Course Code: ESC106A

Course Title: Construction Materials and Engineering Mechanics

Lecture No. 34:
Derivations of Centroid

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Lecture Intended Learning Outcomes

At the end of this lecture, students will be able to:

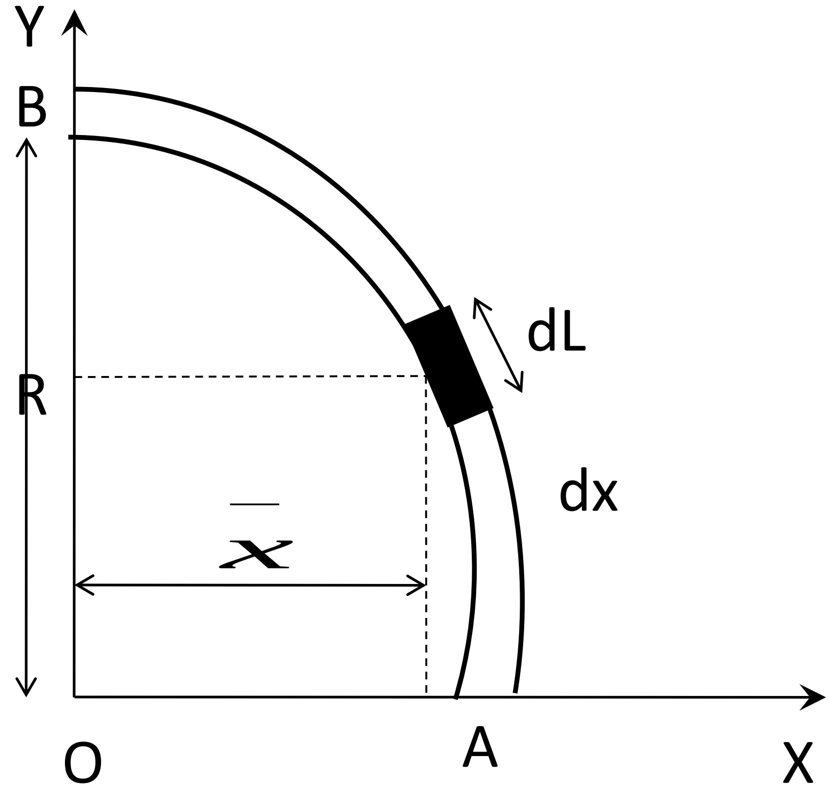
- Determine the centroid of different sections using the principle of moment balance of elements.

Contents

Centroid of Quadrant of ring, Centroid of arc of a circle, centroid of semi circle, Centroid of solid right circular cone, Centroid of solid hemisphere, Centroid of Various shapes.

Centroid of the Quadrant of a Ring (line/rod)

Determine the CG of quadrant AB of the arc of a circle of radius R as shown in Figure.



Solution:

The equation of the curve AB is the equation of the circle of radius R.

Differentiating,

$$x^2 + y^2 = R^2$$

$$2xdx + 2ydy = 0; 2ydy = -2xdx; dy = \frac{-xdx}{y}$$

Centroid of the Quadrant of a ring by integration method

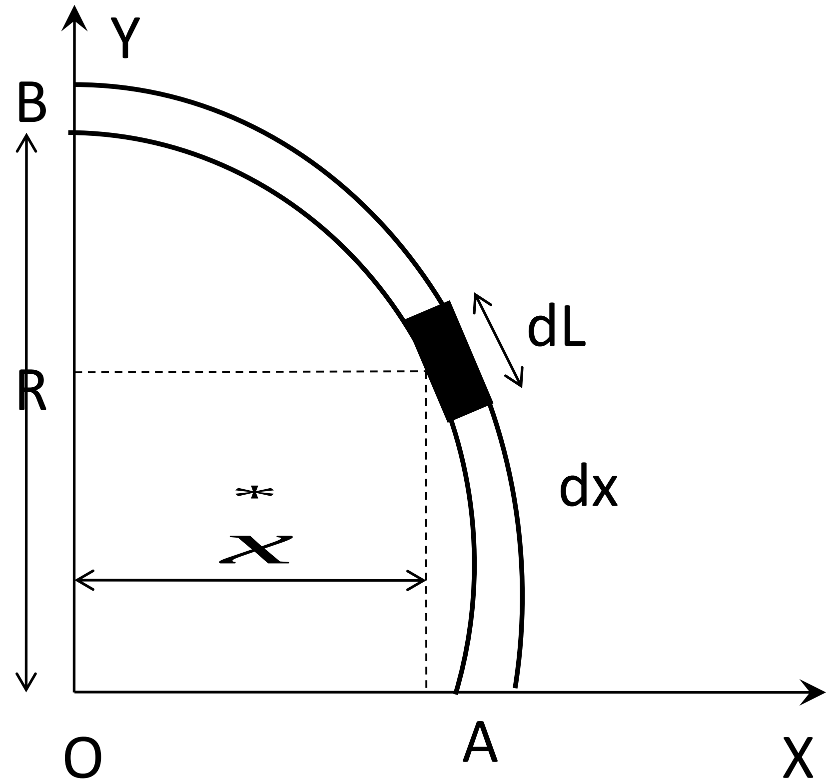
Consider an elemental length dL whose CG is at $\left(\overset{*}{x}, \overset{*}{y}\right)$

$$\bar{y} = \frac{\int \overset{*}{y}\, dL}{\int dL} \text{.....................}(1)$$

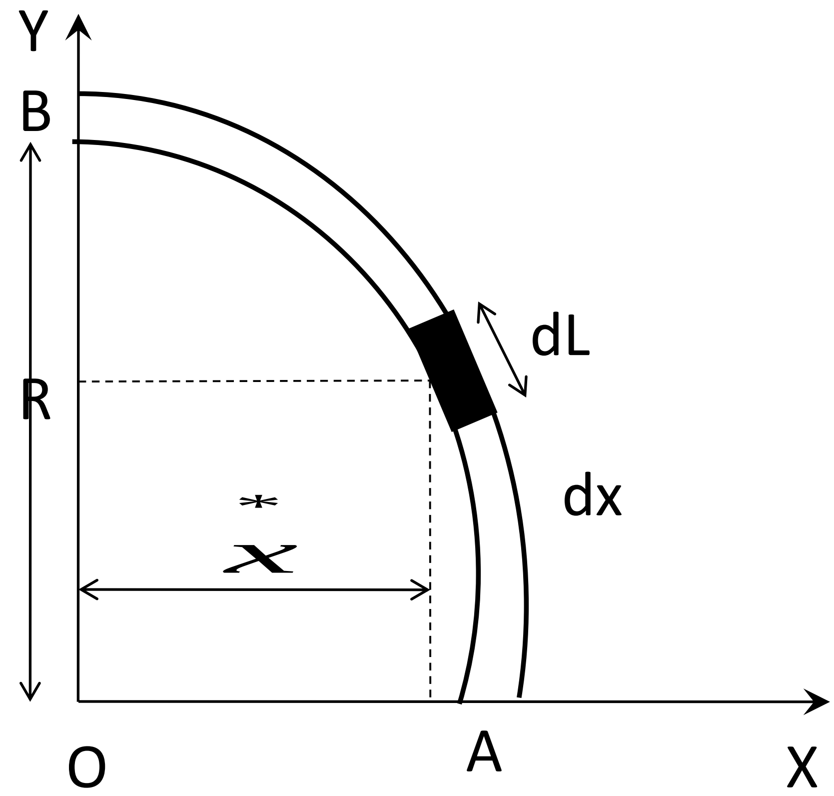
$$dL = \sqrt{dx^2 + dy^2}$$

$$dL = \sqrt{dx^2 + \left(\frac{-xdx}{y}\right)^2}$$

$$dL = dx\sqrt{1+\frac{x^2}{y^2}} = dx\sqrt{\frac{R^2}{y^2}} = \frac{R}{y}dx$$



Centroid of the quadrant of a ring by integration method



Substituting dL in Eqn (1)

$$\bar{y} = \frac{\int y^{*} \frac{R}{y} dx}{\int dL}$$

$$\bar{y} = \frac{\int R dx}{\int dL} = \frac{R \int_0^R dx}{\int dL} = \frac{R[x]_0^R}{\left(\frac{2\pi r}{4}\right)} = \frac{2R}{\pi}$$

Due to symmetry $\bar{x} = \frac{2R}{\pi}$

Centroid of an arc of a circle (Line/wire)

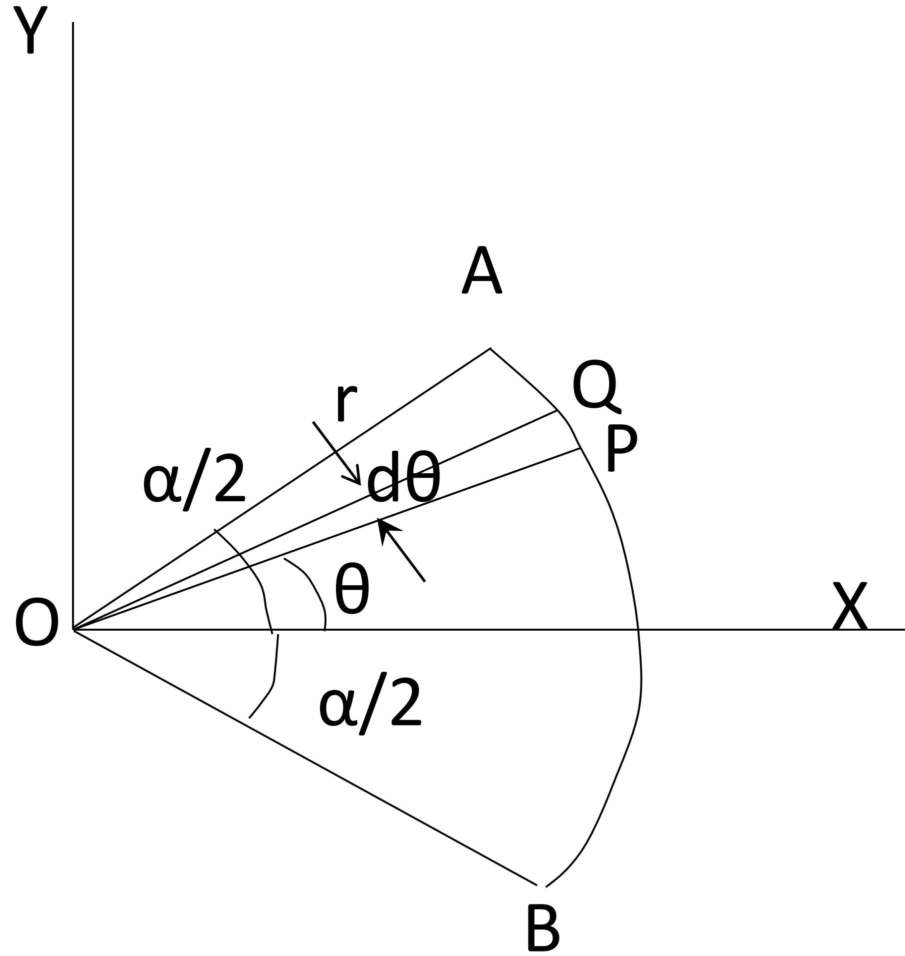
Let arc AB subtend an angle α at the centre O. Let OX be the bisector of angle AOB. By symmetry, CG will lie on OX . So y=0.

$$PQ = rd\theta$$

$$\bar{y}_{PQ} = r\cos\theta$$

$$\bar{x} = \frac{\int_{-\alpha/2}^{\alpha/2} rd\theta . r\cos\theta}{\int_{-\alpha/2}^{\alpha/2} rd\theta} = \frac{r\int_{-\alpha/2}^{\alpha/2} \cos\theta . d\theta}{\int_{-\alpha/2}^{\alpha/2} d\theta}$$

$$\bar{x} = r\frac{\left[\sin\theta\right]_{-\alpha/2}^{\alpha/2}}{\left[\theta\right]_{-\alpha/2}^{\alpha/2}} = 2r\frac{\sin\left(\alpha/2\right)}{\alpha}$$



Centroid of sector of a circle (Area)

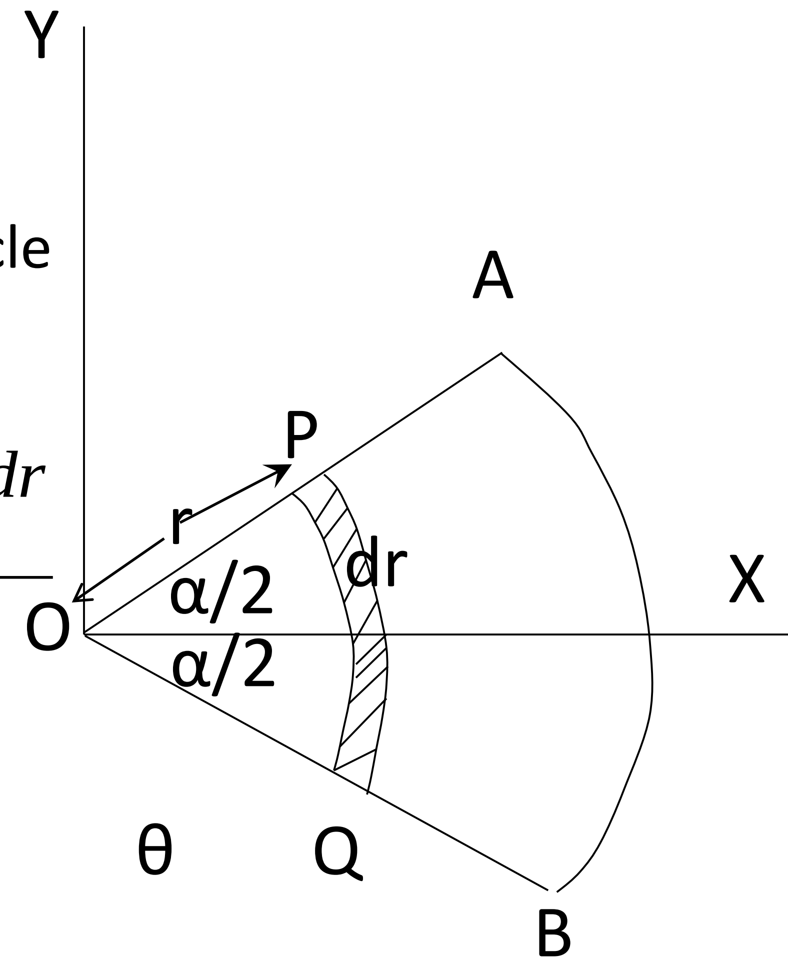
Let α be the angle of sector AOB. Let OX be the bisector of angle AOB. By symmetry, CG will lie on OX . So y=0.

$$Area, PQ = r\alpha dr$$

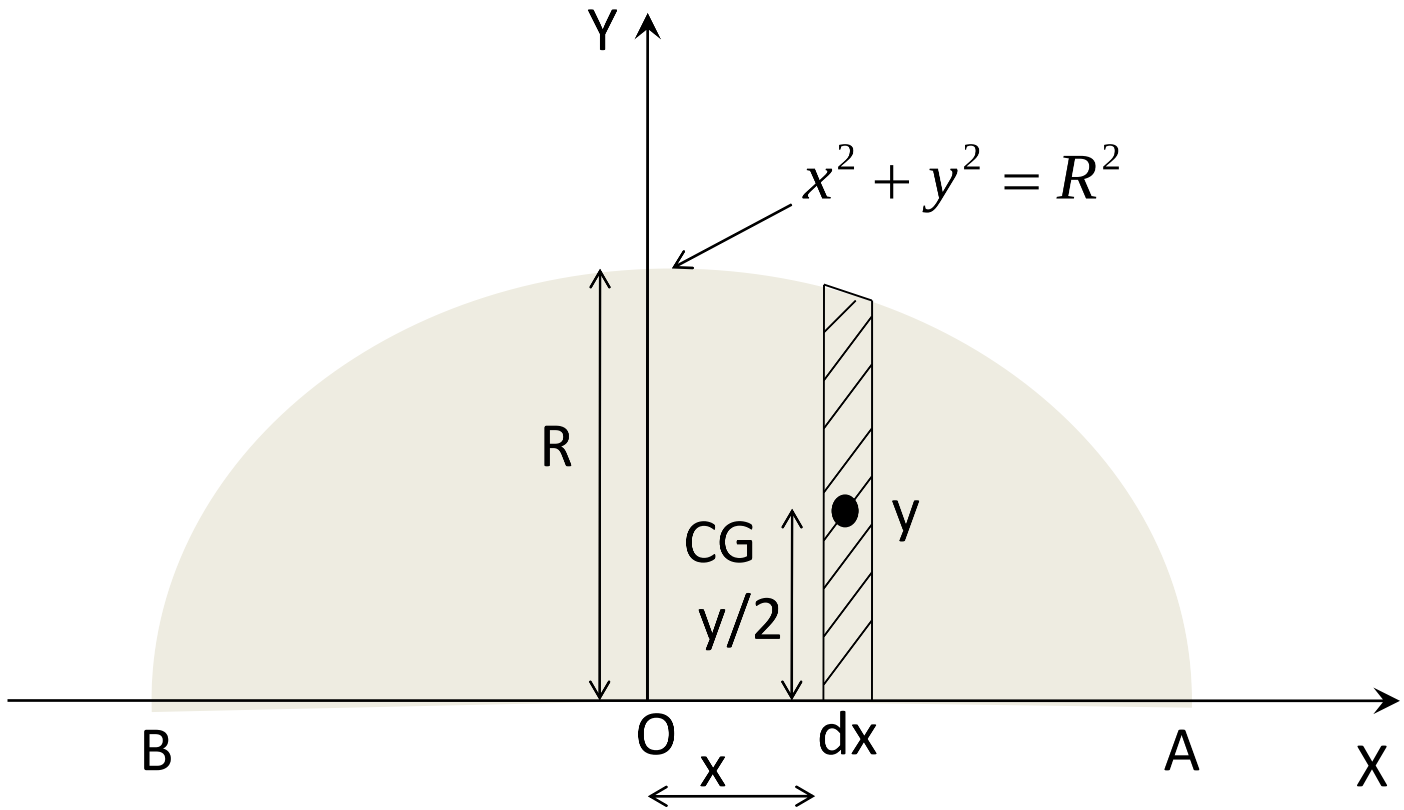
$$\bar{x}_{PQ} = 2r\frac{\sin\left(\alpha/2\right)}{\alpha} \Rightarrow \text{CG of an arc of a circle}$$

$$\bar{x} = \frac{\int_0^r r\alpha dr\, 2r\frac{\sin\left(\alpha/2\right)}{\alpha}}{\int_0^r r\alpha dr} = \frac{2\sin\left(\alpha/2\right)\int_0^r r^2 dr}{\alpha\int_0^r r dr}$$

$$\bar{x} = \frac{2\sin\left(\alpha/2\right)}{\alpha}.\frac{2}{3}r = \frac{4r}{3\alpha}.\sin\left(\alpha/2\right)$$



Centroid of a semi-circle (Area)



Consider a strip parallel to y-axis $dA = y.dx$

Centroid of a semi-circle

Moment of area dA about x-axis

$$dM_x = dA.\frac{y}{2} = \frac{y}{2}ydx = \frac{y^2}{2}dx$$

Moment of total area A about x-axis

$$M_x = \int \frac{y^2}{2}dx = \int_{-R}^{R} \frac{y^2}{2}dx = \int_{-R}^{R} \frac{R^2 - x^2}{2}dx$$

$$M_x = \frac{1}{2}\left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R} = \frac{1}{2}\frac{4R^3}{3} = \frac{2R^3}{3}$$

Centroid of a Semi-circle

Let the distance of CG of total area from x-axis be $\bar{y}$

Total area of semi circle = $\dfrac{\pi R^2}{2}$

$$M_x = \bar{y}.\frac{\pi R^2}{2}$$

Equating,

$$\frac{2R^3}{3} = \bar{y}.\frac{\pi R^2}{2}$$

$$\bar{y} = \frac{4R}{3\pi}$$

Hence centroid of semi-circle is

$$\left(0, \frac{4R}{3\pi}\right)$$

Compare this with the centroid of the sector of the circle.

Note:

Consider a strip parallel to x-axis and locate the CG.

Centroid of a solid right circular cone

Let ABC be the cone and AD its axis. Consider an element PQ cut-off by two planes parallel to the base BC at y and y+dy from A and having centre at M. PQ is considered as a circular plate of radius r.

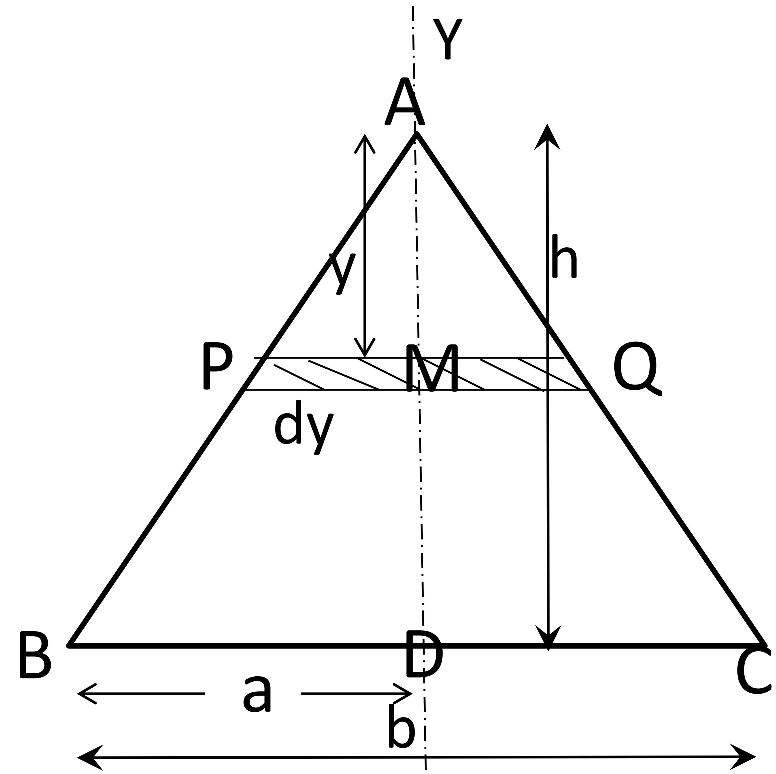
$$AD = h$$

$$BC = b$$

$$\Delta ABC, \Delta APQ$$

$$\therefore \frac{AM}{PM} = \frac{AD}{BD}$$

$$\frac{y}{r} = \frac{h}{a} \Rightarrow r = \frac{ay}{h}$$



Centroid of a solid right circular cone

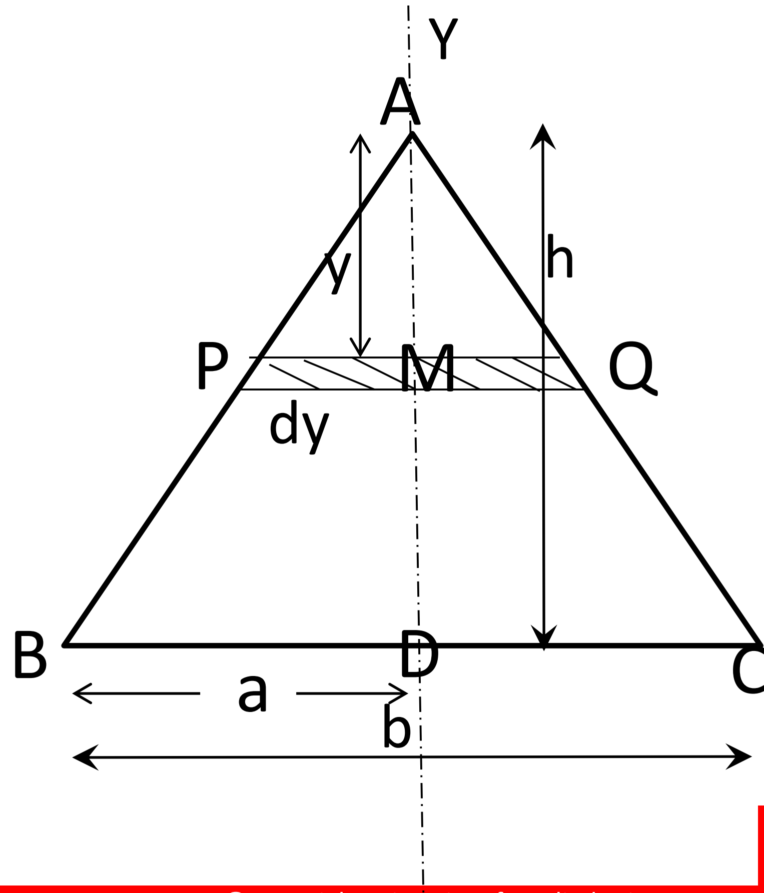
If w is weight of unit volume of the material, weight of PQ is:

$$W_{PQ} = w.\pi r^2 dy$$

$$W_{PQ} = w.\pi \left(\frac{ay}{h}\right)^2 dy$$

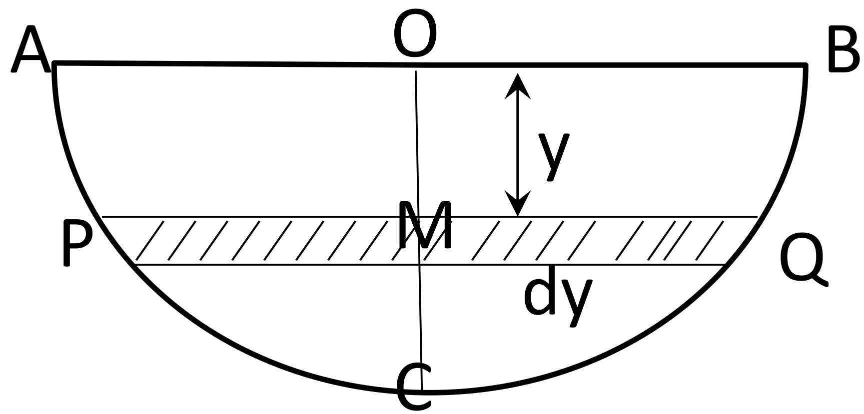
$$\bar{y} = \frac{\sum_{y=0}^{y=h} w.\pi \left(\frac{ay}{h}\right)^2 dy.y}{\sum_{y=0}^{y=h} w.\pi \left(\frac{ay}{h}\right)^2 dy} = \frac{\sum_{y=0}^{y=h} y^3 dy}{\sum_{y=0}^{y=h} y^2 dy}$$

$$\bar{y} = \frac{\left[y^4/4 \right]}{y^3/3} = \frac{3}{4} h \text{ from the apex}$$



Centroid of a solid hemisphere

Let ACB be the hemisphere of radius r and OC its central radius. Consider an elementary circular plate PQ cut off by planes parallel to AB



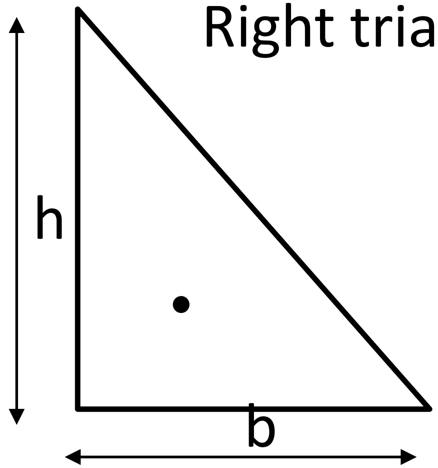
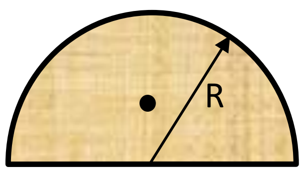
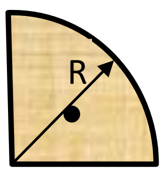
$$PM^2 = OP^2 - OM^2$$

$$PM^2 = r^2 - y^2$$

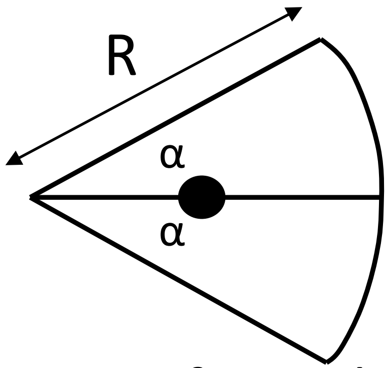
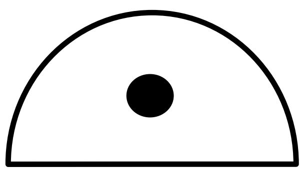
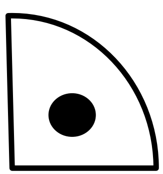
$$W_{PQ} = \pi\left(r^2 - y^2\right) dy.w$$

$$\bar{y} = \frac{\sum_{y=0}^{y=r} \pi\left(r^2 - y^2\right) dy.w.y}{\sum_{y=0}^{y=r} \pi\left(r^2 - y^2\right) dy.w.} = \left[\frac{\frac{r^2 y^2}{2} - \frac{y^4}{4}}{r^2 y - \frac{y^3}{3}}\right]_0^r ; \Rightarrow \bar{y} = \frac{3r}{8}$$

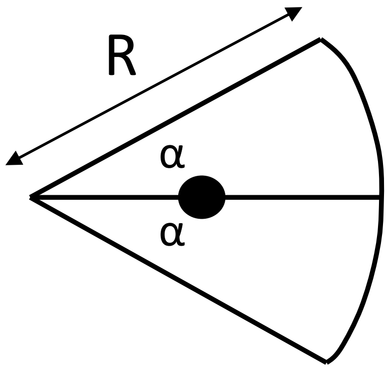
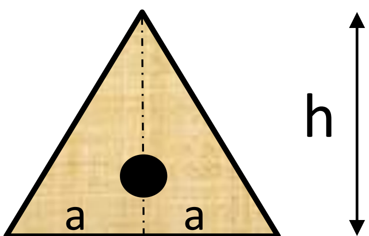
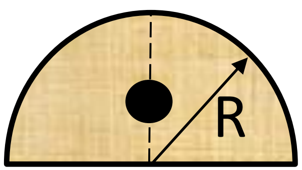
Centroid of various shapes

Shape	$\bar{x}$	$\bar{y}$	Area
Right triangle 	$\frac{1}{3}b$ From base	$\frac{1}{3}h$ From base	$\frac{1}{2}bh$
 Semi Circular area	0 From centre	$\frac{4R}{3\pi}$ From centre	$\frac{\pi R^2}{2}$
 Quarter Circular area	$\frac{4R}{3\pi}$ From centre	$\frac{4R}{3\pi}$ From centre	$\frac{\pi R^2}{4}$

Centroid of various shapes

Shape	$\bar{x}$	$\bar{y}$	Area/Length
 Sector of a circle	$\frac{2r\sin\alpha}{3\alpha}$ From centre	0 From centre	αr^2
 Semi Circular arc	0 From centre	$\frac{2R}{\pi}$ From centre	πR
 Quarter Circular arc	$\frac{2R}{\pi}$ From centre	$\frac{2R}{\pi}$ From centre	$\frac{\pi R}{2}$

Centroid of various shapes

Shape	$\bar{x}$	$\bar{y}$	Length/Volume
 Arc of a circle	$\frac{r\sin\alpha}{\alpha}$ From centre	0 From centre	$2\alpha r$
 Solid triangular cone	0 From centre	$\frac{1}{4}h$ From base	$\frac{\pi a^2 h}{3}$
 Solid hemisphere	0 From centre	$\frac{3R}{8}$ From centre	$\frac{2\pi R^3}{3}$

Summary

- The bodies having a continuous shape, moments are summed (integrated) using differential elements
- The centroid is the location of the geometric center for the body.
- Centroid is determined using a moment balance of geometric elements such as line, area, or volume segments.