Lecture 37
Taylor Series, Maclaurin Series and Uniform Convergence

Dr. Mahesha Narayana

Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Formulate an expansion of a complex valued function through Taylor and Maclaurin series
- Distinguish between convergence and uniform convergence
- Determine the uniform convergence of Taylor and Maclaurin series

Topics

- Taylor series
- Taylor's theorem
- Region of convergence of Taylor series
- Maclaurin series of elementary functions

Taylor series and Maclaurin series

Brook Taylor
English mathematician
1685—1731

Colin Maclaurin
Scottish mathematician
1698—1746

Coefficients of Taylor Series

Consider a power series representation of a function $f(z)$ for $|z - z_0| < R$, $R \neq 0$, in the form

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z)_0^3 + \cdots$$

It follows that

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + \cdots$$

$$f''(z) = \sum_{k=2}^{\infty} k(k-1) a_k (z - z_0)^{k-2} = 2\cdot 1 a_2 + 3\cdot 2 a_3(z - z_0) + \cdots$$

$$f'''(z) = \sum_{k=3}^{\infty} k(k-1)(k-2) a_k (z - z_0)^{k-3} = 3\cdot 2\cdot 1 a_3 + \cdots$$

From the above, at $z = z_0$ we have

$$\boxed{a_0 = f(z_0),\ a_1 = \frac{f'(z_0)}{1!},\ a_2 = \frac{f''(z_0)}{2!}, \ldots, a_n = \frac{f^{(n)}(z_0)}{n!}, \ldots}$$

When $n = 0$, we interpret the zeroth derivative as $f(z_0)$ and $0! = 1$. Now we have

$$\boxed{f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k}$$

This series is called as Taylor series of $f(z_0)$ centered at z_0.

A Taylor series with center $z_0 = 0$

$$\boxed{f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k}$$

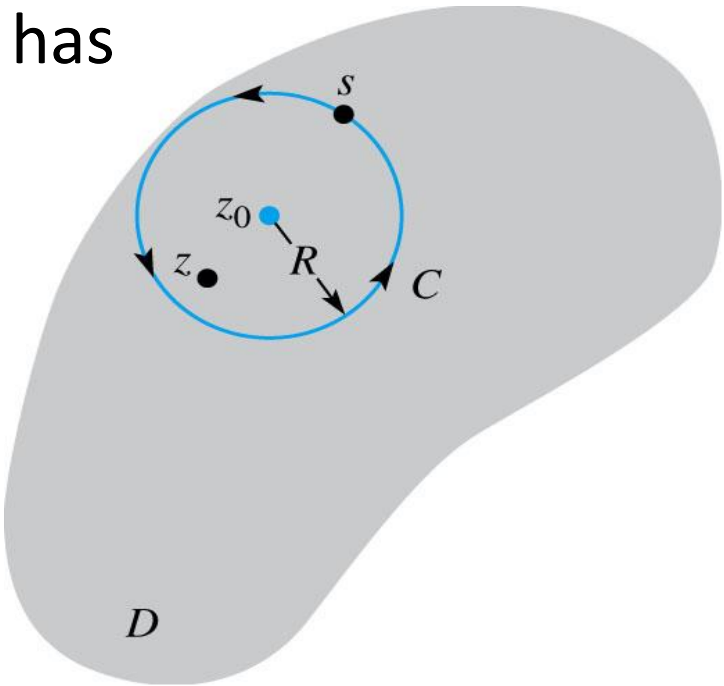
is reffered as Mclaurin series

Taylor's Theorem

Let $f(z)$ be analytic within a domain D and let z_0 be a point in D. Then $f(z)$ has the series representation

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

valid for the largest circle C with center at z_0 and radius R that lies entirely within D.



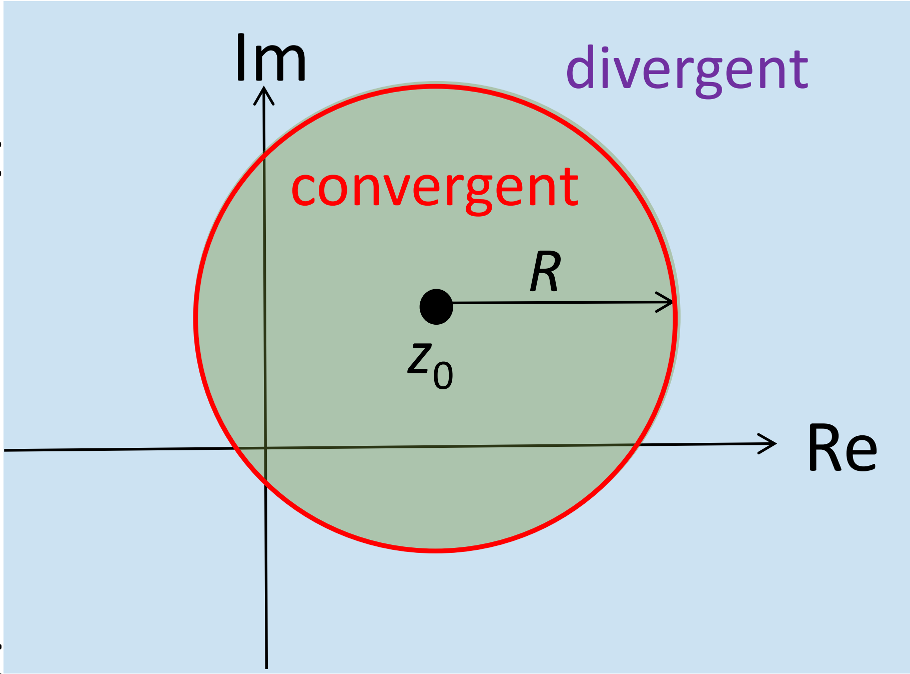
Region of convergence for Taylor series

$$\sum_{k=0}^{\infty} a_k (z-z_0)^k = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z)_0^3 + \cdots$$

Here, Coefficients a_k's are complex constants, z is a complex variable and z_0 is the centre.

There exists a radius R, such that

1. the Taylor series converges if $|z - z_0| < R$,
2. the Taylor series diverges if $|z - z_0| > R$.
3. The series may or may not converge for $|z - z_0| = R$



Note: The region of convergence has the shape of a disc. The radius is called the radius of convergence

Maclurin Series of Elementary Functions

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

Example – 1

Find the Maclurin series of $f(z) = 1/(1 - z)^2$.

Solution

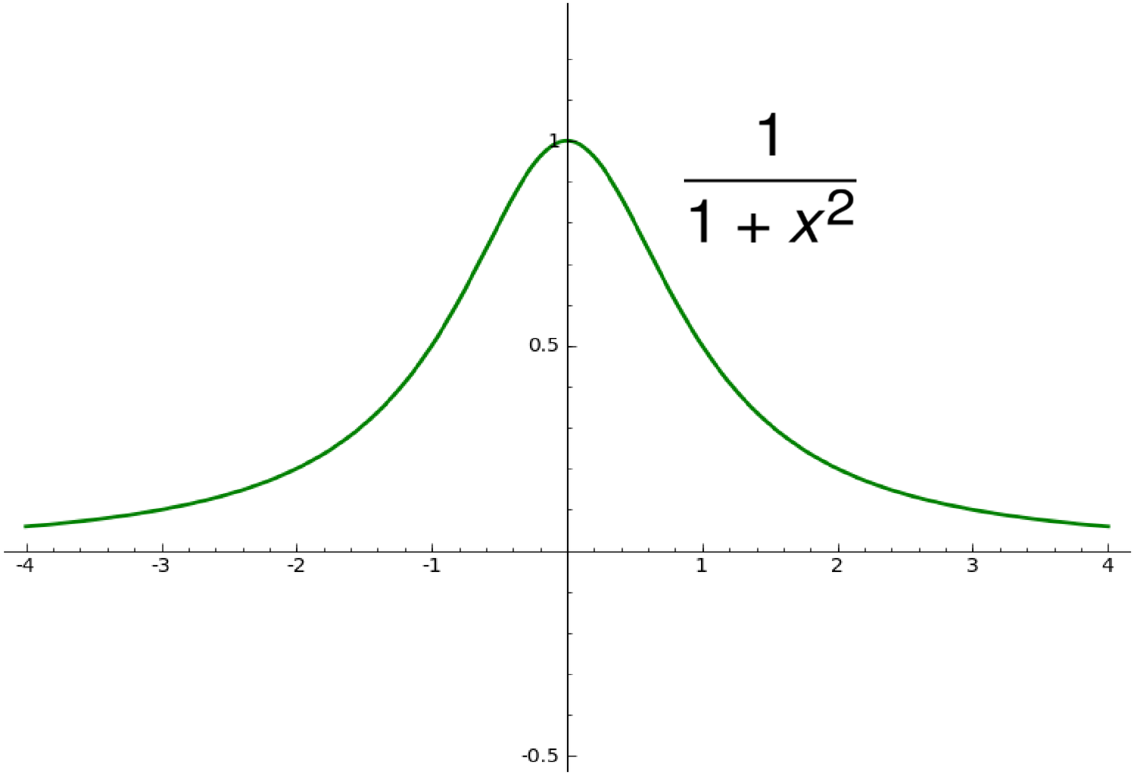
For $|z| < 1$, $$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

Differentiating both sides of above equation

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \ldots = \sum_{k=1}^{\infty} kz^{k-1}$$

Example – 2

What is the Maclaurin series of ?



The Taylor series may not converge even if the function is well-defined !!!

- Use geometric series

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

- It converges if $|x| < 1$, but diverges for $|x| > 1$.

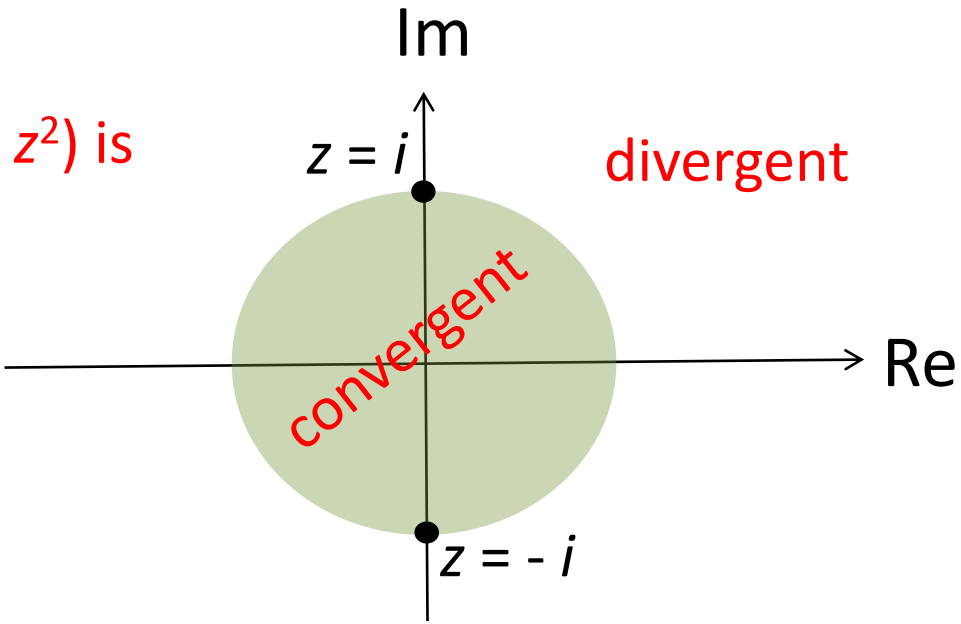
Why does it fail to converge?

The reason is revealed if we extend the domain to complex number

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = 1 - z^2 + z^4 - z^6 + z^8 - \cdots$$

where z may be a complex number.

The value of $1/(1 + z^2)$ is infinite if $z = \pm i$



Problem-1

Obtain the Taylor's series Laurents series which represents the function $\frac{z^2-1}{(z+2)(z+3)}$ in the regions

(i) $|z| < 2$ (ii)$2 < |z| < 3$ (iii) $|z| > 3$

Solution: $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)}$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Problem-1........

When $|z| < 2$, then $\frac{|z|}{2} < 1$

$$f(z) = 1 + \frac{3}{2}\left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$- \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

Problem-1…….

$$= 1 + \frac{3}{2}\sum_0^\infty (-1)^n \frac{z^n}{2^n} - \frac{8}{3}\sum_0^\infty (-1)^n \frac{z^n}{3^n}$$

$$= 1 + \sum_0^\infty (-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right] \frac{z^n}{3^n}$$

$$= 1 + \sum_0^\infty (-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right] z^n$$

This is Taylor's series valid for $|z|<2$

Example-2

Expand $\frac{1}{z^2-3z+2}$ for $0 < |z| < 2$ by using Maclaurin's series

Solution: $f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-2)(z-1)}$

Then $f(z) = \frac{1}{(z-2)(z-1)}$

$$f(z) = (1-z)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$\sum_0^\infty z^n - \frac{1}{2}\sum_0^\infty \left(\frac{z}{2}\right)^n$$

$$= \sum_0^\infty \left(1 - \frac{1}{2^{n+1}}\right) z^n$$

Session Summary

- Every analytic function $f(z)$ can be expanded in Taylor series as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \qquad \text{with } |z - z_0| < R.$$

- This series converge for all z in the open disk with center z_0 and radius R which is equal to the distance from z_0 to the nearest singularity $f(z)$.

- If $f(z)$ is entire (analytic for all z) then the series converges for all z.