

Commutative Algebra

Fall Term

December 13, 2025

To all who find beauty in logic.

Syllabus

We are going to take a brief peek into the field of algebraic number theory in this ongoing seminar. Our ultimate goal is to master some basic tools and techniques, for example, the Dedekind domain and the ramification theory.

In the first part of our seminar, we will have a review on some rudiments from the ring theory and homological algebra. We shall simply follow Atiyah's *An Introduction to Commutative Algebra*.

In the second part, we will briefly discuss some basic concepts in algebraic number theory, like the ring \mathcal{O}_K , the Dedekind domains, primary decomposition and the ramification theory.

This project is maintained on GitHub at

<https://github.com/AlohomoraPZX/Commutative-Algebra>

under the MIT License.

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Chapter 1

Rudiments

In this section, we briefly recall some basic concepts in abstract algebra and homological algebra (especially when things happen in $R\text{-Mod}$ category).

1.1 Homological Algebra

1.1.1 Projective and Injective Objects

Recall that in homological algebra we already knew that functor $\text{Hom}(M, -) : \mathcal{A} \rightarrow \mathbf{Ab}$ is left exact for any $M \in \text{Ob}(\mathcal{A})$, since $\text{Hom}(M, -)$ preserves limits. A natural question is whether it is actually an exact functor or not. The following example shows that the functor can fail to be right exact.

Example 1.1.1. Consider the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Let $M = \mathbb{Z}/2\mathbb{Z}$, the following sequence

$$\text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\times 2} \text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\text{mod } 2} \text{End}(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

cannot be exact at all. Indeed, $\mathbb{Z}/2\mathbb{Z}$ is a torsion module, but \mathbb{Z} is torsion-free, hence $\text{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ could only be $\{0\}$. But $|\text{End}(\mathbb{Z}/2\mathbb{Z})|$ has 2 elements, which is a contradiction.

So naturally, it comes to us that when does $\text{Hom}(M, -)$ be exact? The question leads to the definition of projective and injective objects.

Definition 1.1.1. Let \mathcal{A} be an abelian category, an object $M \in \text{Ob}(\mathcal{A})$ is called **projective** (resp. **injective**), if $\text{Hom}(M, -)$ (resp. $\text{Hom}(-, M)$) is exact.

The name actually comes from the following properties:

Proposition 1.1.1. *An object $M \in \text{Ob}(\mathcal{A})$ is projective if and only if for any epimorphism $f : X \rightarrow Y$ and morphism $g : M \rightarrow Y$, there exists some $\varphi : M \rightarrow X$ such that the diagram*

$$\begin{array}{ccccc} & & M & & \\ & \swarrow \varphi & \downarrow g & & \\ X & \xrightarrow{f} & Y & \longrightarrow & 0 \end{array}$$

commutes.

Remark 1.1.1. *This property is often referred to as ‘the lifting property’ of projective modules. Notice that the uniqueness of φ is not required.*

Proof. (\Rightarrow) $\text{Hom}(M, -)$ preserves epimorphisms since it is right exact and preserves cokernels, hence we obtain

$$\text{Hom}(M, X) \xrightarrow{f_*} \text{Hom}(M, Y) \longrightarrow 0$$

Since $\text{Hom}(M, X)$ and $\text{Hom}(M, Y)$ are abelian groups, hence f_* is surjective. Therefore, for every $g \in \text{Hom}(M, Y)$, there exists some $\varphi \in \text{Hom}(M, X)$ such that $f \circ \varphi = f_*(\varphi) = g$, which shows the commutativity of the diagram.

(\Leftarrow) Now suppose we have the sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

exact, it suffices to show the $\text{Hom}(M, -)$ one is also exact.

The surjectiveness of g_* is a direct result of the lifting property. As for f_* , since $g_* \circ f_* = (g \circ f)_* = 0$, we obtain $\text{Im } f_* \subset \text{Ker } g_*$. It suffices to show $\text{Ker } g_* \subset \text{Im } f_*$. Suppose $\beta \in \text{Hom}(M, Y)$ such that $g \circ \beta = 0$, consider the following diagram:

$$\begin{array}{ccccccc} & & & M & & & \\ & & \swarrow \alpha & \searrow \beta & \searrow \gamma & & \\ & & X & \xrightleftharpoons[f]{\quad} & Y & \xrightarrow{g} & Z \longrightarrow 0 \\ & \nwarrow \pi & \downarrow p & \nearrow \kappa & \uparrow \iota & \nearrow \delta & \\ \text{Coim } f & \xleftarrow{\cong} & \text{Im } f & \xrightleftharpoons[\cong]{\quad} & \text{Ker } g & \longrightarrow & 0 \end{array}$$

By the universal property of kernel, there is a unique $\delta : M \rightarrow \text{Ker } g$ and $\eta : X \rightarrow \text{Ker } g$ such that $\beta = \iota \circ \delta$ and $f = \iota \circ \eta$.

We claim that η is surjective. The exactness of the original sequence yields the canonical morphism $\text{Im } f \rightarrow \text{Ker } g$ to be isomorphic, hence

$$f = \iota \circ \eta = \iota \circ (X \twoheadrightarrow \text{Coim } f \xrightarrow{\cong} \text{Ker } g) \Leftrightarrow \eta = (X \twoheadrightarrow \text{Ker } g)$$

by the injectiveness of ι .

Now, the lifting property of M gives an $\alpha \in \text{Hom}(M, X)$ which commutes the red diagram. Since $f \circ \alpha = (\iota \circ \eta) \circ \alpha = \iota \circ \delta = \beta$, we conclude that $\beta = f_*(\alpha) \in \text{Im } f_*$, which in turn shows that $\text{Ker } g_* \subset \text{Im } f_*$ and completes the proof. \square

The analogue to the result above is *the extension property* of injective modules, which can be stated as following:

Proposition 1.1.2. *An object $M \in \text{Ob}(\mathcal{A})$ is injective if and only if for any monomorphism $f : X \rightarrow Y$ and morphism $g : X \rightarrow M$, there exists some $\varphi : Y \rightarrow M$ such that the diagram*

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y \\ & & \downarrow g & \swarrow \varphi & \\ & & M & & \end{array}$$

commutes. We say g is extended to φ by f .

1.1.2 Flat Modules

1.1.3 Derived Functors

1.2 Ring Theory

1.2.1 Radical of Ideals

1.2.2 Localization

Chapter 2

Hilbert's Nullstellensatz

In this chapter, we introduce an important theorem in algebraic geometry: Hilbert's Nullstellensatz.

2.1 Zariski Topology