Lecture 40
Residue Integration Method

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Intended learning Outcomes

At the end of this lecture, student will be able to:

- State and explain Cauchy’s residue theorem
- Evaluate some complex integrals using Cauchy’s residue theorem

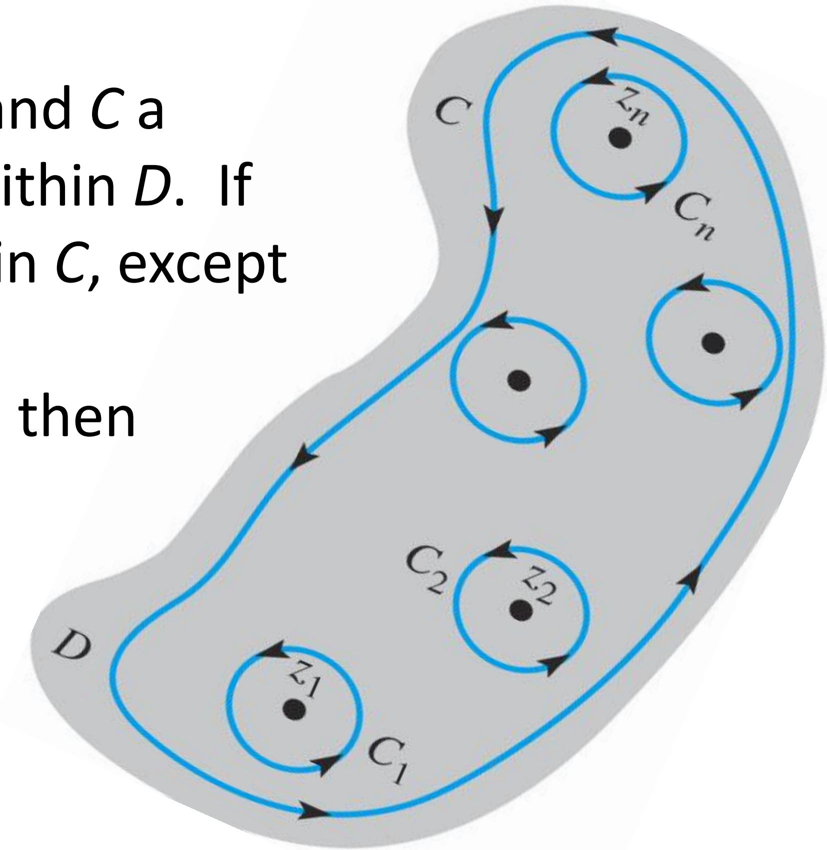
Topics

- Cauchy’s residue theorem

Cauchy's Residue Theorem

Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function $f(z)$ is analytic on and within C, except at a finite number
of singular points $z_1, z_2, \ldots, z_n$ within C, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_k\right)$$



Computing Residues

Some examples

$$\text{Res}\left(\frac{1}{4z+1};\, z=-\frac{1}{4}\right)=\lim_{z\to -1/4}\left(z+\frac{1}{4}\right)\frac{1}{4z+1} \;=\frac{1}{4}$$

$$\text{Res}\left(\frac{1}{\sin z};\, z=0\right)=\lim_{z\to 0} z\,\frac{1}{\sin z} \;=1$$

$$\text{Res}\left(\frac{\ln z}{z^2+4};\, z=2e^{i\pi/2}\right)=\lim_{z\to 2e^{i\pi/2}}(z-2i)\,\frac{\ln z}{z^2+4} \;=\lim_{z\to 2e^{i\pi/2}}\frac{\ln z}{z+2i}=\frac{\ln 2+i\pi/2}{4i}$$

$$\text{Res}\left(\frac{z}{\sin^2 z};\, z=\pi\right)=\lim_{z\to\pi}\frac{d}{dz}\left[(z-\pi)^2\frac{z}{\sin^2 z}\right]$$

$$=\lim_{z\to\pi}\left[(-\pi+z)\left(-\pi+3z+2(\pi-z)\,z\cot z\right)\csc^2 z\right] \;=1$$

Computing Residues

$$\cot x = \frac{1}{x} - \frac{x}{3} + O\left(x^3\right)$$

$$\frac{1}{z+2} = \frac{1}{2}\left(1 - \frac{z}{2} + O\left(z^2\right)\right)$$

$$\operatorname{Res}\left(\frac{\cot \pi z}{z(z+2)}\,;\, z=0\right)$$

$$= a_{-1}\ of\ \left[\frac{1}{z}\cdot\frac{1}{2}\left(1 - \frac{z}{2} + O\left(z^2\right)\right)\left[\frac{1}{\pi z} - \frac{\pi z}{3} + O\left(z^3\right)\right]\right] \quad = -\frac{1}{4\pi}$$

Alternatively, $z = 0$ is a pole of 2nd order :

$$\operatorname{Res}\left(\frac{\cot \pi z}{z(z+2)}\,;\, z=0\right) = \lim_{z\to 0}\frac{d}{dz}\left(z^2\,\frac{\cot \pi z}{z(z+2)}\right)$$

$$= \lim_{z\to 0}\left[-\frac{z\cot \pi z}{(2+z)^2} + \frac{\cot \pi z}{2+z} - \frac{\pi z \csc^2 \pi z}{2+z}\right] \quad = -\frac{1}{4\pi}$$

$$\operatorname{Res}\left(e^{-1/z}\,;\, z=0\right) = a_{-1}\ of\ \left[1 - \frac{1}{z} + O\left(z^{-2}\right)\right] \quad = -1$$

Example-1

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}dz$ where

the contour C is the circle $|z| = 2$

Solution

The only pole lying within the circle C: $|z| = 2$ is $z = 1$, the other singularity $z = 3$ lies outside C,

Thus we have

$$\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i \operatorname{Res}(f(z), 1)$$

$$= 2\pi i\left(-\frac{1}{4}\right)$$

$$= -\frac{\pi}{2}i$$

Example-2

Evaluate the following integral using residue theorem

$$\int_c \frac{4-3z}{z(z-1)(z-2)}dz$$

Where c is the circle $|z| = 3/2$

Solution:

The poles of the function f(z)are given by equating

The denominator to zero.

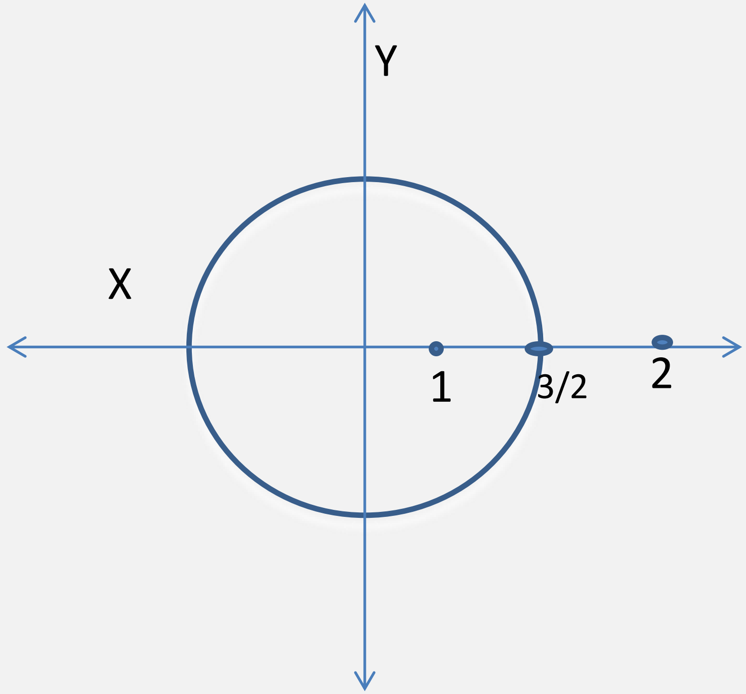
Therefore, the function has poles at z=0,z=1 and z=2 of which given circle encloses the pole at z=0 and z=1

Residue of f(z) at the simple pole z=0 $\lim_{z \to 0} \frac{4-3z}{z(z-1)(z-2)} = 2$

Example-2 (cont.)

Residue of f(z)at the simple pole z=1 is

$$\lim_{z\to 1} \frac{4-3z}{z(z-1)(z-2)} = -1$$



By Cauchy's integral formula

$$\int_c f(z)dz = 2\pi i \times (sum\, of\, the\; residue\;\; with\;\; in\;\; c)$$

$$= 2\pi i \times (2-1) = 2\pi i$$

Example-3

Evaluate $\oint_C \frac{e^{2z}dz}{(z-1)(z-2)}$

Where C is the circle IzI=3.

Solution

$$\oint_C e^{2z}\left(\frac{1}{z-2}-\frac{1}{z-1}\right)dz=\int_C \frac{e^{2z}}{z-2}dz-\int_C \frac{e^{2z}}{z-1}dz$$

$$=2\pi ie^4-2\pi ie^2$$

$$=2\pi i(e^4-e^2)$$

Example-4

Evaluate $f(z)=\int_C \frac{\cos \pi z}{z^2-1}dz$

$$\int_C \frac{\cos \pi z}{z^2-1}dz=\frac{1}{2}\int_C \left(\frac{1}{z-1}-\frac{1}{z+1}\right)\cos \pi z dz$$

$$=\frac{1}{2}\int_C \left(\frac{\cos \pi z}{z-1}\right)dz-\frac{1}{2}\int_C \left(\frac{\cos \pi z}{z+1}\right)dz$$

Example-4(Cont.)

$$= \frac{1}{2}\left\{(2\pi i \cos\pi(1) - (2\pi i \cos\pi(-1)\right\}$$

$$= 0$$

Examples

Evaluate the Integral of $f(z)$ around the positively oriented circle $|z|$=3 when

(a)f(z) $= \frac{(3z+2)^2}{z(z-1)(2z+5)}$ Ans:9 πi

(b)$f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$ Ans:-3 πi

(c)f(z)=$\frac{z^3 e^{\frac{1}{z}}}{1+z^3}$ Ans :2 πi

Session Summary

- **Cauchy Residue Theorem**,

  Let *D* be a simply connected domain and *C* a simple closed contour lying entirely within *D*. If a function $f(z)$ is analytic on and within *C*, except at a finite number of singular points $z_1, z_2, \ldots, z_n$ within *C*, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_k\right)$$