Lecture 3
Row Operations

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Perform elementary row operation on a matrix
- Reduce a matrix in row echelon form and reduced row echelon form
- Apply reduced row echelon form to find the inverse of the square matrix
- Compute the rank of matrix
- Apply MATLAB to find the reduced row echelon form

Topics

- Elementary row operations
- Reduced row echelon form
- Gauss elimination method
- Gauss Jordan method
- Inverse of the matrix by row operations
- Rank

Motivation for Row operation

Find an inverse of an matrix

$$\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Can you find out the inverse and determinant of higher order matrices?

$$\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 3 & 2 & 1 \\ 1 & 4 & 3 & 3 & -1 \\ 1 & 3 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & -2 & -1 & 2 & 2 \end{bmatrix}$$

Elementary Operations (Elementary Transformations of a matrix)

An elementary transformations is an operation of any one of the following types:

- **Interchange:** $(R_i \Leftrightarrow R_j)$
- **Scaling**: The multiplication of the elements of any row R_i by any non-zero scalar quantity k is denoted by $(k.R_i)$
- **Replacement**: Addition of constant multiplication of the elements of any row R_j (or column) to the corresponding elements of any other row R_i (or column) denoted by $(R_i + k.R_j)$

Row Echelon form

Definition: A matrix *A* is said to be in row echelon form if

- Every row of *A* which has all its entries 0 occurs below every row which has non-zero entry

- The first nonzero entry in each nonzero row is 1

- Row *k* does not consist entirely of zeros, the number of leading zero entries in row *k + 1* is greater than the number of leading zero entries in row *k*

Examples $\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Reduced Row Echelon form

Definition: A matrix *A* is said to be in reduced row echelon form if

- The matrix is in row echelon form
- The first nonzero entry in each row is the only nonzero entry in its column

Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example
Row-Echelon and Reduced Row-Echelon form

- All matrices of the following types are in **row-echelon form** (any real numbers substituted for the *'s.) :

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

- All matrices of the following types are in **reduced row-echelon form** (any real numbers substituted for the *'s.) :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Row echelon form

We shall give a step-by-step that can be used to reduce any matrix into row-echelon form

Let

$$A=\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step 1 Locate the leftmost column that does not consist entirely of zeros

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Leftmost nonzero column

Row echelon form…

- **Step2** Interchange the top row with another row ($R_1 \Leftrightarrow R_2$), to bring a nonzero entry to top of the column found in Step 1

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1th and 2th rows in the preceding matrix were interchanged.

Row echelon form….

- **Step 3** If the entry that is now at the top of the column found in Step1 is a, multiply the first row by 1/a in order to introduce a leading 1 $(R_1 = \frac{1}{2} R_1)$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

The 1st row of the preceding matrix was multiplied by 1/2.

- **Step 4** Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros $(R_3 = R_3 - 2R_1)$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-2 times the 1st row of the preceding matrix was added to the 3rd row.

Row echelon form…

Step 5 Now cover the top row in the matrix and begin again with Step1 applied to the sub-matrix that remains. Continue in this way until the entire matrix is in row-echelon form

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 \quad 0 \quad -2 \quad 0 \quad 7 \quad 12 \\ 0 \quad 0 \quad -5 \quad 0 \quad -17 \quad -29 \end{bmatrix}$$

Leftmost nonzero column in the submatrix

$$(R_2 = -\frac{1}{2} R_1)$$

$$\begin{bmatrix} \boxed{1 \quad 2 \quad -5 \quad 3 \quad 6 \quad 14} \\ 0 \quad 0 \quad 1 \quad 0 \quad -\frac{7}{2} \quad -6 \\ 0 \quad 0 \quad 5 \quad 0 \quad -17 \quad -29 \end{bmatrix}$$

The 1st row in the submatrix was multiplied by -1/2 to introduce a leading 1.

Row echelon form...

- **Step5 (cont.)** $(R_3 = R_3 - 5R_2)$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

-5 times the 1st row of the submatrix was added to the 2nd row of the submatrix to introduce a zero below the leading 1

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

The top row in the submatrix was covered, and we returned again Step1

Leftmost nonzero column in the new submatrix

$$R_3 = 2R_3$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The first (and only) row in the new submetrix was multiplied by 2 to introduce a leading 1

- The **entire** matrix is now in **row-echelon form**

Reduced row echelon form

Step 6 Beginning with last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

7/2 times the 3rd row of the preceding matrix was added to the 2nd row

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

-6 times the 3rd row was added to the 1st row

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the 2nd row was added to the 1st row

The **last** matrix is in **reduced row-echelon form**

Row Echelon form and reduced row Echelon form

- Step1~Step5: the above procedure produces a row-echelon form and is called Gaussian elimination

- Step1~Step6: the above procedure produces a reduced row-echelon form and is called Gauss-Jordan elimination

- Every matrix has **a unique reduced row-echelon** form but a row-echelon form of a given matrix is not unique

Row Echelon form and reduced row Echelon form…

- The process of using row operations to transform a linear system into row echelon form is called Gaussian elimination

- The process of using row operations to transform a linear system into reduced row echelon form is called Gauss-Jordon method

Example 1

Let us consider matrix is

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

Here first entry of R_1 is already 1.

Step 1 By elementary row transformation $(R_2 \to R_2 - 2R_1)$ and $(R_4 \to R_4 - 2R_1)$ the matrix *A* will become

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

Example 1...

- **Step 2** Multiplying the 2nd row by -1 $\left(R_2 \to (-1)R_2\right)$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

- **Step 3** $\left(R_3 \to R_3 - 5R_2\right), \left(R_4 \to R_4 - 4R_2\right)$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Example 1...

- **Sept 4** $\left(R_3 \Leftrightarrow R_4\right)$

$$\sim\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 5** $\left(R_3 \to \frac{1}{6} R_3\right)$

$$\sim\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the row echelon form obtained by elementary row operations.

Example 1…

- **Step 6** $\left(R_2 \to R_2 + 3R_3\right)$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Step 7** $\left(R_1 \to R_1 + 2R_2\right)$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the reduced row echelon form obtained by elementary row operations.

Inverse of square matrix by reduced row echelon form

Let $$A=\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

As we know $A=IA$, where I is identity matrix. Then $I = A^{-1} A$.

Step 1 $A=IA$

$$\left[\begin{array}{cccc|cccc} 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

Inverse of square matrix by reduced row echelon form…

Step 2 $(R_1 \Leftrightarrow R_2)$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 2 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

Step 3 $(R_3 \to R_3 - R_1), (R_4 \to R_4 + R_1)$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 4 & 0 & 1 & 0 & 1 \end{array}\right]$$

Inverse of square matrix by reduced row echelon form…

Step 4 $(R_3 \Leftrightarrow R_2)$

$$\begin{bmatrix} 1 & 1 & -1 & -2 & | & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & | & 0 & -1 & 1 & 0 \\ 0 & 2 & 1 & 3 & | & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 4 & | & 0 & 1 & 0 & 1 \end{bmatrix}$$

Step 5 $(R_3 \to R_3 - 2R_2), (R_4 \to R_4 - 2R_2)$

$$\begin{bmatrix} 1 & 1 & -1 & -2 & | & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & | & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -3 & | & 1 & 2 & -2 & 0 \\ 0 & 0 & -1 & -2 & | & 0 & 3 & -2 & 1 \end{bmatrix}$$

Inverse of square matrix by reduced row echelon form…

Step 6 $\left(R_3 \to (-1)R_3\right), \left(R_4 \to (-1)R_4\right)$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & -3 & 2 & -1 \end{array}\right]$$

Step 7 $\left(R_4 \to R_4 - R_3\right)$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & -1 \end{array}\right]$$

Inverse of square matrix by reduced row echelon form…

Step 8 $\left(R_3 \to (-1)R_4\right)$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 & -1 & -2 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

Step 9 $(R_1 \to R_1 + 2R_4), \left(R_2 \to R_2 - 3R_4\right), \left(R_3 \to R_3 - 3R_4\right)$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & -1 & 0 & -2 & 3 & 0 & 2 \\ 0 & 1 & 1 & 0 & 3 & -4 & 1 & -3 \\ 0 & 0 & 1 & 0 & 2 & -5 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

Inverse of square matrix by reduced row echelon form…

Step 10 $(R_1 \to R_1 + R_3), (R_2 \to R_2 - R_3)$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & -2 & 2 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -5 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

Step 11 $(R_1 \to R_1 - R_2)$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -3 & 3 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 & -5 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$I = A^{-1} A$

Hence

$$A^{-1} = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Rank of A Matrix

Definition: The rank of matrix A is said to be r if it

- There is at least one non-zero minor of order r
- Every minor of A of higher than r is zero

In other words, the rank of a matrix is the order of any highest order non-vanishing minor of the matrix.

Methods to find the rank of a matrix

- Upper triangular method
- Echelon form
- Row reduced echelon form

Properties of rank

- The rank of transpose of a matrix is the same as that of the original matrix
- The rank of every non-singular of order n is n
- Elementary transformations do not change the rank of matrix
- The rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix
- The rank of a matrix in upper triangular form is equal to the number of non-zero rows of the matrix
- The rank of a matrix in normal form is equal to the order of of r-rowed unit matrix

Example 2

$$A=\begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$$ Rank(A)=?

Reducing the matrix A in upper triangular matrix by using elementary row transformation

Step 1 $(R_2 \to R_2 + 2R_1),(R_3 \to R_3 - R_1)$

$$\sim\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 7 & 13 & -9 \end{bmatrix}$$

Step 2 $(R_3 \to R_3 - R_2)$

$$\sim\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 7 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 2…

This is upper triangular matrix, then by the upper triangular method, the rank of matrix is equal to number of non-zero rows. Here the number of non-zero rows is 2, therefore the rank of the matrix A = 2

Step 3 $(R_2 \to (\frac{1}{7})R_2)$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 1 & \frac{13}{7} & \frac{-9}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now this matrix reduces to the Echelon form, Here number of non-zero rows is 2, therefore the rank of matrix A=2.

Example 2...

Step 4 $(R_1 \to R_1 - 2R_2)$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{5}{7} & -\frac{10}{7} \\ 0 & 1 & \frac{13}{7} & \frac{-9}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now this matrix reduces to the reduced row Echelon form, Here number of non-zero rows is 2, therefore the rank of matrix A=2.

Example 3

$$A=\begin{bmatrix}2 & 3 & -1 & 1\\ 1 & -1 & -2 & -4\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$$

Rank of A ?

Using Elementary row transformation reducing to reduced row Echelon form, then find rank of the matrix A

Step 1 $(R_1 \Leftrightarrow R_2)$

$$\sim\begin{bmatrix}1 & -1 & -2 & -4\\ 2 & 3 & -1 & 1\\ 3 & 1 & 3 & -2\\ 6 & 3 & 0 & -7\end{bmatrix}$$

Step 2 $(R_2 \to R_2 - 2R_1), (R_3 \to R_3 - 3R_1),$
$(R_4 \to R_4 - 6R_1)$

$$\sim\begin{bmatrix}1 & -1 & -2 & -4\\ 0 & 5 & 3 & 9\\ 0 & 4 & 9 & 10\\ 0 & 9 & 12 & 17\end{bmatrix}$$

Example 3...

Step 3 $\left(R_3 \to R_3 - \frac{4}{5}R_2\right), \left(R_4 \to R_4 - \frac{9}{5}R_1\right)$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 9 \\ 0 & 0 & \frac{33}{5} & \frac{14}{5} \\ 0 & 0 & \frac{33}{5} & \frac{4}{5} \end{bmatrix}$$

Step 4 $\left(R_2 \to \frac{1}{5}R_2\right), (R_4 \to R_4 - R_3)$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{3}{5} & \frac{9}{5} \\ 0 & 0 & \frac{33}{5} & \frac{14}{5} \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Example 3...

Step 5 $\left(R_3 \to \frac{5}{33} R_3\right), \left(R_4 \to (-\frac{1}{2}) R_4\right)$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & \frac{3}{5} & \frac{9}{5} \\ 0 & 0 & 1 & \frac{14}{33} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is in row Echelon form and the number of non-zero row is 4, therefore the rank of matrix

$\rho(A)=4$

Example 3...

Step 6 $\left(R_1 \to R_1 + 4R_4\right), \left(R_2 \to R_2 - \frac{9}{5}R_4\right), \left(R_3 \to R_3 - \frac{14}{33}R_4\right)$

$$\sim \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & \frac{3}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 7 $\left(R_1 \to R_1 + 2R_3\right), \left(R_2 \to R_2 - \frac{3}{5}R_3\right)$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3...

Step 7 $\left(R_1 \to R_1 + R_2\right)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This matrix is in reduced row Echelon form and the number of non-zero rows are 4, therefore the rank of matrix

$\rho(A)=4$

Matlab Code

- in-built function for Reduced row echelon form is

rref(A)

'A' is a given matrix

- in-built function for rank of a matrix is

rank(A)

'A' is a given matrix

Session Summary

- Elementary row operations are interchange, scaling and replacement
- The inverse of a matrix can be also calculated by Gauss Jordon method
- The number of non-zero rows in echelon form is the rank of the matrix
- The rank of matrix can be calculated by anyone of the following methods:
  - Upper triangular method (row operation only)
  - Echelon form and row reduced echelon form (row operation only)
  - Normal form (row and column operation)
- The rank of transpose of a matrix is same as the rank of matrix