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ARMS RACE

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Also Check

You can also view our presentation on the topic at ArmsRace-presentation
Or
View our repository which holds all data at GitHub/DE-Arms-Race.

ABSTRACT

Wars and conflicts are raging every day, and every country cares only about having stronger army, and to help analyzing the situation since world war II till now, many models have been invented. These models are depending many differential equations that uses the data in our world to achieve accurate results to understand and predict the state, and we hope in this humble report to illustrate the concept of an "Arms Race Model".

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1. Introduction:

An arms race, in its original usage, is a competition between two or more states to have the best armed forces. Each party competes to produce more weapons, larger military, superior military technology, etc. in a technological escalation.
Back in the days of the cold war between the United States of America and the Soviet Union, the greatest interest was in the "ARM RACE" between the two countries. And since the end of the war, most of the scientists had focused on the regional antagonisms to analyze it.
And before we get more further in the explanation of our subject we must mention who the pioneer in this field was and what did he contribute with.
"Lewis fry Richardson".

1.1 About him:

Richardson -shown in fig (1)-, -who was born in 1881 and died in 1953- was an English mathematician, meteorologist, pacifist, physicist and psychologist and of course the pioneer in mathematical techniques and its relation in wars and it's causes and weather forecasting. Richardson's Quaker beliefs entailed an ardent pacifism that exempted him from military service during World War I as a conscientious objector, though this subsequently disqualified him from having any academic post. Richardson worked from 1916 to 1919 for the Friends' Ambulance Unit attached to the 16th French Infantry Division. After the war, he rejoined the Meteorological Office but was compelled to resign on grounds of conscience when it was amalgamated into the Air Ministry in 1920. He subsequently pursued a career on the fringes of the academic world before retiring in 1940 to research his own ideas. His pacifism had direct consequences on his research interests. (1)

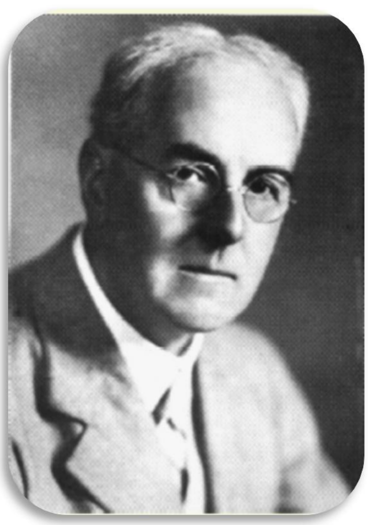

Figure 1:"Lewis Richardson". (1)

1.2 His contributions:

In weather forecasting:

His book - that was published in 1992 - "Weather Pioneering by Numerical Processes" was the first book to suggest techniques of integration to be applied to atmospheric motion.

In Analysis of war:

Richardson used his mathematical talent in the development of the pacifist principles, order to understand the basis of the conflict between the USA and the USSR during the cold war and by his work he is now considered the initiator of analyzation the war and its causes between any countries.
Richardson started his work using differential equations mainly and by considering the armament of only two countries.
He designed a system what can be described as an idealized system of equations. That system where later called the "Richardson's Arms Race Model ".

2. "Richardson's Model" of Arms race:

Here we will just talk about the idea of the model and the way it was designed with but its explanation will be introduced later accompanied by the differential equations used in the system.
According to Richardson the conflict between two nations can be determined and analyzed by a set of differential equations. These equations state the probability of any nation in entering a war and it depends on many factors, He found that the nation's stockpiles of weaponry and available ammo building up is directly proportional to the amount of arms, it's rival possesses and to the hate felt towards its rival and is negatively proportional to the amount of weapons it already has. He also stated that a probability of escalating a war could be represented as a function in common border length between the two nations! And when he tried to analyze the causes of a war he discovered also that economics, religion and language could be a reason to start a war, which of course shows how many people could be so inhuman and cruel.
And here we quote from Richardson himself when he wrote: "There is in the world a great deal of brilliant, witty political discussion which leads to no settled convictions. My aim has been different: Namely to examine a few notions by quantitative techniques in the hope of reaching a reliable answer."
The model is studying the war between two nations from three different cases. (2)

2.1 Richardson's Cases:

The equilibrium case:

Here in this case we simulate the situation in which every possible conditions that could lead the two nations into equilibrium point is achieved. The meaning here of an equilibrium point here

means that the possibility of the conflict between the two nations to change into a war is an unchanging percentage.
However, to achieve accurate answers we cannot neglect the small conflicts that could change the fear constants appropriately, and which will cause shifting the equilibrium point further and further out.
On the long term in this case, with the increase in the number of conflicts may change into what is known as: "A Runaway Arms Race".
As for an example to this case: back at the cold war at some certain points the United States and the Soviet Union resembled that model. Despite it was never fully escalated into a runaway arms race.
This case considered as the most important one.

The runaway race case:

In this case the two nations are sufficiently hostile to each other - don't need more reasons to start a war - so they will simply enter a runaway arms race directly regardless of any starting conditions.
But as we mentioned before that this is just a simulation in the model, as by speaking of this case in worldly terms, this is a hardly possible model as the budget will be a problem to at least on the countries may be both, as a result to that such an arms race will either bankrupt the nation with less money or it will lead to what is known as "An All - Out War".

The friendly case:

The third case is in fact is the total opposite to the previous case, as in this case the two nations are have enough reasons and common benefits to stay friendly towards each other so they will never enter an arms race.
As an example for that is the state of friendship between the United States and Canada.
The final case is the most unusual and improbable in the real world, because by putting that case in the real world terms the two nations will either mutually disarm enter an arms race, and that depends of course on the starting points of their arms stockpiles. (3)

2.2 A more illustrating example:

To understand the subject far better we shall introduce an easier and more realistic example that shows the situation from another point of view.
First we will consider two competitive companies - we will call them as A and B - and they both are sharing the same market and selling the same product with the same price - and it will be called as P - this common market is characterized as an oligopoly - which means it's controlled by a certain company - , we shall also mention that no other company is selling the product P by the same price as companies A and B do , and also the market is closed - means there is no international trading - , and finally we shall mention the advertising costs of the two companies as S .
So to simulate the previous cases we shall use the above example as the two companies represent the two nations in conflict, the product profits of one company represent the reasons of the

conflict and finally the costs of advertising represent the stock piles of armory each company produce.
By taking this data into consideration we shall get that:
First case: if the product profits of the two companies is the same then the two companies are in no need to do any advertising so the advertising costs shall remain the same. (That represents the stability of the equilibrium point between the nations).
Second case: if one of the two companies' profits from selling the product became more than the other, in this case the other company starts to make advertises to make people notice it's product, on the other case the other company starts to respond with other advertises just to make sure they keep their leadership (and that represent the Runaway Arms Race, but of course it's here just an advertising race). That considers the most possible case to become true.
Third case: if the two companies had no reason to compete at all because their friendly situation so they either start to withdraw their products from the market to allow the other company to charge on the market - sounds really impossible - or they work all together and unite their companies together and oligopoly the market. (5)

2.3 The Model's Equations:

Since we have explained above the principles upon which Richardson had constructed his model, here we will introduce the model's mathematical theories. Richardson made many researches and worked on them until he reached his model and published it in 1963, the model was constructed to be a system of differential equations that can be used to analyze the conflict between two nations or more.
The basic system is:

$$\frac{dx}{dt} = ay - mx + r$$

$$\frac{dy}{dt} = bx - ny + s$$

The system can also be expanded to describe the state of a conflict between more than two nations for ex. For a three nation conflict the system will be:

$$\frac{dx}{dt} = ay + bz - mx + r$$

$$\frac{dy}{dt} = cx + hz - ny + s$$

$$\frac{dz}{dt} = ex + fy - oz + s$$

And to fully understand how the system is working we will be satisfied by working the basic system by its two equations.

Each differential equations simulates the rate of change of arms buildup for a certain country.

Also, X and Y are functions of time that represent the amounts of arms that nation (1) and (2) respectively have at a certain time (t).

As for (a) and (b) they are constants that are known as “Fear Constants”, they represent the desire of a nation to increase the amount of arms it has at a rate that is proportional to the amount of arms that the opposite nation has.

Moving to the constants (m) and (n) that are known as the “Fatigue Factors” and they represent the desire of the country to decrease the amount of the arms stockpiles it has at a rate that is proportional to what it possesses – “notice that the rate is not proportional to what the opponent possess “-.

Finally, we get to the (r) and (s) which are known for the “Grievance Constants” and they represent every other factor like the revenge motive, ambition, external pressure or etc., it also could be called as the “Leftovers”, those factors aren’t directly related to the arms stockpiles but they are important and have influence in our calculations, that’s why we are keen on taking them into consideration.

It’s very important to note that: to have a meaningful answer from the model then only constants (r) and (s) can be negative or zero, because having (a) or (b) with negative values means a negative fear and a negative value for (m) or (n) indicates that the nation wants to build up its arms stockpiles at every increasing rate. (6)

Now to find a solution to the system we have:

$$\frac{dx}{dt} = ay - mx + r$$

(System 1)

$$\frac{dy}{dt} = bx - ny + s$$

To solve the system with all these variables and unknowns will take much time and effort so to ease the process we are going to study the simplest case of the model and that is by assuming that no factors are influencing on the state between the two countries except for the fear constants; in other words, by putting m, n, r and s = 0.

The system will transform to:

$$\frac{dx}{dt} = ay$$

(System 2)

$$\frac{dy}{dt} = bx$$

Let X= F(t) and Y= G(t) then:

$$\frac{dF}{dt} = a * G(t)$$

(System 3)

$$\frac{dG}{dt} = \mathbf{b} * \mathbf{F}(\mathbf{t})$$

From system (3) we can deduce the following:

$$F(t) = \frac{1}{b} * \frac{dG}{dt} \qquad (1)$$

$$F(t) = a * \int G(t)\, dt \qquad (2)$$

From equations (1) and (2) we get:

$$\therefore \frac{1}{b} * \frac{dG}{dt} = a * \int G(t)\, dt$$

By differentiating the two sides:

$$\boldsymbol{a * b * G(t) = \ddot{G}(t)}$$

$$\therefore \boldsymbol{\ddot{G}(t) - ab * G(t) = 0} \qquad \textbf{(3)}$$

Now we have a second order, linear differential equation that can easily be solved by Cauchy Euler's Equation to reach:

$$\boldsymbol{X(t) = c_1 * e^{(\sqrt{ab})t} + c_2 * e^{(-\sqrt{ab})t}} (4)$$

By repeating the same previous steps on Y(t) from system (3):

$$\boldsymbol{G(t) = \frac{1}{b} * \frac{dF}{dt}} \qquad \textbf{(5)}$$

$$\boldsymbol{G(t) = a * \int F(t)\, dt} \qquad \textbf{(6)}$$

From equations (5) and (6) we get:

$$\therefore \boldsymbol{\frac{1}{b} * \frac{dF}{dt} = a * \int F(t)\, dt}$$

By differentiating the two sides:

$$\boldsymbol{a * b * F(t) = \ddot{F}(t)}$$

$$\therefore \boldsymbol{\ddot{F}(t) - ab * F(t) = 0} \qquad \textbf{(7)}$$

Now we have a second order, linear differential equation that can easily be solved by Cauchy Euler's Equation to reach:

$$\boldsymbol{Y(t) = c_3 * e^{(\sqrt{ab})t} + c_4 * e^{(-\sqrt{ab})t}} (8)$$

By computing the unknown constants in equations (4) and (8) by advanced mathematical processes we get the following:

$$\boldsymbol{X(t) = \frac{1}{2}\alpha e^{(-\sqrt{ab})t}\left(e^{(2\sqrt{ab})t} + 1\right) + \frac{\beta\sqrt{a}e^{(-\sqrt{ab})t}(e^{(2\sqrt{ab})t} - 1)}{2\sqrt{b}}} \qquad \textbf{(9)}$$

$$Y(t) = \frac{\alpha\sqrt{b}e^{(-\sqrt{ab})t}(e^{(2\sqrt{ab})t} - 1)}{2\sqrt{a}} + \frac{1}{2}\beta e^{(-\sqrt{ab})t}\left(e^{(2\sqrt{ab})t} + 1\right) \qquad (10)$$

Where:α and β are two coefficients that are equal to X(t) and Y(t) respectively at t_0 (i.e. when t= 0).
From the above equations we deduce that if the two functions X(t) and Y(t) are both growing or tending to ∞ then, it means that both countries are running an arms race.

As for the system of equation 1:

$$\frac{dx}{dt} = ay - mx + r$$

(System 1)

$$\frac{dy}{dt} = bx - ny + s$$

We can reach a solution that shows a very important point that can't be neglected; it's what we called before as the "Equilibrium point".

We reached a solution where: (mn-ab ≠ 0)

$$X = X_0 = \frac{as+nr}{mn-ab}$$

$$Y = Y_0 = \frac{br+ms}{mn-ab}$$

These two equations represent the coordinates of the equilibrium point but the equations must be proven mathematically to determine whether or not the equilibrium solution of the system is stable, and from the proof we deduced that:

1) if (mn-ab = 0): there is no equilibrium point to the system which means the two countries are in a runaway arms race or mutual disarmament.
2) if (mn-ab > 0): itmeans that the balanced solution is stable.
3) if (mn-ab < 0): it means that the balanced solution is unstable.

If now you are wondering did that model was tried or not?!

Well the answer is simply YES.
Actually they did really fail first on a case that we will be mentioned below but, in 1990 the model was applied to the conflict between India and Pakistan that was later escalated to an arms race and a war after that, and it was a complete success and gave very accurate results. (8)

2.4 Richardson's Fault:

As we simply illustrated the model above we must mention the reason why we needed to work more on the model and improve it , as in 1981-1985 in fact before the success of the model in the Indian-Pakistan case it unfortunately failed when it was applied to a conflict between Greece and Turkey , as when "Majesky and Jones" tried to apply the model to this case it have given mixed results , later "Kollias in 1991" applied the model again on certain intervals of time during the conflict, However the results turned to be very poor and didn't indicate the presence of an arms race at these periods.
The problem with the model is that Richardson didn't take the budget constraint into consideration. Unfortunately, not considering the budget constraint as a parameter in the calculations causes a great divert in the results causing many errors.
Therefore, many attempts were made to correct the model by mentioning all the possible factors that Richardson hadn't.

There is also another problem in understanding the model and that is: if the hostility/peace terms are negative and all the remaining coefficients are zero then the system will indicate X, Y are less than zero, and that just gives no physical meaning.

After demonstrating the problems in Richardson's model; we will attempt to take it all into consideration in the suggested model. (9)

3. New Models and Researches:

Despite the deficiency in Richardson's model, a lot of models have been manufactured and modified by equations that were inspired from Richardson's work.
One the most famous Reports in this topic is a report that is known as "Prisoners' Dilemma Model" but before further explanation we must talk about a little game first!

3.1 "The Prisoners' Dilemma Game":

Well, the Dilemma is a game that was framed by "Flood and Dresher"in 1950 and it is based on what is known as "Game Theory".
The Game is about two completely persons, and the game shows that these two persons may not cooperate together even if it is in their interest to do so.

For example: Imagine two criminals that have been arrested for a crime they were doing together, the two criminals are being investigated with in separate rooms.
Now, every criminal got an offer, and have three choices
First: if he confessed that his partner was guilty, he will be set free and his partner will get a 10 years' prison.
Second: if he refused to confess and they both kept silent, they will both go to prison for 6 months.

And Third: if they both took the deal and confessed on each other, each will be put in prison for 5 years.

Now, the tricky part–and the funny one if we shall say- is that each criminal doesn't know what the else's response will be and the contact between them is cut so, they are both in curiosity of what they should do.

Of course the answer is obvious that they both should refuse the deal and be free in 6 months' time, but the mistrust between those two individuals would possibly make them confess on each other. And that is the mystery in the game. (9)

3.2 The Dilemma's Model:

Now the big question is:"How this is related to the subject we are introducing?!"

Well, the game is really can be applied to simulate or represent the status of the world, one of the crucial problems between nations and governments is "Mistrust"or the lack of cooperation in other words, as a matter of fact how can they approve on international friendship when every country is aiming to overcome the others.

One of the most important results of that divergent in the world is the "Arms Race". Imagine if we applied the game on two countries for example: The United States and the Soviet Union (representing the two criminals in the game), and they are offered to save the money they are spending on the weapon industry and increasing the armory stockpiles (representing the offer of the two criminals refusing the deal and approving they won't turn each other in).

Now like the game, the two countries have three choices:

First: one of them refuse to trust the other and start a Runaway Arms Race and precede the other in its stockpile and perhaps start a war -that it can easily win- to secure its position in the world, (and that case represent the criminal that take the deal and get free while his partner enters the prison for 10 years).

Second: if both countries trust each other and agreed to be internationally friends for the common favor, and start to use the money they spend on arms industry, in improving the life of their citizens. (And that case represents the two criminals doing what is in their favor and refusing the deal).

And Third: if both countries refused the peace and act hostile –like they always do.! - And they both went to a Runaway Arms race which could be later escalate into war. (And this case represents the two criminals confessing on each other).

Achieving the Second case very difficult, although it's the right and obvious choice to make, because by working together and if every country worked to achieve world peace, then all that is being spent every year can be used in educating more children, destroying poverty and hunger and improving every human life.

The problem of cooperation in general and arms races in particular, can be modeled with game theory. Game theory is an approach to the study of interdependent Decision-making, often called strategic interaction, developed by mathematicians and economists.

One game in particular, the prisoners' dilemma, has received the most attention as a model of how strategic interaction in the anarchic international system.

Creates incentives for governments to enter into arms races and complicates their abilities to effectively end arms races. (10)

4. Mathematical Rules:

At the very beginning to get through the topic of arms race model we need to go over some mathematical rules which will help us a lot during this topic. As long as arms race is an important topic we need to figure out a lot of theories to understand the whole topic which deals with the economic state, the variables the country needs to pay attention and how to calculate them in a scientific method not randomly, etc.
Differential games, convex sets, objective function, optimal control theory and calculus of variations are the mathematical theories which we will get through to understand this interesting topic. From the differential games and how to apply them to win a war to the calculus of variations which concerns basically with the maximum and minimum value which can be applied on the war to get the maximum victory.

4.1 Differential games:

Group of problems related to the analysis of conflict. This method is back to Refuses Isaacs (1951) and has a lot of ways of solution back to different mathematicians like Professor Nash.

4.2 Two-Person Zero-Sum Differential Games:

Consider a state equation

$$\frac{\mathrm{dx}}{\mathrm{dt}} = \mathrm{f}(x, u, v, t)$$

Assuming that the variables are scalar for the human time, (u) and(v) denotes the controls applied by the first country (x_1) and the second country (x_2) from the Richardson's equation. Where $(u\ )$ and (v) are convex sets in E.

4.3 Affine space $\boldsymbol{E}$:

A kind of Euclidean spaces which is not concerns with distances or angles only the parallelism, consists of vector spaces.

Some properties:

Associativity:

$$\forall \boldsymbol{V}, \boldsymbol{W} \in \vec{\boldsymbol{A}}, \forall \boldsymbol{a} \in \boldsymbol{A}, (\boldsymbol{a} + \boldsymbol{v}) + \boldsymbol{W} = \boldsymbol{a} + \ (\boldsymbol{v} + \boldsymbol{w})$$

Right identity:

$$\forall a \in A, a + 0 = a$$

May be affine space is pure mathematics but we will need it in the following problems, so It relates to the topic indirectly.

4.4 Convex sets:

A convex is a subset (v) of an affine space (E) for any two points (a, b) belongs to (v) then we have $(c)$where:

$$C = (1 - \mu)a + \mu b$$

$0 \leq \ \mu \ \geq \ 1 \ (\mu \ \in \ R)$, which means if $\mu = 0$ then $c = a$ and if $c = 1$ then $c = b$.

4.5 Optimal control system:

An important theory used in economic growth, logistics and exhaustion of natural resources, so this theory will help a lot in the case of war as we want to calculate the economic growth for example, so imagine we are about to start a war without paying attention for these things!
The theory of optimal control depends basically on the calculus variations which is a very branching topic we will go briefly over it discussing what helps our topic

4.6 Calculus of variations:

A very vital theory concerns with the maximum and minimum of an integral of the form

$$J(x) = \int_{t_0}^{t_1} f\big(t, x(t), \dot{x}(t)\big) dt \qquad (1)$$

$$x(t_0) = x_0 \qquad x(t_1) = x_1$$

The fundamental equation of calculus of variations is the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial f}{\partial x} = 0$$

THEORM:
Let Y be a curve in a plane, $f = f(x(t), \dot{x}(t), t)$ is a differentiable function, applying the Euler – Lagrange method on (f)

$$l(Y) = \int_{t_0}^{t_1} \left(\left[\frac{\partial F}{\partial X} - \frac{d}{dX}\left(\frac{\partial F}{\partial X}\right) \right] h + \frac{\partial F}{\partial X} h \right) dt$$

Back to optimal theory with this type of calculus of variations

$$\max/\min \int f(t, x, \dot{x},)\, dt$$

$x(t_0) = x_0$, $x(t_1) = x_1$ Assuming that $\dot{x} = u$

$$\max/\min \int F(t, x, u)\, dt$$

$$x(t_0) = x_0 \quad , \quad x(t_1) = u$$

Which is so obvious that this theory has a vital role in arms race applied on the maximum or minimum variables with respect to time.

For Example:

If we have a function of three variables may be money, food and another variable depends on the money
$\dot{x} = u$, so using this theory we will be able to figure out the max or minimum value of this function.

After passing over these mathematics problems we need to go over a famous model of arms race may be it is a little primitive but it's effective and worthy. It's not used only in the case of this topic it uses the differential equations to predict weather too. (11)

5. A suggested Model:

In order to solve the problem of the budget we will introduce terms called "the carrying capacity" terms, these terms are based on the maximum and the minimum values we have explained earlier.
After inserting the carrying capacity terms in the old model the system will change to be in this form:

$$\frac{dx}{dt} = (1 - {}^{x}/_{x_{max}})(ay - mx + r)$$

(system 4)

$$\frac{dy}{dt} = (1 - {}^{y}/_{y_{max}})(bx - ny + s)$$

The constants in the previous equations represents the same quantities they did in the Richardson's model (system 2).

As for the added terms x_{max} and y_{max}; they represent the maximum military expenditures for the two nations in conflict. These carrying capacity terms introduce two new terms that represent the solution to our problem and that is a budget term, where ${}^{dx}/_{dt}$ and ${}^{dy}/_{dt}$ approach zero as x and y approach their budget constraints.

We will test this model using data from researches in the results section.

6. Modeling Application of Arms Race

We benefit from this theory in some recent developments in time-series econometrics. Conflict between Greece and Turkey and India and Pakistan are example for application of arms race. There's little evidence for a Richardson type arms race for Greece and Turkey, India and Pakistan show a stable interaction with a well determined equilibrium.

6.1 The Greece-Turkey Arms Race

There is considerable debate over the Greece-Turkey arms race as previous studies have given mixed results. Majesky and Jones (1981) use analysis, tested for interdependence in the military expenditures of Greece and Turkey for 1949-1975 and their results was very interesting. Kollias (1991) applied the classical Richardson model for the two countries over the periods 1950-1986 as well as over 1974-1986 (the period after the Turkish invasion of Cyprus).

But his results were very poor to proof theory of arms race. However, by employing specific indices of military capabilities, he found that Greek military expenditure depends on Turkish military expenditure and on the relative size of the arms forces.

Also, Kollias and Markakis (1997), using co integration and causality tests, found evidence between Greece and Turkey over the period 1950-1995 could be used as application for arms race.

In 1995 by using ratio of armed forces and military expenditures per soldier which examined the hypothesis of an arms-race between Greece and Turkey over 1962-1990 and found that Turkey's quantitative advantage is the most significant external security determinant of Greek military expenditure. But, there are studies that do not provide strong evidence of an arms-race between the two countries.

In 1990 Georgiou tested the hypothesis of an arms-race over the period 1958-1987 but could not find any evidence of the existence of an Arms-race, but in 1996 Georgiou, Kapopoulos and

Lazaretto use a vector auto regression specification ended up with similar conclusions for the period 1960-1990. (11)

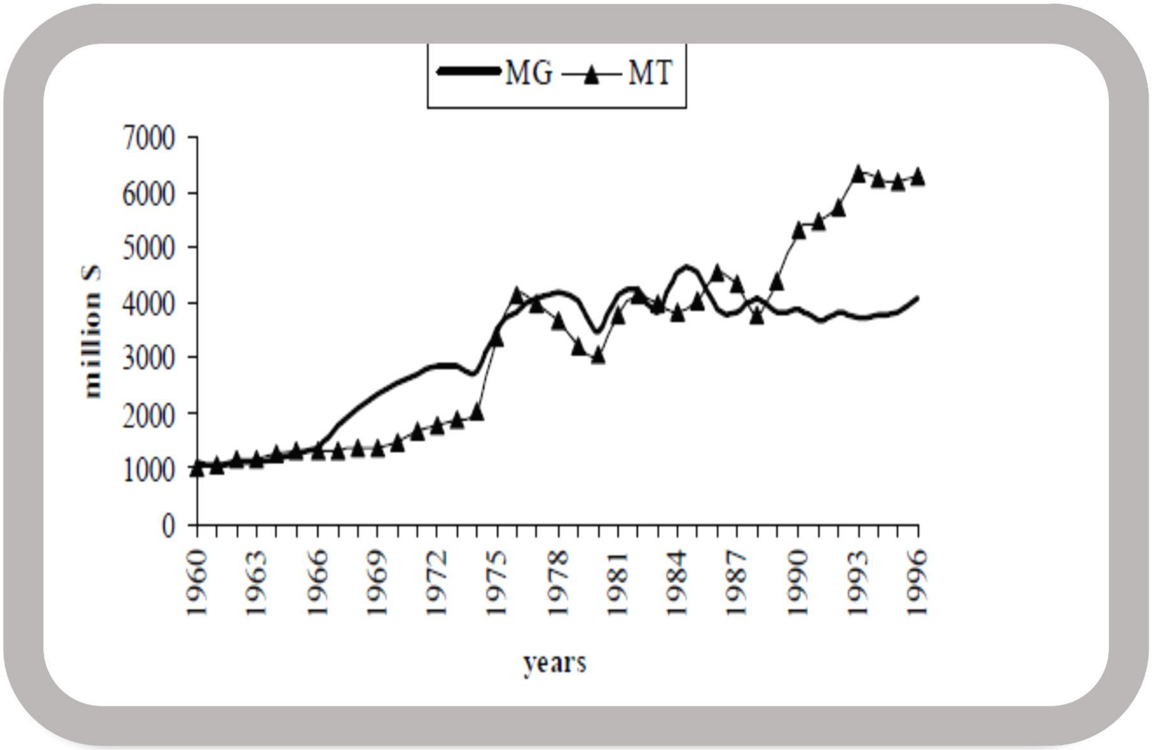

Figure 2:"Military Expenditure for Greece and Turkey". (11)

The Turkish invasion of Cyprus had a marked impact on Greek Turkish relations and this is modeled by a step dummy CD, which takes the value of 0.5 in 1974, of 1.0 for 1979 and zero otherwise.

Starting from a lag length of 5, using the logarithms of military expenditure and using CD as an exogenous variable, adjusted Likelihood Ratio (LR) tests and the Schwarz Bayesian Criterion (SBC) indicate a first order VAR, though the Akanke Information Criteria (AIC) and unadjusted LR tests suggest longer lags. Given the length of the time series we continue with a VAR (1) and investigate whether or not the variables are co integrated.

Using a first order VAR with unrestricted intercepts and restricted trends over 1961 to 1996, trace and eigenvalue tests suggested one co integrating vector at the 5% level between the logarithms of military expenditures. Normalizing on the log of Greek military spending, LMG, the co-integrating vector was:

$$\mathbf{LMG} = -\mathbf{29.1}\ \boldsymbol{LMT},$$

One possible source of misspecification is failure to take account of the budget constraint. To consider this a VAR model in the logarithms of military expenditure and GDP (which were calculated from SIPRI figures for military expenditure and shares), was estimated. All variables were tested for unit roots and were found to be and non-nested tests again suggested that the logarithmic equation fit better than the untransformed equation. In this case the SBC indicated a second order VAR. A joint Likelihood ratio test for the exclusion of the GDP variables:

$X_8^2 = 42.9$, which is well above the 5% critical value, suggesting that income is important.

Assuming unrestricted intercepts and restricted trends in a VAR (2) with the Cyprus Dummy over 1962-1996, trace and eigenvalue tests again suggested one co integrating vector. Again normalizing on Greek Military expenditure gave the co integrating vector:

$$LMG = 4.01\, LMT + 0.70\, LYG – 1.38\, LYT$$

Plus, trend and Cyprus dummy. This has the right signs but an implausibly large coefficient on Turkish military expenditure. It is possible that this is a relation in shares and this was tested by imposing the implied over-identifying restrictions. This gave:

$$(LMG – LYG) = 4.98\,(LMT – LYT)$$

Again the coefficient on the logarithm of the share of Turkish military expenditure is implausibly large. The share restrictions were rejected by the data $X_2^2 = 12.8$ The VECM estimates for the just identified system (t ratios in brackets) are:

$$\Delta G_t = 2.42 + 0.1\Delta G_{t-1} - 0.06\Delta T_{t-1} - 0.58\Delta Y_{t-1} - 0.26\Delta TY_{t-1} + 0.1Z_{t-1} + 0.12CD_t$$

$$R^2 = 0.29\,, SER = 0.09$$

$$\Delta T_t = 5.1 - 0.27\Delta G_{t-1} + 0.37\Delta T_{t-1} - 0.59\Delta Y_{t-1} - 0.62\Delta TY_{t-1} + 0.21Z_{t-1} + 0.24CD_t$$

$$R^2 = 0.69\,, SER = 0.07$$

There are also equations for GDP which we do not report. For the Greek equation, four of the seven coefficients are significant at the 5% level. The coefficients on the error correction term lagged Z, measure the speed at which any disequilibrium is removed, though the adjustment in the Greek case is in the wrong direction. The specification easily passes the tests for first order serial correlation, functional form, normality and heteroscedasticity. The equation for Turkey is a better specification in terms of the coefficient estimates, with six of the seven significant, but fails the test for the functional form $\chi 2(1) = 7.68$.

Again there seems to be some evidence of a co integration, but not in the form of a long run arms race model. Experiments with a variety of different formulations did not reveal a robust arms race relationship. The results tended to be difficult to interpret and extremely sensitive to sample and specification. Given their antagonistic interaction and their reactions to each other's military preparations, there has undoubtedly been an arms race between Greece and Turkey. But that arms race has not taken the form of stable Richardson type reaction functions. Given all the other factors intervening in their interaction, this is not perhaps surprising. (12)

6.2 The India-Pakistan Arms Race

Unlike Greece and Turkey where the literature is ambiguous, previous studies of India and Pakistan e.g. Deger and Sen (1990) have found evidence of a Richardson type arms race in the sub-continent. Given the results for Greece-Turkey, the starting point for India and Pakistan was a VAR in military expenditures and GDP. The GDP figures were again calculated from the SIPRI figures for military expenditures (see figure 2,3 for the levels of military expenditure) and shares.

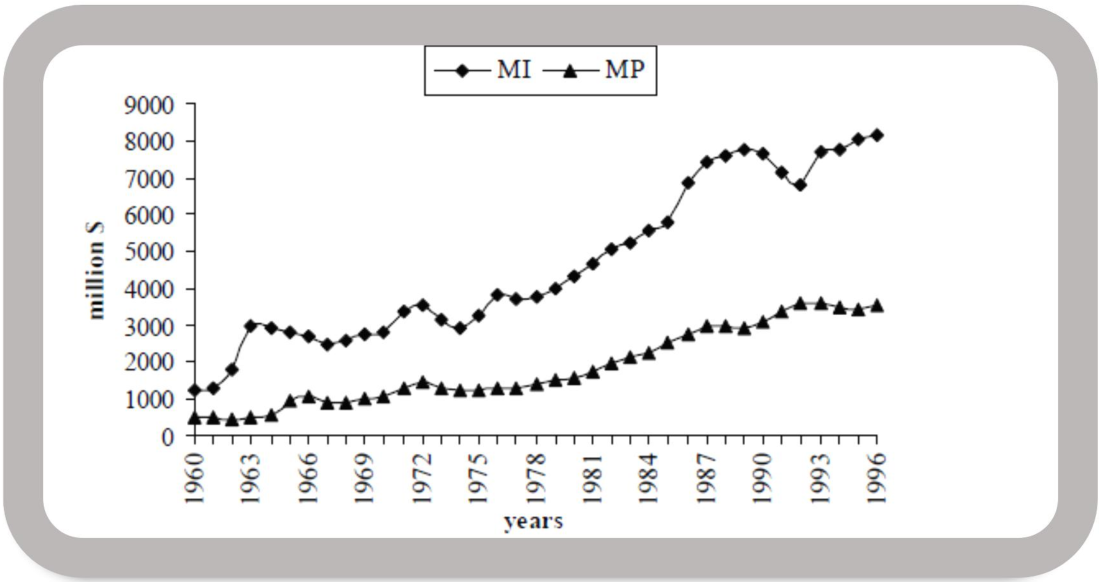

Figure 3:"Military Expenditure for India and Pakistan". (13)

Both model selection criteria and Likelihood Ratio tests indicate that a second order VAR is appropriate. Both non-nested tests and likelihood criteria indicated that a linear model fits better than a logarithmic one. In the linear second order VAR in the four variables, the Likelihood Ratio test statistic for the hypothesis that the income variables do not appear in the military expenditure equations is 15.4.

This is just below the asymptotic 5% critical value for a $X_8^2 = 42.9$, which would be appropriate if the variables were stationary. Given that the small sample critical values for non-stationary variables would be rather larger, this suggests that we can treat the income variables as Granger non-causal with respect to military expenditures and work with a two variable VAR. There is an element of judgment in this, since the theory suggests that income should be included to capture the budget constraint and some income variables are individually significant in the Indian military expenditure equation.

Using unrestricted intercepts and no trends (which were insignificant) in a second order VAR, 1962 1996, the trace and eigenvalue tests both clearly suggest one co integrating vector at the 5% level. The eigenvalue test statistic is 26.3, 5% critical value 14.9, trace 26.4 and 17.9. Normalizing on Indian Military Expenditure the co integrating vector is $MIt = \quad 2.008\ Pt$ which indicates that the long-run relationship is for India to spend about twice the Pakistani level.

Apart from the intercept in the Pakistan equation and lagged Pakistani spending in the Indian equation all the coefficients are significant. Both equations pass tests for first order serial correlation, non-linearity, normality and heteroscedasticity at the 1% level; though Pakistan fails on normality and India on heteroscedasticity at the 5% level. In terms of changes the degree of explanation is low, though the equations explain over 95% of the levels of military expenditure. The coefficients on lagged Z, which measure the speed at which disequilibria are removed, are both of the correct sign and indicate that India adjusts to disequilibrium faster than Pakistan. There seems to be a degree of over-reaction, represented by the negative coefficients on the other countries lagged changes. Convergence back to equilibrium is cyclical, but quite rapid, with adjustment complete in about six years. There is quite a high positive correlation (r=0.46) between the errors in the two equations, indicating a degree of instantaneous feedback. Unlike the Greece-Turkey case, the results seem quite robust and not very sensitive to sample or the details of specification. (13)

7. Discussions

7.1 If you want peace, prepare for war?

At least for the case of symmetric states, the model and analysis here are largely consistent with the Roman adage "if you want peace, prepare for war." The reason states build is to deter attack (or coercion) by other states that would like to expand or otherwise change the status quo away from what the state likes. Arms build ups are costly and inefficient. The states could do better if they could commit themselves not to arm, or not to use arms for attack or coercion. Either possibility would enable a stable outcome at zero or lower levels of arms.

In the symmetric case, however, even if arms build ups are costly they are not themselves "dangerous." They do not raise the probability of war by any sort of dynamic internal to the buildup itself. Indeed, it is failing to race that would be dangerous, if states start from or newly find themselves in a position where current arms are inadequate for deterrence.

Contrary to the Roman adage, or going beyond it, the analysis does make clear that while build ups in the symmetric case are not in themselves dangerous, the anticipation of a costly arms race could be dangerous, by inclining states to fight in the present rather bear the costs of arming to a stable level. In this complete information model this is "all or nothing" in the sense that either the states fight at the outset or they do not. Note, however, that the mechanism is more general in that the greater the anticipated costs of getting to stability, the more willing the states would be to run risks of armed conflict in crisis bargaining over other issues. This is a potential source of pressure for war that is almost completely missed in standard Realist discussions.

7.2 When and why are arms races "dangerous"?

As noted above, from the perspective of a state in the model, the biggest danger would be failing to arm against a greedy adversary, as this would lead to attack or coercion if they are starting from low enough arms levels. Nonetheless, armed conflict can arise in equilibrium in these models even when the states are well aware of this danger and are trying to avoid it.

It is important to distinguish between two different types of equilibrium armed conflict that emerge in these models. First, war may occur at the outset if the states anticipate that the amount of arming needed to get to stability is large enough that simply fighting is the better option. In this case, armed conflict is due to a commitment problem that is "robust" in the sense even if we were to change the extensive form of the game to allow for bargaining between the states, that wouldn't help. The problem in this case is not getting to a distribution of resources that both prefer to fighting, but that it is too costly to get to a level of arms such that peace is self-enforcing.

Second, war may occur if, at a mixed strategy point or when one state has a large enough building advantage, one state builds while the other does not. The advantaged state can then fight to "lock in," in expected terms, a stream of payoffs that is better than a status quo of v every period, and the disadvantaged state may as well fight because this provides some chance of surviving to enjoy future consumption. In this second case, allowing for bargaining and resource transfers could potentially allow the states to avoid a violent armed conflict. Once one state has a military advantage that makes it dissatisfied with the status quo, the two states are in the situation of a "dissatisfied" and "satisfied" state as appear in many bargaining models and there will be agreements on transfers that both sides would prefer to a costly fight.

Bargaining may fail, of course, whether due to private information about willingness to fight, or commitment problems related to whether the stronger state can guarantee to uphold a deal in the future (Ferron, 1995, 2007; Powell, 1999). So we should probably interpret this type of armed conflict in these arms race models as arising as follows: Arming can produce an imbalance of power, which can make one state prefer armed conflict to the status quo, which in turn raises the odds of armed conflict if bargaining over redistribution fails for whatever reason.

Alternatively, bargaining may not fail, so that resources are redistributed but there is no destructive fight. If so, then what happens is coercion rather than war. Although this is bad for the coerced state, the outcome is not necessarily inefficient.

But how is it that arming might produce a dangerous imbalance of power and thus conflict or coercion? As we have seen, in the case of symmetric states imbalances do not arise endogenously if the states start from a position of equality (or rough equality if at higher levels of arms). With symmetric states the only conflict risk from arms races is of the first sort. Possibly the most interesting result is that when states differ in their "greed" or ability to build, arms races may necessarily involve states mixing on whether to build, so that power imbalances and thus war or coercion can arise endogenously.

It is still not clear, however, how to interpret these mixed strategies. Surely state leaders never consciously randomize over decisions to build more weapons, or increase the size of the army.

Harsanyi (1973) showed that for any mixed strategy Nash equilibrium in almost any complete information game there is a game with incomplete information such that players choose pure strategies but the equilibrium distribution on strategies is the same as in the mixed strategy case. This suggests interpreting mixed strategy equilibria as artifacts of too "coarse" a model: if we allow for some small private variation in preferences, then the players are choosing pure strategies in light of their particular preferences, while they are uncertain about what the other players are doing.

Applied to the arms race model, the interpretation is that, depending on their idiosyncratic preferences, the status quo state's leaders may or may not want to take a chance that the adversary will keep building, even though they know war or coercion could follow if they don't build. Like-wise, depending on idiosyncratic, privately known leadership preferences, the greedy state may or may not plunge ahead with its armaments program, hoping but not certain that the security seeker will not keep up.

The difference between a mixed strategy point and a pure strategy point in the arms race models considered here is that in the former, the strategic situation forces a "tough call" by the leadership, because it has to be unclear what the adversary will do. The mixed strategy result suggests that, in an arms race between asymmetric states, a time may come when it cannot be common knowledge that the adversary will continue to build, or will not build. This is why arms races between asymmetric adversaries may endogenously generate a risk of armed conflict or coercion.

7.3 Implementation of the model in Advertising:

During the explanation of the model used an illustrating example that simulated the case of a conflict between two countries as a race in the advertising field between two firms.

Actually, the example wasn't invented from our imagination! in a matter of fact, the example is from an important application to our models in the real world.

As after the growth of the industry in nearly every society, the biggest companies started to study the market and tried to find the best ways to conquer it.

One of the most important tools that was and still is used is the applying of arms race models on the state of the market.

Richardson model and the models that followed it later can really analyze and predict the state of any firm just by substituting its data in the suitable system and give marvelous results and solutions to allow the firm to overcome the others by advertising the right thing in the right time by the right cost.

Arms race models have been a great source of analyzation of any conflict or any political issue, we just hope that one day we stop using it wars and conflicts one day. (14)

8. Results:

We will now demonstrate the results we simulated from Richardson's model using numbers that represent the reality we live.

In the next figure we show the situation that resembles the competition between two countries and explains how an arms race between them is escalating by tending the functions to ∞.

The numbers we are assuming are: ($a = 1$, $b=2$, $\alpha=1$, $\beta=2$)

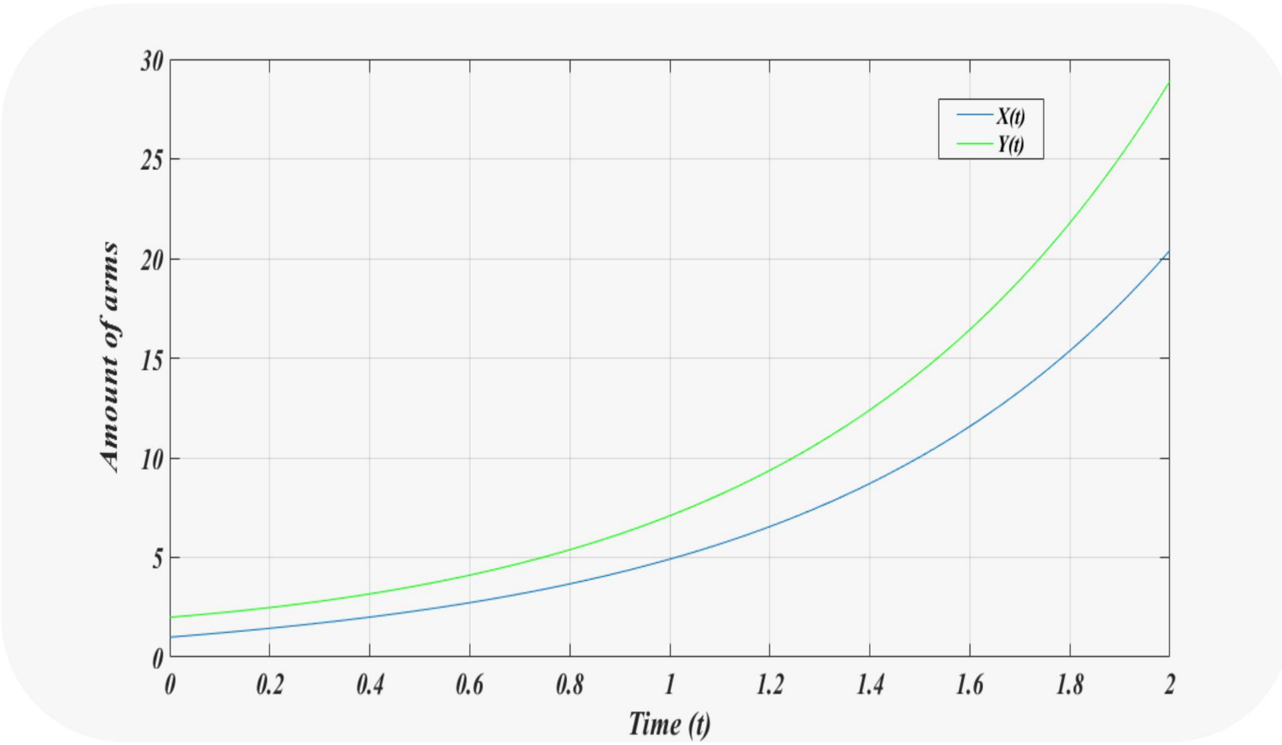

Figure 4: "A simple case of Richardson's model".

The chart here shows us that the second nation is overcoming the first nation in the arms race.

Now we will move to another point of study:

It's very important to determine when the system of equations will intersect or in other words at what time and at which values of a, b, α and β will the armory building up of the first nation will surpass that of the second nation.

The answer of that question can be deduced from figure 4 and from the solution of the equation, as from noticing the image of the curves in fig. 4 and from the symmetry found in the solution of the model {i.e. equation (9) (10)}, we can now find that by using the previous values of a and b at fig. 4 and by interchanging the values of α and β we will reach the answer of our question, which can be demonstrated in the next figure where: $a=1$, $b=2$, $\alpha=2$, $\beta=1$.

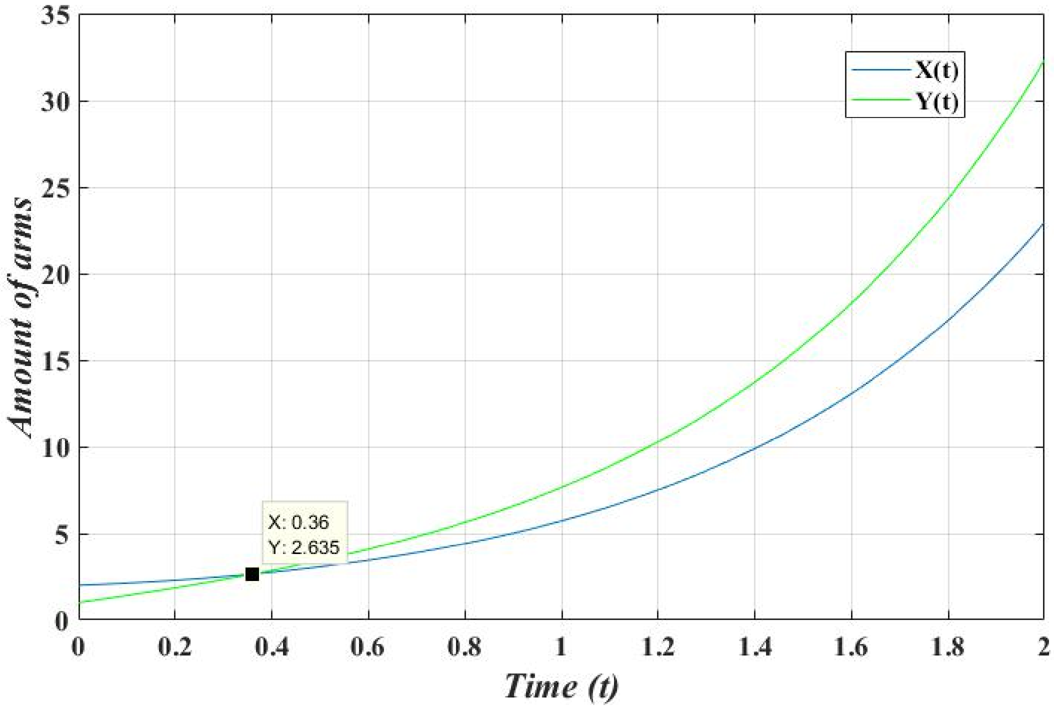

Figure 5 : "A case to find the military expenditure between two nations".

From the graph we can deduce that the first nation will overcome the second one in their arms race at time t= 0.36 and in order to figure out the difference between the military expenditure of the two nation we will calculate the enclosed area between the two curves X and Y.

The calculation will be done by doing the integration process to the two functions; using the time intervals from t= 0 to t= 0.36 where it's the time that corresponds the intersection of the two curves (i.e. the equality of the armory building up levels of the two countries).

$$\therefore \int_{0}^{0.36} \frac{1}{2}\left(2e^{-\sqrt{2}t}\right)\left(e^{2\sqrt{2}}+1\right)+\frac{e^{-2\sqrt{2}t}\left(e^{2\sqrt{2}t}-1\right)}{2\sqrt{2}}=0.8228$$

$$\therefore \int_{0}^{0.36} \frac{1}{2}\left(e^{-\sqrt{2}t}\right)\left(e^{2\sqrt{2}}+1\right)+\frac{2e^{-2\sqrt{2}t}\left(e^{2\sqrt{2}t}-1\right)}{2\sqrt{2}}=0.6457$$

We deduce that the difference between the two areas =0.1771 and that is equal to the military expenditure between the two countries.

Now, to show the effect of the modifications that have been added we will test the suggested model and compare it's results to that of Richardson's, we will assign these values to our coefficients to verify our theories where:

a = 1, b = 1.2, m = 0.9, n = 0.8, r = 1, s = -2, $x_{max} = 7,\ y_{max} = 9$.

By substituting these values, the Richardson model will be:

$$\frac{dx}{dt} = y - 0.9x + 1$$

$$\frac{dy}{dt} = 2x - 0.8y - 2$$

But our model will be:

$$\frac{dx}{dt} = (1 - {}^{x}\!/_{7})(y - 0.9x + 1)$$

$$\frac{dy}{dt} = (1 - {}^{y}\!/_{9})(2x - 0.8y - 2)$$

As shown in the next figure Richardson's model predict that X and Y will either approach infinity or lead to disarmament if time decreases.

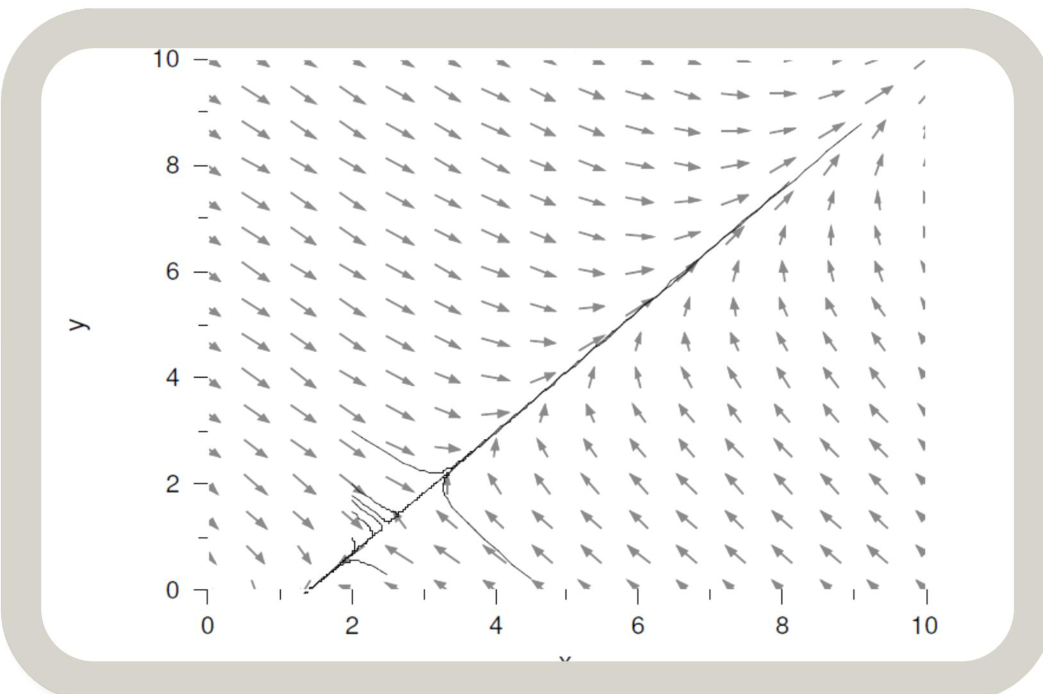

Figure 6: "Richardson model Showing the amount of arms as time decreases". (15)

That was actually the most important problem in the Richardson model may be more important than the problem of the budget because that problem made the model give unreasonable answers and results.

However, our model's results shown in the next figure predicts that both X and Y are approaching an equilibrium point or disarmament as time decreases.

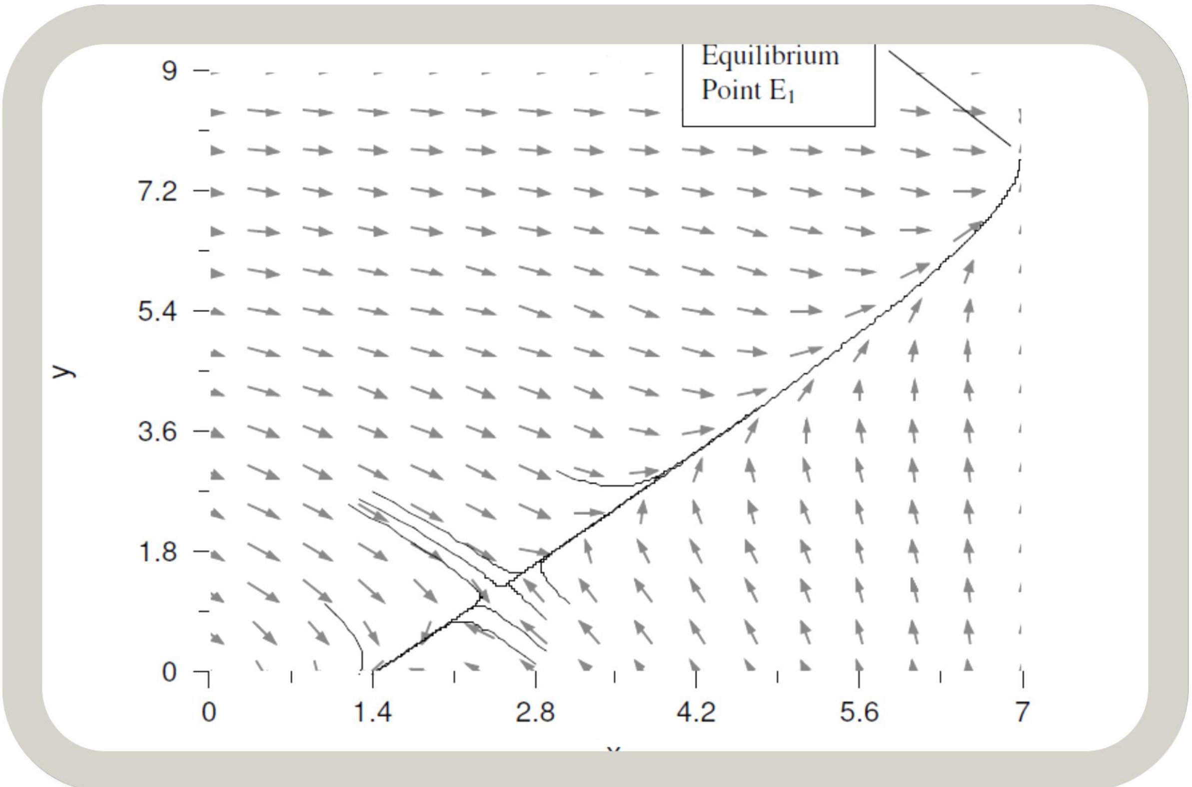

Figure 7: "the Suggested model showing the amount of arms as time decreases". (15)

Now to calculate the value of the equilibrium point to decide whether the system is stable or not we will need to drive lines from the model's system after substituting with the previous values we have used; we will get these four equations:

$$x = 7,$$

$$y = 9,$$

$$y = 1.5x - 2.5,$$

$$y = 0.9x - 1.$$

By graphing these equations, we get the following figure:

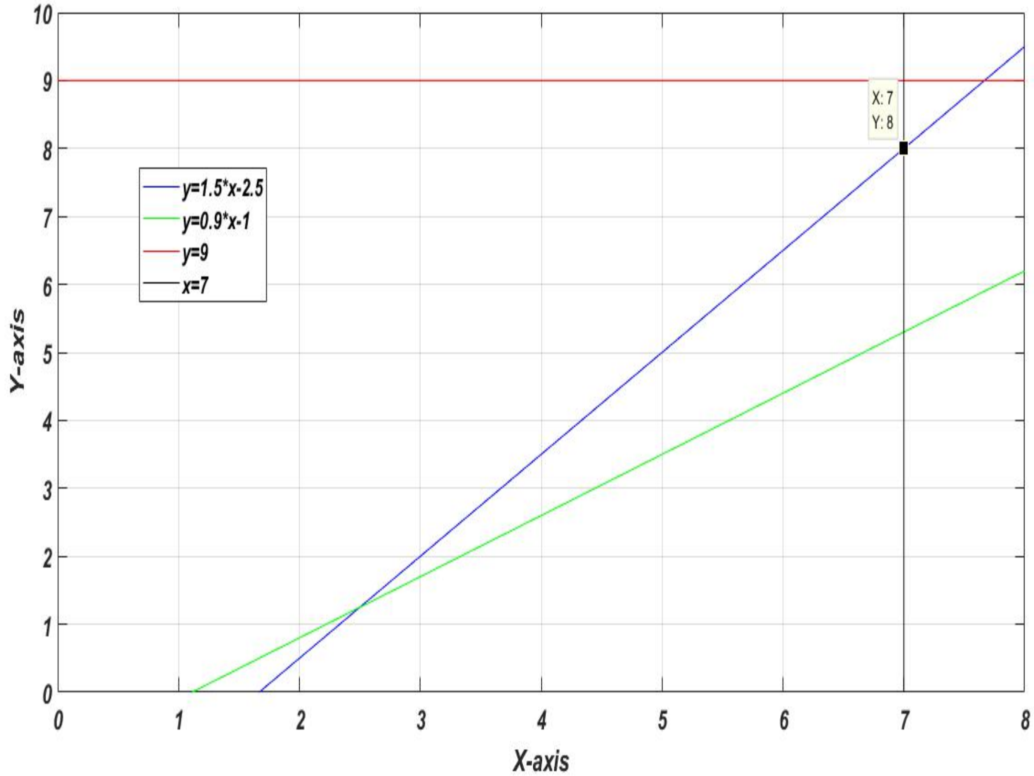

Figure 8: "A graph to deduce the value of the equilibrium point".

From the figure we deduce that the equilibrium point is the intersection of the equations:

$$x = 7, \quad y = 1.5x - 2.5$$

And now to study the stability we will use the jacobian matrix for the system as follows:

Let $F = {}^{dx}/_{dt}$, $G = {}^{dy}/_{dt}$ then the jacobian will be:

$$j = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$$

$$\therefore j = \begin{bmatrix} \dfrac{-ay + mx - r}{x_{max}} - \left(1 - \dfrac{x}{x_{max}}\right) m & \left(1 - \dfrac{x}{x_{max}}\right) a \\ \left(1 - \dfrac{y}{y_{max}}\right) b & \dfrac{-by + nx - s}{y_{max}} - \left(1 - \dfrac{y}{y_{max}}\right) n \end{bmatrix}$$

By substituting with the values we assigned:

$$j = \begin{bmatrix} -.03857 & 0 \\ 0.1333 & -0.01111 \end{bmatrix}$$

By computing the Eigen values : $\therefore \lambda = -\ 0.0111,\ -\ 0.3857$, and since that the real part is less than 0, then the equilibrium point is a stable point.

9. Conclusion:

As we reached the end of our report or we may say the end of our journey through these models that made a difference in the history course, we really have seen how Mathematics and it's principles can introduce to us a group of differential equations that can help us to understand our political world better, we may also say from what we have learned that: This world is so cruel and have no mercy or pity in it, and despite that every country is inviting people to promote for world peace they really only care about being on the top of the others even if it means to consume all the income to go on an "Arms Race".

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Appendix:

Fig 4:

```
>> a=1;
>> b=2;
>> c=1;
>> d=2;
>> m= sqrt(a*b);
>> t= 0:0.01:2;
```

```
>> x= 0.5*(exp(m*t))*(c+d*sqrt(a/b))+0.5*(exp(-m*t))*(c-d*sqrt(a/b));
>> y= 0.5*(exp(m*t))*(c*sqrt(b/a)+d)+0.5*(exp(-m*t))*(-c*sqrt(b/a)+d);
>> plot(t,x);
>> hold on
>> plot(t,y,'g');
>>lgd= legend('X(t)','Y(t)');
```

Fig 5:

```
>> a=1;
>> b=2;
>> c=2;
>> d=1;
>> m= sqrt(a*b);
>> t= 0:0.01:2;
>> x= 0.5*(exp(m*t))*(c+d*sqrt(a/b))+0.5*(exp(-m*t))*(c-d*sqrt(a/b));
>> y= 0.5*(exp(m*t))*(c*sqrt(b/a)+d)+0.5*(exp(-m*t))*(-c*sqrt(b/a)+d);
>> plot(t,x);
>> hold on
>> plot(t,y,'g');
>>lgd= legend('X(t)','Y(t)');
```

Fig 8:

```
>> x= -10:1:1
>> y1= 1.5*x-2.5;
>> y2= .9*x-1;
>> plot(x,y1,'b');
>> hold on
>> plot (x,y2,'g');
>> plot ([-100 100],[9 9],'r');
>> plot ([7 7],[-100 100],'k');
>>lgd = legend (‘y=1.5x-2.5’,’y=0.9x-1’,’y=9’,’x=7’);
```