Lecture 4-5
Linear system

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Intended learning outcomes

At the end of this lecture, student will be able to:

- Distinguish between homogeneous and non-homogeneous linear equations
- Represent the system of linear equations in matrix form
- Solution of linear system by Gauss elimination method and Gauss-Jordon method
- Solve the system of linear equations based on existence of solution

Topics

- Linear system
- Solution of homogenous system of linear equation
- Solution of non-homogenous system of linear equation
- Consistency and existence of solution of system
- MATLAB Program

Motivation for Linear Systems

Mixture Problems: A 50% alcohol solution is to be mixed with a 10% alcohol solution to create an 8-ounce mixture of a 32% alcohol solution. How much of each is needed?

Let x represent the amount of 50% alcohol solution needed and let y represent the amount of 10% alcohol solution needed. Then, the system of linear equations is

$$x + y = 8 \qquad (1)$$

$$0.5x + 0.1y = 0.32(8) \qquad (2)$$

The solution of the given system of linear equations is

x = 4.4 and y = 3.6

Linear Equations

- Any straight line in *xy*-plane can be represented algebraically by an equation of the form:

$$a_1x + a_2y = b$$

- General form: define a linear equation in the *n* variables is:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

  – Where $a_1, a_2, \ldots, a_n,$ and *b* are real constants
  – The variables $x_1, x_2, \ldots, x_n$ in a linear equation are sometimes called unknowns

Homogeneous linear equations

Suppose

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\
a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n = 0 \\
\;.\qquad\quad .\qquad\quad \ldots\quad .\qquad\quad . \\
\;.\qquad\quad .\qquad\quad \ldots\quad .\qquad\quad . \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0
\end{array}
$$

is a system of m homogeneous linear equations in n unknowns x_i where *i=1,2.....n.* The coefficients a_{ij} are the scalars, where i=1,2....n and j=1,2....m.

Matrix representation of homogeneous linear equations

The system of linear equation can be written in matrix form as

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \qquad x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{bmatrix}_{n\times 1} \qquad 0=\begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ 0 \end{bmatrix}_{m\times 1}$$

$$A\boldsymbol{x}=\boldsymbol{0}$$

here matrix *A* is called the coefficient matrix of the system of equations, ***x*** is called the matrix of unknowns and *0* is the zero matrix

Example 1

Suppose a system of homogeneous linear equations is given as

$$x + 3y - 2z = 0$$

$$2x - y - 4z = 0$$

$$x - 11y + 14z = 0$$

We write in matrix form

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & -4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix of the system of linear equations is written as

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Non-homogenous linear equations

Suppose

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
\end{aligned}
$$

is a system of *m* non-homogeneous linear equations in *n* unknowns x_i where $i=1,2\dots n$ and a_{ij} and b_j are real numbers. At least one of the b_j are not zero.

Example

$$
\begin{aligned}
x - y + 2z &= 3 \\
x + 2y + 3z &= 5 \\
3x - 4y - 5z &= -13
\end{aligned}
$$

Matrix representation of non-homogeneous linear equations and Augmented matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m\times n}, \quad x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ x_n \end{bmatrix}_{n\times 1}, \quad b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ b_m \end{bmatrix}_{m\times 1}$$

$$A\boldsymbol{x}=\boldsymbol{b}$$

Augmented matrix

$$[A:B]=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & :b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & :b_2 \\ a_{31} & a_{32} & \dots & a_{3n} & :b_3 \\ . & . & \dots & . & :. \\ a_{m1} & a_{m2} & \dots & a_{mn} & :b_m \end{bmatrix}$$

Solution of the Linear Systems

Solve: $x+3=0 \Rightarrow x=-3$ Single variable

Solve: $\begin{aligned} x+y&=1 \\ x-y&=1 \end{aligned} \Rightarrow x=1, y=0$ Two variable

Its easy

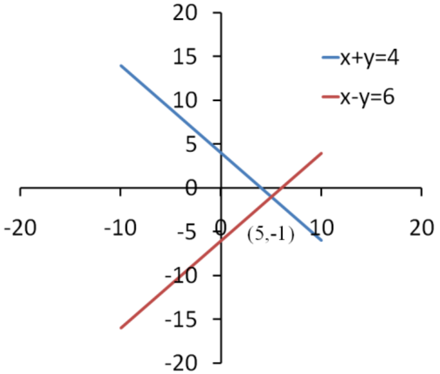
Solve:
$$\begin{aligned} x-y+2z&=3 \\ x+2y+3z&=5 \\ 3x-4y-5z&=-13 \end{aligned}$$

It is simple, but take time

Solve:
$$\begin{aligned} x-y+2z+w&=3 \\ x+2y+3z+3w&=5 \\ 3x-4y-5z+5w&=13 \\ 4x+2y+4z+w&=4 \end{aligned}$$

Feeling difficulty

Solution of the linear system….



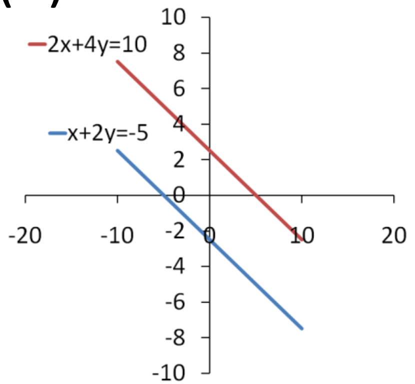
Example 2 find the solution of linear equations

$$x + y = 4 \qquad (1)$$

$$x - y = 6 \qquad (2)$$

Solution: The solution of linear equation (1) and (2)

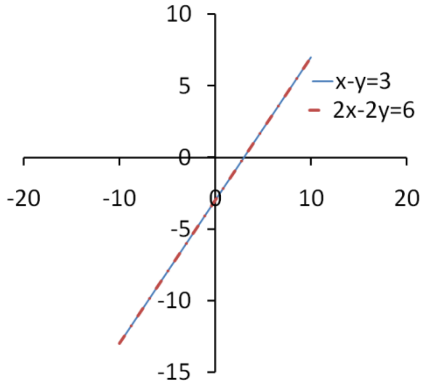
$$x = 5, y = -1$$



Example 3 find the solution of linear equations

$$2x + 4y = 10$$

$$x + 2y = -5$$



Example 4 find the solution of linear equations

$$x - y = 3$$

$$2x - 2y = 6$$

Solution of the linear equations…

Example 5 Find the solution of the linear equation

$$x - y = 3$$

Here we have many solution of this linear equation.

We can assign an arbitrary value to *x* and solve for *y* , or choose an arbitrary value for *y* and solve for *x* .If we follow the first approach and assign *x* =*t* an arbitrary value ,we obtain

$$y = t - 3$$

Where *t* is called parameter and *x* is free variable. Then the solution is x=t, y=t-3

Solution of the linear equations…

Example 6 Find the solution of the linear equation

$$x_1 - 4x_2 + 7x_3 = 5.$$

we can assign arbitrary values to any two variables and solve for the third variable.

$$x_1 = t_1, x_2 = t_2, \text{ then } x_3 = \frac{5 - t_1 + 4t_2}{7}.$$

Gauss elimination method

Example (10): solve the linear equations by Gauss elimination method

$$x - y + 2z = 3$$
$$x + 2y + 3z = 5$$
$$3x - 4y - 5z = -13$$

Step 1 In the matrix form, the equations are written in the following form

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}$$

Step 2 The augmented matrix [A:B]

$$[A:B] = \begin{bmatrix} 1 & -1 & 2 & :3 \\ 1 & 2 & 3 & :5 \\ 3 & -4 & -5 & :-13 \end{bmatrix}$$

Using Gauss elimination method to reduce in row echelon form

Gauss elimination method…

Step 3 $(R_2 \to R_2 - R_1), (R_3 \to R_3 - 3R_1)$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 3 \\ 0 & 3 & 1 & : & 2 \\ 0 & -1 & -11 & : & -22 \end{bmatrix}$$

Step 4 $\left(R_2 \to \frac{1}{3} R_2\right)(R_3 \to (-1)R_3)$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 3 \\ 0 & 1 & \frac{1}{3} & : & \frac{2}{3} \\ 0 & 1 & 11 & : & 22 \end{bmatrix}$$

Step 5 $(R_3 \to R_3 - R_2)$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 3 \\ 0 & 1 & \frac{1}{3} & : & \frac{2}{3} \\ 0 & 0 & \frac{32}{3} & : & \frac{64}{3} \end{bmatrix}$$

Gauss elimination method...

Step 6 $\left(R_3 \to \left(\frac{3}{32}\right)R_3\right)$

$$\sim\begin{bmatrix} 1 & -1 & 2 & : & 3 \\ 0 & 1 & \frac{1}{3} & : & \frac{2}{3} \\ 0 & 0 & 1 & : & 2 \end{bmatrix}$$

Step 7 This is row echelon form, using back substitution method

$$\sim\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ \frac{2}{3} \\ 2 \end{bmatrix}$$

$$x-y+2z=3 \qquad (1)$$

$$y+z/3=2/3 \qquad (2)$$

$$z=2 \qquad (3)$$

Gauss elimination method...

Putting the value of z in equation (2)

$Y=0$

Putting the value of y and z in equation (1)

$X=-1$

The solution of given system of linear equation is

$X=-1$, $y=0$, $z=2$

Gauss-Jordan Elimination method

- Solve by Gauss-Jordan Elimination

$$\begin{array}{l} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ \qquad\qquad 5x_3 + 10x_4 \qquad + 15x_6 = 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 - 18x_6 = 6 \end{array}$$

- Solution:

Step 1 The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & : & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & : & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & : & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & : & 6 \end{bmatrix}$$

Gauss-Jordan Elimination…

- **Step 2** Adding -2 times the 1st row to the 2nd and 4th rows gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & : & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & : & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & : & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & : & 6 \end{bmatrix}$$

- **Step 3** Multiplying the 2nd row by -1 and then adding -5 times the new 2nd row to the 3rd row and -4 times the new 2nd row to the 4th row gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & : & 0 \\ 0 & 0 & 1 & 2 & 0 & -3 & : & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & : & 2 \end{bmatrix}$$

Gauss-Jordan Elimination…

- **Step 4** Interchanging the 3rd and 4th rows and then multiplying the 3rd row of the resulting matrix by 1/6 gives the row-echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & : & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & : & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & : & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

- **Step 5** Adding -3 times the 3rd row to the 2nd row and then adding 2 times the 2nd row of the resulting matrix to the 1st row yields the reduced row-echelon form

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & : & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & : & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Gauss-Jordan Elimination…

- **Step 6** The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\ x_3 + 2x_4 \qquad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

- **Step 7** We assign the x_4 and x_5 free variables, let x_4 = s and x_5 = t. Then the general solution is given by the formulas:

$$x_1 = -3r - 4s - 2t,\ \ x_2 = r,\ \ x_3 = -2s,\ \ x_4 = s,\ \ x_5 = t,\ \ \ x_6 = \tfrac{1}{3}$$

Solution for homogeneous linear equations
Example 11

$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0$$

We write in matrix form

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix can be written as

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$$

Using Gauss elimination method

Example 11...

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix} \quad (R_2 \to R_2 - 3R_1), (R_3 \to R_3 - 7R_1)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} \quad (R_3 \to R_3 - 2R_2), \left(R_2 \to \left(-\frac{1}{2}\right)R_2\right)$$

This is row echelon form, thus using back substitution

$$x + 2y + 3z = 0 \qquad (1)$$

$$y + \frac{5}{2}z = 0 \qquad (2)$$

$$z = 0 \qquad (3)$$

Example 11...

Putting the value of z in equation (2)

$$y=0$$

Putting the value of y and x in equation (1)

$$x=0$$

Therefore the solution of system of homogeneous linear equations is

$$x=0, y=0, z=0$$

Example 12

$$x + 3y - 2z = 0$$

$$2x - y - 4z = 0$$

$$x - 11y + 14z = 0$$

We write in matrix form

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

Using Gauss elimination method

Example 12...

$$\sim\begin{bmatrix}1 & 3 & -2\\0 & -7 & 8\\0 & -14 & 16\end{bmatrix}\quad (R_2\to R_2-2R_1),(R_3\to R_3-R_1)$$

$$\sim\begin{bmatrix}1 & 3 & -2\\0 & -7 & 8\\0 & 0 & 0\end{bmatrix}\quad (R_3\to R_3-2R_2)$$

$$\sim\begin{bmatrix}1 & 3 & -2\\0 & 1 & -\dfrac{8}{7}\\0 & 0 & 0\end{bmatrix}\quad \left(R_2\to\left(-\frac{1}{7}\right)R_2\right)$$

This is row echelon form, now using back substitution

Example 12...

$$x + 3y - 2z = 0$$

$$y - \frac{8}{7}z = 0$$

Thus $\quad y = \frac{8}{7}z, x = -\frac{10}{7}z$

Choose z=c (arbitrary value). Then the solution of system of homogeneous linear equation is

$$x = -\frac{10}{7}c,\ y = \frac{8}{7}c,\ z = c$$

Homogeneous linear equations

Suppose
$$\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\
a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n = 0 \\
. \qquad . \qquad\qquad . \qquad . \\
. \qquad . \qquad\qquad . \qquad . \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0
\end{array}$$

is a system of *m* homogeneous linear equations in *n* unknowns $x_1, x_2, x_3, \ldots\ldots\ldots x_n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ a_{31} & a_{32} & \ldots & a_{3n} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} \qquad 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$$

Ax=0

Here matrix *A* is called the coefficient matrix of the system of equations, *x* is called matrix of unknowns and *0* is the zero matrix

Existence of solutions of homogeneous linear equations

Let A is the coefficient matrix of homogeneous linear equations of m equations and n unknowns and r is the rank of coefficient matrix A then

- if $r=n$, the equation $Ax=0$ is consistent and has unique solution $x_1 =0, x_2 =0, x_3 =0, \ldots\ldots\ldots x_n =0$ (Null or Trivial solution)
- if $r<n$, the equation $Ax=0$ will have infinite number of solutions
- $x=0$ is always a solution

Thus a homogeneous system of linear equations is always consistent and has either a trivial solution or an infinite number of solutions

Example 1

Find all the solutions of the linear homogeneous equations

$$\begin{aligned} 2x-3y+z&=0\\ x+2y-3z&=0\\ 4x-y-2z&=0 \end{aligned}$$

The matrix representation of the given homogeneous linear equations

$$A=\begin{bmatrix} 2 & -3 & 1\\ 1 & 2 & -3\\ 4 & -1 & -2 \end{bmatrix}, X=\begin{bmatrix} x\\ y\\ z \end{bmatrix}, O=\begin{bmatrix} 0\\ 0\\ 0 \end{bmatrix}$$

The coefficient matrix

$$A=\begin{bmatrix} 2 & -3 & 1\\ 1 & 2 & -3\\ 4 & -1 & -2 \end{bmatrix}$$

It can be solved by using Gauss elimination method

Example 1…

$$A=\begin{bmatrix}2 & -3 & 1\\ 1 & 2 & -3\\ 4 & -1 & -2\end{bmatrix} \xrightarrow[\text{method}]{\text{Gauss elimination}} \begin{bmatrix}1 & 2 & -3\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$$

Rank (A)=3=number of unknowns

Therefore, the given system of linear equations is consistent and has only trivial solution.

x=0,y=0,z=0

Example 2

Find all the solutions of the linear homogeneous equations

$$
\begin{aligned}
3x + 4y - z \quad - \quad 6w &= 0 \\
2x + 3y + 2z - \quad 3w &= 0 \\
2x + y \quad - 14z - 9w &= 0 \\
x \quad + 3y + 13z + 3w &= 0
\end{aligned}
$$

The given system of equations is can be written in matrix form as

$$A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient of matrix of the linear homogeneous system

$$A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$$

Using Gauss elimination method for solving this system of linear equation

Example 2...

$$A = \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix} \xrightarrow[\text{method}]{\text{Gauss elimination}} \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank (A)=2<4 (number of unknowns). Therefore, the given system of linear equations has infinitely solutions.

The number of free variables $=n\text{-}r= 4\text{-}2 = 2$

By Back substitution, the equivalence matrix *A* can be written as

$$\begin{aligned} x+3y+13z+3w &= 0 \\ y+8z+3w &= 0 \end{aligned}$$

Example 2...

Let $z=c_1$ and $w=c_2$ then

$$x = 11c_1 + 6c_2, y = -8c_1 - 3c_2$$

Then the solution of linear system of homogeneous equations is

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 11c_1 + 6c_2 \\ -8c_1 - 3c_2 \\ c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 11 \\ -8 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Non-homogeneous linear equations

Suppose

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n &= b_3 \\
. \qquad . \qquad \ldots \quad . \qquad & \quad . \\
. \qquad . \qquad \ldots \quad . \qquad & \quad . \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

is a system of *m* non-homogeneous linear equations in *n* unknowns $x_1, x_2, x_3, \ldots\ldots\ldots x_n$

$$
A = \begin{bmatrix} a_{11} & a_{12} & \ldots. & a_{1n} \\ a_{21} & a_{22} & \ldots. & a_{2n} \\ a_{31} & a_{32} & \ldots. & a_{3n} \\ \ldots & \ldots. & \ldots. & \ldots. \\ \ldots & \ldots. & \ldots. & \ldots. \\ a_{m1} & a_{m2} & \ldots. & a_{mn} \end{bmatrix}_{m\times n} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}_{n\times 1} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \ldots \\ \ldots \\ b_m \end{bmatrix}
$$

$$A\boldsymbol{x}=\boldsymbol{b}$$

Here matrix *A* is called the coefficient matrix of the system of equations, *x* is called matrix of unknowns and *b* is the column matrix

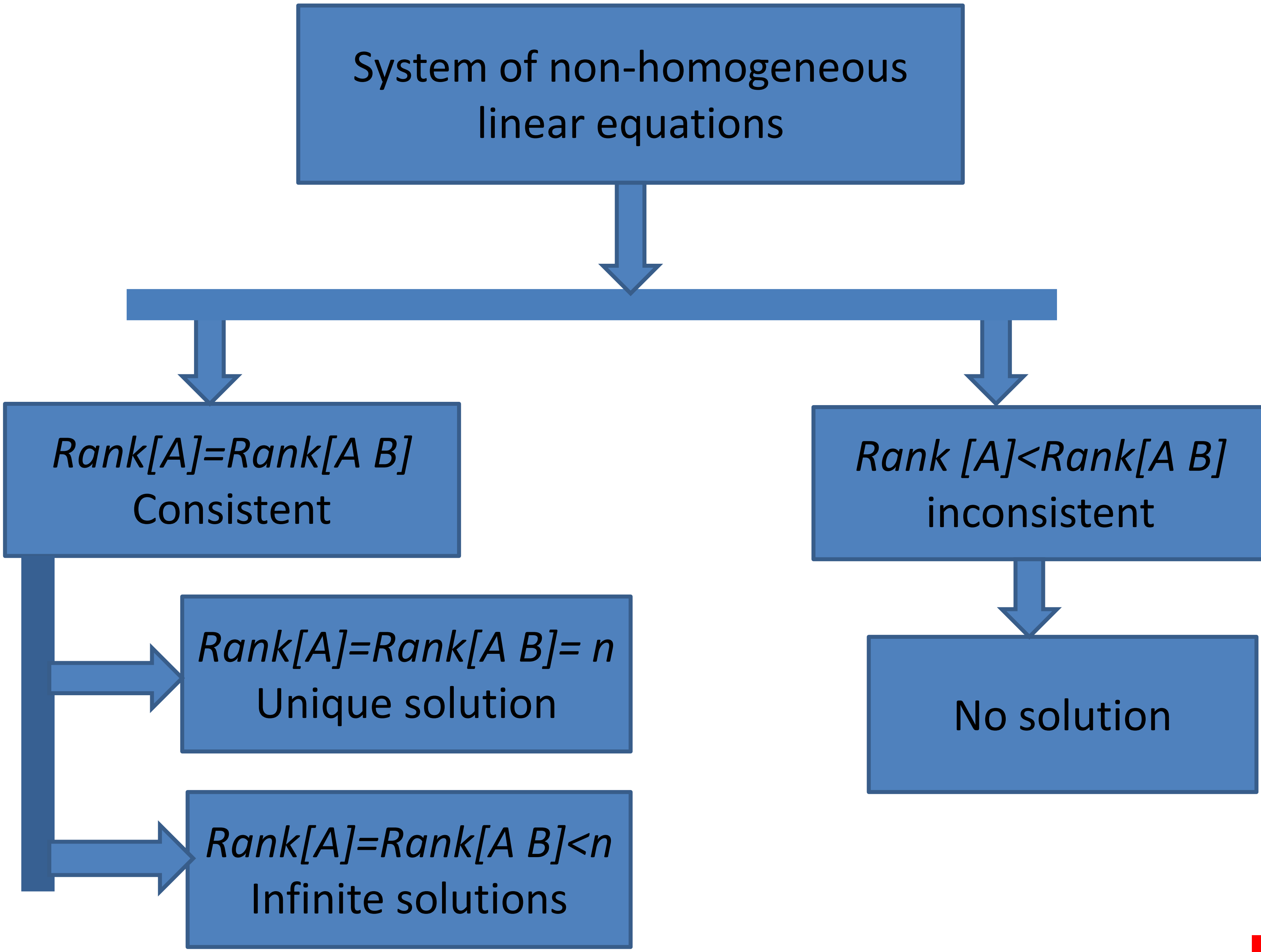
Existence of the solutions

Given the linear system $AX = B$ and the augmented matrix $[A\ B]$

- If $rank[A] = rank[A\ B] = n$ (number of unknowns), then the system of linear equations $AX=B$ is consistent and has a unique solution

- If $rank[A] = rank[A\ B] < n$ (number of unknowns), then the system of linear equations $AX=B$ is consistent and has infinite number of solutions

- If $rank[A] < rank[A\ B]$, then the system of linear equations is inconsistent and has no solution


System of non-homogeneous linear equations
Rank[A]=Rank[A B] Consistent
Rank [A]<Rank[A B] inconsistent
Rank[A]=Rank[A B]= n Unique solution
Rank[A]=Rank[A B]<n Infinite solutions
No solution

Example 3

Check the consistency of the equations

$$2x - y + 3z = 8$$
$$-x + 2y + z = 4$$
$$3x + y - 4z = 0$$

The given system of linear equation is written as in matrix form

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix}$$

The Augmented matrix

$$[A:B] = \begin{bmatrix} 2 & -1 & 3 & : & 8 \\ -1 & 2 & 1 & : & 4 \\ 3 & 1 & -4 & : & 0 \end{bmatrix}$$

Solve by using Gauss elimination method and reduce to echelon form

Example 3...

$$[A:B]=\begin{bmatrix} 2 & -1 & 3 & : & 8 \\ -1 & 2 & 1 & : & 4 \\ 3 & 1 & -4 & : & 0 \end{bmatrix} \xrightarrow{\text{Gauss elimination method}} \begin{bmatrix} 1 & -2 & -1 & : & -4 \\ 0 & 1 & \frac{5}{3} & : & \frac{16}{3} \\ 0 & 0 & 1 & : & 2 \end{bmatrix}$$

$$\text{Rank}[A] = \text{Rank}[A\ B] = 3(\text{number of unknowns})$$

Thus system of linear equation is consistent and has unique solution. Using Back substitution

$$x - 2y - z = -6 \qquad (1)$$

$$y + \frac{5}{3} z = 16/3 \qquad (2)$$

$$z = 2 \qquad (3)$$

Solving these linear equations, we get the solution of given system of linear equations $x=0$, $y=2$, $z=2$

Example 4

Check the consistency of the equations and find the solution if exist

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\
6x_1 + 7x_2 + 8x_3 + 9x_4 &= 10 \\
11x_1 + 12x_2 + 13x_3 + 14x_4 &= 15 \\
16x_1 + 17x_2 + 18x_3 + 19x_4 &= 20 \\
21x_1 + 22x_2 + 23x_3 + 24x_4 &= 25
\end{aligned}
$$

The given system of linear equation is written as in matrix form

$$
A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \\ 21 & 22 & 23 & 24 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 15 \\ 20 \\ 25 \end{bmatrix}
$$

The Augmented matrix

$$
[A:B] = \begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 6 & 7 & 8 & 9 & : & 10 \\ 11 & 12 & 13 & 14 & : & 15 \\ 16 & 17 & 18 & 19 & : & 20 \\ 21 & 22 & 23 & 24 & : & 25 \end{bmatrix}
$$

Solve by using Gauss elimination method

Example 4...

The row echelon matrix of the augmented matrix [A : B] is given as

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & 4 & : & 5 \\ 0 & 1 & 2 & 3 & : & 4 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Rank(A)=Rank[A B]=2<4(number of unknowns)

The system of linear equations is consistent and have infinite number of solution.

Example 4…

Using Back substitution

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ x_2 \;+\; 2x_3 + 3x_4 &= 4 \end{aligned}$$

Number of free variables=*n*-*r*=4-2=2

Let x_3 and x_4 are free variables and $x_3 = k_1$, $x_4 = k_2$ solving for x_1 and x_2

$$\begin{aligned} x_1 &= -3 + k_1 + 2k_2 \\ x_2 &= 4 - 2k_1 - 3k_2 \end{aligned}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 + k_1 + 2k_2 \\ 4 - 2k_1 - 3k_2 \\ k_1 \\ k_2 \end{bmatrix}$$

Example 5

Check the consistency of the equations and find the solution if exist

$$2x+6y=-11$$

$$6x+20y-6z=-3$$

$$6y-18z=-1$$

The given system of linear equation is written as in matrix form

$$A=\begin{bmatrix}2 & 6 & 0\\ 6 & 20 & -6\\ 0 & 6 & -18\end{bmatrix}\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\quad B=\begin{bmatrix}-11\\ -3\\ -1\end{bmatrix}$$

The Augmented matrix

$$[A:B]=\begin{bmatrix}2 & 6 & 0 & :-11\\ 6 & 20 & -6 & :-3\\ 0 & 6 & -18 & :-1\end{bmatrix}$$

Solve by using Gauss elimination method

Example 5...

$$[A:B]=\begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \xrightarrow[\text{method}]{\text{Gauss elimination}} \begin{bmatrix} 1 & 3 & 0 & : & -\frac{11}{2} \\ 0 & 1 & -3 & : & 15 \\ 0 & 0 & 0 & : & -91 \end{bmatrix}$$

$$\text{Rank}[A] < \text{Rank}[A\ \ B]$$

The last equation of this system is *0x+0y+0z=-91*. Therefore the given system of linear equations are inconsistent and they do not have any solutions.

Matlab Code

- To find the unique solution of linear system in MATLAB in-built command

```
>> A\B
>>inv(A)*B
```

Where A is the coefficient matrix and B is the RHS of linear system

Matlab code

```
function[] = sol_linearsys(A, B)
   n = length(A);
   M = [A B];
   rA = rank(A);
   rM = rank(M);
   rfA=rref(M);
   if rank(A)==rank(M)
      fprintf('\n The system is consistent and the Rank of A is : %d and Rank of [A B] is
%d\n',rA, rM);
   else
      fprintf('\n The system is inconsistat\n');
   end
   if (rA == n && rM==rA)
      fprintf('\n The system has unique solution\n');
      X = A\B
   elseif (rA<n && rM==rA)
      fprintf('The system is infinite solution');

   elseif (rA<rM)
      fprintf('\n The system is inconsistant\n');
   end
    display(rfA)
    end
```

Summary

- The system of linear equation can be written in two form homogenous and non-homogeneous
- The solution of linear system may exist or may not exist
- The determinant zero of coefficient matrix can not be solved by Cramer’s Rule
- The system of homogeneous linear equations can not be solved by Cramer’s Rule
- To solve a linear system we can reduce the augmented matrix to row echelon form or reduced row echelon form