Lecture 6
Vector space, subspace, Basis, Dimension, Row Space, Column Space

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Intended Learning Outcomes

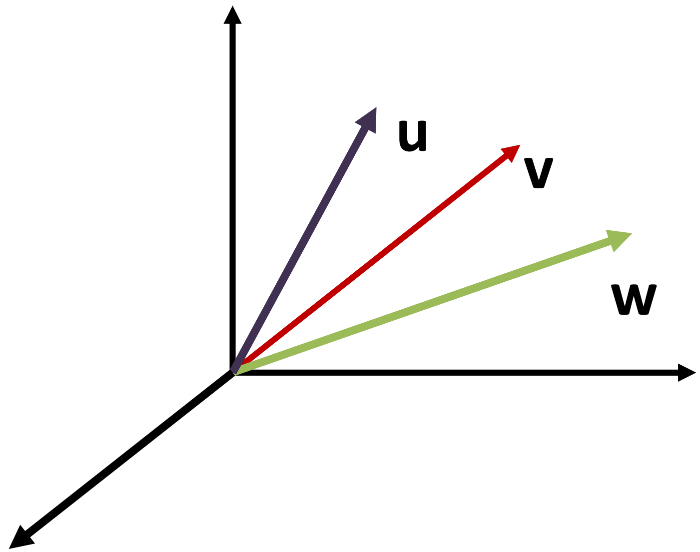
At the end of this lecture, student will be able to:

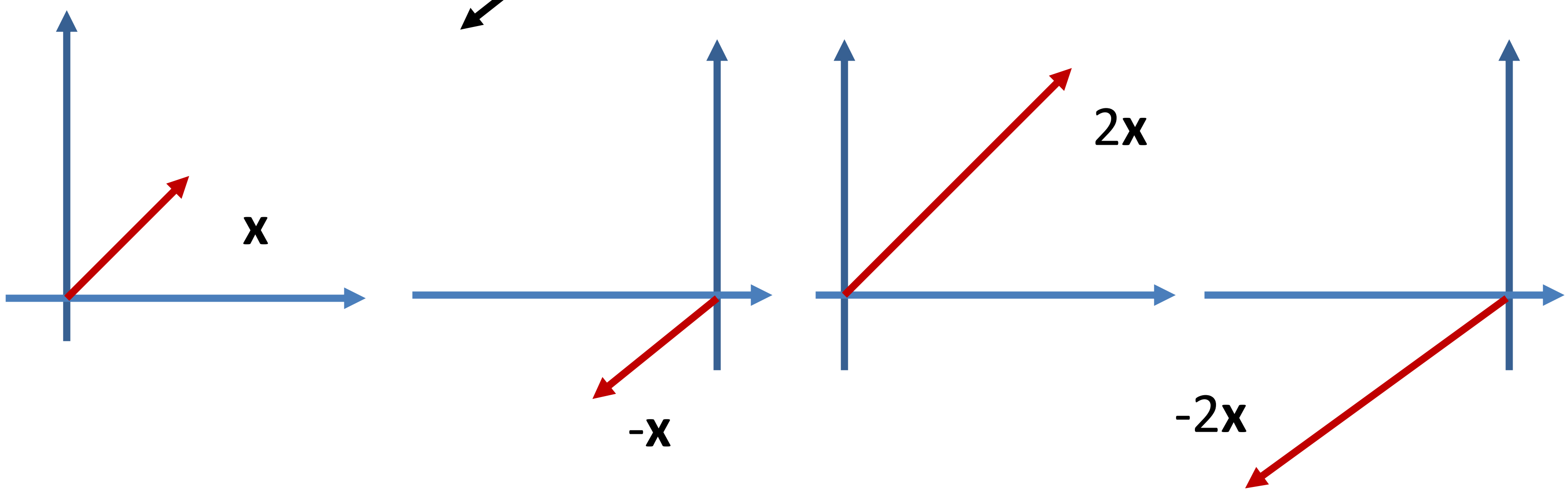
- Illustrate the principles of vector spaces
- Illustrate the concept of subspaces
- Write the vectors in linear combination form
- Find the span of the vector space
- Distinguish linearly independent and linearly dependent
- Define and illustrate basis and dimension of a vector space
- Illustrate row space, column space and null space

Topics

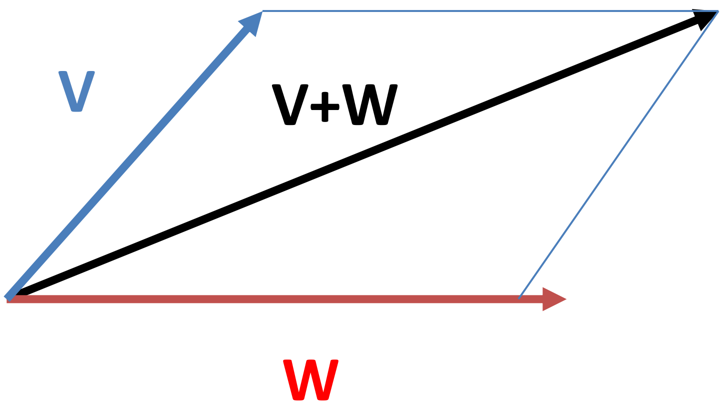
- Vectors in R^n
- Vector operations
- Vector space
- Subspace
- Linear dependence and independence
- Basis and Dimension
- Null space, row space, column space
- MATLAB Code

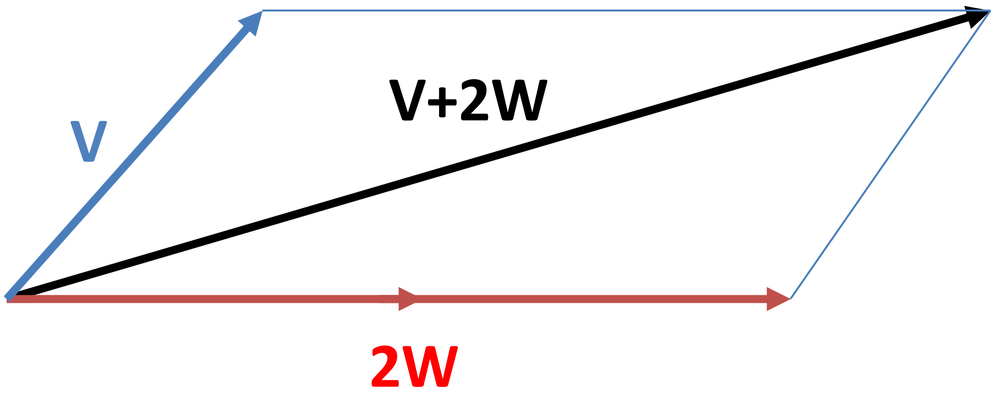
Motivation

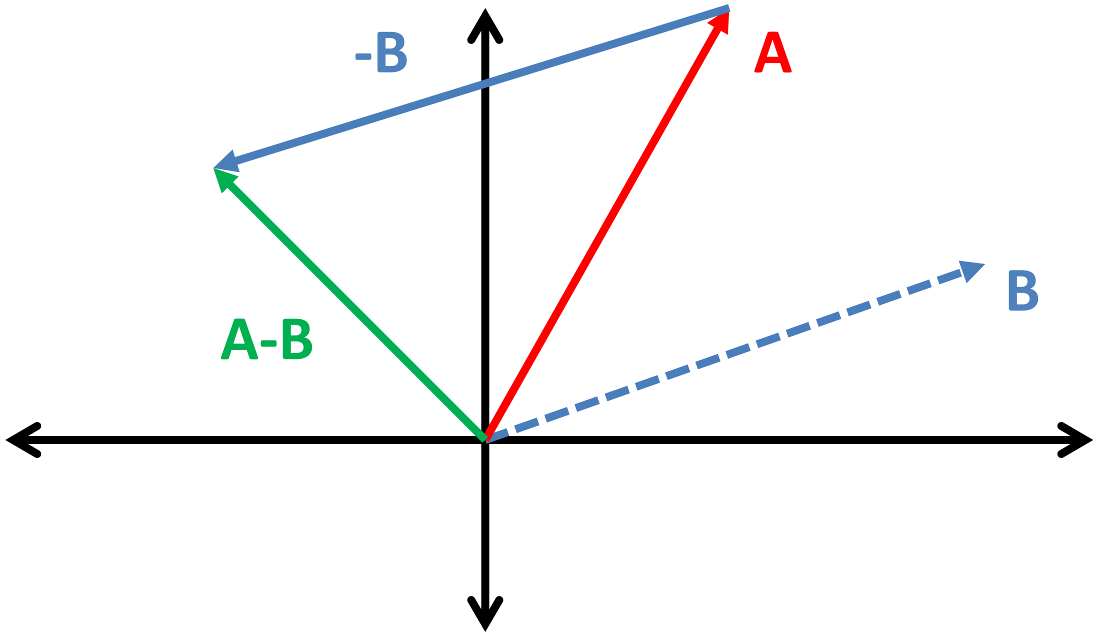




Motivation…

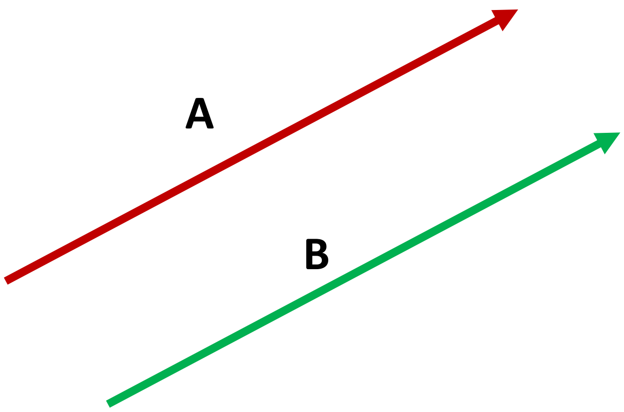




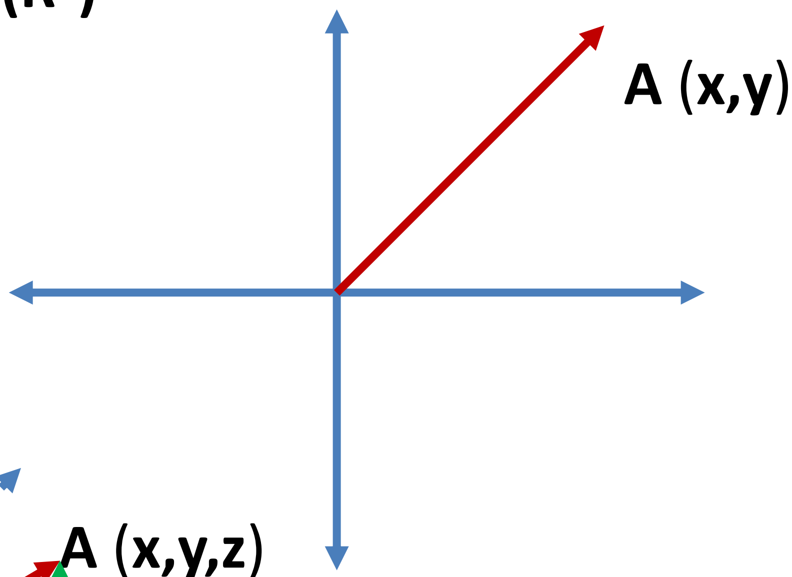


Vector

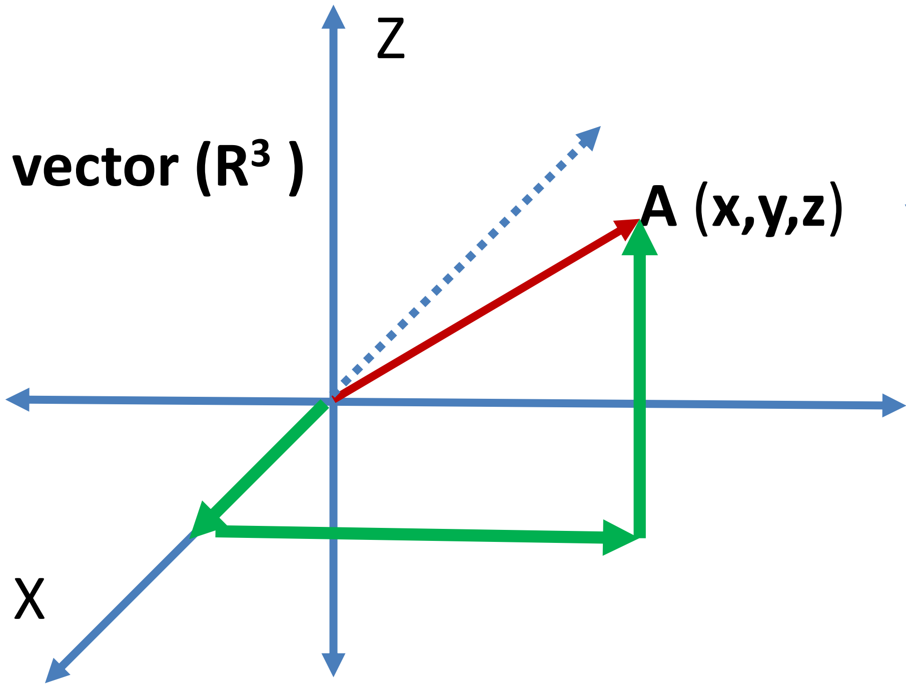
One dimensional vector (R)



Two dimensional vector (R^2)



Three dimensional vector (R^3)



Vectors in R^n

- An ordered *n*-tuple:

  a sequence of *n* real number $(x_1, x_2, \cdots, x_n)$

- *n*-space: R^n

  the set of all ordered n-tuple

$n = 1$ R^1 = 1-space
= set of all real number

$n = 2$ R^2 = 2-space
= set of all ordered pair of real numbers (x_1, x_2)

$n = 3$ R^3 = 3-space

= set of all ordered triple of real numbers (x_1, x_2, x_3)

Vector Operation

$\mathbf{u}=(u_1,u_2,\cdots,u_n),\ \mathbf{v}=(v_1,v_2,\cdots,v_n)$ (two vectors in R^n)

- Equal:

  $\mathbf{u}=\mathbf{v}$ if and only if $u_1=v_1,\ u_2=v_2,\cdots,u_n=v_n$

- Vector addition (the sum of **u** and **v**):

  $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$

- Scalar multiplication (the scalar multiple of **u** by *c*):

  $c\mathbf{u}=(cu_1,cu_2,\cdots,cu_n)$

- Negative:

  $-\mathbf{u}=(-u_1,-u_2,-u_3,\ldots,-u_n)$

- Difference:

  $\mathbf{u}-\mathbf{v}=(u_1-v_1,u_2-v_2,u_3-v_3,\ldots,u_n-v_n)$

- Zero vector:

  $\mathbf{0}=(0,0,\ldots,0)$

Vector Space

Let V be a non empty set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every ***u, v***, and ***w*** in V and every scalar (real number) c and d, then (V, +, .) is called a **vector space**.

- For the vector addition + :

  $\forall\ \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u} \in V$

  1. $\boldsymbol{v} + \boldsymbol{w} \in V$ (Closure)
  2. $\boldsymbol{v} + \mathbf{w} = \boldsymbol{w} + \mathbf{v}$ (Commutativity)
  3. $(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{u} = \boldsymbol{v} + (\boldsymbol{w} + \boldsymbol{u})$ (Associativity)
  4. $\exists\ \mathbf{0} \in V$ s.t. $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$ (Additive identity)
  5. $\exists\ -\boldsymbol{v} \in V$ s.t. $\boldsymbol{v} - \boldsymbol{v} = \mathbf{0}$ (Additive Inverse)

Vector Space...

- For the scalar multiplication :

  $\forall$ $\boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in$ R,

  6. $a\,\boldsymbol{v} \in V$ (Closure)
  7. $(a + b)\,\boldsymbol{v} = a\,\boldsymbol{v} + b\,\boldsymbol{v}$ (Distributivity)
  8. $a(\boldsymbol{v} + \boldsymbol{w}) = a\,\boldsymbol{v} + a\,\boldsymbol{w}$
  9. $(a \times b)\,\boldsymbol{v} = a(b\,\boldsymbol{v}) = a\,b\,\boldsymbol{v}$ (Associativity)
  10. $1\,\boldsymbol{v} = \boldsymbol{v}$

$(V, \ +, \ \bullet)$ is called a vector space

Example 1

Let K be an arbitrary field. The set of all n-tuples of elements of K with vector addition and scalar multiplication defined by

$$(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n)$$

$$k(u_1, u_2, \cdots, u_n) = (ku_1, ku_2, \cdots, ku_n)$$

Where $u, v, k \in K$. Then V is a vector space over K.

Subspace

Definition: Let V is a vector space over a field F and a W is a non-empty subset of V. Then, W is said to be subspace of vector space V if:

- $\boldsymbol{u}, \boldsymbol{v} \in W$ $then$ $\boldsymbol{u} + \boldsymbol{v} \in W$
- $\boldsymbol{u} \epsilon W$ and $k \epsilon F$ then $k\boldsymbol{u} \epsilon W$

Then W is said to be subspace of V

- Trivial subspace

Every vector space V has at least two subspaces.

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

Example 2

Let $S = \{A \epsilon R^{2\times 2} \mid a_{12} = -a_{21}\}$. Show that S is subspace.

i. If $A \in S$, then A must be of the form $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$

and hence $\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix} \Rightarrow \alpha A \in S$

ii. If A, $B \in S$, then they must be of the form

$$A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}, B = \begin{bmatrix} d & e \\ -e & f \end{bmatrix}$$

$$A + B = \begin{bmatrix} a+d & b+e \\ -(b+e) & c+f \end{bmatrix} \Rightarrow A + B \in S$$

Hence, S is subspace of vector space V

Linear combination

A vector ***v*** in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ in *V* if ***v*** can be written in the form

$$\boldsymbol{v} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3 + \ldots\ldots \alpha_n \boldsymbol{u}_n$$

Where $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots \alpha_n$ are scalars

Example 12 Show that ***w***=(1,1,1) is a linear combination of $\boldsymbol{u}_1 = ((1,2,3), \boldsymbol{u}_2 = (0,1,2)\ and\ \boldsymbol{u}_3 = (-1,0,1)$

Sol. Let $W = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3$

$$(1,1,1) = \alpha_1(1,2,3) + \alpha_2(0,1,2) + \alpha_3(-1,0,1)$$

$$(1,1,1) = (\alpha_1 - \alpha_3, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 + \alpha_3)$$

Linear combination….

$$\alpha_1 - \alpha_3 = 1$$

$$2\alpha_1 + \alpha_2 = 1$$

$$3\alpha_1 + 2\alpha_2 + \alpha_3 = 1$$

We write in matrix form,

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \ \boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$[\boldsymbol{A} : \boldsymbol{B}] = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{Solving by Gauss-Jordon method}} [\boldsymbol{A} : \boldsymbol{B}] \sim \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha_1 = 1 + t, \alpha_2 = -1 - 2t, \alpha_3 = t$$

Put t=1

$$\boldsymbol{W} = 2\boldsymbol{u}_1 - 3\boldsymbol{u}_2 + \boldsymbol{u}_3$$

Span of a set: span (S)

If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k\}$ is a set of vectors in a vector space V, then **the span of S** is the set of all linear combinations of the vectors in S,

$$span\ (\boldsymbol{S}) = \{\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \dots\dots + \alpha_n \mathbf{v}_n\}$$

- a spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called **a spanning set** of the vector space

Example 3

Show that the set $S = \left\{(1,1,1)^T, (1,1,0)^T, (1,0,0)^T\right\}$ spans R^3

Sol. We must determine whether a vector $\boldsymbol{u}=(u_1,u_2,u_3)^T$ can be written in the linear combination of $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$

$$\boldsymbol{u} = \alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \alpha_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This leads to system of equation

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= u_1 \\ \alpha_1 + \alpha_2 \qquad &= u_2 \\ \alpha_1 \qquad\qquad &= u_3 \end{aligned}$$

Solving by Gauss elimination method

Example 3…

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}=\begin{bmatrix}u_3\\ u_2-u_3\\ u_1-u_2\end{bmatrix}$$

Thus,

$$\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=u_3\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+(u_2-u_3)\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}+(u_1-u_2)\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

So the three vectors span R^3

Example 4

Show that the set $S = \{(1,2,4)^T, (2,1,3)^T, (4,-1,1)^T\}$ spans R^3

Sol. We must determine whether a vector $\boldsymbol{u} = (u_1, u_2, u_3)$ can be written in the linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$

$$u \in R^3 \Rightarrow u = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \qquad (1)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ 2\alpha_1 + \alpha_2 - \alpha_3 \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 \end{bmatrix}$$

$$\alpha_1 + 2\alpha_2 + 4\alpha_3 = u_1 \qquad (2)$$
$$2\alpha_1 + \alpha_2 - \alpha_3 = u_2 \qquad (3)$$
$$4\alpha_1 + 3\alpha_2 + \alpha_3 = u_2 \qquad (4)$$

Example 4...

Solving equation (2),(3) and (4) by Gauss elimination method

Augmented matrix

$$\begin{bmatrix} 1 & 2 & 4 : u_1 \\ 2 & 1 & -1 : u_2 \\ 4 & 3 & 1 : u_3 \end{bmatrix} \xrightarrow[\text{method}]{\text{GE}} \begin{bmatrix} 1 & 2 & 4 : & u_1 \\ 0 & 1 & 3 : & \dfrac{2u_1 - u_2}{3} \\ 0 & 0 & 0 : & 2u_1 + 5u_2 - 3u_3 \end{bmatrix}$$

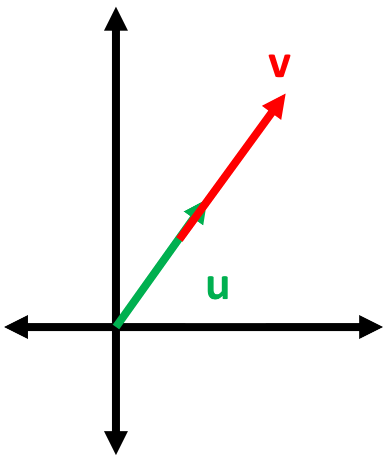
$$2u_1 + 5u_2 - 3u_3 \neq 0$$

then the system is inconsistent. Hence, for most choices of u_1 , u_2 and u_3,it is impossible to express $(u_1 , u_2 , u_3)^T$ as a linear combination of $(1, 2, 4)^T$, $(2, 1, 3)^T$, and $(4,-1, 1)^T$.

Thus S does not span R^3

Linearly dependence

Definition: The set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \ \ldots, \boldsymbol{v}_{\boldsymbol{n}}\}$ is said to be **linearly dependent** if



- All the vectors are of the same order
- There exists n scalars $\lambda_1, \lambda_2, \lambda_3 \ldots\ldots\ldots\ldots \lambda_n$ (not all zero), such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \ldots\ldots\ldots\ldots \lambda_n v_n = 0 \quad \text{----(1)}$$

Where **0** denotes the *n*-vector whose components are zero

Equation (1) is called a **linear dependence relation** among $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{\boldsymbol{n}}$ when the weights $(\lambda_1, \ldots \ldots \ \lambda_n)$ are not all zero.

Example 5

Show that the vectors X_1 =(1,2,4), X_2 = (3,6,12) are linearly dependent.

Sol. Let λ_1, λ_2 are scalars, then

$$\lambda_1 \boldsymbol{X}_1 + \lambda_2 \boldsymbol{X}_2 = 0$$
$$\lambda_1 (1,2,4) + \lambda_2 (3,6,12) = 0$$

$$(\lambda_1 + 3\lambda_2, 2\lambda_1 + 6\lambda_2, 4\lambda_1 + 12\lambda) = 0$$

$$\lambda_1 + 3\lambda_2 = 0$$
$$2\lambda_1 + 6\lambda_2 = 0$$
$$4\lambda_1 + 12\lambda = 0$$

The coefficient matrix of the linear equations

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 4 & 12 \end{bmatrix}$$

Example 5...

$$\left(R_1 \to R_2 - 2R_1\right), \left(R_3 \to R_3 - 4R_1\right)$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using Back substitution

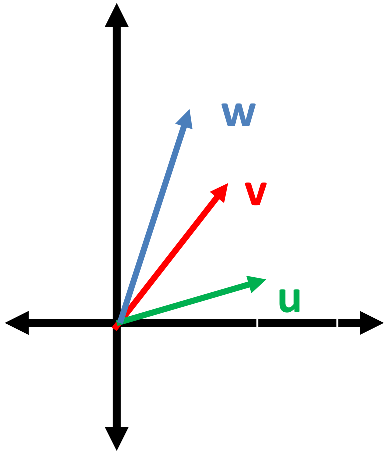
$$\lambda_1 + 3\lambda_2 = 0$$
$$\lambda_1 = -3\lambda_2$$

Put $\lambda_2 = 1 \ then \ \lambda_1 = -3$
$\lambda_1 \neq 0 \ and \ \lambda_2 \neq 0$

Hence X_1 and X_2 are linearly dependent vectors

Linearly independence

Definition: The set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \ \dots, \boldsymbol{v}_n\}$ is said to be **linearly independent** if the vector equation



$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \dots\dots\dots\dots \lambda_n v_n = 0$$

Has only trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = \dots\dots\dots\dots \lambda_n = 0$

Where 0 denotes the n-vector whose components are zero

Equation is called a **linear independence relation** among $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ when the all weights are zero.

Example 6

Show that the three vectors ***u***=(1,0,0), ***v***=(0,1,0) and ***w***=(0,0,1) are linearly independent

Sol. Let k_1, k_2, k_3 are three numbers such that

$$k_1 u + k_2 v + k_3 w = 0$$

$$k_1(1,0,0) + k_2(0,1,0) + k_3(0,0,1) = 0$$

$$(k_1, k_2, k_3) = (0,0,0)$$

$$\Rightarrow k_1 = k_2 = k_3 = 0$$

Thus $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent

Example 7

Show that the row vectors of the matrix

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$ are linearly independent

Sol. Let k_1, k_2, k_3 are three numbers and

$$X_1 = (1,2,-2)', X_2 = (-1,3,0)', X_3 = (0,-2,1)$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, X_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = 0 \qquad (1)$$

$$k_1 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 7...

$$\begin{aligned} k_1 - k_2 \quad\quad\quad &= 0 \\ 2k_1 + 3k_2 - 2k_3 &= 0 \\ -2k_1 \quad\quad + k_3 &= 0 \end{aligned}$$

This is system of homogeneous linear equations, then coefficient matrix of the linear system is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Gauss elimination method}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 5 & -2 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$\begin{aligned} k_1 - k_2 &= 0 \\ 5k_2 - 2k_3 &= 0 \\ \frac{1}{5}k_3 &= 0 \end{aligned}$$

$$\Rightarrow k_3 = 0, \; then \;\; k_1 = k_2 = 0$$
$$\Rightarrow k_1 = k_2 = k_3 = 0$$

Hence the given row vector of the matrix are linearly independent

Linearly dependence and independence of vectors by rank method

- If the rank of matrix of the given vectors is equal to number of vectors, then the vectors are linearly independent

- If the rank of matrix of the given vectors is less than the number of vectors, then the vectors are linearly dependent

Example 8

Show that the set of vectors X=[1, 2, -3, 4], Y=[3,-1,2,1] and Z=[1,-5,8,-7] are linearly dependent

Sol. Let us form a matrix of given vectors

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 1 \\ 1 & -5 & 8 & -7 \end{bmatrix}$$

Using Gauss elimination method

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 2 & 1 \\ 1 & -5 & 8 & -7 \end{bmatrix} \xrightarrow[\text{method}]{\text{Gauss elimination}} \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 11 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here the rank of matrix=2 < number of vectors. Hence, vectors are linearly dependent

Example 9

Show using the matrix that the vectors: [2, 5, 2, -3], [3, 6, 5, 2], [4, 5, 14, 14], [5, 10, 8, 4] are linearly independent

Sol. Here the given vectors are [2, 5, 2, -3], [3, 6, 5, 2], [4, 5, 14, 14], [5, 10, 8, 4]. Let us form a matrix of given vectors

$$\boldsymbol{A} = \begin{bmatrix} 2 & 5 & 2 & -3 \\ 3 & 6 & 5 & 2 \\ 4 & 5 & 14 & 14 \\ 5 & 10 & 8 & 4 \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 2 & 5 & 2 & -3 \\ 3 & 6 & 5 & 2 \\ 4 & 5 & 14 & 14 \\ 5 & 10 & 8 & 4 \end{bmatrix}$$

$$\xrightarrow{\text{Gauss elimination method}} \begin{bmatrix} 1 & 1 & 3 & 5 \\ 0 & 3 & -4 & -13 \\ 0 & 0 & \frac{10}{3} & -\frac{5}{3} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The rank of matrix A=4=number of vectors

Hence, the vectors are linearly independent

Basis

Definition: Let V be a vector space. The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3.......\boldsymbol{v}_n$ form basis for a vector space V if and only if

1. $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3.......\boldsymbol{v}_n$ are linearly independent
2. $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3.......\boldsymbol{v}_n$ span V

$$\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3.......\alpha_{\boldsymbol{n}} \boldsymbol{v}_{\boldsymbol{n}}$$

$(\alpha_1, \alpha_2, \alpha_3.......\alpha_{\boldsymbol{n}})$ is called coordinates of the basis V.

Dimension of Basis

Definition: The number of vectors presents in a basis of a vector space V is called the dimension of V. It is denoted as dim(V)

Example (1): Dimension of the vector space V_4 *(R)* is 4, since four vectors (1,0,0,0),(0,1,0,0), (0,0,1,0) and (0,0,0,1) form a basis of V_4

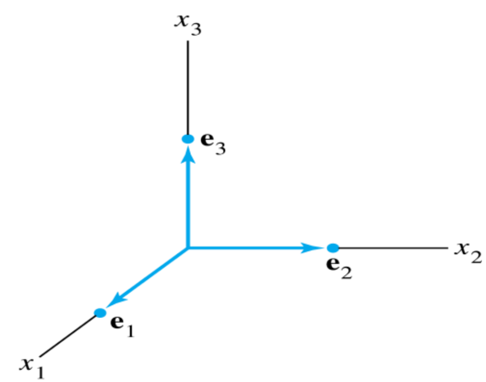
Example (2): Dimension of the vector space V_n *(R)* is *n*, since there are n number of vectors in a basis of V_n

Standard Basis

The set $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is called the **standard basis** for R^n . See the following figure.

Where

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}, \boldsymbol{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}, \ldots\ldots \boldsymbol{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ 1 \end{bmatrix}$$



The standard basis for $\mathbb{R}^3$.

As particular case, the standard basis of V (F) is {(1,0),(0,1)} and that of V (F) is {(1,0,0),(0,1,0),(0,0,1)

Example 1

Find the coordinate vector of W={-12,20} relative to the basis V_1 ={(-1,2)} and V_2 = {(4,-6)}

Solution. We have to find out α_1 and α_2 so that

$$\boldsymbol{W} = \alpha_1 \boldsymbol{V}_1 + \alpha_2 \boldsymbol{V}_2$$

$$(-12,20) = \alpha_1(-1,2) + \alpha_2(4,-6)$$
$$(-12,20) = (-\alpha_1 + 4\alpha_2, 2\alpha_1 - 6\alpha_2)$$

$$-\alpha_1 + 4\alpha_2 = -12$$
$$2\alpha_1 - 6\alpha_2 = 20$$

Solving these equations, we get α_1 =4 and α_1 =-2. Therefore W is the linear combination of V_1 and V_2 and we can write

$$\boldsymbol{W} = 4V_1 - 2V_2$$

Hence the coordinate vector of (-12,20) is (4,-2)

Example 2

Find the coordinate vector of w relative to the basis

$\boldsymbol{S} = \{(1,1,1),(1,1,0),(1,0,0)\}$ of V (R) when W=(4,-3,2)

Sol. Now let $\boldsymbol{W} = \alpha_1 \boldsymbol{V}_1 + \alpha_2 \boldsymbol{V}_2 + \alpha_3 \boldsymbol{V}_3$

$$(4,-3,2) = \alpha_1 (1,1,1) + \alpha_2 (1,1,0) + \alpha_3 (1,0,0)$$

$$(4,-3,2) = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1)$$

$$\begin{aligned} &\alpha_1 + \alpha_2 + \alpha_3 = 4 \\ &\alpha_1 + \alpha_2 = -3 \\ &\alpha_1 = 2 \end{aligned}$$

Solving these equations, we get α_1 =2 and α_2 =-5 and α_3 = 7.
Therefore W is the linear combination of V_1, V_2 and V_3 and we can write

$$\boldsymbol{W} = 2\boldsymbol{V}_1 - 5\boldsymbol{V}_2 + 7\boldsymbol{V}_3$$

Hence the coordinate vector of (4,-3,2) is (2,-5,7)

Example 3

Show that the vectors (1,0,0),(1,1,0) and (1,1,1) form a basis for R^3

Sol. Let $\boldsymbol{u_1}$ = (1,0,0), $\boldsymbol{u_2}$ = (1,1,0), $\boldsymbol{u_3}$ = (1,1,1). Let

$\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3 \in \boldsymbol{R}^3$ ***and*** $\alpha_1, \alpha_2, \alpha_3 \in \boldsymbol{R}$ Such that

$$\alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 + \alpha_3 \boldsymbol{u}_3 = 0 \qquad (1)$$

$$\alpha_1(1,0,0) + \alpha_2(1,1,0) + \alpha_3(1,1,1) = 0$$

$$(\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3) = (0,0,0)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \qquad (2)$$

$$\alpha_2 + \alpha_3 = 0 \qquad (3)$$

$$\alpha_3 = 0 \qquad (4)$$

Example 3…

Solving these equations, $\alpha_1 = \alpha_2 = \alpha_3 = 0$ the non-zero values of $\alpha_1, \alpha_2, \alpha_3$ do not exist which can satisfy

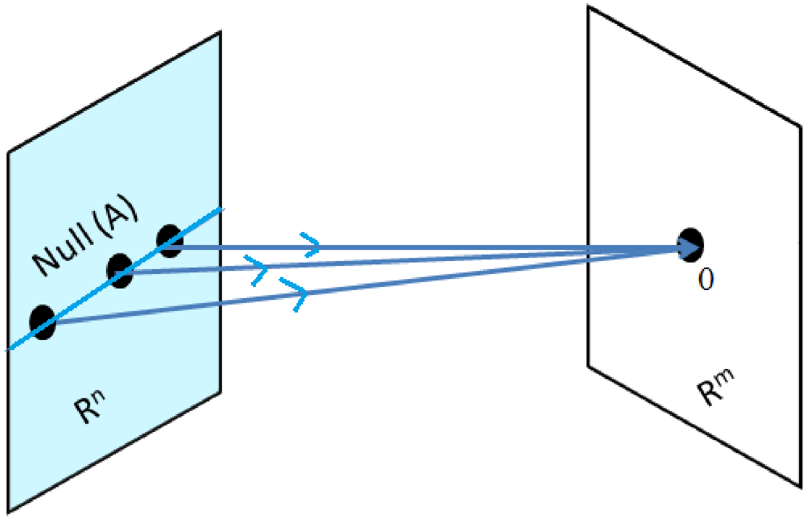
Thus $\boldsymbol{u}_1$ = (1,0,0), $\boldsymbol{u}_2$ = (1,1,0), $\boldsymbol{u}_3$ = (1,1,1) are linearly independent.

Also the dimension of vector space is R^3. Hence any set of three linearly independent vectors is form a basis for R^3

Null Space and Nullity of a Matrix

Definition: The null space of an $m \times n$ matrix *A*, written as Nul (*A*), is the set of all solutions of the homogeneous equation $A\mathrm{x} = 0$
In set notation,

$$Nul(A) = \{x : x \in R^n \ \ and \ \ Ax = 0\}$$



Nullity: The dimension of the null space of the matrix *A* is called the nullity of *A* and is denoted as nullity (*A*) or the number of free variables in the solution *A***x**=*0*

Example 4

Consider the following system of homogeneous equations

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0 \end{aligned} \qquad (1)$$

In the matrix form, the system is written as A**x**=0

$$\boldsymbol{A} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$$

The set of all ***x*** that satisfy equation(1) is called the solution set of the system (1). We call this set of x that satisfy *A**x**=0* is the null space of the matrix *A.*

Let

$$u = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Then $\boldsymbol{Au} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus u is in Nul (A)

Example 5

Find a spanning set for null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Step I find the general solution of Ax=0 by reducing A into reduced row echelon form

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By back substitution $\begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ x_3 + 2x_4 - 2x_5 &= 0 \end{aligned}$

$$\begin{aligned} x_1 &= 2x_2 + x_4 - 3x_5 \\ x_3 &= -2x_4 + 2x_5 \end{aligned}$$

Example 5…

The free variables are x_2 , x_4 , x_5,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$= x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$

Every linear combination of ***u,v*** and ***w*** is an element of Nul (A). The dimension of null space of the matrix A is 3. Thus the nullity of the matrix A =3

Row space

Definition: if $r_1, r_2, r_3.....r_n$ are the rows of the matrix *A* then subspace of R^n that is spanned by the row vectors of *A* is called row space. It is denoted as row(*A*)

$$Row\ space\ of\ \ A = \{r_1, r_2, r_3.......r_n\}$$

THEOREM: if two matrices are row equivalent, then their row space are the same. If *B* is in echelon form, the non-zero rows of *B* form a basis for the row space of *A* as well as for that of *B.*

Example 6

Let

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$\begin{aligned} r_1 &= (-2,-5,8,0,-17) \\ r_2 &= (1,3,-5,1,5) \\ r_3 &= (3,11,-19,7,1) \\ r_4 &= (1,7,-13,5,-3) \end{aligned}$$

The row space of *A* is the subspace of *R* spanned by $\{r_1, r_2, r_3, r_4\}$. That is

$$Row(A) = span\,\{r_1, r_2, r_3, r_4\}.$$

Column space

Definition: if $c_1, c_2, c_3.....c_m$ are the columns of the matrix *A*, then subspace of R^n that is spanned by the column vectors of *A* is called Column space . It is denoted as *col*(*A*)

$$Column\ space\ of\ \ A = \{c_1, c_2, c_3.........c_m\}$$

Example 7

Find the bases for the row space, the column space and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Matrix *A* reduce to row echelon form

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three rows of *B* form a basis for the row space of the matrix *A*. Thus, basis for row space is

$$B_R = \{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}$$

Example 7…

For the column space, it is observe that the pivots are in columns 1, 2 and 4 only. Hence Columns 1,2, and 4 of A (not B) form a basis for Col(*A*)

Basis for Col(*A*)= $\left\{\begin{bmatrix}-2\\1\\3\\1\end{bmatrix},\begin{bmatrix}-5\\3\\11\\7\end{bmatrix},\begin{bmatrix}0\\1\\7\\5\end{bmatrix}\right\}$

The reduced row echelon form of matrix *A*

$$A \sim B \sim C = \begin{bmatrix}1 & 0 & 1 & 0 & 1\\0 & 1 & -2 & 0 & 3\\0 & 0 & 0 & 1 & -5\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

The equation $A\boldsymbol{x}=0$ equivalent to equation $C\boldsymbol{x}=0$

Example 7

$$\begin{aligned} x_1 + \quad x_3 \quad + x_5 &= 0 \\ x_2 - 2x_3 \quad + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

So

$$\begin{aligned} x_1 &= -x_3 - x_5 \\ \Rightarrow x_2 &= 2x_3 - 3x_5 \\ x_4 &= 5x_5 \end{aligned}$$

x_3 and x_5 are free variables, then the solution of the Ax=0 can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

Basis for Nul(A)= $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

Rank and Rank nullity theorem

Rank of a Matrix

- The row rank of matrix is equal to the dimension of the row space of the matrix
- The column rank of matrix is equal to the dimension of the column space of the matrix

Rank nullity theorem

Consider a matrix *A* then

Rank(A)+null(A)=number of columns of *A*

Example 8

Determine the null space, row space and column space of the matrix

$$A=\begin{bmatrix} 1 & 3 & 4 & -2 & -1 \\ 3 & 7 & 6 & 2 & 1 \\ 2 & 4 & 2 & 4 & 2 \\ 1 & 1 & -2 & 6 & 3 \end{bmatrix}$$

Sol. We solve the following system of equations to find the null space

$$\begin{aligned} x_1+3x_2+4x_3-2x_4-x_5&=0 \\ 3x_1+7x_2+6x_3+2x_4+x_5&=0 \\ 2x_1+4x_2+2x_3+4x_4+2x_5&=0 \\ x_1+x_2-2x_3+6x_4+3x_5&=0 \end{aligned}$$

We solve this by using Gauss elimination method to reduce in echelon form

Example 8...

The coefficient matrix

$$A=\begin{bmatrix}1 & 3 & 4 & -2 & -1\\ 3 & 7 & 6 & 2 & 1\\ 2 & 4 & 2 & 4 & 2\\ 1 & 1 & -2 & 6 & 3\end{bmatrix}$$

$$\left(R_2 \to R_2 - 3R_1\right),\left(R_3 \to R_3 - (R_2 - R_1)\right)$$
$$\left(R_4 \to R_4 - (R_2 - 2R_1)\right)$$

$$\begin{bmatrix}1 & 3 & 4 & -2 & -1\\ 0 & -2 & -6 & 8 & 4\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$x_1 + 3x_2 + 4x_3 - 2x_4 - x_5 = 0$$
$$-2x_2 - 6x_3 + 8x_4 + 4x_5 = 0$$

The rank (A)=2, then

Number of free variables=5-2 = 3

Let $x_3 = k_1$, $x_4 = k_2$, $x_5 = k_3$

$$x_1 = 5k - 10k_2 - 5k_3$$

$$x_2 = -3k_1 + 4k_2 + 2k_3$$

Example 8...

The null space of A consists of the following vectors

$$X=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}5k_1-10k_2-5k_3\\-3k_1+4k_2+2k_3\\k_1\\k_2\\k_3\end{bmatrix}=k_1\begin{bmatrix}5\\-3\\1\\0\\0\end{bmatrix}+k_2\begin{bmatrix}-10\\4\\0\\1\\0\end{bmatrix}+k_3\begin{bmatrix}-5\\2\\0\\0\\1\end{bmatrix}=k_1u+k_2v+k_3w$$

The null space (A)={u,v,w} and the nullity of A=3. The dimension of the null space is 3.

Thus by rank nullity theorem

Rank(A)=Number of column of A-nullity(A)

= 5-3

=2

Example 8

The row space of A, Row(A)={r_1 ,r_2}, where

$$r_1 = \begin{bmatrix}1 & 3 & 4 & -2 & -1\end{bmatrix},\ r_2\begin{bmatrix}0 & 1 & 3 & 4 & -2\end{bmatrix}$$

The column space of A, col(A)={c_1 ,c_2}, where

$$c_1 = \begin{bmatrix}1\\3\\2\\1\end{bmatrix},\ c_2\begin{bmatrix}3\\7\\4\\1\end{bmatrix}$$

Linear Transformation

Definition: If *T: V→W* is a function from a vector space *V* into a vector space *W,* then *T* is called a ***linear transformation*** from *V* to *W* if for all vectors **u** and **v** in *V* and all scalars c

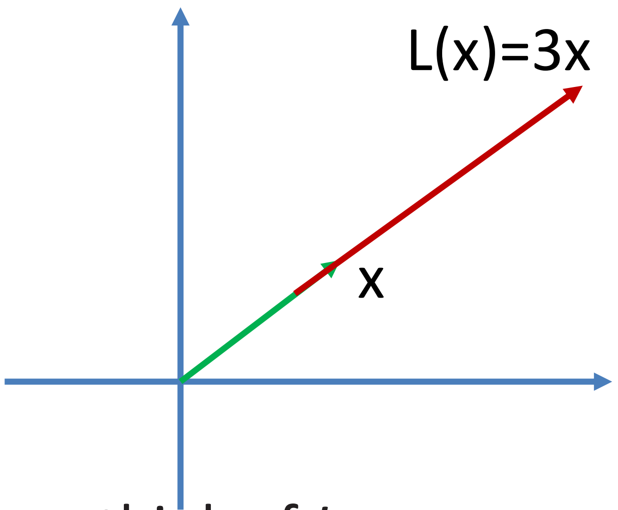
- $T\,(\boldsymbol{u}+\boldsymbol{v}) \;=\; T\,(\boldsymbol{u}) \;+\; T\,(\boldsymbol{v})$
- $T\,(c\boldsymbol{u}) \;=\; cT\,(\boldsymbol{u})$

In the special case where *V=W*, the linear transformation *T:V→V* is called a ***linear operator*** on *V*.

Example 9

Prove that $L: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ s.t $L(\boldsymbol{x}) = 3\boldsymbol{x}$ is a linear transformation.

Sol. Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{R}^2$. Since



$$\boldsymbol{L}(\alpha\boldsymbol{x}) = 3(\alpha\boldsymbol{x}) \Rightarrow \boldsymbol{L}(\alpha\boldsymbol{x}) = \alpha(3\boldsymbol{x})$$

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{x}+\boldsymbol{y}) &= 3(\boldsymbol{x}+\boldsymbol{y}) \\ &= 3\boldsymbol{x}+3\boldsymbol{y} \\ \boldsymbol{L}(\boldsymbol{x}+\boldsymbol{y}) &= \boldsymbol{L}(\boldsymbol{x})+\boldsymbol{L}(\boldsymbol{y}) \end{aligned}$$

it follows that *L* is a linear operator. We can think of *L* as a stretching by a factor of 3. In general, if α is a positive scalar, the linear operator $F(\mathbf{x}) = \alpha\mathbf{x}$ can be thought of as a stretching or shrinking by a factor of α.

Dilation and Contraction operators

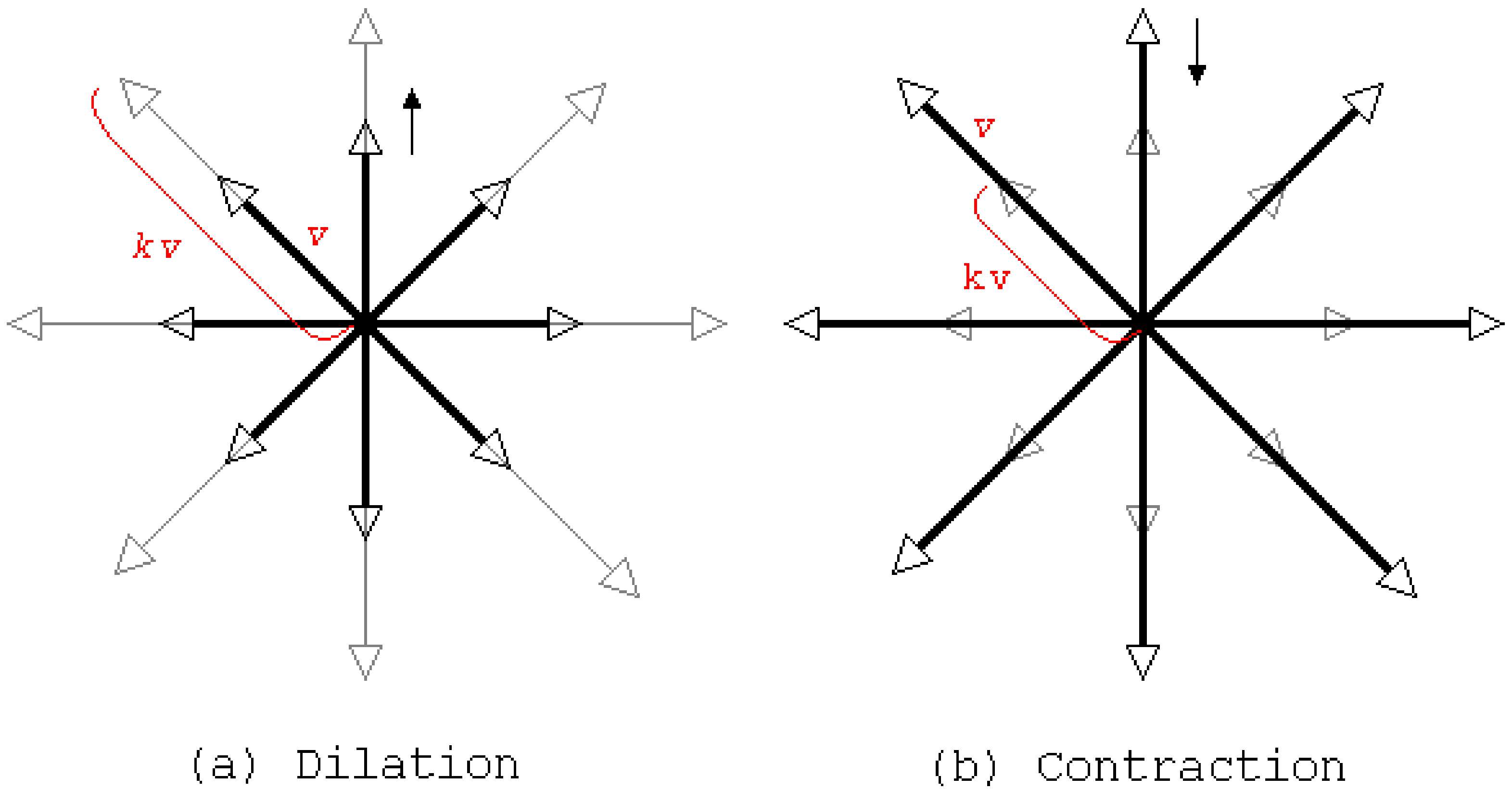
Let V be any vector space and k any fixed scalar. The function $T:V \rightarrow V$ defined by

$$T(\mathbf{v}) = k\,\mathbf{v}$$

is linear operator on V.

- **Dilation:** $k > 1$
- **Contraction:** $0 < k < 1$

Dilation and Contraction operators



Matrix representation for Linear Transformations

Two representations of the linear transformation $T:R^3 \to R^3$:

$$(1)T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2)T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Three reasons for matrix representation of a linear transformation:

- It is simpler to write.
- It is simpler to read.
- It is more easily adapted for computer use.

Standard matrix representation for a linear transformation

Theorem : Let $\boldsymbol{T} : \boldsymbol{R}^n \rightarrow \boldsymbol{R}^m$ be a linear transformation and $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ are the basis of R^n such that

$$\boldsymbol{T}(\boldsymbol{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \boldsymbol{T}(\boldsymbol{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \cdots, \quad \boldsymbol{T}(\boldsymbol{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

Then the $m \times n$ matrix whose n columns correspond to $T(e_i)$

$$\boldsymbol{A} = \left[\boldsymbol{T}(\boldsymbol{e}_1) \mid \boldsymbol{T}(\boldsymbol{e}_2) \mid \cdots \mid \boldsymbol{T}(\boldsymbol{e}_n) \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $\boldsymbol{T}(\mathbf{v}) = \boldsymbol{A}\mathbf{v}$ for every $\mathbf{v}$ in $\boldsymbol{R}^n$.

A is called the standard matrix for $\boldsymbol{T}$.

Proof

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n$$

$$\begin{aligned} T \text{ is a L.T.} \Rightarrow T(\mathbf{v}) &= T(v_1\vec{e}_1 + v_2\vec{e}_2 + \cdots + v_n\vec{e}_n) \\ &= T(v_1\vec{e}_1) + T(v_2\vec{e}_2) + \cdots + T(v_n\vec{e}_n) \\ &= v_1T(\vec{e}_1) + v_2T(\vec{e}_2) + \cdots + v_nT(\vec{e}_n) \end{aligned}$$

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

Proof...

$$
= v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}
$$

$$
= v_1 T(e_1) + v_2 T(e_2) + \cdots + v_n T(e_n)
$$

Therefore, $T(\mathbf{v}) = A\mathbf{v}$ for each $\mathbf{v}$ in R^n

Example 10

Find the standard matrix for the L.T. $\boldsymbol{T : R^3 \to R^2}$ define by

$$T(x, y, z) = (x - 2y, 2x + y)$$

Sol:

Vector Notation

$$T(e_1) = T(1, 0, 0) = (1, 2)$$

$$T(e_2) = T(0, 1, 0) = (-2, 1)$$

$$T(e_3) = T(0, 0, 1) = (0, 0)$$

Matrix Notation

$$T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T(e_3) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 10...

$$A = [T(e_1) \mid T(e_2) \mid T(e_3)]$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- Check:

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x+y \end{bmatrix}$$

$$\text{i.e. } T(x, y, z) = (x-2y, 2x+y)$$

- Note:

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{matrix} \leftarrow & 1x-2y+0z \\ \leftarrow & 2x+1y+0z \end{matrix}$$

Example 11

Let $L : R^2 \to R^3$ be the linear transformation defined by

$$L(x) = (x_2, x_1 + x_2, x_1 - x_2)^T$$

Find the matrix representations of *L* with respect to the ordered bases $\{u_1, u_2\}$ and $\{b_1, b_2, b_3\}$, where $u_1 = (1,2)^T, u_2 = (3,1)^T$ And $b_1 = (1,0,0)^T, b_2 = (1,1,0)^T, b_3 = (1,1,1)^T$

We must compute *L(***u**1*)* and *L(***u**2*)* and then transform the matrix *(***b**1*,* **b**2*,* **b**3 | *L(***u**1*),L(***u**2*))* to reduced row echelon form:

$$L(u_1) = (2,3,-1)^T \qquad L(u_2) = (1,4,2)^T$$

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -1 & -3 \\ 0 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

Example 11...

The matrix representing *L* with respect to the given ordered bases is

$$\boldsymbol{A} = \begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$$

We can verify that

$$\begin{aligned} \boldsymbol{L}(\boldsymbol{u}_1) &= -\boldsymbol{b}_1 + 4\boldsymbol{b}_2 - \boldsymbol{b}_3 \\ \boldsymbol{L}(\boldsymbol{u}_2) &= -3\boldsymbol{b}_1 + 2\boldsymbol{b}_2 + 2\boldsymbol{b}_3 \end{aligned}$$

Matlab Code

- To find the Null space in MATLAB in-built command

```
>>ns = null(A,'r')
```

Matlab Code

```
function [cs,ns,rs] = threeb(A)
    [V, pivot] = rref(A);
    r = length(pivot);
    cs = A(:,pivot);
    ns = null(A,'r');
    rs = V(1:r,:);
end
```

Session Summary

- For a set of vectors to be a vector space, it has to satisfy ten conditions
- A subset of a vector space is a subspace if it closed under vector addition and scalar multiplication
- The set $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ is a **spanning set** for *V* if and only if every vector in *V* can be written as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$
- A set of vectors is linearly independent if the linear combination of vectors is zero provided that all the scalars are zero
- A set of vectors is linearly dependent if the linear combination of vectors is zero for some non-zero scalars