Lecture 27
Conformal Mapping_1

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Intended Learning Outcomes

At the end of this lecture, student will be able

- Find conformal mappings
- Solve application oriented problems using conformal mappings

Topics

- Conformal Mapping

Example-1

- The transformation $w = z^2$

Solution: Consider the transformation $w = z^2$

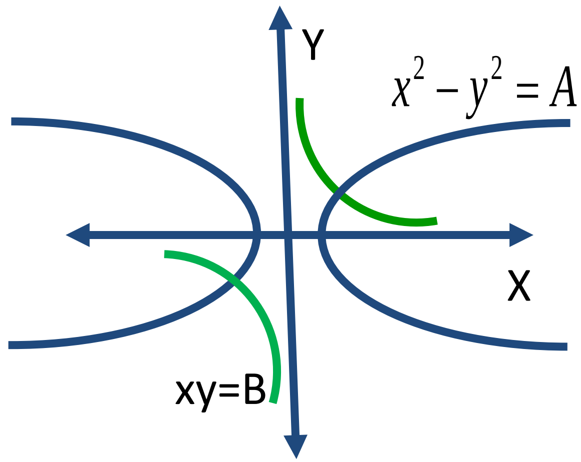
Put z=x+iy and w=u+iv , we have

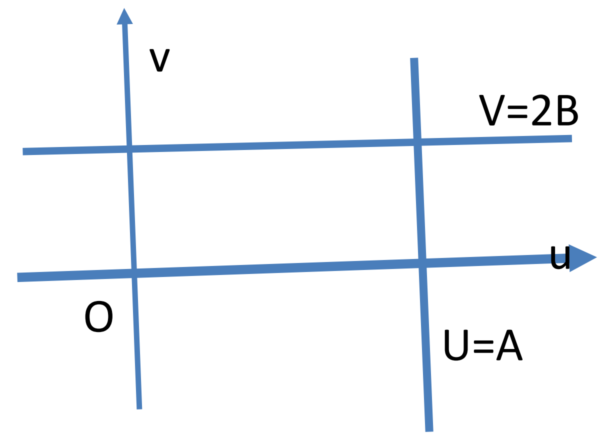
$$(u+iv) = (x+iy)^2 = (x^2 - y^2) + i2xy, \;\; so \; that$$

$$u = x^2 - y^2, \quad v = 2xy$$

We note that u=constant(say A) if $x^2 - y^2 = A$ which represents A rectangular hyperbola, and v=constant (say 2B) if xy =B which is also a rectangular hyperbola

Example-1 (cont.)





Thus rectangular hyperbola $x^2 - y^2 = A$ and xy=B in the z-plane transforms to the straight lines u=A and v=2B in the w-plane

Next , consider a line parallel to the x-axis :x=a

From equation(2), we have $u = a^2 - y^2, \ v = 2ay$, from which we get $v^2 = 4a^2y^2 = -4a^2(u - a^2)$

Example-1 (cont.)

This represents a parabola in the w-plane.

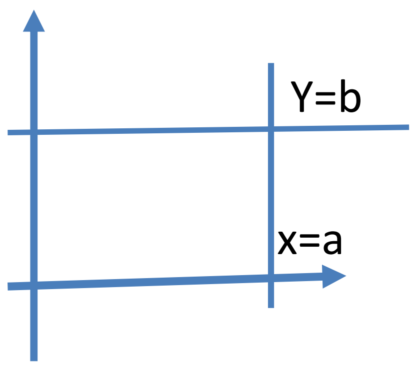
Consider a line parallel to x-axis:y=b ,

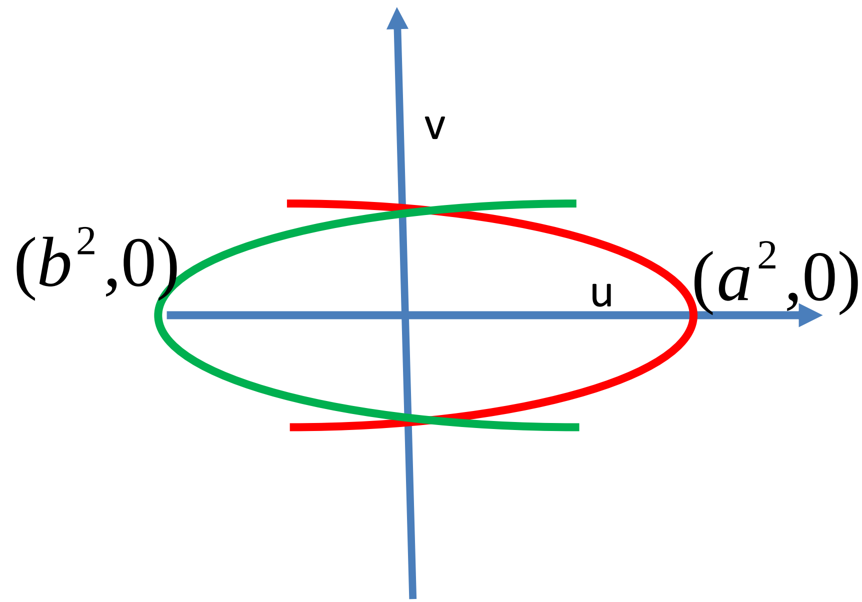
from (2) $u = x^2 - b^2, \quad v = 2xb$ which yield

$$v^2 = 4x^2b^2 = 4b^2(u + b^2)$$

This represents a parabola in the w-plane

Example-1 (cont.)





Thus the transformation $w = z^2$ transforms straight lines parallel to the y-axis to parabolas having the negative u-axis as their axis and the straight lines parallel to the x-axis to parabolas having the positive u-axis as their axis

Example-2

Show the transformation $w = z^2$ transforms the circle $|z-a| = c$ to a cardioid or limacon.

On the circle $|z-a| = c$ we have $z - a = c\,e^{i\theta}$. Therefore, the corresponding w is

$$w = z^2 = (a + c\,e^{i\theta})^2 = a^2 + c^2\,e^{2i\theta} + 2ac\,e^{i\theta}$$

Which gives

$$w - a^2 + c^2 = c^2 e^{2i\theta} + 2ac\ e^{i\theta} + c^2 = c\,e^{i\theta}(c\,e^{-i\theta} + c\,e^{i\theta} + 2a)$$

$$= 2c\ e^{i\theta}(c\cos\theta + a)$$

Example-2(cont.)

Setting $w - a^2 + c^2 = R\, e^{i\phi}$, we find that

$$R = 2c\,(a + c\cos\phi), \quad \phi = \theta, \qquad \text{or equivalently}$$

$$R = 2c\,(a + c\cos\phi)$$

This is the polar equation of the image of the circle $|z - a| = c$

If a=c , this polar equation becomes $R = 2c^2\,(1 + \cos\phi)$ which represents a cardioid in the w-plane. If $a \neq c$, it represents a curve called the Limacon

Session Summary

- A complex function w = $f(z)$ gives a **mapping of its domain** in the complex z-plane onto its **range of values** in the complex w-plane
- *If* $f(z)$ is analytic, this mapping is **conformal,** that is**, angle-preserving**, i.e., The angle between any two intersecting curves and the corresponding angle between their image curves are the same.