Lecture 20
Lagrange's Method of Multipliers_II

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Intended learning Outcomes

At the end of this lecture, student will be able to:

- State and explain Lagrange Method of Multipliers
- Apply the Lagrange Method of Multipliers to maximize/minimize the given function subject to equality constraints

Topics

- Lagrange method of multipliers
- Maximum and minimum by Lagrange method of multipliers
- Examples

Lagrange's multipliers

- Let us try to maximize the volume function $V = xyz$ subject to the constraint $2xz + 2yz + xy = 12$

- Solution: We wish to maximize the function $V = xyz$ subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

- Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that:

$$\nabla V = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 12$$

Lagrange's multipliers

- This gives the following equations
  - $V_x = \lambda g_x$
  - $V_y = \lambda g_y$
  - $V_z = \lambda g_z$
  - $2xz + 2yz + xy = 12$

- The above equations becomes:
  - $yz = \lambda(2z + y)$……………………………1
  - $xz = \lambda(2z + x)$……………………………2
  - $xy = \lambda(2x + 2y)$…………………………..3
  - $2xz + 2yz + xy = 12$…………………….4

Lagrange's multipliers

- In this example, we notice that if we multiply Equation 2 by x, Equation 3 by y, and Equation 4 by z, then left sides of the equations will be identical, which gives the following equations
  - $xyz = \lambda(2xz + xy)$……………….5
  - $xyz = \lambda(2yz + xy)$……………….6
  - $xyz = \lambda(2xz + 2yz)$……………..7
- We observe that $\lambda \neq 0$ because $\lambda = 0$, would imply $yz = xz = xy = 0$ from Equations 1, 2 and 3. This would contradict Equation 4
- Therefore, from Equations 5 and 6, we have
  - $2xz + xy = 2yz + xy$…………….8

Lagrange’s multipliers

- From Equations 6 and 7, we have
  - $2yz + xy = 2xz + 2yz$
- which yields us $2xz = xy$, since $x \neq 0$, $y = 2z$
- If we now put $x = y = 2z$ in Equation 6, we get:
  - $4z^2 + 4z^2 + 4z^2 = 12$
- Since x, y, and z are all positive, we therefore have $z = 1$, and so $x = 2$ and $y = 2$.

Lagrange's multipliers

- Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$

- We are asked for the extreme values of f subject to the constraint

$$g(x, y) = x^2 + y^2 = 1$$

- Using Lagrange multipliers, we solve the equations $\nabla f = \lambda \nabla g$

- These can be written as:
  - $f_x = \lambda g_x$
  - $f_y = \lambda g_y$
  - $g(x, y) = 1$

Lagrange's multipliers

- The above equation can be written
  - $2x = 2x\lambda$................*1*
  - $4y = 2y\lambda$................*2*
  - $x^2 + y^2 = 1$............3
- From Equation 1, we have $x = 0$ or $\lambda = 1$
  - If $x = 0$, then Equation 3 gives $y = \pm 1$
  - If $\lambda = 1$, then $y = 0$ from Equation 2;
    so, then Equation 3 gives $x = \pm 1$.
- Therefore, f has possible extreme values at the points

$$(0, 1), (0, -1), (1, 0), (-1, 0)$$

- Evaluating f at these four points, we find that:
  - $f(0, 1) = 2 \quad f(0, -1) = 2 \quad f(1, 0) = 1 \quad f(-1, 0) = 1$

Lagrange's multipliers

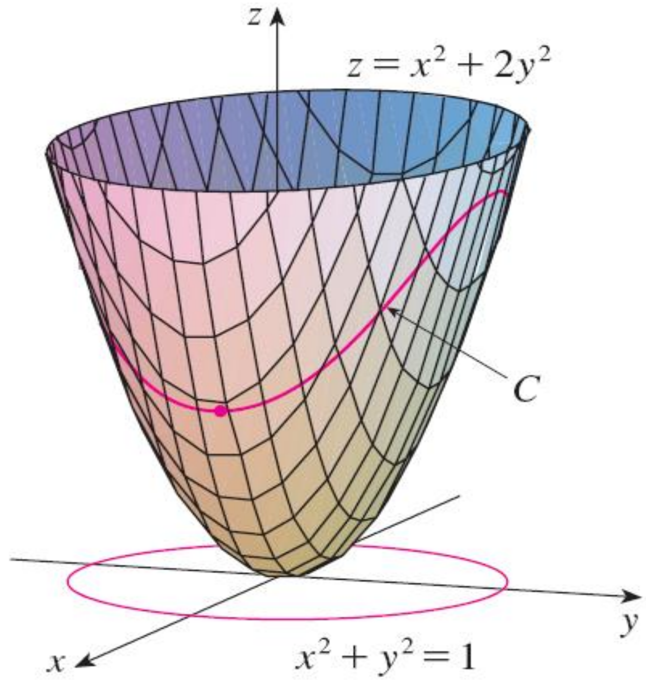
- Therefore, the maximum value of f on the circle $x^2 + y^2 = 1$ is:

$$f(0, \pm 1) = 2$$

- The minimum value is:

$$f(\pm 1, 0) = 1$$

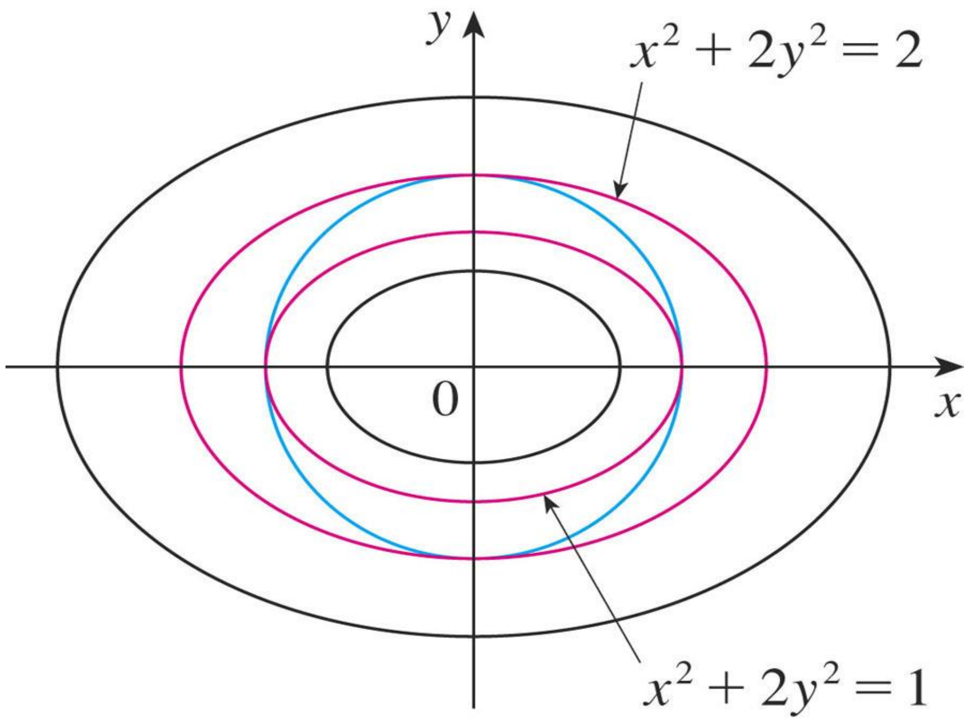
- One can check these values by plotting and is given in the figure



Lagrange's multipliers

- The geometry behind the use of Lagrange multipliers in second Example is shown here

- The extreme values of $f(x, y) = x^2 + 2y^2$ correspond to the level curves that touch the circle $x^2 + y^2 = 1$



Lagrange's multipliers

- Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point (3, 1, –1).
- The distance from a point (x, y, z) to the point (3, 1, –1) is:

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

- However, the algebra can be made simpler if we instead maximize and minimize the square of the distance, hence

$$d^2 = f(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

- The constraint is that the point (x, y, z) lies on the sphere, that is,

$$g(x, y, z) = x^2 + y^2 + z^2 = 4$$

Lagrange's multipliers

- According to the method of Lagrange multipliers, we solve

$$\nabla f = \lambda \nabla g, \; g = 4$$

- Which yields us the following equations
  - $2(x - 3) = 2x\lambda$……….*1*
  - $2(y - 1) = 2y\lambda$ ………*2*
  - $2(z + 1) = 2z\lambda$ ……….*3*
  - $x^2 + y^2 + z^2 = 4$ ………*4*
- The simplest way to solve these equations is to solve for *x*, *y*, and *z* in terms of *λ* from Equations 1, 2 and 3, and then substitute these values into Equation 4

Lagrange's multipliers

- From Equation 1, we have
- $x - 3 = x\lambda$ or $x(1-\lambda) = 3$ or $x = \dfrac{3}{1-\lambda}$
- Similarly, Equations 2 and 3 gives:
- $y = \dfrac{1}{1-\lambda}$ and $z = \dfrac{-1}{1-\lambda}$
- Note that $1 - \lambda \neq 0$ because $\lambda = 1$ is impossible from Equation 1
- Upon substituting for x, y and z, Equation 4 takes the form

$$\frac{3^2}{(1-\lambda)^2} + \frac{1^2}{(1-\lambda)^2} + \frac{(-1)^2}{(1-\lambda)^2} = 4$$

Lagrange's multipliers

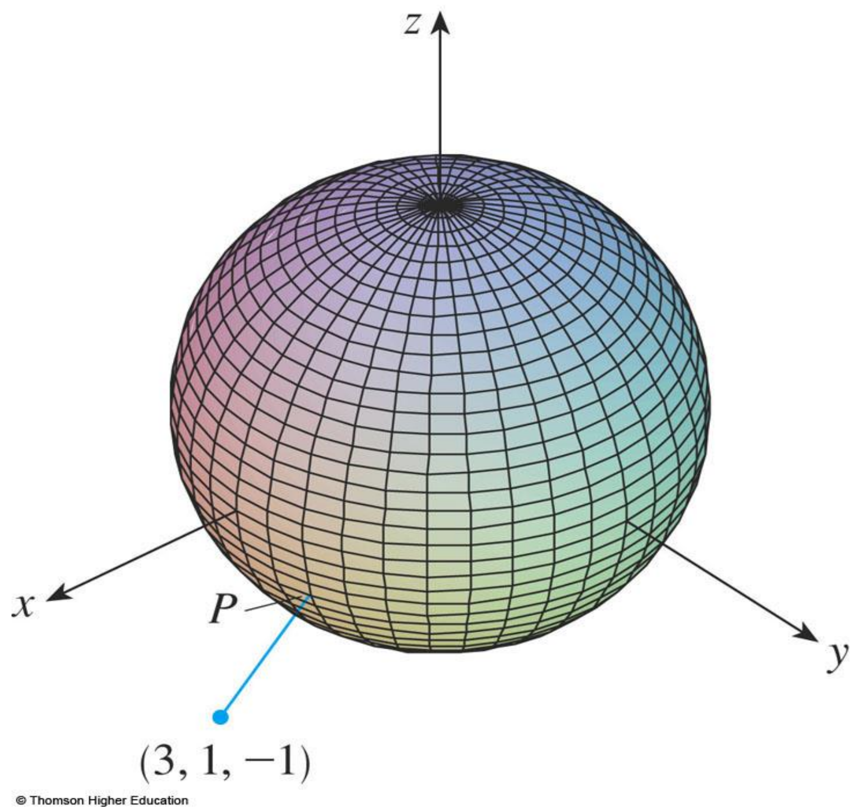
- This gives $(1-\lambda)^2 = 11/4$, $1-\lambda = \pm\sqrt{11}/2$
- Thus $\lambda = 1 \pm \dfrac{\sqrt{11}}{2}$
- These values of λ then give the corresponding points (x, y, z):

$$\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) \text{ and } \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

- Thus, the closest point is: $\left(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}\right)$
- The farthest is: $\left(-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11}\right)$

Lagrange’s multipliers

- The figure shows the sphere and the nearest point in Example 3



Session Summary

•In mathematical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and minima of function subject to equality constraints.

- Procedure for Applying the Method of Lagrange Multipliers

Step 1. Write the problem in the form:

Maximize (minimize) $f(x, y)$ subject to $g(x, y) = k$

Step 2. Simultaneously solve the equations

$$f_x(x,y) = \lambda g_x(x,y)$$
$$f_y(x,y) = \lambda g_y(x,y)$$
$$g(x,y) = k$$

Step 3. Evaluate *f* at all points found in step 2. If the required maximum (minimum) exists, it will be the largest (smallest) of these values.