Lecture 9
Newton Raphson Method

Dr. Mahesha Narayana

Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Illustrate the steps involved in Newton-Raphson method
- Analyze the rate of convergence of the Newton-Raphson method

Topics

- Newton-Raphson method
- Convergence of Newton-Raphson method
- Drawbacks of Newton-Raphson method
- MATLAB Program

Motivation: Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.

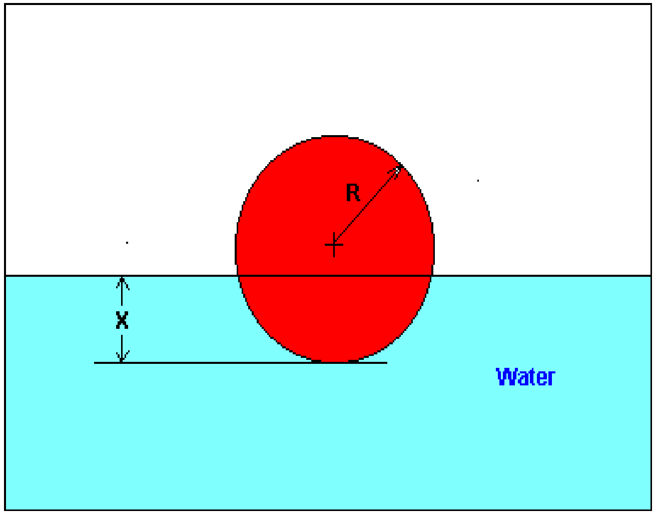


Figure 3 Floating ball problem.

Example 1 cond…

- According to Newton's third law of motion, every action has an equal and opposite reaction. In this case, the weight of the ball is balanced by the buoyancy force (Figure 1)
- Weight of ball = Buoyancy force (1)
- The weight of the ball is given by

Weight of ball = (Volume of ball) x (Density of ball) x (Acceleration due to gravity)

$$= \left(\frac{4}{3}\pi R^3\right)\rho_b g \qquad (2)$$

- where R is radius of ball (m), ρ_b is density of ball (kg/m^3), g is acceleration due to gravity (m/s^2)

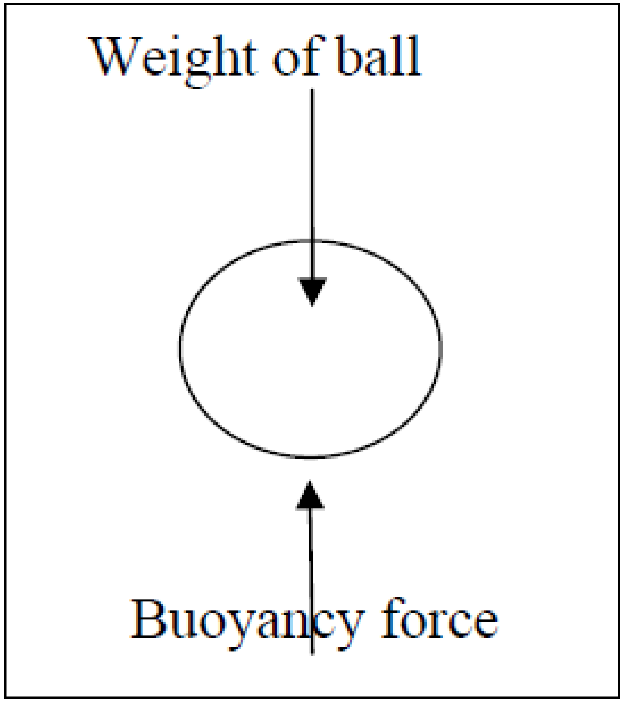
- The buoyancy force is given by

Buoyancy force = weight of water displaced

= (volume of ball under water)x(density of water)x(Acceleration due to gravity)

$$= \pi x^2\left(R - \frac{x}{3}\right)\rho_w g \qquad (3)$$

Example 1 cond.



- where x is depth to which ball is submerged
- Now substituting Equations (2) and (3) in Equation (1)
- We get

$$4R^3 \frac{\rho_b}{\rho_w} - 3x^2R + x^3 = 0$$

- where $\frac{\rho_b}{\rho_w}$ is specific gravity of the ball.
- Given $R = 0.055$ and $\frac{\rho_b}{\rho_w} = 0.6$
- We get

$$x^3 - 0.165\,x^2 + 3.993 \times 10^{-4} = 0$$

Or

$$f(x) = x^3 - 0.165\,x^2 + 3.993 \times 10^{-4}$$

Example 1 Cont.

The equation that gives the depth *x* in meters to which the ball is submerged under water is given by

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

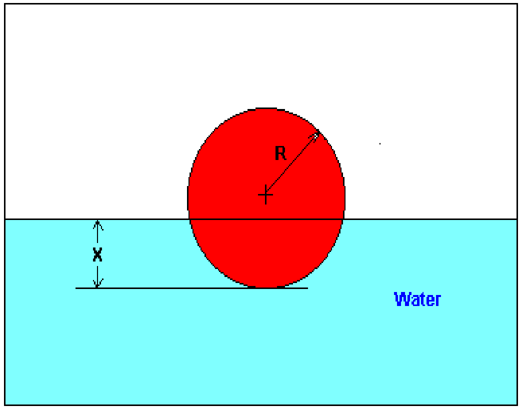


Figure 3 Floating ball problem.

Use the Newton's method of finding roots of equations to find

a) the depth 'x' to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.

b) The absolute relative approximate error at the end of each iteration, and

c) The number of significant digits at least correct at the end of each iteration.

Newton-Raphson method

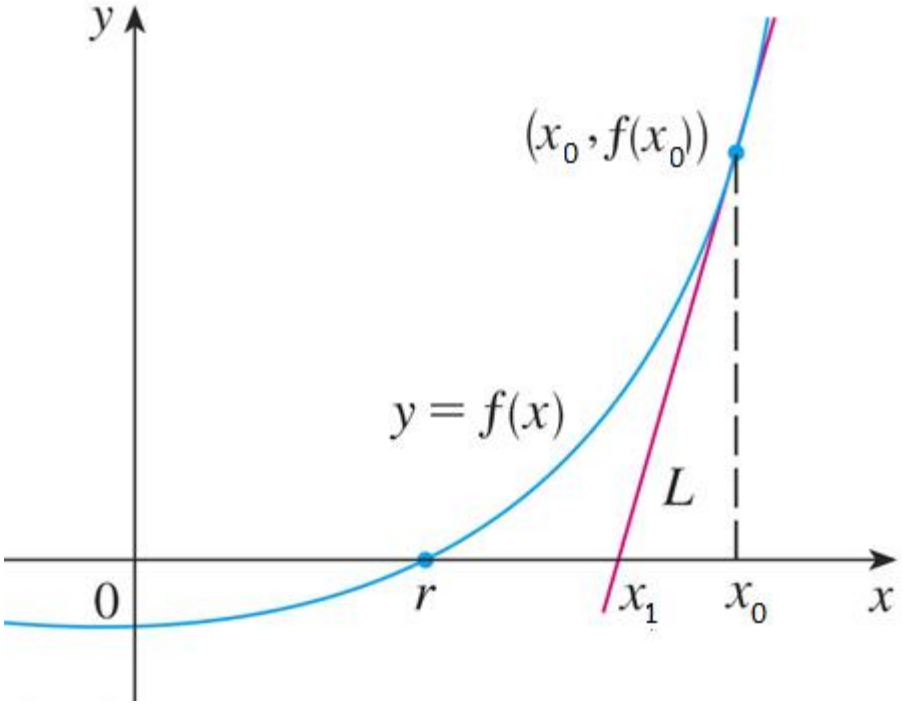
- We will explain how the method works, for two reasons:
  - To show what happens inside a calculator or computer
  - As an application of the idea of linear approximation

Newton-Raphson method

- The geometry behind Newton-Raphson's method is shown here. r represents the root of the equation

- We start with a first approximation x_0 , which is obtained by one of the following methods:

  – Guessing
  – A rough sketch of the graph of f
  – A computer-generated graph of f

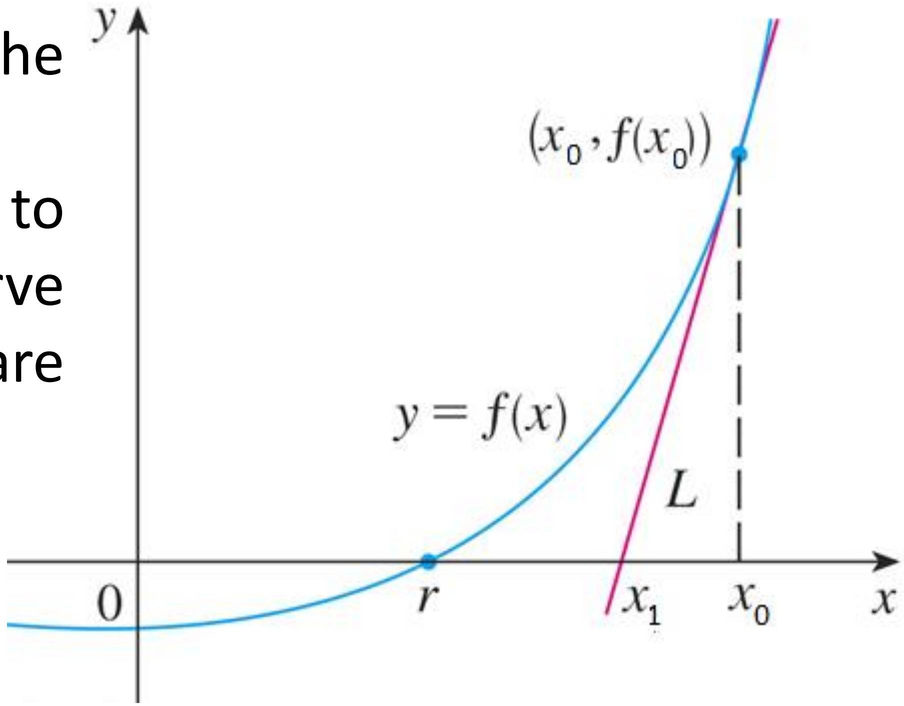


Newton-Raphson method

- Consider the tangent line *L* to the curve $y = f(x)$ at the point $(x_0, f(x_0))$ and look at the *x*-intercept of *L*, labeled x_1

Here's the idea behind the method

- The tangent line is close to the curve
- So, its *x*-intercept, x_1, is close to the *x*-intercept of the curve (namely, the root *r* that we are seeking)
- As the tangent is a line, we can easily find its *x*-intercept



Newton-Raphson method

- To find a formula for x_1 in terms of x_0, we use the fact that the slope of L is $f'(x_0)$.

- So, its equation of tangent line is:

$$y - f(x_0) = f'(x_0)(x - x_0)$$

As the x-intercept of L is x_2, we set $y = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This is first approximation

Second approximation

- As the x-intercept of L is x_2, we set $y = 0$ and obtain:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

- If $f'(x_1) \neq 0$, we can solve this equation for x_2:

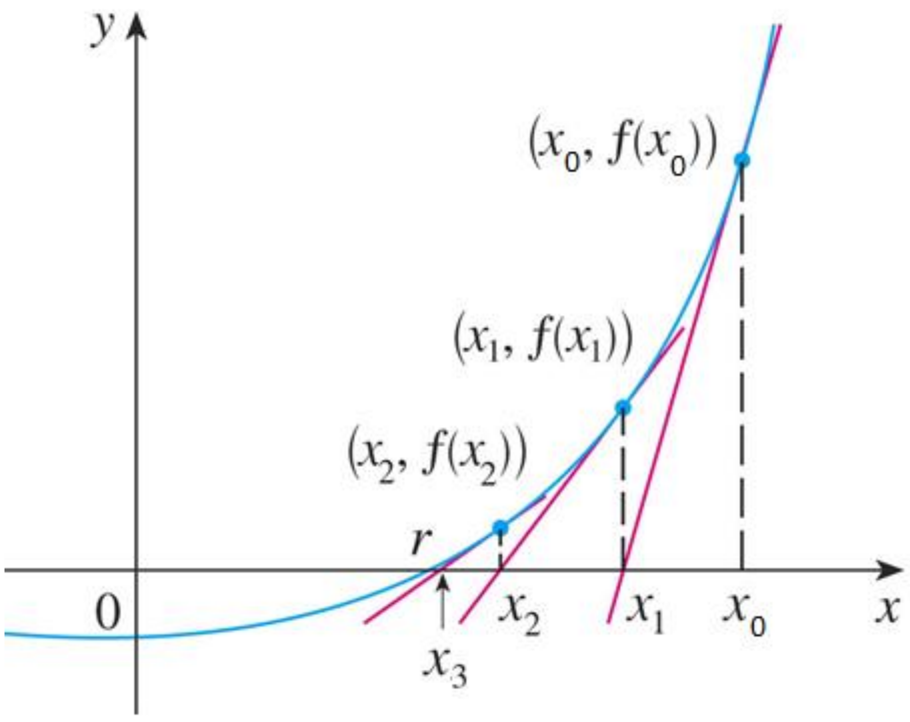
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

  – We use x_2 as a second approximation to r

Third approximation

- Next, we repeat this procedure with x_2 replaced by x_3, using the tangent line at $(x_2, f(x_2))$
  - This gives a third approximation:

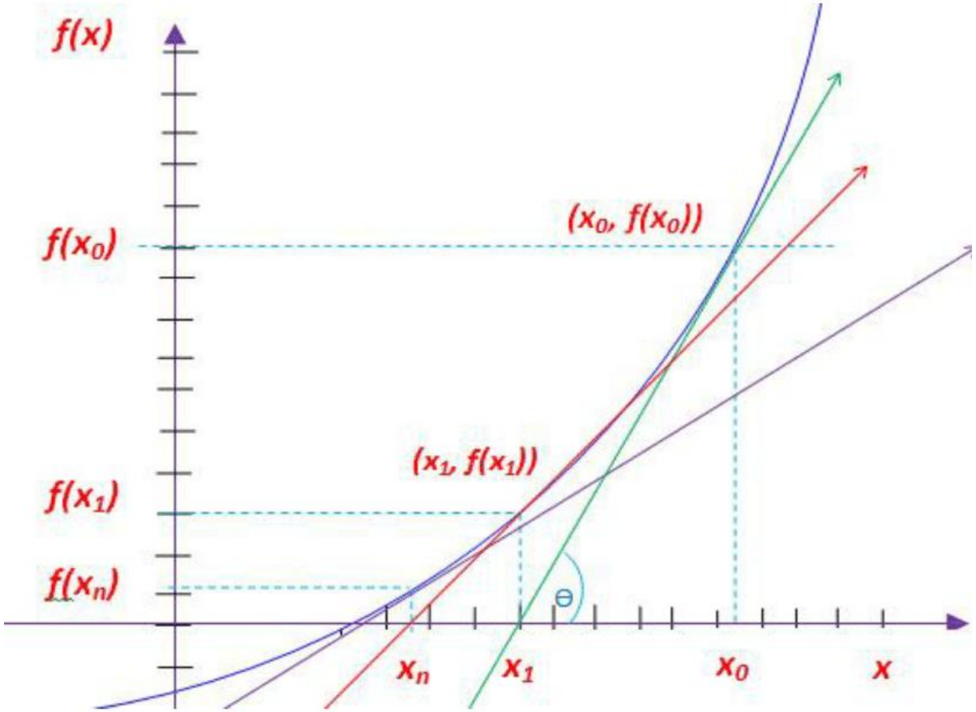
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$



Successive approximation

- If we keep repeating this process, we obtain a sequence of approximations $x_0, x_1, x_2, x_3, \ldots$ In general, if the nth approximation is x_n and $f'(x_n) \neq 0$, then the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Newton- Raphson method

- For the derivation of the formula used for solving a one-dimensional problem, we simply make a first-order Taylor series expansion of the function F(x)

$$f(x+h) = f(x) + hf'(x) \qquad (1)$$

Let us use the following notation for the x-values:

$$x_k = x, \quad x_{k+1} = x + h$$

Then, eq. (1) may be rewritten as

$$f(x_{k+1}) = f(x_k) + (x_{k+1} - x_k) f'(x_k)$$

Solving for x_{k+1} ,$f(x_{k+1})=0$

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

This general Newton-Raphson method to find roots of a equation

Convergence

- If the numbers x_n become closer and closer to r as n becomes large, then we say that the sequence converges to r and we write:

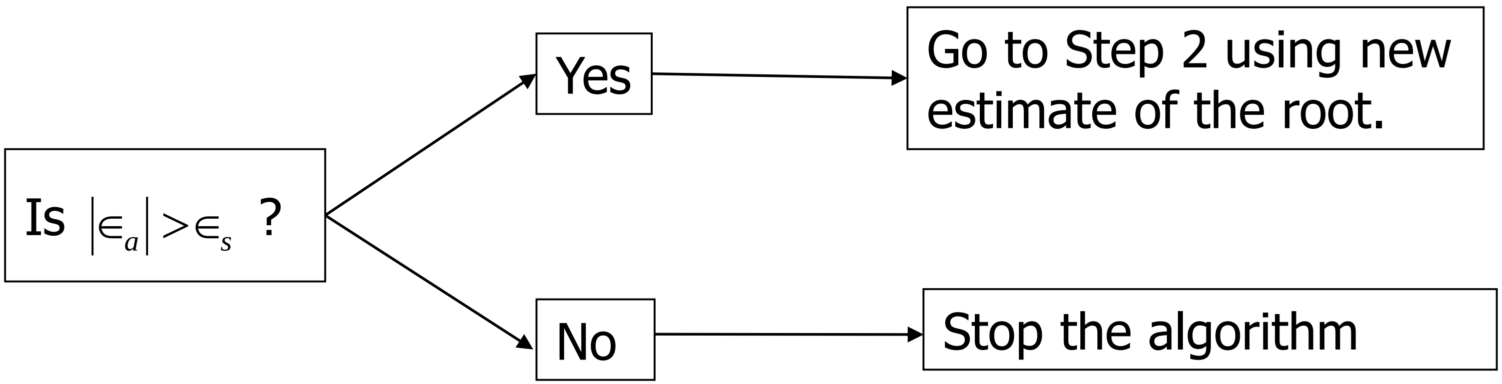
$$\lim_{n \to \infty} x_n = r$$

Let $\varepsilon > 0$. then the sequence x_n is said to be convergence if

$$\varepsilon_a = \left| \frac{x_{n+1} - x_n}{x_{n+1}} \right| < \varepsilon$$

Convergence…

Compare the absolute relative approximate error with the pre-specified relative error tolerance $\in_s$.



Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

Convergence of Newton-Raphson method

Suppose x_r is a root of $f(x)=0$ and x_n is an estimate of x_r such that

$$\left|x_r - x_n\right| = \delta << 1$$

Then by Taylor expansion we have,

$$0 = f(x_r) = f(x_n + \delta) = f(x_n) + f'(x_n)(x_r - x_n) + \frac{f''(\xi)}{2}(x_r - x_n)^2 \qquad (1)$$

For some ξ between x_r and x_n

Now by Newton method, we know that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \Rightarrow f(x_n) = f'(x_n)\left(x_n - x_{n+1}\right) \qquad (2)$$

Convergence of Newton-Raphson method...

Using (2) in (1), we get

$$0 = f'(x_n)(x_r - x_{n+1}) + \frac{f''(\xi)}{2}(x_r - x_n)^2 \qquad (3)$$

Let $e_{n+1} = (x_r - x_{n+1})$ and $e_n = x_r - x_n$

Where e_n and e_{n+1} denotes the error in the solution at n^{th} and $(n+1)^{th}$ iterations.

$$\therefore \quad e_{n+1} = -\frac{f''(\xi)}{2(x_n)} \sim e_n^2$$

$$\Rightarrow e_{n+1} \propto e_n^2$$

Hence Newton-Raphson method is said to have quadratic convergence

Convergence…

- The sequence of successive approximations converges to the desired root for functions of the type illustrated in the previous figure

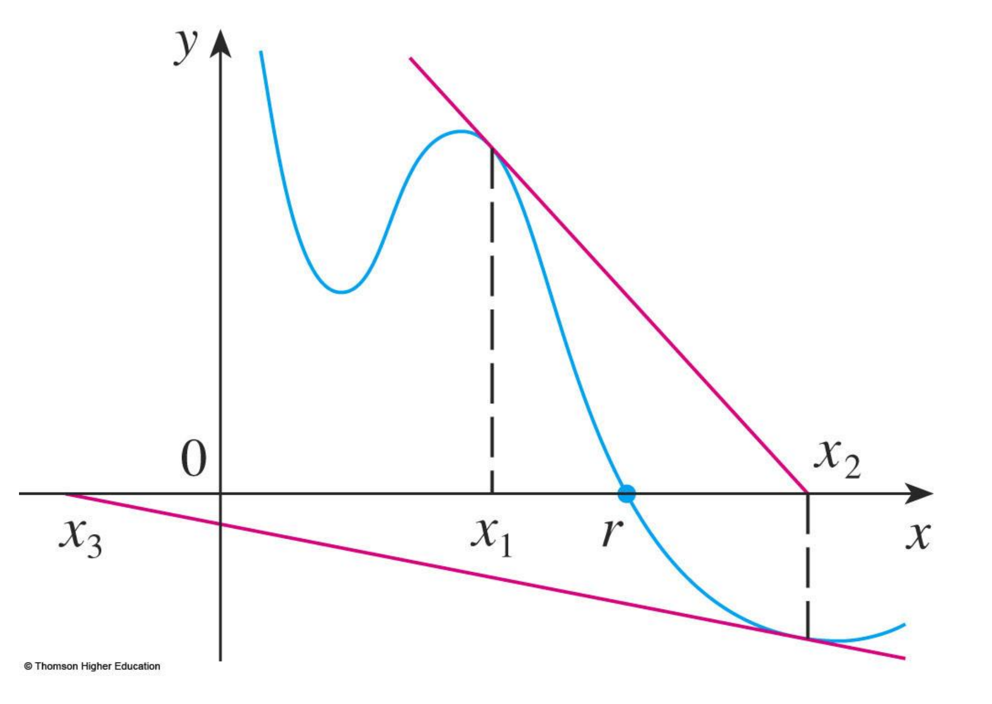
- However, in certain circumstances, it may not converge

Non-convergence

- Consider the situation shown here

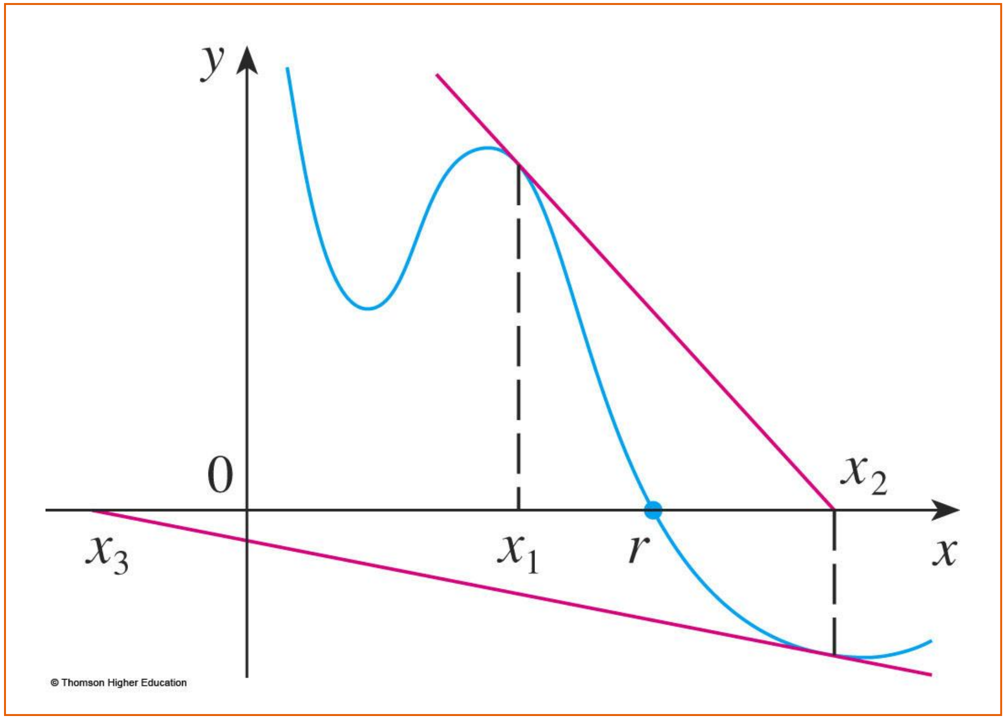
- You can see that x_2 is a worse approximation than x_1

  - This is likely to be the case when $f'(x_1)$ is close to 0



Non-convergence

- It might even happen that an approximation falls outside the domain of *f*, such as x_3
  - Then, Newton's method fails
  - In that case, a better initial approximation x_1 should be chosen



Drawbacks

1. Divergence at inflection points
   Selection of the initial guess or an iteration value of the root that is close to the inflection point of the function $f(x)$ may start diverging away from the root in ther Newton-Raphson method

For example, to find the root of the equation

$$f(x) = (x-1)^3 + 0.512 = 0$$

The Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$$

Drawbacks…

- Table 1 shows the iterated values of the root of the equation

- The root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$

- Eventually after 12 more iterations the root converges to the exact value of $x=0.2$

Drawbacks – Inflection Points

Table 1 Divergence near inflection point

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	−30.119
7	−19.746
18	0.2000

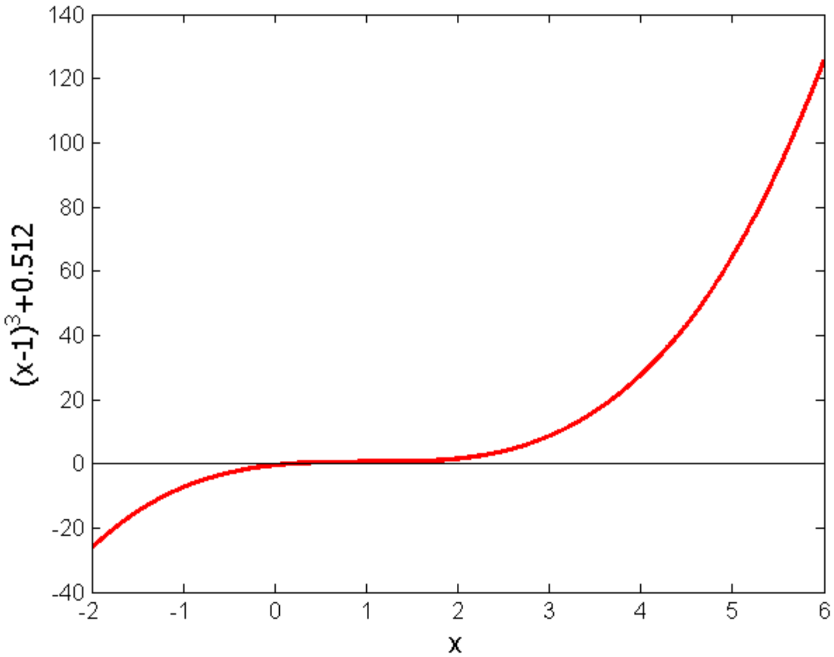


Figure 8 Divergence at inflection point for $f(x)=(x-1)^3+0.512=0$

Drawbacks – Division by Zero

2. Division by zero

For the equation

$$f(x) = x^3 - 0.03x^2 + 2.4\times10^{-6} = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{x_i^3 - 0.03x_i^2 + 2.4\times10^{-6}}{3x_i^2 - 0.06x_i}$$

For $x_0 = 0 \text{ or } x_0 = 0.02$, the denominator will equal zero.

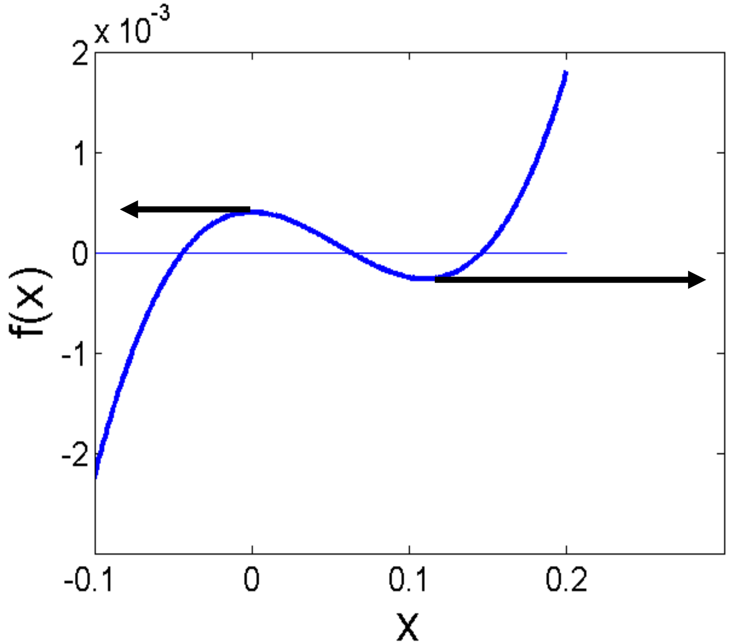


Figure 9 Pitfall of division by zero or near a zero number

Drawbacks – Oscillations near local maximum and minimum

3. **Oscillations near local maximum and minimum**

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum.

Eventually, it may lead to division by a number close to zero and may diverge.

For example for $f(x) = x^2 + 2 = 0$ the equation has no real roots.

Drawbacks – Oscillations near local maximum and minimum…..

Table 3 Oscillations near local maxima and minima in Newton-Raphson method.

Iteration Number	x_i	$f(x_i)$	$\lvert\in_a\rvert\%$
0	–1.0000	3.00	
1	0.5	2.25	300.00
2	–1.75	5.063	128.571
3	–0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	–0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96

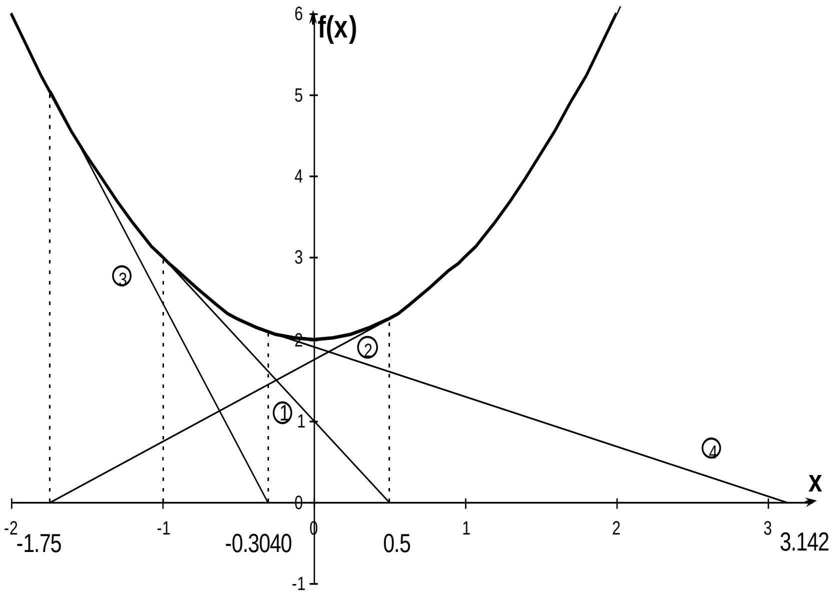


Figure 10 Oscillations around local minima for $f(x) = x^2 + 2$

Drawbacks – Root Jumping

4. **Root Jumping**
In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

For example
$$f(x) = \sin x = 0$$

Choose $x = 2\pi = 6.2831853$

It will converge to
$$x_0 = 2.4\pi = 7.539822$$
instead of $x = 0$

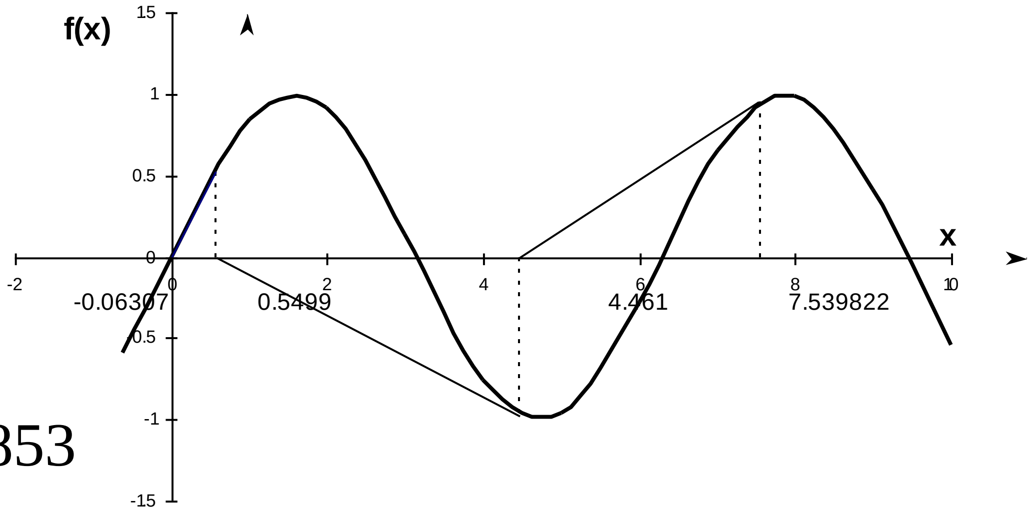


Figure 11 Root jumping from intended location of root for $f(x) = \sin x = 0$

Newton-Raphson method

Example 1: find the root of the equation

$$x^3 - 2x - 5 = 0$$

We apply Newton-Raphson method with

$$f(x) = x^3 - 2x - 5, \quad f'(x) = 3x^2 - 2$$

Newton himself used this equation to illustrate his method

We chose $x_0 = 2$ after some experimentation because

$$f(1) = -6, \quad f(2) = -1, \quad f(3) = 16$$

Thus the roots of equation lie between x=2 and x=3

Example 1...

The Newton-Raphson method to find the roots of equation is given as

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \qquad (1)$$

We write the problem in this form

$$x_n = x_{n-1} - \frac{x_n^3 - 2x_{n-1} - 5}{3x_{n-1}^2 - 2} \qquad (2)$$

First approximation: put n=1

$$x_1 = x_0 - \frac{x_0^3 - 2x_0 - 5}{3x_0^2 - 2}$$

Put $x_0 = 2$

$$\boxed{x_1 = 2.1}$$

Example 1…

Second approximation: put n=2

$$x_2 = x_1 - \frac{x_1^3 - 2x_1 - 5}{3x_1^2 - 2}$$

Put x_1 = 2.1

$$\boxed{x_2 = 2.0496}$$

Third approximation: put n=3

$$x_3 = x_2 - \frac{x_2^3 - 2x_2 - 5}{3x_2^2 - 2}$$

Put x_2 = 2.0496

$$\boxed{x_3 = 2.096}$$

Example 1…

$$x_4 = 2.095$$

$$x_5 = 2.09452$$

$$x_6 = 2.094567$$

$$x_7 = 2.094551$$

$$x_8 = 2.094551$$

$$x_9 = 2.094551$$

We conclude that the roots of the non-linear equation is 2.094551

Example 1...

- Suppose that we want to achieve a given accuracy—say, to eight decimal places—using Newton's method

  – How do we know when to stop?

- The rule of thumb that is generally used is that we can stop when successive approximations x_n and x_{n+1} agree to eight decimal places

- Notice that the procedure in going from n to $n + 1$ is the same for all values of n

- It is called an iterative process

Example 2

- Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.
  - First, we observe that finding $\sqrt[6]{2}$ is equivalent to finding the positive root of the equation $x^6 - 2 = 0$
  - So, we take $f(x) = x^6 - 2$
  - Then $f'(x) = 6x^5$

Example 2…

- So, Formula 2 (Newton's method) becomes:

$$x_n = x_{n-1} - \frac{x_n^6 - 2}{6x_n^5}$$

Choosing $x_1 = 1$ as the initial approximation, we obtain:

$$x_2 \approx 1.6666667$$

$$x_3 \approx 1.12644368$$

$$x_4 \approx 1.12249707$$

$$x_5 \approx 1.12246205$$

$$x_6 \approx 1.12246205$$

As x_5 and x_6 agree to eight decimal places, we conclude that $\sqrt[6]{2} \approx 1.12246205$ to eight decimal places

Example 3

- Find, correct to six decimal places, the root of the equation

  $\cos x = x$

  - We rewrite the equation in standard form: $\cos x - x = 0$

  - Therefore, we let $f(x) = \cos x - x$
  - Then, $f'(x) = -\sin x - 1$

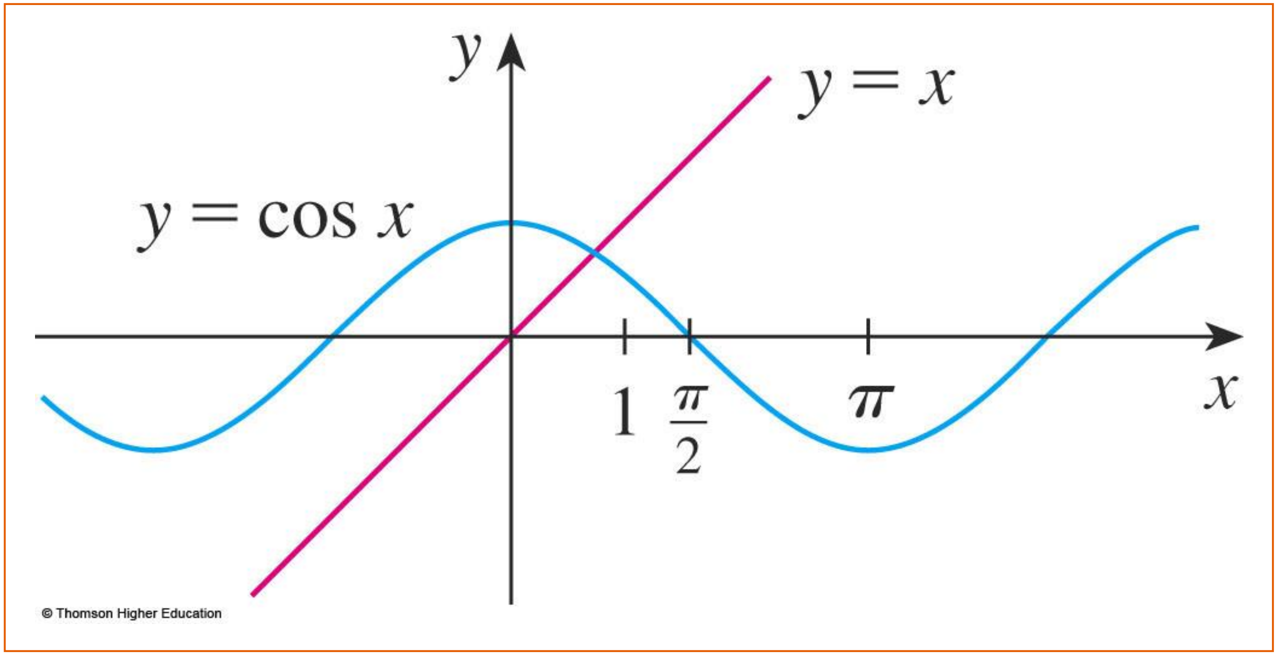
Then by Newton Raphson method

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1}$$

$$= x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

Example 3…

- To guess a suitable value for x_1, we sketch the graphs of $y = \cos x$ and $y = x$
  - It appears they intersect at a point whose x-coordinate is somewhat less than 1



Example 3...

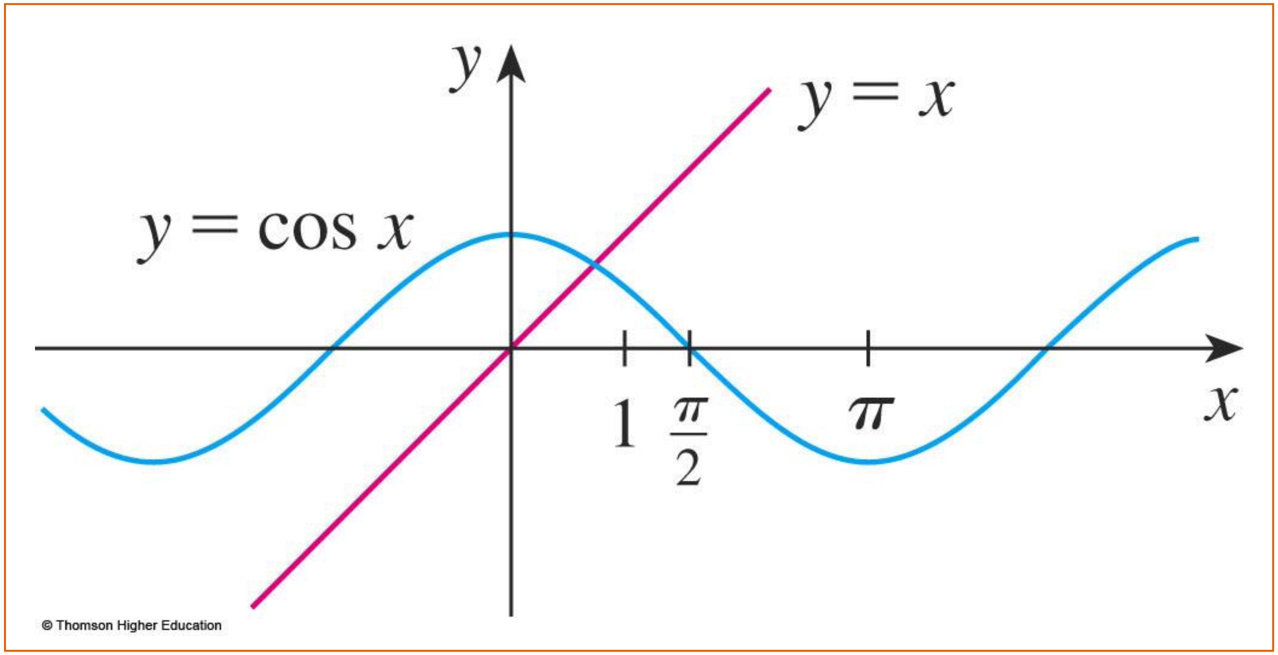
- So, let's take $x_1 = 1$ as a convenient first approximation
  - Then, remembering to put our calculator in radian mode, we get:

$$x_2 \approx 0.75036387$$
$$x_3 \approx 0.73911289$$
$$x_4 \approx 0.73908513$$
$$x_5 \approx 0.73908513$$

  - As x_4 and x_5 agree to six decimal places (eight, in fact), we conclude that the root of the equation, correct to six decimal places, is 0.739085

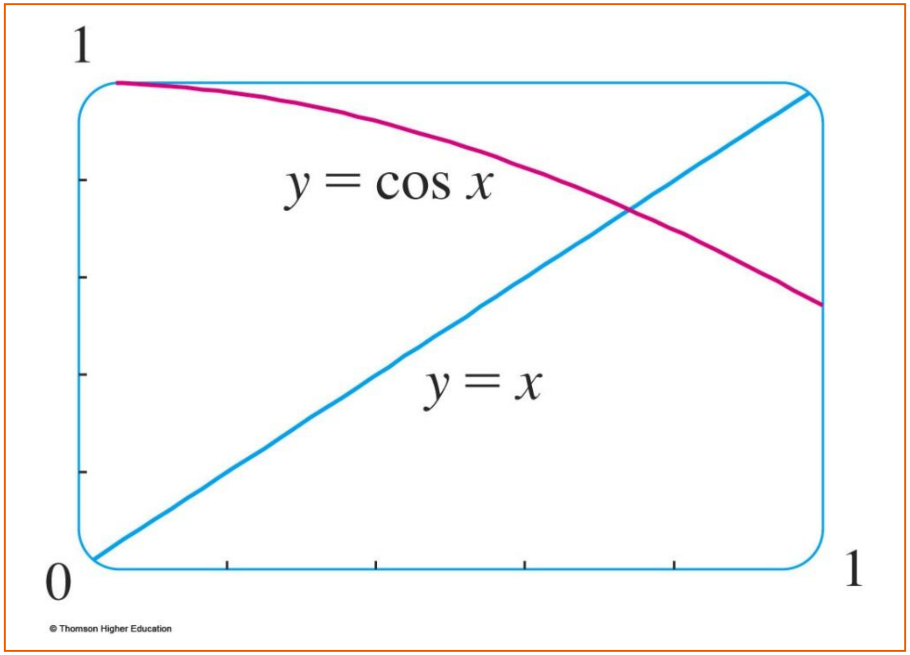
Example 3…

- Instead of using this rough sketch to get a starting approximation for the method in the example, we could have used the more accurate graph that a calculator or computer provides



Example 3...

- This figure suggests that we use $x_1 = 0.75$ as the initial approximation



Example 3…

- Then, Newton's method gives:

$$x_2 \approx 0.73911114$$

$$x_3 \approx 0.73908513$$

$$x_4 \approx 0.73908513$$

  – So we obtain the same answer as before—but with one fewer step

Newton's method Vs. Graphic devices

- You might wonder why we bother at all with Newton's method if a graphing device is available
  - Isn't it easier to zoom in repeatedly and find the roots as we did in Section 1.4?
  - If only one or two decimal places of accuracy are required, then indeed the method is inappropriate and a graphing device suffices
  - However, if six or eight decimal places are required, then repeated zooming becomes tiresome

Newton’s method Vs. Graphic devices…

- It is usually faster and more efficient to use a computer and the method in tandem

- You start with the graphing device and finish with the method

Failure of Newton-Raphson

- **Poor choice of starting value**

  If your initial starting value is not close to the root or near a turning point it may diverge away from the root. It may converge on another root, but this is classed as failure if it is not the root you wanted to find

Matlab Code

```
function [x0,err] = newraph(x0)
  maxit = 100;
  tol = 10^(-6);
  err = 100;
  Numit = 0;
  xold =x0;
  while (err > tol && Numit <= maxit)
  Numit = Numit + 1;
  f = funkeval(xold);
  df = dfunkeval(xold);
  xnew = xold - f/df;
  if (Numit > 1)
  err = abs((xnew - xold)/xnew);
  end
  fprintf('Numit = %f \t f = %f \t xnew = %f \t err = %f \n',Numit, f, xnew, err);
  xold = xnew;
  end
```

Matlab Code

```
   x0 = xnew;
   if (Numit >= maxit)
   % you ran out of iterations
   fprintf('Sorry. You did not converge in %i
iterations.\n',maxit);
   fprintf('The final value of x was %e \n', x0);
end

 function f = funkeval(x)
 f = 5*cos(x)-4*log(x+1)+x^2;
 function df = dfunkeval(x)
 df = -5*sin(x)-4/(x+1)+2*x;
```

Session Summary

- Newton-Raphson method is given by $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$
- If the algorithm converges, the convergence rate is faster and is of quadratic convergence
- Disadvantages: The algorithm will diverge
  - If the initial guess is close to an inflection point
  - If there are oscillations near local extremum
  - If the function oscillates and has a number of roots