Lectures 12-13
Derivatives of Composite and Implicit Functions

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Define composite and implicit functions
- Find derivatives of composite functions and implicit functions
- Evaluate higher order partial derivatives

Topics

- Composite functions
- Implicit functions
- Differentiation of Composite and Implicit function
- Chain rule for differentiation of composite function

Motivation

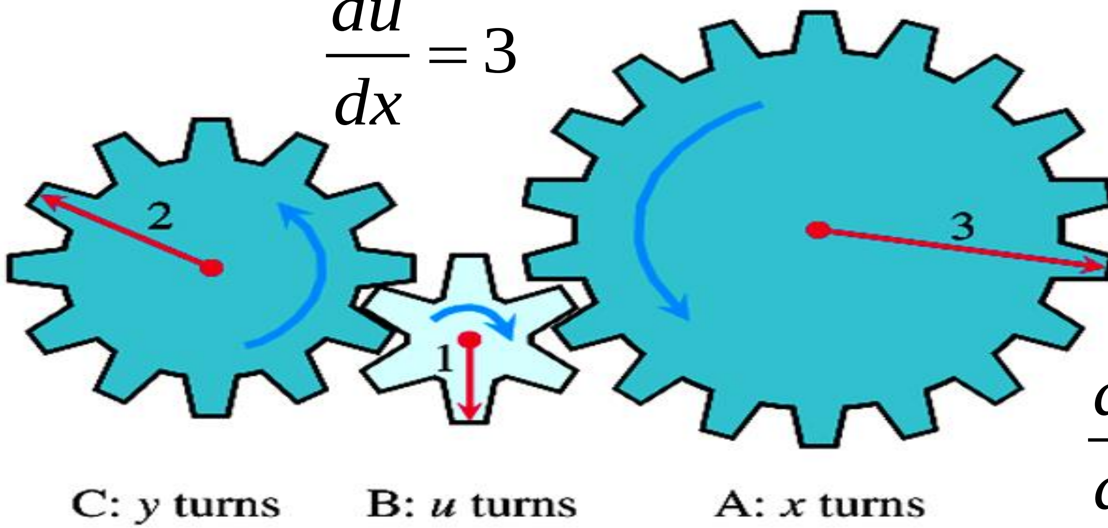
When gear A makes x turns, gear B makes u turns and gear C makes y turns., u turns 3 times as fast as x

y turns ½ as fast as u

So y turns 3/2 as fast as x

$$\frac{du}{dx} = 3$$

$$\frac{dy}{du} = \frac{1}{2}$$



$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Note: The Rates are multiplied

Composite Functions

- **Definition:** A function y is a **composite** of functions *f* and *g* if

$$y(x) = f\,[g(x)]$$

- The domain of *m* is the set of all numbers *x* such that *x* is in the domain of *g* and *g*(*x*) is in the domain of *f*.

- When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

Composite Functions and the Chain Rule

If y = f(u) and u = g(x) then y = f(g(x)) and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

multiply rates

$$\frac{dy}{dx} = \frac{d}{dx}\left(f(g(x)) \right) = f'(g(x)) \cdot g'(x)$$

multiply rates

Chain Rule

- The **chain rule**, that enables us to compute the derivatives of many composite functions of the form f(g(x)).

Chain Rule: If $y = f(u)$ and $u = g(x)$ define the composite function

$y = f[g(x)]$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \quad \text{provided } \frac{dy}{du} \text{ and } \frac{du}{dx} \text{ exist.}$$

or **y’(x) = f’[g(x)] g’(x)** provided that f’[g(x) and g’(x) exist

$$y' = \underbrace{f'(u)}_{\substack{\text{Derivative of} \\ \text{outer function}}} \cdot \underbrace{u'}_{\substack{\text{Derivative of} \\ \text{inner function}}}$$

Chain Rule

The chain rule can be extended to compositions of three or more functions.

Given: Y = f(w), w = g(u), and u = h(x)

Then
$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

The Chain Rule for Functions of Two Variables

- For functions of more than one variable, the Chain Rule has several versions.
- Each gives a rule for differentiating a composite function.

Case 1: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

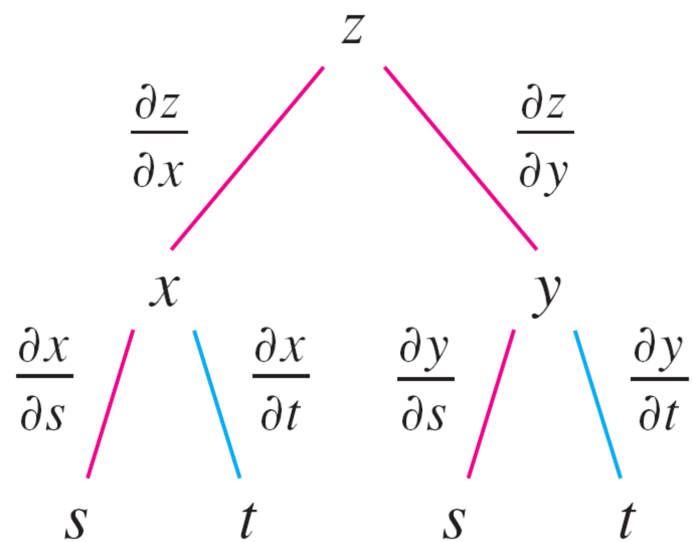
The Chain Rule for Functions of Two Variables

Case 2: Suppose z = f(x, y) is a differentiable function of x and y, where x = g(s, t) and y = h(s, t) are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Observe

- s and t are independent variables
- x and y are called intermediate variables
- z is the dependent variable



Example-1

$$z = xy^2 + x^2 y \quad where \quad x = at \;, y = 2at$$

$$\{z \to (x, y) \to t\} \Rightarrow z \to t \;\& \frac{dz}{dt}$$

is the total derivative

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} \quad \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (y^2 + 2xy)a + (2xy + x^2)2a$$

$$= 18a^3t^2 \qquad ....(i) \quad \text{(substitute for } x \;\&\; y\text{)}$$

Example-1 (cont.)

Now by direct substitution we have

$$z = xy^2 + x^2 y$$ substitute for x &y

$$= 6a^3t^3$$ differentiate w.r.t t ,

$$\frac{dz}{dt} = 18a^3t^2 \qquad ....(ii)$$

From (i) and (ii) the result is verified

Example-2

- If $Ifu = f\left(\frac{x}{y},\frac{y}{z},\frac{z}{x}\right),\ prove\ that\ \ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

Here we need to convert the given function u into a composite function $u = f(p,q,r)\ \ where\ \ p=\frac{x}{y},\ q=\frac{y}{z},\ r=\frac{z}{x}$

$$\{u \to f(p,q,r) \to (x,y,z)\} \Rightarrow u \to x,y,z$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial u}{\partial q}\frac{\partial q}{\partial x}+\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial p}\frac{1}{y}+\frac{\partial u}{\partial q}.0+\frac{\partial u}{\partial r}\left(\frac{-z}{x^2}\right)$$

Example-2(cont.)

$$x\frac{\partial u}{\partial x}=\frac{x}{y}\frac{\partial u}{\partial p}-\frac{z}{x}\frac{\partial u}{\partial r} \qquad ....(1)$$

Similarly by symmetry we can write

$$y\frac{\partial u}{\partial y}=\frac{y}{z}\frac{\partial u}{\partial q}-\frac{x}{y}\frac{\partial u}{\partial p} \qquad ....(2)$$

$$z\frac{\partial u}{\partial z}=\frac{z}{x}\frac{\partial u}{\partial r}-\frac{y}{z}\frac{\partial u}{\partial q} \qquad ....(3)$$

Add. (1),(2)and (3), we get $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$

Example-3

If $u = f(x-y, y-z, z-x)$ *show that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Let $u = f(p,q,r)$ *where* $p = x-y, \quad q = y-z, \quad r = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}.1 + \frac{\partial q}{\partial x}.0 + \frac{\partial u}{\partial r}(-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}$$

Similarly we have by symmetry,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \quad \ldots\ldots(i) \qquad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \quad \ldots\ldots(i)$$

Example-3(cont.)

Thus by adding (1), (2) and (3) we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

Why Implicit Differentiation?

- When an applied problem involves an equation not in explicit form, implicit differentiation is used to locate extrema or to find rates of change.

- Up to this point, all of our functions have been expressed as y = f(x), that is y is expressed explicitly in terms of the independent variable x. However, not all functions are expressed in this form.
  - Explicit functions:
    1) $y = 3x – 2$
    2) $y = x^2 + 5$
  - Implicit functions:
    1) $y^2 + 2yx \ 4x^2 = 0$
    2) $y^5 - 3y^2x^2 + 2 = 0$

Process for Implicit Differentiation

- To find dy/dx
- Differentiate both sides with respect to x (y is assumed to be a function of x, so d/dx)
- Collect like terms (all dy/dx on the same side, everything else on the other side)
- Factor out the dy/dx and solve for dy/dx

Implicit Differentiation

- If F is differentiable, we can apply the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x and obtain:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

- we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.
  - This means that $F(x, y, f(x, y)) = 0$, for all (x, y) in the domain of f.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Implicit Differentiation

- Find y' if $x^3 + y^3 = 6xy$
  - The given equation can be written as:

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

  - So,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Implicit Differentiation for Functions of Two Variables

- Find $\partial z/\partial x$ and $\partial z/\partial y$ if

$$x^3 + y^3 + z^3 + 6xyz = 1$$

  – Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$, then we have:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

Summary

- Implicit functions are those is which dependent variable cannot be expressed explicitly in terms of the independent variable

- When performing implicit differentiation we must find the derivative of y as a Chain Rule since y is a function of x and multiply by y'

- Chain rule can also be operated on functions involving more than one variable