5. Given $\bar{e} \rightarrow$ eigenvector of C with highest eigenvalue λ_0

$\therefore$ $C\bar{e} = \lambda_0 \bar{e} \Rightarrow \bar{e}^t C e = \lambda_0 \bar{e}^t \bar{e} = \lambda_0$

$\Rightarrow e^t C e e^t = \lambda_0 e^t \Rightarrow \boxed{e^t C = \lambda_0 e^t}$ $\rightarrow$ (1)

Now we want ~~to~~ find direction f, s.t,

constraints

$\bar{f} \perp \bar{e} \Rightarrow f^t e = e^t f = f e^t = e f^t = 0.$

$f^t C f$ is maximised

$f f^t = 1$ ($\bar{f}$ is a unit vector direct^n.)

$J(\bar{f}) = f^t C f - \lambda_1 (f f^t - 1) - \lambda_2 (f^t e - 0)$ (By lagrange's multiplier)

$\therefore$ To maximise it, $\frac{\partial J(\bar{f})}{\partial f} = 0$

$$\therefore \frac{\partial J(\bar{f})}{\partial f} = Cf - \lambda_1 f - \lambda_2 e = 0 \rightarrow (2)$$

$$\therefore e^t \frac{\partial J(\bar{f})}{\partial f} = e^t C f - \lambda_1 e^t f - \lambda_2 e^t e = 0$$

But ~~for~~ from eqn (1), $e^t C = \lambda_0 e^t$ & $e^t e = 1$

$\therefore$ $\lambda_0 e^t f - \lambda_1 e^t f - \lambda_2 = 0$, But $e^t f = 0$

$0 - 0 - \lambda_2 = 0$

$\therefore$ $\boxed{\lambda_2 = 0}$

$\therefore$ Putting this in eqn (2), $Cf = \lambda_1 f$

$\therefore$ f is an eigenvector of Covariance matrix C with $\lambda_1 \rightarrow$ eigenvalue.

$\therefore$ $f^t C f = \lambda_1 f^t f = \lambda_1$

$\therefore$ To maximise $f^t C f$ now, we need to maximise λ_1.

$\lambda_0 \rightarrow$ highest eigenvalue & all ~~eigenvectors ($\Rightarrow$ eigenvalues)~~ eigenvalues are distinct.

$\Rightarrow$ We need to take λ_1 as the 2nd highest eigenvalue to maximise $f^t C f$ ~~now~~ with the given constraints.

For $\bar{f} \perp \bar{e}$ s.t. $f^t C f$ is maximised, $\bar{f}$ = eigenvector of C corresponding to 2nd largest eigenvalue λ_1