Lecture 10
Gauss-Siedel Method

Dr. Mahesha Narayana

Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Illustrate the steps involved in Gauss-Jacobi method and Gauss-Seidel method
- Verify the strictly diagonally dominant for the system
- Solve the linear system by Gauss-Jacobi method and Gauss-Seidel method
- Solve systems of nonlinear equations with successive iterations

Topics

- Jacobi iteration method
- Gauss-Siedel method
- Diagonally Dominant Coefficient Matrix
- MATLAB Program

Motivation

Importance of the Gauss-Jacobi and Gauss-Siedel method to solve system of linear equations because

- Iterative or approximate methods provide an alternative to the elimination methods. The Gauss-Seidel method is the most commonly used iterative method
- Gauss-Siedel method is an iterative method to solve linear and non-linear system of equation
- The Gauss-Seidel Method allows the user to control round-off error
- Elimination methods such as Gaussian Elimination and LU Decomposition are prone to round-off error

Jacobi Iteration method

Definition: Jacobi iteration method is a first iteration method *that* used to solve linear system of equations. We assume that the diagonal elements a of the matrix A, are non-zero. We write the system of linear equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

Jacobi Iteration method…

The Guass-Jacobi iteration method or simply Jacobi iteration method is defined as

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \ldots\ldots.a_{1n}x_n^{(k)}\right)\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(k)} + a_{23}x_3^{(k)} + \ldots\ldots.a_{2n}x_n^{(k)}\right)\right]$$

.

.

.

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)}\ldots\ldots\ldots\ldots a_{n,n-1}x_{n-1}^{(k)}\right)\right]$$

Starting with initial guess $x^{(0)} = \left[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots\ldots\ldots\ldots x_n^{(0)}\right]^T$, generate the sequence of iterates $x^{(1)}, x^{(2)}, x^{(3)}, \ldots\ldots\ldots, x^{(n)}$ which in the limit converges to the exact solution **x**.

Jacobi Iteration method…

We can also write the method in matrix form. Let

$$A=L+D+U$$

Where L is strictly lower triangular matrix, D is diagonal matrix, and U is strictly upper triangular matrix of the matrix A. For example, for a 3x3 system, we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} = L + D + U$$

We write the system of equation Ax=b as

$$(L+D+U)x = b \Rightarrow Dx = -(L+U)x + b$$

$$Dx^{(k+1)} = -(L+U)x^{(k)} + b$$

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$$

Example 1

Perform three iterations of the Gauss-Jacobi iteration method for solving the system of equations

$$\begin{bmatrix} 6 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 1 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 8 \end{bmatrix}$$

Take the initial approximation as $x^{(0)} = [1.3, -1.9, 0.8]^T$. Compare with the exact solution $x_1 = 1, x_2 = -2, x_3 = 1$

We write the Jacobi iteration method is as

$$x_1^{(k+1)} = \frac{1}{6}\left[6 - x_2^{(k)} - 2x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{4}\left[-4 - x_1^{(k)} - 3x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{8}\left[8 - 2x_1^{(k)} - x_2^{(k)}\right], \; k = 0,1,2,3....$$

Example 1…

Starting with $x_1^{(0)} = 1.3, x_2^{(0)} = -1.9, x_3^{(0)} = 0.8$, we get the following results

K=0

$$x_1^{(1)} = \frac{1}{6}\left[6 - x_2^{(0)} - 2x_3^{(0)}\right] = 1.0500,\quad x_2^{(1)} = \frac{1}{4}\left[-4 - x_1^{(0)} - 3x_3^{(0)}\right] = -1.9250$$

$$x_3^{(1)} = \frac{1}{8}\left[8 - 2x_1^{(0)} - x_2^{(0)}\right] = 0.9125$$

K=1

$$x_1^{(2)} = \frac{1}{6}\left[6 - x_2^{(1)} - 2x_3^{(1)}\right] = 1.0167,\quad x_2^{(2)} = \frac{1}{4}\left[-4 - x_1^{(1)} - 3x_3^{(1)}\right] = -1.9469$$

$$x_3^{(2)} = \frac{1}{8}\left[8 - 2x_1^{(1)} - x_2^{(1)}\right] = 0.9781$$

K=2

$$x_1^{(3)} = \frac{1}{6}\left[6 - x_2^{(2)} - 2x_3^{(2)}\right] = 1.9984,\quad x_2^{(3)} = \frac{1}{4}\left[-4 - x_1^{(2)} - 3x_3^{(2)}\right] = -1.9878$$

$$x_3^{(3)} = \frac{1}{8}\left[8 - 2x_1^{(2)} - x_2^{(2)}\right] = 0.9892$$

Comparing with the exact solution, the errors in magnitude are 0.0016, 0.0122 and 0.0108. The maximum absolute error is 0.0122.

Gauss-Seidel Method

Algorithms

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero. Rewrite each equation solving for the corresponding unknown

Starting with initial guess $x^{(0)} = \left[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots\ldots\ldots\ldots x_n^{(0)}\right]^T$, generate the sequence of iterates $x^{(1)}, x^{(2)}, x^{(3)}, \ldots\ldots\ldots\ldots, x^{(n)}$ which in the limit converges to the exact solution **x**.

Gauss-Seidel Method…

If we use the latest available values of each variable then the method is called Gauss-Siedel iteration method

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \ldots\ldots\ldots a_{1n}x_n^{(k)}\right)\right]$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \ldots\ldots\ldots a_{2n}x_n^{(k)}\right)\right]$$

.

.

.

$$x_n^{(k+1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} \ldots\ldots\ldots\ldots a_{n,n-1}x_{n-1}^{(k+1)}\right)\right]$$

Starting with initial approximation $x^{(0)} = \left[x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \ldots,\ldots\ldots\ldots x_n^{(0)}\right]^T$ generate the sequence of iterates $x^{(1)}, x^{(2)}, x^{(3)}, \ldots\ldots\ldots\ldots x^{(n)}$, which in the limit converges to the exact solution **x**.

Calculate the Absolute Relative Approximate Error

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a pre-specified tolerance for all unknowns.

Example 3

Perform three iterations of the Gauss-Siedel iteration method for solving the system of equations

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 2 \\ 5 & 4 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 11 \end{bmatrix}$$

Take the initial approximation as $x_i^{(0)} = \dfrac{b_i}{a_{ii}}, i = 1,2,3$. Compare with the exact solution $x = [1,-1,1]^T$

We write the Jacobi iteration method as

$$x_1^{(k+1)} = \frac{1}{6}\left[6 - 2x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{5}\left[-3 - 2x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{10}\left[11 - 5x_1^{(k+1)} - 4x_2^{(k+1)}\right],\ k = 0,1,2,3....$$

Example 3...

The initial approximation is $x_1^{(0)} = 1.5, x_2^{(0)} = -0.6, x_3^{(0)} = 1.1$. For k=0,1,2, we have following results

$$x_1^{(1)} = \frac{1}{6}\left[6 - 2x_3^{(0)}\right] = 0.95, \quad x_2^{(1)} = \frac{1}{5}\left[-3 - 2x_3^{(0)}\right] = -1.04$$

$$x_3^{(1)} = \frac{1}{10}\left[11 - 5x_1^{(1)} - 4x_2^{(1)}\right] = 1.0451$$

$$x_1^{(2)} = \frac{1}{6}\left[6 - 2x_3^{(1)}\right] = 0.9795, \quad x_2^{(2)} = \frac{1}{5}\left[-3 - 2x_3^{(0)}\right] = -1.0164$$

$$x_3^{(2)} = \frac{1}{10}\left[11 - 5x_1^{(2)} - 4x_2^{(2)}\right] = 1.0168$$

$$x_1^{(3)} = \frac{1}{6}\left[6 - 2x_3^{(0)}\right] = 0.9916, \quad x_2^{(3)} = \frac{1}{5}\left[-3 - 2x_3^{(2)}\right] = -1.0067$$

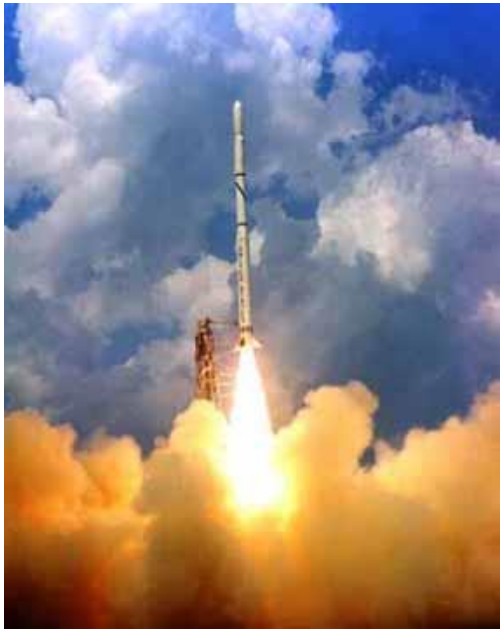
$$x_3^{(3)} = \frac{1}{10}\left[11 - 5x_1^{(3)} - 4x_2^{(3)}\right] = 1.0069$$

Comparing with the exact solution, the errors in magnitude are 0.0084, 0.0067 and 0.0069. The maximum absolute error is 0.0084.

Example 4

Table 1 Velocity vs. Time data.

Time,	Velocity
5	106.8
8	177.2
12	279.2



The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 \text{ , } 5 \le \mathrm{t} \le 12.$$

Example 4...

Using a Matrix template of the form

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

The system of equations becomes

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Initial Guess: assume an initial guess of

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Example 4...

Rewriting each equation

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_2 = \frac{177.2 - 64a_1 - a_3}{8}$$

$$a_3 = \frac{279.2 - 144a_1 - 12a_2}{1}$$

Example 4…

Applying the initial guess and solving for a_i

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Initial Guess

$$a_1 = \frac{106.8 - 5(2) - (5)}{25} = 3.6720$$

$$a_2 = \frac{177.2 - 64(3.6720) - (5)}{8} = -7.8510$$

$$a_3 = \frac{279.2 - 144(3.6720) - 12(-7.8510)}{1} = -155.36$$

When solving for a_2, how many of the initial guess values were used?

Example 4...

Finding the absolute relative approximate error

$$\left|\in_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

At the end of the first iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

$$\left|\in_a\right|_1 = \left|\frac{3.6720 - 1.0000}{3.6720}\right| x100 = 72.76\%$$

$$\left|\in_a\right|_2 = \left|\frac{-7.8510 - 2.0000}{-7.8510}\right| x100 = 125.47\%$$

$$\left|\in_a\right|_3 = \left|\frac{-155.36 - 5.0000}{-155.36}\right| x100 = 103.22\%$$

The maximum absolute relative approximate error is 125.47%

Example 4...

Iteration 2

Using

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3.6720 \\ -7.8510 \\ -155.36 \end{bmatrix}$$

from iteration 1

the values of a_i are found:

$$a_1 = \frac{106.8 - 5(-7.8510) - 155.36}{25} = 12.056$$

$$a_2 = \frac{177.2 - 64(12.056) - 155.36}{8} = -54.882$$

$$a_3 = \frac{279.2 - 144(12.056) - 12(-54.882)}{1} = -798.34$$

Example 4…

Finding the absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{12.056 - 3.6720}{12.056}\right| x100 = 69.543\%$$

$$\left|\in_a\right|_2 = \left|\frac{-54.882 - (-7.8510)}{-54.882}\right| x100 = 85.695\%$$

$$\left|\in_a\right|_3 = \left|\frac{-798.34 - (-155.36)}{-798.34}\right| x100 = 80.540\%$$

At the end of the second iteration

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 12.056 \\ -54.882 \\ -798.54 \end{bmatrix}$$

The maximum absolute relative approximate error is 85.695%

Example 4...

Repeating more iterations, the following values are obtained

Iteration	a_1		a_2		a_3	
1	3.6720	72.767	−7.8510	125.47	−155.36	103.22
2	12.056	69.543	−54.882	85.695	−798.34	80.540
3	47.182	74.447	−255.51	78.521	−3448.9	76.852
4	193.33	75.595	−1093.4	76.632	−14440	76.116
5	800.53	75.850	−4577.2	76.112	−60072	75.963
6	3322.6	75.906	−19049	75.972	−249580	75.931

Notice – The relative errors are not decreasing at any significant rate.

Also, the solution is not converging to the true solution of given linear equations

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.690 \\ 1.0857 \end{bmatrix}$$

Gauss-Seidel Method: Pitfall

What went wrong?

Even though done correctly, the answer is not converging to the correct answer

This example illustrates a pitfall of the Gauss-Siedel method: not all systems of equations will converge.

Is there a fix?

One class of system of equations always converges: One with a *diagonally dominant* coefficient matrix.

Diagonally dominant: [A] in [A] [X] = [C] is diagonally dominant if:

$$\left|a_{\text{ii}}\right| \geq \sum_{\substack{j=1 \\ j\neq \text{i}}}^{n}\left|a_{ij}\right|$$ for all 'i' and $$\left|a_{ii}\right| > \sum_{\substack{j=1 \\ j\neq i}}^{n}\left|a_{ij}\right|$$ for at least one 'i'

Diagonally Dominant Coefficient Matrix

Diagonally dominant: The coefficient on the diagonal must be at least equal to the sum of the other coefficients in that row and at least one row with a diagonal coefficient greater than the sum of the other coefficients in that row.

Which coefficient matrix is diagonally dominant?

$$[A]=\begin{bmatrix} 2 & 5.81 & 34 \\ 45 & 43 & 1 \\ 123 & 16 & 1 \end{bmatrix} \qquad [B]=\begin{bmatrix} 124 & 34 & 56 \\ 23 & 53 & 5 \\ 96 & 34 & 129 \end{bmatrix}$$

Most physical systems do result in simultaneous linear equations that have diagonally dominant coefficient matrices.

Examples

Which of the following systems of linear equations has a strictly diagonally dominant coefficient matrix?

1. $$\begin{aligned} 3x_1 - x_2 &= -4 \\ 2x_1 + 5x_2 &= 2 \end{aligned}$$

2. $$\begin{aligned} 4x_1 + 2x_2 - x_3 &= -1 \\ x_1 \qquad + 2x_3 &= -4 \\ 3x_1 - 5x_2 + x_3 &= 3 \end{aligned}$$

3. $$\begin{aligned} x_1 - 5x_2 &= -4 \\ 7x_1 - x_2 &= 6 \end{aligned}$$

What about example 3?

$R_1 \Leftrightarrow R_2$ Now check

$$\begin{aligned} 7x_1 - x_2 &= 6 \\ x_1 - 5x_2 &= -4 \end{aligned}$$

Example 5

Given the system of equations

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

The coefficient matrix is:

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Will the solution converge using the Gauss-Siedel method?

Example 5…

Checking if the coefficient matrix is diagonally dominant

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

The inequalities are all true and at least one row is *strictly* greater than:

Therefore, the solution should converge using the Gauss-Siedel Method

Example 5...

Rewriting each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Example 5...

The absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.50000 - 1.0000}{0.50000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923 - 1.0000}{3.0923}\right| \times 100 = 67.662\%$$

The maximum absolute relative error after the first iteration is 100%

Example 5...

After Iteration 1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

Substituting the x values into the equations

$$x_1 = \frac{1 - 3(4.9000) + 5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28 - (0.14679) - 3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76 - 3(0.14679) - 7(4.900)}{13} = 3.8118$$

After Iteration 2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

Example 5...

Iteration 2 absolute relative approximate error

$$\left|\in_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.61\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.889\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.874\%$$

The maximum absolute relative error after the first iteration is 240.61%

This is much larger than the maximum absolute relative error obtained in iteration #1. Is this a problem?

Example 5…

Repeating more iterations, the following values are obtained

Iteration	a_1	$\left|\in_a\right|_1 \%$	a_2	$\left|\in_a\right|_2 \%$	a_3	$\left|\in_a\right|_3 \%$
1	0.50000	100.00	4.9000	100.00	3.0923	67.662
2	0.14679	240.61	3.7153	31.889	3.8118	18.876
3	0.74275	80.236	3.1644	17.408	3.9708	4.0042
4	0.94675	21.546	3.0281	4.4996	3.9971	0.65772
5	0.99177	4.5391	3.0034	0.82499	4.0001	0.074383
6	0.99919	0.74307	3.0001	0.10856	4.0001	0.00101

The solution obtained $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$ is close to the exact solution of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Example 6

Given the system of equations

$$3x_1 + 7x_2 + 13x_3 = 76$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$12x_1 + 3x_2 - 5x_3 = 1$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Rewriting the equations

$$x_1 = \frac{76 - 7x_2 - 13x_3}{3}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{1 - 12x_1 - 3x_2}{-5}$$

Example 6...

Conducting six iterations, the following values are obtained

Iteration	a_1	$\left\|\in_a\right\|_1 \%$	A_2	$\left\|\in_a\right\|_2 \%$	a_3	$\left\|\in_a\right\|_3 \%$
1	21.000	95.238	0.80000	100.00	50.680	98.027
2	-196.15	110.71	14.421	94.453	-462.30	110.96
3	-1995.0	109.83	-116.02	112.43	4718.1	109.80
4	-20149	109.90	1204.6	109.63	-47636	109.90
5	2.0364×10^5	109.89	-12140	109.92	4.8144×10^5	109.89
6	-2.0579×10^5	109.89	1.2272×10^5	109.89	-4.8653×10^6	109.89

The values are not converging.

Does this mean that the Gauss-Seidel method cannot be used?

Example 6…

The Gauss-Seidel Method can still be used

The coefficient matrix is not diagonally dominant

$$[A]=\begin{bmatrix} 3 & 7 & 13 \\ 1 & 5 & 3 \\ 12 & 3 & -5 \end{bmatrix}$$

But this is the same set of equations used in example #2, which did converge.

$$[A]=\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

If a system of linear equations is not diagonally dominant, check to see if rearranging the equations can form a diagonally dominant matrix.

Gauss-Seidel Method

Not every system of equations can be rearranged to have a diagonally dominant coefficient matrix.

Observe the set of equations

$$x_1 + x_2 + x_3 = 3$$

$$2x_1 + 3x_2 + 4x_3 = 9$$

$$x_1 + 7x_2 + x_3 = 9$$

Which equation(s) prevents this set of equation from having a diagonally dominant coefficient matrix?

Matlab Code

```
function [GST, MaxIter ] = Gauss_Seidal(A,C)
n = length(C);
   X = zeros(n,1);
   Error_eval = ones(n,1);

   %% Check if the matrix A is diagonally dominant
   for i = 1:n
      j = 1:n;
      j(i) = [];
      B = abs(A(i,j));
      Check(i) = abs(A(i,i)) - sum(B); % Is the diagonal value greater than the remaining
row values combined?
      if Check(i) < 0
         fprintf('The matrix is not strictly diagonally dominant at row %2i\n\n',i)
      end
   end
```

Matlab Code

```
%% Start the Iterative method
   iteration = 0;
   while max(Error_eval) > 0.001
      iteration = iteration + 1;
      Z = X;  % save current values to calculate error later
      for i = 1:n
         j = 1:n; % define an array of the coefficients' elements
         j(i) = [];  % eliminate the unknow's coefficient from the remaining coefficients
         Xtemp = X;  % copy the unknows to a new variable
         Xtemp(i) = [];  % eliminate the unknown under question from the set of values
         X(i) = (C(i) - sum(A(i,j) * Xtemp)) / A(i,i);
      end
      Xsolution(:,iteration) = X;
      Error_eval = sqrt((X - Z).^2);
   end

   %% Display Results
   GST = [1:iteration;Xsolution]';
   MaTrIx = [A X C];
end
```

Session Summary

- Gauss Jacobi and Gauss-Seidel method is an iterative procedure

- This method is suitable for physical applications where we can make an initial guess

- If the matrix is strictly diagonally dominant, then both the gauss-Jacobi and Gauss-Siedel iterations methods converge for any initial approximations

- Gauss-Seidel iteration converges more rapidly than the Jacobi iteration since it uses the latest updates

- The rearrangement of the equations also plays important role for the convergent of the solutions