Lecture 5
Taylor’s Theorem and Taylor’s Series

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- State and construct Taylor's Theorem
- Apply Taylor Series to expand standard functions

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Topics

- Approximation of functions
  - Trigonometric functions
  - Polynomial functions
- Taylor's theorem
- Taylor's series

Motivation

- Approximating of differential functions more precisely by polynomials of higher degrees.

- The geometrical idea of the tangent line of a differential function

- Evaluate the integral of the form $\int \sin(x^2)$dx and $\int e^{-x^2}$dx

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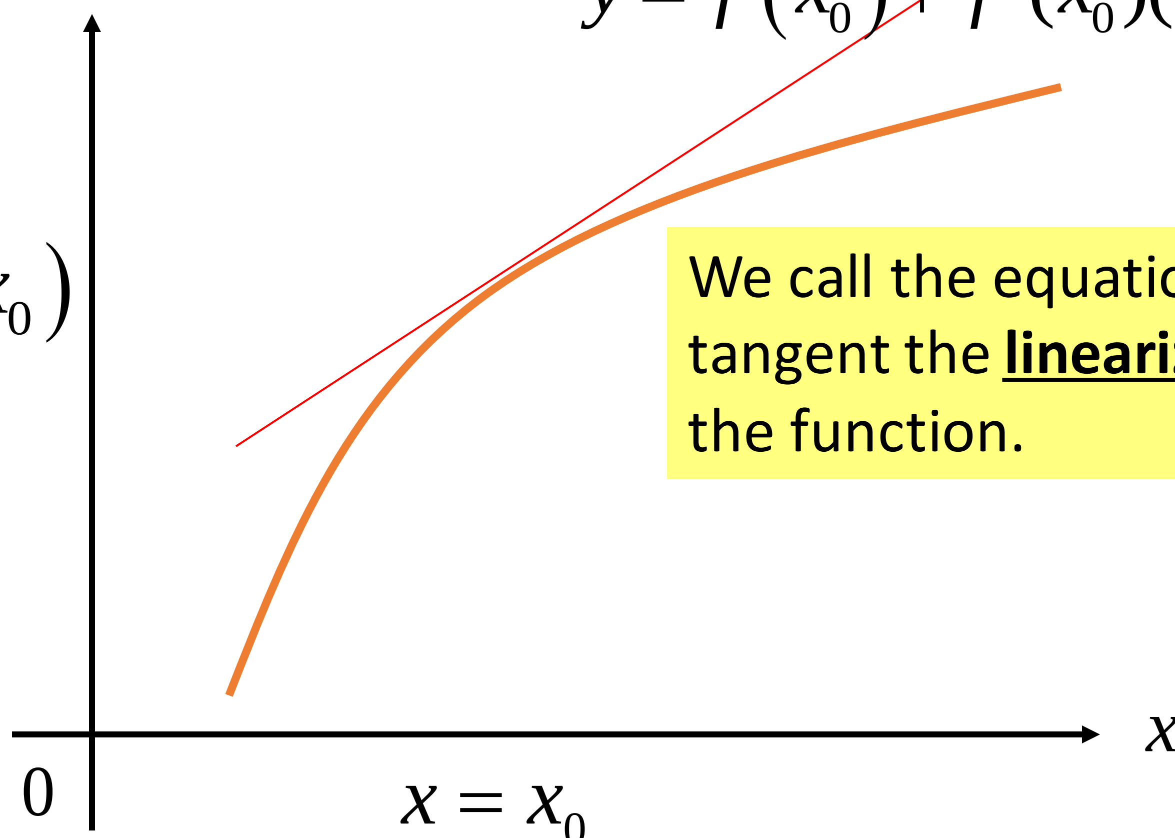
Approximation of the function

For a function *f* (*x*) that is differentiable at $x=x_0$, the tangent is a close **approximation** of the function in a neighborhood of the tangent point x_0.

$$y = f\left(x_0\right) + f\,'(x_0)(x - x_0)$$

$$f\left(x\right) = f\left(x_0\right)$$



We call the equation of the tangent the <u>**linearization**</u> of the function.

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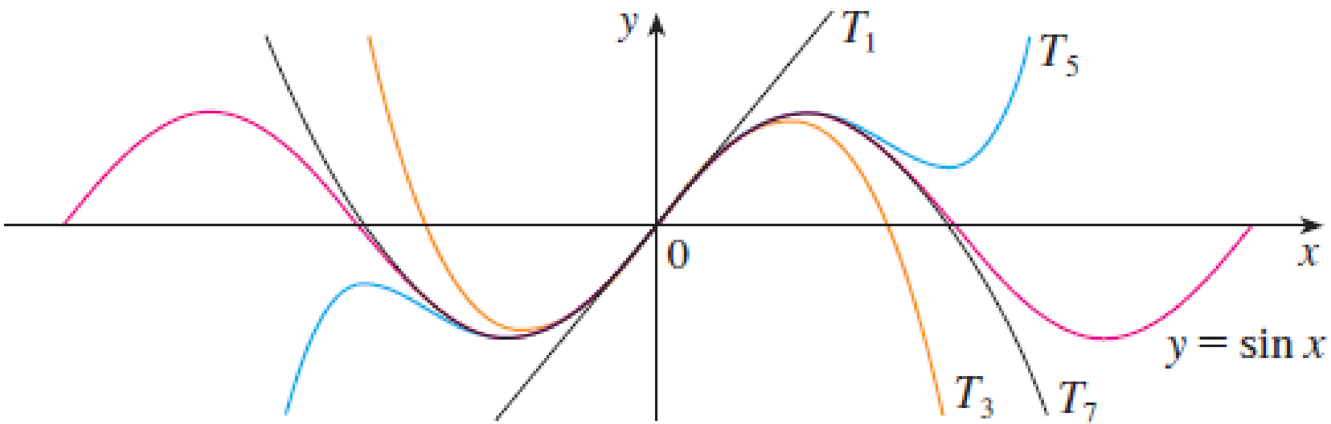
Approximation of the function Sin x

- $T_1(x) = x$
- $T_3(x) = x - \frac{x^3}{3!}$
- $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
- $T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$



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Polynomial Approximations

- Example

$$f(x) = \frac{1}{1-x}$$

$$p_1(x) = a_0 + a_1 x$$

$$f(0) = \frac{1}{1-0} = 1 \Rightarrow a_0 = f(0) = 1$$

$$f'(0) = \frac{1}{(1-x)^2} = 1 \Rightarrow a_1 = f'(0) = 1$$

$$\Rightarrow p_1(x) = 1 + x$$

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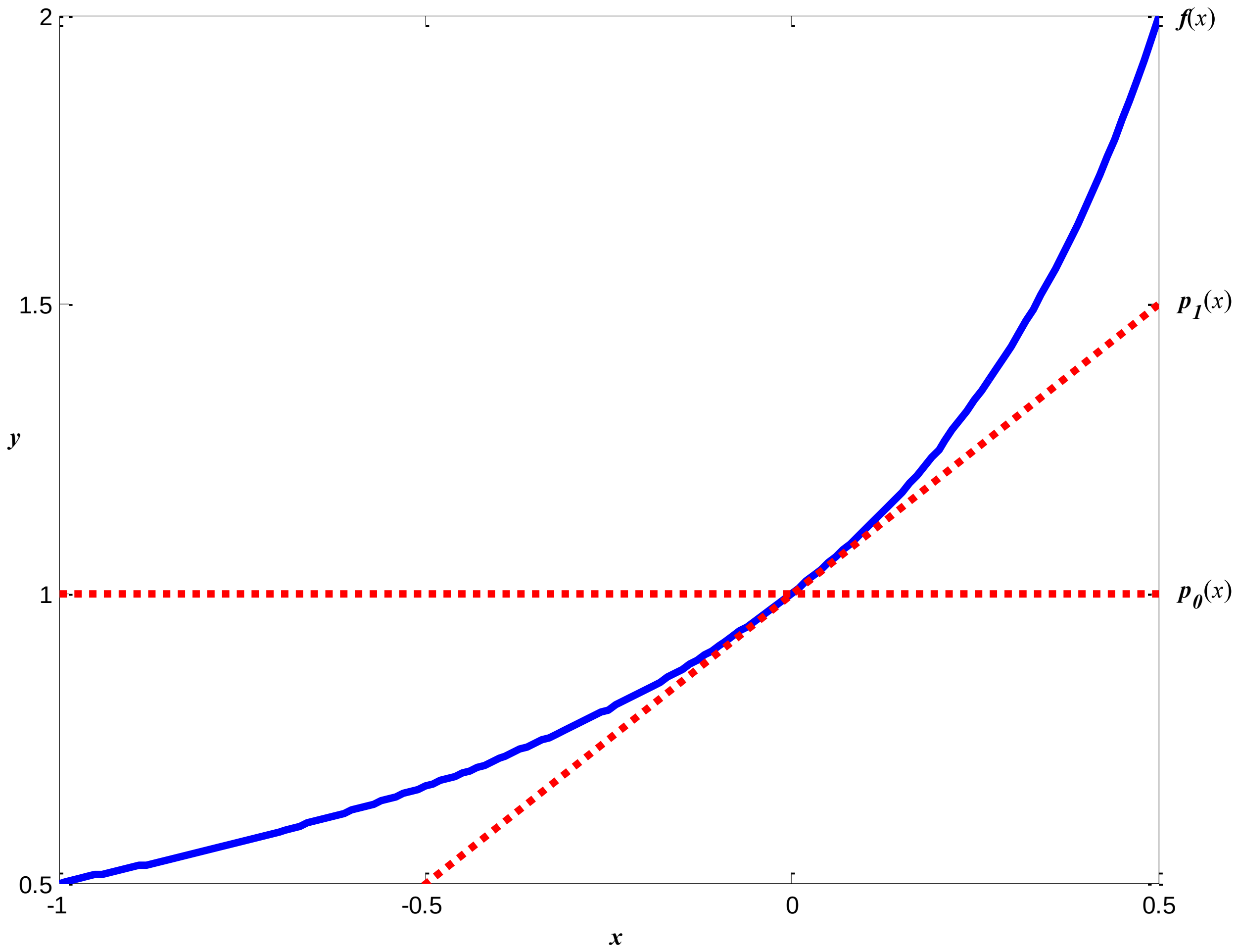
Polynomial Approximations (Graph)



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Taylor's Theorem

Statement : If f has derivatives of all orders in an open interval *I* containing x_0, then for each positive integer *n* and for each *x* in *I*:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + \ldots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} + \ldots$$

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Example 1

Obtain the power series expansion of $f(x) = cosx$ about $\frac{\pi}{3}$. Hence find an approximate value of 61°

Solution: Required Taylor's series expansion is given by

$$f(x) = f\left(\frac{\pi}{3}\right) + \sum_{n=1}^{\infty} \frac{\left(x-\frac{\pi}{3}\right)^3}{n!} f^n(\pi/3) \quad .... \quad \text{(i)}$$

We note that

$$f(\pi/3) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f(x) = \cos(x + n\pi/2)$$

Therefore, (i) becomes

$$\cos x = 1/2 + \sum_{n=1}^{\infty} \frac{(x-\pi/3)}{n!} \cos(\pi/3 + n\pi/2)$$

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Example 1 contd...

$$\Rightarrow \cos x = 1/2 - \frac{\sqrt{3}}{2}(x - \pi/3) - \frac{1}{4}(x - \pi/3)^2 + \frac{\sqrt{3}}{12}(x - \pi/3)^3$$

Put $x = 61^\circ$ so that $x - \frac{\pi}{3} = 1^\circ = \frac{\pi}{180}$ radian

$\therefore$ above equation becomes

$$cos61° = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{4}\left(\frac{\pi}{180}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\pi}{180}\right)^3$$

$cos61°$ =0.4848

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Example 2

Obtain the power series expansion of about $f(x) = \log(cosx)$ about the point $x = \frac{\pi}{3}$ up to the fourth-degree term

Solution: Taylor's series expansion is given by

$$f(x) = f\left(\frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right) f'(\pi/3) + \frac{\left(x-\frac{\pi}{3}\right)^2}{2!} f''(\pi/3) + \frac{\left(x-\frac{\pi}{3}\right)^3}{3!} f'''(\pi/3)$$

$$\frac{\left(x-\frac{\pi}{3}\right)^4}{4!} f'^{v}(\pi/3) + \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad \text{(i)}$$

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Example 2 (cont.)

Now since $f(x) = \log(cosx)\,, we\ have$

$$f\left(\frac{\pi}{3}\right) = \log\left(\cos\left(\frac{\pi}{3}\right)\right) = \log(1/2)$$

$$f'(x) = -tanx \Longrightarrow f'\left(\frac{\pi}{3}\right) = -\sqrt{3}$$

$$f''(x) = -\sec^2 x, \quad \Rightarrow f''(\pi/3) = -4$$

$$f'''(x) = -2\sec^2 x \tan x \ \Rightarrow f'''(\pi/3) = -8\sqrt{2}$$

$$f^{iv}(x) = -4\sec^2 x\tan^2 x - 2\sec^4 x \Rightarrow f^{iv}(x) = -80$$

Using these in expression (i), we get

$$\log(\cos x) = \log(1/2) - \sqrt{3}(x - \pi/3) - 2(x - \pi/3)^2 - \frac{4}{\sqrt{3}}(x - \pi/3)^3$$

$$-\frac{10}{3}(x - \pi/3)^4$$

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Example 3

Obtain Taylor's series expansion of the function $f(x) = lnx$ about $x = 1$

Solution:

$$f(x) = \ln(x) \quad \Rightarrow f(x_0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow f'(x_0) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow f''(x_0) = -\frac{1}{1^2} = -1$$

$$f'''(x) = \frac{2}{x^3} \quad \Rightarrow f'''(x_0) = \frac{2}{1^3} = 2$$

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Example 3 (contd.)

Taylor's series formula is given by

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!}$$

$$+ \ldots + f^{(n)}(x_0)\frac{(x - x_0)^n}{n!} + \ldots$$

$$\Rightarrow \ln(x) = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{2!(x-1)^3}{3!}$$

$$+ \ldots + (n-1)!(-1)^{n-1}\frac{(x-1)^n}{n!} + \ldots$$

$$\Rightarrow \ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$$

$$+ \ldots + (-1)^{n-1}\frac{(x-1)^n}{n} + \ldots$$

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Summary

- Formula for Taylor's series expansion of a function $f(x)$ in the neighborhood of x_0 is given by solution

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \frac{f'''(x_0)(x - x_0)^3}{3!} + \cdots$$

- Taylor series are used to estimate the value of functions (at least theoretically - now days we can usually use the calculator or computer to calculate directly)

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