Lecture 26
Cauchy-Riemann Equation in Polar Form

Dr. Mahesha Narayana

Intended learning Outcomes

At the end of this lecture, student will be able to:

- State Cauchy-Riemann equations for analytic function in polar form
- Apply Cauchy-Riemann equations to verify the analyticity of complex valued functions
- Illustrate harmonic function and discuss its properties

Topics

- Cauchy-Riemann equation in polar co-ordinates

Cauchy – Riemann Equations in Polar Form

Given $f(z) = u(r,\theta) + iv(r,\theta)$ is an analytic function then the Cauchy – Riemann equations in polar form are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

Note: The derivatives of $f(z)$ in polar form are given by

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\frac{1}{r}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)$$

Harmonic function in polar form

If $f(z) = u(r,\theta) + iv(r\,\theta)$ be an analytic function then u and v satisfies Laplace's equation in the polar form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

Example 2

Ex: Verify that $u = \frac{1}{r^2}(\cos 2\theta)$ is harmonic . find also an analytic function.

Soln: $\frac{\partial u}{\partial r} = (-\frac{2}{r^2})\cos 2\theta$; $\frac{\partial u}{\partial \theta} = (-\frac{2}{r^2})\sin 2\theta$

$$\frac{\partial^2 u}{\partial r^2} = \frac{6}{r^4}\cos 2\theta \quad ; \quad \frac{\partial^2 u}{\partial \theta^2} = (-\frac{4}{r^2})\cos 2\theta$$

Then the Laplace equation in polar form is given by,

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{6}{r^4}\cos 2\theta - \left(\frac{2}{r^4}\right)\cos 2\theta - \left(\frac{4}{r^2}\right)\cos 2\theta = 0$$

Hence u-satisfies the laplace equation and hence is harmonic.

Let us find required analytic function f(z) = u+iv.
We note that from the theory of differentials,

$$dv = \frac{\partial v}{\partial r}\partial r + \frac{\partial v}{\partial \theta}\partial \theta$$

Using C-R equations ,

Example 2 (cont.)

$$= \left(-\frac{1}{r}\frac{\partial u}{\partial r} \right) \partial r + \left(r\frac{\partial u}{\partial r} \right) \partial \theta$$

$$= \left(-\frac{2}{r^3}\sin 2\theta \right) dr - \left(\frac{2}{r^2}\cos 2\theta \right) d\theta$$

$$= d\left(-\frac{2}{r^2}\sin 2\theta \right)$$

From this $V = -\frac{1}{r^2}\sin 2\theta + c$

$$f(Z) = u + iv = \left(\frac{1}{r^2}\cos 2\theta \right) + i\left(-\frac{1}{r^2}\sin 2\theta \right) + c$$

$$= \frac{1}{r^2}\left[\cos 2\theta - i\sin 2\theta \right] + ic$$

$$= \frac{1}{r^2}e^{-2i\theta} + ic = \frac{1}{(re^{i\theta})^2} + ic$$

$$f(Z) = \frac{1}{z^2} + ic.$$

Example 3

Ex 2:Find an analytic function f(z)= u+iv given that

$$v=\left(r-\frac{1}{r}\right)\sin\theta \quad r\neq 0$$

$$so\ln: \quad \frac{\partial v}{\partial r}=\left(r+\frac{1}{r^2}\right)\sin\theta \quad : \quad \frac{\partial v}{\partial \theta}=\left(r-\frac{1}{r}\right)\cos\theta$$

To find u using the differentials

$$\mathrm{du}=\frac{\partial u}{\partial r}\partial r+\frac{\partial u}{\partial \theta}\partial\theta \quad ,$$

Using C-R equations $ru_r=v_\theta \quad rv_r=-u_\theta$

$$=\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\partial r+\left(r\frac{\partial v}{\partial r}\right)\partial\theta$$

$$=\frac{1}{r}\left(r-\frac{1}{r}\right)\cos\theta \; dr - r\left(r+\frac{1}{r^2}\right)\sin\theta \; d\theta$$

$$=d\left[\left(r+\frac{1}{r}\right)\cos\theta\right]$$

Example 3 (Cont.)

$$u = \left(r + \frac{1}{r} \right) \cos\theta + c$$

$$f(z) = u + iv$$

$$= \left(r + \frac{1}{r} \right) \cos\theta + c + i\left(r - \frac{1}{r} \right) \sin\theta$$

$$= r(\cos\theta + i\sin\theta + \frac{1}{r}(\cos\theta - i\sin\theta) + c$$

$$f(z) = re^{i\theta} + \frac{1}{r}e^{i\theta} = z + \frac{1}{z} + c$$

Example 4

Ex : Construction an analytic function given $u = r^2 \cos 2\theta$,

(Milne Thomson Method)

$$u = r^2 \cos 2\theta \cdots\cdots\cdots\cdots\cdots (1)$$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta \qquad \frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta$$

$$f^1(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

Using C-R equations $ru_r = v_\theta, rv_r = -u_\theta$

$$f^1(z) = e^{-i\theta}\left[2r\cos 2\theta + i\left(-\frac{1}{r}\right)(-2r^2 \sin 2\theta)\right]$$

$$= e^{-i\theta}[2r\cos 2\theta + i2r\sin 2\theta)]$$

$$= 2re^{-i\theta}[\cos 2\theta + i\sin 2\theta)]$$

Now put $r = z, \quad and \quad \theta = 0$

$$f^1(z) = 2z$$ on integrating

$$f(z) = z^2 + c.$$

Session Summary

C-R equations in Polar form: $\boxed{\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}, \quad \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}}$