Lecture 33
Cauchy's Integral Formula-1

Dr. Mahesha Narayana

Intended learning Outcomes

At the end of this lecture, student will be able to:

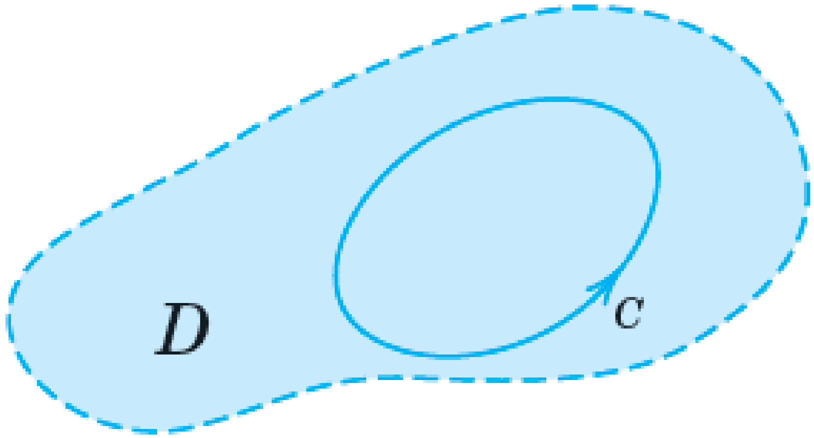
- State Cauchy’s integral theorem and its utility
- Apply Cauchy’s integral theorem to evaluate complex integrals

Topics

- Cauchy theorem
- Extension of Cauchy theorem
- Cauchy's integral formula
- Cauchy inequality

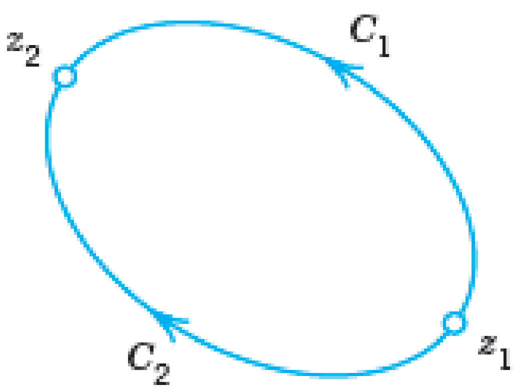
Cauchy's Theorem

Let *f*(*z*) be an analytic in a simply connected domain D, then for every closed path C in D we have



$$\oint_C f(z)\,dz = 0$$

Independence of Path of Integration

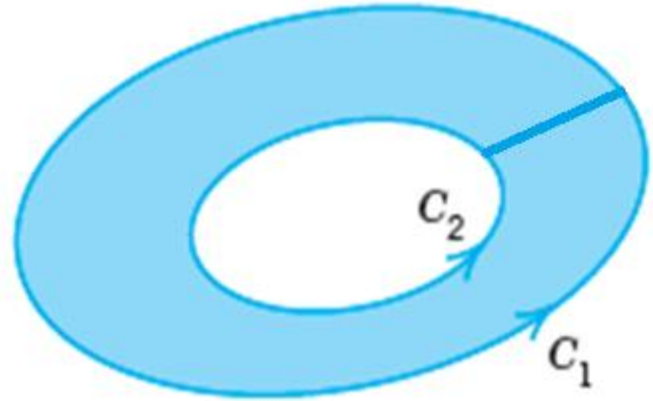


If *f*(*z*) is analytic in a simply connected domain D then the integral of *f*(*z*) is independent of path of integration

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

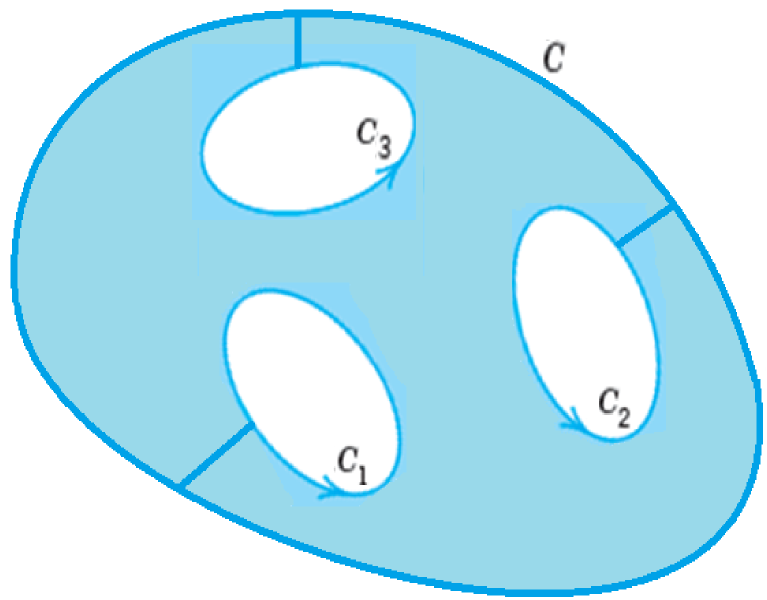
Extension of Cauchy’s Theorem

If C_1 and C_2 are two simple closed curves such that C_2 lies entirely within C_1 and if $f(z)$ is analytic on C_1 , C_2 and ir the region bounded by C_1 and C_2 ther



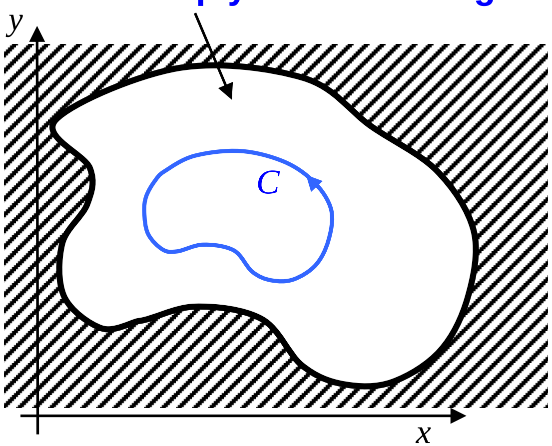
$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

If C is a simple closed curve enclosing non overlapping simple closed curves C_1 , C_2 and C_3 and $f(z)$ is analytic in the annular region between C and these curves then



$$\oint_{C} f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$

Cauchy's Theorem



- Cauchy's Theorem : If $f(z)$ is analytic in R then

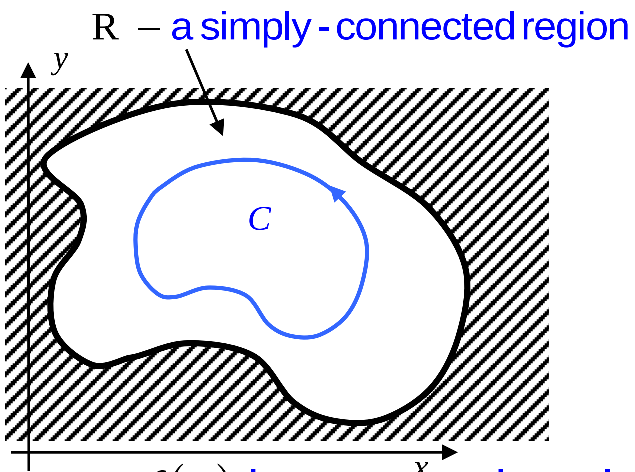
$$\oint_C f(z)\,dz = 0$$

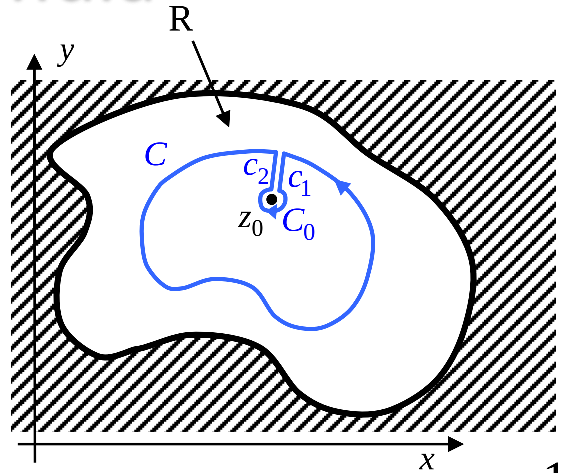
- First, note that if $f(z) = w = u + iv$, then

$$\oint_C f(z)\,dz = \oint_C udx - vdy \; + i \oint_C vdx + udy \;\; ;$$

now use a well-known vector analysis result to prove

Cauchy Integral Formula





- $f(z)$ is assumed analytic in R but we multiply by a factor $\dfrac{1}{(z-z_0)}$ that is analytic *except* at z_0 and consider the integral around C

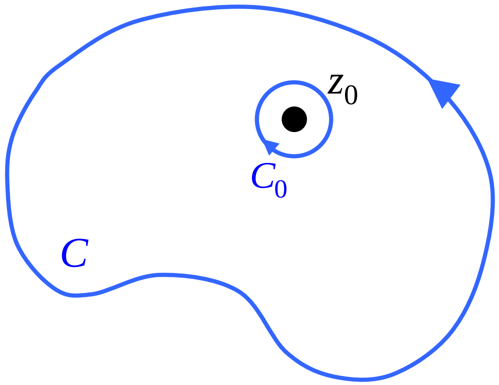
$$\int_C \frac{f(z)}{(z-z_0)}dz$$

- To evaluate, consider the path $C+c_1+c_2+C_0$ shown that encloses a simply - connected region for which the integrand is analytic on and inside the path :

$$\int_{C+\cancel{c_1}+\cancel{c_2}+C_0} \frac{f(z)}{z-z_0}dz = 0 \quad \Rightarrow \quad \int_C \frac{f(z)}{z-z_0}dz = -\int_{C_0} \frac{f(z)}{z-z_0}dz$$

Cauchy Integral Formula, cont'd

$$\int_{C} \frac{f(z)}{z-z_0}\,dz = -\int_{C_0} \frac{f(z)}{z-z_0}\,dz$$



- Evaluate the C_0 integral on a circular path, $z-z_0 = re^{i\theta}$, $dz = rie^{i\theta}d\theta$:

$$\int_{C_0} \frac{f(z)}{(z-z_0)}\,dz \overset{r\to 0}{=} f(z_0)\int_{2\pi}^{0} \frac{\cancel{r}\,i\,\cancel{e^{i\theta}}\,d\theta}{\cancel{r}\,\cancel{e^{i\theta}}} = -2\pi i f(z_0) \quad \text{for } r\to 0$$

$$\Rightarrow \quad \int_{C} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0) \quad \Rightarrow \quad \boxed{f(z_0) = \frac{1}{2\pi i}\oint_{C} \frac{f(z)}{z-z_0}\,dz}$$

Cauchy Integral Formula

- The value of $f(z)$ at z_0 is completely determined by its values on C !

Cauchy Integral Formula, cont'd

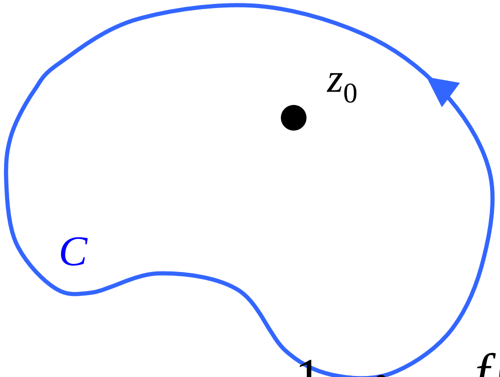
- Note that if z_0 is outside C, the integrand is analytic inside C; hence by the Cauchy integral theorem,

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = 0$$

- In summary,

$$\boxed{\oint_C \frac{f(z)}{z-z_0}\,dz = \begin{cases} 2\pi i\, f(z_0), & z_0 \text{ inside } C \\ 0, & z_0 \text{ outside } C \end{cases}}$$

Derivative Formulas



- Since $f(z)$ is analytic in C, its derivative exists; let's express it in terms of the Cauchy formula,

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\, dz$$

- $$f(z_0 + \Delta z) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0 - \Delta z}\, dz$$

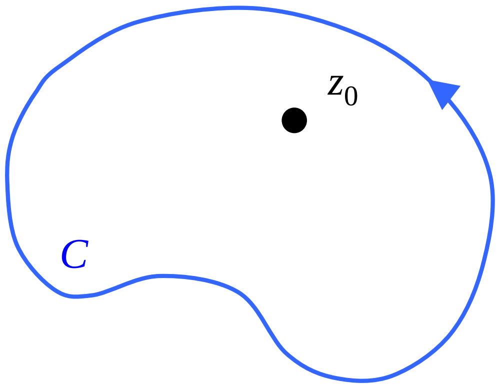
$$\Rightarrow \quad \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i \Delta z}\oint_C \left(\frac{f(z)}{z - z_0 - \Delta z} - \frac{f(z)}{z - z_0} \right) dz$$

$$= \frac{1}{2\pi i \cancel{\Delta z}}\oint_C f(z) \left(\frac{\cancel{\Delta z}}{(z - z_0 - \Delta z)(z - z_0)} \right) dz$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{1}{2\pi i}\oint_C f(z) \left(\frac{1}{(z - z_0 - \Delta z)(z - z_0)} \right) dz$$

$$\Rightarrow \quad f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_0)^2}\, dz$$

We've also just proved we can differentiate w.r.t. z_0 under the integral sign!



- Similarly,

$$f''(z_0) = \frac{2}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

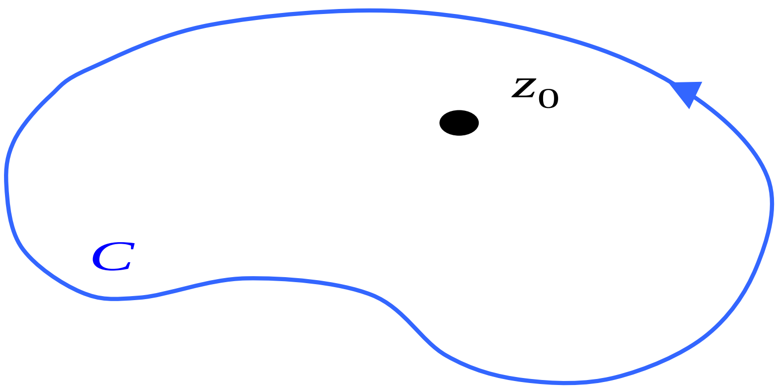
- In general,

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

or $$f^{(n)}(z_0) = \frac{1}{2\pi i}\oint_C f(z)\frac{d^n}{dz_0^n}\left(\frac{1}{z-z_0}\right)dz$$

- $\Rightarrow$ If $f(z)$ is analytic in C, then its derivatives of all orders exist, and hence they are analytic as well.

Derivative Formulas, cont'd



- Similarly,

$$f''(z_0) = \frac{2}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz$$

- In general,

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$

or $$f^{(n)}(z_0) = \frac{1}{2\pi i}\oint_C f(z)\frac{d^n}{dz_0^n}\left(\frac{1}{z-z_0}\right)dz$$

- $\Rightarrow$ If $f(z)$ is analytic in C, then its derivatives of all orders exist, and hence they are analytic as well.

Cauchy's Inequality

- Suppose $f(z)$ is (a) analytic in, (b) bounded ($|f(z)| < M$) on, and (c) has a convergent power series representation,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

within a circle of radius R about the origin. Then $|a_n| \le \dfrac{M}{R^n}$.

- By the Cauchy Integral Formula,

$$\frac{1}{2\pi} \int_{|z|=R} \frac{f(z)}{z^{m+1}} dz = \frac{1}{2\pi} \sum_{n=0}^{\infty} a_n \int_{|z|=R} z^{n-m-1} dz = \frac{1}{\cancel{2\pi}} \cancel{2\pi} i a_m$$

$$\Rightarrow \quad |a_m| = \frac{1}{2\pi} \left| \int_{|z|=R} \frac{f(z)}{z^{m+1}} dz \right| \le \frac{1}{2\pi} \int_{|z|=R} \frac{|f(z)|}{|z^{m+1}|} |dz| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{M}{R^{m+\cancel{1}}} \cancel{R} d\theta = \frac{M}{R^m}$$

$$\overset{m \to n}{\Rightarrow} \quad \boxed{|a_n| \le \frac{M}{R^n}}, \qquad M \equiv \max_{|z|=R} |f(z)|$$

Session Summary

- **Cauchy's integral theorem** states that if $f(z)$ is analytic in a simply connected domain D, then for every closed path C in D

- If $f(z)$ is analytic the complex line integral is independent of the path joining end points of the curve.

$$\oint_C f(z)\,dz = 0.$$