6. $A_{m\times n}$, $P = A^TA$, $Q = AA^T$

a) $y^tPy = y^tA^TAy = (Ay)^TAy = \|Ay\|^2 \geq 0$

Similarly, $z^tQz = z^tAA^Tz = z^t(A^t)^tA^tz = (A^tz)^TA^Tz$

(Square magnitude of a vector is ≥ 0) $= \|A^Tz\|^2 \geq 0$

Let λ & μ be eigenvalues of P & Q respectively with x & y as the eigenvectors respectively.

$\therefore$ $Px = \lambda x$ & $Qy = \mu y$

$\therefore$ $x^TPx = \lambda x^Tx = \lambda\|x\|^2$ $\quad y^TQy = \mu y^Ty = \mu\|y\|^2$

But we showed earlier that for any x & any y,

$x^TPx \geq 0$ & $y^TQy \geq 0$

$\Rightarrow$ $\lambda\|x\|^2 \geq 0$ & $\mu\|y\|^2 \geq 0$

But $\|x\|^2$ & $\|y\|^2 \geq 0$ $\Rightarrow$ $\lambda \geq 0$ & $\mu \geq 0$ also.

Hence Proved that eigenvalues of P & Q (λ & μ resp.) are non-negative.

b) $P \rightarrow$ eigenvector u, eigenvalue λ.
$Q \rightarrow$ eigenvector v, eigenvalue μ.
$\boxed{u \in \mathbb{R}^n \quad v \in \mathbb{R}^m}$

$\therefore$ $Pu = \lambda u$
$\therefore$ $APu = \lambda Au$
$\therefore$ $A(A^TA)u = \lambda Au$
$\therefore$ $(AA^T)Au = \lambda Au$ $\leftarrow$ (By Associativity) $\rightarrow$
$\therefore$ $Q(Au) = \lambda(Au)$
$\therefore$ Au is eigenvector of Q with eigenvalue λ

$Qv = \mu v$
$\therefore$ $A^TQv = \mu A^Tv$
$\therefore$ $A^T(AA^T)v = \mu A^Tv$
$(A^TA)A^Tv = \mu A^Tv$
$\therefore$ $P(A^Tv) = \mu(A^Tv)$
$\therefore$ A^Tv is eigenvector of P with eigenvalue μ.

c) $v_i \rightarrow$ eigenvector of Q. Let x_i be eigenvalue corresponding to it.

$\therefore$ $Qv_i = x_iv_i$
$\therefore$ $(AA^T)v_i = x_iv_i$
$\therefore$ $A(A^Tv_i) = x_iv_i$

$\therefore A \dfrac{A^T v_i}{\|A^T v_i\|} = x_i \dfrac{v_i}{\|A^T v_i\|}$

$\therefore A u_i = x_i \dfrac{v_i}{\|A^T v_i\|} \Rightarrow A u_i = \gamma_i v_i$, where $\gamma_i = \dfrac{x_i}{\|A^T v_i\|}$

& $u_i = \dfrac{A^T v_i}{\|A^T v_i\|}$

Now to show, $\gamma_i \geq 0$,

$x_i \rightarrow$ eigenvalue of Q which is ≥ 0 as shown in part a).

& $\|A^T v_i\| \rightarrow$ magnitude of a vector which is ≥ 0.

$\therefore \gamma_i = \dfrac{x_i}{\|A^T v_i\|} \geq 0$ i.e. it is non-negative.

d) First to show, $u_i^T u_j = 0$ & $v_i^T v_j = 0 \ \forall\ i \neq j$

$\left.\begin{array}{l} \therefore P u_1 = \lambda_1 u_1 \\ \& \ P u_2 = \lambda_2 u_2 \end{array}\right\}$ $u_2^T P u_1 = \lambda_1 u_2^T u_1$

$\hookrightarrow = \underset{(\text{As } P = P^T)}{u_2^T P^T u_1} = (P u_2)^T u_1 = (\lambda_2 u_2)^T u_1 = \lambda_2 u_2^T u_1$

$\therefore$ As $\lambda_1 \neq \lambda_2$, $u_2^T u_1 = 0$. Similarly it can be shown for $v_2^T v_1 = 0$.

$U = [v_1 | v_2 \ldots | v_m]$, $v_i \in \mathbb{R}^m \Rightarrow U \in \mathbb{R}^{m \times m}$

$V = [u_1 | u_2 \ldots | u_m]$, $u_i \in \mathbb{R}^n \Rightarrow V \in \mathbb{R}^{n \times n}$

As P & Q are <u>symmetric</u> ($P = P^T$ & $Q = Q^T$), the eigenvectors u_i & v_i of P & Q respectively will be <u>orthonormal</u>.

Hence we can say, the matrices U & V are also orthonormal

Now from part c), $A u_i = \gamma_i v_i$. for some $\gamma_i \geq 0$

Let $n \leq m$, (without any loss of generality)

then $A u_i = \gamma_i v_i$ for $i \in [1, n]$

& $A u_i = 0$ for $i \in [n+1, m]$

$\therefore$ In matrix form, we can write, $AV = U\Gamma$,

where Γ is a diagonal matrix with maximum of n non-zero values along its diagonal.

$\therefore AV.V^T = U\Gamma V^T \Rightarrow \underset{(\text{As } V \text{ is orthonormal})}{AI} = U\Gamma V^T \Rightarrow \boxed{A = U\Gamma V^T}$