Lecture 2
Rolle's Mean Value Theorem

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

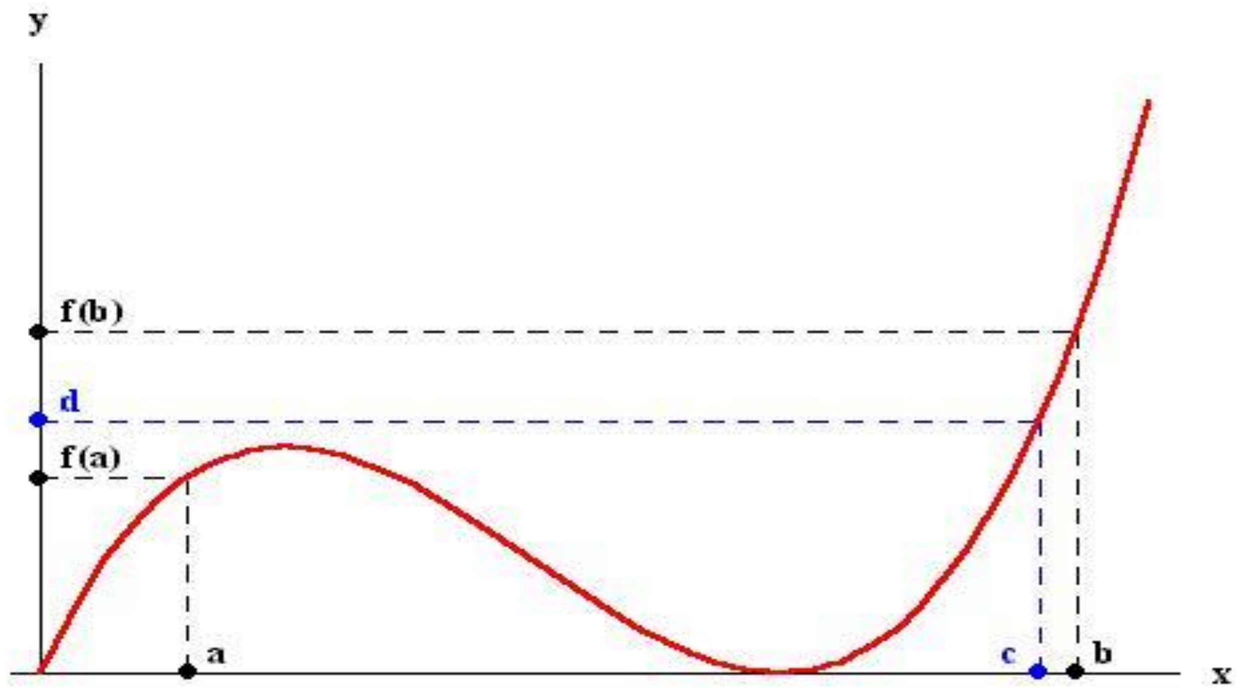
- State and Rolle's mean value theorem
- Discuss the geometrical interpretation of this theorem
- Apply Rolle's mean value theorems to specific problems

Topics

- Intermediate mean value theorem
  - Roll’s mean value theorem
  - Geometrical meaning of mean value theorem

The Intermediate Value Theorem

- If $f(x)$ is a continuous function on $[a, b]$, then for every d between d between f(a)and f(b), there exists a value c in between a and b such that f(c) = d.



Motivation for Rolle's Theorem

- If you bike up a hill, then back down, at some point your velocity was stationary.

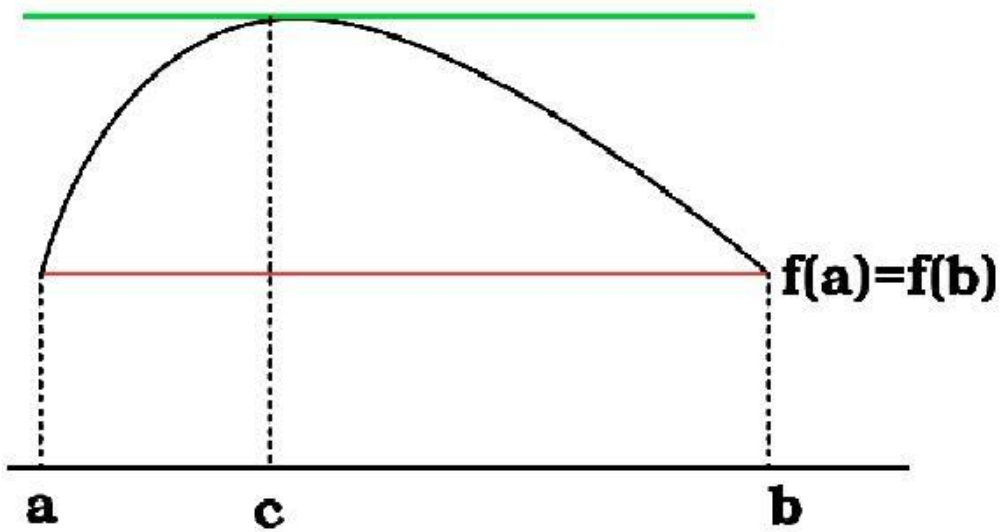
Mathematical Statement of Rolle's Mean Value Theorem

Let f(x) be a real function defined in the closed interval [a, b] such that

- f(x) is continuous in the closed interval [a, b]
- f(x) is differentiable in the open interval (a, b).
- f(a) = f(b), then there is a point c in the open interval (a, b), such that $f'(c) = 0$

Geometrical Meaning

- There are no gaps in the curve $y = f(x)$ from $(a, f(a))$ and $(b, f(b))$, hence the function is continuous
- There exists unique tangent for every intermediate point between a and b
- Also the ordinates of a and b are same, then by Rolle's theorem, there exists atleast one point c in between a and b such that the tangent at c is parallel to x-axis



Example 1

Verify Rolle's theorem for the function $f(x) = sinx$

In the interval $[0,2\pi]$

Solution: Given that $f(x) = sinx \Longrightarrow f'(x) = cosx$

We notice that (i) $f(x)$ is differentiable in$(0, 2\,\pi)$

(ii)continuous in $[0,2\pi]$

(iii)$f(0) = f(2\,\pi)$

Therefore $f(x)$ satisfies all three conditions of Roll's theorem , evidently

Now, $f'(x) = cosx$ $\therefore cosx = 0 \Longrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

$$\therefore c = \frac{\pi}{2}, \frac{3\pi}{2} \in (0, 2\,\pi)$$

Thus, for the given function Rolle's theorem is verified

Example 2

Verify Rolle's theorem for the function $f(x) = log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a, b]$, $b > a > 0.$

Solution: Given that $f(x) = log\left[\frac{x^2+ab}{x(a+b)}\right]$-----------------------------------(1)

The given function may be rewritten as

$f(x) = \log(x^2 + ab) - logx - \log(a + b)$--------------(2)

$f(a) = \log(a^2 + ab) - loga - \log(a + b)$

$f(b) = \log(b^2 + ab) - logb - \log(a + b)$

Thus $f(a) = f(b)$.

Diff. Equation (2) with respect to x, we have

$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{x^2-ab}{x(x^2+ab\ )}$-------------------------------*(3)*

Example 2 (Contd...)

we notice that (i) $f(x)$ is differentiable in (a, b)

(ii) continuous in $[a, b]$

(iii) $f(a) = f(b)$

The given function satisfies all the three conditions of Rolle's theorem.

Therefore there exists a point $c \in (a, b)$ such that $f'(c) = 0$

$$\Longrightarrow c^2 - ab = 0$$

$\Longrightarrow \quad c = \sqrt{ab}$ (note that c is the geometric mean of a and b)

Thus, for the given function Rolle's theorem is verified

Summary

- Rolle's theorem, guarantees that there exists at least one point in (a,b) such that tangent at that point is parallel to x-axis
- $\mathrm{f}'(\mathrm{c}) = 0$ need not imply local extrema in the (a,b)
- If the differentiability fails at an interior point of the interval, the conclusion of the Rolle's theorem may not hold
- The converse of the Rolle's mean value theorem need not be true,