Lecture 14
Homogenous functions and Euler's Theorem

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Illustrate homogeneous functions
- Verify Euler's theorem for Homogeneous functions

Topics

- Homogeneous function
- Euler's theorem
- Examples

Motivation

- In Economics production function is
- $Y = (K + L)^2$
- production function satisfies Euler's Theorem
- Demand function is homogeneous of degree zero
- Expand two variable function by using Taylor's series

Homogeneous Functions

- A function f(x, y) of two variables is called homogeneous of degree α if $f(tx, ty) = t^{\alpha} f(x, y)$

•One can usually spot such functions and their degrees with some practice.

- **Example:** $f(x, y) = x^2 + 5xy + y^2$ is homogeneous of degree 2, since

$$
\begin{aligned}
f(tx, ty) &= (tx)^2 + 5(tx)(ty) + (ty)^2 \\
&= t^2x^2 + t^2(5xy) + t^2y^2 \\
&= t^2(x^2 + 5xy + y^2) \\
&= t^2 f(x, y).
\end{aligned}
$$

Homogeneous Functions

- Euler's Theorem

For a function F(x,y) which is homogeneous of degree n, then

$$\frac{\partial F}{\partial x}x+\frac{\partial F}{\partial y}y=nF$$

- If F is a Homogeneous function of degree n then

$$\frac{\partial^2 F}{\partial x^2}x^2+\frac{\partial^2 F}{\partial y^2}y^2+2xy\frac{\partial^2 F}{\partial y\partial x}=n(n-1)F$$

Example-1

$$u = \frac{x^3 + y^3}{\sqrt{x - y}}, \quad \text{prove that}$$

$$(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}u \qquad (ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{15}{4}u$$

We note that the given $u = \dfrac{x^3(1 + y^3/x^3)}{\sqrt{x}\ \sqrt{1 - y/x}} = x^{5/2}\varphi(y/x)$

u is a homogeneous function with degree 5/2 . Therefore, from Euler's theorem ,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}u \left(Since\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 5/2(5/2 - 1)u = \frac{15}{4}u$$

Example-2

If $u=\sin^{-1}\left(\dfrac{x-y}{x+y}\right)^{1/2}$, $\quad$ prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$

From the given u, we have

$$\sin u=\left(\frac{x-y}{x+y}\right)^{1/2} = f \quad say$$

We note that $f=\left(\dfrac{x-y}{x+y}\right)^{1/2}=\dfrac{\sqrt{x}\,(1-y/x)^{1/2}}{\sqrt{x}\,(1-y/x)}=x^{0}\,\dfrac{(1-y/x)^{1/2}}{(1-y/x)}$

Therefore, f is homogeneous with degree zero.

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0.f=0,\ \ or\ \ x\frac{\partial(\sin u)}{\partial x}+y\frac{\partial(\sin u)}{\partial y}=0,$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial x}=0\Rightarrow\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Example-3

- If $u = \cos^{-1}\left(\frac{x}{y}\right) + \tan^{-1}(y/x)$ show that,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Proof: We can write u as

$$u = \cos^{-1}(x/y) + cot^{-1}(x/y)$$

$$u = y^0\{\cos^{-1}(x/y) + cot^{-1}(x/y)\}$$

We have Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Putting $n = 0$ we get $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

Example-4

- If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$ show that $x\frac{\partial u}{\partial x}$ +y$\frac{\partial u}{\partial y}$+z$\frac{\partial u}{\partial z}$=0

Proof: We can write u as

$$u = \frac{x}{y/x+z/x} + \frac{y/x}{z/x+1} + \frac{z/x}{1+y/x} = x^0\{g(y/x, z/x)\}$$

We have Euler's theorem,

+y$\frac{\partial u}{\partial y}$+z$\frac{\partial u}{\partial z}$=$nu$

Putting $n = 0$ we get, $x\frac{\partial u}{\partial x}$ +y$\frac{\partial u}{\partial y}$+z$\frac{\partial u}{\partial z}$=0

Summary

- A function $F(x,y)$ of the variables x and y is called Homogeneous of degree n iff for any parameter t

$$F(tx,ty)=t^n F(x,y)$$

- For a function $F(x,y)$ which is homogeneous of degree n, then

$$\frac{\partial F}{\partial x}x+\frac{\partial F}{\partial y}y=nF$$

- If F is a Homogeneous function of degree n then