Lecture 36
Power Series

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Intended learning Outcomes

At the end of this lecture, student will be able to:

- Analyze various power series
- Evaluate the radius of convergence of power series

Topics

- Power series
- Convergence of power series
- Divergence of power series
- Region of convergence
- Radius of convergence
- Properties of power series

Power Series

A power series in powers of $z - z_0$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots$$

where z is a complex variable, a_0, a_1, a_2, … are complex constants (may be real) called the coefficients of the power series and z_0 is a complex constant called the centre of the series

If $z_0 = 0$ then the power series takes the form

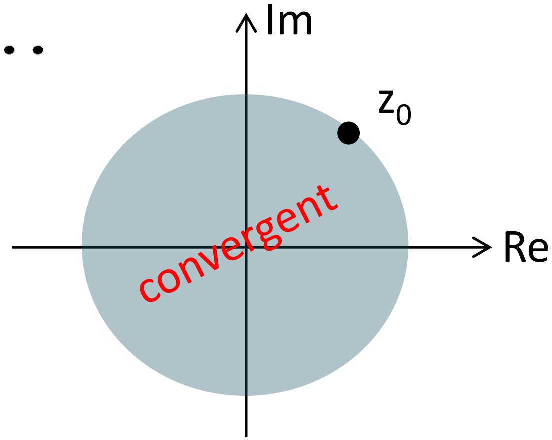
$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots$$

Convergence of Power Series

Let

$$p(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots$$

be a power series, whose coefficients are complex numbers.



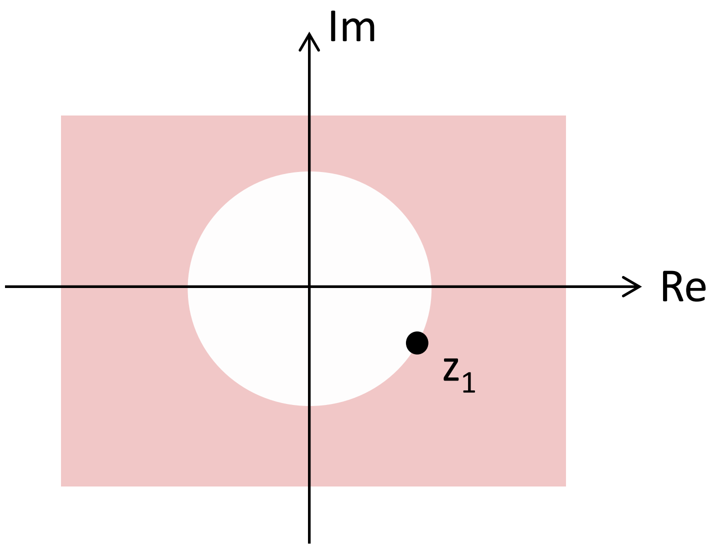
If $p(z)$ converges at $z = z_0$, then $p(z)$ converges absolutely for all z in the circle $|z| < |z_0|$.

Divergence of Power Series

If the power series

$$p(z)=\sum_{n=0}^{\infty}a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots$$

diverges at $z = z_1$, then $p(z)$ diverges whenever $|z| > |z_1|$.



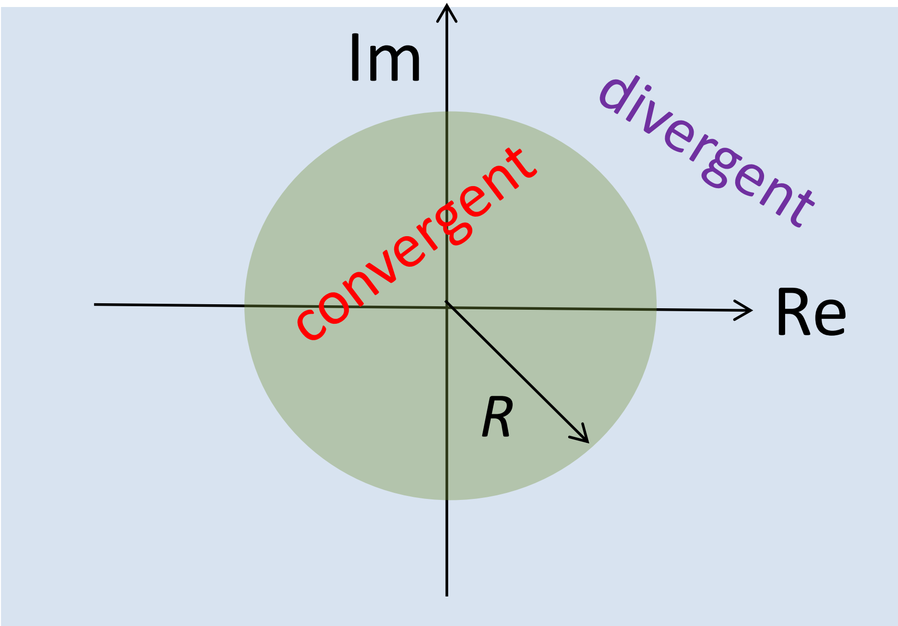
Region of Convergence

For any power series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots$$

there exists a radius *R*, such that

1. the power series converges if $|z - c| < R$,
2. the power series diverges if $|z - c| > R$.



The region of convergence of a power series has the shape of a disc. The radius is called the radius of convergence.

Special Cases

1. Convergence Everywhere

The power series may converges for all values of z then we say it is convergent everywhere.

In this case $R = \infty$. The whole complex plane is the region of convergence.

2. Convergence only at Origin

The power series converges only at the origin $z = 0$.

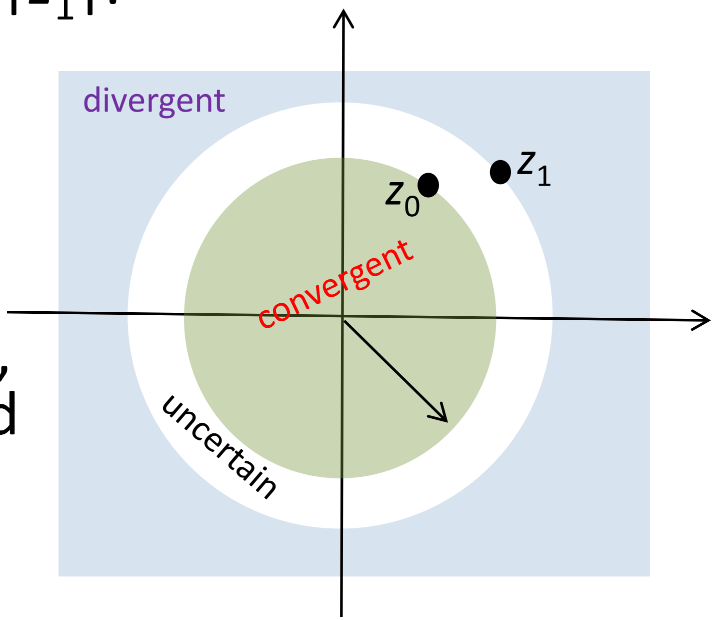
In this case $R = 0$. The region of convergence contains a single point.

3. Existence Uncertain Region

Let the power series converges for some complex number z_0 and diverges for some complex number z_1. Then we must have $|z_0| \leq |z_1|$.

If $|z_0| = |z_1|$, then the results follows immediately. If $|z_0| < |z_1|$, there is an annulus of uncertainty $|z_0| < z < |z_1|$.

Pick a complex number z' such that
$|z'| = (|z_0| + |z_1|) / 2$.
If the power series converges at z', then the area of convergence can be enlarged to $|z| < |z'|$. If the power series diverges at z', then the area of divergence can be enlarged to $|z| > |z'|$. In any case, we shrink the annulus of uncertainty.



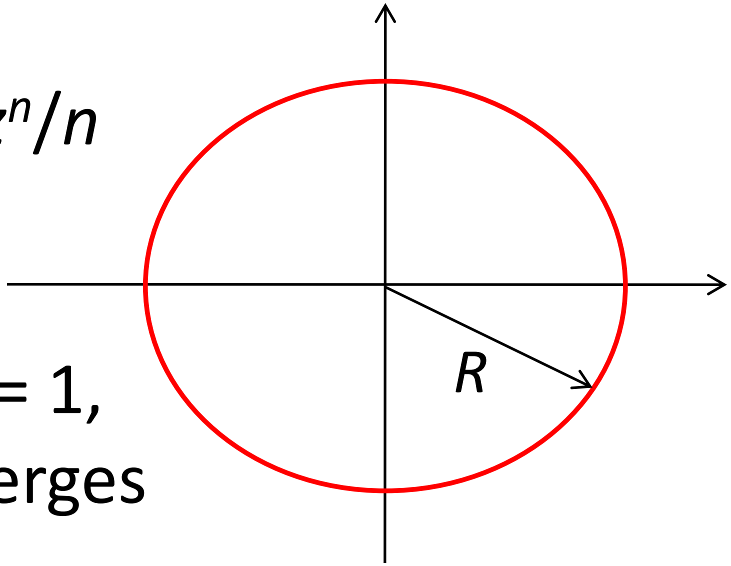
If we repeat this argument infinitely many times, the difference between $|z_0|$ and $|z_1|$ can be made arbitrarily small. This will give the desired radius of convergence.

4. Convergence on the boundary

On the circle of convergence $|z - c| = R$, a power series may or may not converges.

For example, All three series Σz^n, $\Sigma z^n/n$ and $\Sigma z^n/n^2$ have the same radius of convergence $R = 1$. But

1. Σz^n diverges everywhere on $|z| = 1$,
2. $\Sigma z^n/n$ diverges at $z = 1$ and converges at $z = -1$,
3. $\Sigma z^n/n^2$ converges everywhere on $|z| = 1$.



Radius of Convergence in terms of the Coefficients of the Power Series

Define $L = \lim_{n \to \infty} |a_{n+1} / a_n|$ with respect to a power series $\Sigma a_n z^n$ then we have the following situations:

1. If $L = 0$ then $R = \infty$, that is, the power series $\Sigma a_n z^n$ converges for all z.
2. If $L \neq 0$ (hence $L > 0$), then

$$R = 1/L = \lim_{n \to \infty} |a_n / a_{n+1}|$$

3. If L does not exist, i.e., $|a_{n+1} / a_n| \to \infty$ then $R = 0$, which means that the power series $\Sigma a_n z^n$ converges for all $z = 0$.

Three properties for power series

- A power series can be differentiated term by term within the region of convergence.
- A power series can be integrated term by term within the region of convergence
- The order of summing the terms in a power series does not matter. We can re-arrange the order of summation freely for power series.

All of these properties are not true for series in general.

Session Summary

- A **power series** is of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ \dots$$

  where z_0 is the centre of the series. This series converges for $|z-z_0| < R$ and diverges $|z-z_0| > R$, where R is the **radius of convergence**.

- The radius of convergence can be obtained from

$$R = \frac{1}{L} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$