Lecture 38
Laurent Series

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Define meromorphic function
- State Laurent’s theorem
- Expand some meromorphic functions in Laurent series

Topics

- Isolated singularity
- Laurent series
- Laurent’s theorem

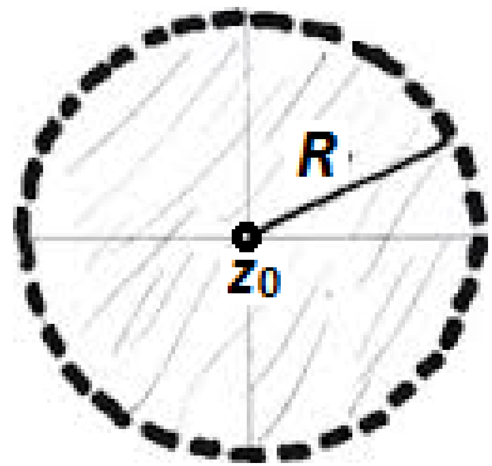
Isolated Singularity

The point z_0 is said to be an *isolated singularity*, if there exists some deleted neighborhood or punctured open disk

$0 < |z - z_0| < R$ throughout which is analytic.

Example: The function $f(z) = \dfrac{z}{z^2 + 4}$

has two singularities at $z = 2i$ and $z = -2i$



A function $f(z)$ with one or more isolated singularities is called as a said to be an *MEROMORPHIC* function.

A New Kind of Series

About an isolated singularity, it is possible to represent *f*(*z*) by a new kind of series involving both negative and positive integer powers of $z - z_0$, that is

$$f(z) = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

$$\boxed{f(z) = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k}$$

The part with negative powers is called the *principal part* and will converge for $|1/(z - z_0)| < r^*$ or $|z - z_0| > 1/r^* = r$. The part with nonnegative powers is called the *analytic part* and will converge for $|z - z_0| < R$. Hence the sum of these parts converges when $r < |z - z_0| < R$.

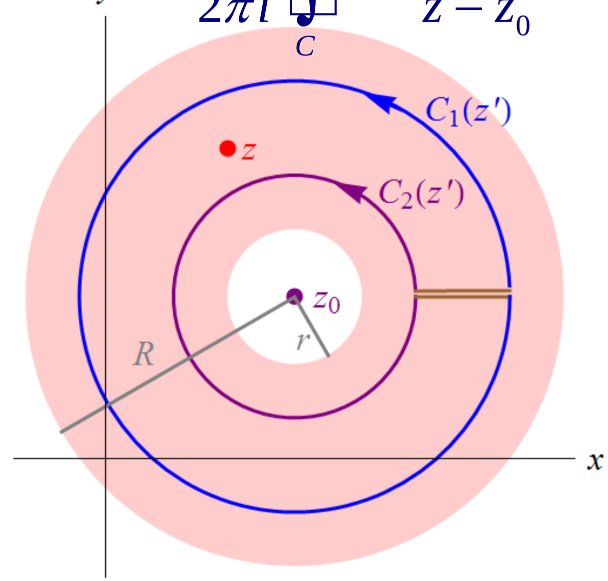
Laurent Series

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \, \frac{f(z)}{(z - z_0)^{n+1}}$$

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}$$

Let f be analytic within an annular region

$$r \le |z - z_0| \le R$$



$$f(z) = \frac{1}{2\pi i} \left[\oint_{C_1} - \oint_{C_2} \right] dz' \, \frac{f(z')}{z' - z}$$

$$\frac{1}{z' - z} = \frac{1}{z' - z_0 - (z - z_0)} = \frac{1}{z' - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{z' - z_0} \right)^n$$

$$\frac{1}{z' - z} = -\frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{z' - z_0}{z - z_0} \right)^n$$

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} dz' \, \frac{f(z')}{(z' - z_0)^{n+1}} + \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{(z - z_0)^{n+1}} \oint_{C_2} dz' \, (z' - z_0)^n f(z'$$

Laurent Series

$$\boxed{f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint_{C_1} dz' \frac{f(z')}{(z'-z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\oint_{C_2} dz' (z'-z_0)^n f(z')}$$

$$\sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\oint_{C_2} dz' (z'-z_0)^n f(z') = \sum_{n=-\infty}^{-1}(z-z_0)^n \oint_{C_2} dz' \frac{f(z')}{(z'-z_0)^{n+1}}$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

Laurent series

$$a_n = \frac{1}{2\pi i}\oint_C dz' \frac{f(z')}{(z'-z_0)^{n+1}}$$

C within f's region of analyticity

Example

The function $f(z) = (\sin z)/z^3$ is not analytic at $z = 0$ and hence can not be expanded in a Maclaurin series. We find that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

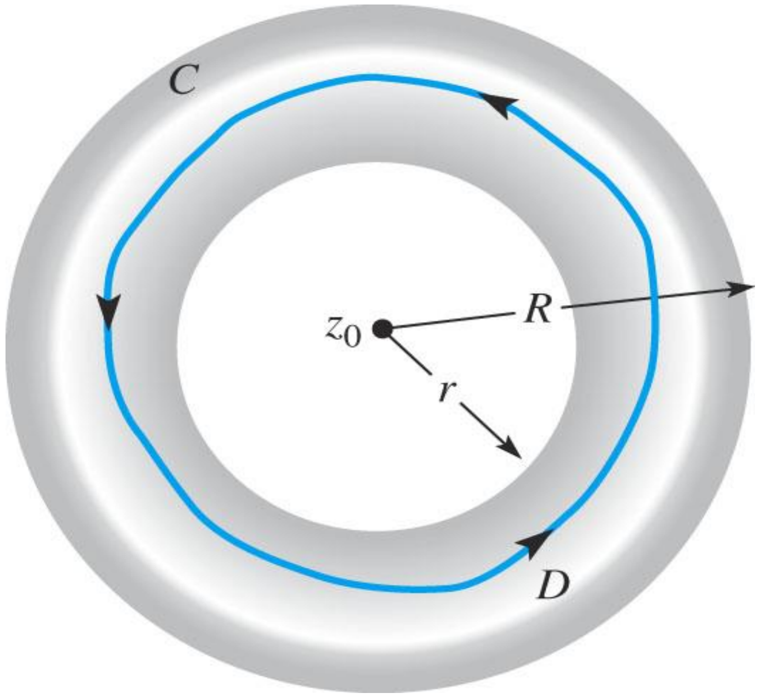
converges for all z. Thus

$$f(z) = \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots$$

This series converges for all z except $z = 0$, $0 < |z|$.

Laurent's Theorem

Let $f(z)$ be analytic within the annular domain D defined by $r < |z - z_0| < R$. Then $f(z)$ has the series representation



$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

valid for $r < |z - z_0| < R$.

The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \; k = 0, \pm 1, \pm 2, \ldots$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior.

Example – 1

Expand $f(z)=\dfrac{8z+1}{z(1-z)}$ in a Laurent series valid for $0 < |z| < 1$.

Solution: We can write

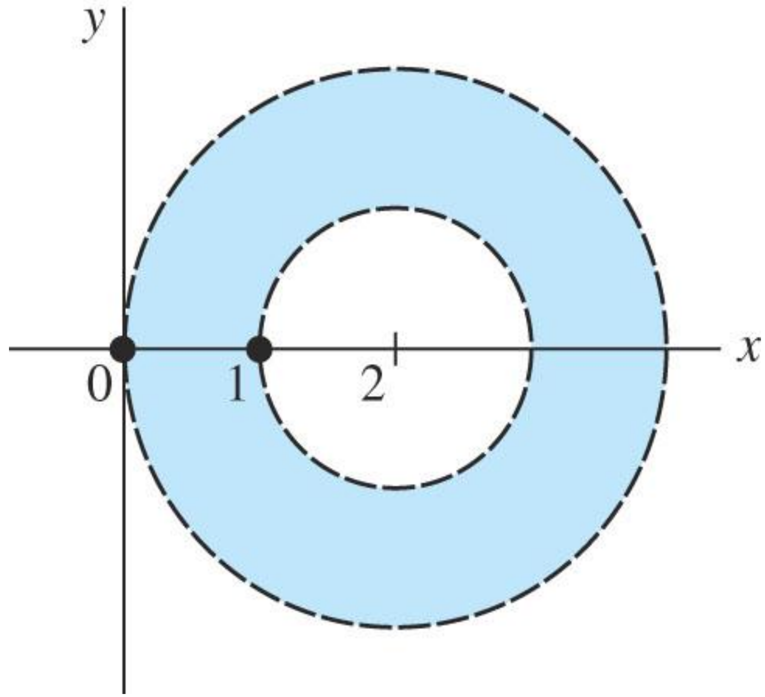
$$f(z)=\frac{8z+1}{z(1-z)}=\left(\frac{8z+1}{z}\right)\left(\frac{1}{1-z}\right)$$

$$=\left(8+\frac{1}{z}\right)\left[1+z+z^2+\ldots\right]$$

$$=\frac{1}{z}+9+9z+9z^2+\ldots$$

Example – 2

Expand $f(z)=\dfrac{1}{z(1-z)}$ in a Laurent series valid for $1<|z-2|<2$.



Solution: The center $z = 2$ is a point of analyticity of $f(z)$. We want to find two series involving integer powers of $z-2$; one converging for $1<|z-2|$ and the other converging for $|z-2|<2$.

Let us write

$$f(z)=\frac{1}{z-1}-\frac{1}{z}=f_1(z)+f_2(z)$$

$$f_1(z)=\frac{1}{z-1}=\frac{1}{1+z-2}=\left(\frac{1}{z-2}\right)\left\{1+\frac{1}{z-2}\right\}^{-1}$$

$$=\left(\frac{1}{z-2}\right)\left[1-\frac{1}{z-2}+\left(\frac{1}{z-2}\right)^2-\left(\frac{1}{z-2}\right)^3+\ldots\right]$$

$$=\frac{1}{z-2}-\frac{1}{(z-2)^2}+\frac{1}{(z-2)^3}-\ldots$$

This series converges for $|1/(z-2)| < 1$ or $1 < |z-2|$.

$$f_2(z)=-\frac{1}{z}=-\frac{1}{2+z-2}=\left(-\frac{1}{2}\right)\left\{1+\left(\frac{z-2}{2}\right)\right\}^{-1}$$

$$=-\frac{1}{2}\left[1-\frac{z-2}{2}+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3+\ldots\right]$$

This series converges for $|(z-2)/2| < 1$ or $|z-2| < 2$.

Example-3

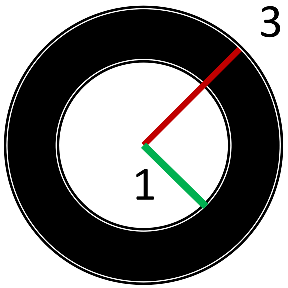
Expand $f(z)=\dfrac{1}{(z+1)(z+3)}$ in Laurent series valid for

$(i)\ 1<|z|<3$ $\qquad$ $(ii)\ |z|>3$

$(iii)\ 0<|z+1|<2$ $\qquad$ $(iv)\ |z|<1$

$$f(z)=\frac{1}{(z+1)(z+3)}=\frac{1}{2}\left(\frac{1}{z+1}-\frac{1}{z+3}\right)$$

$$(i)\ \ 1<|z|<3 \quad \Rightarrow \quad \frac{1}{|z|}<1 \ \ and \ \ \frac{|z|}{3}<1$$



$$f(z)=\frac{1}{2}\left(\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{1}{3\left(\frac{z}{3}+1\right)}\right)$$

Example-3(Cont.)

$$f(z)=\frac{1}{2}\left(\frac{\left(1+\frac{1}{z}\right)^{-1}}{z}-\frac{\left(\frac{z}{3}+1\right)^{-1}}{3}\right)$$

$$=\frac{1}{2}\left[\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}\right)-\frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}\right)\right]$$

$$=\left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}\right)-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}$$

Session Summary

- A **Laurent series** is a series of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \text{ where } \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^n} ds$$

  and *n* is an integer.

- The Laurent series converges in a neighborhood of z_0 except at z_0 itself, i.e., it converges in the annular region $0 < |z - z_0| < R$.

- The of series of the negative powers in this Laurent series is called the **principal part** of $f(z)$ at z_0 , while that of the positive powers called as **analytic part**.