Lecture 9
Absolute Convergence of Improper Integrals

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Illustrate absolute convergence
- Analyse and test the convergence of improper integrals

Topics

- Convergent and divergent
- Graphical explanation of convergence
- Absolute convergence of improper integrals

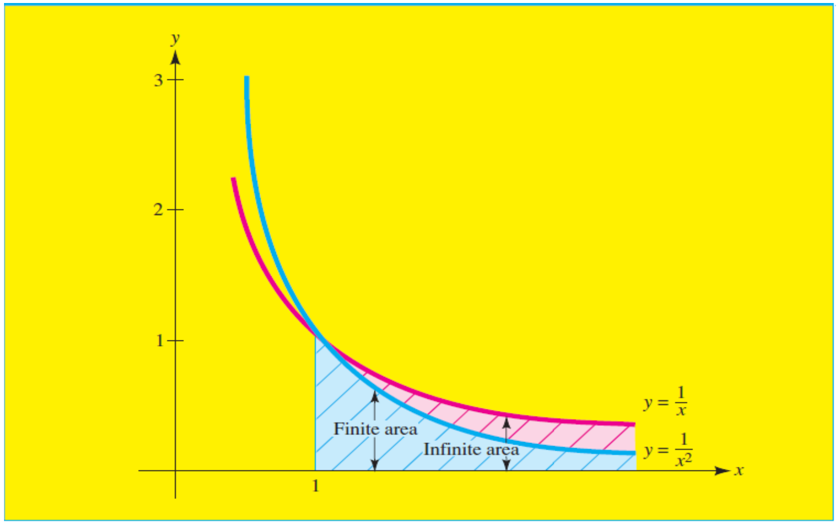
Definition of Convergent and Divergent

- If the limit defining the improper integral is a finite number, the integral is said to **converge**. Otherwise the integral is said to **diverge**.

Example: $\int_{1}^{\infty} \frac{1}{x^{2}} dx = 1$ is convergent.

Example: $\int_{1}^{\infty} \frac{1}{x} dx = +\infty$ is divergent.

Graphical Explanation of Convergence



$$f(x) = \frac{1}{x} \quad \text{-------(A)}$$

$$f(x) = \frac{1}{x^2} \quad \text{-------(B)}$$

- Notice that the improper integral of the function *f*(*x*) in example (B) is converged, while that of the function *f*(*x*) in example (A) is diverged.

- In geometric terms, this says that the area to the right of $x = 1$ under the curve $y = 1/x^2$ is finite, while the corresponding area under the curve $y=1/x$ is infinite. The reason for the difference is that, as *x* increases, approaches zero more quickly than does (see Figure).

Absolute Convergence of Improper Integrals

- Since most of the tests of convergence for improper integrals are only valid for positive functions, it is legitimate to wonder what happens to improper integrals involving non positive functions.

- So consider a function $f(x)$ (not necessarily positive) defined on $[a,b]$

- It is easy to see that both functions $f(x)$ and $|f(x)|$ will exhibit the same kind of improper behavior.

Natural Conclusion

- The improper integral $\int_a^b f(x)dx$ is said to be absolutely convergent if $\int_a^b |f(x)|dx$ is convergent

- If the integral $\int_a^b |f(x)|dx$ is convergent, then the integral $\int_a^b f(x)dx$ is also convergent.

Caution!!

- We have to be careful the converse is not true. Indeed, the improper integral

$$\int_{o}^{+\infty} \frac{\sin(x)}{x} dx$$

is convergent while the improper integral

$$\int_{0}^{+\infty} \frac{|\sin(x)|}{|x|}$$

is divergent.

Example 1

- Show that the improper integral $\int_{-\infty}^{\infty} \frac{sinx}{1+x^2} dx$ converges.

Solution: We consider the absolute convergence of the given integral,

we have $\left| \int_{-\infty}^{\infty} \frac{sinx}{1+x^2} dx \right| \leq \int_{-\infty}^{\infty} \left|\frac{sinx}{1+x^2}\right| dx$

$$\lim_{a\to\infty} \int_{-a}^{c} \left|\frac{sinx}{1+x^2}\right| dx + \lim_{b\to\infty} \int_{c}^{b} \left|\frac{sinx}{1+x^2}\right| dx = I_1 + I_2$$

Now $I_1 = \lim_{a\to\infty} \int_{-a}^{c} \left|\frac{sinx}{1+x^2}\right| dx \leq \int_{a}^{c} \frac{1}{1+x^2} dx = \lim_{a\to\infty} [\tan^{-1}\mathrm{c} - \tan^{-1}\mathrm{a}]$

$$= \tan^{-1}\mathrm{c} + \frac{\pi}{2}$$

Example 1(Cont.)

$$I_2 = \lim_{b\to\infty} \int_c^b \left|\frac{sinx}{1+x^2}\right| dx \ \le \lim_{b\to\infty} \int_c^b \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} c] = \frac{\pi}{2} - \tan^{-1} c$$

Hence $|I_1| = I_1 + I_2 \le \pi$. Therefore the given integral converges

Example 2

- Show that $\int_1^\infty \frac{cosx}{\sqrt{(1+x^3)}}$ dx is absolutely convergent

Solution: Let $f(x)$=$\frac{cosx}{\sqrt{(1+x^3)}}$ $x \geq 1$ then

$$|f(x)| = \left|\frac{cosx}{\sqrt{(1+x^3)}}\right| < \frac{1}{x^{\frac{3}{2}}} \quad , x \geq 1$$

The integral $\int_1^\infty \frac{dx}{x^{\frac{3}{2}}}$ is convergent, since $p = \frac{3}{2} > 1$

Hence by comparison test the given integral is convergent

Absolutely and Conditionally convergent

- An improper integral of f is absolutely convergent (or converges absolutely) if the improper integral of |f| also converges.

- If an improper integral converges but does not converge absolutely, it is said to converge conditionally.

Summary

- An improper integral of f is absolutely convergent (or converges absolutely) if the improper integral of |f| also converges.

- If an improper integral converges but does not converge absolutely, it is said to converge conditionally.