Lectures 7-8
Indefinite, Definite and Improper Integrals, Absolute Convergence

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Intended Learning Outcomes

At the end of this lecture, student will be able to:

- Distinguish between indefinite integrals and definite integrals
- Differentiate between proper and improper integrals
- Classify and evaluate improper integrals

Topics

- Anti-derivative
- Fundamental theorem of integral calculus
- Indefinite integral, definite integral
- Improper integral
- Types of improper integrals
- Convergence of improper integrals

Motivation

- The most common application of such integrals is in probability and statistics

- Some quantity is modeled by a probability distribution which is supported on the entire real line, such as the normal distribution ("bell curve").

Anti derivatives

Theorem

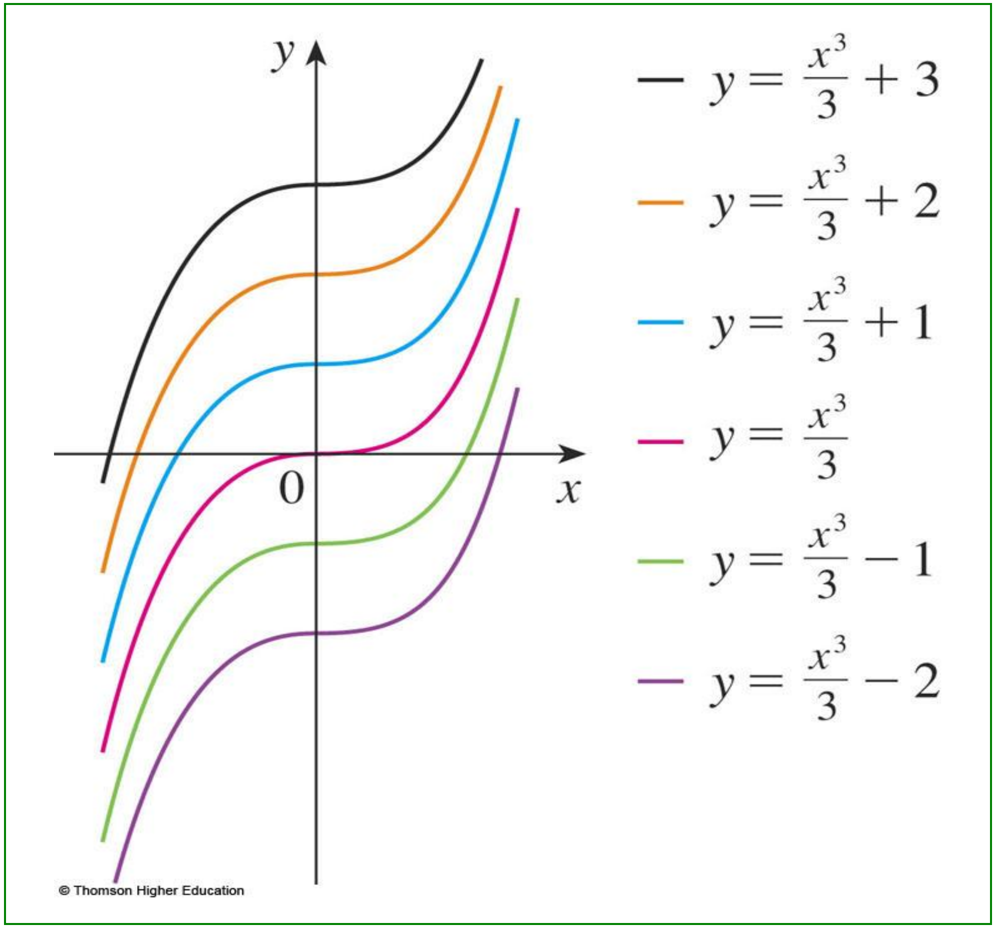
- If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Family of Functions

- By assigning specific values to *C*, we obtain a family of functions.
  - Vertical translates of the graph.
  - This makes sense, as each curve must have the same slope at any given value of *x*.



Notation for Antiderivatives

- The symbol $\int f(x)\,dx$ is traditionally used to represent the most general ***antiderivative of f on an open interval*** and is called the ***indefinite integral of f*** .

- Thus, $F(x) = \int f(x)dx$ means $F'(x) = f(x)$

Indefinite Integral

The expression: $\int f(x)dx$

read "the indefinite integral of *f* with respect to *x*," means to find the set of all antiderivatives of *f*.

$$\int \underbrace{f(x)}_{\text{Integrand}} dx$$

x is called the variable of integration

Integral sign

Indefinite Integral

- For example, we can write

$$\int x^2 dx = \frac{x^3}{3} + C \quad \text{because} \quad \frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

  - Thus, we can regard an indefinite integral as representing an entire family of functions (one antiderivative for each value of the constant *C*).

Constant of Integration

Every antiderivative F of f must be of the form $F(x) = G(x) + C$, where C is a constant.

Example:

$$\int 6x\,dx = \underbrace{3x^2 + C}$$

Represents every possible antiderivative of $6x$.

Fundamental Theorem of Calculus

Let f be a continuous function on $[a, b]$. If F is any antiderivative of f defined on $[a, b]$, then the definite integral of f from a to b is defined by

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

$\int_a^b f(x)dx$ is read "the integral, from a to b of $f(x)\,dx$."

- The function f is called the integrand
- The numbers a and b are called the limits of integration,
- The variable x is called the variable of integration.

Notation

In the notation $\int_a^b f(x)\,dx$,

- $f(x)$ is called the integrand
- a and b are called the limits of integration; a is the lower limit and b is the upper limit
- For now, the symbol dx has no meaning by itself; is all one symbol.
- The dx simply indicates that the independent variable is x

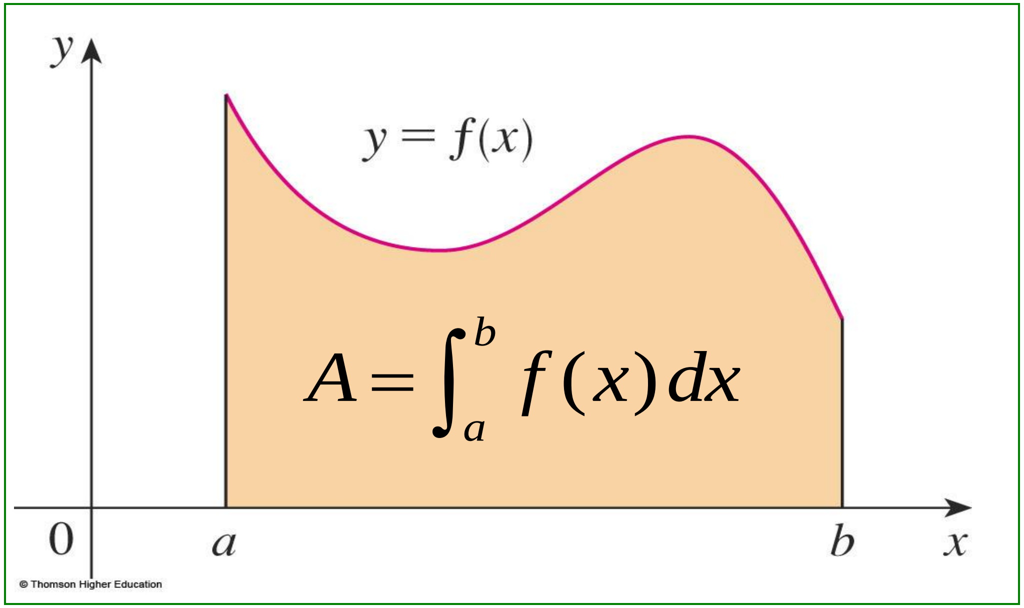
The Definite Integral

- The procedure of calculating an integral is called ***integration***.
- The **definite integral** $\int_a^b f(x)dx$ **is a number**. It does not depend on *x.*
- Also note that the variable *x* is a "dummy variable."

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(r)dr$$

Definite Integral As Area

- If f is a ***positive*** function defined for $a \le x \le b$, then the definite integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b



Definite Integral as Area

- If f is a ***negative*** function for $a \le x \le b$, then the area between the curve $y = f(x)$ and the x-axis from a to b, is the ***negative*** of

$$\int_a^b f(x)\,dx.$$

$$\text{Area from } a \text{ to } b = -\int_a^b f(x)\,dx$$

NOTE!!!

Distinguish carefully between definite and indefinite integrals.

- A *definite integral is a number*

- An *indefinite integral is a function* (or family of functions). The connection between them is given by the Evaluation Theorem:

$$\int_a^b f(x)dx = \int f(x)dx \Big]_a^b$$

IMPROPER INTEGRALS

Improper Integral

TYPE-I:

Infinite Limits of Integration

Example

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

TYPE-II:

Discontinuous Integrand

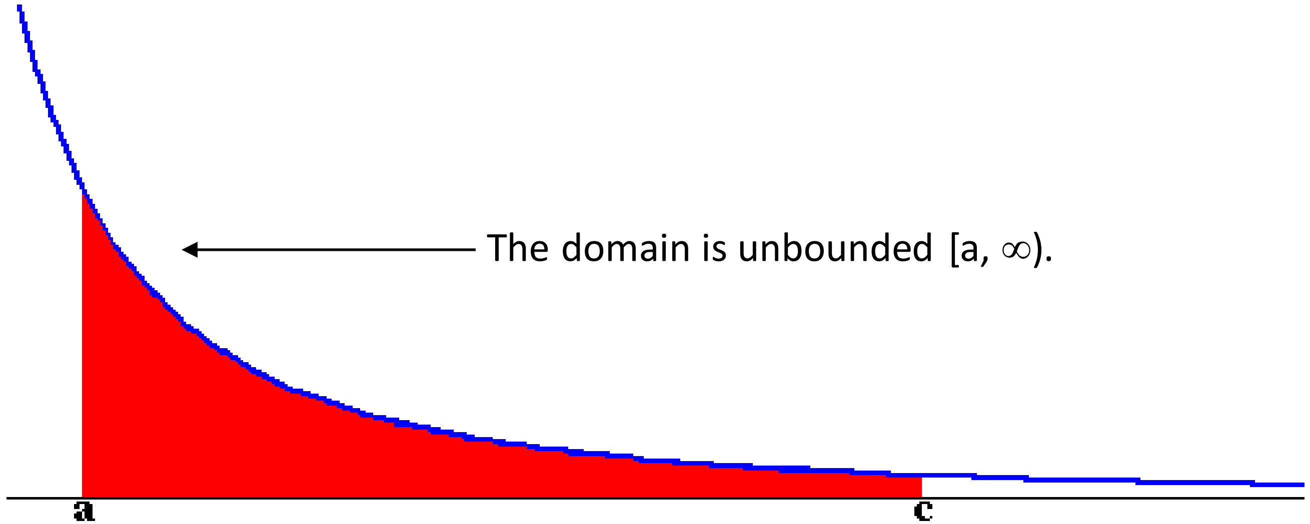
Integrands with Vertical Asymptotes

Example

$$\int_{-1}^{1} \frac{1}{x^2}\, dx$$

Definition of an Improper Integral (1st Kind)

The integral $\int_a^b f(x)$ is called first kind improper integral if the interval [*a*, *b*] becomes unbounded (that is *a* = ∞ or *b* = ∞).



Definition of an Improper Integral of Type 1

$$\int_a^{\infty} f(x)\, dx = \lim_{b \to \infty}\left(\int_a^b f(x)\, dx \right)$$

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a \to -\infty}\left(\int_a^b f(x)\, dx \right)$$

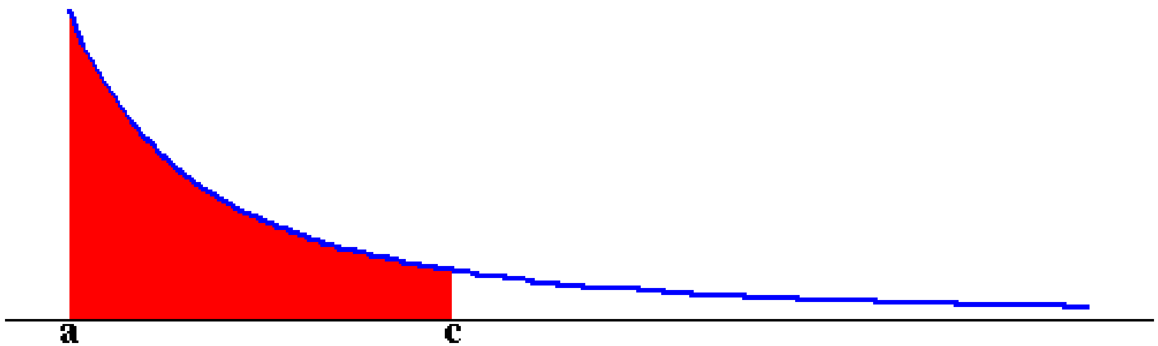
The improper integrals

$$\int_a^{\infty} f(x)\, dx \qquad \int_{-\infty}^{a} f(x)\, dx$$

- Converges if the corresponding limit exists
- Divergent if the limit does not exists

Limit of a sum of an Improper Integral

The same as for the Type I, we considered a positive function just for the sake of illustrating what we are doing. The following picture gives a clear idea about what we will do (using the area approach)

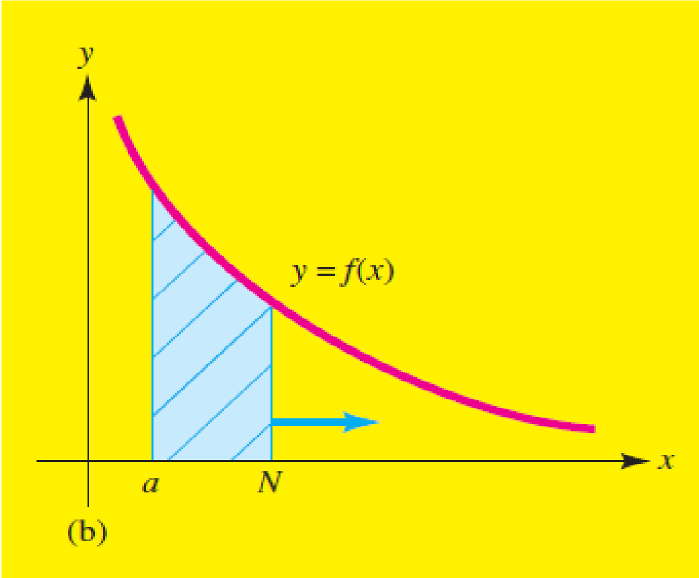


So, we have $\int_a^\infty f(x)\,dx = \underset{c \to \infty}{Lt} \int_a^c f(x)\,dx$

An Example for Unbounded Interval

Problem:

Evaluate $\int_1^{\infty} \frac{dx}{x^2}$ if it converges



Solution:

For any fixed b > 1, $\int_1^b f(x)dx$ is the area between the curve y = $1/x^2$, the x axis, x=1 and x=b

$$\underset{b \to \infty}{Lt} \int_1^b \frac{dx}{x^2} = \underset{b \to \infty}{Lt} \left[\left(-x^{-1} \right) \right]_1^b$$

$$= \underset{b \to \infty}{Lt} \left(-\frac{1}{b} + 1 \right) = 1$$

Examples

2. Evaluate the following improper integral

$$\int_0^\infty \frac{dx}{a^2+x^2}, \quad a>0,$$

Solution:

$$\int_0^\infty \frac{dx}{a^2+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{a^2+x^2} = \lim_{b\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{b}{a}\right] = \frac{\pi}{2a}$$

Therefore improper integral converges to $\pi/2a$

3. Discuss the convergence of the integral $\int_{-\infty}^{\infty} x e^{-x^2}\,dx$

Solution: We write

$$\int_{-\infty}^{\infty} x e^{-x^2}\,dx = \int_{-\infty}^{c} x e^{-x^2}\,dx + \int_{c}^{\infty} x e^{-x^2}\,dx$$

where c is any finite constant, we have

Examples (contd...)

$$I = \lim_{a\to-\infty}\int_a^c x e^{-x^2}\,dx + \lim_{b\to\infty}\int_c^b x e^{-x^2}\,dx$$

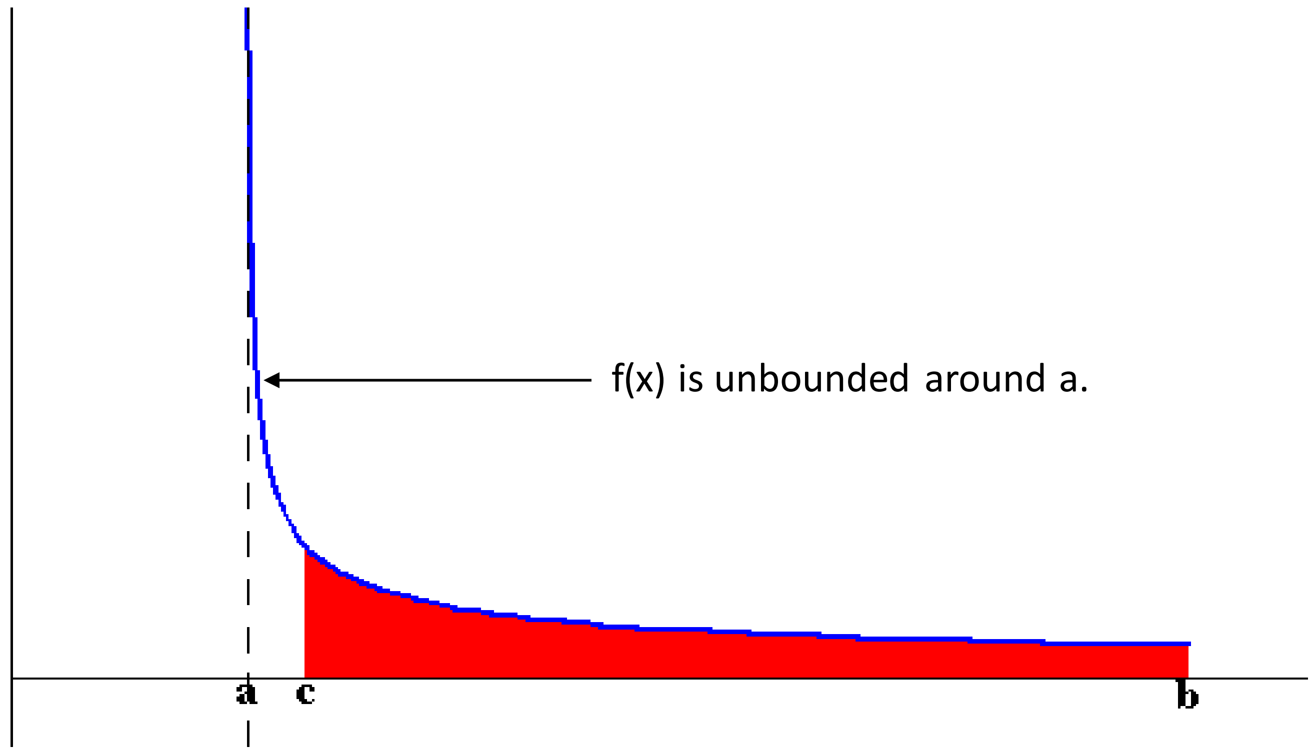
$$= \lim_{a\to-\infty}\left[1/2(e^{-a^2} - e^{-c^2})\right] + \lim_{b\to\infty}\left[1/2(e^{-c^2} - e^{-b^2})\right]$$

$$= 1/2(e^{-c^2} - e^{-c^2}) = 0$$

Therefore, the given improper integral converges to zero

Definition of an Improper Integral (2nd Kind)

The integral $\int_a^b f(x)$ is called second kind improper integral if the function $f(x)$ becomes unbounded around a or b.



Improper integral of type 2

Definition :

If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $(a, c) \cup (c, b)$, *then* $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Examples

1.Evaluate the following improper integrals , if exists $\int_0^4 \frac{dx}{\sqrt{x}}$

Solutions: $$\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0}\int_\varepsilon^4 \frac{dx}{\sqrt{x}} = 2\lim_{\varepsilon\to 0}\left(2-\sqrt{\varepsilon}\right) = 4$$

Therefore , the improper integral converges to 4

2. Evaluate $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$, if exists

$$\int_0^4 \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon\to 0}\int_0^{2-\varepsilon} \frac{dx}{\sqrt{4-x^2}}$$

$$= \lim_{\varepsilon\to 0}\sin^{-1}\left(1-\frac{\varepsilon}{2}\right) = \sin^{-1}1 = \pi/2$$

Comparison Test

If if $0 \leq f(x) \leq g(x)$ *for all* x, *then*

(i) $\int_a^{\infty} f(x)dx$ *converges if* $\int_a^{\infty} g(x)\, dx$ *converges*

(ii) $\int_a^{\infty} g(x)\, dx$ *diverges if* $\int_a^{\infty} f(x)dx$ *diverges*

Convergence of the improper integral

Discuss the convergence of the improper integral

$$\int_a^\infty \frac{dx}{x^p}$$

Solution: $\int_1^b \frac{dx}{x^p} \quad = \frac{1}{1-p}\left[\mathrm{x}^{1-p}\right]_1^{\mathrm{b}} = \frac{1}{1-p}\left[b^{1-p} - 1\right]$

Now, $\lim_{b\to\infty}\left[b^{1-p}\right] = \begin{cases} \infty & if\ p < 1 \\ 0 & if\ p > 1 \end{cases}$

Therefore, the improper integral converges if $p > 1$ and diverges if p < 1

For p$=$ 1, we have $\lim_{b\to\infty} \int_1^\infty \frac{dx}{x}$ = $\lim_{b\to\infty}\{lnx\}_1^{\mathrm{b}} = \lim_{b\to\infty} lnb$

The given improper integral converges for $p > 1$ and diverges for $p \leq 1$

Convergence of the improper integral

Discuss the convergence of the improper integral $\int_a^b \frac{dx}{(x-a)^p}$ $\quad p > 0.$

Solution: The integrand has infinite discontinuity at $x = a$, we write

$$\int_a^b \frac{dx}{(x-a)^p} = \lim_{\epsilon \to 0} \int_{a+\epsilon}^b \frac{dx}{(x-a)^p} = \frac{1}{p-1} \lim_{\epsilon \to 0} \left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\epsilon^{p-1}}\right]$$

$$= \left\{\frac{1}{(1-p)(b-a)^{p-1}}\right\} \quad \text{if } p < 1$$

and ∞ if $p > 1$

For $p = 1$, we get $\int_a^b \frac{dx}{(x-a)}$ $=\lim\limits_{\epsilon \to 0} \int_{a+\epsilon}^b \frac{dx}{(x-a)}$ $=\lim\limits_{\epsilon \to 0} ln\left(\frac{b-a}{\epsilon}\right) = \infty$

Therefore , the improper integral converges for $p \leq 1$

and diverges for $p \geq 1$

Summary

- If f(x) is a nonnegative, continuous function on [a, b], then $\int_a^b f(x)\,dx$ is equal to the area of the region under the graph of f(x) on [a, b].
- A definite integral is a number whereas an indefinite integral is a function (or family of functions).
- If the interval of integration is not bounded, we have an improper integral of the first kind.
- If f is not bounded on the interval of integration, we have an improper integral of the second kind.
- An improper integral of f is absolutely convergent (or converges absolutely) if the improper integral of |f| also converges.
- If an improper integral converges but does not converge absolutely, it is said to converge conditionally.