Lecture 28
Conformal Mapping_2

Dr. Mahesha Narayana

Intended learning Outcomes

At the end of this lecture, student will be able to:

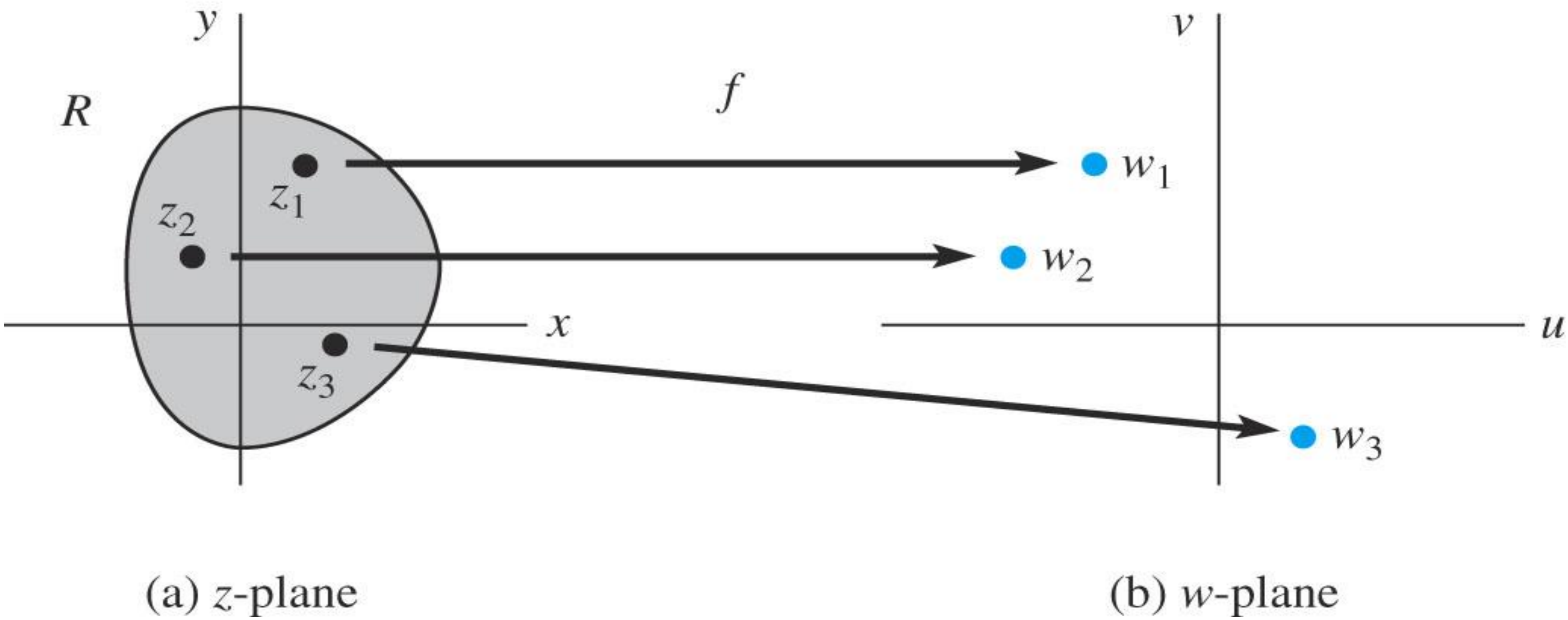
- Illustrate conformal mapping
- Discuss the properties of standard conformal mappings
- Solve application oriented problems using conformal mappings

Topics

- Complex function as mapping
- Exponential mapping
- Reciprocal function
- Translation and rotation
- Magnification
- Power function
- Conformal mapping

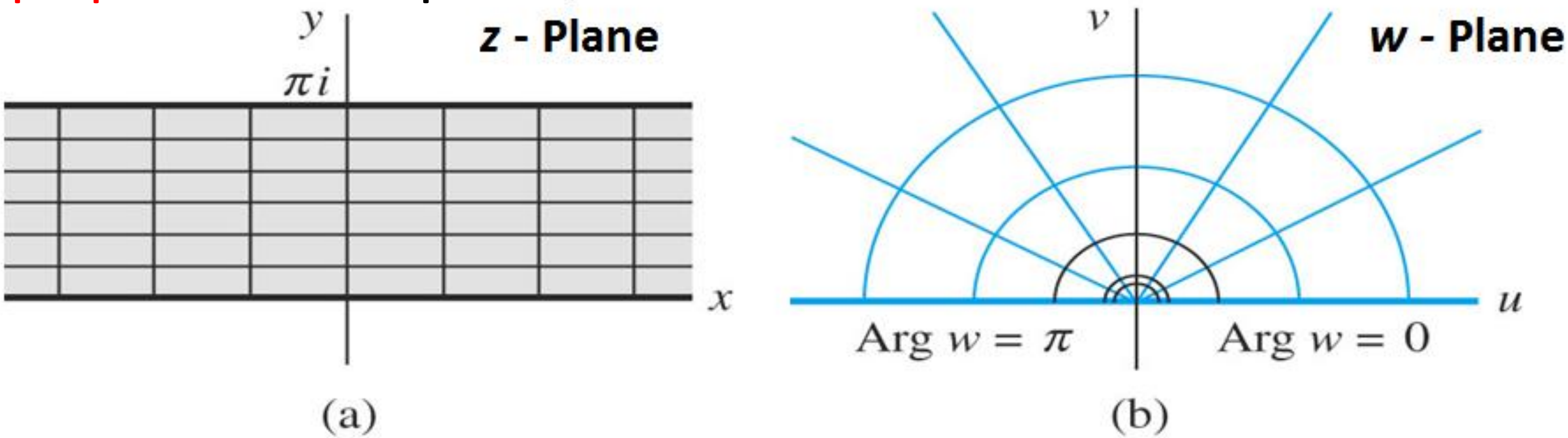
Complex Functions as Mapping

The complex function $w = f(z) = u(x, y) + iv(x, y)$ may be considered as the planar transformation. We also call $w = f(z)$ is the image of z under f.



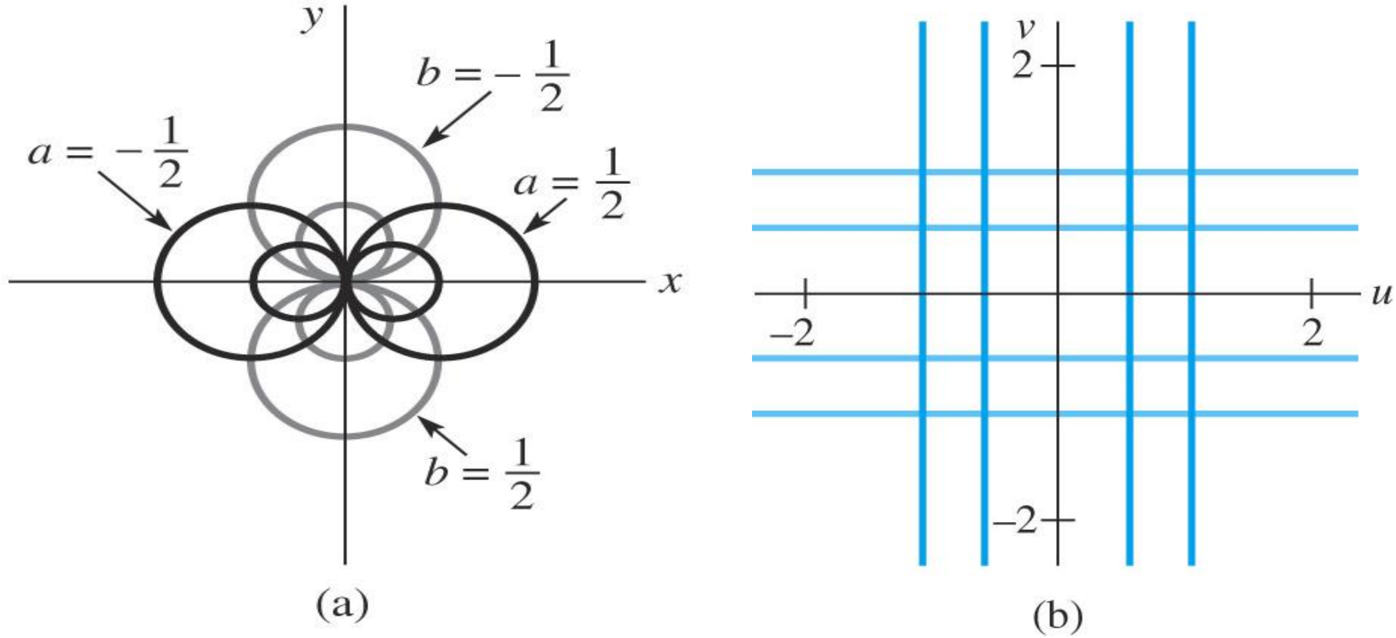
Exponential Mapping $f(z) = e^z$

- Consider $\mathrm{Re}(z) = \alpha$ (a constant) which represents lines parallel to imaginary axis, then $z = \alpha + iy$, $0 \le y < \infty$, $w = f(z) = e^{\alpha}\, e^{iy}$. This represents a semicircle with center $w = 0$ and radius $r = e^{\alpha}$ in w – plane.
- Consider $\mathrm{Im}(z) = \beta$ (*a* constant) which represents lines parallel to real axis, then $z = x + i\beta$, $-\infty \le x \le \infty$, $w = f(z) = e^{x} e^{i\beta}$. This represents a ray with $\mathrm{Arg}\ w = b$ and $|w| = e^{x}$ in w – plane.



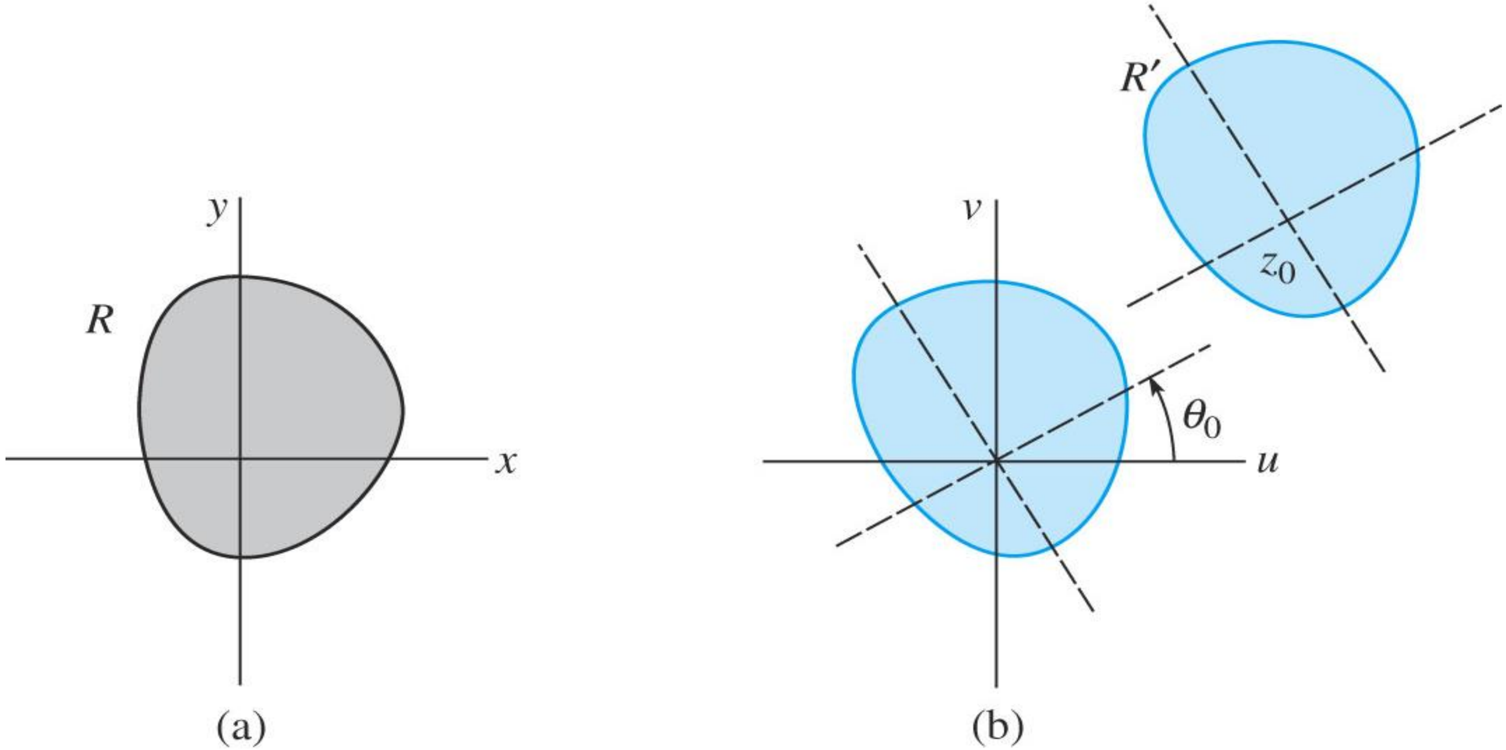
Reciprocal function $f(z) = 1/z$

- The function $f(z) = 1/z = u + iv$ has domain $z \neq 0$ with $u(x,y) = x/(x^2+y^2)$ and $v(x,y) = -y/(x^2+y^2)$.
- For $a \neq 0$, $u(x, y) = a$ represents family of circles $(x - \alpha)^2 + y^2 = \alpha^2$ (where $\alpha = ½a$) in w – plane.
- For $b \neq 0$, $v(x, y) = b$ represents family of circles $x^2 + (y + \beta)^2 + = \beta^2$ (where $\beta = ½b$) in w – plane.



Translation and Rotation

- The function $f(z) = z + z_0$ is interpreted as a translation.
- The function $g(z) = e^{i\theta_0} z$ is interpreted as a rotation.



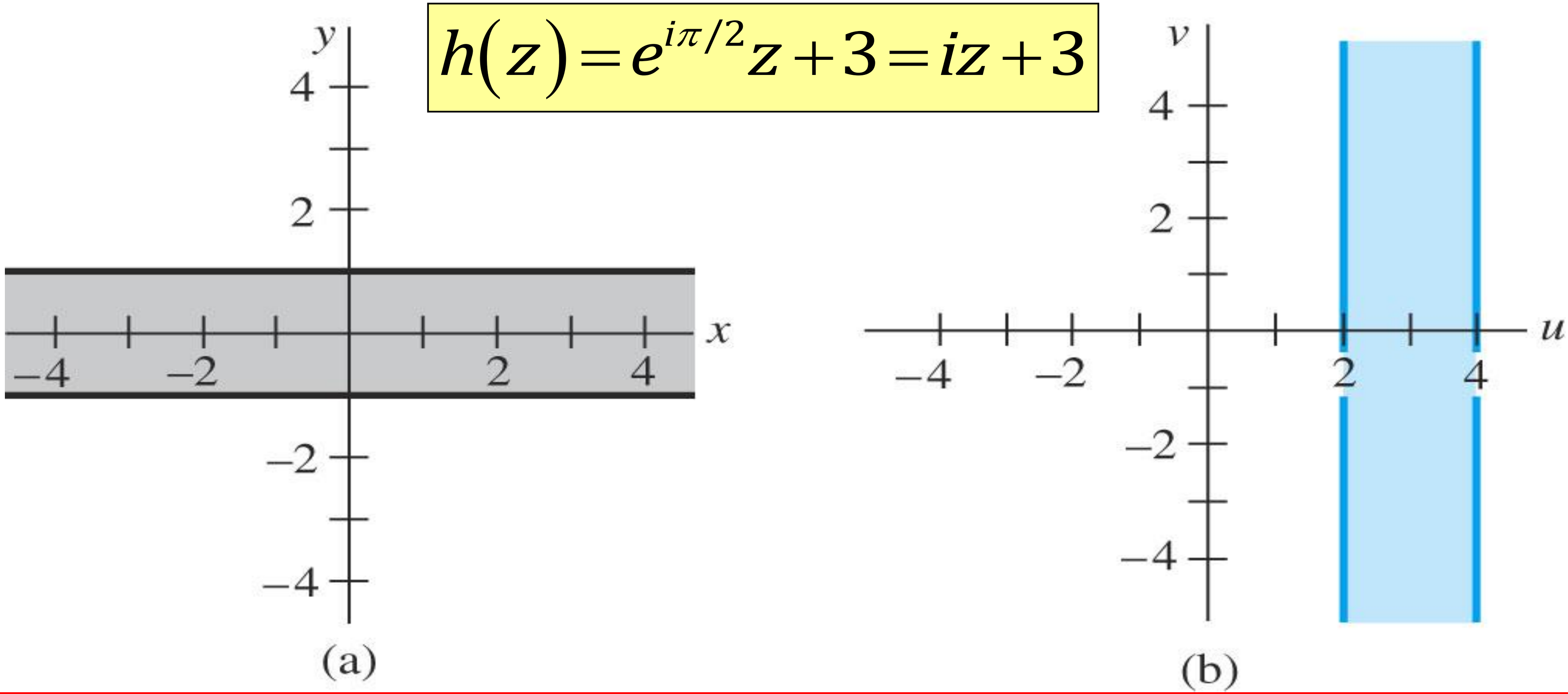
Example

Find a complex function that maps $-1 \le y \le 1$ onto $2 \le x \le 4$.

Solution

We find that $-1 \le y \le 1$ is first rotated through 90° and shifted 3 units to the right. Thus the mapping is

$$h(z) = e^{i\pi/2}z + 3 = iz + 3$$



Magnification

A magnification is the function $f(z) = \alpha z$, where α is a fixed positive real number. Note that $|w| = |\alpha z| = \alpha\, |z|$.

Examples

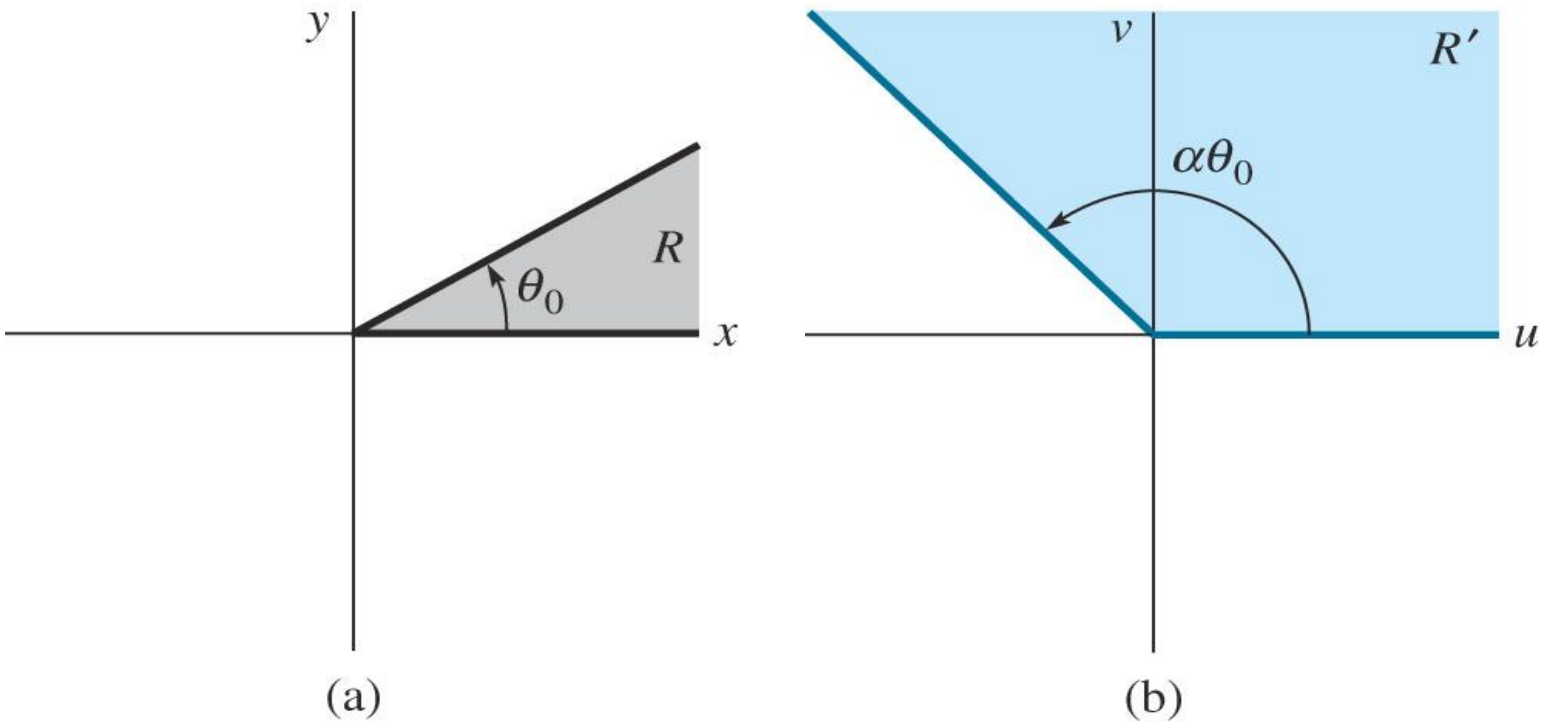
1. Consider $g(z) = az + b$ where $a = r_0 e^{i\theta_0}$ then the vector is rotated through θ_0, magnified by a factor r_0, and then translated using b

2. The function as $f(z) = \frac{1}{2}z + (1 + i)$ maps the disk $|z| \leq 1$ onto the disk $|w - (1 + i)| \leq \frac{1}{2}$.

Power Function

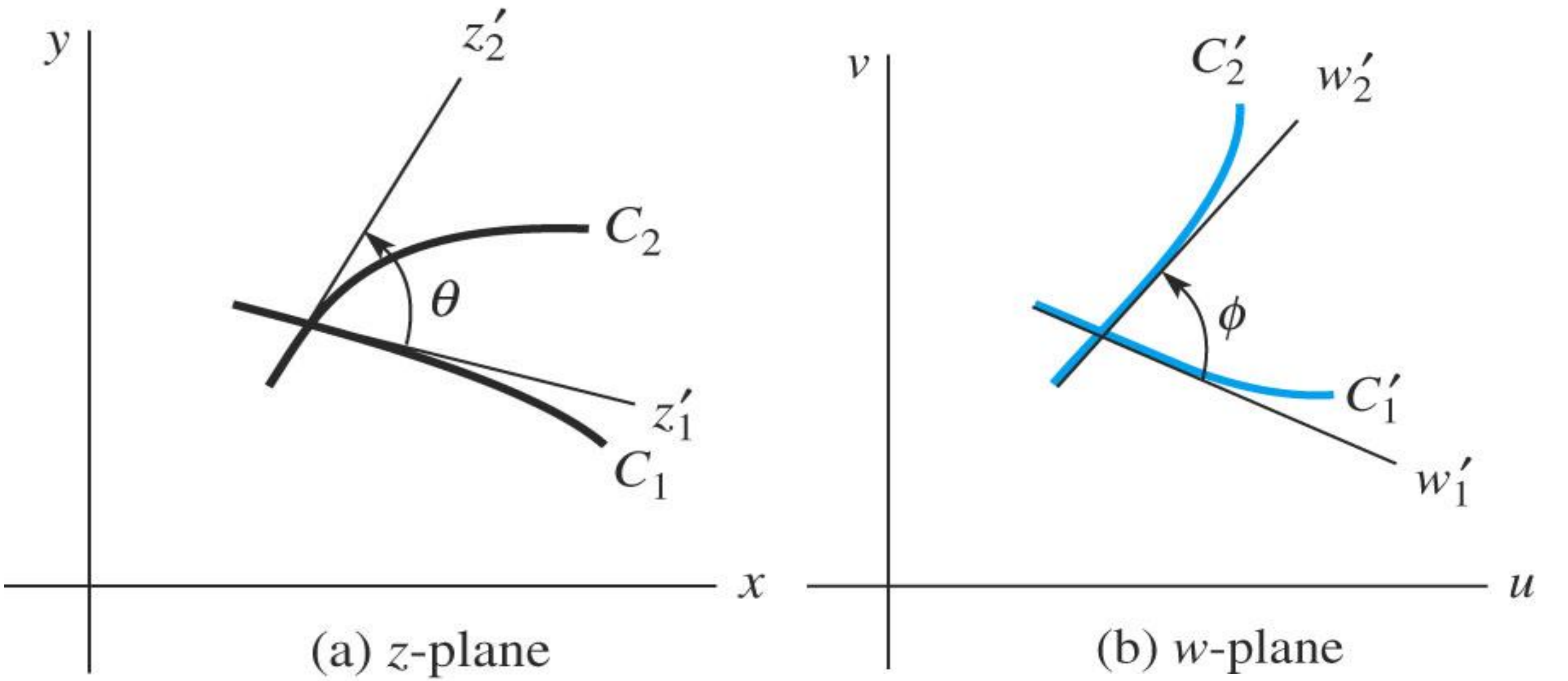
A complex function $f(z) = z^{\alpha}$ where α is a fixed positive number, is called a real power function.

If $z = re^{i\theta}$, then $w = f(z) = r^{\alpha}e^{i\alpha\theta}$.



Conformal Mappings

A complex mapping $w = f(z)$ defined on a domain D is called conformal t $z = z_0$ in D when $f(z)$ preserves the angle between two curves in D that intersect at the point $z = z_0$.



Important Result

If $f(z)$ is analytic in the domain D and $f'(z) \neq 0$, then f is conformal at $z = z_0$

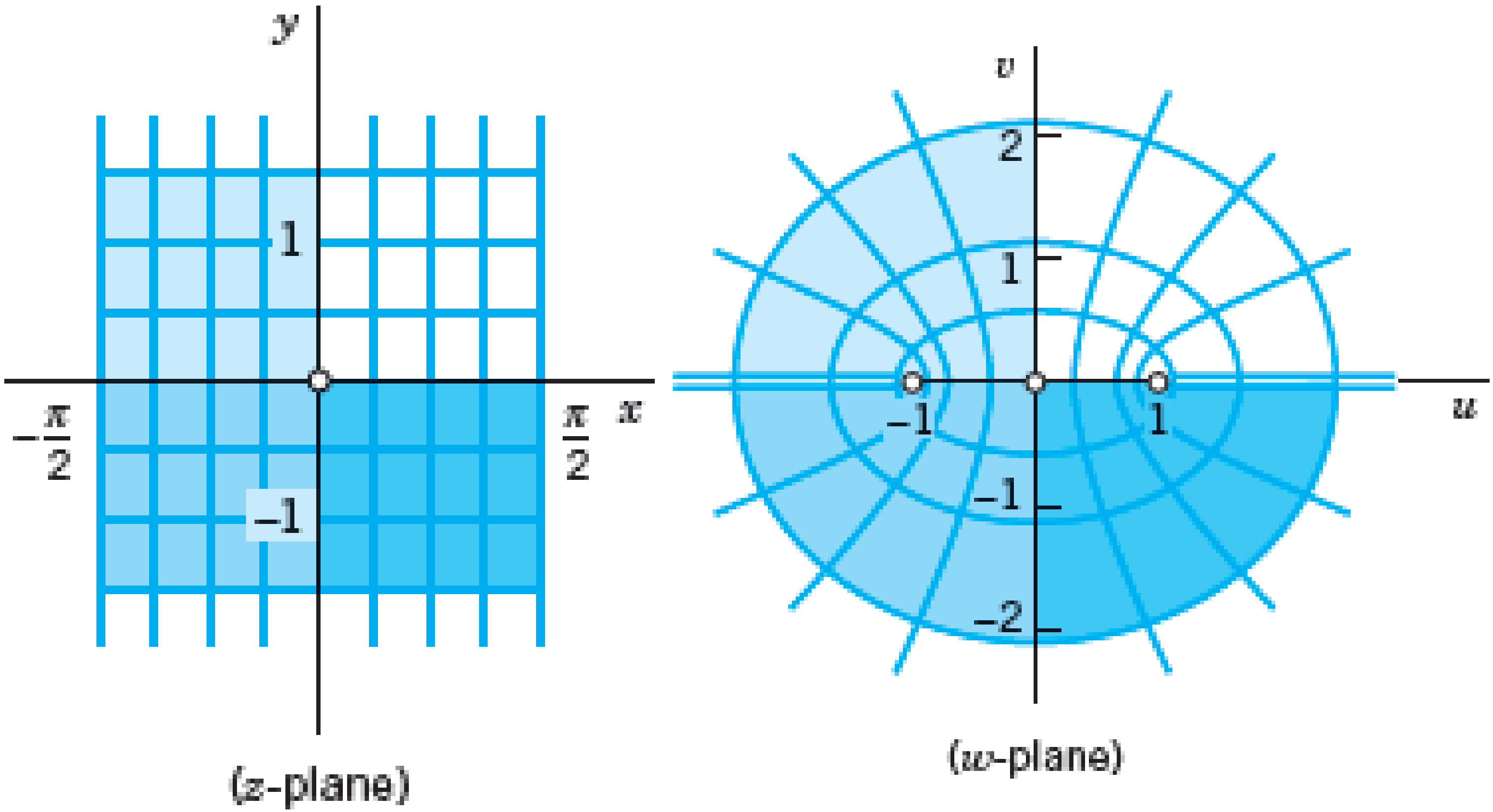
Examples

1. The analytic function $f(z) = e^z$ is conformal at all points, since $f'(z) = e^z$ is never zero.

2. The analytic function $g(z) = z^2$ is conformal at all points except $z = 0$, since $g'(z) = 2z \neq 0$.

The mapping $f(z) = \sin z$

- The vertical strip $-\pi/2 \le x \le \pi/2$ is called the fundamental region of the trigonometric function $w = \sin z$.
- A vertical line $x = a$ in the interior of the region can be described by $z = a + it$, $-\infty \le t \le \infty$. Then
  $u + iv = \sin(a + it) = \sin a \cosh t + i \cos a \sinh t.$
  Since $\cosh^2 t - \sinh^2 t = 1$, we have
  $$\frac{u^2}{\sin^2 a} - \frac{v^2}{\cos^2 a} = 1$$
- A horizontal line $y = b$ in the interior of the region can be described by $z = t + ib$, $-\infty \le t \le \infty$. Then
  $$\frac{u^2}{\cosh^2 b} + \frac{v^2}{\sin^2 b} = 1$$


y
1
−1
x
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
(z-plane)
v
2
1
−1
−2
u
(w-plane)

Session Summary

- A complex function w = $f(z)$ gives a **mapping of its domain** in the complex z-plane onto its **range of values** in the complex w-plane. *If* $f(z)$ is analytic, this mapping is **conformal,** that is**, angle-preserving**, i.e., The angle between any two intersecting curves and the corresponding angle between their image curves are the same.

- **Linear fractional transformations,** also called ***Möbius transformations*** map the extended complex plane onto itself.

- They solve the problems of mapping half-planes onto half-planes or disks, and disks onto disks or half-planes.