

Course Code: ESC106A

**Course Title: Construction Materials and Engineering
Mechanics**

**Lecture No. 34:
Derivations of Centroid**

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Lecture Intended Learning Outcomes

At the end of this lecture, students will be able to:

- Determine the centroid of different sections using the principle of moment balance of elements.



Contents

Centroid of Quadrant of ring, Centroid of arc of a circle, centroid of semi circle, Centroid of solid right circular cone, Centroid of solid hemisphere, Centroid of Various shapes.



Centroid of the Quadrant of a Ring (line/rod)

Determine the CG of quadrant AB of the arc of a circle of radius R as shown in Figure.

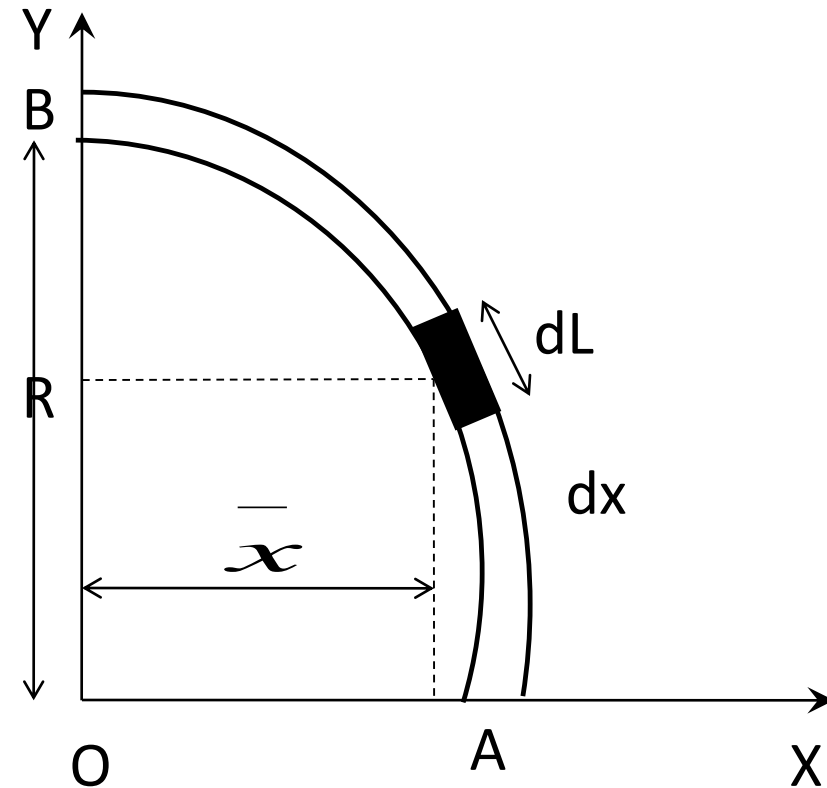
Solution:

The equation of the curve AB is the equation of the circle of radius R.

Differentiating,

$$x^2 + y^2 = R^2$$

$$2x dx + 2y dy = 0; 2y dy = -2x dx; dy = \frac{-x dx}{y}$$



Centroid of the Quadrant of a ring by integration method

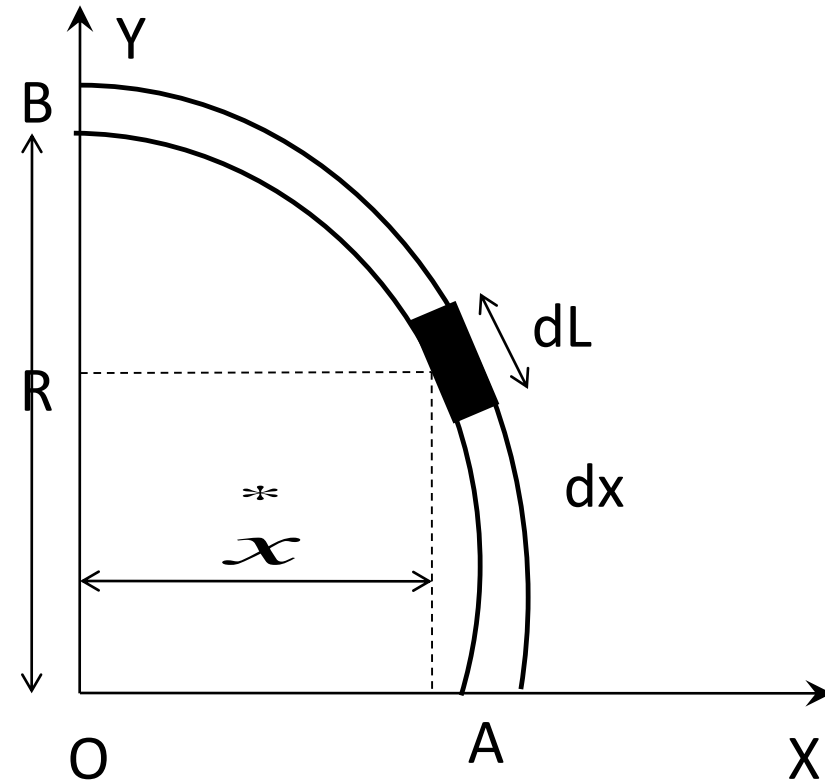
Consider an elemental length dL whose CG is at (x^*, y^*)

$$\bar{y} = \frac{\int y^* dL}{\int dL} \dots \dots \dots (1)$$

$$dL = \sqrt{dx^2 + dy^2}$$

$$dL = \sqrt{dx^2 + \left(\frac{-x dx}{y} \right)^2}$$

$$dL = dx \sqrt{1 + \frac{x^2}{y^2}} = dx \sqrt{\frac{R^2}{y^2}} = \frac{R}{y} dx$$



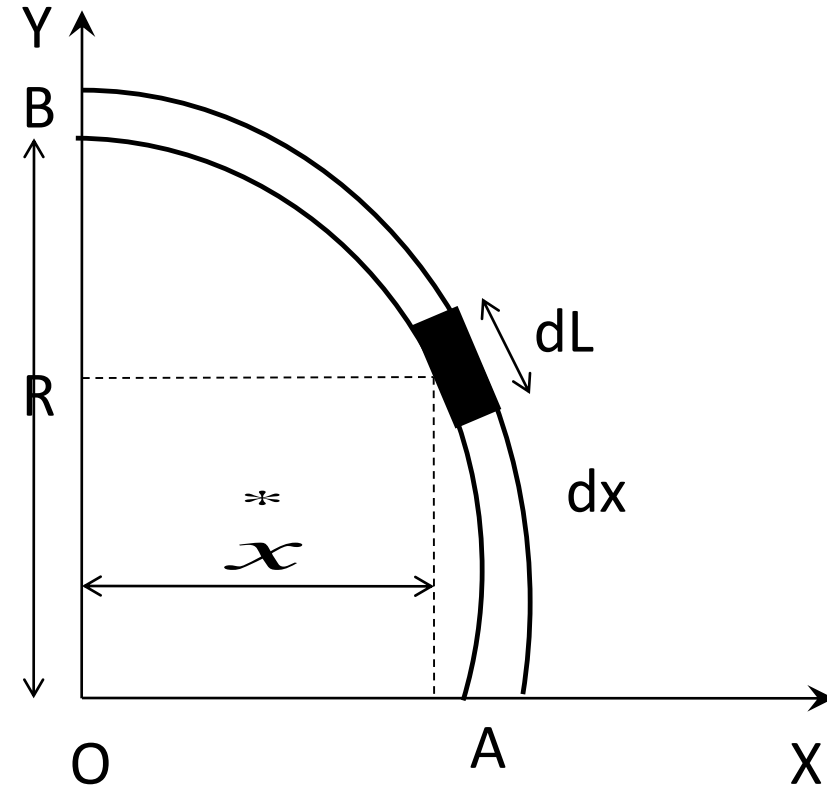
Centroid of the quadrant of a ring by integration method

Substituting dL in Eqn (1)

$$\bar{y} = \frac{\int y^* \frac{R}{y} dx}{\int dL}$$

$$\bar{y} = \frac{\int R dx}{\int dL} = \frac{R \int_0^R dx}{\int dL} = \frac{R[x]_0^R}{\left(\frac{2\pi r}{4}\right)} = \frac{2R}{\pi}$$

Due to symmetry $\bar{x} = \frac{2R}{\pi}$



Centroid of an arc of a circle (Line/wire)

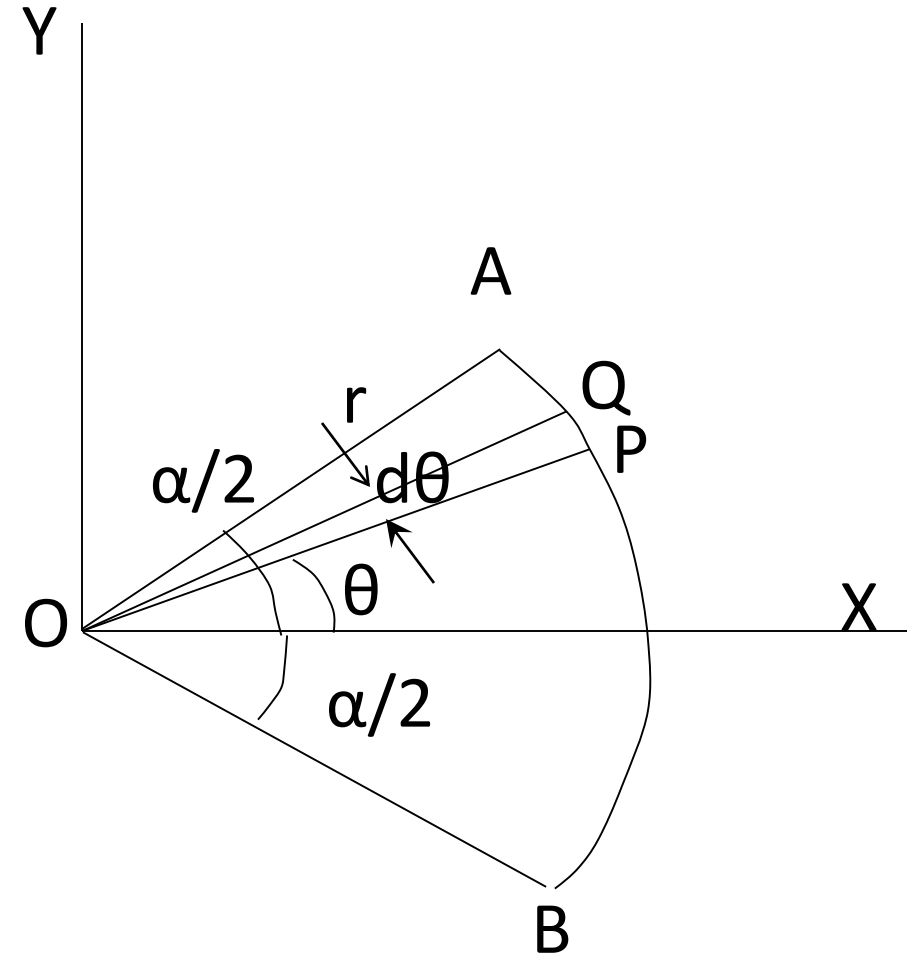
Let arc AB subtend an angle α at the centre O. Let OX be the bisector of angle AOB. By symmetry, CG will lie on OX. So $y=0$.

$$PQ = r d\theta$$

$$y_{PQ} = r \cos \theta$$

$$\bar{x} = \frac{\int_{-\alpha/2}^{\alpha/2} r d\theta \cdot r \cos \theta}{\int_{-\alpha/2}^{\alpha/2} r d\theta} = \frac{r \int_{-\alpha/2}^{\alpha/2} \cos \theta \cdot d\theta}{\int_{-\alpha/2}^{\alpha/2} d\theta}$$

$$\bar{x} = r \frac{[\sin \theta]_{-\alpha/2}^{\alpha/2}}{[\theta]_{-\alpha/2}^{\alpha/2}} = 2r \frac{\sin(\alpha/2)}{\alpha}$$



Centroid of sector of a circle (Area)

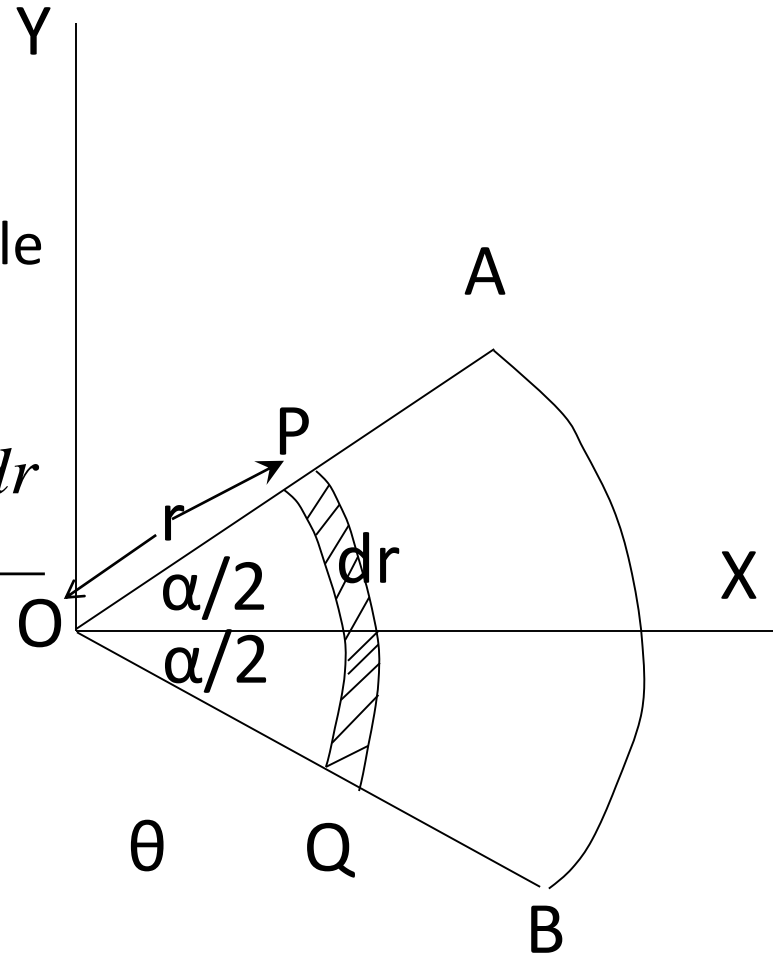
Let α be the angle of sector AOB. Let OX be the bisector of angle AOB. By symmetry, CG will lie on OX. So $y=0$.

$$\text{Area, } PQ = r\alpha dr$$

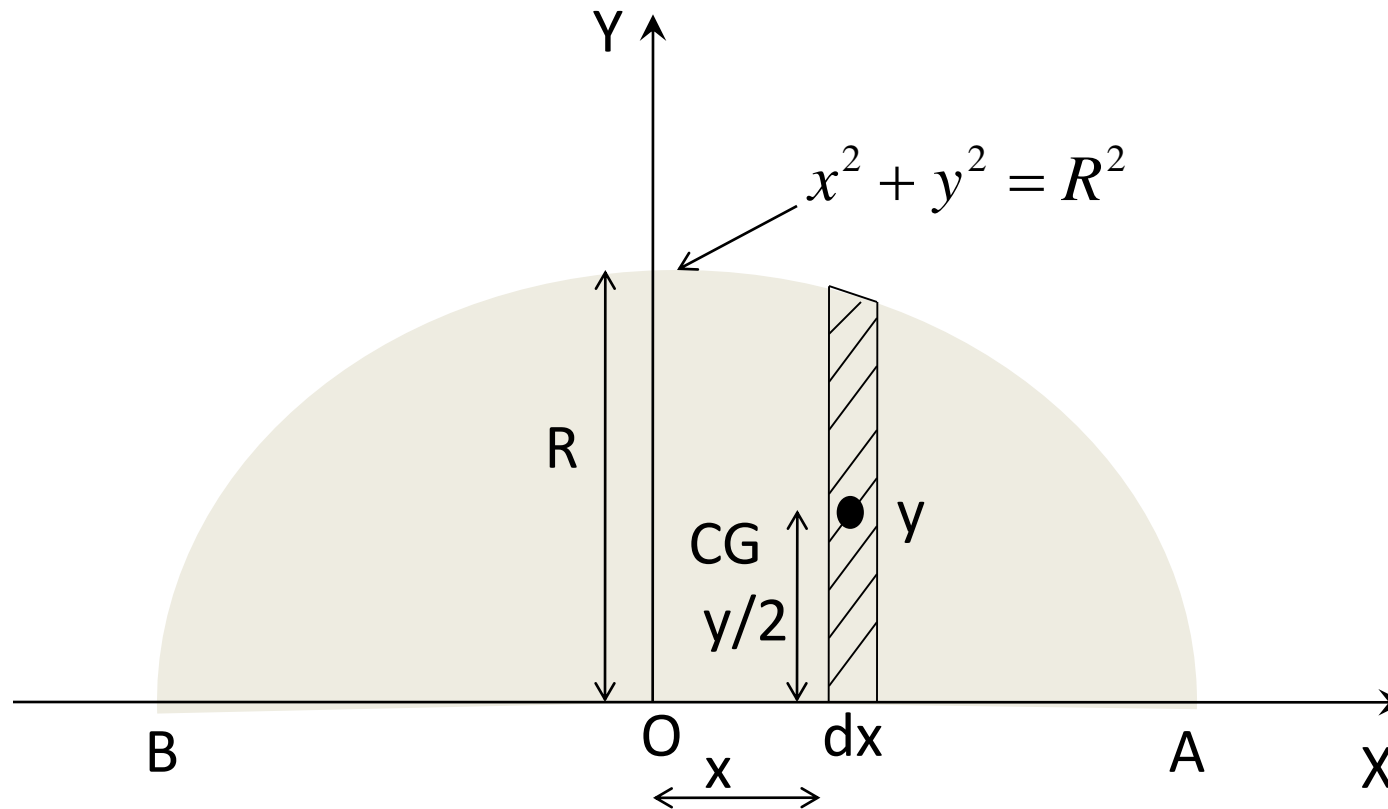
$$\bar{x}_{PQ} = 2r \frac{\sin(\alpha/2)}{\alpha} \Rightarrow \text{CG of an arc of a circle}$$

$$\bar{x} = \frac{\int_0^r r\alpha dr 2r \frac{\sin(\alpha/2)}{\alpha}}{\int_0^r r\alpha dr} = \frac{2 \sin(\alpha/2) \int_0^r r^2 dr}{\alpha \int_0^r r dr}$$

$$\bar{x} = \frac{2 \sin(\alpha/2)}{\alpha} \cdot \frac{2}{3} r = \frac{4r}{3\alpha} \cdot \sin(\alpha/2)$$



Centroid of a semi-circle (Area)



Consider a strip parallel to y-axis $dA = y \cdot dx$

Centroid of a semi-circle

Moment of area dA about x -axis

$$dM_x = dA \cdot \frac{y}{2} = \frac{y}{2} y dx = \frac{y^2}{2} dx$$

Moment of total area A about x -axis

$$M_x = \int \frac{y^2}{2} dx = \int_{-R}^R \frac{y^2}{2} dx = \int_{-R}^R \frac{R^2 - x^2}{2} dx$$

$$M_x = \frac{1}{2} \left[R^2 x - \frac{x^3}{3} \right]_{-R}^R = \frac{1}{2} \frac{4R^3}{3} = \frac{2R^3}{3}$$



Centroid of a Semi-circle

Let the distance of CG of total area from x-axis be \bar{y}

$$\text{Total area of semi circle} = \frac{\pi R^2}{2}$$

$$M_x = \bar{y} \cdot \frac{\pi R^2}{2}$$

Equating,

$$\frac{2R^3}{3} = \bar{y} \cdot \frac{\pi R^2}{2}$$

$$\bar{y} = \frac{4R}{3\pi}$$

Hence centroid of semi-circle is

$$\left(0, \frac{4R}{3\pi}\right)$$

Compare this with the centroid of the sector of the circle.

Note:

Consider a strip parallel to x-axis and locate the CG.



Centroid of a solid right circular cone

Let ABC be the cone and AD its axis. Consider an element PQ cut-off by two planes parallel to the base BC at y and $y+dy$ from A and having centre at M. PQ is considered as a circular plate of radius r .

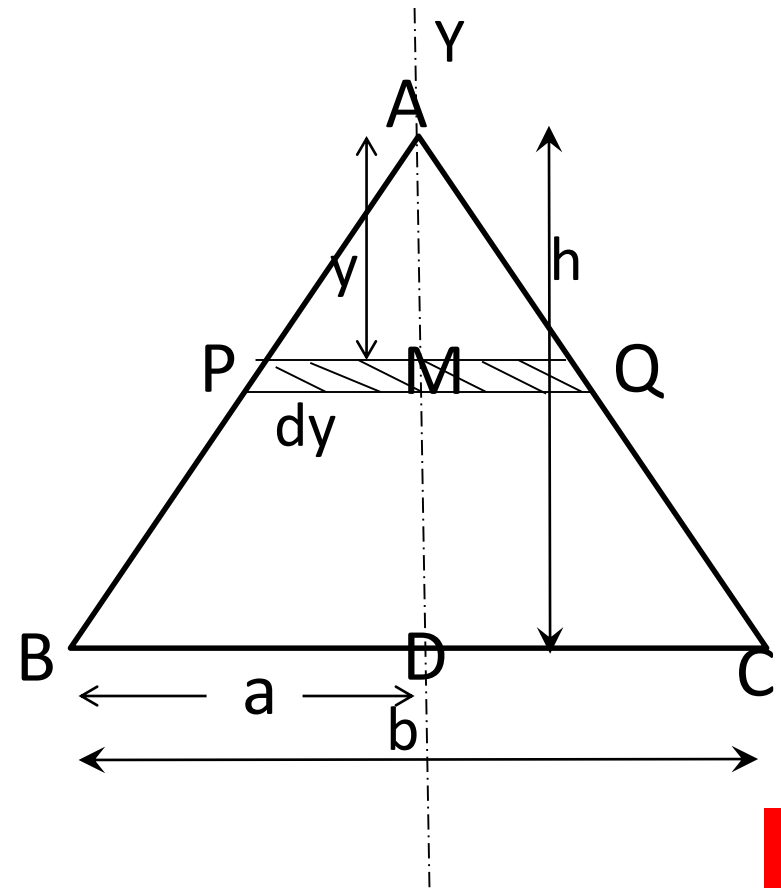
$$AD = h$$

$$BC = b$$

$$\Delta ABC, \Delta APQ$$

$$\therefore \frac{AM}{PM} = \frac{AD}{BD}$$

$$\frac{y}{r} = \frac{h}{a} \Rightarrow r = \frac{ay}{h}$$



Centroid of a solid right circular cone

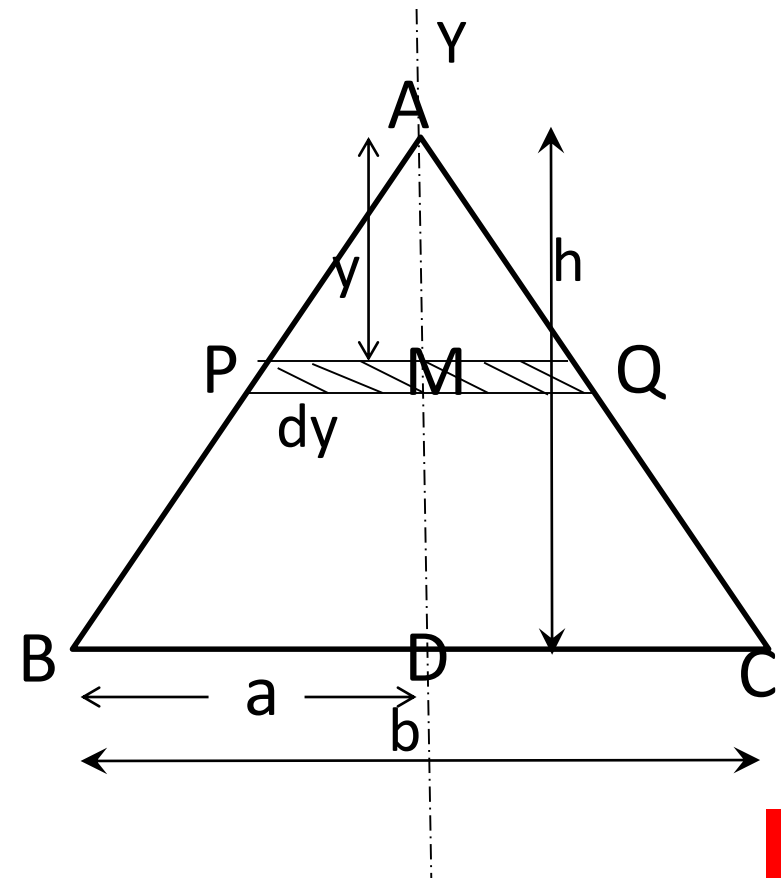
If w is weight of unit volume of the material, weight of PQ is:

$$W_{PQ} = w \cdot \pi r^2 dy$$

$$W_{PQ} = w \cdot \pi \left(\frac{ay}{h} \right)^2 dy$$

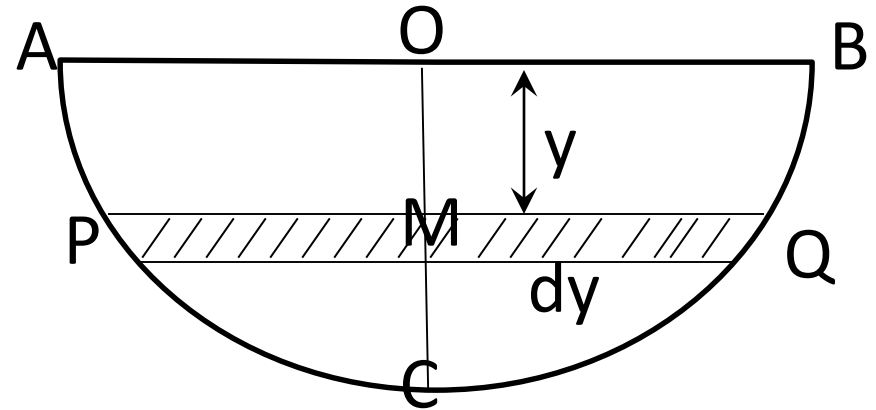
$$\bar{y} = \frac{\sum_{y=0}^{y=h} w \cdot \pi \left(\frac{ay}{h} \right)^2 dy \cdot y}{\sum_{y=0}^{y=h} w \cdot \pi \left(\frac{ay}{h} \right)^2 dy} = \frac{\sum_{y=0}^{y=h} y^3 dy}{\sum_{y=0}^{y=h} y^2 dy}$$

$$\bar{y} = \frac{\left[\frac{y^4}{4} \right]}{\frac{y^3}{3}} = \frac{3}{4} h \text{ from the apex}$$



Centroid of a solid hemisphere

Let ACB be the hemisphere of radius r and OC its central radius. Consider an elementary circular plate PQ cut off by planes parallel to AB



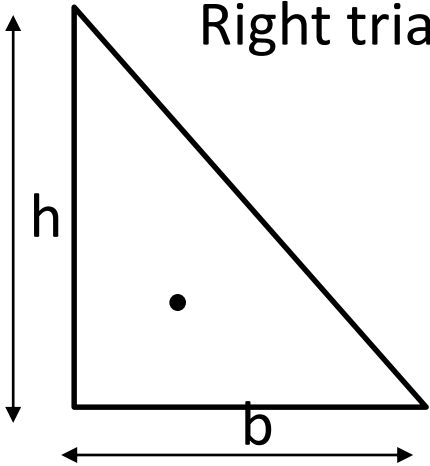

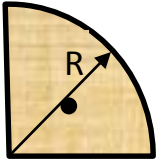
$$PM^2 = OP^2 - OM^2$$

$$PM^2 = r^2 - y^2$$

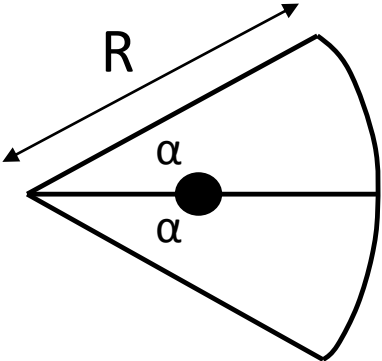
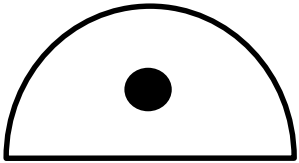
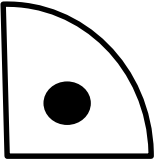
$$W_{PQ} = \pi (r^2 - y^2) dy \cdot w$$

$$\bar{y} = \frac{\sum_{y=0}^{y=r} \pi (r^2 - y^2) dy \cdot w \cdot y}{\sum_{y=0}^{y=r} \pi (r^2 - y^2) dy \cdot w} = \left[\frac{\frac{r^2 y^2}{2} - \frac{y^4}{4}}{r^2 y - \frac{y^3}{3}} \right]_0^r \Rightarrow \bar{y} = \frac{3r}{8}$$

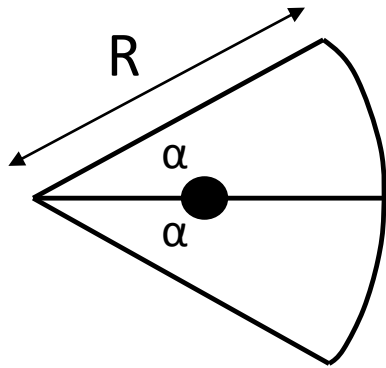
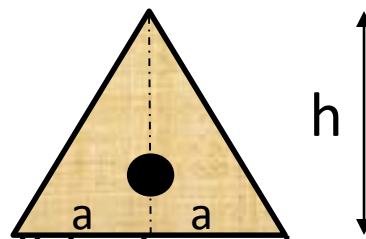
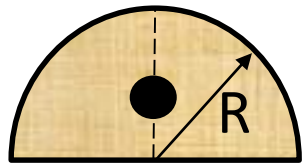
Centroid of various shapes

Shape	\bar{x}	\bar{y}	Area
 <p>Right triangle</p>	$\frac{1}{3}b$ From base	$\frac{1}{3}h$ From base	$\frac{1}{2}bh$
 <p>Semi Circular area</p>	0 From centre	$\frac{4R}{3\pi}$ From centre	$\frac{\pi R^2}{2}$
 <p>Quarter Circular area</p>	$\frac{4R}{3\pi}$ From centre	$\frac{4R}{3\pi}$ From centre	$\frac{\pi R^2}{4}$

Centroid of various shapes

Shape	\bar{x}	\bar{y}	Area/Length
 <p>Sector of a circle</p>	$\frac{2r \sin \alpha}{3\alpha}$ <p>From centre</p>	<p>0</p> <p>From centre</p>	αr^2
 <p>Semi Circular arc</p>	<p>0</p> <p>From centre</p>	$\frac{2R}{\pi}$ <p>From centre</p>	πR
 <p>Quarter Circular arc</p>	$\frac{2R}{\pi}$ <p>From centre</p>	$\frac{2R}{\pi}$ <p>From centre</p>	$\frac{\pi R}{2}$

Centroid of various shapes

Shape	\bar{x}	\bar{y}	Length/Volume
 Arc of a circle	$\frac{r \sin \alpha}{\alpha}$ From centre	0 From centre	$2\alpha r$
 Solid triangular cone	0 From centre	$\frac{1}{4}h$ From base	$\frac{\pi a^2 h}{3}$
 Solid hemisphere	0 From centre	$\frac{3R}{8}$ From centre	$\frac{2\pi R^3}{3}$

Summary

- The bodies having a continuous shape, moments are summed (integrated) using differential elements
- The centroid is the location of the geometric center for the body.
- Centroid is determined using a moment balance of geometric elements such as line, area, or volume segments.

