Part 5

9.

(a) We'll assume $V \geq 0$, as it was told in the forums that this is a valid assumption.

Let x be a value bigger than V, We'll show that $0 = p_1$ strictly dominates $p_1 = x$.

If $p_2 > x$, then the utility of candidate 1 if $p_1 = 0$ is 0 (because $x > V \geq 0$) and the utility of candidate 1 in the case that $p_1 = x$ is $-x$, which is negative (due to the assumption that $V \geq 0$).

If $p_2 = x$, then the utility of candidate 1 if $p_1 = 0$ is 0, and the utility of candidate 1 in the case that $p_1 = x$ is $\frac{V}{2} - x$ which is less than 0.

If $0 < p_2 < x$, then the utility of candidate 1 in the case that $p_1 = 0$ is 0, and the utility of candidate 1 in the case that $p_1 = x$ is $V - x$ which is negative.

If $p_2 = 0$, then the utility of candidate 1 in the case that $p_1 = 0$ is V and the utility of candidate 1 in the case that $p_1 = x$ is $V - x$, which is less than V (due to the assumption that $V \geq 0$).

We've seen that for every p_2 and $x > V$, the utility of player 1 would be bigger if $p_1 = 0$ than it would be if $p_1 = x$, and therefore there is no action profile where $p_1 > V$ is a best-response for candidate 1.

(b) Let's take a look at the function of x, $u_1(x, p_2)$, we'll shrink our domain to $\{0\} \cup [V]$.

If $p_2 > 0$, out of the values that are less than p_2, $x = 0$ maximizes the function $u_1(x, p_2)$, and yields $u_1(x, p_2) = 0$,

If $p_2 < V$, out of the values that are bigger than p_2, $p_2 + 1 = x$ maximizes the function $u_1(x, p_2)$ and yields $u_1(x, p_2) = V - p_2 - 1$, also:

$$u_1(p_2, p_2) = \frac{V}{2} - p_2 < V - p_2 - 1 = u_1(p_2 + 1, p_2) \Leftrightarrow$$
$$2 < V$$

and therefore $p_1 = p_2 + 1$ would be best out of every $p_1 \geq p_2$, this will also be the only general best option for p_1 as long as $p_2 \leq V - 2$ due to the fact that we have seen that for every $p_2 > 0$ and $x < p_2$, 0 is the maximal value of $u_1(x, p_2)$ (and is achieved only for $x = 0$) and because if $p_2 = 0$, p_1 can't be smaller than p_2.

If $p_2 = V - 1$, then $x = p_2 + 1 = V$ and $x = 0$ maximize the function $u_1(x, p_2)$.

If $p_2 = V$, then,

$$u_1(V, V) = \frac{V}{2} - V = -\frac{V}{2} < 0$$

so because we've seen that for every $p_2 > 0$ and $x < p_2$, 0 is the maximal value of $u_1(x, p_2)$ (and is achieved only for $x = 0$), if $p_2 = V$, $x = 0$

maximizes $u_1(x, p_2)$.
The discussion above shows that:

$$(*)\ BR_1(p_1, p_2) = \begin{cases} \{p_2 + 1\}, & p_2 < V - 1 \\ \{0, p_2 + 1\}, & p_2 = V - 1 \\ \{0\}, & p_2 = V \end{cases}$$

and by symmetry:

$$(**)\ BR_2(p_1, p_2) = \begin{cases} \{p_1 + 1\}, & p_1 < V - 1 \\ \{0, p_1 + 1\}, & p_1 = V - 1 \\ \{0\}, & p_1 = V \end{cases}$$

so if (p_1, p_2) is a PNE by Claim 1.4, we get that, if $p_2 < V - 1$ then $p_1 = p_2 + 1$ but then due to $(**)$ p_2 must be 0, otherwise p_2 would have to be $p_1 + 1 = p_2 + 2$, contradiction, but $p_2 = 0$ also leads to contradiction because then due $(**)$ $p_1 = V - 1$ or $p_1 = V$, while we've also seen that $p_1 = p_2 + 1 = 1$.
That prove that p_2 can't be smaller than $V - 1$, if (p_1, p_2) is a PNE.
We'll go on with the assumption that (p_1, p_2) is a PNE.
If $p_2 = V - 1$, then p_1 is 0 or V (By $(*)$) but because we have seen that p_2 can't be smaller than $V - 1$ in a PNE and due to symmetry, $p_1 = V$, but then $(**)$ and claim 1.4 give us a contradiction, and that prove that in a PNE, p_2 can't be $V - 1$.
If $p_2 = V$, then $(*)$ shows that $p_1 = 0$, but we've seen that p_2 can't be smaller than $V - 1$ in a PNE, so due to symmetry, neither can p_1, and that shows that $p_2 \neq V$ in a PNE.
We've seen that p_2 can't have a value in $[V] \cup \{0\}$ in a PNE, so due to section (a), there are no PNEs in the game.
(c) If one of the players has won then social welfare will be:

$$V - p_1 - p_2$$

where one of p_1, p_2 is positive, if there's a tie then social welfare will be:

$$V - 2p_1$$

if $p_1 = p_2 = 0$ then the tie social welfare reaches it's maximum, and would be bigger than any social welfare that we can get in a case that one of the players wins.
We would like to reduce the question to mechanism design and say that the policy makers should compute a VCG to make that outcome a PNE, but VCG maximizes social value and not social welfare, thus, we'll suggest another external pressure.
The policy makers can change the amount candidate i pays when he decides an integer price $p_i \geq 0$, s.t., he'll pay $(V + 1)p_i$.
Now, the winner of the election will get utility $V - (V + 1)p_i$, the loser j get utility $-(V + 1)p_j$, and when there's each candidate gets utility $\frac{V}{2} - (V + 1)p_i$.

This means that $p_i > 0$ is strictly dominated by $p_i = 0$, because $p_i > 0$ will always make u_i negative and p_i will yield 0 or $\frac{V}{s}$, so both candidates will be motivated to declare a price of 0, which will make $(0,0)$ a PNE.

10.

(a) The graph is exactly the graph depicted by Figure 1 in the assignment:

$$G = \left(\{A, B, \dots, I\}, \left\{\{A, B\}, \{A, D\}, \{B, E\}, \{B, C\}, \{C, F\}, \{D, E\}, \{D, G\}, \{E, F\}, \{E, H\}, \{F, I\}, \{G, H\}, \{H, I\}\right\}\right)$$

The valuation function is the constant function $v(e) \coloneqq 1$.

A stable outcome, (M, d), in G corresponds to a stable assignment of schools in this way:

A town, v, will be assigned a school iff $M(v) \neq \perp$, and in that case $d(v)$ will be the distance of v from the assigned school along the road between v and $M(d)$.

(b) Each school must serve exactly two adjacent towns and each town can only send their students to a single school, so because the number of towns is odd, a valid placement of schools must leave at least one town without school.

(c) G is bipartite, we can divide its nodes into the two sets $\{A, C, E, G, I\}$ and $\{B, D, F, H\}$ so we can reduce the exchange network instance we got to a matching market instance (and I'm talking about the generalized version of matching market, for any number of buyers and goods, as we have seen in the previous assignment) where $\{B, D, F, H\}$ are the buyers and they have value 1 for every good, the goods are $\{A, C, E, G, I\}$.

It's easy to see that a market equilibrium of such matching market would translate to a valid outcome if the prices of items will be their division function.

Also, because a market equilibrium in a matching market maximizes social value, every node in $\{B, D, F, H\}$ must be matched, from here it was easy to see that:

$$\left(\{\{D, A\}, \{B, C\}, \{F, I\}, \{H, G\}, d\}\right)$$

where:

$$d(x) = \begin{cases} 1, & x \in \{B, D, F, H\} \\ 0, & else \end{cases}$$

is stable.

The outcome is stable because the surplus of every edge is zero (the division of its end in $\{B, D, F, H\}$ is 1, and the division of its other end is 0.)

11.

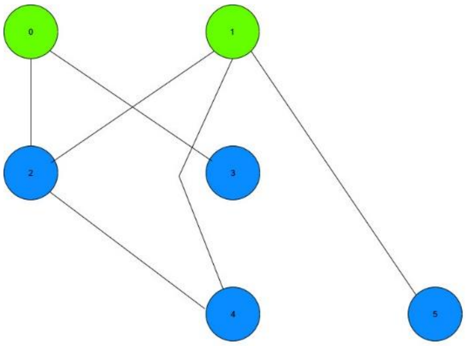
(a)

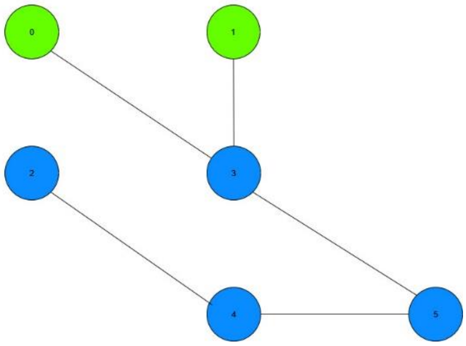
$$S = \{0,1\}$$

(b) To make the computation easier to follow I draw two graphs, one is $\left(V, E_f\right)$ and the other is (V, E_c).

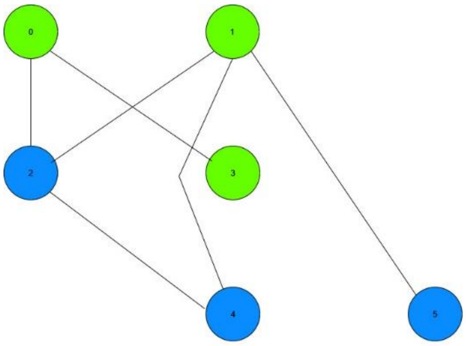
I marked the nodes that play BOTA with green and the nodes that play Age

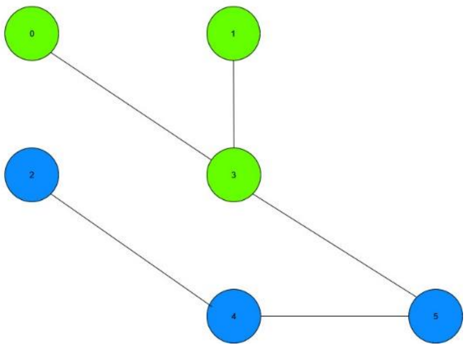
of Agents with blue.
We start with:



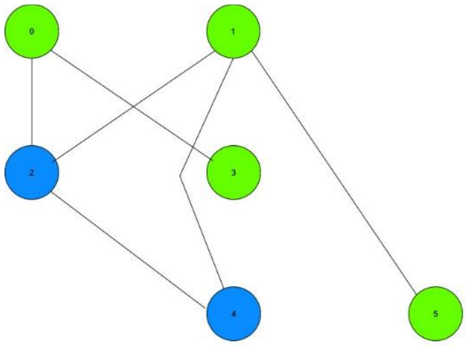


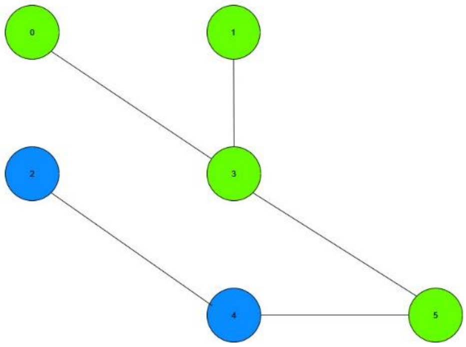
At this point, the only node that may be infectable is 3 because that's the only node that has neighbors that play BOTA in (V, E_c), clearly a fraction bigger than $t_c = 0.49$ of its neighbors in (V, E_c) play BOTA ($\frac{2}{3}$ of its neighbors) and a fraction bigger than $t_f = 0.65$ of its neighbors in $\left(V, E_f\right)$ play BOTA (1 of its neighbors), so we can infect it with BOTA:





Now the only that may be infectable is 5 because that's the only non-BOTA-player that has neighbors that play BOTA in (V, E_c). A fraction bigger than $t_c = 0.49$ of its neighbors play BOTA in (V, E_c) ($\frac{1}{2}$ of its neighbors) and a fraction bigger than $t_f = 0.65$ of its neighbors play BOTA in (V, E_f) (1 of its neighbors), so we can infect 5 with BOTA:





The only node that may be infectable is 4, because that's the only non-BOTA-player in (V, E_c) that has neighbors that play BOTA, however only $\frac{1}{2}$ of its neighbors in (V, E_f) play BOTA, that's less than $t_f = 0.65$ and therefore the node is uninfectable, which means my choice of S doesn't jointly-cascade w.r.t. (V, E_f, E_c) and the given parameters t_f, t_c.

(c) In part (a) we've found a sub-set of V, S s.t.

- S is cascading w.r.t. (V, E_f) with adoption threshold t_f.
- S is cascading w.r.t. (V, E_c) with adoption threshold t_c.

So, by theorem 5.1,

- There does not exist a set of nodes $T \subseteq V \setminus S$ having density $1 - t_f$ w.r.t. (V, E_f) with adoption threshold t_f.
- There does not exist a set of nodes $T \subseteq V \setminus S$ having density $1 - t_c$ w.r.t. (V, E_c) with adoption threshold t_c.

respectively.

Therefore, there does not exist $T \subseteq V \setminus S$ s.t. either T has density $1 - t_f$ w.r.t. (V, E_f) or T has density $1 - t_c$ w.r.t. (V, E_c), so if Bitdidle claim held S would jointly-cascade, but we've seen in part (b) that S doesn't jointly-cascade and therefore Bitdidle's claim doesn't hold.