Lectures 10 -11
Partial Derivatives-limits and Continuity, Total differentiation and Derivatives

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Illustrate the principals of limit and continuity of functions of two variables
- Illustrate the principal of partial derivatives of functions of two variables
- Apply the concepts of total derivatives in errors and approximations

Topics

- Partial derivatives of a function
- Limit and continuity of a function
- Clairaut's theorem
- Total differentiation

Limit of a Function

- Let $z = f(x, y)$ be a function of two variables defined in a domain D . Let $P(x_0, y_0)$ be a point in D. If for a given real number $\epsilon > 0$, however small, we can find real number $\delta > 0$ such that for every point (x, y).

  In the δ-neighborhood of $p(x_0, y_0)$

  $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x - x_0) + (y - y_0)^2} < \delta$

- The function $f(x, y)$ may or may not be defined at (x_0, y_0) . If $f(x, y)$

  is not defied at $p(x_0, y_0)$ then we write

  $|f(x, y) - L| < \epsilon$ whenever $0 < \sqrt{(x - x_0) + (y - y_0)^2} < \delta$

This definition is called $\delta - \epsilon$ approach to study the existence of limits

Limit of a function

- Notice that:

- $|f(x, y) - L|$ is the distance between the numbers $f(x, y)$ and L

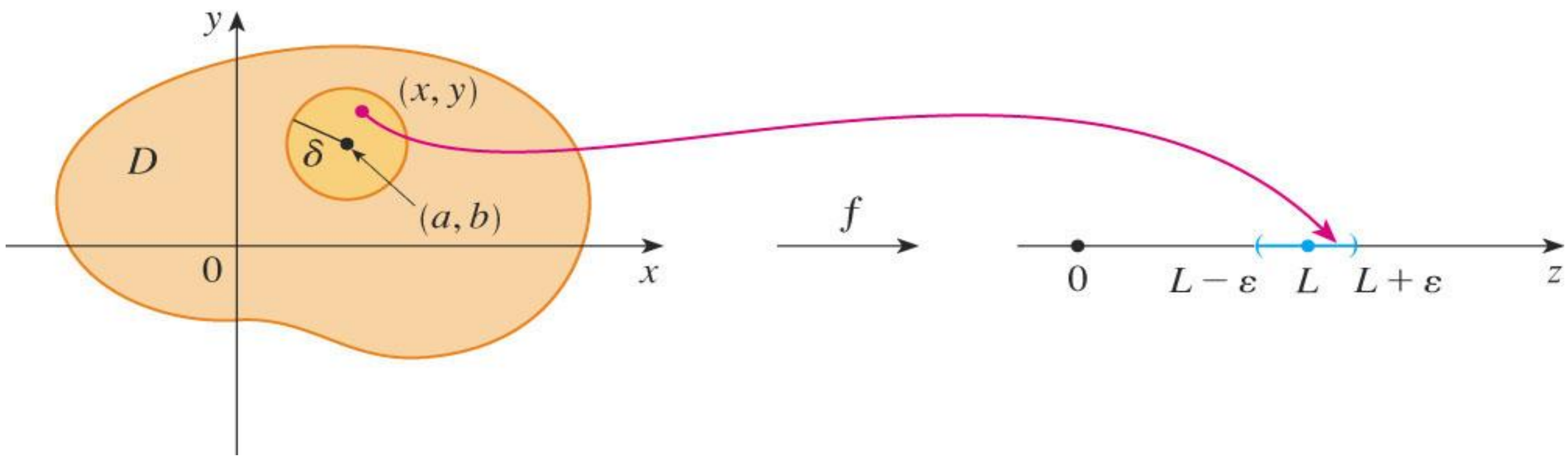
- $\sqrt{(x-a)^2 + (y-b)^2}$ is the distance between the point (x, y) and the point (a, b).

- It does not refer to the direction of approach.
- the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).

Limit of a function

- If any small interval $(L - \varepsilon, L + \varepsilon)$ is given around L, then we can find a disk D_δ with center (a, b) and radius $\delta > 0$ such that:

- f maps all the points in D_δ [except possibly (a, b)] into the interval $(L - \varepsilon, L + \varepsilon)$.



Example 1

Using $\delta - \epsilon$ approach , show that $\lim_{(x,y)\to(2,1)} (3x + 4y) = 10$.

Solution: Given that $f(x, y) = 3x + 4y$ is defined at (2,1), we have

$|f(x, y) - 10| = |3x + 4y - 10|$

$= |3(x - 2) + 4(y - 1) - 10| \leq 3|x - 2| + 4\,|y - 1|$

If we take $|x - 2| < \delta$ and $|y - 1| < \delta$, we get

$|f(x, y) - 10| < 7\,\delta < \epsilon$, which is satisfied when $\delta < \frac{\epsilon}{7}$

Example 2

Using $\delta - \epsilon$ approach , show that $\lim_{(x,y)\to(1,1)} (x^2 + 2y) = 3.$

Solution: Given that $f(x,y) = (x^2 + 2y)$ is defined at (1,1). We have

$$|f(x,y) - 3| = |x^2 + 2y - 3| = |(x - 1 + 1)^2 + 2(y - 1 + 1) - 3|$$

$$= |(x-1)^2 + 2(x-1) + 2(y-1)|$$

$$\leq |(x-1)^2| + 2|x-1| + 2|y-1|$$

If we take $|x - 1| < \delta$ and $|y - 1| < \delta$, we get

$|f(x,y) - 3| < \delta^2 + 4\delta < \epsilon$ which is satisfied when

$(\delta + 2)^2 < \epsilon + 4$ or $\delta < \sqrt{\epsilon + 4}$ -2

Continuity

- A function $z = f(x, y)$ is said to be continuous at a point (x_0, y_0), If
  - (i) $f(x, y)$ is defined at the point (x_0, y_0)
  - (ii) $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists, and
  - (iii) $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.
- If any one of the above conditions is not satisfied, then the function is said to be discontinuous at the point (x_0, y_0)
- A function $f(x, y)$ is continuous at (x_0, y_0) if

$$|f(x,y) - L| < \epsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0) + (y - y_0)^2} < \delta$$

Continuity

- The intuitive meaning of continuity is that, if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount

- This means that a surface that is the graph of a continuous function has no hole or break

- Using the properties of limits, you can see that sums, differences, products, quotients of continuous functions are continuous on their domains

Example 1

Show that the following functions are continuous at the point $(0,0)$

$$f(x) = \begin{cases} \frac{2x^4+3y^4}{x^2+y^2}, & (x,y) \neq 0 \\ 0, & ,(x,y) = (0,0) \end{cases}$$

Solution: $x = rcos\theta, y = rsin\theta.$ Then

$r = \sqrt{x^2 + y^2} \neq 0$ we have

$$|f(x,y) - f(0,0)| = \left|\frac{2x^4+3y^4}{x^2+y^2}\right| = \left|\frac{r^4(2\cos^4\theta+3\sin^4\theta)}{r^2(\cos^2\theta+2\sin^2\theta)}\right|$$

$$< r^2 2|\cos^4\theta| + 3\,|\sin^4\theta| < 5\ r^2 < \epsilon$$

Example1 (Cont.)

$r = \sqrt{x^2 + y^2} \leq \sqrt{\epsilon \backslash 5}$

If we choose $\delta \leq \sqrt{\epsilon \backslash 5}$, we find that

$|f(x, y) - f(0,0)| \leq \epsilon$, whenever $0 \leq \sqrt{x^2 + y^2} \leq \delta$

Therefore $\lim_{(x,y) \to (x_0, y_0)} f(x, y) = f(0,0) = 0$

Hence $f(x, y)$ is continuous at $(0,0)$

Partial Derivatives of a Function of Two Variables

Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

provided the limits exist.

Notation for First Partial Derivatives

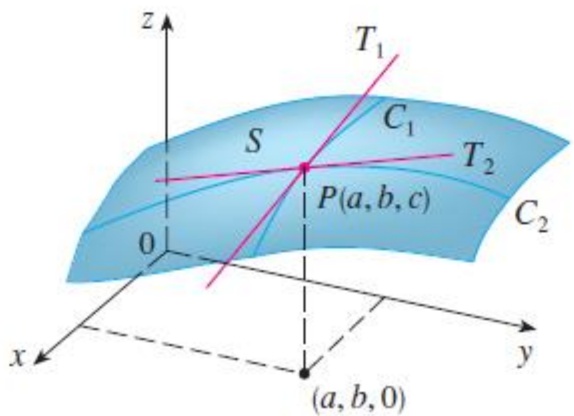
Notation for First Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) =$$

and

$$\frac{\partial}{\partial y} f(x, y) =$$



The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,\, b)} = f_x(a, b) \qquad \text{and} \qquad \left.\frac{\partial z}{\partial y}\right|_{(a,\, b)} = f_y(a, b).$$

Alternative Notations for Partial Derivative

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Alternative Notations for Partial Derivative

The **second partial derivatives** of f. If z=f (x, y), we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Clairaut's Theorem

- Suppose f is defined on a disk D that contains the point (a, b).

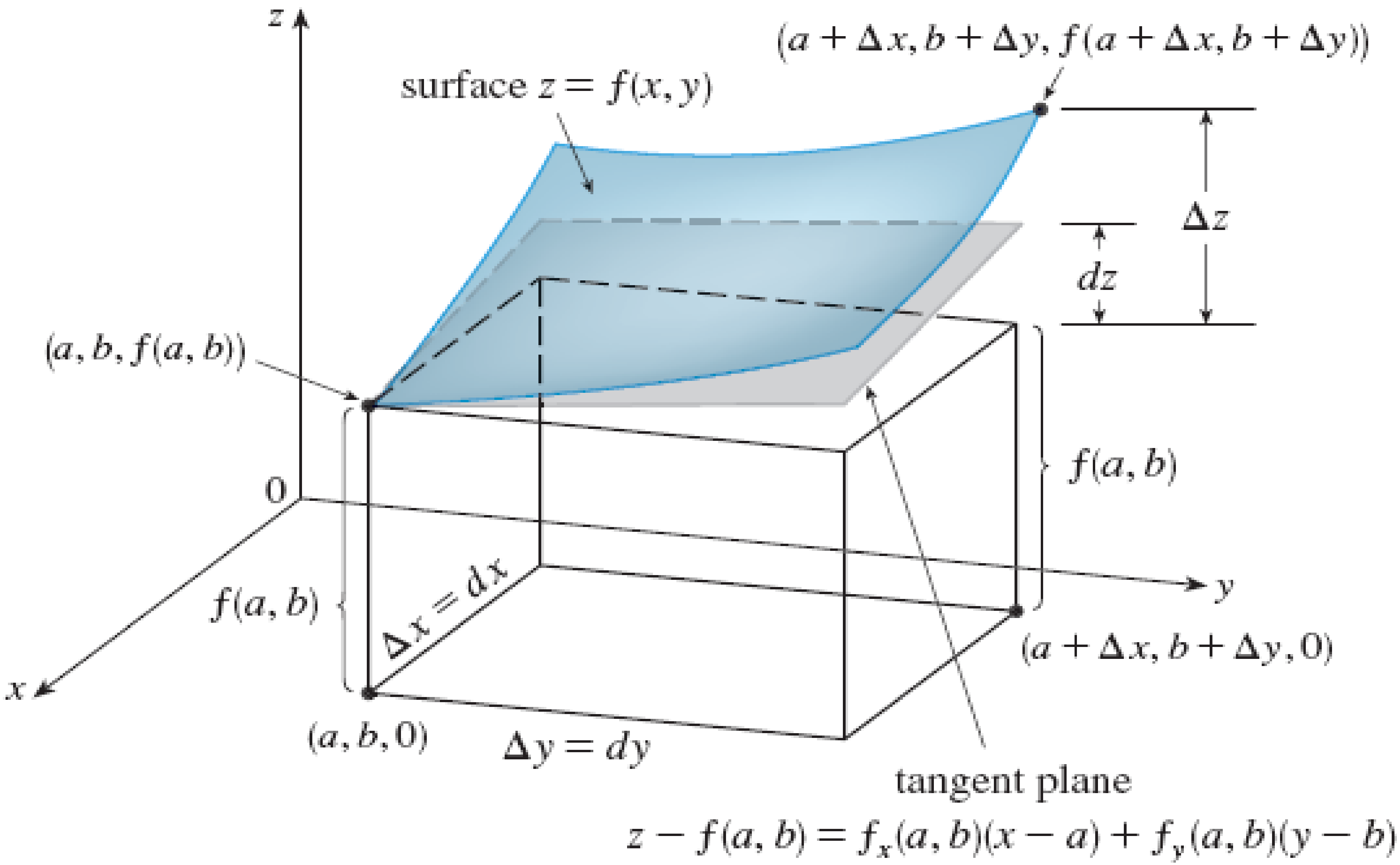
If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Total Differential

- For a differentiable function of two variables, z= f (x ,y), we define the **differentials dx** and **dy** to be independent variables; that is, they can be given any values. Then the **differential dz**, also called the **total differential**, is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$


z
$(a+\Delta x, b+\Delta y, f(a+\Delta x, b+\Delta y))$
surface $z = f(x, y)$
Δz
dz
$(a, b, f(a, b))$
$f(a, b)$
0
y
$f(a, b)$
$\Delta x = dx$
x
$(a+\Delta x, b+\Delta y, 0)$
$(a, b, 0)$
$\Delta y = dy$
tangent plane
$z - f(a, b) = f_x(a, b)(x-a) + f_y(a, b)(y-b)$

Total Differentials

- For such functions the **linear approximation** is

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

- and the linearization L (x, y, z) is the right side of this expression.
- If w=f (x, y, z), then the **increment** of w is

$$\Delta w = f(x+\Delta x, y+\Delta y, z+\Delta z) - f(x,y,z)$$

- The **differential dw** is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial a}dz$$

The chain rule (general version)

Suppose that u is a differentiable function of the n variables x_1, $x_2, \cdots, x_n$ and each x_j is a differentiable function of the m variables t_1, $t_2, \cdots, t_m$ Then u is a function of $t_1, t_2, \cdots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{dt_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i=1,2,\cdots,m$.

Tangent plane

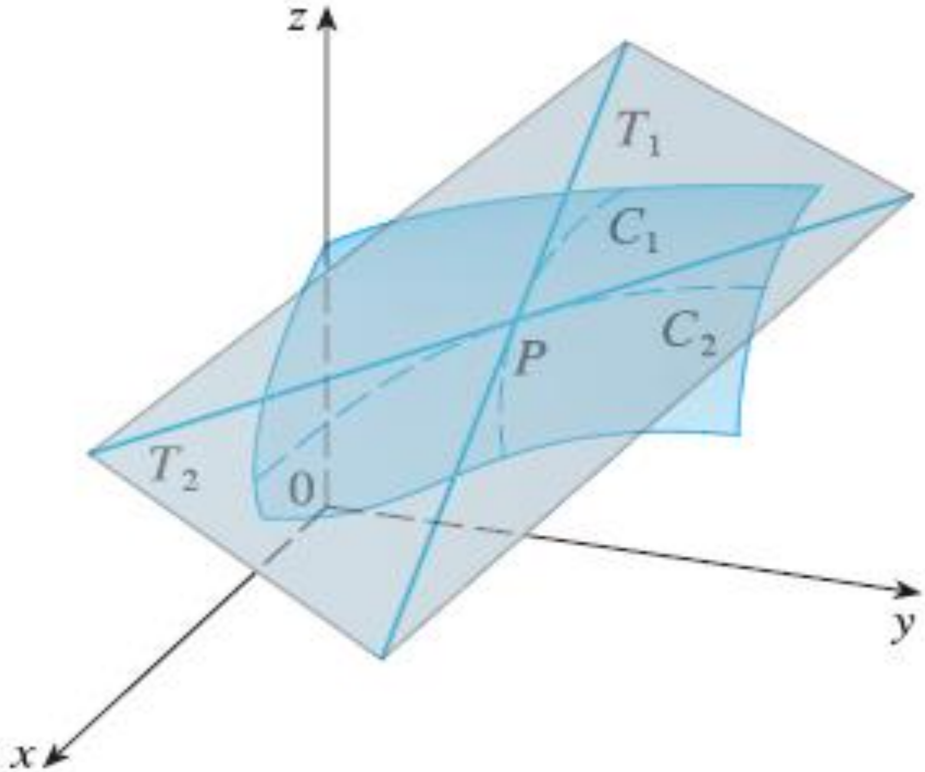


FIGURE 1

The tangent plane contains the tangent lines T_1 and T_2

Linearization

1. Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface z=f (x, y) at the point P (x_o, y_o, z_o) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The linear function whose graph is this tangent plane, namely

2. $L(x, y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is called the **linearization** of *f* at (a, b) and the approximation

3. $f(x, y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$ is called the **linear approximation** or the **tangent plane approximation** of *f* at *(a, b)*

Example on total derivative

If $u = e^x \sin(yz), \quad where \quad x = t^2, \; y = t-1, \; z = 1/t,$

$\text{find} \; du/dt \quad at \; t = 1$

Solution: $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \qquad ....(i)$$

From the given u, x, y, z, we get $\frac{\partial u}{\partial x} = e^x \sin yz, \quad \frac{\partial u}{\partial y} = e^x z \cos yz, \quad \frac{\partial u}{\partial z} = e^x y \cos yz$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = \frac{-1}{t^2}$$

Putting these into (i), we get

$$\frac{du}{dt} = e^{t^2}\left[2t \sin(1-1/t)\right] + (1/t^2)\cos(1-1/t)]$$

$$At \;\; t = 1, \;\; this \;\; becomes \;\; \frac{du}{dt} = e$$

Examples

1.Find the differentials of the function $f(x,y) = x\cos y - y\cos x$

Solution: $\frac{\partial f}{\partial x} = \cos y + y\sin x, \quad \frac{\partial f}{\partial y} = -x\sin y - \cos x$

Therefore , the differential of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (\cos y + y\sin x)dx - (x\sin y + \cos x)dy$$

2.Find the differentials of the function $f(x,y,z) = e^{xyz}$

Solution: $\frac{\partial f}{\partial x} = e^{xyz}yz, \quad \frac{\partial f}{\partial y} = e^{xyz}zx, \quad \frac{\partial f}{\partial z} = e^{xyz}xy,$

The differential of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = e^{xyz}(yzdx + zxdy + xydz)$$

Summary

- The general definition of the total derivative is:

$$df\left(x,y\right)=\frac{\partial f\left(x,y\right)}{\partial x}dx+\frac{\partial f\left(x,y\right)}{\partial y}dy$$

- The general rule, with a function of several variables is:
  - Calculate the partial derivatives for each of the variable, keeping the other variables constant
  - Add them up to get the total derivative