Lecture 35
Sequences, Series and Convergence Tests

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Differentiate between sequence and series of complex numbers
- Explain the tests to verify convergence of a sequence/series
- Apply the standard tests to test and verify the convergence of complex sequence/series

Topics

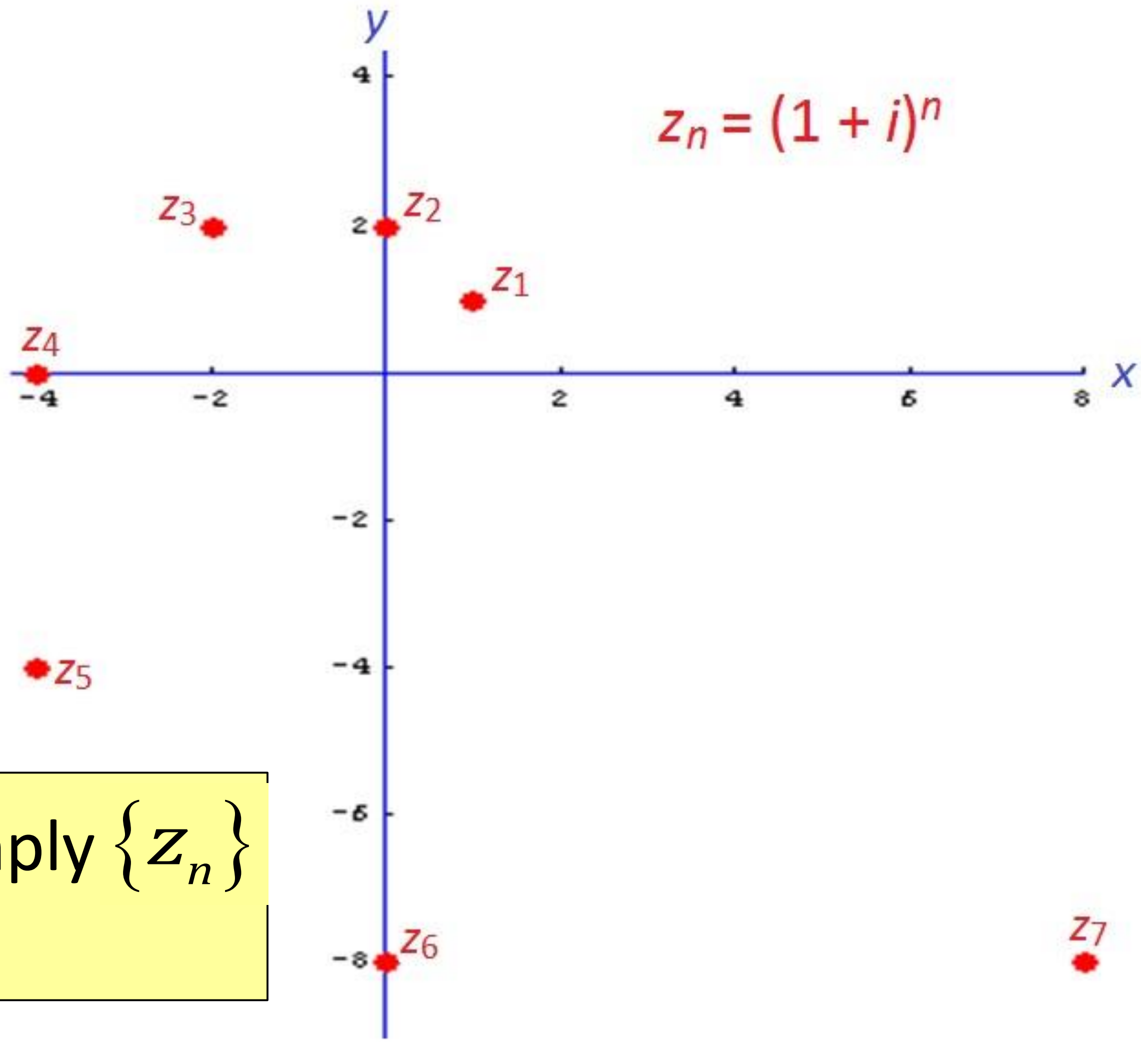
- Sequence
- Convergence
- Series
- necessary and sufficient conditions for convergence
- Absolute and conditional convergence
- Cauchy’s criterion applied to geometric series

Sequence

A sequence of complex numbers is a mapping from the set of natural numbers to the set of complex numbers,

i.e., $f : \mathbb{N} \to \mathbb{C}$

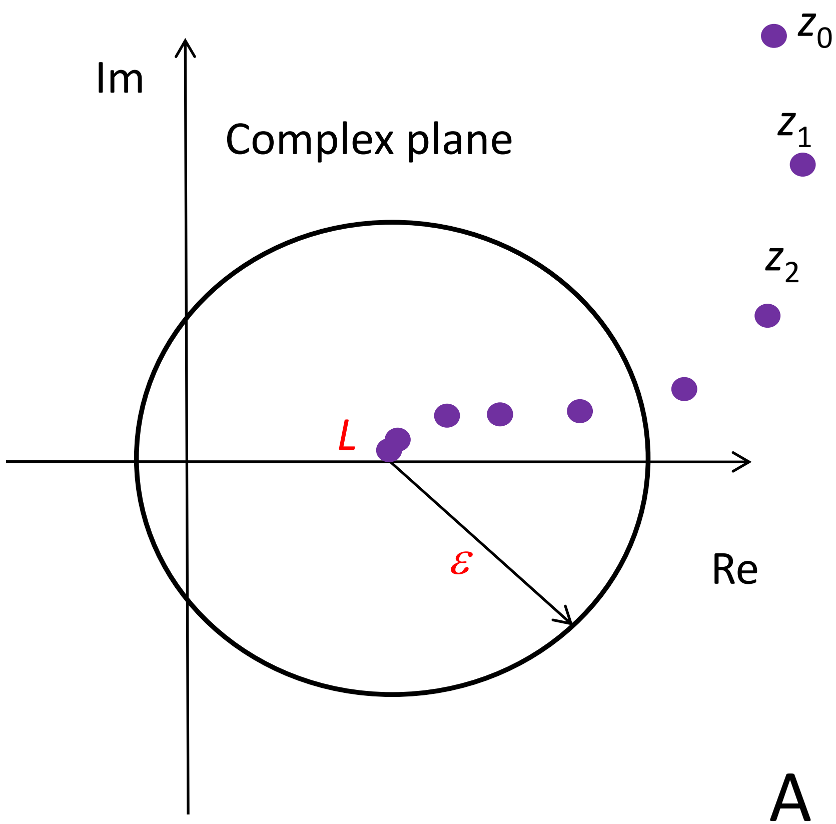
such that $f(n) = z_n$



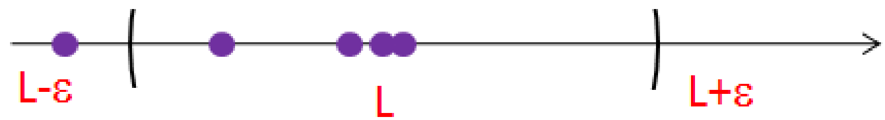
We write $\{z_n\}_{n=1}^{\infty}$ or simply $\{z_n\}$ to denote the sequence

Convergence

Complex infinite sequence



Real infinite sequence



A sequence $\{z_n\}$ is said to converge to a limit L if for every ε > 0 there exit N such that $|z_n - L| < \varepsilon$ for all $n > N$

A sequence that does not converge is said to be Divergent

Series

Consider an infinite series of complex numbers given by

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \ldots$$

Define $\{S_n\}$ to be the sequence of partial sums, where

$$S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \ldots + z_n$$

The infinite series is said to be convergent if there is a number L such that, for every arbitrarily small $\varepsilon > 0$, we can find an integer N such that

$$|S_n - L| < \varepsilon \text{ for all } n \geq N$$

The number L is called the limit of the infinite series.

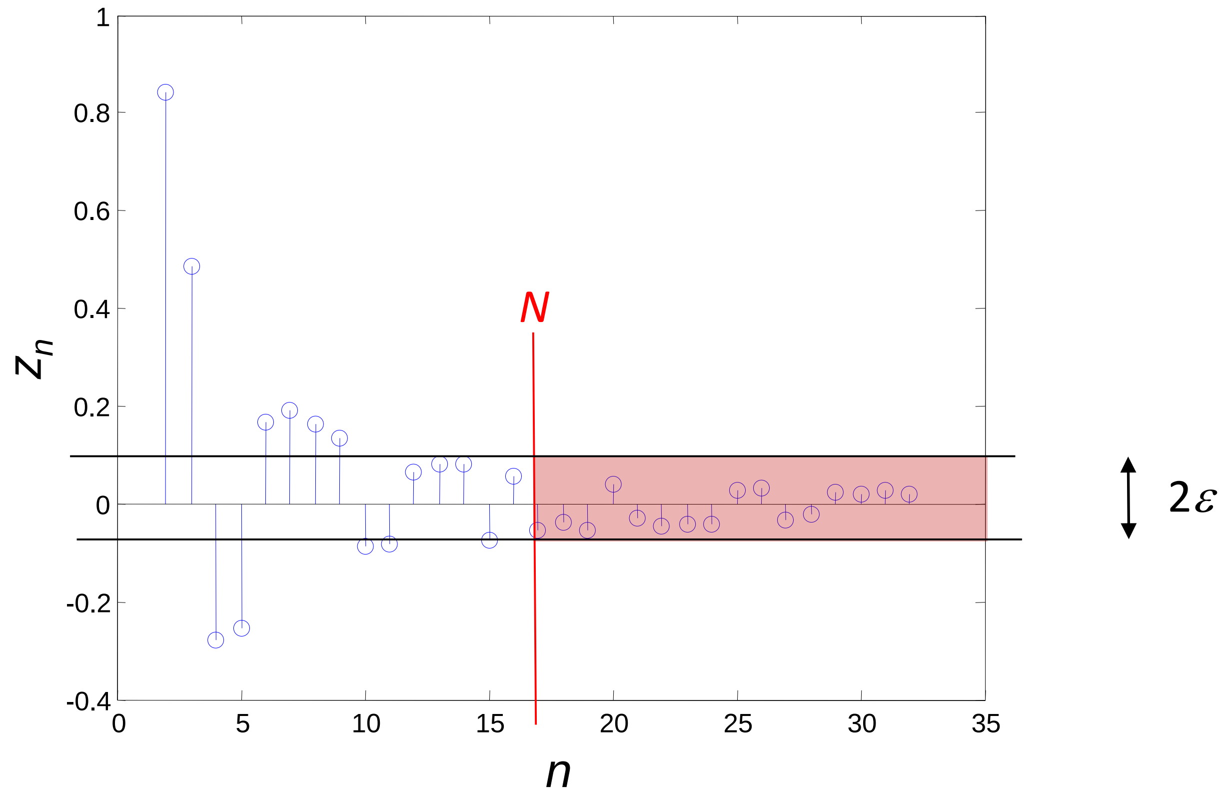
If no such L exists, the infinite series is said to be DIVERGENT.

A necessary condition for convergence

If a series $\Sigma\, Z_n$ converges, then $|\, Z_n \,|$ converges to zero.

This means that for any arbitrarily small $\varepsilon>0$, we can find a sufficiently large integer N such that $|\, a_i \,| < \varepsilon$ for all $n \geq N$.

This condition is not sufficient because we have the divergent series $\Sigma\, 1/n$.



The converse is false

Conversely if $|z_n|$ approaches 0 as n approaches infinity, the series $z_1 + z_2 + z_3 + \ldots$ may or may not converges.

- The harmonic series

  $$1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + \ldots$$

  is divergent.

- However,

  $$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots$$

  is convergent.

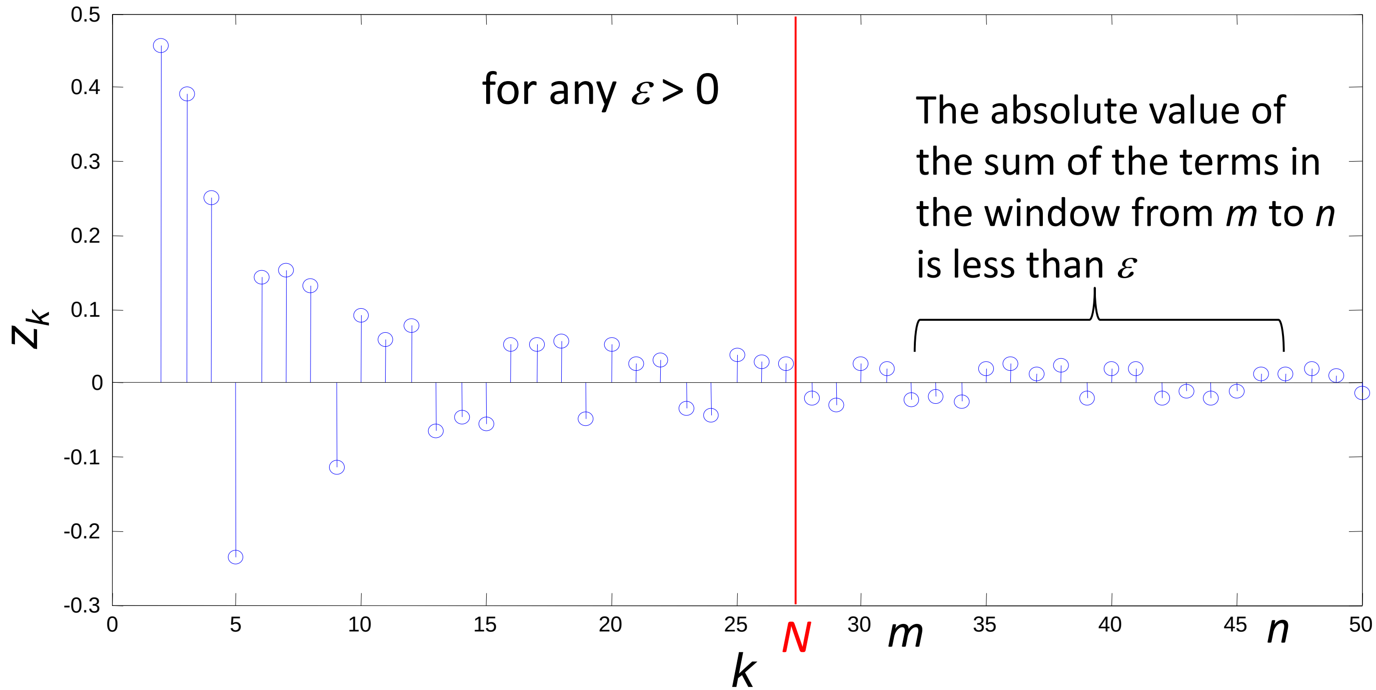
Augustin – Louis Cauchy (1789-1857)

- French mathematician
- Introduced the epsilon-delta argument in calculus.
- The Cauchy-Riemann condition in complex analysis.

Cauchy Criterion for Convergence of a Series

- This is a necessary and sufficient condition for convergence of series.

- Given any arbitrarily small $\varepsilon > 0$, we can find a sufficiently large integer N such that with $m < n$
$| z_m + z_{m+1} + z_{m+2} + \ldots + z_n | < \varepsilon$ for all $m, n \geq N$.



Absolute and Conditional Convergence

- An infinite series $z_1 + z_2 + z_3 + \ldots$ is called absolutely convergent if $|z_1| + |z_2| + |z_3| + \ldots$ is convergent.

- An infinite series $z_1 + z_2 + z_3 + \ldots$ is called conditionally convergent if $z_1 + z_2 + z_3 + \ldots$ is convergent, but $|z_1| + |z_2| + |z_3| + \ldots$ is divergent.

Examples

$$\sum_{i=1}^{\infty} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

divergent

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

conditionally convergent

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2^i} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \ldots$$

absolutely convergent

Geometric Series

A geometric series of complex numbers of the form

$$S_n = \sum_{k=1}^{\infty} a^k = 1 + a^1 + a^2 + \dots$$

Converges to the sum $\dfrac{1}{1-a}$ if |a|<1 and

Diverges if |a| ≥ 1

Cauchy's criterion applied to geometric series

- It is known that the series $1+\tfrac{1}{2}+\left(\tfrac{1}{2}\right)^2+\left(\tfrac{1}{2}\right)^3+\ldots$ is convergent, but we can still apply Cauchy's convergence criterion as an illustration.
- Let ε be an arbitrarily small and positive real number.
- Let N be the smallest integer such that $(½)^N < \varepsilon$.
- Then for any integer m and $n > N$, with $m < n$, we have

$$\left(\tfrac{1}{2}\right)^m+\left(\tfrac{1}{2}\right)^{m+1}+\ldots+\left(\tfrac{1}{2}\right)^n=\left(\tfrac{1}{2}\right)^m\left\{\frac{1-\left(\tfrac{1}{2}\right)^{n-m+1}}{1-\tfrac{1}{2}}\right\}$$

$$=\left(\tfrac{1}{2}\right)^{m-1}<\left(\tfrac{1}{2}\right)^N<\varepsilon$$

- Hence by Cauchy's convergence criterion the geometric series converges.

Comparison Test

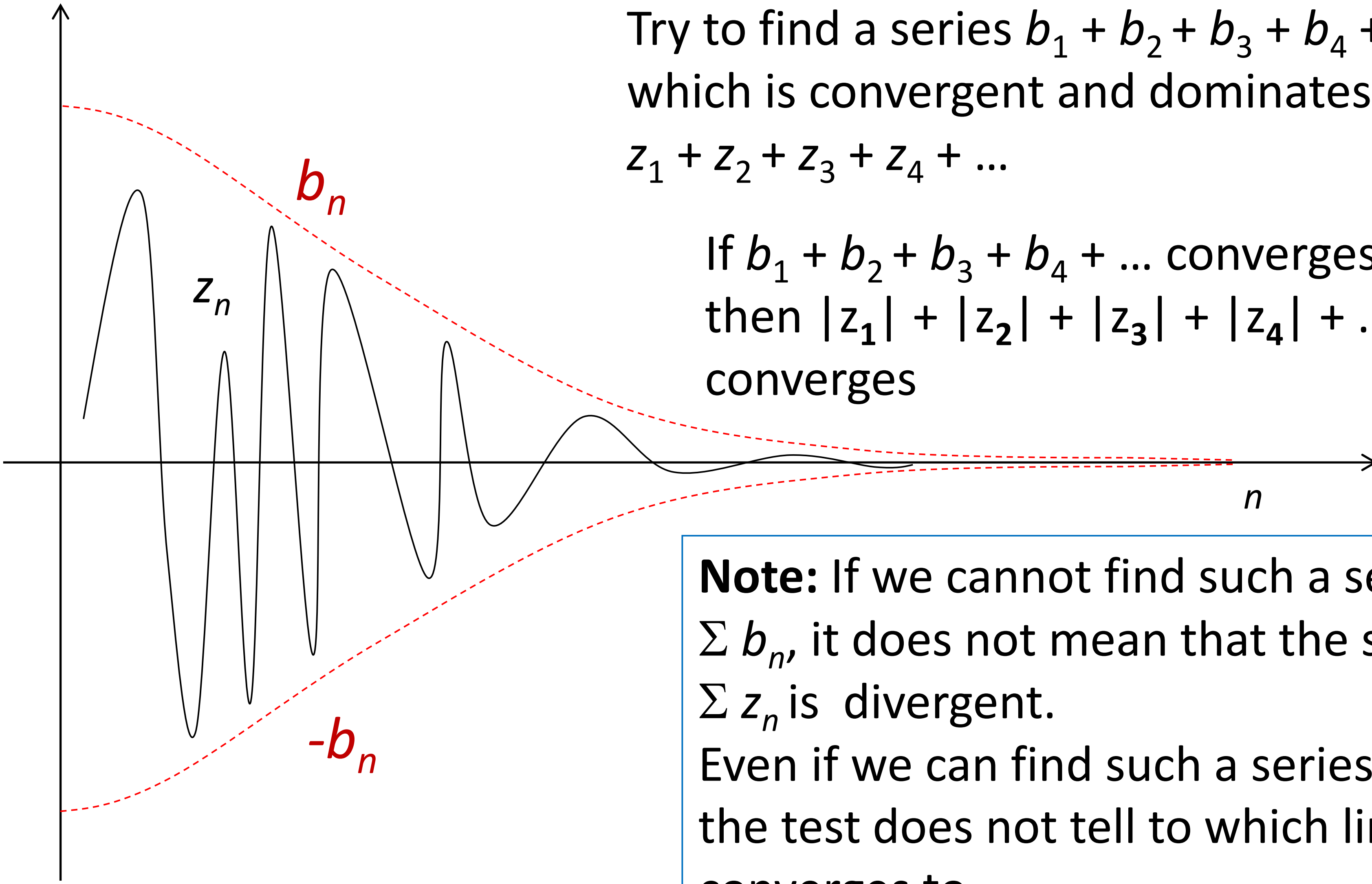
Let $z_1 + z_2 + z_3 + z_4 + \ldots$ be a given infinite series.

If we can find non-negative real numbers $b_1, b_2, b_3, b_4, \ldots$ such that

1. $|z_n| \leq b_n$ for all n, and (z_n is dominated by b_n)
2. $b_1 + b_2 + b_3 + b_4 + \ldots$ converges,

then $z_1 + z_2 + z_3 + z_4 + \ldots$ converges absolutely.

To check whether $z_1 + z_2 + z_3 + z_4 + \ldots$ converges?



Try to find a series $b_1 + b_2 + b_3 + b_4 + \ldots$ which is convergent and dominates $z_1 + z_2 + z_3 + z_4 + \ldots$

If $b_1 + b_2 + b_3 + b_4 + \ldots$ converges then $|z_1| + |z_2| + |z_3| + |z_4| + \ldots$ converges

Note: If we cannot find such a series $\Sigma\, b_n$, it does not mean that the series $\Sigma\, z_n$ is divergent.
Even if we can find such a series $\Sigma\, b_n$, the test does not tell to which limit it converges to.

Ratio test

If an infinite series $z_1 + z_2 + z_3 + \ldots$, with all terms nonzero, is such that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$$

For simplicity, we assume that the limit exists.

Then

1. The series converges if $L < 1$.
2. The series diverges if $L > 1$.
3. No conclusion if $L = 1$.

Root test

If an infinite series $z_1 + z_2 + z_3 + \ \ldots$, with all terms nonzero, is such that

$$\lim_{n\to\infty} |z_n|^{1/n} = L$$

For simplicity, we assume that the limit exists.

Then

1. The series converges absolutely if $L < 1$.
2. The series diverges if $L > 1$.
3. No conclusion if $L = 1$.

Example

The power series

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

converges for all complex numbers z. The radius of convergence is infinity.

It is because for each *z*, the ratio of consecutive terms

$$\left| \frac{\frac{z^{n+1}}{(n+1)!}}{\frac{z^n}{n!}} \right| = \left| \frac{z}{n+1} \right| \to 0$$

as *n* approaches infinity. By the ratio test, this series converges for every complex number *z*.

Example

- Test for the convergence

Session Summary

- **Ratio Test:** If an infinite series $z_1 + z_2 + z_3 + \ \dots$, with all terms nonzero, is such that

$$L = \lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right|$$

  then the series (i) converges if $L < 1$, (ii) diverges if $L > 1$ and (iii) No conclusion if $L = 1$.

- **Root Test:** If an infinite series $z_1 + z_2 + z_3 + \ \dots$, with all terms nonzero, is such that

$$L = \lim_{n \to \infty} \left| (z_{n+1})^{1/n} \right|$$

  then the series (i) converges absolutely if $L < 1$, (ii) diverges if $L > 1$ and (iii) No conclusion if $L = 1$.

- If there is a sequence $b_1, b_2, b_3, b_4, \dots$ such that $|z_n| \leq b_n$ for all n, and $b_1 + b_2 + b_3 + b_4 + \dots$ converges, then $z_1 + z_2 + z_3 + z_4 + \dots$ converges absolutely.