Lecture 18
Maximum and Minimum values of Functions_II

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Define and explain the significance of maxima and minima
- Determine the maximum and minimum of a function at different points

Topics

- Maximum and minimum of functions
- Examples

Maximum and Minimum Values

- In last session, we illustrated how to use partial derivatives to locate maxima and minima of functions of two variables.

- Look at the hills and valleys in the graph of f shown in the below Figure 1.

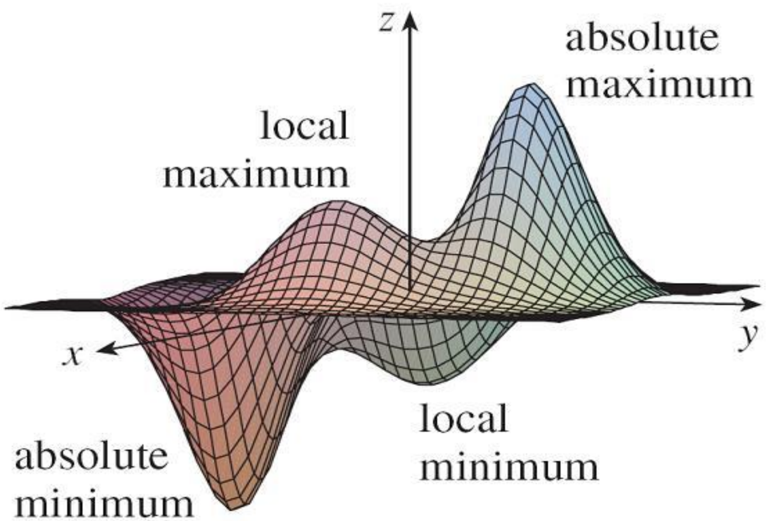

Figure 1

Test for Maximum and Minimum Values

- The following test, is analogous to the Second Derivative Test for functions of one variable

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

- In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b)

Example 1

- Find the extreme value for the function $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

- Solution: Consider the first order partial derivatives, to find the critical points

$$f_x(x, y) = 2x - 2 \text{ and } f_y(x, y) = 2y - 6$$

- These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is $(1, 3)$

Example 1 cont….

- Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y.

- Therefore $f(1,3) = 4$ is a local minimum, and in fact it is the absolute minimum of f.

- This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex $(1,3,4)$ shown in below Figure 2.

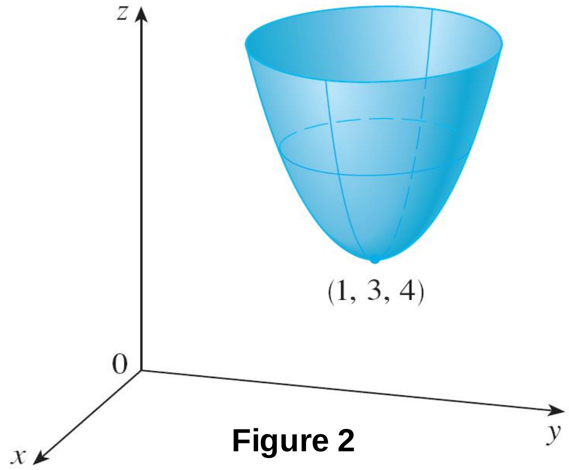

Figure 2

$z = x^2 + y^2 - 2x - 6y + 14$

Example 2

- Find the extreme value for the function $f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x$

- Solution: Consider the first order partial derivatives, to find the critical points

$$f_{x}(x, y) = 6x^2 + 6y^2 - 150 \quad \text{and} \quad f_{y}(x, y) = 12xy - 9y^2$$

- For stationary points we need $f_x = 0$ and $f_y = 0$, i. e., $x^2 + y^2 = 25$ and $y(4x - 3y) = 0$
- The second of these equations implies **either** that $y = 0$ or that

Example 2 cont....

- If $y = 0$ then the first equation implies that $x^2 = 25$ so that $x = \pm 5$ giving $(5, 0)$ and $(-5, 0)$ as stationary points

- If $4x = 3y$ then $x = 3y/4$ and so the first equation becomes $y^2 = 16$, so that $y = \pm 4$ we have $f\ (x, y) \geq 4$ for all values of x and y

- So we have two further stationary points $(3, 4)$ and $(-3, -4)$

- Thus in total there are four stationary points $(5, 0), (-5, 0), (3, 4)$ and $(-3, -4)$

Example 2 cont....

- Lets start with $(5, 0)$. For this stationary point, $f_{xx}f_{yy} - f^2{}_{xy} = 60^2 > 0$ so it is either a max or a min. But $f_{xx} = 60 > 0$ and $f_{yy} = 60 > 0$. Hence $(5, 0)$ is a minimum

- Now deal with $(-5, 0)$. For this stationary point, $f_{xx}f_{yy} - f^2{}_{xy} = (-60)^2 > 0$ so it is either a max or a min. But $f_{xx} = -60 < 0$ and $f_{yy} = -60 < 0$. Hence (–5, 0) is a maximum

- Now deal with (3, 4). For this stationary point, $f_{xx}f_{yy} - f^2{}_{xy} = -3600 < 0$ so (3, 4) is a saddle

Examples

- Find and classify the critical points of the function $f(x, y) = x^3 + 3y - y^3 - 3x$
- Solution: Critical points are $(1, 1), (1, -1), (-1, 1), (-1, -1)$

- Find the maximum value of the function $x^3 y^2 (1 - x - y) = 0$ for $x, y > 0$

Summary

- Suppose the second partial derivatives of *f* are continuous $f_x(a,b)$ and f_y (a, b)=0 [that is, (a, b) is a critical point of f].

(a)If D>0 and f_{xx} (a, b)>0 , then f (a, b) is a local minimum
(b)If D>0 and f_{xx} (a, b)<0, then f (a, b) is a local maximum
(c) If D<0, then f(a, b) is not an extreme value
(d) If D=0, then at (a, b), the case is doubtful and needs further investigation

Where

$$D = AC - B^2, \ A = f_{xx}, \ B = f_{xy}, \quad C = f_{yy}$$