Lecture 19
Lagrange's Method of Multipliers_I

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Intended learning Outcomes

At the end of this lecture, student will be able to:

- State and explain Lagrange Method of Multipliers
- Apply the Lagrange Method of Multipliers to maximize/minimize the given function subject to equality constraints

Topics

- Lagrange method of multipliers
- Maximum and minimum by Lagrange method of multipliers
- Examples

Motivation

- A function of two variables is to be optimized subject to a restriction or **constraint** on the variables

- We present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

- The method of Lagrange multipliers gives a set of necessary conditions to identify optimal points of <u>equality constrained</u> optimization problems

- This is done by converting a constrained problem to an equivalent unconstrained problem with the help of certain unspecified parameters known as <u>Lagrange multipliers.</u>

Lagrange Method of Multipliers

- The classical problem formulation
  - minimize $f(x, y)$
  - Subject to $g(x, y) = 0$
- can be converted to
  - minimize $F(x, ,y, \lambda) = f(x, y) - \lambda\, g(x, y)$
- where
  - $F(x, y, \lambda)$ is the Lagrangian function
  - λ is an unspecified positive or negative constant called the <u>Lagrangian Multiplier</u>

Lagrange Method of Multipliers

The Method of Lagrange Multipliers

To find a maximum or minimum value of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$:

- Form a new function:

$$F(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

- The variable λ (lambda) is called a Lagrange multiplier.

Lagrange Method of Multipliers

- Find the first partial derivatives F_x, F_y, and F_λ.

- Solve the system

$$F_x = 0, \quad F_y = 0, \quad \text{and} \quad F_\lambda = 0,$$

Let (a, b, λ) represent a solution of this system. We normally must determine whether (a, b, λ) yields a maximum or minimum of the function f.

NOTE: The method of Lagrange multipliers can be extended to functions of three (or more) variables.

Lagrange Method of Multipliers

Example 1: Find the maximum value of $A(x, y) = xy$ subject to the constraint $x + y = 20$.

First note that $x + y = 20$ is equivalent to $x + y - 20 = 0$.

Step 1. We form the new function, F, given by

$$F(x, y, \lambda) = xy - \lambda \cdot (x + y - 20).$$

Lagrange Method of Multipliers

Example 1 (continued):

Step 2. We find the first partial derivatives:

$F_x = y - \lambda$

$F_y = x - \lambda$

$F_\lambda = -(x + y - 20)$

Step 3. We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned} y - \lambda &= 0 \\ x - \lambda &= 0 \\ -(x + y - 20) &= 0 \end{aligned}$$

Lagrange Method of Multipliers

Example 1 (concluded):

From the first two equations, we can see that x = λ = y.

Substituting x for y in the last equation, we get

$$\begin{aligned} x + x - 20 &= 0 \\ 2x &= 20 \\ x &= 10 \end{aligned}$$

Thus, y = x = 10. The maximum value of A subject to the constraint occurs at (10, 10) and is

$$\begin{aligned} A(10,10) &= 10 \cdot 10 \\ &= 100 \end{aligned}$$

Example-2

Find the extreme values of $f(x,\ y,\ z) = 2x + 3y + z$ such that $x^2 + y^2 = 5$ and $x + z = 1$

Solution : Consider the auxiliary function

$F(x, y, z, \lambda_1, \lambda_2)$=$2x + 3y + z + \lambda_1(x^2 + y^2 - 5) + \lambda_2\ (x + z - 1)$.

For the extreme , we have the necessary conditions

$$\frac{\partial F}{\partial x} = 2 + 2\lambda_1 \mathrm{x} + \lambda_2 = 0; \frac{\partial F}{\partial y} = 3 + 2\lambda_1 \mathrm{y} = 0; \frac{\partial F}{\partial z} = 1 + \lambda_2 = 0;$$

From these equations , we get

λ_2=-1, $3 + 2\lambda_1 y = 0$, and $2 + 2\lambda_1 \mathrm{x} + \lambda_2$

$\Longrightarrow \mathrm{x} = -1/2\lambda_1$ and $y = -3/2\lambda_1$

$\Longrightarrow \frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5$ or $\lambda_1^2 = \frac{1}{2}$ or $\lambda_1 = \frac{1}{\sqrt{2}}$

Example-2 (Cont.)

For $\lambda_1 = \frac{1}{\sqrt{2}} \Longrightarrow x = -\frac{\sqrt{2}}{2}, \quad y = -\frac{3\sqrt{2}}{2}, \quad z = 1 - x = \frac{(2+\sqrt{2})}{2}$

and $f(x, y, z) = -\sqrt{2} - \frac{9\sqrt{2}}{2} + \frac{(2+\sqrt{2})}{2}$ $=1 - 5\sqrt{2}$

For $\lambda_1 = -\frac{1}{\sqrt{2}}$, we get $x = -\frac{\sqrt{2}}{2} y = \frac{3\sqrt{2}}{2}, \quad z = 1 - x = \frac{(2-\sqrt{2})}{2}$

and $f(x, y, z) = 1 + 5\sqrt{2}$

Session Summary

•In mathematical optimization, the method of Lagrange multipliers is a strategy for finding the local maxima and minima of function subject to equality constraints.

- Procedure for Applying the Method of Lagrange Multipliers

Step 1. Write the problem in the form:

Maximize (minimize) $f(x, y)$ subject to $g(x, y) = k$

Step 2. Simultaneously solve the equations

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = k$$

Step 3. Evaluate f at all points found in step 2. If the required maximum (minimum) exists, it will be the largest (smallest) of these values.