Lecture 17
Maximum and Minimum Values of Functions_I

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Define and explain the significance of maxima and minima
- Determine the maximum and minimum of a function at different points

Topics

- Maximum and minimum of functions
- Geometrical meaning of maximum and minimum
- Examples

Maximum and Minimum Values of Functions

- A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).]

- The number f (a, b) is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f (a, b) is a **local minimum value**.

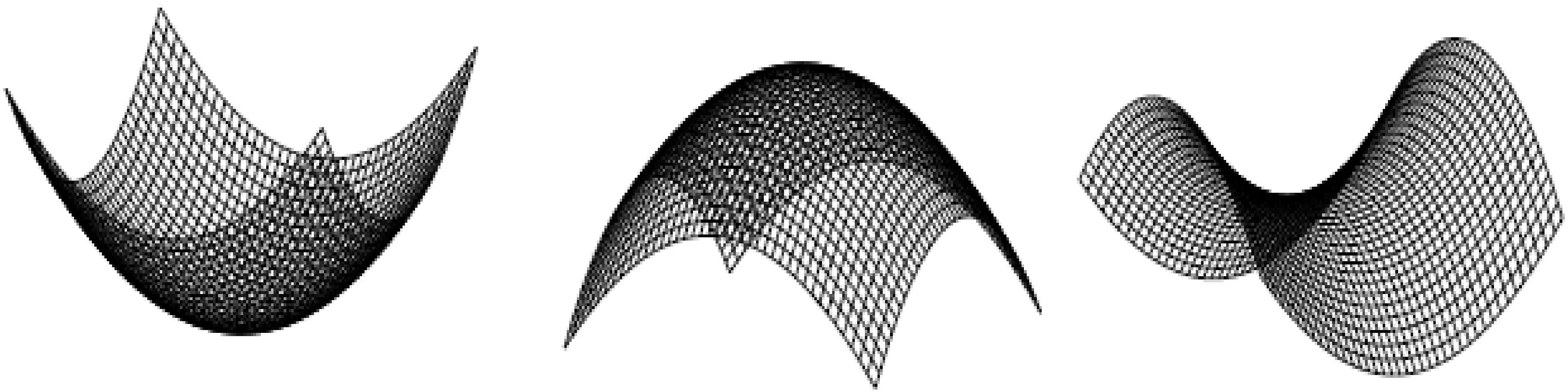
Maximum and Minimum Values of Functions

- A point (a, b) is called a **critical point** (or *stationary point*) of f if f_x (a, b)=0 and f_y (a, b)=0, or if one of these partial derivatives does not exist.

- If f has a local maximum or minimum at (a, b) and the first order partial derivatives of f exist there, then f_x(a, b)=0 and f_y(a, b)=0.

- Note that the function f must have continuous second order partial derivatives

Geometrical Meaning



(a) A local minimum (b) A local maximum (c) A saddle point

General working procedure

SOLVING AN APPLIED EXTREMA PROBLEM

1. Read the problem carefully. Make sure you understand what is given and what is unknown .
2. If possible, sketch a diagram. Label the various parts
3. Decide on the variable that must be maximized or minimized. Express that variable as a function of one other variable.
4. Find the domain of the function.

General working procedure

SOLVING AN APPLIED EXTREMA PROBLEM

1. Find the critical points for the function from step 3.
2. If the domain is a closed interval, evaluate the function at the end points and at each critical number to see which yields the absolute maximum or minimum
3. If the domain is an open interval, apply the critical point theorem when there is only one critical number
4. If there is more than one critical number, evaluate the function at the critical numbers and also find the limit as the end points of the interval are approached to determine if an absolute maximum or minimum exists at one of the critical points

Maximum and Minimum Values of Functions

- Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that f_x (a, b) and f_y (a, b)=0 [that is, (a, b) is a critical point of f]. Let

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [\, f_{xy}(a,b)]^2$$

(a)If D>0 and f_{xx} (a, b)>0 , then f (a, b) is a local minimum

(b)If D>0 and f_{xx} (a, b)<0, then f (a, b) is a local maximum

(c) If D<0, then f(a, b) is not an extreme value

(d) If D=0, then at (a, b), the case is doubtful and needs further investigation

Maximum and Minimum Values of Functions

- In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

- If D=0, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

- To remember the formula for D it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xy} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

Chapter 11, 11.7, P648

Maximum and Minimum Values of Functions

Extreme Value Theorem for Functions of Two Variables

- If f is continuous, then f attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some points $(x_{1,}y_1)$ and (x_2,y_2) in some real domain D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of in D
2. Find the extreme values of f on the boundary of D
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value

Working Procedure to solve the given problem

- We first find the stationary points (x,y) such that $f_x = 0 \quad f_y = 0$
- We find $A = f_{xx}\,, \quad B = f_{yy}, \quad C = f_{yy}$ and evaluate these at all stationary points & also compute the corresponding value of $AC - B^2$
- (a)A Stationary point (x_0, y_0) is a maximum point if

  $AC - B^2 > 0 \;\;\&\;\; A < 0.\;\; f(x_0, y_0)$is a maximum value .

(b) A stationary point (x_1, y_1) is a minimum point if

$AC - B^2 > 0 \;\;\&\;\; A > 0.\;\; f(x_1, y_1)$ is a minimum value

Note: Following cases gives Saddle pints

$$AC - B^2 < 0, \quad AC - B^2 = 0$$

Example 1

- Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$

  Solution: We first locate the critical points by deriving f_x , f_y and equate to zero $f_x = 4x^3 - 4y = 0$ and $f_y = 4y^3 - 4x = 0$

  solving the equations, we get critical points:

  (0, 0), (1, 1) and (–1, –1)

  Next, we calculate the second partial derivatives and $D(x, y)$:

  $f_{xx} = 12x^2$, $f_{xy} = -4$ and $f_{yy} = 12y^2$,

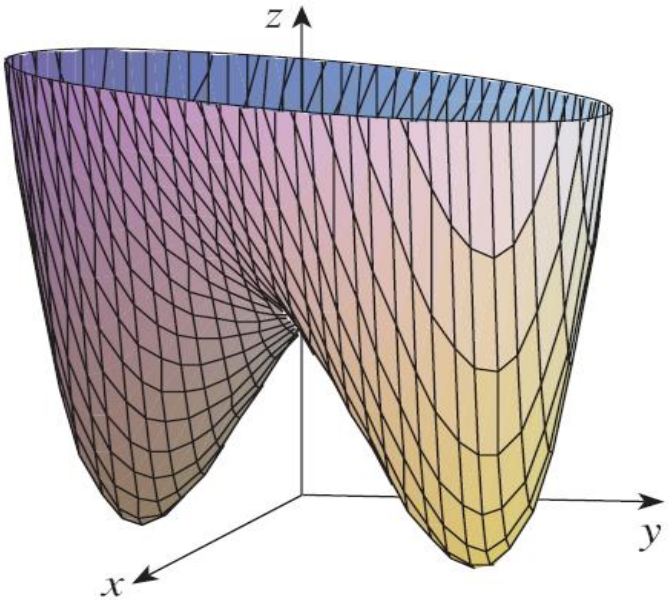
$$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2$$
$$= 144x^2y^2 - 16$$

– $D(0, 0) = -16 < 0$, it follows that the origin is a saddle point. That is, f has no local maximum or minimum at (0, 0)

– $D(1, 1) = 128 > 0$ and $f_{xx}(1, 1) = 12 > 0$, we see that at (1,1) has a local minimum.

– $D(-1, -1) = 128 > 0$ and $f_{xx}(-1, -1) = 12 > 0$.

So $f(-1, -1) = -1$ is also a local minimum.



Example 2

- Find the extreme values of the function

$$f(x,y)=x^3+y^3-3x-12y+20$$

Solution: $f_x=3x^2-3,\quad f_y=3y^2-12\quad Since\ D=AC-B^2$

We shall find the points (x,y) such that

$$f_x=0\quad and\quad f_y=0$$

$$i.e\ 3x^2-3=0\ \ and\ \ 3y^2-12=0\ \ or\ \ x^2-1=0\ \ and\ \ y^2-4=0$$

$$ie.,\ x=\pm1,\ \ y=\pm2\ \ \therefore(1,2),(1,-2),(-1,2),(-1,-2)$$

$$are\ the\ stationary\ po\text{int}\ s$$

$$Let\ \ A=f_{xx,}\quad B=f_{xy},\quad C=f_{yy}$$

Example 2 (cont.)

	(1,2)	(1,-2)	(-1,2)	(-1,-2)	
A=6x	6>0	6	-6	-6<0	
B=0	0	0	0	0	
C=6y	12	-12	12	-12	
$D = AC - B^2$	72>0	-72<0	-72<0	72>0	
Conclusion	Min.pt	Saddle pt.	Saddle pt	Max. pt	

Maximum value of $f(x,y)$: $f(-1,2) = -1 - 8 + 3 + 24 + 20 = 38$

Minimum value of $f(x,y)$: $f(1,2) = 1 + 8 - 3 - 24 + 20 = 2$

Thus the extreme values of the given function are 38 and 2

Example 3

Show that $z = f(x,y) = x^3 + y^3 - 3xy + 1$ *is* minimum *at* (1,1)

$z_x = 3x^2 - 3y; \quad z_y = 3y^2 - 3x$

$Let \;\; A = z_{xx,} \;\; B = z_{xy} \;\; C = z_{yy} \;\; \therefore A = 6x, \;\; B = -3, \;\; C = 6y$

Now , at (11) $z_x = 0$ and $z_y = 0$

$Also \;\; A = 6, B = -3, \;\; C = 6 \qquad \therefore AC - B^2 = 27 > 0$

At (11) $z_x = 0$ and $z_y = 0 \quad AC - B^2 > 0, A = 6 > 0 \Rightarrow z(x,y)$ is min. at (1, 1)

Summary

- Suppose the second partial derivatives of *f* are continuous f_x (a, b) and f_y (a, b)=0 [that is, (a, b) is a critical point of f].

(a)If D>0 and f_{xx} (a, b)>0 , then f (a, b) is a local minimum
(b)If D>0 and f_{xx} (a, b)<0, then f (a, b) is a local maximum
(c) If D<0, then f(a, b) is not an extreme value
(d) If D=0, then at (a, b), the case is doubtful and needs further investigation

Where

$$D = AC - B^2, \;\; A = f_{xx}, \;\; B = f_{xy}, \qquad C = f_{yy}$$