Lecture 3
Lagrange's Mean Value Theorem

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Intended Learning Outcomes

At the end of this Lecture, student will be able to:

- State Lagrange's mean value theorem
- Discuss the geometrical interpretation of this theorem
- Apply Lagrange's mean value theorems to specific problems

Topics

- Lagrange's mean value theorem
- Geometrical meaning of Lagrange's mean value theorem
- Applications of Lagrange's mean value theorem

Motivation for Lagrange's Theorem

- If you drive between points A and B, at some time your speedometer reading was the same as your average speed over the drive.

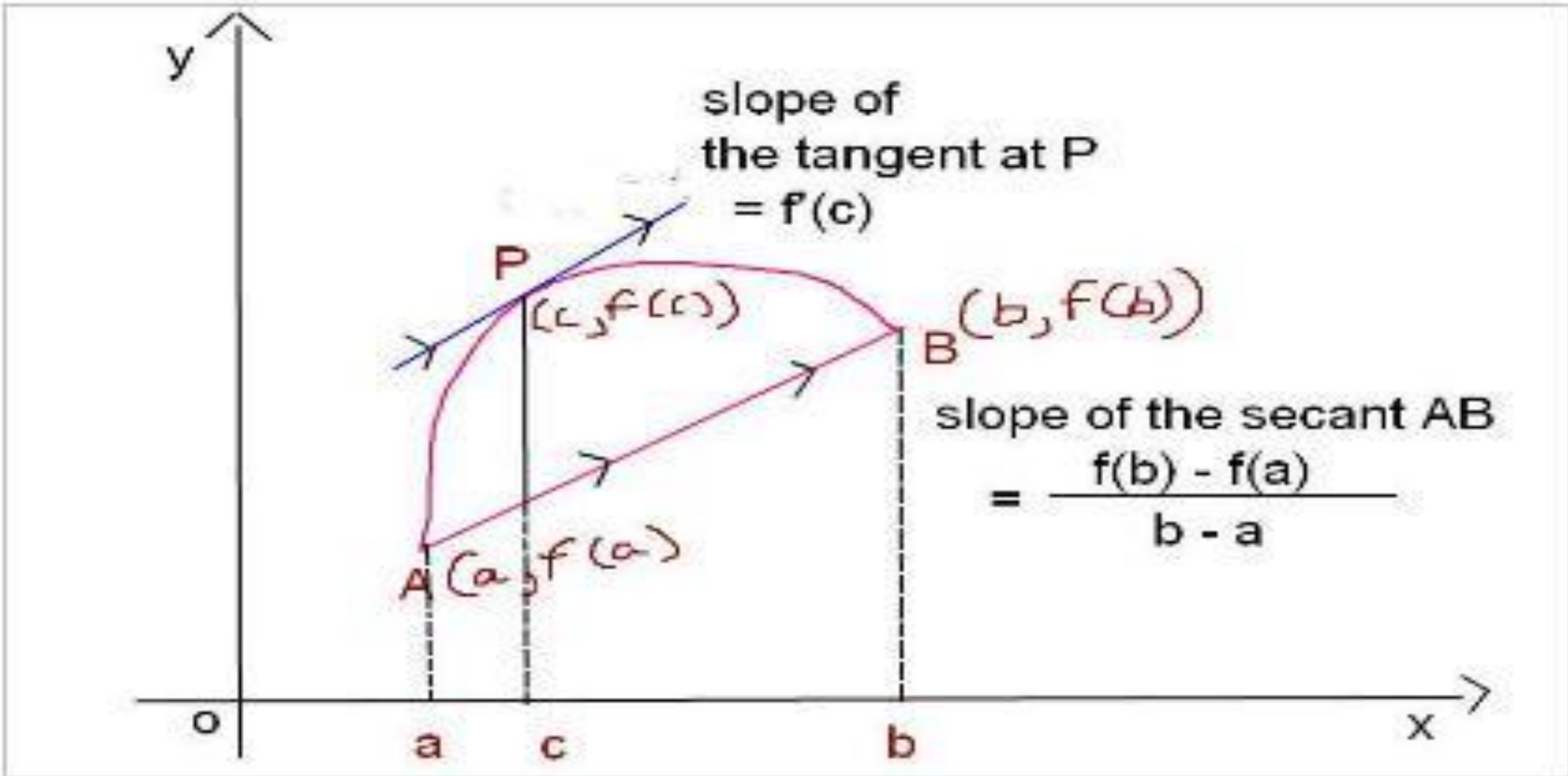
Mathematical Statement of Lagrange's Mean Value Theorem

Let f(x) be a real function defined in the closed interval [a, b] such that

- f(x) is continuous in the closed interval [a, b]
- f(x) is differentiable in the open interval (a, b)
- then there exists atleast one point c in the open interval (a,b), such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Graphical Interpretation



Geometrical Meaning

- There are no gaps in the curve $y = f(x)$ from $(a, f(a))$ and $(b, f(b))$, hence the function is continuous
- There exists unique tangent for every intermediate point between a and b
- then by Lagrange's mean value theorem, there exists atleast one point $(c, f(c))$ in between $(a, f(a))$ and $(b, f(b))$ such that tangent at $(c, f(c))$ is parallel to a straight line joining the points $(a, f(a))$ and $(a, f(a))$

Example 1

To illustrate the Mean Value Theorem with a specific function, let's consider

$$f\ (x)\ =\ x^3 - x, a\ =\ 0, b\ =\ 2.$$

Solution: Given that $f\ (x)\ =\ x^3 - x \implies f'(x) = 3x^2 - 1$

We notice that (i) $f(x)$ is differentiable in $(0,2)$

(ii) $f(x)$ is continuous in $[0,2]$

Therefore, by the Mean Value Theorem, there is a number c in $(0, 2)$ such that $f\ (2) - f\ (0)\ =\ f\ '(c)(2 - 0)$

Example 1 (Contd...)

Now $f\ (2)\ =\ 6$,

$f\ (0)\ =\ 0$, and

$f\ '(x)\ =\ 3x^2 - 1$, so this equation becomes

$6\ = 2(3c^2 - 1)$

$6 =\ 6c^2 - 2$

which gives $c^2 = \frac{4}{3}$ that is, $c = \pm\frac{2}{\sqrt{3}}$

$\Longrightarrow c = -\frac{2}{\sqrt{3}} \notin (0,2), \quad c = 2/\sqrt{3} \in (0,2)$

$\therefore$ Mean Value Theorem is verified

Example 2

Suppose that $f\ (0)\ = -3$ and $f\ '(x) \leq 5$ for all values of x. How large can $f\ (2)$ possibly be?

Solution: Given that f is differentiable (and therefore continuous) everywhere.

In particular, we can apply the Mean Value Theorem on the interval $[0, 2]$.

There exists a number c such that $f\ (2) - f\ (0)\ = f\ '(c)(2 - 0)$

Example 2 (Contd...)

so $f\,(2) = f\,(0) + 2f\,'(c) = -3 + 2f'\,(c)$

We are given that $f\,'\,(x) \leq 5$ for all x, so in particular we know that $f\,'(c) \leq 5$.

Multiplying both sides of this inequality by 2, we have $2f'(c) \leq 10$, so

$$f\,(2) = -3 + 2f\,'(c) \leq -3 + 10 = 7$$

The largest possible value for $f\,(2)$ is 7.

Example 3

Verify Lagrange's mean value theorem for the function

$f(x) = x(x-1)(x-2)$ in $\left[0, \frac{\pi}{2}\right]$.

Solution: Given that $f(x) = x(x-1)(x-2)$

$$\Longrightarrow f'(x) = 3x^2 - 6x + 2$$

We notice that (i) $f(x)$ is differentiable in $\left(0, \frac{\pi}{2}\right)$

(ii) $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

Also we find that $f(0) = 0$ and $f\left(\frac{1}{2}\right) = 3/8$

Thus f(x) satisfies both the conditions of the Lagrange's mean value theorem

Example 3(cont.)

$$\therefore f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Longrightarrow 3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2}}$$

$$\Longrightarrow 12c^2 - 24c + 5 = 0$$

$$\Longrightarrow c = 1 \pm 0.764$$

So that $c = 1.764$ or $c = 0.236$

Among these two values of c , only the value 0.236 belongs to the interval $\left(0, \frac{\pi}{2}\right)$. Thus the required value of $c = 0.236$

Example 4

Employing the Lagrange's mean value theorem, prove that

$\frac{b-a}{\sqrt{(1-a^2)}} < (\sin^{-1} b - \sin^{-1} a) < \frac{b-a}{\sqrt{(1-b^2)}}$

where a <b<1 deduce that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$

Solution: Let $f'(x) = \sin^{-1} x$ then $f'(x) = \frac{1}{\sqrt{1-x^2}}$

for $x = \pm 1$. Employing the mean value theorem to f(x) in the interval $[a, b]$, we get

Example 4 (Cont.)

$$\frac{\sin^{-1} b - \sin^{-1} a}{b-a} = \frac{1}{\sqrt{1-c^2}} \quad for \quad a < c < b < 1 \ldots. \ (i)$$

Since a<c<1, we have $\sqrt{1-a^2} > \sqrt{1-c^2}$ so that

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} \quad \ldots\ldots\ldots.(ii)$$

Since c<b<1, we have $\sqrt{1-c^2} > \sqrt{1-b^2}$ so that

$$\frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}} \quad \ldots\ldots\ldots.(iii)$$

Example 4 (Cont.)

From (ii) and (iii), we get

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

Using (i), this inequality reads

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}} \quad ....(iv)$$

Example 4 (Cont.)

For $a < b < 1$

Taking $a = 1/2$ and $b = 3/5$ in (iv), we get

$$\frac{3/5-1/2}{\sqrt{1-1/4}} < \{\sin^{-1}(3/5)-\sin(1/2)\} < \frac{3/5-1/2}{\sqrt{1-9/25}}$$

$$\Rightarrow \frac{1/10}{\sqrt{3}/2} < \{\sin^{-1}(3/5)-\pi/6\} < \frac{1/10}{4/5}$$

$$or \quad \frac{5}{\sqrt{3}} + \frac{\pi}{6} < Sin^{-1}(3/5) < 1/8 + \frac{\pi}{6}$$

Expressions (Iv) and (v) are the required results

Application of the Mean Value Theorem for Derivatives

Example 1.

You are driving a car at 55 mph when you pass a police car with radar. Five minutes later, 6 miles down the road you pass another police car with radar and you are still going 55mph. He pulls you over and gives you a ticket for speeding citing **the mean value theorem** as proof.

- Let $t = 0$ be the time you pass PC1. Let s = distance traveled.

Five minutes later is 5/60 hour = 1/12 hr. and 6 mi later, you pass PC2. There is some point in time c where your average velocity is defined by

$$\frac{f(b)-f(a)}{b-a}$$

$$\text{Average Vel.} = \frac{s(1/12)-s(0)}{(1/12-0)} = \frac{6mi}{1/12hr} = \textbf{72 mph}$$

Summary

- According to the Lagrange's Mean Value there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval [a, b]

- Mathematically, Mean value theorem geometrically implies slope chord joining the end point is equal to the slope of the tangent at some point $p \in (a,b)$
- In physical terms, the mean value theorem says that the average velocity of a moving object during an interval of time is equal to the instantaneous velocity at some moment in the interval.