Lecture 6
Maclaurin’s Series

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- State and construct Maclaurin Series
- State Exponential, Logarithmic and Binomial Series
- Apply Maclaurin’s Series to expand standard functions

Topics

- Taylor’s theorem
- Maclaurin ’s expansion
- Binomial series

Taylor's Theorem

If f has derivatives of all orders in an open interval *I* containing x_0, then for each positive integer *n* and for each *x* in *I*:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \ldots + f^{(n)}(x_0)\frac{(x-x_0)^n}{n!} + \ldots$$

Note: If $x_0 = 0$, the series is the Maclaurin's series for $f(x)$

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \ldots.$$

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Example 1

Use the function $f(x) = \sin x$ to form the Maclaurin' s series and determine the interval of convergence.

Solution:

A Maclaurin' s series is given by

$$f(x) = f(0) + \frac{x}{1!}f'(x) + \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}f'''(x) + \cdots \quad \text{(i)}$$

Successive differentiation of $f(x)$ yields

$f(x) = \sin x$ $\quad$ $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\quad$ $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\quad$ $f''(0) = -\sin 0 = 0$

Example 1 (Contd...)

$f'''(x) = -\cos x$ $\qquad$ $f'''(x) = -\cos 0 = -1$

$f^{(iv)}(0) = \sin x$ $\qquad$ $f^{(iv)}(0) = 0$

$f^{v}(x) = cosx$ $\qquad$ $f^{v}(0) = 1$

Substitute the derivatives in the formula (1), we have

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Example 2

Expand tan x Using the Maclaurin 's expansion

$$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2 \Rightarrow y_1(0) = 1$$

$$y_2 = 2yy_1 \qquad \Rightarrow y_2(0) = 0$$

$$y_3 = 2(y_1{}^2 + yy_2) \Rightarrow y_3(0) = 2$$

$$y_4 = 2(3y_1y_2 + yy_3) \Rightarrow y_4(0) = 0$$

$$y_5 = 2(3y_2{}^2 + 4y_1y_3 + yy_4) \Rightarrow y_5(0) = 16$$

$$\therefore \quad \tan x = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \ldots$$

Approximating Logarithms

- The Maclaurin series $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots;(-1<x\le 1)$

taking the top equation minus the bottom gives

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots;(-1<x\le 1)$$

$$\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots;(-1\le x<1)$$

$$\ln\left(\frac{1+x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\ldots\right);-1<x<1$$

Approximating Logarithms

- This new series can be used to compute the natural log of any positive number y by letting

$$y = \frac{1+x}{1-x}$$

or equivalently

$$x = \frac{y-1}{y+1}$$

and noting that $-1 < x < 1$.

Approximating Logarithms

- For example, to compute ln2 we let y = 2 in $x = \frac{y-1}{y+1}$ which yields x = 1/3. Substituting this value in

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right); -1 < x < 1$$

gives

$$\ln 2 = 2\left[\frac{1}{3} + \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} + \frac{(\frac{1}{3})^7}{7} + \ldots\right]$$

Binomial Series

- If m is a real number, then the Maclaurin series for $(1 + x)^m$ is called the binomial series; it is given by

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots + \frac{m(m-1)(m-k+1)}{k!}x^k + \ldots$$

Binomial Series

- Consider the function $f(x) = (1+x)^k$
    - This produces the binomial series
- We seek a Maclaurin series for this function
    - Generate the successive derivatives
    - Determine $f^{(n)}(0) = ?$
    - Now create the series using the pattern

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(c)}{2!} \cdot x^2 + \ldots$$

Binomial Series

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- In the case where m is a nonnegative integer, the function

  $f(x) = (1 + x)^m$ is a polynomial of degree m, so

$$f^{m+1}(0) = f^{m+2}(0) = f^{m+3}(0) = \ldots = 0$$

- The binomial series reduces to the familiar binomial expansion

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots + x^m$$

Binomial Series

- It can be proved that if m is not a nonnegative integer, then the binomial series converges to $(1 + x)^m$ if $|x| < 1$. Thus, for such values of x

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots + \frac{m(m-1)(m-k+1)}{k!}x^k + \ldots$$

or in sigma notation

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\ldots(m-k+1)}{k!}x^k ; |x| < 1$$

Example 1

- *Find the binomial series for*

(a) $\dfrac{1}{(1+x)^2}$ *(b)* $\dfrac{1}{\sqrt{1+x}}$

(a) Substitution m = -2 in the formula yields

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \ldots$$

$$= 1 + -2x + \frac{3!}{2!}x^2 - \frac{4!}{3!}x^3 + \ldots \qquad = 1 + -2x + 3x^2 - 4x^3 + \ldots$$

$$\sum_{k=0}^{\infty} (-1)^k (k+1) x^k$$

Example 1 contd...

(b) Substitution m = -1/2 in the formula yields

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 + (-1/2)x + \frac{(-1/2)(-3/2)}{2!}x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^3 + \dots$$

Summary

- If the Taylor series is **centered at zero**, then that series is also called a **Maclaurin's series**

- Formula for Maclaurin's series solution

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!} + \ldots$$