Lecture 31
Line Integral-1

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Intended Learning Outcomes

At the end of this session, student will be able to:

- Define line integral in complex plane
- Solve problems on line integral

Topics

- Complex line integral

I. Complex line integral in terms of real integrals

II. Complex line integral in terms of parametric form

- Properties of line integral
- Simple closed curve-Jordon curve
- Simply connected domain

Complex Line Integral

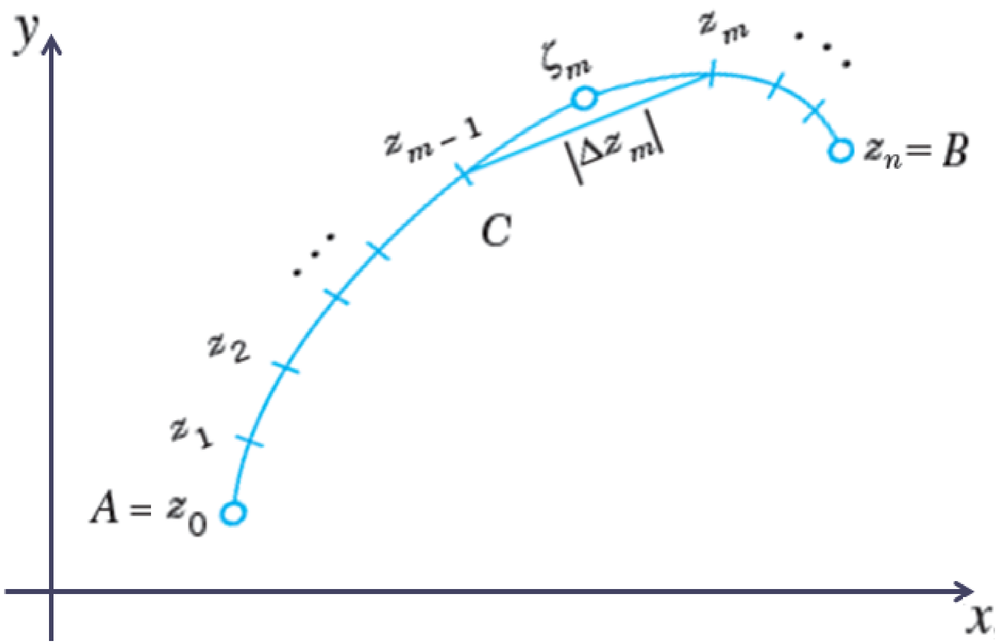
Consider a continuous function *f*(*z*) of the complex variable *z* = *x* + *iy* defined at all points of a curve *C* extending from *A* to *B*. The complex line integral of the function is defined as

$$\int_A^B f(z)\,dz = \lim_{n\to\infty}\sum_{m=1}^{n} f(\varsigma_m)\Delta z_m$$

where max | Δz_m | $\to 0$

We can also denote the integral as

$$\int_A^B f(z)\,dz = \int_C f(z)\,dz$$



Complex Line Integral in terms of Real Integrals

Let $f(z) = u(x, y) + iv(x, y)$ be a function of a complex variable $z = x + iy$ defined over a region R and C be a curve in the region. Then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$
$$= \int_C (udx - vdy) + i\int_C (vdx + udy)$$

That is, a complex line integral can be written a line integral of real valued functions

Complete Line Integral in parametric form

Suppose $f(z) = u(x, y) + iv(x, y)$ be a function of a complex variable $z(t) = x(t) + iy(t)$ where $a \le t \le b$ then the line integral can be written as

$$\int_a^b f(z)z'(t)dt = \int_a^b \left[\left(ux' - vy'\right) + i\left(vx' + uy'\right) \right] dt$$

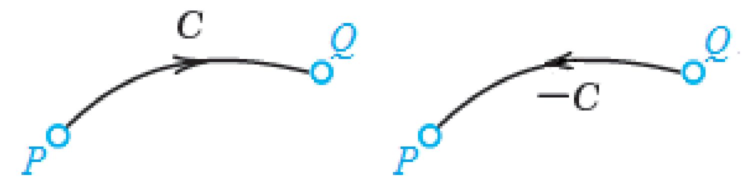
Here, primes indicate differentiation with respect to the parametric variable t

Properties of line integrals

Linearity Property

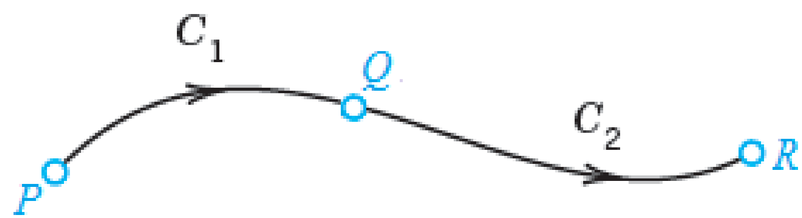
$$\int_C \left[k_1 f(z) + k_1 g(z)\right] dz = k_1 \int_C f(z)\,dz + k_2 \int_C g(z)\,dz$$

Sense Reversal



$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

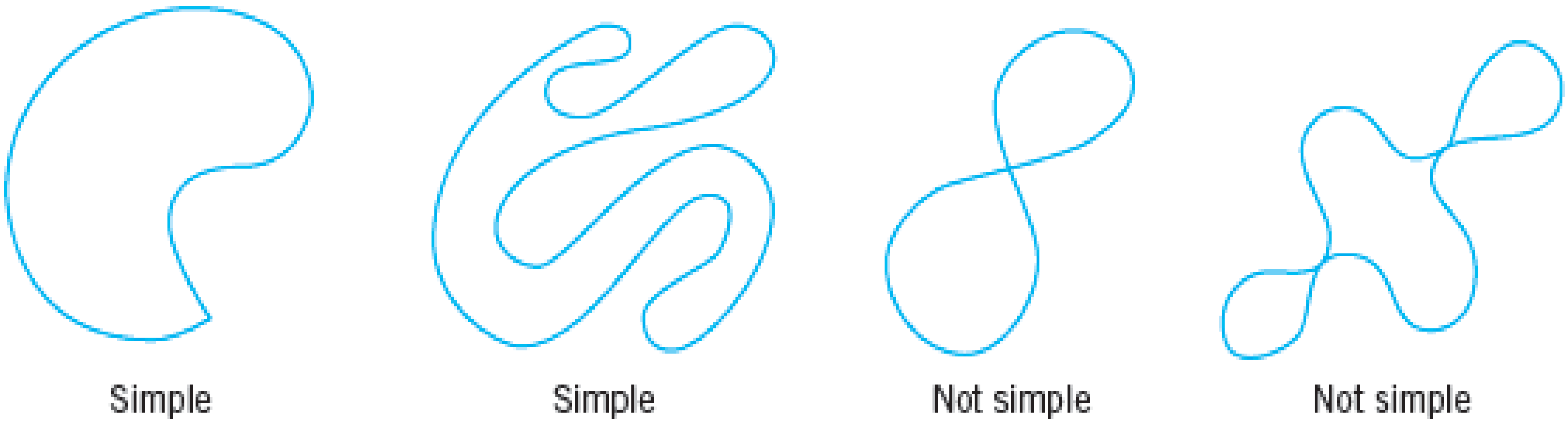
Partitioning of Path



$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

Simple Closed Curve – Jordan Curve

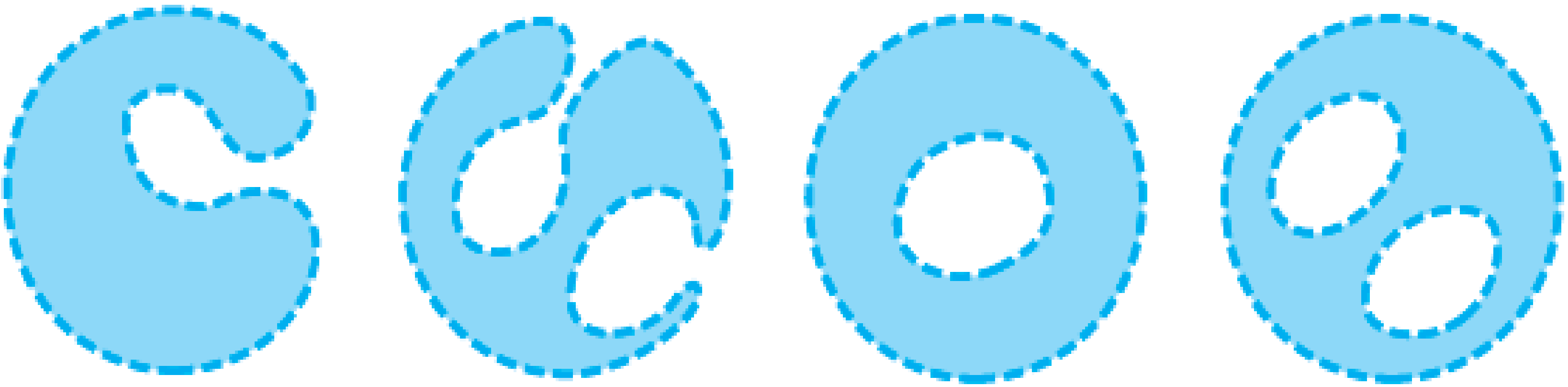
A closed curve that does not intersect or touches itself is called as simple closed curve or a Jordan curve



Simply Connected Domain

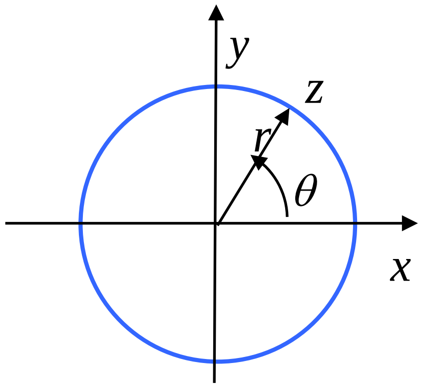
A region D of complex plane in which every simple closed curve encloses points of D is called as a Simply Connected Domain

A domain that is not simply connected is called Multiply Connected Domain



Simply connected | Simply connected | Doubly connected | Triply connected

Example



- Evaluate $\oint_C z^n dz$: where

$$C: \ x = r\cos\theta,\ y = r\sin\theta, \quad 0 \le \theta \le 2\pi$$

$$\Rightarrow \quad z = r\cos\theta + i\,r\sin\theta = re^{i\theta},$$

$$\Rightarrow \ dz = rie^{i\theta}d\theta,$$

$$\Rightarrow \ \oint_C z^n dz = \int_0^{2\pi}\left(re^{i\theta}\right)^n rie^{i\theta}d\theta$$

$$= \ ir^{n+1}\int_0^{2\pi} e^{i\theta(n+1)}d\theta \ = \cancel{i}r^{n+1}\left.\frac{e^{i\theta(n+1)}}{\cancel{i}(n+1)}\right|_0^{2\pi}$$

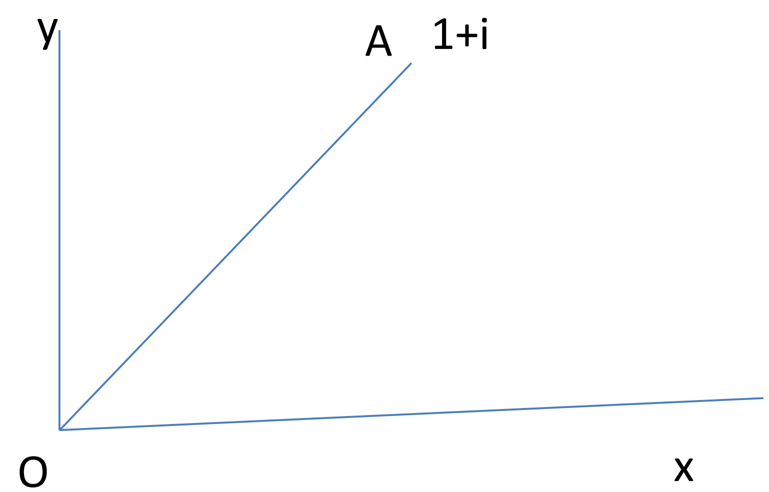
$$= \ r^{n+1}\frac{e^{i2\pi(n+1)}-1}{(n+1)} \quad = \begin{cases} 0, & n \ne -1 \\ 2\pi i, & n = -1 \end{cases}$$

Useful result and a special case of the "residue theorem"

Example-1

Evaluate the integral $\int_{z=0}^{1+i}(x^2 - iy)dz$ along the following curves :
(i)The straight line $y = x$ (ii) The parabola $y = x^2$
(i)The parametric equation of the given straight line $y = x$

Are $x = t, y = t,$ so that $z = x + iy = t + it$.
As z varies from 0 to $1 + i,$ the parameter t increases from 0 to 1. the given straight line denoted by oA is shown in figure



Example-1..........

Therefore the given line integral is

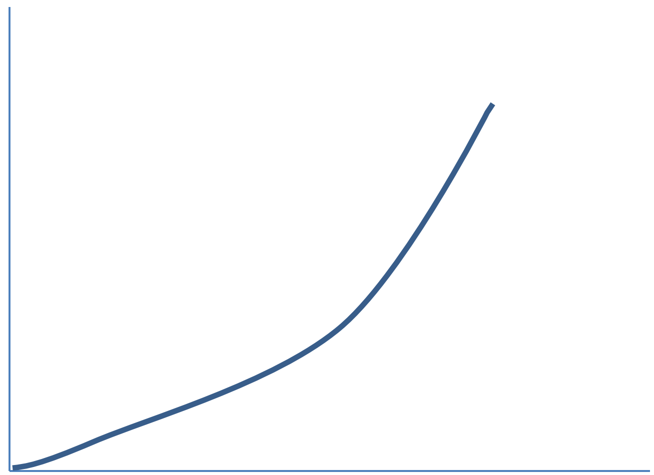
$$I=\int_{z=0}^{z=1+i}(x^2-iy)dz=\int_{z=0}^{z=1+i}(x^2-iy)(dx+idy)$$

$$=\int_0^1(t^2-it)(dt+idt)=(1+i)\int_0^1(t^2-it)dt$$

$$=\frac{1}{6}(5-i)$$

Example-1…

(ii) The parametric equation of the given parabola $y = x^2$ are $x = t$ $y = t^2$ so that $z = (x + iy)$=$t(1 + it)$. As z varies from 0 to $(1 + i\,)$, the parameter t increases from 0 to 1 , the path of the integration is shown below



Therefore, along the given parabola, the given integral is

$$I = \int_{z=0}^{1+i} (x^2 - iy)dz = \int_{z=0}^{1+i} (x^2 - iy)(dx + idy)$$

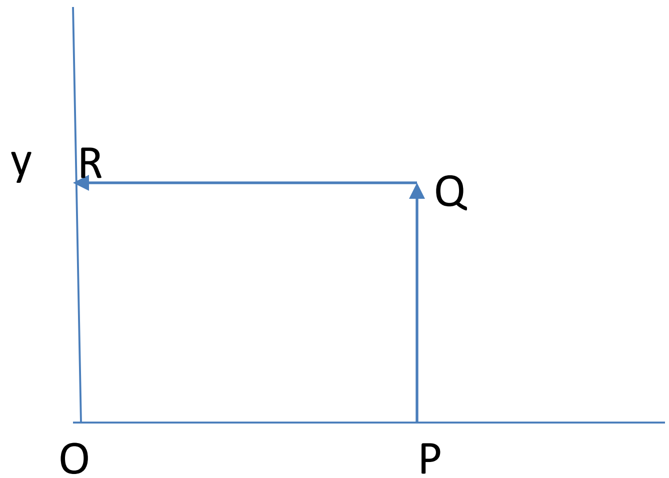
$$= (1 - i) \int_0^1 t^2(1 + 2it)dt$$

$$= \frac{1}{6}(5 + i)$$

Example-2

Evaluate the integral $\int_c |z|^2 dz$, where C is the square having vertices at the origin O and the points P(0,1), Q(1,1), R(0,1)

Solution: Here the given curve C is made up of the line segments Op, PQ,QR and RO shown in figure



Therefore

$\int_C |z|^2 = \int_{op} |z|^2\, dz + \int_{PQ} |z|^2 dz + \int_{QR} |z|^2\, dz + \int_{RQ} |z|^2\, dz$..(i)

Example-2…….

On OP, we have $y = 0$, so that $z = x$,
$0 \leq x \leq 1$, therefore

$$\int_{OP} |z|^2 \, dz = \int_0^1 x^2 dx = \frac{1}{3}$$

On PQ, we have $x = 1$, so that $z = (1 + iy)$, $0 \leq y \leq 1$, therefore

$$\int_{PQ} |z|^2 dz = \int_0^1 (1 + iy)^2 dy = \frac{4}{3} i$$

On QR, we have $y = 1$ so that $z = x + i$, and x
Decreases from 1 to zero .Therefore

$$\int_{QR} |z|^2 dz = -\int_0^1 (x^2 + 1) dx = -\left(\frac{1}{3} + 1\right) = -\frac{4}{3}$$

On RO, we have $x = 0$ so that $z = iy$, and y decreases from 1 to zero. Therefore
Substitute the above integral values in (i), we have
$\int_C |z|^2 = -1 + i$;

Examples

a)Find the value of the integral $\int_c (x+y)dx + x^2 y dy$

(i)Along $y = x^2$, having (0,0), (3,9) end points

(ii)Along $y = 3x$ between the same points.

Do the values depend upon the path

ans: (i) 256/2 (ii) 200/4

(b)Evaluate $\int_c (12z^2 - 4iz)dz$ along the curve C joining the points

(1, 1) and (2 3)

ans: -156+38i

(c)Evaluate the integral $\int_c |z|$ dz , where c is the straight line from

$z = -i$ to $z = i$

Session Summary

- The **complex line integral** of a function taken over a path C is denoted by $\int_C f(z)\mathrm{d}z$ or by $\oint_C f(z)\mathrm{d}z$ if C is closed.