Lecture 7-8
Eigen Values and Eigen Vectors

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Illustrate the eigenvalues and eigenvectors of a given square matrix
- Determine the eigenvalues and corresponding eigenvectors of a given matrix
- Apply eigenvalues and eigenvector in real world problems

Topics

- Eigen values and eigen vectors
- Similar matrices
- Diagonalization of matrices
- Power of matrices
- MATLAB code

Eigenvalue and Eigenvector - Importance

- Matrix Inverse
- Stability and System Performance
- Linear System of Differential Equations
- Diagonalisation of Matrix
- DSP, DIP, Control System Engineering, Communication Engineering, Mechanical Systems, Structural Analysis

Eigen values and Eigen vectors

Let

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & .... & a_{1n} \\ a_{21} & a_{22} & a_{23} & .... & a_{2n} \\ a_{31} & a_{32} & a_{33} & .... & a_{3n} \\ \cdots & \cdots & \cdots & .... & \cdots \\ \cdots & \cdots & \cdots & .... & \cdots \\ a_{n1} & a_{n2} & a_{n3} & .... & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ \cdots \\ y_n \end{bmatrix}$$

$$A\boldsymbol{x}=\boldsymbol{b} \qquad (1)$$

Where A is the coefficient matrix, $\boldsymbol{x}$ is a column vector of unknown and $\boldsymbol{b}$ is a column vector. Here column $\boldsymbol{x}$ is transformed into the column vector $\boldsymbol{b}$ by means of square matrix A.

Eigen Values And Eigen Vectors…

Let $\boldsymbol{x}$ be a such vector which transforms into $\lambda\boldsymbol{x}$ by means of transformation (1). Suppose the linear transformation $\boldsymbol{b}=A\boldsymbol{x}$ transforms $\boldsymbol{x}$ into a scalar multiple of itself i.e. $\lambda\boldsymbol{x}$

$$AX = Y = \lambda X \Rightarrow \mathrm{A}X - \lambda \mathrm{I}X = 0$$

$$\Rightarrow (A - \lambda I)X = 0$$

$$BX = 0 \;\; where \;\; B = A - \lambda \mathrm{I}$$

$$X = B^{-1}0 = 0$$

It is obvious that the zero vector $\boldsymbol{x}$=0 is a solution, but the Eigen vector cannot be zero.

Eigen values and Eigen vectors…

The necessary and sufficient condition for equations to posses a non-zero solution (***x*** ≠0) is that the coefficient matrix *B* should be of rank less than number of unknowns *n*. But this will be possible if and only if the matrix *B* is singular i.e. ***B*** does not have an inverse, or equivalently ***|B|=0***, or ***|A-λI|=0***

Characteristic matrix: The characteristic matrix of the matrix *A* is denoted as [A- λI] and defined as

$$[A-\lambda \mathrm{I}]=\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

Characteristic polynomial and characteristics equation

Characteristic polynomial:

The determinant of the matrix [A-λI] is called the characteristic polynomial

$$|A-\lambda \mathrm{I}|=\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

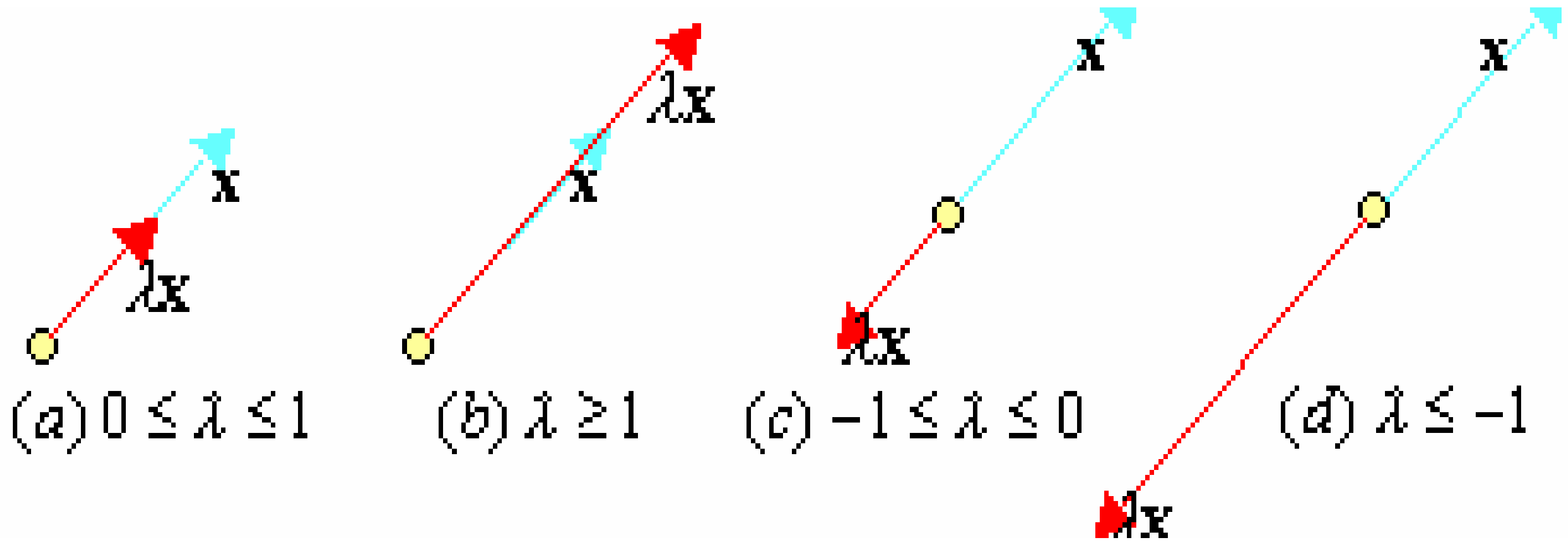
Characteristic equation:

$$|A-\lambda \mathrm{I}|=0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix}=0 \qquad (1)$$

The solutions of equation (1) are called characteristic roots, eigen values, or latent values of the matrix *A*.

Geometrical Interpretation Eigenvalue and Eigen vector

- Linear Operator *Ax* compresses or stretches *x* by a factor '*λ*'
- Reversal of direction in case '*λ*' is negative
- Eigenvalue tantamount to 'Scaling Factor'



Properties of Eigenvalues and Eigenvectors

Property 1: The sum of the eigenvalues of a matrix equals the trace of the matrix.

Property 2: Any square matrix *A* and its transpose *A'* have the same Eigen values

Property 3: The product of eigen values of a matrix is equal to the determinant of the matrix *A*

Property 4: A matrix is singular if and only if it has a zero eigenvalue.

Property 6: If λ is an eigenvalue of A and A is invertible, then $1/\lambda$ is an eigenvalue of matrix A^{-1}.

Property 7: If λ is an eigenvalue of A then $k\lambda$ is an eigenvalue of kA where k is any arbitrary scalar.

Property 8: If λ is an eigenvalue of A then λ^k is an eigenvalue of A^k for any positive integer k.

Example 1

Find the eigenvalues of $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$

Characteristic polynomial $=|A\text{-}\lambda I|$

$$|A - \lambda \mathrm{I}| = \begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix} = (2-\lambda)(-5-\lambda)+12$$

$$|A - \lambda \mathrm{I}| = -(10 - 5\lambda + 2\lambda - \lambda^2) + 12$$
$$|A - \lambda \mathrm{I}| = \lambda^2 + 3\lambda + 2$$

Characteristic equation

$$|A - \lambda \mathrm{I}| = 0$$
$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$
$$\Rightarrow \lambda = -1, -2$$

The two eigenvalues: $-1, -2$

Example 2

Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Sol. The Characteristic equation of the matrix *A*

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^3 = 0$$

Here the Eigen value λ=2 is of multiplicity 3.

Note*:* The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity *k*.

Example 3

Find the eigen values $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ and hence find the eigen values of A^{25} and $A+2I$

Sol. $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(-1-\lambda) = 0$

$$(3-\lambda)(1+\lambda) = 0 \Rightarrow \lambda_1 = -1, \quad \lambda_2 = 3$$

The eigen values of matrix A = -1,3. Hence the eigen values of A^{25} corresponding to eigen values -1 and 3 are (-1) and 3^{25} , respectively

Similarly, the eigen values of $A+2I$ corresponding to the eigen values -1 and 3 are 1 and 5, which are calculated as

$$A + 2I = -1 + 2 = 1$$
$$A + 2I = 3 + 2 = 5$$

Example 4

The matrix A defined as $A=\begin{bmatrix}1 & 2 & -3\\ 0 & 3 & 2\\ 0 & 0 & -2\end{bmatrix}$ Find the eigen values of $3A^3+5A^2-6A+2I$

Sol.

$$|A-\lambda \mathrm{I}|=0 \Rightarrow \begin{vmatrix}1-\lambda & 2 & -3\\ 0 & 3-\lambda & 2\\ 0 & 0 & -2-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)(3-\lambda)(-2-\lambda)=0$$

$$\Rightarrow \lambda=1,3,-2$$

The Eigen values of A^3 = 1,27,-8. The Eigen values of A^2 = 1,9,4

The Eigen values of A = 1,3,-2. The Eigen values of I = 1,1,1

Example 4…

The Eigen values of $3A^3+5A^2-6A+2I$

First Eigen value $=3(1)+5(1)-6(1)+2(1)=4$

Second Eigen value$=3(27)+5(9)-6(3)+2(1)=110$

Third Eigen value$=3(-8)+5(4)-6(-2)+2(1)=10$

Example 5

Show that the eigen values of triangular matrix are just the diagonal elements

Sol. Let us consider the triangular matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{13} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$

$$|A - \lambda \mathrm{I}| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} - \lambda & a_{23} & a_{24} \\ 0 & 0 & a_{13} - \lambda & a_{34} \\ 0 & 0 & 0 & a_{44} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)(a_{44} - \lambda) = 0$$

$$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}, a_{44}$$

are the eigen values of the upper triangular matrix *A*

Eigen vectors

If λ is a characteristic root of *nxn* matrix *A*, then a non-zero vector *X* such that

$$AX = \lambda X$$

is called a characteristic vector or eigen vector of *A* corresponding to the characteristic root λ

A non-zero characteristic vectors or eigen vectors of the matrix *A* corresponding eigen value is the solution of homogeneous linear equations generated from

$$(A - \lambda \mathrm{I})X = 0$$

Note: λ is a characteristic root of a matrix *A* if and only if there exists a non-zero vector *X* such that $AX=\lambda X$

Example 6

Show that the vector (1,1,2) is an eigen vector of the matrix *A* corresponding to the eigen value 2.

Sol. Let ***x***=(1,1,2) and the matrix $A=\begin{bmatrix}3 & 1 & -1\\ 2 & 2 & -1\\ 2 & 2 & 0\end{bmatrix}$. Now

$$Ax=\begin{bmatrix}3 & 1 & -1\\ 2 & 2 & -1\\ 2 & 2 & 0\end{bmatrix}\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}=\begin{bmatrix}2\\ 2\\ 4\end{bmatrix}=2\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}=\lambda x$$

Hence ***x*** is an eigen vector of the matrix *A*

Properties of the eigen vectors

- The eigen vector ***X*** of a matrix is not unique
- If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\lambda_n$ be the distinct eigen values of an *nxn* matrix then corresponding eigen vectors $X_1, X_2, X_3 \ldots\ldots X_n$ form a linearly independent set
- If two are more eigen values are equal it may or may not be possible to get linearly independent eigen vectors corresponding to the equal roots
- Two eigen vectors are called orthogonal vectors if $X_1^T X_2 = 0$
- Eigen vectors of a symmetric matrix corresponding to different eigen values are orthogonal

Example 7

Find the eigen values and eigen vectors of matrix

The characteristic equation of matrix *A*

$$A=\begin{bmatrix}3 & 1 & 4\\ 0 & 2 & 6\\ 0 & 0 & 5\end{bmatrix}$$

$$|A-\lambda I|=\begin{bmatrix}3-\lambda & 1 & 4\\ 0 & 2-\lambda & 6\\ 0 & 0 & 5-\lambda\end{bmatrix}=0\Rightarrow(3-\lambda)(2-\lambda)(5-\lambda)=0$$

The eigen values of the given matrix are λ=2,3,5.

The eigen vector of the matrix *A* corresponding to the eigen value is given by the non-zero solution of the equation

*(A-λI)**x**=0*

$$\begin{bmatrix}3-\lambda & 1 & 4\\ 0 & 2-\lambda & 6\\ 0 & 0 & 5-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix} \qquad (1)$$

Example 7...

When $\lambda=2$, the eigen vector corresponding to the eigen value

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by Gauss elimination method

$$x_1 + x_2 + 4x_3 = 0$$
$$x_3 = 0$$

Let $x_2 = k \; then \;\; x_1 = -k$

$$X_1 = \begin{bmatrix} -k \\ k \\ 0 \end{bmatrix} = k\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Example 7...

When λ=3, the eigen vector corresponding to the eigen value

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by Gauss elimination method

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 + 4x_3 = 0$$
$$x_3 = 0$$

$$x_2 = 0 \; then \;\; x_1 = k$$

$$X_2 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example 7...

When $\lambda=5$, the eigen vector corresponding to the eigen value

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -2x_1 + x_2 + 4x_3 &= 0 \\ -3x_2 + 6x_3 &= 0 \end{aligned}$$

$$x_3 = k \ \ then \ \ x_2 = 2k \ \ and \ \ x_1 = 3k$$

$$X_3 = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$X = [X_1, X_2, X_3]$ Where $X_1 = k\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, X_2 = k\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_3 = k\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Example 8

Find the eigen values and eigen vectors of matrix $A=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation of matrix *A*

$$|A-\lambda \mathrm{I}|=0 \Rightarrow \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix}=0 \Rightarrow (\lambda-1)^2(\lambda-4)=0$$

$$\Rightarrow \lambda=1,1,4$$

The eigen values are 1,1,4.

Eigen vector corresponding to eigen value λ=1 is

$$X_1=\begin{bmatrix} k_1 \\ k_2 \\ k_2-k_1 \end{bmatrix}=k_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}+k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to eigen value λ=4 is $X_2=k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Note: The matrix *A* has only two linearly independent eigen vectors

Example 9

Find the eigen values and eigen vectors of matrix $A=\begin{bmatrix}-3 & -7 & -5\\ 2 & 4 & 3\\ 1 & 2 & 2\end{bmatrix}$

The characteristic equation of matrix *A*

$$|A-\lambda I|=0 \Rightarrow \begin{bmatrix}-3-\lambda & -7 & -5\\ 2 & 4-\lambda & 3\\ 1 & 2 & 2-\lambda\end{bmatrix}=0$$

$$\Rightarrow (\lambda-1)^3=0$$
$$\Rightarrow \lambda=1,1,1$$

The given matrix having only one linearly independent eigen vector corresponding to the eigen value λ=1. Then the eigen vector corresponding to the eigen value λ=1

$$\begin{bmatrix}-3-1 & -7 & -5\\ 2 & 4-1 & 3\\ 1 & 2 & 2-1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=0$$

Example 9...

$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Solve by Gauss elimination method

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + 2x_2 + x_3 = 0$$
$$x_2 - x_3 = 0$$

$$Let\ \ x_3 = k,\ then\ \ x_2 = k,\ \ x_1 = -3k$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Note: The matrix *A* has only one linearly independent eigen vector

Example 10

Find the eigen values and eigen vectors of matrix $A=\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

The characteristic equation of matrix *A*

$$|A-\lambda \mathrm{I}|=0 \Rightarrow \begin{bmatrix} -\lambda & 2 \\ -2 & -\lambda \end{bmatrix}=0$$

$$\Rightarrow \lambda^2+4=0$$
$$\Rightarrow \lambda=\pm i2$$

Here the eigen values are complex number and there are i2,-i2.

Then the eigen vector corresponding to the eigen value *λ*=i2

$$X_1=k\begin{bmatrix} 1 \\ i \end{bmatrix}$$

Then the eigen vector corresponding to the eigen value *λ*=i2

$$X_2=k\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Algebraic multiplicity

Algebraic multiplicity of an eigen value is the number of times of repetition of an eigen value

- In the example 1, the Algebraic multiplicity of eigen values λ=2,3,5 is 1 for all eigen values
- In the example 2, the algebraic multiplicity of eigen value λ=2 is 2 while the algebraic multiplicity of eigen value λ=3 is 1
- In the example 3, the algebraic multiplicity of eigen value λ=1 is 3

Similar matrices

Let A and B are square matrix of order n. Then B is said to be similar of A if there exists a non-singular matrix P such that

$$B = P^{-1}AP$$

- Similar matrices have the same determinant
- Similar matrices have the same characteristic equation and hence same eigenvalues
- If X is eigen vector of A corresponding to the eigenvalue λ, then $P^{-1}X$ is an eigen vector of B corresponding to the eigenvalue λ
- If A is similar to a diagonal matrix D, then the diagonal elements of D are the eigenvalues of A

Example 11

Examine whether *A* is similar to *B*, where

$$A=\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} \quad and \quad B=\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

Sol. Let $P=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

We shall determine *a, b, c* and *d* such that *PA=BP* and then check whether *P* is non-singular

$$PA=BP \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix}=\begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}$$

Example 11...

$$5a-2b=a+2c \qquad or \quad 4a-2b-2c=0$$

$$5a=b+2d \qquad or \quad 5a-b-2d=0$$

$$5c-2d=-3a+4c \qquad or \quad 3a+c-2d=0$$

$$5c=-3b+4d \qquad or \quad 3b+5c-4d=0$$

A solution of this system of linear equations is *a*=1, *b*=1, *c*=1, *d*=2.

Therefore, we get $P=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \ and \ |P| \neq 0$ which is non-singular. Hence, the matrices *A* and *B* are similar.

Example 12

Examine whether *A* is similar to *B*, where $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ *and* $B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$

Sol. Let $P=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. We shall determine *a, b, c* and *d* such that *PA=BP* and then check whether P is non-singular

$$PA=BP \Rightarrow \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}a+c & b+d\\ c & d\end{bmatrix}$$

$$\begin{aligned} a &= a+c \qquad or \quad c=0\\ b &= b+d \qquad or \quad d=0 \end{aligned}$$

Therefore, we get $P=\begin{bmatrix}a & b\\ 0 & 0\end{bmatrix}$ *and* $|P|=0$ which is singular.
Hence, the matrices A and B are not similar.

Diagonalizable Matrix

Definition: A matrix *A* is said to be diagonalizable if it is similar to a diagonal matrix.

Thus a matrix *A* is diagonalizable if there exists an invertible matrix P such that

$$D = P^{-1}AP \quad or \quad AP = PD \qquad (1)$$

Where *D* is diagonal matrix and *P* is model matrix. Matrix *B* is then said to similar to *A*.

From equation (1), we can obtain

$$A = PDP^{-1}$$

$$A^2 = PD^2P^{-1}, \ Similarly \ \ A^m = PD^mP^{-1}$$

Procedure for matrix Diagonalisation

Step 1: Find n linearly independent eigenvectors of A, say P_1 , P_2 ,P_3 …… ,P_n corresponding to eigen values λ_1 , λ_2 , λ_3 , ….. λ_n which are not necessary distinct

Step 2: Form the matrix P having P_1 , P_2 ,P_3 …… ,P_n as its column vectors

Step 3: The matrix $P^{-1}AP$ will then be the diagonal matrix D with λ_1 , λ_2 , λ_3 , ….. λ_n as its successive diagonal entries

Example 13

Show that the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ is diagonalizable.

Sol. The characteristic equation of the matrix A is

$$|A - \lambda \mathrm{I}| = 0 \quad \Rightarrow \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow \lambda = 1,2,3$$

The eigen values of the matrix are 1,2,3.

The eigen vector corresponding to eigen value λ =1

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Example 13 …

The eigen vector corresponding to eigen value λ =2

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vector corresponding to eigen value λ =2

$$X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence the model matrix P is given by

$$P = [X_1, X_2, X_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \; and \; P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Then $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$P^{-1}AP = D = diag(1,2,3)$$

Hence the matrix *A* is diagonalizable

Example 14

Examine whether the matrix $A=\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ is diagonalizable.

Sol. The characteristic equation of the matrix *A* is

$$|A-\lambda \mathrm{I}|=0 \;\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)(2-\lambda)^2=0$$

$$\Rightarrow \lambda=1,2,2$$

The eigen values of the matrix are 1,2,2.

The eigen vector corresponding to eigen value λ =1 is $X_1=\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

The eigen vector corresponding to eigen value λ =2

$$[A-2I]X=\begin{bmatrix} 1-2 & 2 & 2 \\ 0 & 2-2 & 1 \\ -1 & 2 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0$$

Example 14...

The eigen vector corresponding to eigen value λ =2

$$\Rightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$X_2 = k \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

By using Gauss elimination method

Example 14…

- The rank of coefficient matrix = 2. the algebraic multiplicity for $\lambda=2$ is 2 and the geometric multiplicity for $\lambda=2$ is 1

- The algebraic multiplicity does not coincide with geometric multiplicity. Therefore, it has only one linearly independent vector. We have another linearly independent vector from the eigen value $\lambda=1$

- Since the matrix has only two linearly independent eigen vectors, the matrix is not diagonalizable.

Example

1. Examine whether the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable.

2. The eigenvectors of a 3x3 matrix A corresponding to the eigen values 1,1,3 are $[1,0,-1]^T$, $[0,1,-1]^T$ and $[1,1,0]^T$ respectively. Find the matrix *A.*

Power of a matrix (by Diagonalisation)

Thus a matrix A is diagonalizable if there exists an invertible matrix P such that

$$D = P^{-1}AP \qquad (1)$$

From equation (1), we can obtain

$$\mathrm{A} = \mathrm{PDP}^{-1}$$

$$\mathrm{A}^2 = \mathrm{PD}^2\mathrm{P}^{-1}$$

$$\text{In general } \mathrm{A}^m = \mathrm{PD}^m\mathrm{P}^{-1}$$

Example 15

Find the matrix *P* which transform the matrix $A=\begin{bmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{bmatrix}$ to diagonal form. Hence find A^4.

The characteristic equation of the matrix *A* is

$$|A-\lambda \mathrm{I}|=0 \Rightarrow \begin{bmatrix}1-\lambda & 0 & -1\\ 1 & 2-\lambda & 1\\ 2 & 2 & 3-\lambda\end{bmatrix}=0$$

$$\lambda^3-6\lambda^2+11\lambda-6=0$$

The eigenvalues of the matrix A are 1,2,3. The corresponding eigenvector are

$$X_1=\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}, X_2=\begin{bmatrix}-2\\ 1\\ 2\end{bmatrix}, X_3=\begin{bmatrix}-1\\ 1\\ 2\end{bmatrix}$$

Example 15...

Then the model matrix P

$$P=\begin{bmatrix}1 & -2 & -1\\ -1 & 1 & 1\\ 0 & 2 & 2\end{bmatrix}, P^{-1}=-\frac{1}{2}\begin{bmatrix}0 & 2 & -1\\ 2 & 2 & 0\\ -2 & -2 & -1\end{bmatrix}$$

Then

$$P^{-1}AP=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}=D$$

Hence matrix A is diagonalizable and

Then $$A^4=PD^4P^{-1}=\begin{bmatrix}1 & -2 & -1\\ -1 & 1 & 1\\ 0 & 2 & 2\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 81\end{bmatrix}\begin{bmatrix}0 & -1 & \frac{1}{2}\\ -1 & -1 & 0\\ 1 & 1 & \frac{1}{2}\end{bmatrix}$$

$$A^4=\begin{bmatrix}-49 & -50 & -40\\ 65 & 66 & 40\\ 130 & 130 & 81\end{bmatrix}$$

Matlab Code

- in-built matlab for eigenvalues and eigenvectors is

```
>> [X, D] = eig(A)
```

Session Summary

- Eigenvalues are the roots of the characteristic equation $\det(A - \lambda I) = 0$
- Eigenvectors are obtained by solving the equation $Ax = \lambda x$
- Determinant of a matrix is the product of its eigenvalues
- If the matrix is non-singular, then all the eigenvalues are non-zero