Dynamics of a Mobile Cart

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1 Project Theory

A typical model describing the motion of a cart being controlled is:

$$M\ddot{d} = K_m u - B\dot{d} \tag{1}$$

where:

- d: cart displacement from origin
- u: control input
- M: mass of cart
- K_m: force-to-control proportionality constant
- B: coefficient of friction

Let $x_1 = d$ be the cart's position, and $x_1 = \dot{d}$ be its velocity. Also, define $T = M/B$ and $K_p = K_m/D$. Then, the above 2nd order ODE can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{K_p}{T} u \tag{2}$$

$$d = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the purpose of control, one changes to z-coordinates as follows:

$$z_1 = x_1 - y_d \qquad\qquad z_2 = x_2$$

where yd is a setpoint being tracked. A controller is also designed and used, leading to the system dynamics being described by (5):

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1/T \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \frac{K_p}{T} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & L \end{bmatrix} \begin{bmatrix} u \\ d - \hat{x}_1 \end{bmatrix} \tag{3}$$

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -1/T\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} + \begin{bmatrix}\frac{K_p}{T}\begin{bmatrix}0\\ 1\end{bmatrix} & L\end{bmatrix}\begin{bmatrix}K[z_1 \;\; z_2]^T\\ d - \hat{x}_1\end{bmatrix} \tag{4}$$

Here, L is called an observation matrix, K a feedback matrix for control, and $\hat{x}_1$ is a control for the cart position (constant).

Note that (5) is a system of coupled differential equations expressed entirely in terms of $[z_1 z_2]$ and constants. Hence, it can be solved using numerical methods to determine the displacement of the cart over time, which is $d = x_1 = z_1 + y_d$.

2 Project Description

In this project, the Forward Euler Step method was used to solve the equation in discrete time, and so control a physical cart in real time.

The equation was solved as follows:

$$\begin{bmatrix}z_1(t+1)\\ z_2(t+1)\end{bmatrix} = \begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix} + \begin{bmatrix}\dot{z}_1(t)\\ \dot{z}_2(t)\end{bmatrix}\Delta t \tag{5}$$

where t is the time index and Δt is the time step.

3 Program

Section 3 shows how this was programmed in C. Note that (5) was implemented on lines 38-39 and 46. Also note that the matrices A and B are:

$$A = \begin{bmatrix}0 & 1\\ 0 & -1/T\end{bmatrix}$$

$$B = \begin{bmatrix}0\\ 1\end{bmatrix}\frac{K_p}{T}$$