Lecture 25
Harmonic functions and Milne Method

Dr. Mahesha Narayana

Intended Learning Outcomes

At the end of this lecture, student will be able to:

- State Harmonic function
- Apply Milne method to construct an analytic function
- Illustrate harmonic function and discuss its properties
- State Orthogonal System

Topics

- Harmonic functions
- Harmonic conjugates
- Milne Thompson method

Motivation

Electrostatics. ϕ is electric potential . The electric field is $E = -\nabla\phi$. In the absence of charged sources $\nabla . E = 0$ then $\nabla^2\phi = 0$

Fluid flow $\nabla\phi = v$ is the velocity of a fluid then $\nabla^2\phi = 0$. Level curves $f(z) = \phi(x, y) + i\psi(x, y)$: ϕ =constant are the equipotential and $\psi =$ constant are the stream lines (direction of motion of the fluid particles)

Heat flow. Steady state system . ϕ =Temperature , $\nabla^2\phi = 0$, Level curves of $f(z) = \phi + i\psi$: $\phi =$ constant are isothermals and $\psi =$ constant represents the direction of the heat flow

Harmonic Functions

Given $f(z) = u(x, y) + iv(x, y)$ is an analytic function then $u(x, y)$ and $v(x, y)$ have continuous second order partial derivatives and they satisfy Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

If a function satisfies Laplace equation it is said to be HARMONIC, hence both $u(x, y)$ and $v(x, y)$ are harmonic functions. In polar form we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

Harmonic Conjugates

Two functions $u(x, y)$ and $v(x, y)$ are said to be harmonic conjugates of each other if they satisfy C – R equations

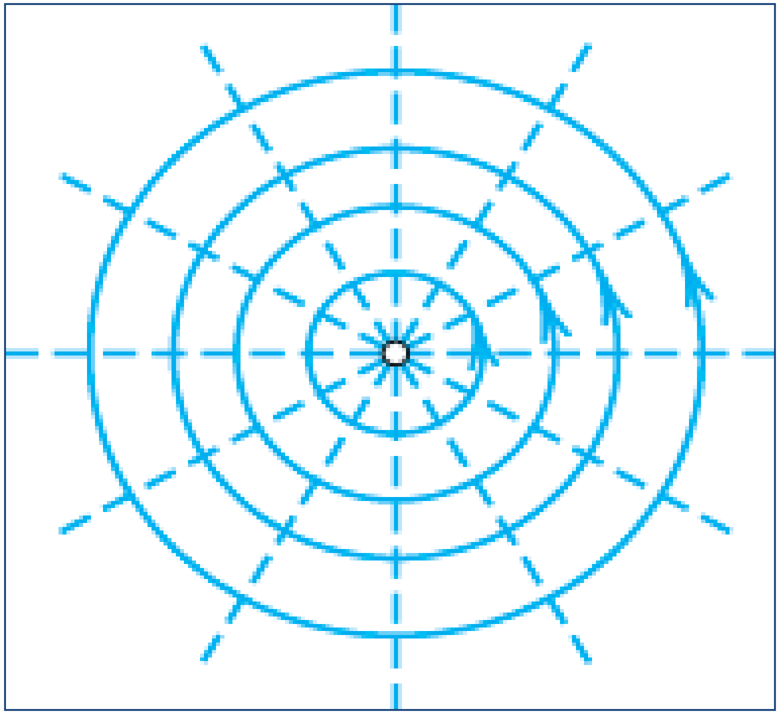
Milne – Thompson method

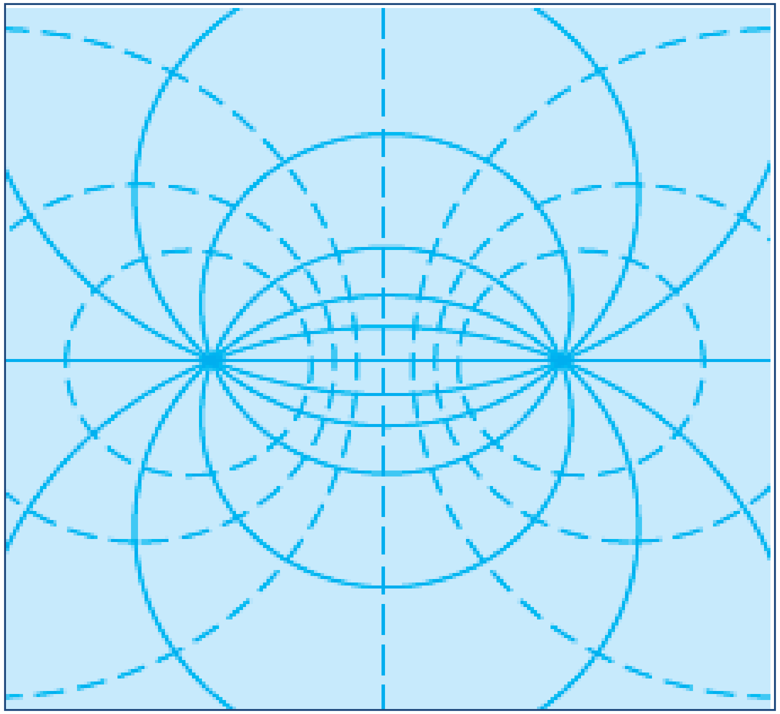
Given either $u(x, y)$ or $v(x, y)$ we can find $f(z) = u(x, y) + iv(x, y)$ as follows

1. Find u_x , u_y or v_x , v_y and consider $f'(z) = u_x + i\, v_x$
2. Using C – R equations write $f'(z) = u_x - i\, u_y$ or $f'(z) = v_y + i\, v_x$
3. Substitute $x = z$ and $y = 0$ to get $f'(z)$ in terms of z
4. Integrate with respect to z to get $f(z)$

Orthogonal System

If $f(z) = u(x, y) + iv(x, y)$ is analytic then the family of curves $u(x, y) = C_1$ and $v(x, y) = C_2$, where C_1 and C_2 are constants form orthogonal





Example-1

Find the analytic function $f(z)$ given $u = e^{-x}\{(x^2 - y^2)cosy + 2xy\ siny\}$

Solution:

$u_x = e^{-x}(2xcosy + 2ysiny)+\{(x^2 - y^2)cosy + 2xy\ siny\}(-e^{-x})$

$u_y = e^{-x}\{(x^2 - y^2)(-siny) + cosy(-2y) + 2x(ycosy + siny)\}$

Consider $f'(z) = u_x + iv_x$

But $\mathrm{v}_x = -\mathrm{u}_\mathrm{y}$ (C R equation)

Putting $x = z,\ y = 0,$

We have

$f'(z)=e^{-z}\ (2z) + z^2(-e^{-z}) - i\ 0 = (2z - z^2)e^{-z}$

$\therefore\ f(z) = \int (2z - z^2)e^{-z}dz + c$

Integrating by applying Bernoulli's rule we have ,

$f(z) = z^2\ e^{-z} + c$

Example-2

Find the analytic function $f(z)$ whose real part is $\frac{sin2x}{cosh2y-cos2x}$

And hence find the imaginary part

Let $u = \frac{sin2x}{cosh2y-cos2x}$ $\quad\therefore\quad u_x = \frac{(cosh2y-cos2x)2cos2x-sin2x\,(2sin2x)}{(coah2y-cos2x)^2}$

$$u_y = -\frac{sin2x(2\,sinh2y)}{(cosh2y-cos2x)^2}$$

Consider $f'(z) = u_x + iv_x = u_x - iu_y$ by C-R equation

Putting $x = z, y = 0$ we have

$f'(z) = -\frac{2}{2\sin^2 z}$=$-cosec^2 z$

$f(z) = cotz + c$

Example-2…………..

We shall separate $\cot(z) = \cot(x+iy)$ into real and imaginary parts to find v

Consider $f(z) = \cot(z)$

$$u + iv = \cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)}$$

$$= \frac{\cos(x+iy)\sin(x-iy)}{\sin(x+iy)\sin(x-iy)}$$

$$= \frac{\frac{1}{2}[\sin(x-iy+x+iy)+\sin(x-iy-x-iy)]}{\frac{1}{2}[\cos(x+iy-x+iy)-\cos(x+iy+x-iy)]}$$

$$\frac{[sin2x + \sin(-2iy)]}{\cos(2iy) - cos2x} = \frac{(sin2x - i\ sinh2y)}{cosh2y - cos2x}$$

(it may be observed that the real part u is the given problem

Thus $v = -\frac{sinh2y}{cosh2y - cos2x}$

Example-3

Find the analytic function $f(z) = u + iv$, given $u - v = e^x(cosy - siny)$
Solution: From the given $u - v$,

We find that

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{\partial (u-v)}{\partial x} = e^x(cosy - siny)$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{\partial (u-v)}{\partial y} = e^x - siny - cosy)$$

By using C R equations, these can be rewritten as

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{\partial (u-v)}{\partial x} = e^x(cosy - siny)$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{\partial (u-v)}{\partial y} = e^x(siny + cosy)$$

Example-3...........

Solving these equations , we get $\frac{\partial v}{\partial y} = e^x cosy$ and

$$\frac{\partial \mathrm{v}}{\partial \mathrm{x}} = \mathrm{e^x siny}$$

Therefore $f'(z) = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x} = \frac{\partial v}{\partial y} + \frac{i\partial v}{\partial x}$

$$= e^x cosy + ie^x siny = e^x e^{iy} = e^{x+iy} = e^z$$

From this, it is readily seen that f(z)=$e^z + c$

, where c is a complex constant. This is the required analytic function

Example-4

Find the analytic function $f(z) = u + iv$, given that $u + v = \frac{2sin2x}{e^{2y}+e^{-2y}-2cos2x}$

Solution:The given $u + v$ can be rewritten as $u + v = \frac{sin2x}{cosh2y-cos2x}$

This gives $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x} = \frac{2(\cos 2x \cosh 2y-1)}{(cosh2y-cos2x)^2}$

$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} = -\frac{2sin2x\ sinh2y}{(cosh2y-cos2x)^2}$

In view of CR equations , these may be written as

$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \frac{2(\cos 2x \cosh 2y-1)}{(cosh2y-cos2x)^2}$ .... (i)

$\frac{\partial u}{\partial y}+\frac{\partial u}{\partial x} = -\frac{2sin2x\ sinh2y}{(cosh2y-cos2x)^2}$ .... (ii)

Example-4…….

Solving these equations , we get

$$\frac{\partial u}{\partial x}=\frac{cos2x\ cosh2y-sin2x\ sinh2y-1}{(cosh2y-cos2x)^2}$$

$$\frac{\partial u}{\partial y}=\frac{1-cos2xcosh2y-sin2xsinh2y}{(cosh2y-cos2x)^2}$$

Therefore , if $f(z)=u+iv$ is analytic , we have

$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}$ (substitute the above expressions in $f'(z)$)

put $x=z$ and $y=0$,

We have $f'(z)=\frac{1}{2}(1+i)cosec^2z=\frac{1}{2}(1+i)\frac{d}{dz}(cotz)$

$$f(z)=\frac{1}{2}(1+i)cotz+c$$

Session Summary

- Harmonic function in Cartesian and polar form

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$