Lecture 1
Functions of Real Variable, Limit of Function, Continuity and Derivatives

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Intended Learning Outcomes

At the end of the lecture, student will be able to:

- Analyze real valued functions and plot the same
- Illustrate the concepts of limit , continuity and differentiability of a real valued function

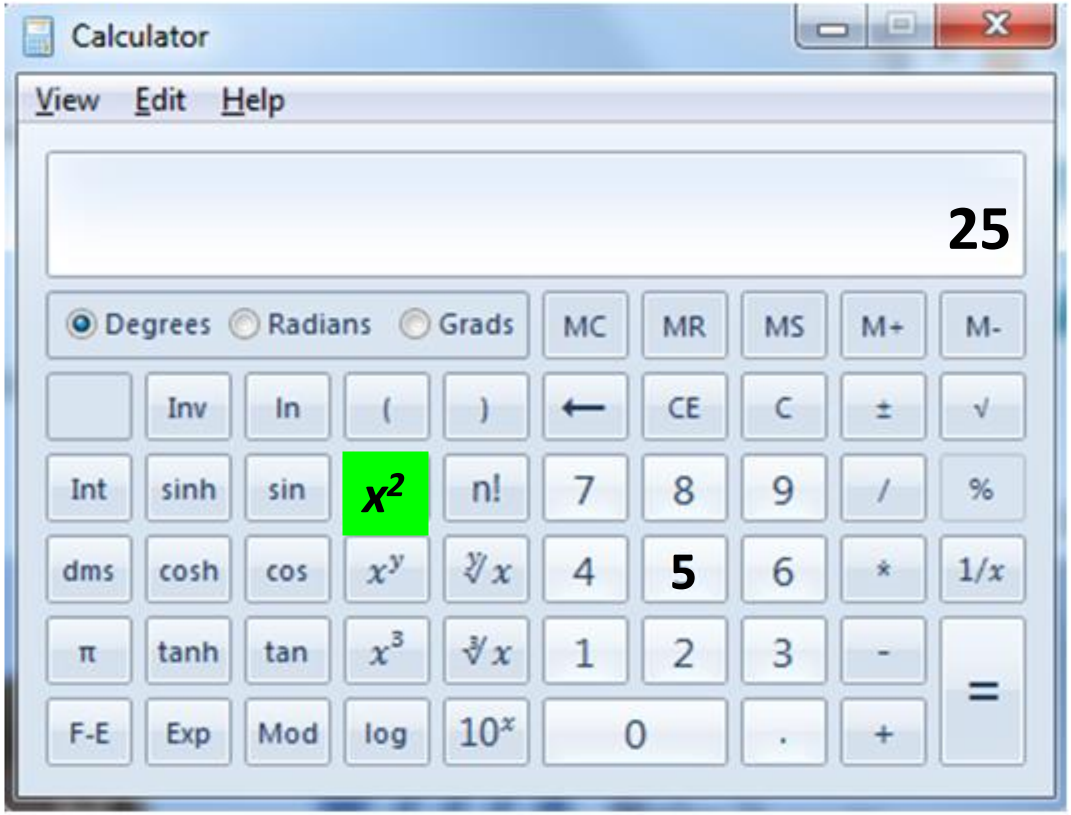
Topics

- **Function**
  - Limit of a function
  - Continuity of a function
  - Continuous and discontinuous functions
- **Types of discontinuity**
  - Discontinuity of first kind
  - Discontinuity of second kind

Function

Input

5



Output

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- The number you entered is the ***input number*** (or x-value on a graph).
- The result is the ***output number*** (or y-value on a graph).
- The $\boldsymbol{x^2}$ key illustrates the idea of a ***function***.

Function….

A *function* is a rule that gives a single output number for every valid input number.

To help remember & understand the definition:

Think of your *input number*, usually your x-coordinate, as a letter.

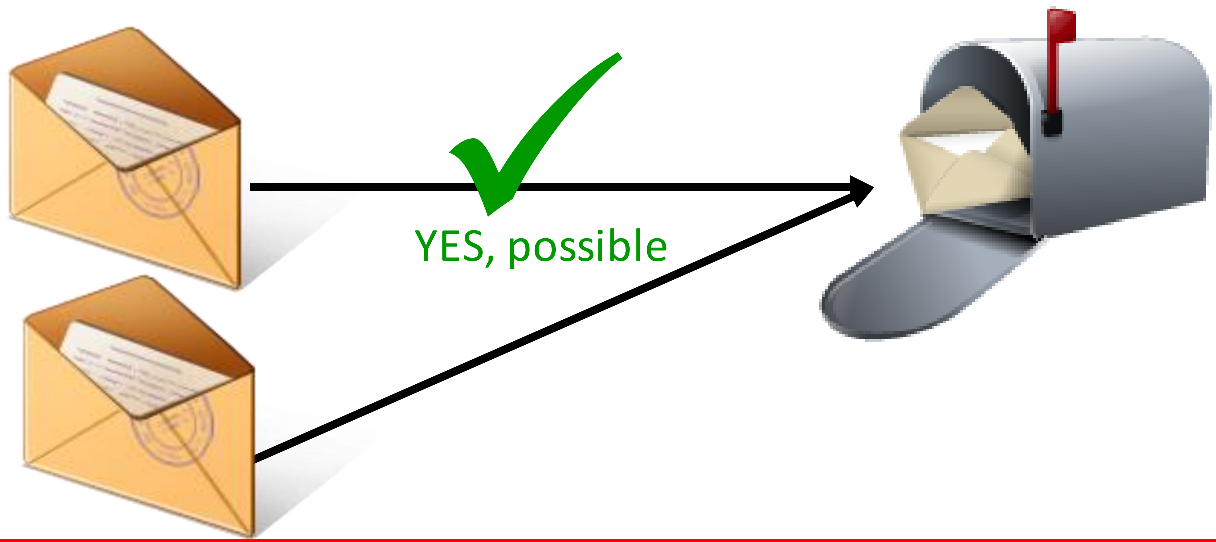
Think of your *output number*, usually your y-coordinate, as a mailbox.

Function….

Input number

Output number

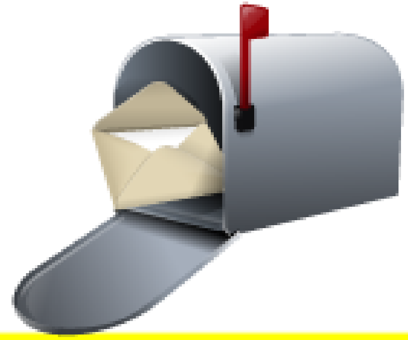
Can you have two different letters going to one mail box?



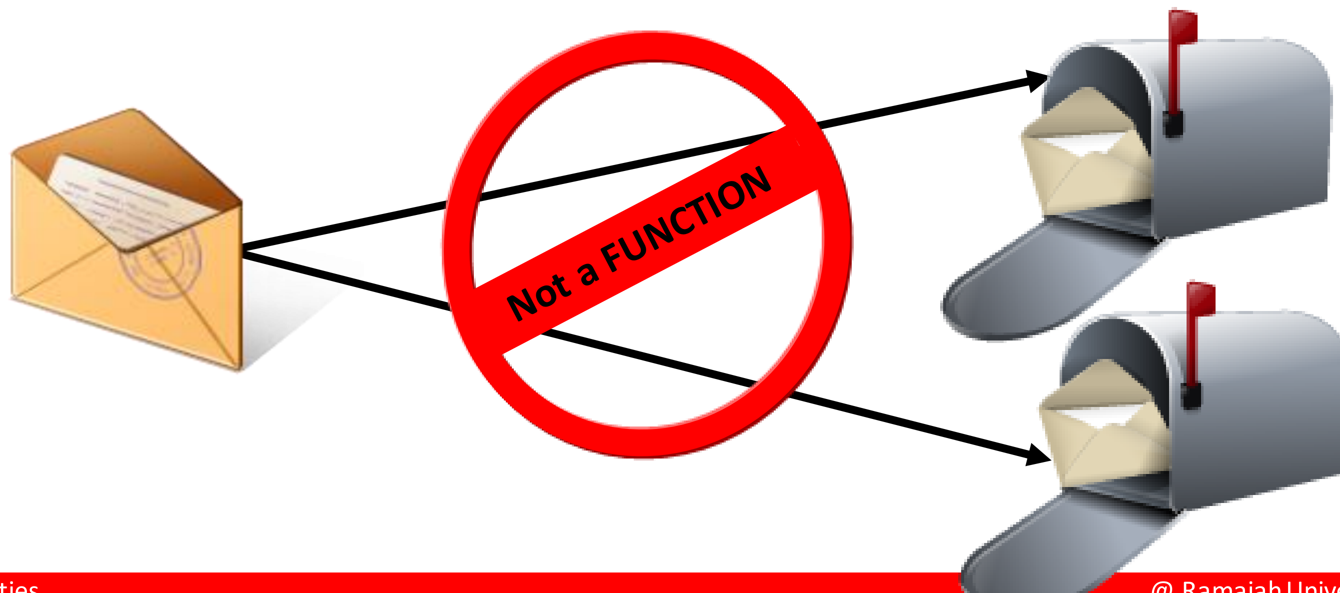
Function….

Input value

Output value



Can you have one letter going to two different mail boxes?



Graph of a function

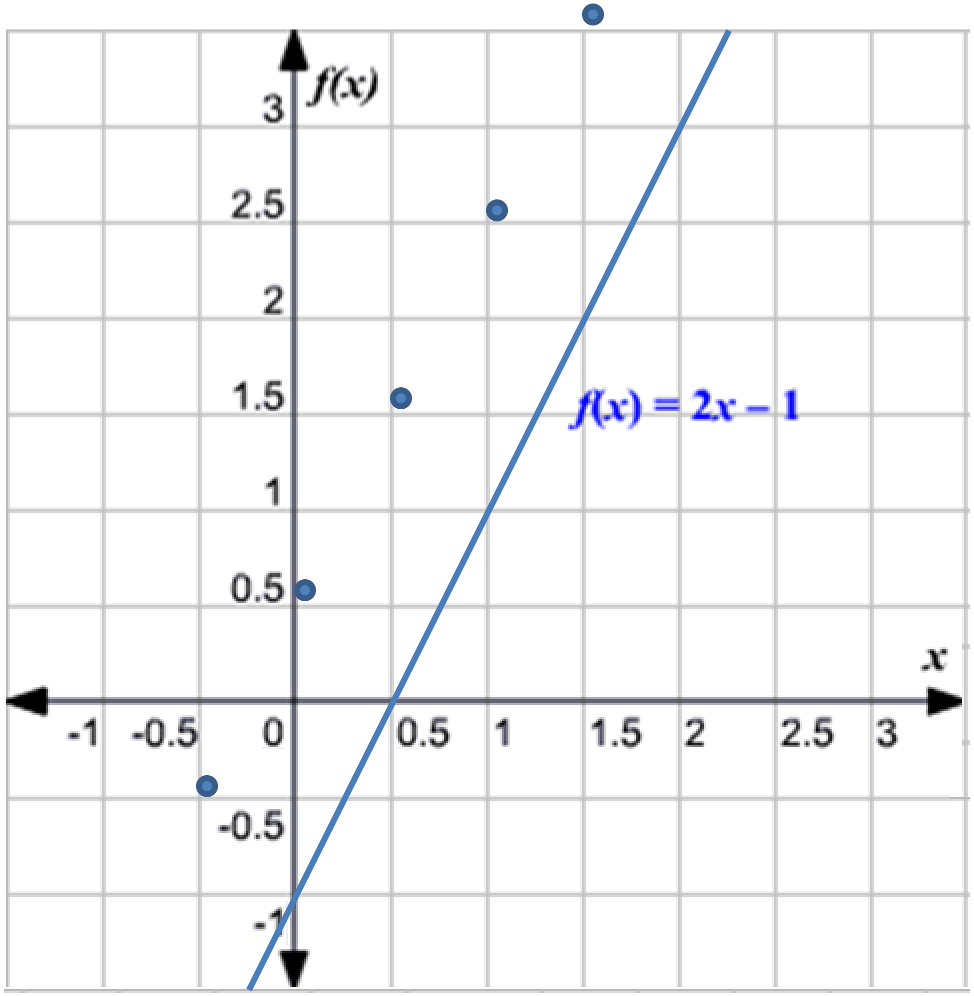
In words: Double the number and subtract 1

As an equation:

$y = 2x - 1$

As a table of values:

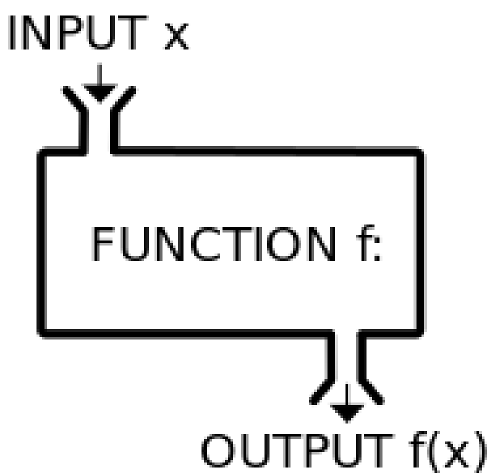
x	y
0	-1
0.5	0
1	1
1.5	2
2	3



Function

Let X and Y be two non-empty sets. If there exists a rule 'f' which associates to every element $x \in X$, a unique element $y \in Y$, then such a rule $'f'$ is known as a function(or Mapping) from the set X to the set Y.

If f is a function from X to Y, then we write $f: X \longrightarrow Y$, which is read as f is a function from X to Y



Real valued functions

Real Valued function:

A function f with domain as any set (say X) and range as set of real numbers is called a real valued function, I.e., $f: X \longrightarrow R$

Function of real variables:

If a function f has domain as set of real numbers then it is called function of real variables, i.e., $f: R \longrightarrow R$

Examples

1. $f(x) = cosx$
2. $f(x) = sinx$

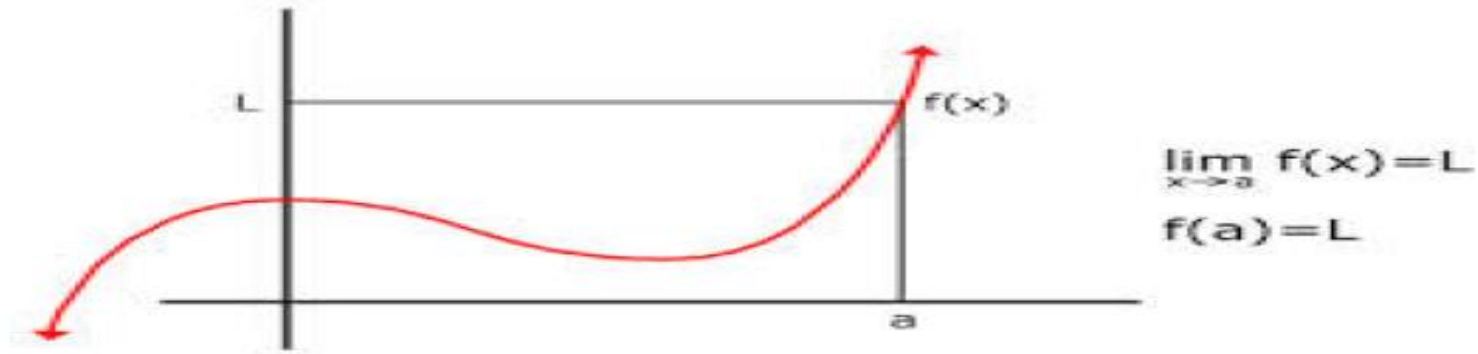
Note: In the sequel, we mainly deal with real valued functions of real variables unless specified

Limit of a function

Formal definition: The limit of $f(x)$ as x approaches a is L

i.e $\lim_{x \to a} f(x) = L$

If and only if, given $\epsilon > 0$, there exists $\delta > 0$ Such that $0 < |x - a| < \delta$ implies that $|f(x) - L| < \epsilon$



Example 1

Evaluate : $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

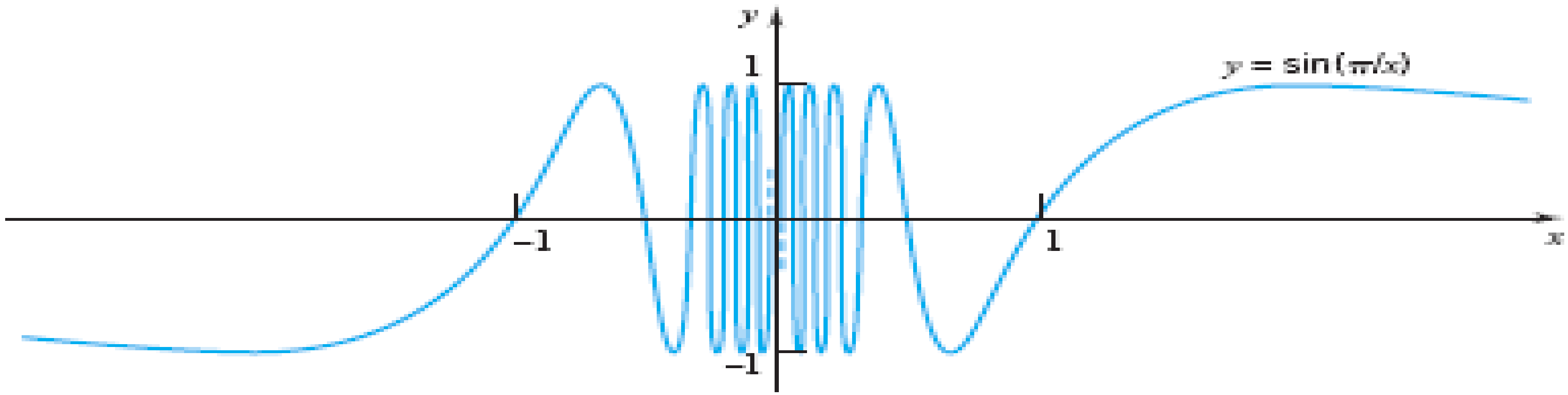
Solution:

The function $f(x) = \frac{x^3-8}{x-2}$ is undefined at $x = 2$. But, as we said before , that doesn't matter. For all $x \neq 2$,

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{(\mathrm{x} - 2)(\mathrm{x}^2 + 2\mathrm{x} + 4)}{x - 2} = 12$$

Example 2

Show that the function $f(x) = \sin(\pi/x)$ have no limit as $x \longrightarrow 0$



The function is not defined at $x = 0$, as you know, that's irrelevant. What keeps f from having a limit as $x \rightarrow 0$ is indicated in figure .

As $x \rightarrow 0, f(x)$ keeps oscillating between $y = 1$ and $y = -1$ and therefore cannot remain close to any one number L.

Left and right hand limit

Left hand limit

For the left hand limit we say that,

$$\lim_{x \to a^-} f(x) = L$$

If for every number $\epsilon > 0$, there exists $\delta > 0$

Such that $|\mathrm{f(x)} - \mathrm{L}| < \epsilon$ when ever $-\delta < |x - a| < 0$

(or $\delta < x - a < 0$).

Right hand limit

For the right-hand limit we say that

$$\lim_{x \to a^+} f(x) = L$$

If for every number $\epsilon > 0$, there exists $\delta > 0$

Such that $|\mathrm{f(x)} - \mathrm{L}| < \epsilon$ when ever $0 < |x - a| < \delta$

(or $0 < x - a < \delta$)

Limit of a function

We say that, $\lim\limits_{x\to a} f(x) = L$, if and only if $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$

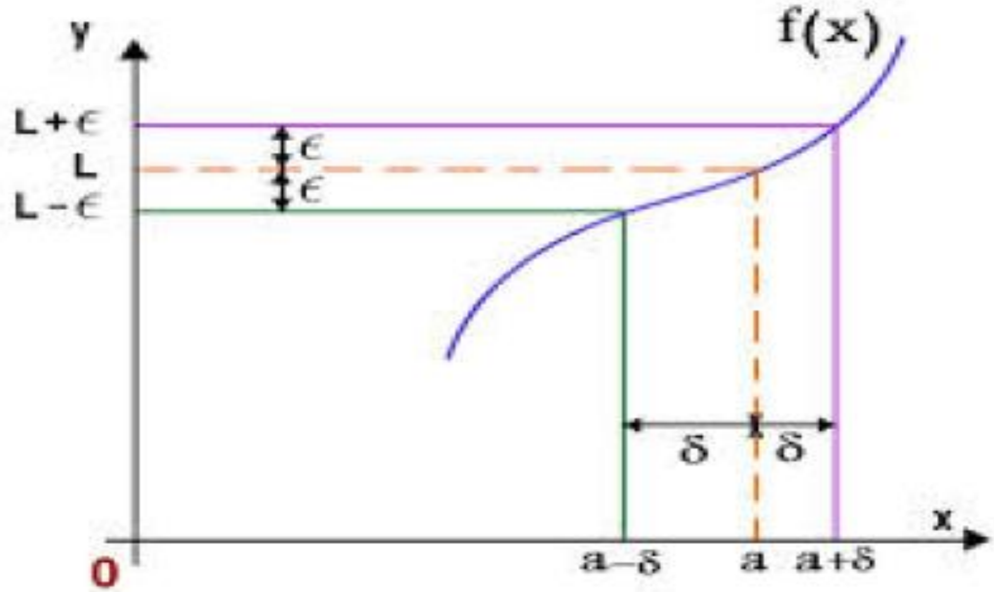


Figure: Limit of a function $f(x) = L$

Continuity of functions

Formal definition

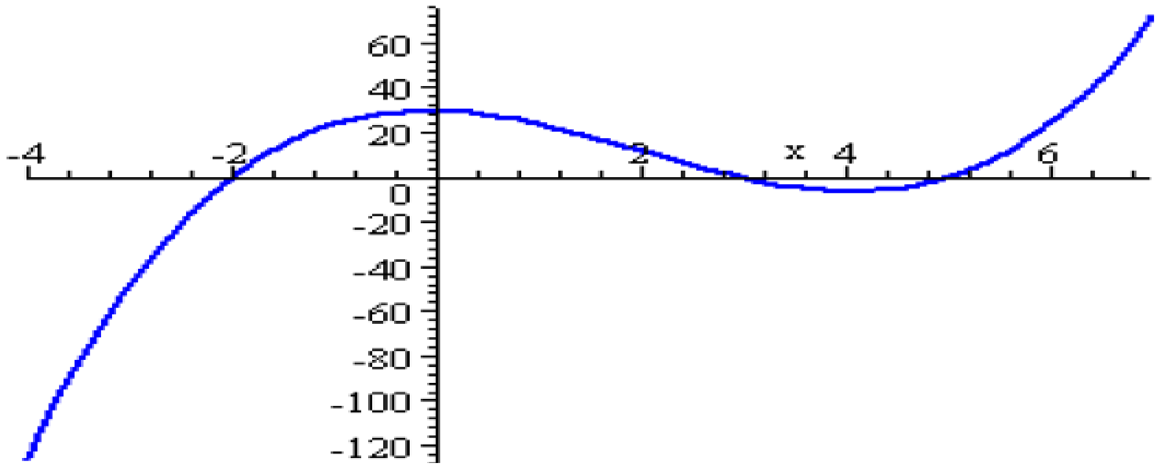
A function f is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$

If a function $f(x)$ is continuous at point $x = a$ then we must have the following conditions:

1. $\lim_{x \to a} f(x) = f(a)$
2. $\lim_{x \to a^-} f(x) = f(a)$
3. $\lim_{x \to a^+} f(x) = f(a)$

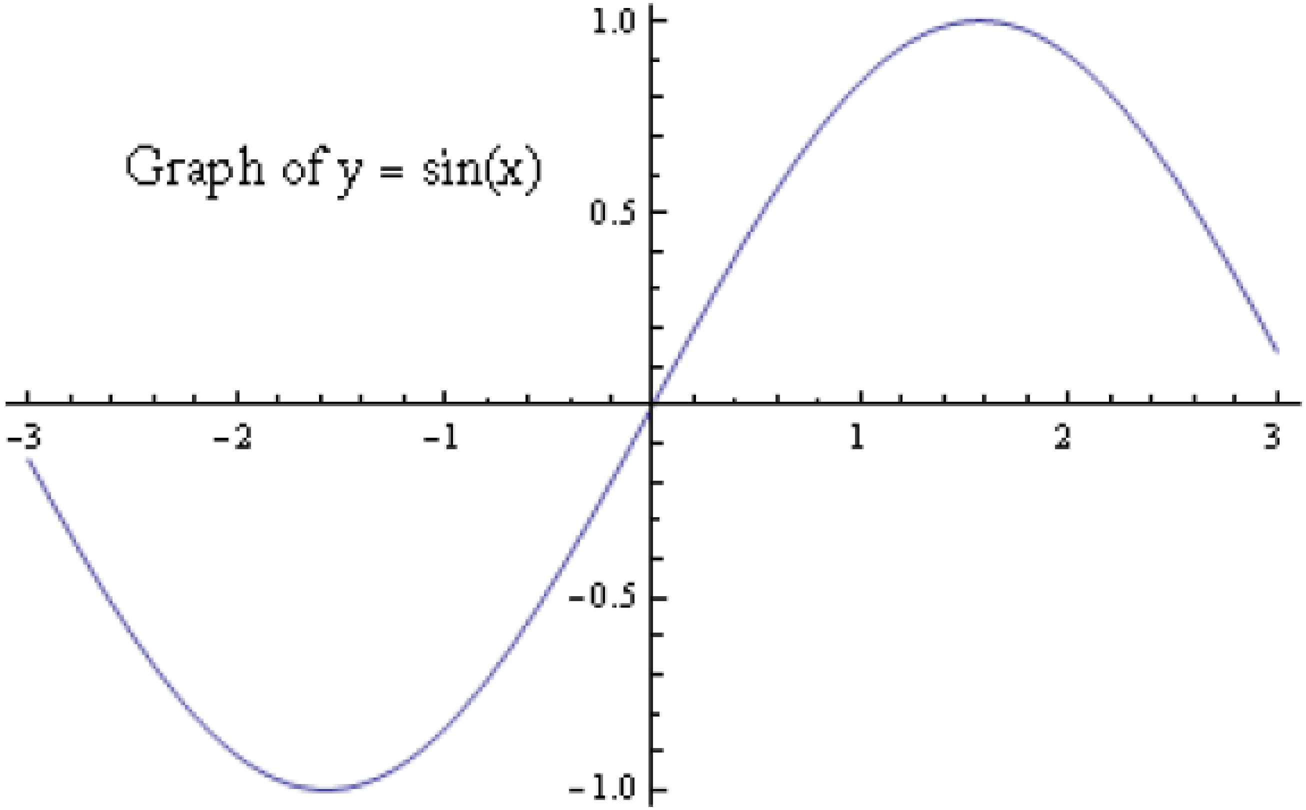
Graph of the continuous function

Consider the graph of the function $f(x) = x^3 - 6x^2 - x + 30$



We can see that there are no gaps in the curve. Any value of x will give us a corresponding value of y . We could continue the graph in the negative and positive directions

Graph of the continuous function……



Discontinuous functions

Definition:

If f is not continuous at $x = a$, then f is said to be discontinuous at this point

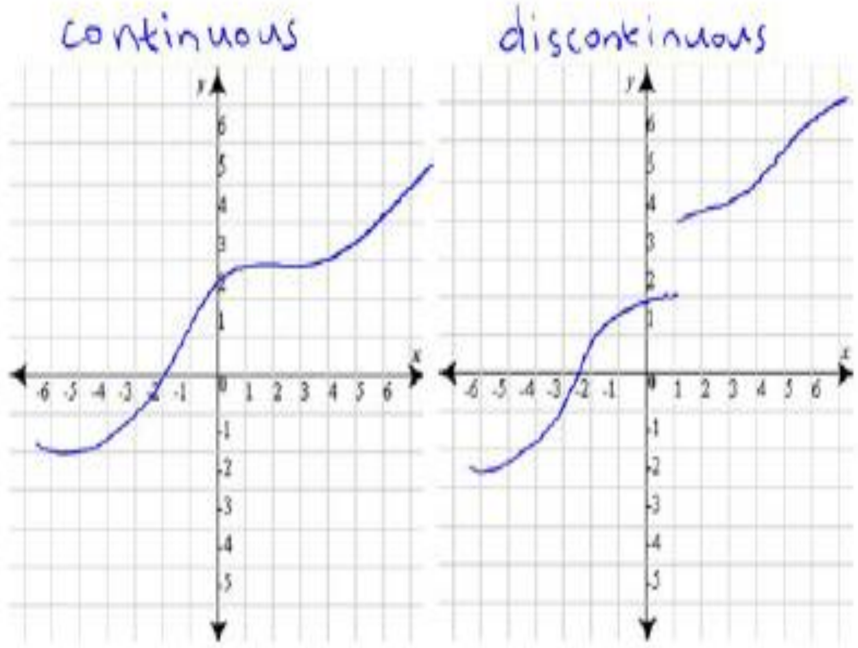


Figure: Continuous and discontinuous function

Discontinuous functions....

Determine the discontinuities, if any, of the following function:

$$f(x) = \begin{cases} 2x + 1, & x \leqq 0 \\ 1, & 0 < x \leqq 1 \\ x^2 + 1, & x > 1. \end{cases}$$

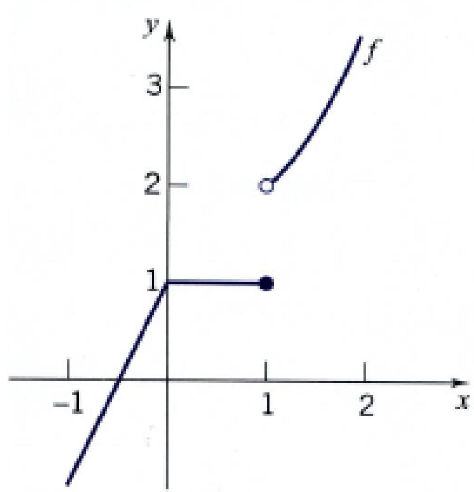
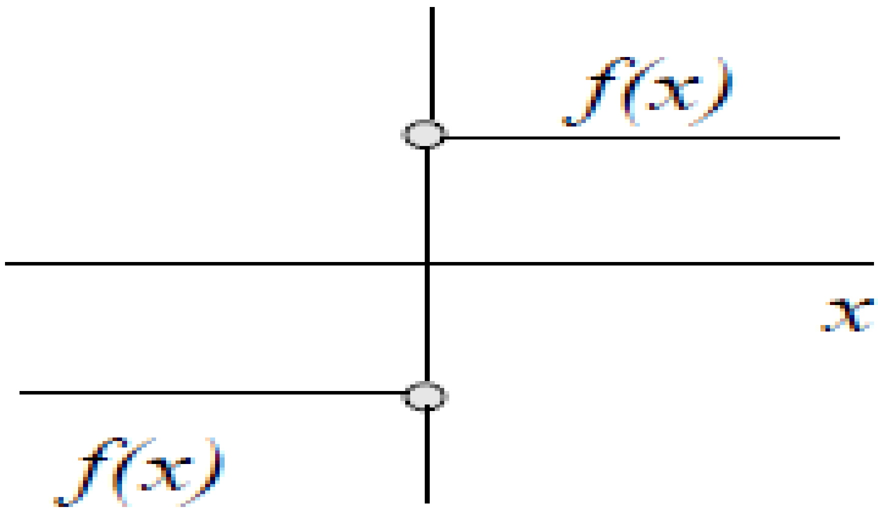


Figure 2.4.8

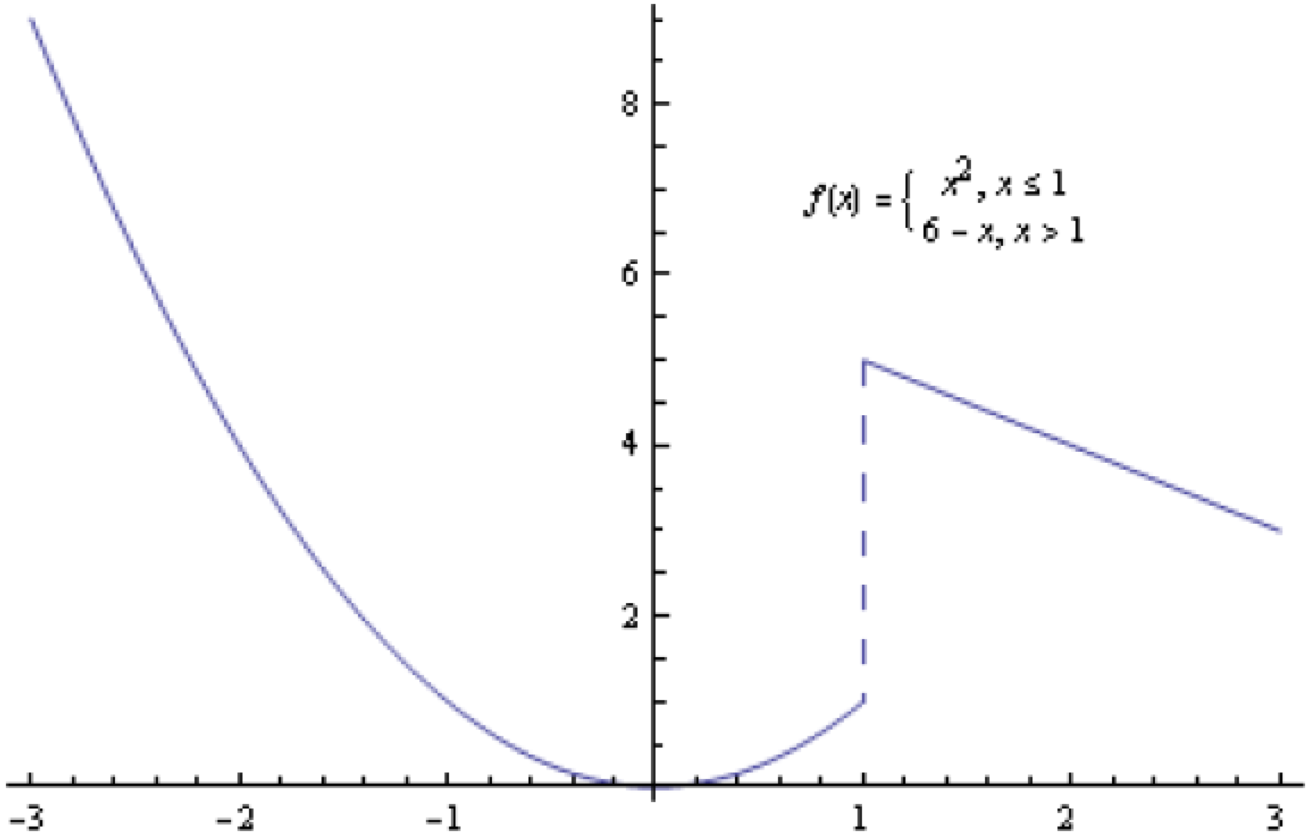
Discontinuous functions….

Example : $f(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

$$\lim_{x \to 0^-} f(x) = -1, \qquad \lim_{x \to 0^+} f(x) = 1,$$



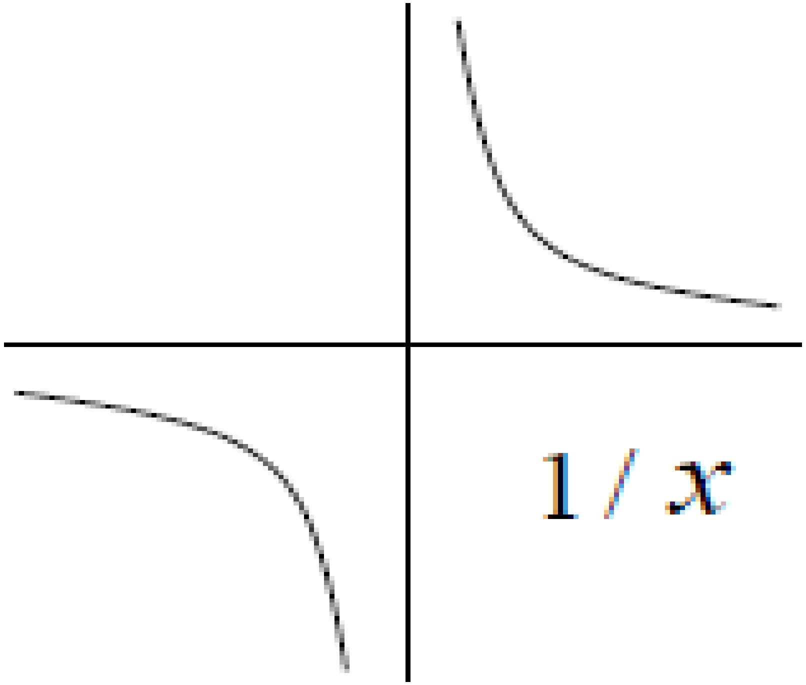
Discontinuous functions….



Discontinuous functions….

Example: $f(x) = \frac{1}{x}$

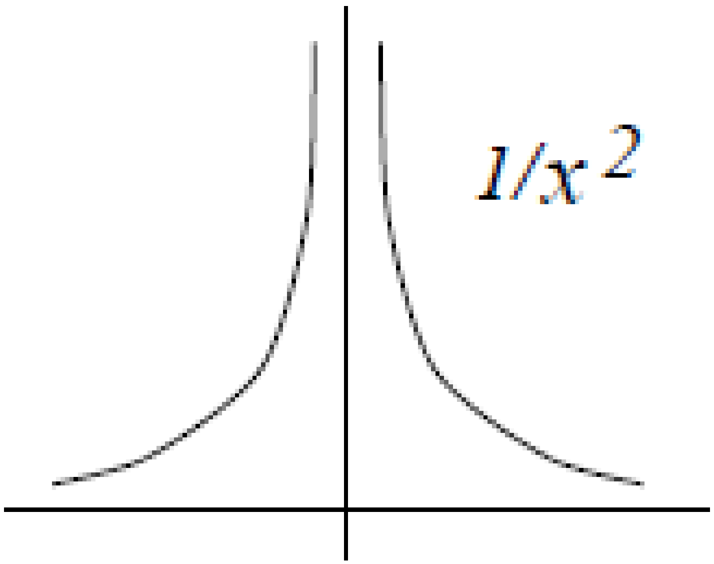
$$\lim_{x \to 0^+} f(x) = \infty, \qquad \lim_{x \to 0^-} f(x) = -\infty,$$



Discontinuous functions....

Example: $f(x) = \frac{1}{x^2}$

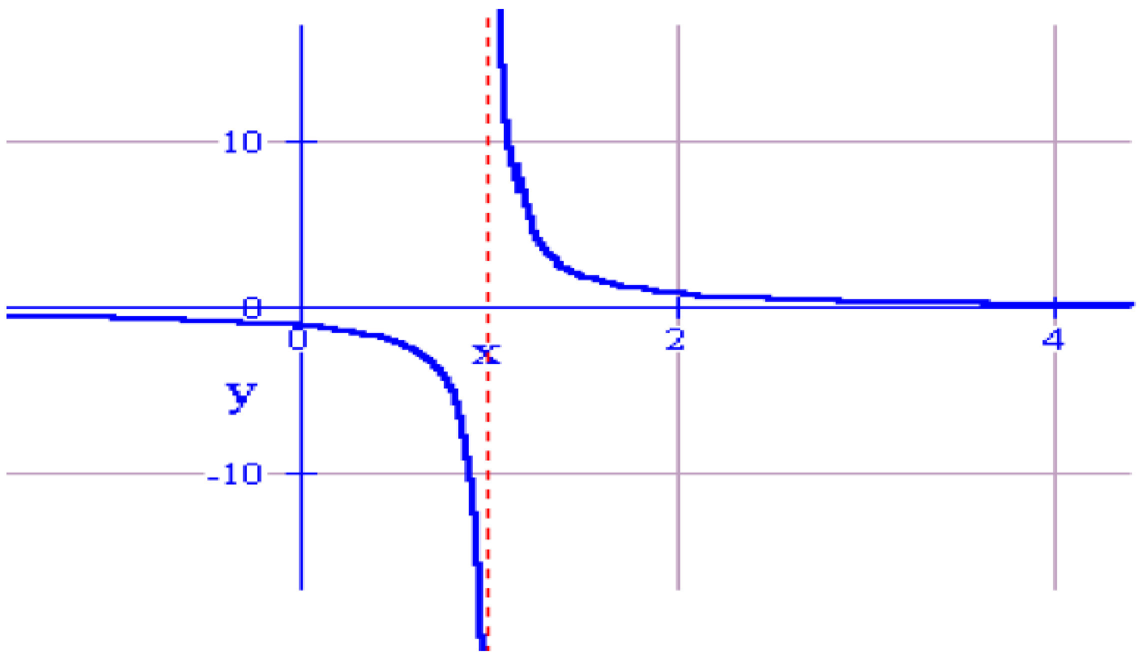
$$\lim_{x \to 0^+} f(x) = \infty, \qquad \lim_{x \to 0^-} f(x) = \infty,$$



Discontinuous functions….

Consider the function $f(x) = \frac{1}{x-1}$.

We note that the curve is not continuous at $x = 1$



Classification of discontinuity points

All discontinuity points are divided into discontinuities of the first and second kind

The function f discontinuity of the first kind at $x = a$ if

1. There exists left-hand limit $\lim_{x \to a^-} f(x) = f(a)$ and right-hand limit $\lim_{x \to a^+} f(x) = f(a)$
2. These one-sided limits are finite

Discontinuity of First Kind

Further there may be the following two options:

1. Removable discontinuity

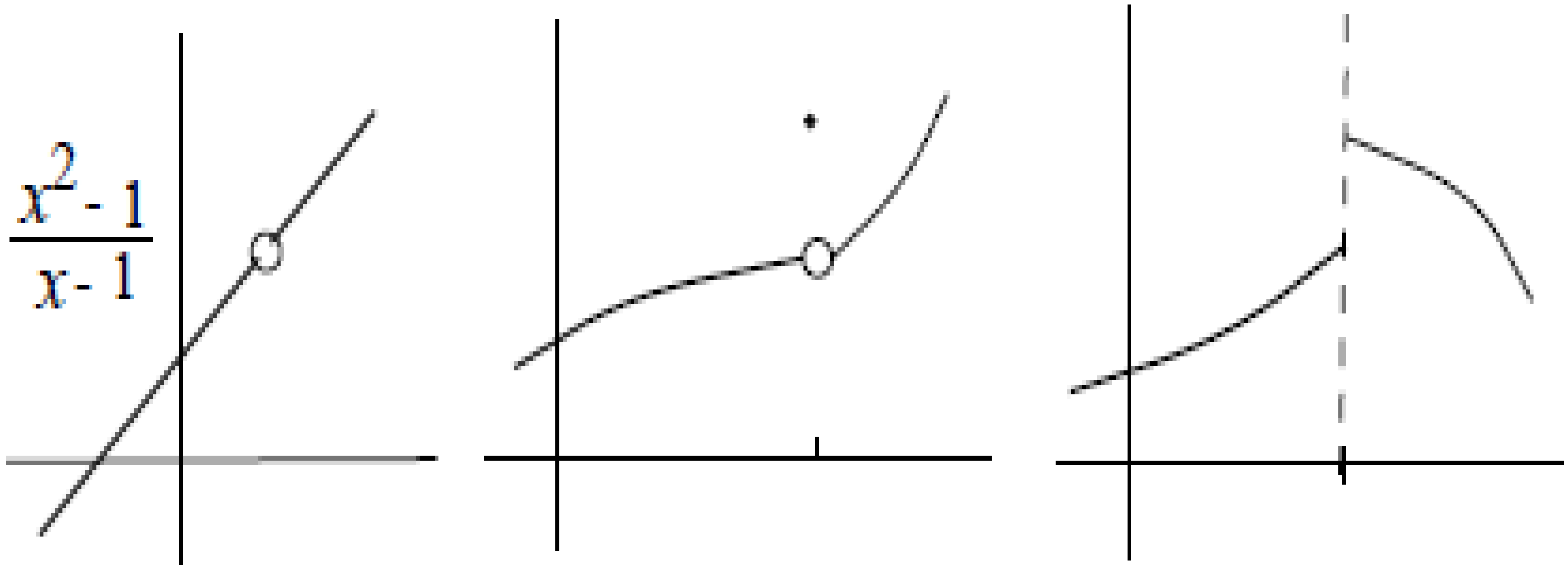
   Left-hand limit $\lim_{x \to a^-} f(x)$ and right-hand limit $\lim_{x \to a^+} f(x)$ are equal.

   But $\lim_{x \to a} f(x) = f(a)$

2. Jump discontinuity

The right-hand limit and the left-hand limit are unequal: $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$

Discontinuity of First Kind…….

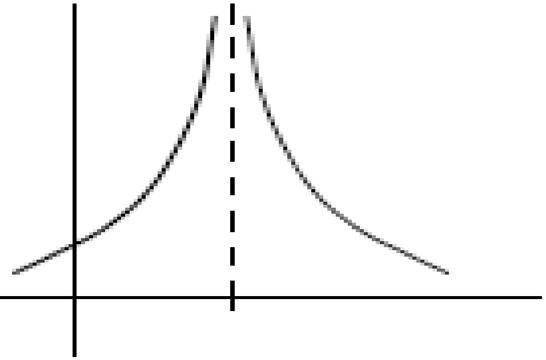


Removable discontinuity

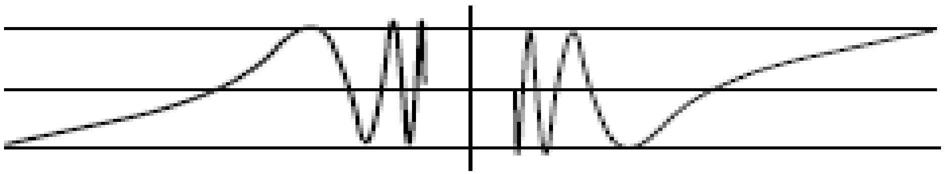
Jump discontinuity

Discontinuity of second Kind

The function f is said to have a discontinuity of the second kind (or a non removable or essential discontinuity) at $x = a,$ if at least one of the one-sided limits either does not exist or is infinite



$infinite$



$essential$

Geometric representation of derivative

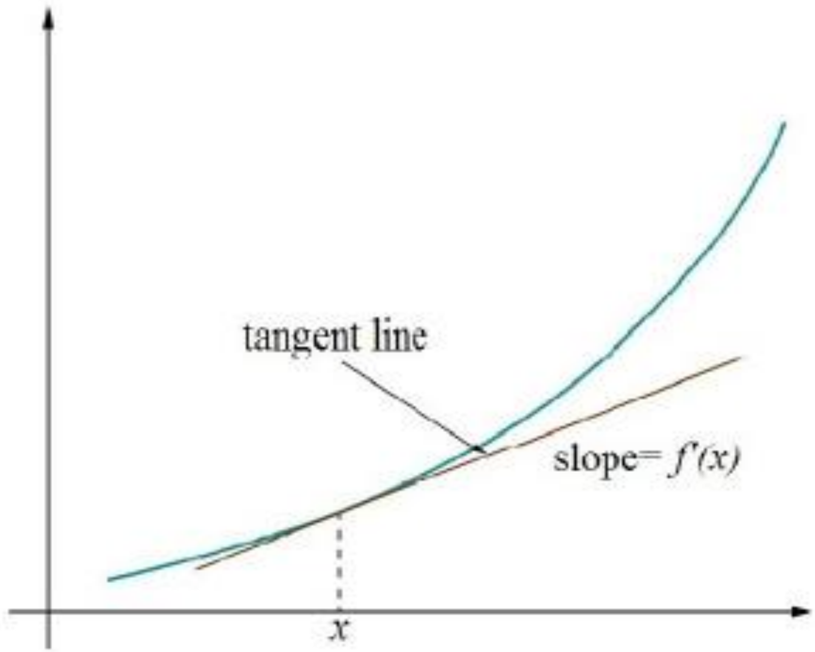


Figure: Derivative: Geometric representation

- Derivative of a function at a point is a slope of a tangent of the function at that point

Continuous function

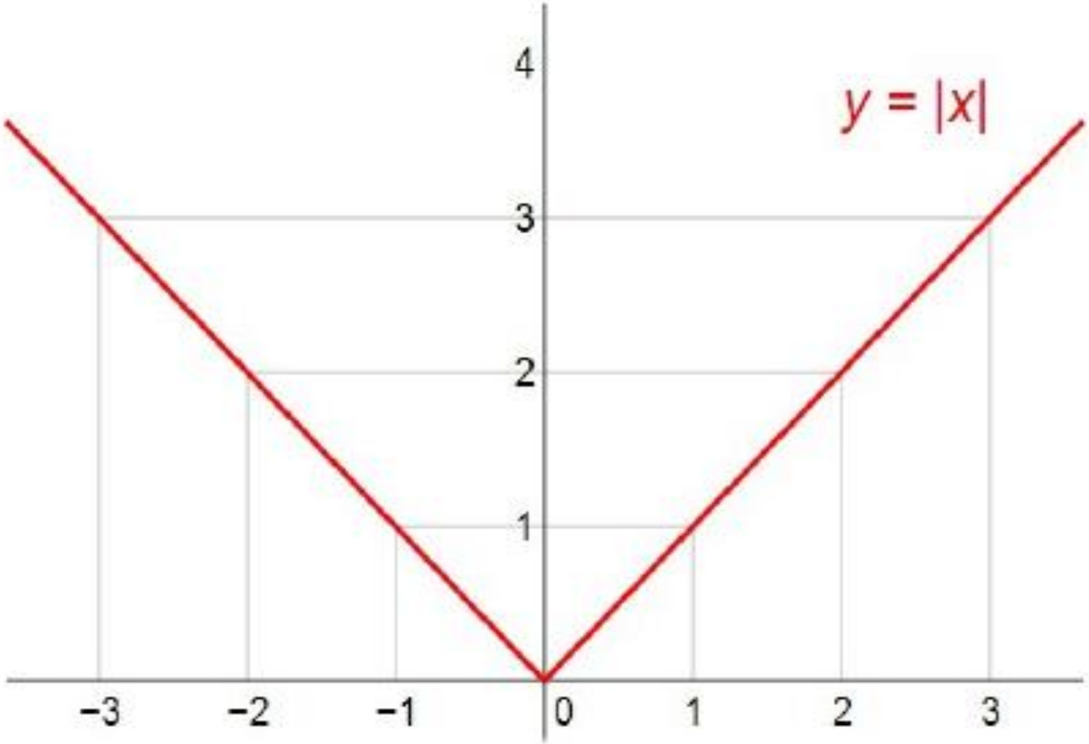


Figure: Continuous function

- Every differentiable function is continuous but the converse is not true !!!

Summary

1. The limit of a function $f(x)$ exists i.e., $\lim_{x\to a} f(x) = L$ if and only if

$$\lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L$$

2. A function f is continuous at $x = a$ if and only if $\lim_{x\to a} f(x) = \mathrm{f(a)} =$

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

3 . All continuous functions need not be differentiable Ex: $f(x) = |x|$, this function is not differentiable at $x = 0$