Lectures 15-16
Taylor's and McLaurin's Theorem for Two Variables

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Distinguish between Euler's and Taylor's theorem
- Extend Taylor's theorem to functions of two variables

Topics

- Taylor's theorem for function of two variables
- McLaurin's theorem for function of two variables

Taylor’s Theorem for function of two variables

- Considering f(x+h,y+k) as a function of a single variable x, we have by

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f + \ldots\ldots.$$

Taking x=a and y=b, the above equations becomes

$$f(\mathrm{a}+h, \mathrm{b}+k) = f(\mathrm{a}, \mathrm{b}) + \left(hf_x(\mathrm{a}, \mathrm{b}) + kf_y(\mathrm{a}, \mathrm{b}) \right)$$

$$+\frac{1}{2!}\left(h^2 f_{xx}(\mathrm{a}, \mathrm{b}) + k^2 f_{yy}(\mathrm{a}, \mathrm{b}) + 2\,\mathrm{hk}\ f_{xy}(\mathrm{a}, \mathrm{b}) \right) + \ldots\ldots.$$

Taylor's Theorem for function of two variables

- Putting a+h =x and b+k=y, so that h=x-a and k=y-b, we arrive at the Taylor's expansion of the function f(x,y) in powers of (x-a) and (y-b)

- This formula is useful in expanding the function in the neighborhood of (a,b)

$$f(x,y) = f(a,b) + \left((x-a) f_x(a,b) + (y-b) f_y(a,b)\right)$$

$$+ \frac{1}{2!}\left((x-a)^2 f_{xx}(a,b) + (y-b)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)\right) + \ldots\ldots$$

Taylor's Theorem for function of two variables

- The Taylor's series can be rewritten as

$$f\left(x,y\right)= f\left(a,b\right)+\sum_{j=1}^{n-1}\frac{1}{j!}\left[\left(x-a\right)\frac{\partial}{\partial x}+\left(y-k\right)\frac{\partial}{\partial y}\right]^{j} f\left(x,y\right)\Big|_{(a,b)} + R_n$$

where

$$R_n = \frac{1}{(n+1)!}\left((x-a)f_x+(y-k)f_y\right)^{n+1} f(a+\theta h, b+\theta k), 0<\theta<1$$

- Note that, if we put a=b=0 in the above, we get

$$f(x,y)= f(0,0)+\left(xf_x(0,0)+yf_y(0,0)\right)$$

$$+\frac{1}{2!}\left(x^2 f_{xx}(0,0)+y^2 f_{yy}(0,0)+2xyf_{xy}(0,0)\right)+\ldots\ldots$$

Example 1

Example: Expand $x^2y+3y-2$ in the powers of (x-1) and (y+2) using Taylor's theorem.

Sol: In the Taylor's expansion, we make the following observation

- a=1, b=-2 and $x^2y+3y-2$
- $f(1,-2)=-10,\ f_x(1,-2)=-4,\ f_y(1,-2)=4,\ f_{xx}(1,-2)=-4,\ f_{xy}(1,-2)=2,\ f_{yy}(1,-2)=0$
- All partial derivatives of third (and higher orders vanish)
- Substituting in the Taylor's expansion and simplifying, we get

$$x^2y+3y-2=-10-4(\mathrm{x}-1)+4(\mathrm{y}+2)-2(\mathrm{x}-1)^2+2(x-1)(y+2)+(\mathrm{x}-1)^2(y+2)$$

Example

Find the linear and quadratic Taylor series polynomial approximations to the function

$f(x,y) = 2x^3 + 3y^3 - 4x^2y$ about the point (1,2).

- Solution:

$f(x,y) = 2x^3 + 3y^3 - 4x^2y$: $f(1,2) = 18$

$f_x(x,y) = 6x^2 - 8xy$; $f_x(1,2) = -10$

$f_y(x,y) = 9\,y^2 - 4x^2$; $f_y(1,2) = 32$

$f_{xx}(x,y) = 12x - 8y$; $f_{xx}(1,2)$ =-4

$f_{yy}(x,y) = 18y$; $f_{yy}(1,2) =$ 36

Example 2 (cont.)

$f_{xy}(x,y) = -8x \qquad f_{xy}(1,2) = -8$

$f_{xxx}(x,y) = 12, \qquad f_{xxy}(x,y) = -8$

$f_{xyy}(x,y) = 0, \qquad f_{yyy}(x,y) = 18$

Linear approximation is given by

$$f(x,y) = f(x_0,y_0) + (x-x_0)f_x + (y-y_0)f_y$$

$$=f(1,2) + (x-1)f_x(1,2) + (y-2)f_y(1,2)$$

$$=18 - 10(x-1) + 32(y-2)$$

Quadratic approximation is given by

$$f(x,y) = f(x_0,y_0) + (x-x_0)f_x + (y-y_0)f_y$$

$$+\frac{1}{2}[(x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0)f_{xy} + (y-y_0)^2 f_{yy}]$$

Example 2 (cont.)

$$f(x,y) = 18 - 10(x-1) + 32(y-2)$$
$$+\frac{1}{2}[-4(x-1)^2 - 16(x-1)(y-2) + 36(y-2)^2]$$

$\Longrightarrow$

$$f(x,y) = 18 - 10(x-1) + 32(y-2)$$
$$-2[(x-1)^2 + 4(x-1)(y-2) - 9(y-2)^2]$$

Example 3

Expand $f(x,y) = 21 + x - 20\,y + 4x^2 + xy + 6y^2$ in Taylour's series of maximum order about the point $(-1,2)$

Solution: $f(-1,2) = 6, f_x(x,y) = 1 + 8x + y,\ f_x = (-1,2) = -5$

$f_y(x,y) = -20 + x + 12y,\ \ f_y(-1,2) = 3$

$f_{xx}(x,y) = 8, \qquad f_{xy}(x,y) = 1, \qquad f_{yy}(x,y) = 12$

$$f(x,y) = f(-1,2) + \left[(x+1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right] f(-1,2)$$

$$+\frac{1}{2!}\left[(x+1)\frac{\partial}{\partial x} + (y-2)\frac{\partial}{\partial y}\right]^2 \mathrm{f}(-1,2)$$

$\Longrightarrow$

$$f(x,y) = 6 - 5(x+1) + 3(y-2) + 4(x+1)^2 + (x+1)(y-2)$$
$$+6(y-2)^2$$

Summary

- Taylor's Theorem is useful in expanding the function in the neighborhood of (a, b)

- Maclaurin's series is the special case of Taylor series where the function is expanded at the points $(0,0)$