Lecture 30
Bilinear Transformation

Dr. Mahesha Narayana

Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Find the Bilinear Transformation
- Solve application oriented problems using Bilinear Transformation

Topics

- Bilinear transformation

Bilinear Transformation

- A transformation T(z) : z → w is called bilinear if it takes the form

$$w = T(z) = (az + b)/ (cz + d)$$

Properties

- A bilinear transformation transforms circles to circles
- There exists a unique bilinear transformation that maps three given distinct points z_1, z_2, z_3 onto three given distinct points w_1, w_2, w_3
- Bilinear transformation preserve (do not alter) the cross ratio of four points

Matrix Methods

We associate the matrix A with transformation $T(z)$ as follows

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Leftrightarrow T(z) = \frac{az+b}{cz+d}$$

If $T(z) = \dfrac{az+b}{cz+d} = w$ then $T^{-1}(w) = \dfrac{dw-b}{-cw+a} = z$

for which we associate the matrix B given by

$$B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \Leftrightarrow T^{-1}(w) = \frac{dw-b}{-cw+a}$$

Clearly, B = adj A

Matrix Methods

Given $T_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$ and $T_2(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$

and corresponding matrices are *A* and *B.*

where, $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

The matrix associated with the composition of T_1 and T_2 is the product of matrices associated with T_1 and $T_{2.}$

If $C \Leftrightarrow T(z) = T_1(T_2(z))$ then $C = AB$.

Example

Given $T(z)=\dfrac{2z-1}{z+2}$ and $S(z)=\dfrac{z-i}{iz-1}$ find $S^{-1}(T(z))$

Solution Let $S^{-1}(T(z)) \Leftrightarrow A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A=adj\begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} -2+i & -1+2i \\ 1-2i & 2+i \end{bmatrix}$$

$$\boxed{S^{-1}(T(z))=\frac{(-2+i)z+(1+2i)}{(1-2i)z+(2+i)}}$$

Triples to Triples

The linear fractional transformation $T(z)$ given by

$$T(z) = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}$$

has a zero at $z = z_1$, a pole at $z = z_3$ and $T(z_2) = 1$.
Thus $T(z)$ maps three distinct complex numbers z_1, z_2, z_3 to $0, 1, \infty$, respectively.

The term $\dfrac{z - z_1}{z - z_3} \dfrac{z_2 - z_3}{z_2 - z_1}$ is called the cross ratio of of the points z, z_1, z_2, z_3.

Triples to Triples

Likewise, The linear fractional transformation given by

$$S(w) = \frac{w - w_1}{w - w_3}\frac{w_2 - w_3}{w_2 - w_1}$$

maps w_1, w_2, w_3 to 0, 1, ∞, and therefore
S^{-1} maps 0, 1, ∞ to w_1, w_2, w_3.

It follows that $w = S^{-1}(T(z))$ maps the triple z_1, z_2, z_3 to the triples w_1, w_2, w_3.

From $w = S^{-1}(T(z))$, we have $S(w) = T(z)$ and

$$\boxed{\frac{w - w_1}{w - w_3}\frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3}\frac{z_2 - z_3}{z_2 - z_1}}$$

Example-1

Construct a linear fractional transformation that maps the points ∞, 0, 1 on the real x-axis to the points 1, i, -1 on the circle $|w| = 1$.

Solution

Since $z_1 = \infty$, the terms $z - z_1$ and $z_2 - z_1$ in the cross-product are replaced by 1. Then

$$\left(\frac{w-1}{w+1}\right)\left(\frac{i+1}{i-1}\right)=\left(\frac{1}{z-1}\right)\left(\frac{0-1}{1}\right)$$

$$S(w)=\frac{-iw+i}{w+1}=\frac{-1}{z-1}=T(z)$$

Example continued...

If we use the matrix method to find $w = S^{-1}(T(z))$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \operatorname{adj}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -i & -1+i \\ -i & 1+i \end{bmatrix}$$

$$w = \frac{-iz-1+i}{-iz+1+i} = \frac{z-1-i}{z-1+i}.$$

Example-2

Find the bilinear transformation which maps the points 0,1,∞ onto the points -5,-1,3 respectively

Solution: Here $z_1 = 0, z_2 = 1, z_3 = \infty$ so that $1/z_3 = 0,\ w_1 = -5, w_2 = -1,\ w_3 = 3$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z-z_1)\left(\dfrac{z_2}{z_3}-1\right)}{\left(\dfrac{z}{z_3}-1\right)(z_2-z_1)}$$

$$\frac{(w+5)(-4)}{(w-3)4} = \frac{(z-0)(-1)}{(-1)(1-0)}$$

$$\Rightarrow z = \frac{3z-5}{z+1}$$

This is the required bilinear transformation

Session Summary

- A transformation T(z) : z → w is called bilinear if it takes the form w = T(z) = (az + b)/ (cz + d)
- Bilinear transformations preserve the cross ratio of four points

$$\boxed{\frac{w-w_1}{w-w_3}\frac{w_2-w_3}{w_2-w_1}=\frac{z-z_1}{z-z_3}\frac{z_2-z_3}{z_2-z_1}}$$