# Lecture 34

# Cauchy's Integral Formula problems

Dr. Mahesha Narayana

# Intended learning Outcomes

At the end of this lecture, student will be able to:

- Apply Cauchy's integral  to evaluate complex integrals

# Topics

- Examples on Cauchy’s integral formula

# Example-1

$$I = \oint_C dz \, \frac{1}{z(z+2)}$$

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}}$$

Solution:

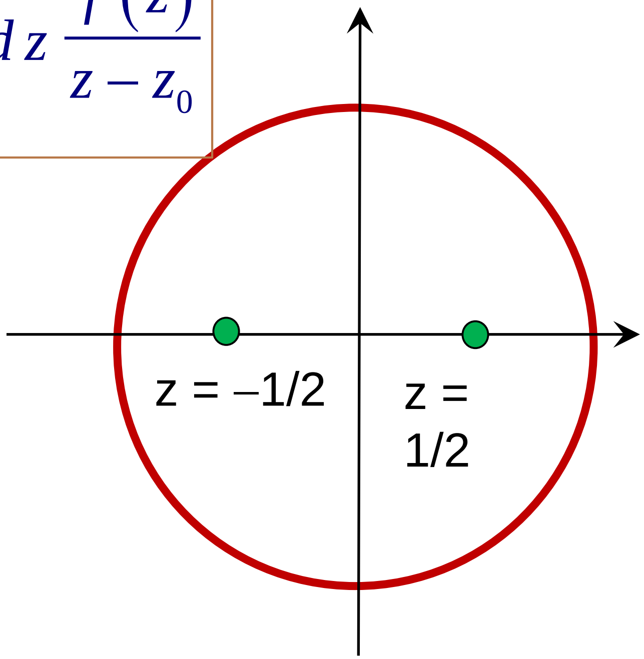
$$\frac{1}{z(z+2)} = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right)$$

$$I = \frac{1}{2}\left[\oint_C dz \, \frac{1}{z} - \oint_C dz \, \frac{1}{z+2}\right] = \frac{1}{2}(2\pi i - 0) = \pi i$$

# Example-2

$$I = \oint_C dz \, \frac{1}{4z^2 - 1}$$

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}}$$




$$\frac{1}{4z^2 - 1} = \frac{1}{(2z+1)(2z-1)}$$

$$= \frac{1}{2}\left(\frac{1}{2z-1} - \frac{1}{2z+1}\right) = \frac{1}{4}\left(\frac{1}{z-1/2} - \frac{1}{z+1/2}\right)$$

$$I = 2\pi i \, \frac{1}{4}(1-1) \quad = 0$$

# Example-3

$$I = \oint_C dz \frac{\sin^2 z}{(z-a)^4} \qquad f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \frac{f(z)}{(z-z_0)^{n+1}}$$

Solution

Let $\quad f(z) = \sin^2 z \qquad f^{(3)}(a) = \dfrac{3!}{2\pi i} \oint_C dz \dfrac{\sin^2 z}{(z-a)^4}$

$$f'(z) = 2\sin z \cos z$$

$$f''(z) = 2\left(\cos^2 z - \sin^2 z\right) \quad \rightarrow \quad I = \frac{2\pi i}{3!} f^{(3)}(a) = -\frac{8\pi i}{3} \sin a \cos a$$

$$f^{(3)}(z) = -8 \sin z \cos z$$

# Example-4

Evaluate $\int_c \frac{e^z}{z^2+1}dz$ over the circular path $|z| = 2.\,i$

***Solution**: Poles of the integrand are given by putting the denominator equal to zero*

$$z^2 + 1 = 0 \implies z^2 = -1 \implies z \mp i$$

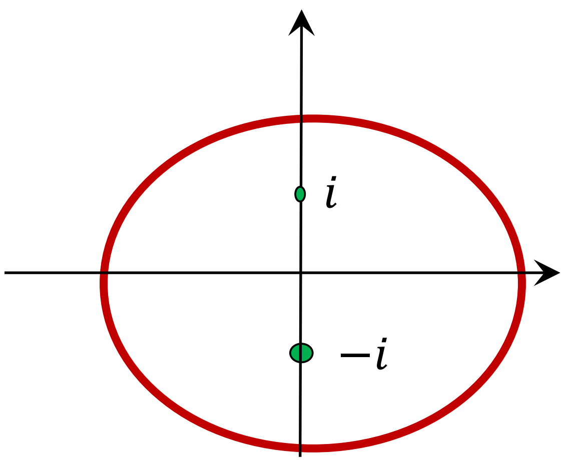
*The integrand has two simple poles at* $z = i$ *and* $z = -i$

Both poles are inside the given circle with center at the origin and radius 2

$$\int_c \frac{1}{2i}\left(\frac{e^z}{z-i} - \frac{e^z}{z+i}\right)dz = \frac{1}{2i}\int_c \frac{e^z}{z-i}\,dz - \frac{1}{2i}\int_c \frac{e^z}{z+i}dz$$

$$= \frac{1}{2i}\{2\pi i(e^z)_{z=i} - 2\pi i\,(e^z)_{z=i}\}$$

$$= \frac{2\pi i}{2i}\{e^i - e^{-i}\} = 2\pi i\, sin1$$



# Session Summary

- **Cauchy's integral theorem** states that if $f(z)$ is analytic in a simply connected domain $D$, then for every closed path $C$ in $D$

$$\oint_C f(z)\,dz = 0.$$

- Cauchy Integral Formula : $\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}}$