Lecture 22
Trigonometric and Hyperbolic Functions

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Explain the trigonometric and hyperbolic functions
- Express the complex function in terms of polar and Cartesian form

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Topics

- Complex trigonometric functions
- Complex hyperbolic functions
- Functions of complex variables

Complex Trigonometric Functions

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z}, \quad \operatorname{cosec} z = \frac{1}{\sin z}$$

$$(\cos z)' = \sin z, \quad (\sin z)' = \cos z, \quad (\tan z)' = \sec^2 z$$

Euler formula holds for complex numbers

also, i.e., $\boxed{e^{iz} = \cos z + i \sin z}$

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Complex Hyperbolic Functions

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}$$

$$\text{sech} z = \frac{1}{\cosh z}, \quad \text{cosech}\, z = \frac{1}{\sinh z}$$

$$(\cosh z)' = \sinh z, \quad (\sinh z)' = \cosh z, \quad (\tanh z)' = \text{sech}^2 z$$

Relation between trigonometric and hyperbolic functions:

$$\cosh iz = \cos z, \quad \sinh iz = i \sin z$$

$$\cos iz = \cosh z, \quad \sin iz = i \sinh z$$

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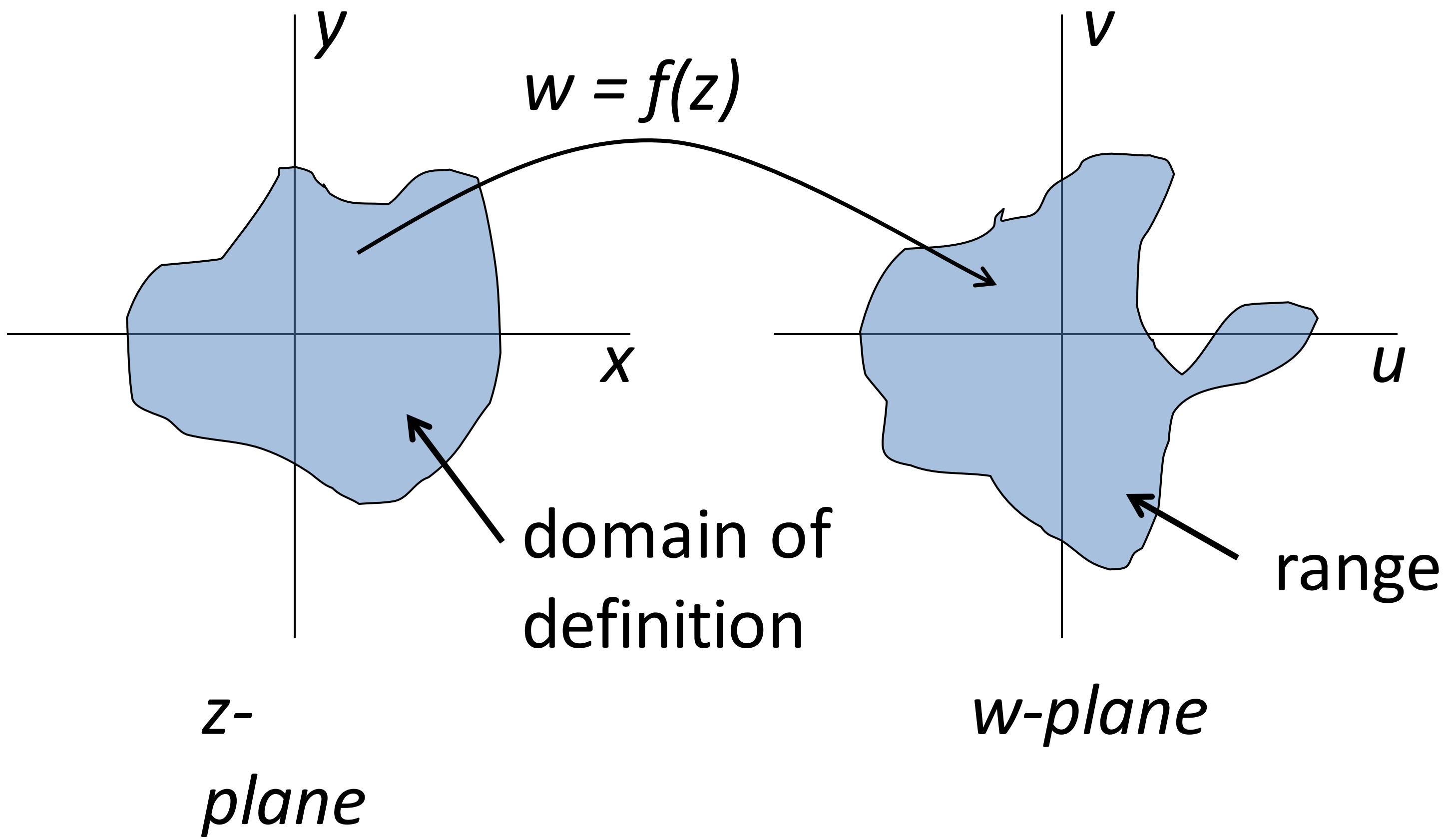
Graph of Complex Function

A complex –valued function f of complex variable z in a set D one and only one complex number w.

We write $w = f(z)$ and call w the image of z under f



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Analogy

Real variable

Complex variable

x

$z = x + i\,y$

reals

Function of real variable

Function of complex variable

$f(x)$

$f(z) = u(x, y) + iv(x, y)$

functions of real variables

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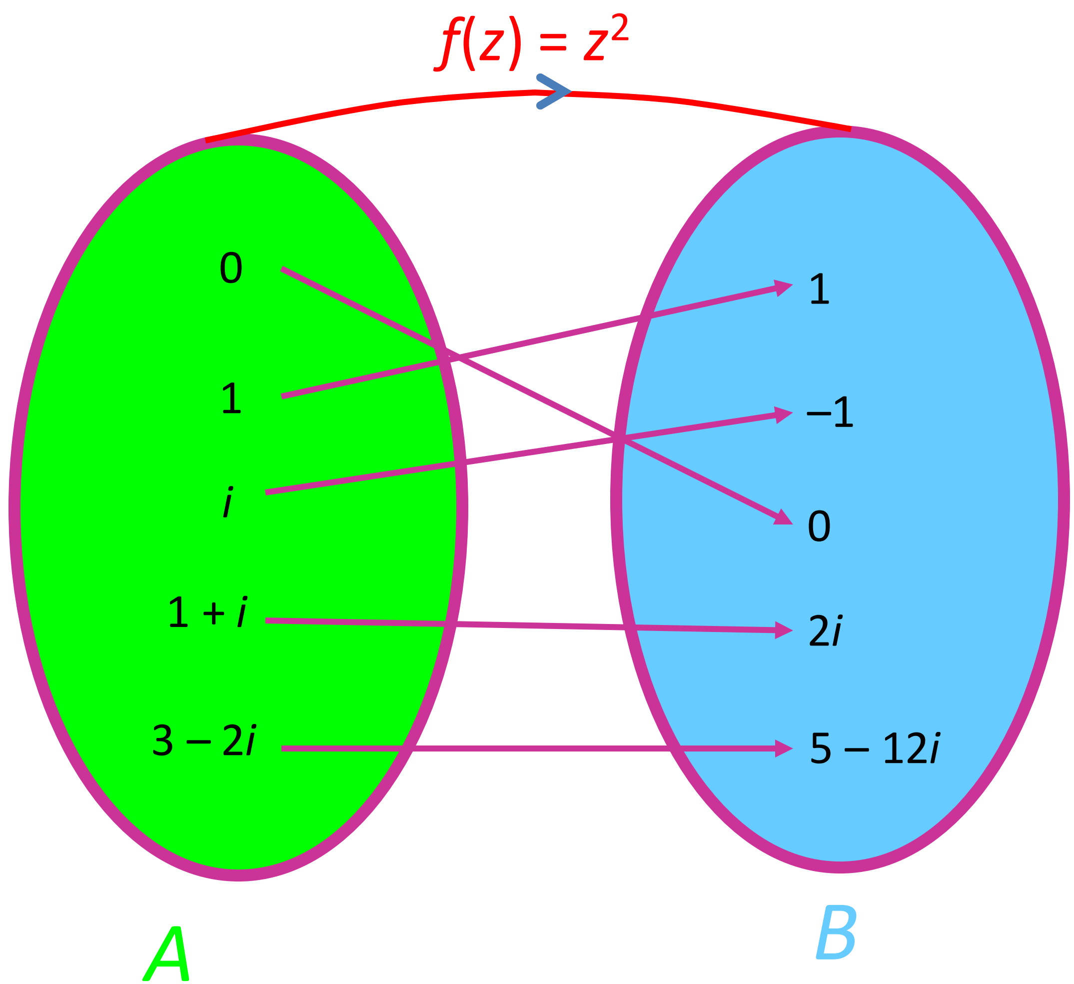
Function of Complex Variable

A function $\boldsymbol{f : A \to B}$ is a ***rule*** that assigns to every complex number z in A *a complex number* w in B, *called the value of* f *at* z. *We write* $w = f(z)$



The set ***A*** is called the **domain** and set ***B*** is called the **range** of the function ***f***.

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Example-1

Express $\sin(z)$ in the form of

$$f(z) = u(x, y) + iv(x, y)$$

Solution: $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

$$= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{-y+ix} - e^{y-ix}}{2i}$$

$$\frac{1}{2i}\{e^{-y}(cosx + isin(x)) - e^{y}(cosx - isinx)\}$$

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Example-1.........

$$= sinx\,\frac{e^{y}+e^{-y}}{2} + \text{icosx}\ \frac{e^{y}+e^{-y}}{2}$$

$$= sinx\ coshy + icosx\ sinhy$$

Express $\cos(z)$ in the form of

$$f(z) = u(x, y) + iv(x, y)$$

Answer: $f(z) = cosxcoshy - isinx\ sinhy$

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Example-2

Write $f(z) = z^4$ the form of $f(z) = u(x, y) + iv(x, y)$

Solution:

Using the binomial formula, we obtain

$$f(z) = (x + iy)^2$$

$$= x^4 + 4x^3\, iy + 6\, x^2\, (iy)^2 + 4x(iy)^3 + (iy)^3$$

so that $u(x, y) = x^4 - 6x^2y^2 + y^4$

and $v(x, y) = 4\, x^3\, y - 4xy^3$

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Example-3

Express $f(z) = z^2$ in both Cartesian and polar form

For the Cartesian form

$$f(z) = f(x + iy) = (x + iy)^2$$

$$f(z) = (x^2 - y^2)i + 2xy$$

so that $u(x, y) = (x^2 - y^2)$

$$v(x, y) = 2\,xy$$

polar form:

Put $z = re^{i\theta}$ in the given equation, we have

$$f\left(r\,e^{i\theta}\right) = r^2 cos2\theta + \mathrm{ir}^2 \mathrm{sin}2\theta$$

so that $u(r, \theta) = r^2 cos2\theta$ and $v(r, \theta) = r^2 sin2\theta$

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Express the following functions in the form of
$f(z) = u(x,y) + iv(x,y)$

a. $f(z) = z^3$

b. $f(z) = \bar{z}^2 + \frac{2-3i}{z}$

c. $f(z) = z^2$

d. $f(z) = tanz$

Express the following functions in the form of
$u(r,\theta) + iv(r,\theta)$

a. $f(z) = z^5 + \bar{z}^5$

b. $f(z) = z^5 + \bar{z}^3$

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Session Summary

- In the Cartesian form $f(z)$ is expressed as $u(x, y) + iv(x, y)$
- In the polar form $f(z)$ is expressed as $u(r, \theta) + iv(r, \theta)$

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