0.410
		80	0.100	0.812	0.120	0.235	0.120	0.315	0.130	0.330
		120	0.100	0.572	0.100	0.155	0.085	0.200	0.085	0.220

APPENDIX D: SIMULATION RESULTS FOR CASE 2-5 IN EXAMPLE 2

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Table S2. *The empirical powers for the AFT models in Case 3 for Example* 2 *at the significance level 5% and 10%.*

							Gaussian Kernel		Laplace Kernel	
	n	p	α	SNR	IPCW	SICM_{L^2}	KLR	SICM_K	KLR	SICM_K
(i)	40	20	0.050	2.185	0.110	0.235	0.430	0.590	0.445	0.615
		40	0.050	1.780	0.095	0.245	0.260	0.565	0.265	0.550
		80	0.050	1.459	0.065	0.220	0.190	0.440	0.195	0.435
		20	0.100	2.185	0.210	0.360	0.600	0.735	0.652	0.730
		40	0.100	1.780	0.165	0.325	0.430	0.695	0.465	0.680
		80	0.100	1.459	0.120	0.315	0.305	0.520	0.310	0.575
	60	30	0.050	2.619	0.080	0.285	0.435	0.800	0.445	0.805
		60	0.050	2.088	0.085	0.250	0.225	0.635	0.230	0.625
		90	0.050	1.931	0.105	0.210	0.175	0.545	0.170	0.570
		30	0.100	2.619	0.170	0.380	0.590	0.880	0.605	0.890
		60	0.100	2.088	0.145	0.355	0.370	0.760	0.380	0.750
		90	0.100	1.931	0.140	0.330	0.255	0.725	0.275	0.755
	80	40	0.050	3.510	0.120	0.480	0.490	0.945	0.570	0.945
		80	0.050	2.788	0.080	0.345	0.210	0.840	0.240	0.840
		120	0.050	2.629	0.050	0.340	0.165	0.775	0.185	0.775
		40	0.100	3.510	0.220	0.680	0.685	0.975	0.760	0.980
		80	0.100	2.788	0.150	0.455	0.370	0.900	0.380	0.900
		120	0.100	2.629	0.125	0.455	0.295	0.860	0.315	0.850
(ii)	40	20	0.050	1.600	0.075	0.185	0.200	0.480	0.230	0.485
		40	0.050	1.154	0.025	0.105	0.105	0.290	0.100	0.270
		80	0.050	0.994	0.035	0.150	0.115	0.290	0.110	0.285
		20	0.100	1.600	0.145	0.265	0.355	0.575	0.370	0.580
		40	0.100	1.154	0.080	0.175	0.190	0.425	0.195	0.390
		80	0.100	0.994	0.105	0.245	0.180	0.405	0.180	0.445
	60	30	0.050	1.900	0.040	0.195	0.170	0.550	0.170	0.585
		60	0.050	1.469	0.055	0.170	0.120	0.385	0.115	0.385
		90	0.050	1.411	0.050	0.170	0.110	0.385	0.115	0.395
		30	0.100	1.900	0.110	0.290	0.270	0.675	0.295	0.700
		60	0.100	1.469	0.110	0.230	0.235	0.510	0.235	0.500
		90	0.100	1.411	0.085	0.290	0.205	0.510	0.220	0.525
	80	40	0.050	2.436	0.070	0.250	0.265	0.745	0.265	0.750
		80	0.050	2.063	0.045	0.215	0.110	0.670	0.115	0.640
		120	0.050	1.908	0.055	0.205	0.120	0.610	0.130	0.600
		40	0.100	2.436	0.155	0.375	0.375	0.835	0.395	0.830
		80	0.100	2.063	0.095	0.320	0.215	0.755	0.210	0.755
		120	0.100	1.908	0.080	0.305	0.225	0.690	0.220	0.690

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Table S3. *The empirical powers for the accelerated hazards models in Case 4 for Example 2 at the significance level 5% and 10%.*

							Gaussian Kernel		Laplace Kernel	
	n	p	α	SNR	IPCW	SICM_{L^2}	KLR	SICM_K	KLR	SICM_K
(i)	40	20	0.050	0.992	0.075	0.150	0.170	0.285	0.185	0.305
		40	0.050	0.757	0.050	0.130	0.115	0.200	0.115	0.215
		80	0.050	0.703	0.050	0.135	0.125	0.215	0.130	0.205
		20	0.100	0.992	0.100	0.220	0.290	0.405	0.290	0.400
		40	0.100	0.757	0.120	0.190	0.205	0.305	0.195	0.290
		80	0.100	0.703	0.080	0.205	0.195	0.335	0.190	0.350
	60	30	0.050	1.139	0.055	0.105	0.110	0.305	0.125	0.310
		60	0.050	0.943	0.075	0.140	0.115	0.235	0.120	0.230
		90	0.050	0.872	0.020	0.095	0.080	0.180	0.065	0.200
		30	0.100	1.139	0.070	0.175	0.210	0.425	0.210	0.430
		60	0.100	0.943	0.155	0.195	0.210	0.310	0.220	0.330
		90	0.100	0.872	0.075	0.150	0.185	0.305	0.180	0.290
	80	40	0.050	1.549	0.065	0.200	0.200	0.430	0.210	0.450
		80	0.050	1.293	0.065	0.150	0.100	0.365	0.110	0.370
		120	0.050	1.384	0.050	0.180	0.120	0.410	0.120	0.405
		40	0.100	1.549	0.140	0.270	0.335	0.550	0.360	0.560
		80	0.100	1.293	0.115	0.210	0.175	0.505	0.180	0.495
		120	0.100	1.384	0.110	0.270	0.185	0.500	0.195	0.505
(ii)	40	20	0.050	1.296	0.065	0.205	0.065	0.365	0.060	0.380
		40	0.050	0.921	0.055	0.125	0.040	0.255	0.040	0.280
		80	0.050	0.820	0.060	0.125	0.040	0.190	0.040	0.210
		20	0.100	1.296	0.145	0.275	0.145	0.455	0.170	0.475
		40	0.100	0.921	0.115	0.225	0.105	0.385	0.120	0.380
		80	0.100	0.820	0.135	0.190	0.115	0.305	0.115	0.305
	60	30	0.050	1.529	0.080	0.210	0.075	0.435	0.085	0.435
		60	0.050	1.337	0.060	0.155	0.055	0.380	0.070	0.390
		90	0.050	1.330	0.050	0.195	0.055	0.370	0.055	0.400
		30	0.100	1.529	0.165	0.305	0.120	0.555	0.140	0.555
		60	0.100	1.337	0.095	0.260	0.125	0.515	0.135	0.505
		90	0.100	1.330	0.150	0.270	0.090	0.500	0.090	0.530
	80	40	0.050	2.391	0.115	0.285	0.075	0.695	0.075	0.710
		80	0.050	1.972	0.100	0.260	0.060	0.550	0.070	0.565
		120	0.050	1.807	0.050	0.230	0.040	0.560	0.030	0.560
		40	0.100	2.391	0.190	0.430	0.165	0.800	0.175	0.800
		80	0.100	1.972	0.165	0.345	0.125	0.680	0.130	0.700
		120	0.100	1.807	0.120	0.330	0.070	0.655	0.065	0.685

Table S4. *The empirical powers for the transformed hazards models in Case 5 for Example 2 at the significance level 5% and 10%.*

							Gaussian Kernel		Laplace Kernel	
	n	p	α	SNR	IPCW	SICM$_{L^2}$	KLR	SICM$_K$	KLR	SICM$_K$
(i)	40	20	0.050	1.091	0.065	0.195	0.070	0.310	0.065	0.320
		40	0.050	0.648	0.030	0.105	0.045	0.175	0.045	0.195
		80	0.050	0.560	0.040	0.090	0.035	0.160	0.030	0.145
		20	0.100	1.091	0.130	0.245	0.105	0.415	0.120	0.435
		40	0.100	0.648	0.095	0.175	0.090	0.270	0.100	0.305
		80	0.100	0.560	0.105	0.145	0.115	0.230	0.115	0.230
	60	30	0.050	1.244	0.055	0.135	0.060	0.355	0.075	0.365
		60	0.050	1.032	0.055	0.115	0.060	0.300	0.065	0.300
		90	0.050	0.926	0.045	0.135	0.050	0.265	0.050	0.275
		30	0.100	1.244	0.115	0.230	0.105	0.470	0.105	0.490
		60	0.100	1.032	0.095	0.230	0.120	0.425	0.130	0.430
		90	0.100	0.926	0.115	0.225	0.085	0.360	0.085	0.385
	80	40	0.050	2.051	0.100	0.250	0.050	0.650	0.050	0.600
		80	0.050	1.384	0.050	0.150	0.100	0.400	0.100	0.350
		120	0.050	0.877	0.045	0.050	0.050	0.200	0.052	0.200
		40	0.100	2.051	0.150	0.350	0.150	0.900	0.100	0.950
		80	0.100	1.384	0.150	0.300	0.100	0.500	0.150	0.500
		120	0.100	0.877	0.050	0.100	0.100	0.300	0.100	0.300
(ii)	40	20	0.050	1.206	0.060	0.175	0.060	0.330	0.070	0.335
		40	0.050	0.878	0.020	0.130	0.055	0.250	0.065	0.270
		80	0.050	0.705	0.020	0.110	0.035	0.251	0.035	0.225
		20	0.100	1.206	0.100	0.250	0.150	0.410	0.160	0.435
		40	0.100	0.878	0.095	0.200	0.140	0.335	0.140	0.335
		80	0.100	0.705	0.080	0.145	0.075	0.285	0.075	0.285
	60	30	0.050	0.769	0.035	0.085	0.040	0.155	0.045	0.155
		60	0.050	0.779	0.045	0.130	0.045	0.230	0.035	0.215
		90	0.050	0.735	0.050	0.105	0.040	0.215	0.045	0.215
		30	0.100	0.769	0.080	0.135	0.090	0.235	0.085	0.255
		60	0.100	0.779	0.095	0.200	0.115	0.305	0.115	0.320
		90	0.100	0.735	0.120	0.195	0.090	0.275	0.085	0.295
	80	40	0.050	1.643	0.050	0.150	0.150	0.450	0.250	0.400
		80	0.050	1.267	0.050	0.200	0.050	0.350	0.050	0.350
		120	0.050	0.616	0.050	0.050	0.050	0.200	0.050	0.200
		40	0.100	1.643	0.100	0.350	0.300	0.600	0.350	0.550
		80	0.100	1.267	0.050	0.200	0.150	0.550	0.100	0.500
		120	0.100	0.616	0.050	0.100	0.050	0.250	0.100	0.250