Lecture 21
Complex Variables

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Intended learning Outcomes

At the end of this lecture, student will be able to:

- State different forms to represent a complex number
- Explain the advantages and applications of different forms
- State DeMoivres theorem
- Apply DeMoivres theorem to find powers and roots of complex variable

Topics

- Complex number
- Complex plane

Motivation

- Complex numbers are a way to combine the idea of number (addition, multiplication, distribution, etc.) with the idea of vectors in the plane

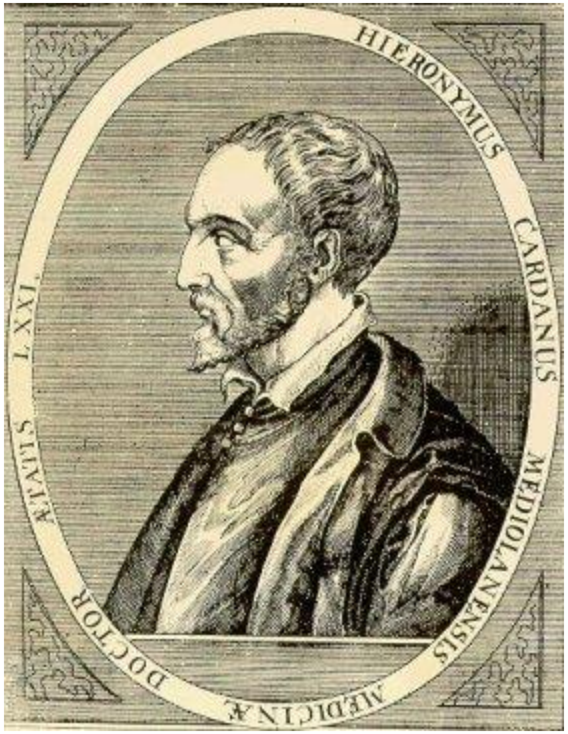
Complex Numbers

Origin of Complex numbers

$$x^2 + 1 = 0$$

$$x = \sqrt{-1}$$

Square root of negative numbers is not real !!!



Gerolamo Cardano (1545)

Set of Complex Numbers

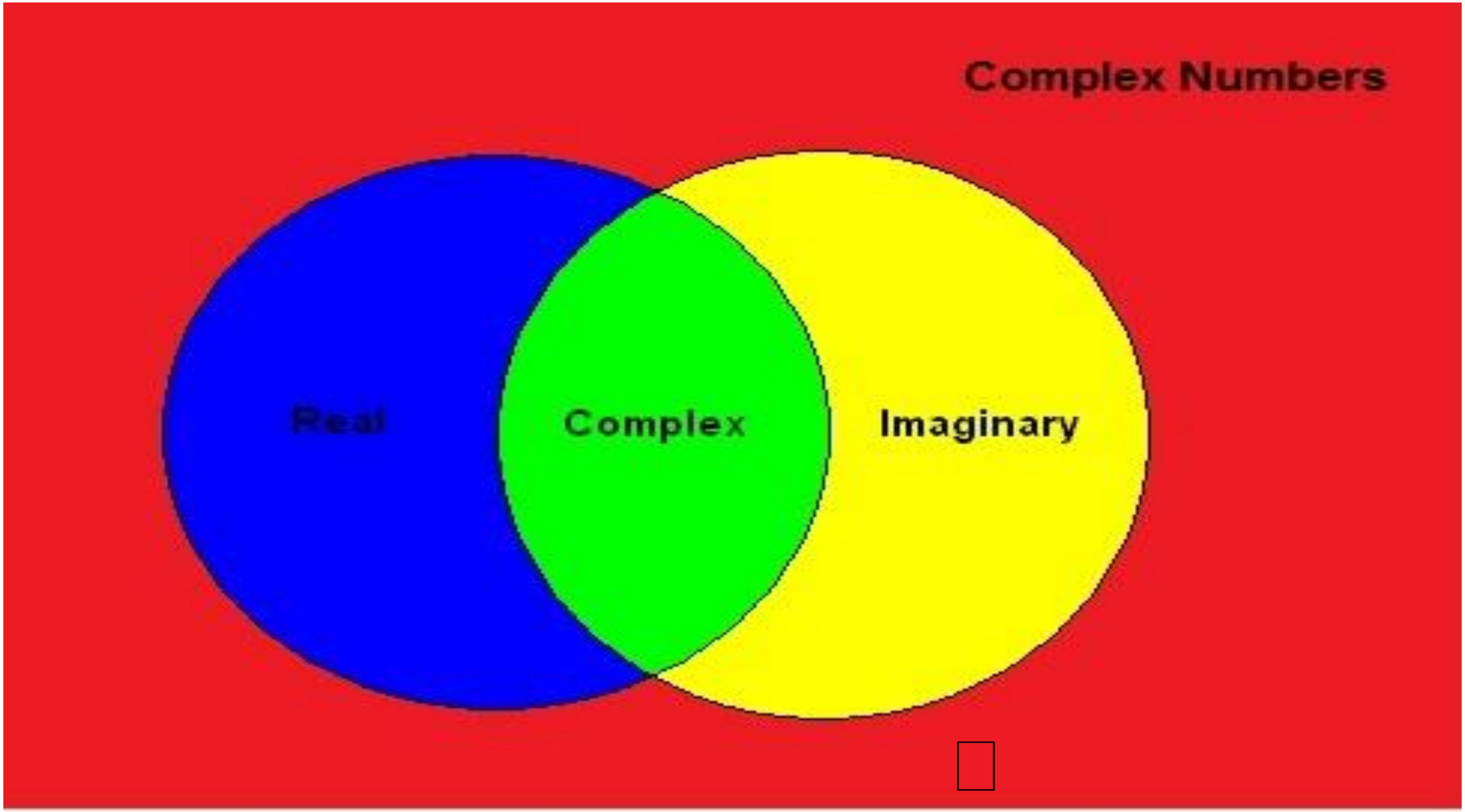
$$\mathbb{C} = Z = x + iy \;/\quad x, y \;\in \mathbb{R}, \qquad i^2 = -1$$

$$i = \sqrt{-1}$$ Imaginary unit

$$x = \mathrm{Re}(z)$$ Real part

$$y = \mathrm{Im}(z)$$ Imaginary part

Complex Numbers



Given two complex numbers

$$z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$$

Equality of two complex numbers is defined as

$$\boxed{z_1 = z_2 \Rightarrow x_1 = x_2,\ y_1 = y_2}$$

i.e., Real and imaginary parts must be equal

Sum / Difference are defined as

$$\boxed{z_1 \pm z_2 = \left(x_1 \pm x_2\right) + i\left(y_1 \pm y_2\right)}$$

i.e., Add/subtract real parts and imaginary parts

Product is defined as

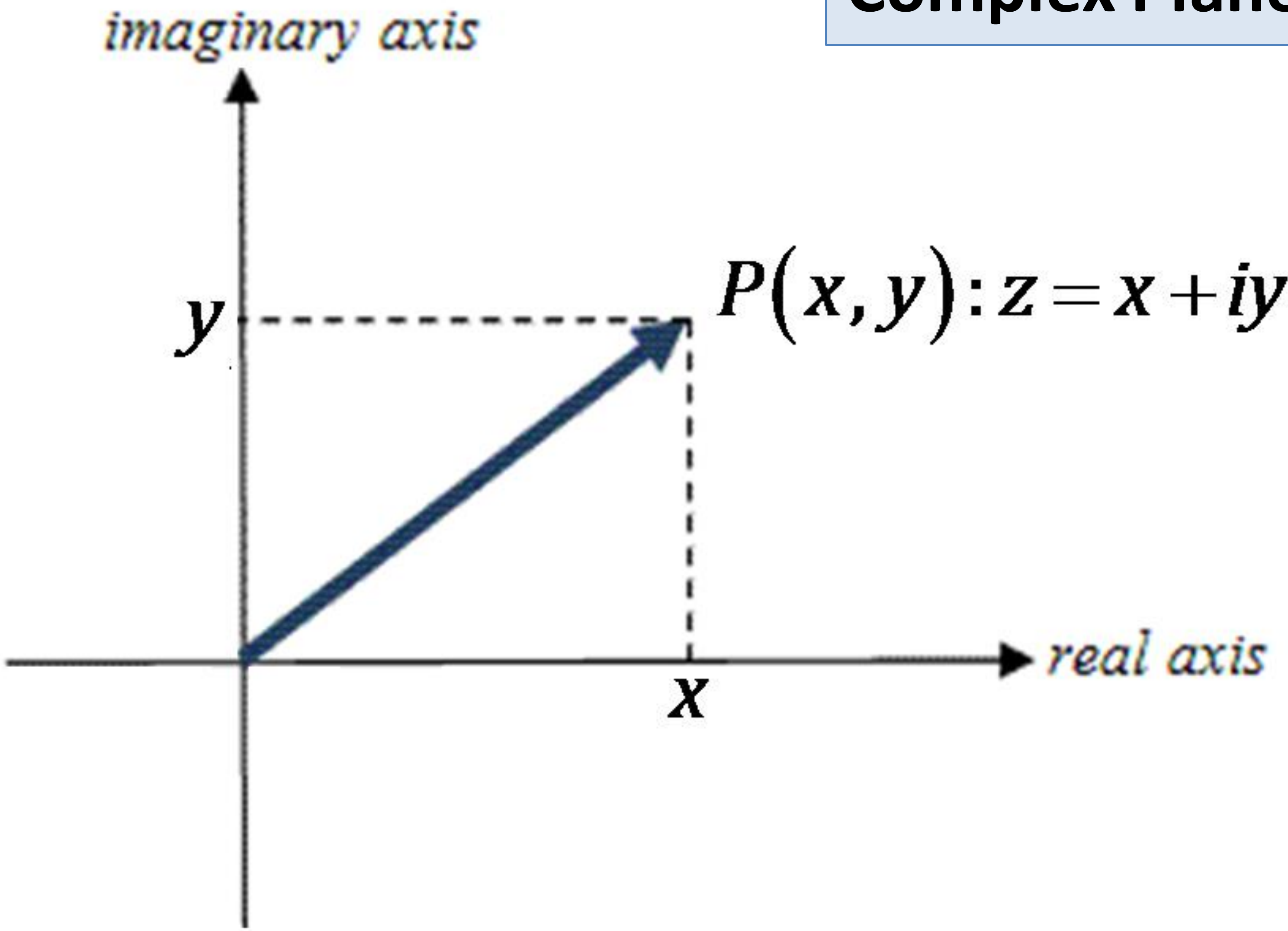
$$\boxed{z_1 z_2 = \left(x_1 x_2 - y_1 y_2\right) + i\left(x_1 y_2 + x_2 y_1\right)}$$

Quotient is defined as

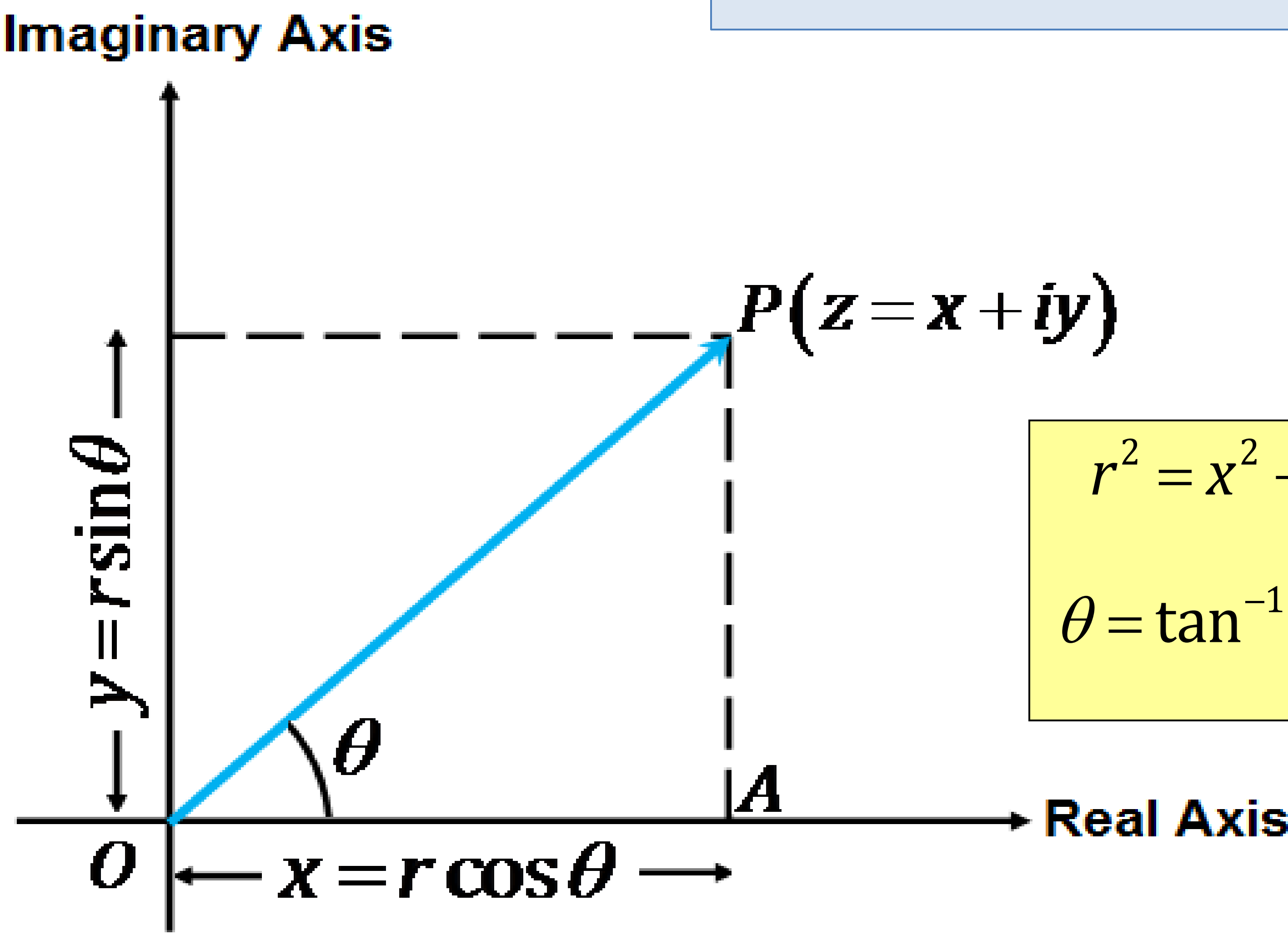
$$\boxed{\frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)} \quad \boxed{z_2 \neq 0}$$

i.e., rationalize the denominator with the conjugate of the denominator

Complex Plane



Polar Coordinate system



$$r^2 = x^2 + y^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Product and Quotient in polar form

Given two complex numbers in polar form

$$z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right),\ z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$$

The product is defined as

$$z_1 z_2 = r_1 r_2\left[\cos\left(\theta_1 + \theta_2\right) + i\sin\left(\theta_1 + \theta_2\right)\right]$$

The quotient is defined as

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

Complex number representations

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

Modulus $\longrightarrow$ $|z| = \sqrt{x^2 + y^2}$

Argument $\longrightarrow$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Equivalence

Point ≡ Complex No. ≡ Vector

$$(x, y) \equiv x + iy \equiv x\hat{i} + y\hat{j}$$

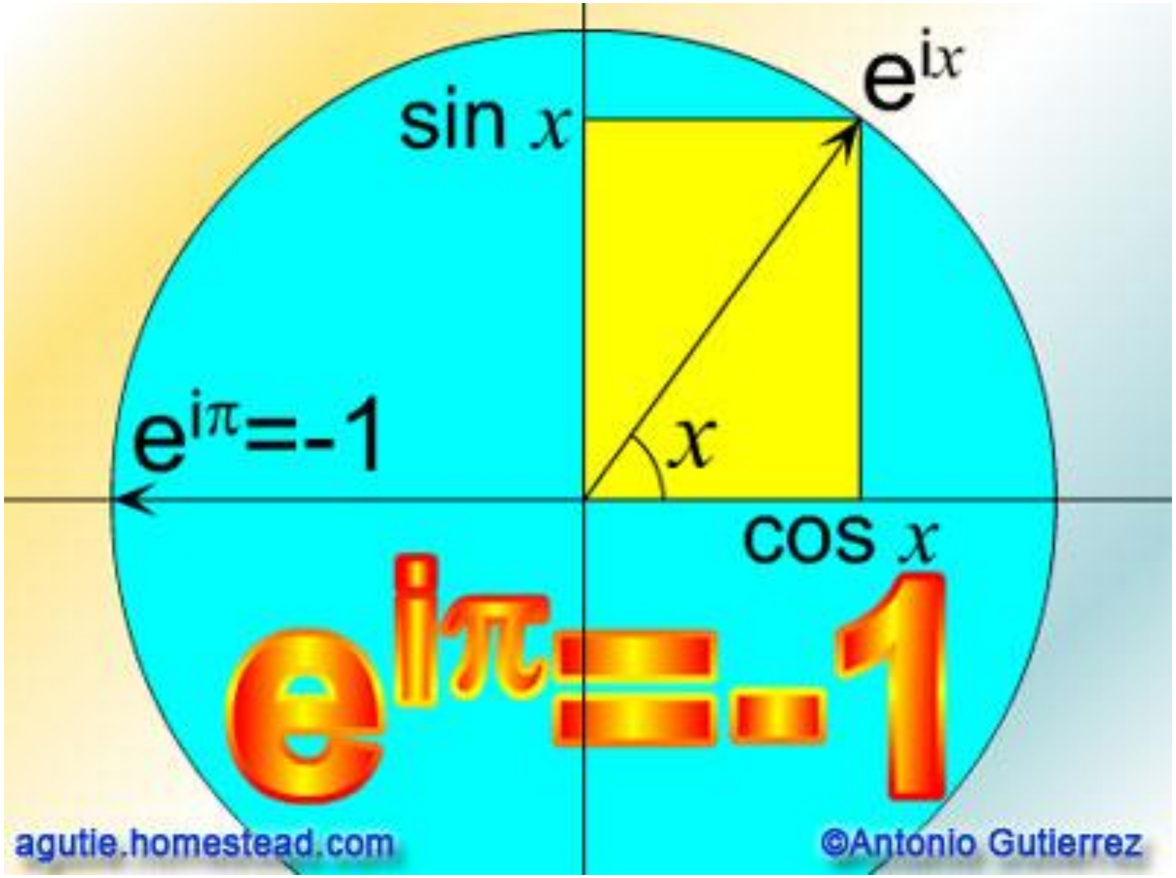
Euler Formula

For any real number *x*, Euler's formula states that the complex exponential function ssatisfies

$$\boxed{e^{ix} = \cos x + i \sin x}$$

If $x = \pi$, we get

$$\boxed{e^{i\pi} = -1}$$



De Moivre's Theorem

Given a complex number $z = r(\cos\theta + i\sin\theta)$

its power is given by

$$\boxed{z^n = r^n(\cos n\theta + i\sin n\theta)}$$

where *n* is a positive integer

This says to raise a complex number to a power, raise the modulus to that power and multiply the argument by that power.

This theorem is used to raise complex numbers to powers. It would be a lot of work to find $\left(-\sqrt{3}+i\right)^4$

$$\left(-\sqrt{3}+i\right)^4 = \left(-\sqrt{3}+i\right)\left(-\sqrt{3}+i\right)\left(-\sqrt{3}+i\right)\left(-\sqrt{3}+i\right)$$

you would need to FOIL and multiply all of these together and simplify powers of *i* --- UGH!

Instead let's convert to polar form and use DeMoivre's Theorem.

$$r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2 \quad \theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$$ but in Quad II $$\theta = \frac{5\pi}{6}$$

$$\left(-\sqrt{3}+i\right)^4 = \left[2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^{\boxed{4}} = 2^{\boxed{4}}\left[\cos\left(\boxed{4}\times\frac{5\pi}{6}\right) + i\sin\left(\boxed{4}\times\frac{5\pi}{6}\right)\right]$$

$$= 16\left[\cos\left(\frac{10\pi}{3}\right) + i\sin\left(\frac{10\pi}{3}\right)\right] = 16\left(-\frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right)i\right)$$

$$= -8 - 8\sqrt{3}i$$

Solve the following over the set of complex numbers:

$$z^3 = 1$$

We know that if we cube root both sides we could get 1 but we know that there are 3 roots. So we want the complex cube roots of 1.

Using DeMoivre's Theorem with the power being a rational exponent (and therefore meaning a root), we can develop a method for finding complex roots. This leads to the following formula:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

$$\text{where } k = 0, 1, 2, \ldots, n-1$$

Let's try this on our problem. We want the cube roots of 1.

We want cube root so our $n = 3$. Can you convert 1 to polar form? (hint: $1 = 1 + 0i$)

$$r = \sqrt{(1)^2 + (0)^2} = 1 \qquad \theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$z_k = \sqrt[3]{1}\left[\cos\left(\frac{0}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{0}{3} + \frac{2k\pi}{3}\right)\right], \text{ for } k = \underbrace{0, 1, 2}$$

We want cube root so use 3 numbers here

Once we build the formula, we use it first with $k = 0$ and get one root, then with $k = 1$ to get the second root and finally with $k = 2$ for last root.

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

$$z_k = \sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2k\pi}{3}\right)\right], \text{ for } k = 0, 1, 2$$

$$z_0 = \sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2(0)\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2(0)\pi}{3}\right)\right] = 1[\cos(0)+i\sin(0)] = \boxed{1}$$

Here's the root we already knew.

$$z_1 = \sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2(1)\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2(1)\pi}{3}\right)\right]$$

$$= 1\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right] = \boxed{-\frac{1}{2}+\frac{\sqrt{3}}{2}i}$$

If you cube any of these numbers you get 1.
(Try it and see!)

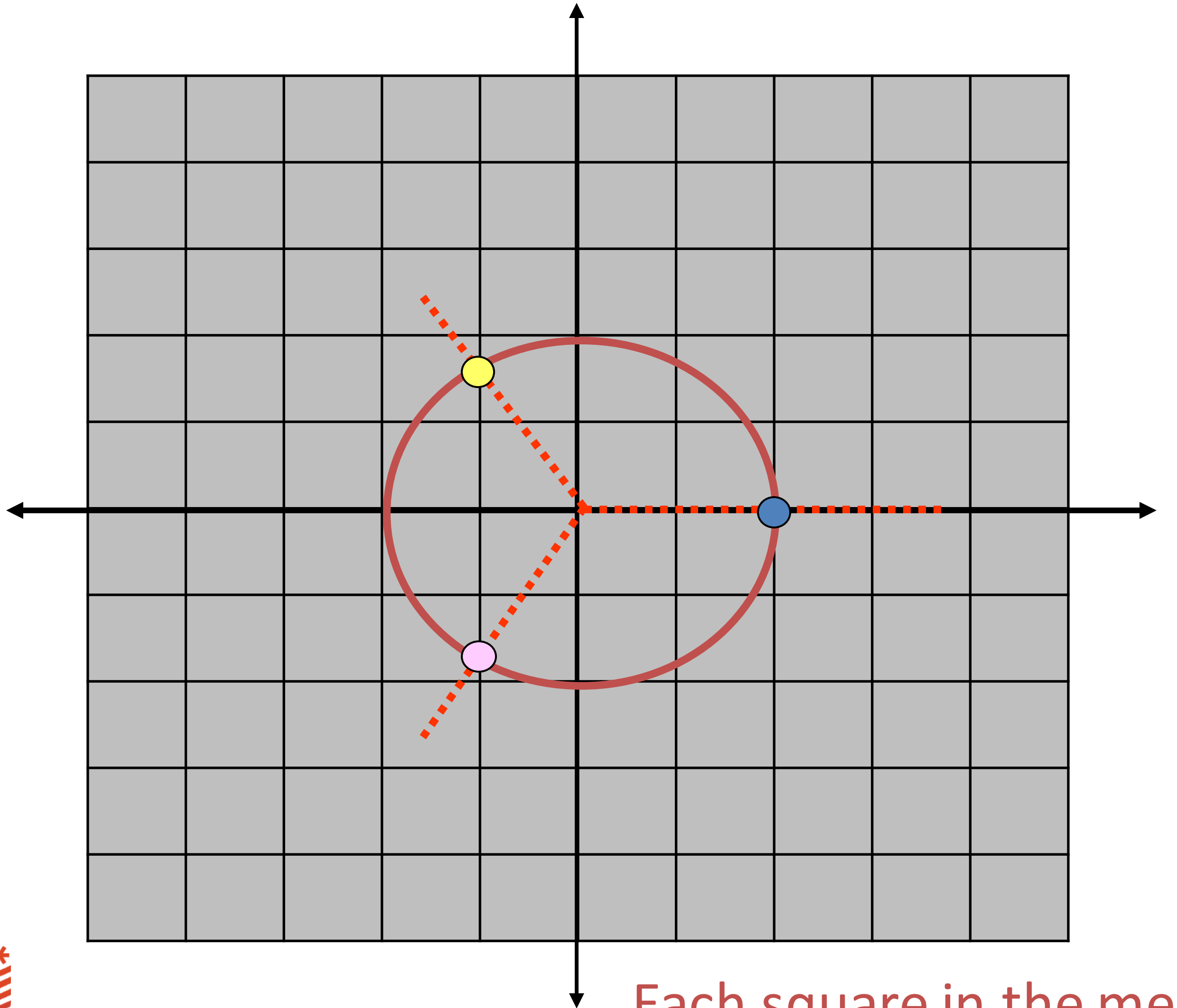
$$z_2 = \sqrt[3]{1}\left[\cos\left(\frac{0}{3}+\frac{2(2)\pi}{3}\right)+i\sin\left(\frac{0}{3}+\frac{2(2)\pi}{3}\right)\right]$$

$$= 1\left[\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right] = \boxed{-\frac{1}{2}-\frac{\sqrt{3}}{2}i}$$

We found the cube roots of 1 were: $1, \; -\frac{1}{2}+\frac{\sqrt{3}}{2}i, \; -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Let's plot these on the complex plane

about 0.9



Notice each of the complex roots has the same magnitude one. Also the three points are evenly spaced on a circle. This will always be true of complex roots.

Each square in the mesh has side = ½ unit

Complex Exponential Function

The complex exponential function is written as e^z or $\exp(z)$ where $z = x + iy$, i.e., $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

Properties

1. If z is real $e^z = e^x$
2. It is analytic for all z
3. Its derivative is itself, i.e., $(e^z)' = e^z$
4. $e^{z_1} e^{z_2} = e^{z_1+z_2}$, $e^{z_1} / e^{z_2} = e^{z_1-z_2}$ $\quad \left|e^{i\theta}\right| = 1$
5. Exponential function is periodic with period $2\pi i$

Session Summary

- There is a one to one correspondence between points in plane, vectors in plane and complex numbers
- A complex number can be written as

$$z = x + iy = r\,e^{i\theta} = r(\cos\theta + i\sin\theta)$$

where r is the modulus and θ is the argument of the complex number

- The powers and roots of the can be calculated using DeMoivres as

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$