Commutative Algebra

Fall Term

December 14, 2025

To all who find beauty in logic.

Syllabus

We are going to take a brief peek into the field of algebraic number theory in this ongoing seminar. Our ultimate goal is to master some basic tools and techniques, for example, the Dedekind domain and the ramification theory.

In the first part of our seminar, we will have a review on some rudiments from the ring theory and homological algebra. We shall simply follow Atiyah's *An Introduction to Commutative Algebra*.

In the second part, we will briefly discuss some basic concepts in algebraic number theory, like the ring $\mathcal{O}_K$, the Dedekind domains, primary decomposition and the ramification theory.

This project is maintained on GitHub at

`https://github.com/AlohomoraPZX/Commutative-Algebra`

under the MIT License.

Contents

Chapter 0

Rudiments

In this section, we briefly recall some basic concepts in abstract algebra and homological algebra (especially when things happen in R-Mod category).

0.1 Homological Algebra

0.1.1 Projective and Injective Objects

Recall that in homological algebra we already knew that functor $\mathrm{Hom}(M, -) : \mathscr{A} \to \mathsf{Ab}$ is left exact for any $M \in \mathrm{Ob}(\mathscr{A})$, since $\mathrm{Hom}(M, -)$ preserves limits. A natural question is whether it is actually an exact functor or not. The following example shows that the functor can fail to be right exact.

Example 0.1.1. *Consider the following exact sequence:*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\bmod 2} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Let $M = \mathbb{Z}/2\mathbb{Z}$, the following sequence

$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\times 2} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{\bmod 2} \mathrm{End}(\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

cannot be exact at all. Indeed, $\mathbb{Z}/2\mathbb{Z}$ is a torison module, but $\mathbb{Z}$ is torison-free, hence $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ could only be $\{0\}$. But $|\mathrm{End}(\mathbb{Z}/2\mathbb{Z})|$ has 2 elements, which is a contradiction.

So natually, it comes to us that when does $\mathrm{Hom}(M, -)$ be exact? The question leads to the definition of projective and injective objects.

Definition 0.1.1. *Let $\mathscr{A}$ be an abelian category, an object $M \in \mathrm{Ob}(\mathscr{A})$ is called* ***projective*** *(resp.* ***injective****), if $\mathrm{Hom}(M, -)$ (resp. $\mathrm{Hom}(-, M)$) is exact.*

The name actually comes from the following properties:

Proposition 0.1.1. *An object $M \in \mathrm{Ob}(\mathscr{A})$ is projective if and only if for any epimorphism $f : X \to Y$ and morphism $g : M \to Y$, there exists some $\varphi : M \to X$ such that the diagram*

$$\begin{tikzcd}
 & M \arrow[dl, "\varphi"'] \arrow[d, "g"] & \\
X \arrow[r, "f"] & Y \arrow[r] & 0
\end{tikzcd}$$

commutes.

Remark 0.1.1. *This property is often referred to as* ***the lifting property*** *of projective objects. Notice that the uniqueness of φ is not required.*

Proof. ($\Rightarrow$) $\mathrm{Hom}(M, -)$ preserves epimorphisms since it is right exact and preserves cokernels, hence we obtain

$$\begin{tikzcd}
\mathrm{Hom}(M, X) \arrow[r, "f_*"] & \mathrm{Hom}(M, Y) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{Hom}(M, X)$ and $\mathrm{Hom}(M, Y)$ are abelian groups, hence f_* is surjective. Therefore, for every $g \in \mathrm{Hom}(M, Y)$, there exists some $\varphi \in \mathrm{Hom}(M, X)$ such that $f \circ \varphi = f_*(\varphi) = g$, which shows the commutativity of the diagram.

($\Leftarrow$) Now suppose we have the sequence

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

exact, it suffices to show the $\mathrm{Hom}(M, -)$ one is also exact.

The surjectiveness of g_* is a direct result of the lifting property. As for f_*, since $g_* \circ f_* = (g \circ f)_* = 0$, we obtain $\mathrm{Im}\, f_* \subset \mathrm{Ker}\, g_*$. It suffices to show $\mathrm{Ker}\, g_* \subset \mathrm{Im}\, f_*$. Suppose $\beta \in \mathrm{Hom}(M, Y)$ such that $g \circ \beta = 0$, consider the following diagram:

$$\begin{tikzcd}
 & & & M \arrow[dll, red, "\alpha"'] \arrow[dl, "\beta"'] \arrow[d, "\gamma"] \arrow[ddl, red, "\delta"] & \\
 & X \arrow[r, "f"] \arrow[dl, two heads, "\pi"'] \arrow[d, two heads, "p"] \arrow[dr, red, "\eta"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \\
\mathrm{Coim}\, f \arrow[r, hook, two heads, "\cong"] & \mathrm{Im}\, f \arrow[r, hook, two heads, "\cong"] \arrow[ur, hook, "\kappa"] & \mathrm{Ker}\, g \arrow[u, hook, "\iota"] \arrow[r, red] & \color{red}{0} &
\end{tikzcd}$$

By the universal property of kernel, there is a unique $\delta : M \to \mathrm{Ker}\, g$ and $\eta : X \to \mathrm{Ker}\, g$ such that $\beta = \iota \circ \delta$ and $f = \iota \circ \eta$.

We claim that η is surjective. The exactness of the original sequence yields the cannonical morphism $\operatorname{Im} f \to \operatorname{Ker} g$ to be isomorphic, hence

$$f = \iota \circ \eta = \iota \circ (X \twoheadrightarrow \operatorname{Coim} f \xrightarrow{\cong} \operatorname{Ker} g) \Leftrightarrow \eta = (X \twoheadrightarrow \operatorname{Ker} g)$$

by the injectiveness of ι.

Now, the lifting property of M gives an $\alpha \in \operatorname{Hom}(M, X)$ which commutes the red diagram. Since $f \circ \alpha = (\iota \circ \eta) \circ \alpha = \iota \circ \delta = \beta$, we conclude that $\beta = f_*(\alpha) \in \operatorname{Im} f_*$, which in turn shows that $\operatorname{Ker} g_* \subset \operatorname{Im} f_*$ and completes the proof. □

The analogue to the result above is *the extension property* of injective objects, which can be stated as following:

Proposition 0.1.2. *An object $M \in \operatorname{Ob}(\mathscr{A})$ is injective if and only if for any monomorphism $f : X \to Y$ and morphism $g : X \to M$, there exists some $\varphi : Y \to M$ such that the diagram*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[dl, "\varphi"] \\
& M &
\end{tikzcd}$$

commutes. We say g is extended to φ by f.

An interesting fact is that projective and injective objects have close relationship with split exact sequences, hence are close with direct sums.

Proposition 0.1.3. *Suppose $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence in an abelian category $\mathscr{A}$, then the sequence splits if C (resp. A) is projective (resp. injective), hence we obtain $B \cong A \oplus C$.*

Proof. Suppose C be a projective object, then we obtain $\varphi : C \to B$ such that the diagram

$$\begin{tikzcd}
& C \arrow[dl, "\varphi"'] \arrow[d, equal, "\mathrm{id}"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

commutes, meaning that $g \circ \varphi = \mathrm{id}_C$, which shows that the sequence splits.

As for the case of injective objects, the proof is a complete analogue. □

Talking about the direct sum, the following proposition shows that the direct summand of project objects is also projective.

Proposition 0.1.4. *Suppose I is an index set, $\{M_i\}_{i\in I}$ is a family of objects in an abelian category $\mathscr{A}$, then $\bigoplus_{i\in I} M_i$ is projective if and only if each M_i is projective.*

Proof. ($\Leftarrow$) Suppose each M_i is projective, we are going to show $N = \bigoplus_{i\in I} M_i$ is also projective. Let $f : X \to Y$ be an epimorphism, $\tilde{g} : \bigoplus_{i\in I} M_i \to Y$, which in fact gives a family of morphisms $g_i : M_i \to Y$ by setting $g_i = \tilde{g} \circ \iota_i$, where $\{\iota_i\}$ are cannonical morphisms. Now consider the diagram:

$$\begin{tikzcd}
& & X \arrow[d, dashed, two heads, "f"] & \\
& & Y \arrow[dd, dashed] & \\
M_i \arrow[r, "\iota_i"] \arrow[uurr, "\varphi_i"] \arrow[urr, dashed, "g_i"] & \bigoplus_{i\in I} M_i \arrow[uur, "\tilde{\varphi}"] \arrow[ur, dashed, "\tilde{g}"] & & M_j \arrow[ll, "\iota_j"] \arrow[uul, "\varphi_j"'] \arrow[ul, dashed, "g_j"'] \\
& & 0 &
\end{tikzcd}$$

By the liftting property of each M_i, we obtain $\varphi_i : M \to X$ such that $g_i = f \circ \varphi_i$. Now, by the universal property of direct sum, we obtain a $\tilde{\varphi} : \bigoplus_{i\in I} M_i \to X$ such that $\varphi_i = \tilde{\varphi} \circ \iota_i$.

We aim to show the whole diagram commutes, which only requires $\tilde{g} = f \circ \tilde{\varphi}$. In fact, we have $(f \circ \tilde{\varphi}) \circ \iota_i = f \circ \varphi_i = g_i$ for each $i \in I$, thus by the universal property, we have $f \circ \tilde{\varphi} = \tilde{g}$, which completes the proof.

($\Rightarrow$) Now suppose $\bigoplus_{i\in I} M_i$ be projective, $g_i : M_i \to Y$. Our goal is to show the existence of some φ_i such that $g_i = f \circ \varphi_i$. Actually, set $g_j = 0$ for any $j \neq i$, which defines a unique $\tilde{g} : \bigoplus_{i\in I} M_i \to Y$ which commutes the diagram in the bottom surface. By the lifting property, we obtain a $\tilde{\varphi} : \bigoplus_{i\in I} M_i \to X$, which commutes the vertical diagram. Now notice that $f \circ (\tilde{\varphi} \circ \iota_i) = g_i$. We may set $\varphi_i = \tilde{\varphi} \circ \iota_i$, which completes the proof. □

Corollary 0.1.1. *Direct summands of a projective object are always projective.*

Now we state a much more general fact about projective and injective objects.

Proposition 0.1.5. *Let $\mathcal{F} : \mathscr{A} \to \mathscr{B}$ be a functor, where $\mathscr{A}, \mathscr{B}$ are both abelian categories. If $\mathcal{F}$ admits a right adjoint $\mathcal{G} : \mathscr{B} \to \mathscr{A}$ which preserves surjectiveness, then $\mathcal{F}$ preserves projectiveness.*

Proof. Suppose $M \in \mathrm{Ob}(\mathscr{A})$ be projective, we aim to show that $\mathcal{F}(M)$ is also projective. Let $f : X \twoheadrightarrow Y$ be an arbitrary epimorphism, where $X, Y \in \mathrm{Ob}(\mathscr{B})$, the adjunction suggests the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(M, \mathcal{G}(X)) \arrow[r, "(\mathcal{G}(f))_*"] \arrow[d, hook, two heads, "\eta_X"'] & \mathrm{Hom}_{\mathscr{A}}(M, \mathcal{G}(Y)) \arrow[d, hook, two heads, "\eta_Y"] \\
\mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(M), X) \arrow[r, "f_*"] & \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(M), Y)
\end{tikzcd}$$

commutes. Suppose $v : \mathcal{F}(M) \to Y$, we obtain a $\eta_Y^{-1} \circ v : M \to \mathcal{G}(Y)$. Since $\mathcal{G}(f)$ is still an epimorphism in $\mathscr{A}$, the liftting property of M gives some $u : M \to \mathcal{G}(X)$ such that the diagram

$$\begin{tikzcd}
 & M \arrow[dl, "u"'] \arrow[d, "\eta_Y^{-1}\circ v"] & \\
\mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(Y) \arrow[r] & 0
\end{tikzcd}$$

commutes. Let $\tilde{u} = \eta_X \circ u \in \mathrm{Hom}_{\mathscr{B}}(\mathcal{F}(M), X)$, the adjuction suggests

$$f_*(\tilde{u}) = f \circ \eta_X(u) = \eta_Y(\mathcal{G}(f) \circ u) = \eta_Y(\eta_Y^{-1} \circ v) = v,$$

which means that the following diagram

$$\begin{tikzcd}
 & \mathcal{F}(M) \arrow[dl, "\eta_X(u)"'] \arrow[d, "v"] & \\
X \arrow[r, "f"] & Y \arrow[r] & 0
\end{tikzcd}$$

commutes, which completes the proof. □

Remark 0.1.2. *Now let me simply explain how this derives the result that direct sum of projective objects is still projective. Consider the coproduct functor* $\mathcal{F} = \bigoplus_{i\in I}(-) : \mathscr{A}^I \to \mathscr{A}$ *defined by* $(M_i)_{i\in I} \mapsto \bigoplus_{i\in I} M_i$*, and the diagonal functor* $\Delta : \mathscr{A} \to \mathscr{A}^I$ *defined by* $M \mapsto (M)_{i\in I}$*. We aim to show that* Δ *is the right adjoint of* $\mathcal{F}$*.*

In fact, suppose $\mathscr{I} \in \mathscr{A}^I$*, the universal property of direct sum suggests that for each* $A \in \mathrm{Ob}(\mathscr{A})$*, we have*

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{A}}(A, \mathcal{F}(\mathscr{I})) = \mathrm{Hom}_{\mathscr{A}}\left(A, \bigoplus_{i\in I} M_i\right) &\cong \bigoplus_{i\in I} \mathrm{Hom}_{\mathscr{A}}(A, M_i) \\
&\cong \mathrm{Hom}_{\mathscr{A}^I}(\Delta(A), \mathscr{I}),
\end{aligned}$$

which suggests adjunction. Now, Δ *preserves surjectiveness since epimorphisms in* $\mathscr{A}^I$ *are defined componentwise, which completes the proof.*

0.1.2 Flat Modules

0.1.3 Derived Functors

0.2 Ring Theory

0.2.1 Radical of Ideals

0.2.2 Localization

0.2.3 Nakayama's Lemma

Chapter 1

Hilbert's Nullstellensatz

In this chapter, we introduce an important theorem in algebraic geometry: Hilbert's Nullstellensatz.

1.1 Zariski Topology