Lecture 24
Cauchy-Riemann Equation_1

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Intended learning Outcomes

At the end of this lecture, student will be able to:

- State mathematical statement of analytic function
- State and prove the Cauchy-Riemann equations in Cartesian and polar form
- Apply Cauchy-Riemann equations to verify the analyticity of complex valued functions

Topics

- Cauchy-Riemann equation
- Sufficient conditions for differentiability
- C-R equations in polar coordinates

Cauchy – Riemann Equations

Theorem: Suppose that $f(z) = u(x, y) + iv(x, y)$ and that f'(z) exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations $u_x = v_y; u_y = -v_x$

Cauchy – Riemann Equations

Proof:

Let $z_0 = x_0 + iy_0 ; \Delta z = \Delta x + i\Delta y$

$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$

$$= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]$$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i\Delta y}$$

Cauchy – Riemann Equations

Note that (Δx, Δy) can be tend to (0,0) in any manner

Consider the horizontally and vertically directions

- Horizontally direction (Δy=0)

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) + i[v(x_0+\Delta x, y_0+\Delta y) - v(x_0, y_0)]}{\Delta x + i0}$$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0, y_0) + i[v(x_0+\Delta x, y_0) - v(x_0, y_0)]}{\Delta x}$$

$$= u_x(x_0, y_0) + i v_x(x_0, y_0)$$

Cauchy – Riemann Equations

Vertically direction (Δx=0)

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0) + i[v(x_0, y_0 + \Delta y) - v(x_0, y_0)]}{0 + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{i\{[u(x_0, y_0 + \Delta y) - u(x_0, y_0)] + i^2[v(x_0, y_0 + \Delta y) - v(x_0, y_0)]\}}{i(i\Delta y)}$$

$$= v_y(x_0, y_0) - iu_y(x_0, y_0)$$

From the above equation, we have

$$u_x = v_y ; u_y = -v_x$$

Consequences-1

- If $f(z) = u + iv$ is an analytic function ,then u and v both satisfy the two dimensional Laplace equation
- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$
- $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0;$

Consequences-2

- If $f(z) = u + iv$ is an analytic function, then the Equations $u(x, y) = c_1$ and $v(x, y) = c2$ represent orthogonal families of curves (that is, the two families of curves are orthogonal trajectories of each other)

Example 1

Example: $f(z) = z^2 = x^2 - y^2 + i2xy$

is differentiable everywhere and that f'(z)=2z. To verify that the Cauchy-Riemann equations are satisfied everywhere,

$$u(x,y) = x^2 - y^2 \qquad v(x,y) = 2xy$$

$$u_x = 2x = v_y \qquad u_y = -2y = -v_x$$

$$f'(z) = 2x + i2y = 2(x+iy) = 2z$$

Exampl-2

$$f(z) = |z|^2$$

$$u(x,y) = x^2 + y^2 \qquad v(x,y) = 0$$

If the C-R equations are to hold at a point (x,y), then

$$u_x = u_y = v_y = -v_x = 0$$

$$\Rightarrow \quad x = y = 0$$

Therefore, f'(z) does not exist at any nonzero point.

Exampe-3

Using C-R equations , show that the function

$f(z) = (e^x cosy + 3) + i(e^x siny - 3)$ is analytic function

Solution:$u = e^x cosy + 3 \quad v = e^x siny - 3$

$\frac{\partial u}{\partial x} = e^x cosy \qquad \frac{\partial u}{\partial y} = -e^x siny, \quad \frac{\partial v}{\partial x} = e^x siny$

$\frac{\partial v}{\partial y}$=$e^x cosy$

Evidently

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$$

∴ C-R equations are satisfied

Hence $f(z)$ is analytic

Sufficient Conditions for Differentiability

Theorem: $f(z) = u(x,y) + i\,v(x,y)$ be defined throughout some ε neighborhood of a point $z_0 = x_0 + iy_0$, and suppose that

a. the first-order partial derivatives of the functions u and v with respect to x and y exist everywhere in the neighborhood;

b. those partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy–Riemann equations

$$u_x = v_y;\, u_y = -v_x \text{ at } (x_0, y_0)$$

Then $f'(z_0)$ exists, its value being $f'(z_0) = u_x + iv_x$ where the right-hand side is to be evaluated at (x_0, y_0).

C-R equations in polar coordinates

Theorem: Let the function $f(z)=u(r,\theta)+iv(r,\theta)$ be defined throughout some ε neighborhood of a nonzero point $z_0=r_0\exp(i\theta_0)$ and suppose that

(a) the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood;

(b) those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form $ru_r = v_\vartheta,\ u_\vartheta = -rv_r$ of the Cauchy-Riemann equations at (r_0, θ_0). Then $f'(z_0)$ exists, its value being

$$f'(z_0) = e^{-i\theta}(u_r(r_0,\theta_0) + iv_r(r_0,\theta_0))$$

Analytic Function

- Analytic vs. Derivative

➢ For a point

Analytic ➔ Derivative ✓

Derivative ➔ Analytic ✗

➢ For all points in an open set

Analytic ➔ Derivative ✓

Derivative ➔ Analytic ✓

Session Summary

- Cauchy–Riemann equations in Cartesian form:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- Cauchy-Riemann equations in polar form: $ru_r = v_\vartheta,\ u_\vartheta = -rv_r$