Project 1
Calculation of Dust Column Density in the Coma of a Comet

Revision 1

1 Background

Comets are small solar system bodies which are composed of volatile ices. As a comet approaches the Sun in its orbit, the surface heats up and solid water ice (the main volatile) sublimates into water vapor. The comet's gravity is too small to retain the water vapor, and so it expands into space at some velocity.

Comets also consist of small particles, colloquially known as *dust*. Cometary dust particles cover a wide range of possible sizes from very small to centimeter scale particles. In the case of the comet explored by the Rosetta Mission, known as 67P/Churyumov-Gerasimenko, there are even some boulder sized objects that escape from the nucleus, but that's something to be considered on another day. Our interest here will be to calculate the velocity achieved by dust grains of different sizes — from microns to perhaps centimeters — in the coma of a comet.

My interest in this particular problem comes from actual observations with the MIRO instrument on the Rosetta Orbiter. MIRO measures the light emitted by dust particles in the coma of 67P, and we would like to analyze our observational results in terms of models of the production and outflow of particles from the surface. MIRO observes at a wavelength of about 1 millimeter, which makes it especially sensitive to the largest particles in the coma.

In Figure 1, I show one of many maps of the emission observed near the comet nucleus that is due to dust. (The emission from the actual nucleus is saturated in this picture, so it just appears black.) The dust we see is concentrated on the right side of the image in the direction of the Sun. This is because, apparently, the large particles observed by MIRO are only lifted from the surface of the comet when the Sun illuminates the surface and sublimates a lot of water ice. The figure to the right in Figure 1 shows how the signal decreases, on the average, with distance from the nucleus. This is a log-log plot and you will note that the decrease is well described by a line with slope -1.64 over the inner few kilometers where we can actually detect the dust. (The observed curve flattens at larger radii because the signal is too weak to be detected beyond about 8 km.)

The signal we see is proportional to the total number of large dust grains in the beam of the antenna along the particular line of sight through the coma. Our group did not expect the answer -1.6 for the slope in this diagram (corresponding to decrease in signal as distance raised to the -1.6 power). For reasons we develop below, we expected -1! So the hunt began to find an explanation. In this project, we consider just the first step we took to find an answer.

The basic theory is a straightforward evaluation of a one-dimensional equation of motion. To begin, even though Comet 67P is definitely not a sphere, we will consider that the nucleus of the comet is a sphere of radius 2 km. We will adopt a value for the mass of the comet that is the same as that measured for 67P. In this case, we represent the mass as the product of Newton's Universal Gravitational constant G with the mass of the comet nucleus M_N. Thus, we say that

$$GM_N = 667\mathrm{m}^3\mathrm{s}^{-2}$$

We represent the result this way because the product GM_N is *MUCH* better determined from the spacecraft navigation data than the constant G itself is determined.

Our calculation begins with dust grains lifted off of the surface of the nucleus by the outflowing gas. Then, as grains move outward in the coma, the gas continues to push and accelerate the grains. Perhaps you will not be surprised that eventually a terminal speed is reached which depends on the size of the dust grain. We will then use this result to compute the quantity that is observed by MIRO: the dust column density at different distances from the surface of the comet nucleus.

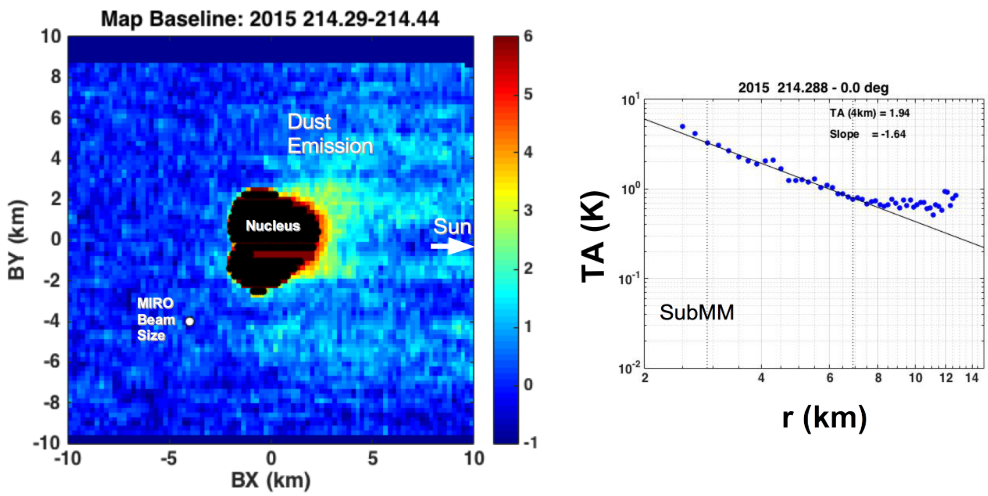

Figure 1: MIRO Observations of Dust in the Coma of Comet 67P.

2 Equation of Motion

The one dimensional equation of motion can be written:

$$M_d \frac{dv}{dt} = F_d + F_g \tag{1}$$

where v is the dust velocity, F_d and F_g are the drag and gravitational forces and M_d is the mass of the particle under consideration.

2.1 Drag Force

The drag force is directed outward and given by

$$F_d = \frac{1}{2} C_D (v_g - v)^2 \rho_g \pi a^2 \tag{2}$$

where C_D is a constant known as the drag coefficient, ρ_g is the gas density, and a is the particle radius.

In our simple case, corresponding to a spherical comet of radius R which is releasing water vapor at a uniform rate, Q molecules per second, with water molecules flowing outward at a constant speed, v_g, we expect that the water gas density will vary with distance, r, from the comet's center as r^{-2}. It is useful to define a gas density at the surface of the comet's nucleus, (i.e. at $r = R$), to be $\rho_g(R)$. In our case, this corresponds to:

$$\rho_g(R) = \frac{M_{H_2O} Q}{4\pi R^2 v_g}$$

where M_{H_2O} is the mass of a water molecule. Given this value, the density at other radii is just:

$$\rho_g(r) = \rho_g(R) \frac{R^2}{r^2}$$

2.2 Gravitational Force

The gravitational force is directed inward and given by

$$F_g = -\frac{GM_N M_d}{r^2} \tag{3}$$

with GM_N is gravitational constant times mass of nucleus and r is the distance from comet COM.

2.3 Maximum Particle Size

At the surface of the nucleus, there is a maximum particle size which may be lifted by the outflowing gas. This occurs when the drag force, which lifts the particle, is equal to the gravitational force on the dust grain. Thus, we can solve for this maximum size, a_{max}, by solving for the condition that $\frac{dv}{dt} = 0 = \frac{F_d}{M_d} + \frac{F_g}{M_d}$ with $v = 0$ at radius R, so that:

$$0 = \frac{1}{2} C_D v_g^2 \rho_g(R) \frac{\pi a_{max}^2}{M_d} - \frac{GM_N}{R^2}$$

Now M_d can be derived from $M_d = \frac{4}{3}\pi a^3 \rho_d$ for a spherical grain, where ρ_d is the density of a spherical grain, we may write the above in terms of a_{max} only:

$$0 = \frac{3}{8} C_D v_g^2 \rho_g(R) \frac{1}{\rho_d a_{max}} - \frac{GM_N}{R^2}$$

The above can then be solved for a_{max}:

$$a_{max} = \frac{3}{8} C_D v_g^2 \rho_{gR} \frac{1}{\rho_d} \frac{R^2}{GM_N} \tag{4}$$

2.4 Summary of Basic Parameters

Parameter	Value	Units	
Nucleus Radius	R	2000	m
Nucleus Mass	GM_N	667	$m^3\ s^{-2}$
Gas Production Rate	Q	7×10^{27}	molecules s^{-1}
Gas Velocity	v_g	1000	$m\ s^{-1}$
Drag Coefficient	C_D	4	Dimensionless
Grain Density	ρ_d	500	$kg\ m^{-3}$

3 Project Calculations

3.1 Part 1 - Solution of Differential Equation

We solve the differential equation (Equation 1), with the following initial conditions. Initial velocity is 0, that is we start grains at rest. Initial position is R, meaning that we start the calculation at the surface of the nucleus. We seek to find position and velocity of dust grains of a particular size, a, as a function of time. That is we we want $r(t)$ and $v(t)$.

We would like to consider a few grain sizes. I suggest writing a function to compute the maximum size of particle that can be lifted, using Equation 4 and then try running cases for 0.9 a_{max}, 0.5 a_{max}, 0.1 a_{max}, 0.01 a_{max}, and 0.001 a_{max}.

My advice is to write and test functions that compute the Drag (Equation 2) and Gravitational (Equation 3) forces. Then use these functions in another function to compute the solution to the basic differential equation, where grain size would be one of the possible arguments. That function should calculate position and velocity and return $r(t)$ and $v(t)$ (and perhaps the t array as well if that is set up in the function).

With a function to compute result for a specific grain size, then we can call it using different grain sizes to compute the velocities for these different cases. You will note that each dust grain is accelerated from 0 and reaches a terminal speed.

3.1.1 Testing the Program

It is important to test the program to be sure it gives correct answers before going on to the next steps in the complete calculation for the project. How do we do that?

One good scheme is to check the functions you use to calculate forces to be sure that they are giving correct results. Be sure that you do that and demonstrate thet your functions work in your writeup.

Another good method is to do a check calculation using your program to compute an answer which can be computed by hand. Think about ways you might do this and be sure to demonstrate them in your program writeup.

3.1.2 Desired Results

- Make a graph of the velocity of the particle versus radius.
- Using all particle size cases, make a graph of the terminal speed achieved as a function of the particle size.

3.2 Part 2 - Determination of Number Density of Particles

If we say that q is the production rate of particles of a specific size, then in an interval dt the total number of particles produced is just $q\ dt$. To make things concrete in what follows, please adopt the case:

$$a = 0.9a_{max}$$

$$q = 100000$$

Consider this number of particles at some distance r from the nucleus. The *number density* of particles will be number divided by volume, so to find number density we must compute the volume of a shell of radius r with a thickness that corresponds to how far the particles will travel in our time interval dt. Obviously that's just the velocity of the particle at radius r times the time interval $v(r)\ dt$, so the volume of our shell is:

$$\text{Volume} = \text{Shell Surface Area} \times \text{Shell Thickness} = 4\pi r^2 v(r) dt$$

Therefore, the number density, n, at radius r is:

$$n(r) = \frac{q\ dt}{4\pi r^2 v(r) dt} = \frac{q}{4\pi r^2 v(r)} \tag{5}$$

You will note that our expression above will have a singularity for the number density of particles right at the surface of the nucleus, since at that position the outward velocity, $v(R)$, is 0. Obviously this is an indication that we expect the particle density n to drop very rapidly as the dust is accelerated away from the surface. For now, let's not worry about this point — we don't need it later — and just graph how the number density varies with distance from the nucleus, starting with the 1st point after the surface value.

3.2.1 Desired Results

- Evaluate Eqaution 5 for all calculated points using the parameters for q and a given above.
- Make a log-log graph of the number density versus radius. You should find that, after terminal velocity is achieved, the number density decreases as r^{-2}, corresponding to a slope of -2 on a log-log plot.

3.3 Calculation of Column Density

When we observe the comet from a distance, we see the amount of dust decrease as we look farther from the nucleus. As noted in our introduction, the MIRO signal is proportional to the amount of dust in the beam of the antenna along a particular line of sight through the coma, so we need to calculate this by adding up all the dust particles that cover a particular area along that line of sight. This is a quantity known as the *column density* and is expressed in units of number of particles divided by the area of the column along the line of sight. We will use the symbol N for the column density to distinguish it from n, the number density (number of particles divided by volume) described in the previous section.

To compute N we must do an integral of n along a specific line of sight through the coma. Let s be the straight line path through the coma, then we may formally write:

$$N = \int_{-\infty}^{\infty} n(s) ds \tag{6}$$

The geometry of the integral along s is shown in Figure 2.

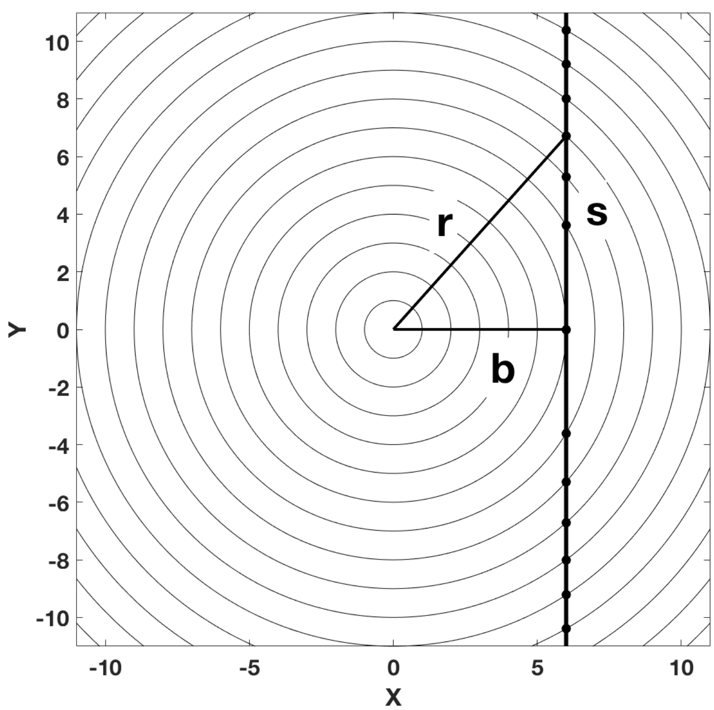

Figure 2: Geometry of calculation of column density N. Circles represent positions where we have calculated $n(r)$ with our differential equation. The vertical line represents a path taken through the coma at some distance b from the nucleus. Note that the position, s, along the path is related to r and b.

To make our integral more concrete, let's do the special case where $n(r) = n_0 \frac{R^2}{r^2}$. Such a model would result from the case where the velocity v is a constant throughout the coma. In this case, it is easy to write down our integral along s at distance b from the nucleus in terms of something we know:

$$N(b) \ = \ \int_{-\infty}^{\infty} n_0 R^2 \frac{ds}{s^2 + b^2}$$

If we note that n(s) does not depend on the sign of s, then we may write this one as:

$$N(b) \ = \ 2\int_{0}^{\infty} n_0 R^2 \frac{ds}{s^2 + b^2}$$

This has the solution

$$N(b) \ = \ \frac{2n_0R^2}{b} \tan^{-1}\left(\frac{s}{b}\right)\Big|_{s=0}^{\infty} \ = \ \frac{2n_0R^2}{b}\left(\frac{\pi}{2} - 0\right) \ = \ \frac{\pi n_0 R^2}{b}$$

and we see that for this special case we expect the column density to decrease with distance b from the nucleus according to b^{-1}.

Now our problem is not *quite* so simple. We must integrate our function $n(r)$ along the line of sight s numerically.

As we calculate this integral, there are a couple of things to consider:

First, we need to pay some attention to the fact that the points where n is evaluated alonf the r direction are not equally spaced along the path through the coma s. So in calculating the integral we will have to account for this.

Second, there is the matter of integrating to ∞. Here the trick is that since n is dropping as r^{-2} there will come a point where adding more values of r to our calculation do not affect the result significantly. Thus, our job will be to be sure to evaluate $n(r)$ out to a large enough value of r that our integral has converged on a stable answer.

3.3.1 Desired Results

- Do this integral for many values of b and see how N varies with b in our model. It is most important, for our project, to see how N varies in the region from about 4km to 10km from the center of the nucleus, since that is where MIRO's observations are made.
- Make a log-log graph of the results, $\log(N)$ versus $\log(b)$.
- Make a graph to show how the *slope* in the previous plot changes with b. You should find that the slope in the $\log(N)$-$\log(b)$ approaches 1 for large b, but has a steeper value for smaller values due to the acceleration of the dust particles in the coma.

4 FInal Interpretation of Results

So, finally, the question to be answered is whether the effect of accelerating the large dust particles can provide an adequate explanation of the observation that the MIRO emission decreases with b as $b^{-1.6}$. Without acceleration of the dust particles, we'd have observed b^{-1}. Can just the expected acceleration account for the difference?

5 Extra Credit Problem

There is an additional effect that our group has been trying to model. If you have time and would like to consider it, then here is a little extra credit work to try.

Our idea is that large particles may sublimate as they move away from the comet since they are heated by the Sun. If they do, then at some point they will become too small to be observed and the MIRO signal will go away.

It is easy to model this. If the change in the number of particles just depends on the number of particles, as in the radioactive decay example in class, then we'd expect the number of particles in each radial shell in our calculation to decrease exponentially with time. Let, τ_0 be the e-folding lifetime of the particles, then at some time t after release from the surface, we'd expect to only see $\exp(-\frac{t}{\tau_0})$ of them remaining.

5.1 Desired Results

Modify your function $n(r)$ using the above decay process and try to find a lifetime τ_0 that will make the slope in our log-log plot match the observed value of -1.6.

6 Write Up your Results in a Jupyter Notebook

The jupyter notebook was invented as a way to write up research results, involving calculations, in a way that documented both the result and the calculation. Please use this format to present both your calculation and your writeup. The writeup should include the following:

- **Introductory Remarks** - describe the physics of the system and introduce any special numerical approach that is required.
- **Describe the Numerical Method** - describe the algorithm that is used to solve the problem. Give example listings of your program to show how it is implemented. Sometimes various development steps are identified in an assignment. In this case, you should be sure to show the program as it appeared at these milestone points.
- **Program Verification** - tell what you did to verify that the program gives correct results.
- **Presentation of Results** - present the results of running the program to demonstrate the behavior of the system under different circumstances. Results might be presented in graphical form or as tables, as appropriate. Be sure that results that are presented are labeled properly, so that the reader can figure out what has been calculated and what is being displayed.
- **Analysis** - sometimes, we will take the results of many runs and derive relationships between the variables of the model and computed quantities. When this occurs, you should present this analysis and comment on the accuracy of the relationship.
- **Discussion** - present a discussion of the physical behavior of the system based on your simulations and answer any special questions posed in the assignment.