Lecture 32
Line Integral -2

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Define line integral in complex plane
- Solve problems on line integral

Topics

- Line integral -2
- Examples

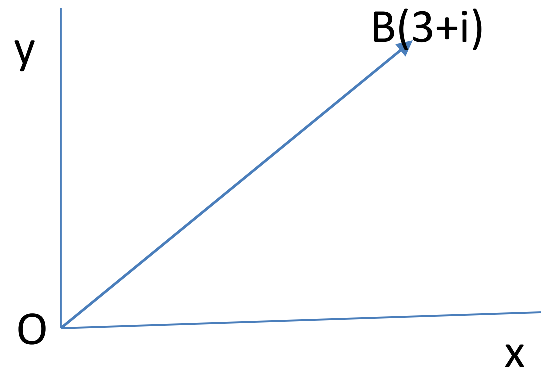
Example-1

Evaluate $\int_c z^2 dz$

(a)Along the straight line from $z = 0$ to $z = 3 + i$

(b)Along the curve made up of two line segments, one from $z = 0 \; to \; z = 3$

And another from $z = 3 \; to \; z = 3 + i$



(a) $\int_c z^2 dz = \int_{z=0}^{3+i} z^2 dz$

Here z varies from 0 to $3 + i$ means that (x, y) varies from (0,0) to (3,1). The equation of the line joining (0,0) and (3,1) is given by

$\frac{y-0}{x-0} = \frac{1-0}{3-0}$ or $y = \frac{x}{3}$

Further $z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy)$ and $dz = dx + idy$

$\int_C z^2 dz = \int_{(0,0)}^{(3\ 1)} \{(x^2 - y^2) + i(2xy)\}(dx + idy)$

$$\int_C z^2 dz = \int_{(0,0)}^{(3\ 1)} \{(x^2 - y^2)dx - 2xydy\} + i \int_{(0,0)}^{(3\ 1)} \{2xydx + (x^2 + y^2)dy\}$$

Example-1……

We have $y = \frac{x}{3}$
Or $x = 3y$ and and we shall convert these integrals into the variables y and integrate wrt y and integrate wrt y from 0 to 1. we also have $dx = 3dy$

$$\int_C z^2 dz = \int_{y=0}^{1} \{(9y^2 - y^2)3dy - 2(3y)ydy\} + i\int_{y=0}^{1} \{2(3y)y.3dy + (9y^2 - y^2)dy\}$$

$$\int_C z^2 dz = \int_{y=0}^{1} (24y^2 - 6y^2)dy + i\int_{y=0}^{1} (18y^2 + 8\,y^2)dy$$

$$= 6 + \frac{26}{3}i$$

Along the given path

Example-2

Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$ along the following paths

(a) The parabola $x = 2t, \quad y = t^2 + 3$

(b)The straight line from $(0,3)$ to $(2,4)$

(a) x varies from 0 to 2 and hence

$$\left.\begin{array}{l} \text{If } x = 0, 2t = 0 \quad \Rightarrow t = 0 \\ \text{If } x = 2, 2t = 2 \quad \Longrightarrow t = 1 \end{array}\right\} \Rightarrow t \text{ varies from 0 to 1}$$

I=$\int_{(0\ 3)}^{(2\ 4)} (2y + x^2)dx + (3x - y)dy$

$$I = \int_0^1 \{2(t^2 + 3) + 4t^2\}2dt\{3(2t) - (t^2 + 3)\}2tdt$$

$$= \frac{33}{2}$$

Example-2................

(b) Equation of the straight line joining (0,3) and (2,4) is given by

$\frac{y-3}{x-0} = \frac{4-3}{2-0} \Longrightarrow x = 2y - 6.$

Hence $dx = 2dy$

$I = \int_3^4 (8y^2 - 39y + 54)$dy

$= \frac{97}{6}$

Examples

Evaluate the following Line Integrals

a. $\int_C Re(z)dz$, C is the parabola $y = 1 + \frac{1}{2}(x-1)^2$ from $(1+i)$ to $3+3i$

b. $\int_C e^z dz$, C is the shortest path from $\frac{\pi}{2i}$ to π

c. $\int_C Im(z^2)dz$ counterclockwise around the triangle with vertices 0, 1 , i

Session Summary

- The **complex line integral** of a function taken over a path C is denoted by $\int_C f(z)\mathrm{d}z$ or by $\oint_C f(z)\mathrm{d}z$ if C is closed.