Lecture 39
Singularities, Zeros, Poles and Residues

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Classify singularities of complex valued functions
- Describe the concept of zero and infinity
- Define residue at a singularity of the complex valued function
- Apply Laurent series to find the residue

Topics

- Singularity
- Types of singularities
- Zeros
- Pole
- Residue

Types of Singularities

Suppose that $z = z_0$ is an isolated singularity of $f(z)$ then

$$f(z)=\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

is the Laurent series of $f(z)$ valid for $r < |z - z_0| < R$. The principal part of is the series

$$\sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k} = \ldots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0}.$$

Based on the number of terms in the principle part we classify the singularities into three kinds

Classification

1. If the principal part is zero, $z = z_0$ is called a *removable singularity*.
2. If the principal part contains a finite number of terms, then $z = z_0$ is called a pole. If the last nonzero coefficient is a_{-n}, $n \geq 1$, then we say it is a pole of order n. A pole of order 1 is commonly called a *simple pole*.
3. If the principal part contains infinitely many nonzero terms, $z = z_0$ is called an *essential singularity*.

Example – 1

For the function $$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$z = 0$ is a removable singularity as the principle part in the Laurent series is absent.

Example – 2

For the function $$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots$$

$z = 0$ is a simple pole as $a_{-1} \neq 0$ and $a_{-2} = a_{-2} = \ldots = 0$.

Example – 3

The Laurent series of $f(z) = 1/z(z-1)$ valid for $1 < |z|$ is

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \dots$$

The point $z = 0$ is an isolated singularity of $f(z)$ and the Laurent series contains an infinite number of terms involving negative integer powers of z.

Does it mean that $z = 0$ is an essential singularity?

The answer is "NO". Since the interested Laurent series is the one with the domain $0 < |z| < 1$, for which we get

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - \dots$$

Thus $z = 0$ is a simple pole for $0 < |z| < 1$.

Zeros

We say that z_0 is a zero of f if $f(z_0) = 0$. An analytic function $f(z)$ has a zero of order n at $z = z_0$ if

$$f(z_0)=0,\ f'(z_0)=0,\ f''(z_0)=0,\ldots,\ f^{(n-1)}(z_0)=0,\ f^{(n)}(z_0)\neq 0.$$

Example

The analytic function $f(z) = z \sin z^2$ has a zero at $z = 0$,

$$f(z)=z\sin z^2 = z^3\left[1-\frac{z^4}{3!}+\frac{z^8}{5!}-+\ldots\right]$$

Here, we have $f(0) = f'(0) = f''(0) = 0$ and $f'''(0) \neq 0$.

Hence $z = 0$ is a *zero of order 3*.

Relation between Poles and Zeros

If the functions f and g are analytic at $z = z_0$ and f has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $F(z) = g(z)/f(z)$ has a pole of order n at $z = z_0$.

Example

Consider the function
$$F(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}.$$

Inspection reveals that the denominator has zeros of order 1 at $z = 1$ and $z = -5$, and a zero of order 4 at $z = 2$. Since the numerator is not zero at these points, $F(z)$ has simple poles at $z = 1$ and $z = -5$ and a pole of order 4 at $z = 2$.

Residues

The coefficient a_{-1} of $1/(z - z_0)$ in the Laurent series is called the residue of the function $f(z)$ at the isolated singularity z_0.

We use this notation $a_{-1} = \text{Res}(f(z), z_0)$

Example

For the function $f(z) = 1/(z - 1)^2(z - 3)$ the singularities are $z = 1, 3$ and $z = 1$ is a pole of order 2.

The coefficient of $1/(z - 1)$ is $a_{-1} = -¼$.

Residues at a Simple Pole

If $f(z)$ has a simple pole at $z = z_0$, then

$$\operatorname{Res}\left(f(z), z_0\right) = \lim_{z \to z_0} f(z)$$

Residues at a Pole of Order *m*

If $f(z)$ has a pole of order m at $z = z_0$, then

$$\operatorname{Res}\left(f(z), z_0\right) = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left\{ (z - z_0)^n f(z) \right\} \right]$$

Example-1

Given that the function $f(z) = 1/(z-1)^2(z-3)$ has a pole of order 2 at $z = 1$. Find the residue of $f(z)$ at $z = 1$.

Solution

$$\operatorname{Res}\left(f(z),\, 1\right) = \frac{1}{1!}\lim_{z\to 1}\left[\frac{d}{dz}\left\{(z-1)^2 f(z)\right\}\right]$$

$$= \lim_{z\to 1}\left[\frac{d}{dz}\left\{\frac{1}{z-3}\right\}\right]$$

$$= \lim_{z\to 1}\left[\frac{-1}{(z-3)^2}\right]$$

$$= -\frac{1}{4}$$

Residue at Simple Pole – Aliter

If f can be written as $f(z)= g(z)/h(z)$ and has a simple pole at z_0 (note that $h(z_0) = 0$ and $g(z_0) \neq 0$), then

$$\mathrm{Res}\left(f(z), z_0\right) = \frac{g(z_0)}{h'(z_0)}$$

This is because

$$\lim_{z \to z_0}\left\{(z - z_0)\frac{g(z)}{h(z)}\right\} = \frac{\lim_{z \to z_0} g(z)}{\lim_{z \to z_0}\left\{\frac{h(z) - h(z_0)}{z - z_0}\right\}} = \frac{g(z_0)}{h'(z_0)}$$

Example

Find the residues at each of simple poles of the function $f(z) = 1/(z^4 + 1)$.

Solution The polynomial $z^4 + 1$ can be factored as $(z - z_1)(z - z_2)(z - z_3)(z - z_4)$.

We see that $z_1 = e^{\pi i/4}$, $z_2 = e^{3\pi i/4}$, $z_3 = e^{5\pi i/4}$, $z_4 = e^{7\pi i/4}$ are simple poles of $f(z)$

$$\operatorname{Res}\left(f(z), z_1\right) = \frac{1}{4{z_1}^3} = \frac{1}{4} e^{-3\pi i/4} = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}} i$$

$$\operatorname{Res}\left(f(z), z_2\right) = \frac{1}{4{z_2}^3} = \frac{1}{4} e^{-9\pi i/4} = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}} i$$

$$\operatorname{Res}\left(f(z), z_3\right) = \frac{1}{4{z_3}^3} = \frac{1}{4} e^{-15\pi i/4} = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} i$$

$$\operatorname{Res}\left(f(z), z_4\right) = \frac{1}{4{z_4}^3} = \frac{1}{4} e^{-21\pi i/4} = -\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} i$$

Example-2

Find the order of each pole and residue at it of

$$\frac{(1-2z)}{z(z-1)(z-2)}$$

Solution: The poles of f(z) are given by z=0,1,2

Residue of f(z) at (z=0) = $\lim_{z\to 0}(z-0)f(z) = \lim_{z\to 0}\frac{z(1-2z)}{z(z-1)(z-2)} = 1/2$

Residue of f(z) at (z=1) = $\lim_{z\to 0}(z-1)f(z) = \lim_{z\to 1}\frac{(z-1)(1-2z)}{z(z-1)(z-2)} = 1$

Residue of f(z) at (z=2)= $\lim_{z\to 0}(z-2)f(z) = \lim_{z\to 2}\frac{(z-2)(1-2z)}{z(z-1)(z-2)} = -3/2$

Example-3

- Determine the residue of $\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its simple poles.

- The poles of f(z) are z=1,1,1,1 ,2,3
- The simple poles of the function are z=2 and z=3

$$R(2) = \lim_{z\to 2} \frac{(z-2)z^3}{(z-1)^4(z-2)(z-3)} = -8$$

$$R(3) = \lim_{z\to 2} \frac{(z-3)z^3}{(z-1)^4(z-2)(z-3)} = 27/16$$

Example-3

- Determine the poles and residue at each pole of the function f(z)=cotz

Solution: $f(z) = \cot z = \cos z$

The poles of the function f(z) are given by

$$Sinz = 0,\ z = n\pi,\quad where\ \ n = 0, \pm 1, \pm 2, \pm 3....$$

Residue of f(z) at $z = n\pi \ is = \dfrac{\cos z}{\dfrac{d}{dz}(\sin z)} = \dfrac{coz}{\cos z} = 1$

Example-4

Determine the poles and residue at each pole of the function $f(z) = \dfrac{z}{\sin z}$

Solution: Poles are determined by putting sinz=0 $= \sin n\pi = 0 \Rightarrow z = n\pi$

Residue $= \left(\dfrac{z}{\cos z}\right)_{z=n\pi} = \dfrac{n\pi}{\cos n\pi} = \dfrac{n\pi}{(-1)^n}$

Session Summary

- If the **Laurent series** ,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \text{ where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^n} ds$$

  1. contains finitely many terms, say m terms, we say z_0 is a pole of order m (A pole of order one is called as simple pole).
  2. contains infinitely many terms then is called as an essential singularity

- The coefficient a_{-1} of $1/(z - z_0)$ in the above Laurent series is called the residue of $f(z)$ at z_0 and we write $a_{-1} = \mathrm{Res}(f(z), z_0)$.

- We say that z_0 is a zero of $f(z)$ if $f(z_0) = 0$.
- An analytic function $f(z)$ is said to have a zero of order n at z_0 if $f(z_0) = f'(z_0) = \ldots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$ in the Taylor series expansion of $f(z)$ about $z = z_0$.