Lecture 23
Limit, Continuity and Analytic Functions

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

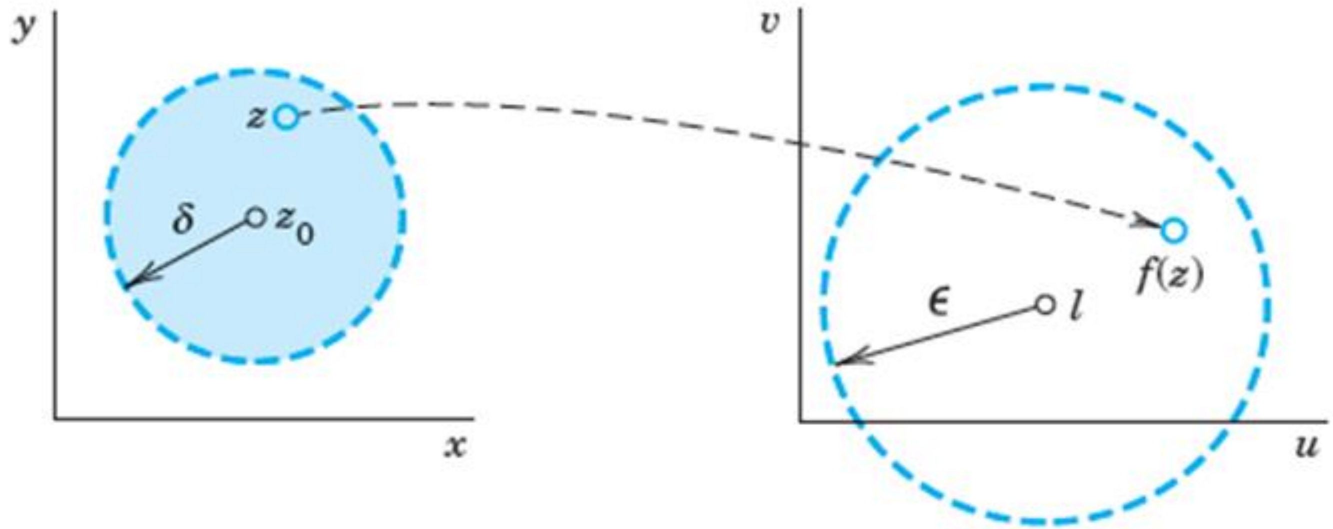
- State Limit and continuity
- State analytic functions

Topics

- Limit of a function
- Properties of Limit
- Continuity
- Differentiability
- Analytic Function

Limit

A function $f(z)$ is said to have a limit l as z approaches z_0 if for every $\varepsilon > 0$ however small there exit a $\delta > 0$ such that $| f(z) - l | < \varepsilon$ whenever $| z - z_0 | < \varepsilon$

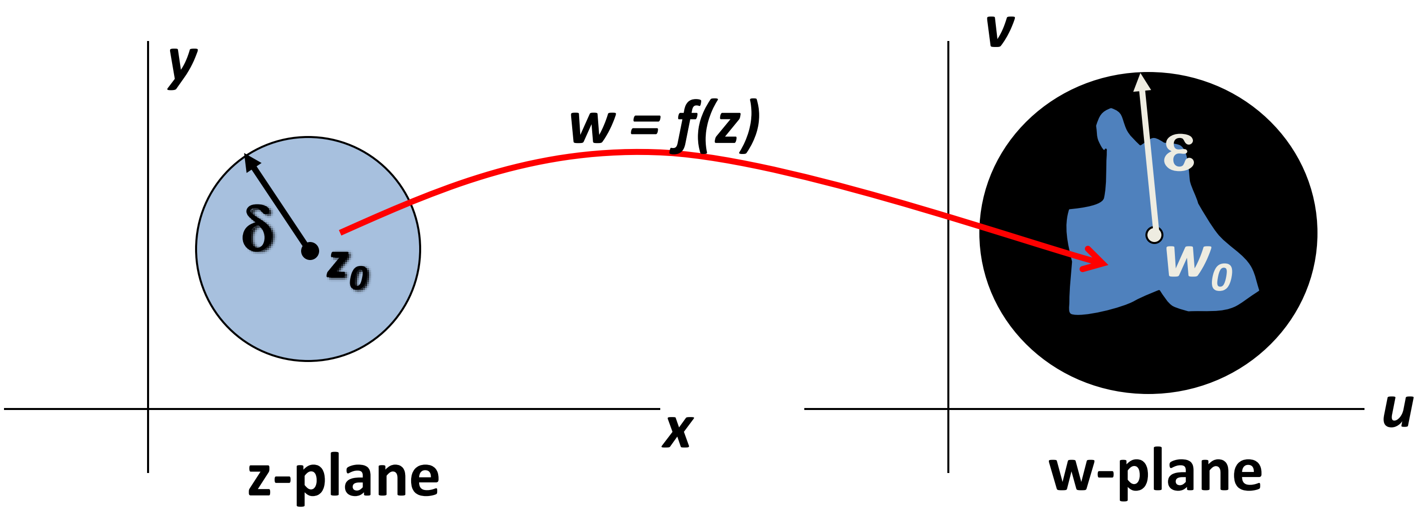


Here, z can approach z_0 any direction

Symbolically we write $\boxed{\lim_{z \to z_0} f(z) = l}$

Limits: Interpretation

We can interpret this to mean that if we observe points *z* within a radius δ ***of*** z_0***, we can find a corresponding disk about*** w_0 ***such that all the points in the disk about*** z_0 ***are mapped into it. That is, any neighborhood of*** w_0 ***contains all the values assumed by f in some full neighborhood of*** z_0***, except possibly*** $f(z_0)$***.***



Properties of Limits

If as $z \to z_0$, $\lim f(z) \to A$ and $\lim g(z) \to B$, then

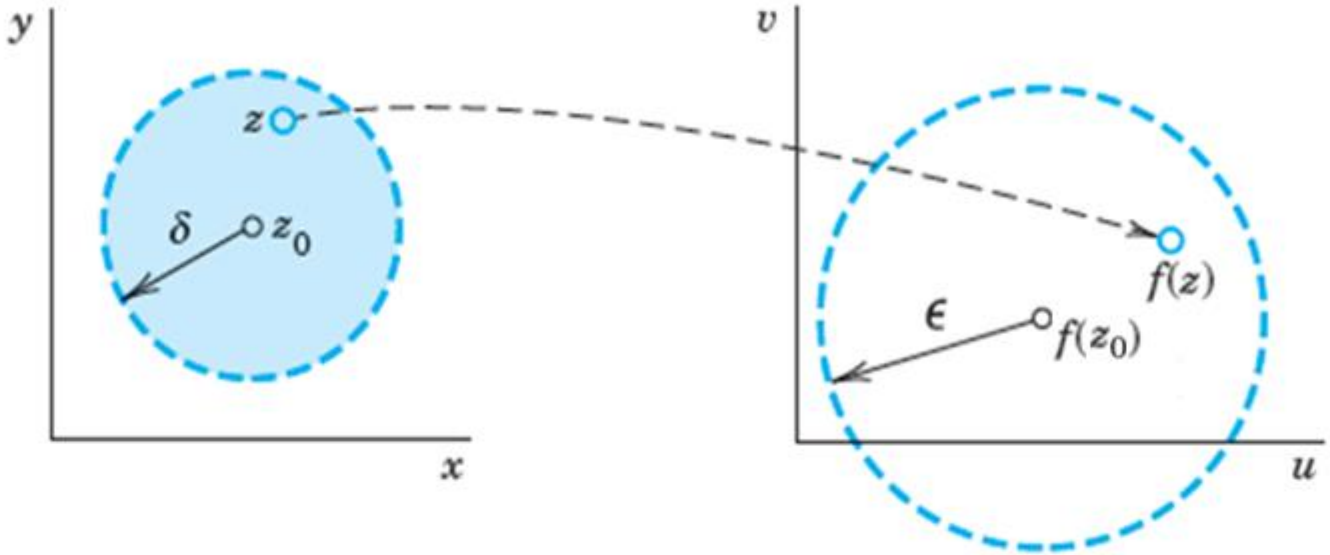
- $\lim[\, f(z) \quad g(z)\,] = A \pm B$
- $\lim f(z)g(z) = AB$, and
- $\lim f(z)/g(z) = A/B$. if $B \neq 0$.

Continuity

A function $f(z)$ is said to be continuous at $z = z_0$ if for

(i) $f(z_0)$ is defined and

(ii) $\lim f(z) = f(z_0)$



Differentiability

A function $f(z)$ is said to be differentiable at $z = z_0$ if the following limit exits

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

We call $f'(z_0)$ as the derivative of $f(z)$ at $z = z_0$.

Also, if we write $\Delta z = z - z_0$ the above definition takes the form

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Properties of Derivatives

$$(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)$$

$$(cf)'(z_0) = cf'(z_0) \text{ for any constant } c.$$

$$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0)$$

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}, \text{ if } g(z_0) \neq 0.$$

$$\frac{d}{dz} f[g(z_0)] = f'[g(z_0)]g'(z_0) \text{ Chain Rule.}$$

Analyticity

A function $f(z)$ is said to be analytic if it is differentiable at z_0 and in the neighborhood z_0

A function $f(z)$ is said to be analytic in a domain D of complex plane if it is differentiable at all points of domain D

From the definition, we note that if $f(z)$ is analytic at a point a then it is differentiable at a. But the converse need not be true

Example-1

- Show that $\lim_{z\to 1+i}(z^2 - 2z + 1) = -1$
- Solution: Let $f(z) = (z^2 - 2z + 1)$
- $= x^2 - y^2 - 2x + 1 + i(2xy - 2y)$

Computing the limits for u and v , we obtain

$$\lim_{(x,y)\to(1,1)} u(x,y) = 1 - 1 - 2 + 1 = -1$$

$$\lim_{(x,y)\to(1,1)} v(x,y) = 2 - 2 = 0$$

Example-2

Show that $\lim\limits_{z\to 1+i} \frac{(z^2-2i)}{z^2 \ -2z+2} = 1+i$

$$\lim_{z\to 1+i} \frac{(z^2-2i)}{z^2 \ \ - 2z + 2} = \frac{(z-1-i)(z+1+i)}{(z-1-i)(z-1+i)}$$

$$= \lim_{z\to 1+i} \frac{(1+i+1+i)}{(1+i-1+i)} \ = 1-i$$

Example-3

- Show that $\lim\limits_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

First, suppose $z \to 0$ along $x - axis$ (for which $y = 0$).
Then $z = \bar{z} = x$ and we have

$$\lim_{z\to 0}\frac{\bar{z}}{z}=\lim_{x\to 0}\frac{x}{x}=1 \quad -------------(i)$$

Next , suppose $z \to 0$ along y$-axis$ (for which x$=$ 0). Then
$z = iy$ and $\bar{z} = -iy$ we have

$$\lim_{z\to 0}\frac{\bar{z}}{z}=\lim_{y\to 0}\frac{-iy}{iy}=-1\text{---------------------(ii)}$$

Since the two limits given by (1) and (ii) are different, the requird limit does not exist

Example-4

- Show that $f(z) = e^{-z}$ is everywhere analytic . Find $f'(z)$

- Solution: $\mathrm{f}'(z) = \lim\limits_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$

- $= \lim\limits_{\delta z \to 0} \frac{e^{(z+\delta z)} - e^{-z}}{\delta z}$

- $= e^{-z} \lim\limits_{\delta z \to 0} \frac{(e^{-\delta z} - 1)}{\delta z}$

Example-4………

$$e^{-\delta z} = 1 - \delta z + \frac{(\delta z)^2}{2} - \frac{(\delta z)^3}{3!} + \ldots.$$

Therefore $\frac{e^{-\delta z} - 1}{\delta z}$ = -1 + $\delta z - \frac{(\delta z)^2}{2!} + \frac{(\delta z)^3}{3!} + \ldots.$

$$\lim_{\delta z \to 0} \frac{e^{-\delta z} - 1}{\delta z} = -1$$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = e^{-z}\,(-1) = -e^{-z}$$

Therefore $f(z)$ is analytic

Problems

1. Show that the function $f(z) = |z|^2$ is differentiable but not analytic at the origin
2. Show that the function $f(z)$=z^2 is continuous and differentiable at every point z

Session Summary

- Definition of limit : $|f(z)-l|<\varepsilon$ whenever $|z-z_0|<\varepsilon$
- Differentiability condition : $f'(z_0)=\lim\limits_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}$