Prof. Stephan Fritzsche
TPI and HI Jena
— Advanced Quantum Theory —

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Winter Term

Home Work (12)

Task 1: Orbital Angular Momentum (3 Points)

a) The z-component of the orbital angular momentum operator is in Cartesian coordinates given by

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Find the form of L_z in spherical coordinates.

b) Show that the eigenfunction of L_z reads

$$\psi_m(\phi) = c\mathrm{e}^{im\phi}.$$

Task 2: Angular momentum operator (3 Points)

For the operators $\hat{\mathbf{j}}^2 = \hat{j}_x^2 + \hat{j}_y^2 + \hat{j}_z^2$ and $\hat{j}_z$, we have

$$\hat{\mathbf{j}}^2 |jm\rangle = \lambda_j |jm\rangle \,, \quad \hat{j}_z |jm\rangle = m |jm\rangle$$

with $\hat{j}_z = -i\left[\hat{j}_x, \hat{j}_y\right]$. The ladder operators are defined by

$$\hat{j}_+ = \hat{j}_x + i\hat{j}_y \,, \quad \hat{j}_- = \hat{j}_x - i\hat{j}_y \,.$$

with

$$\hat{j}_+ |jm_{\max}\rangle = 0 \,, \quad \hat{j}_- |jm_{\min}\rangle = 0 \,.$$

a) Use these operators to show that $m_{\max} = -m_{\min}$.

b) Assume $m_{\max} = j$ to show that $\lambda_j = j(j+1)$.

Task 3: Ladder Operators (2 Points)

Let $\boldsymbol{j} = (j_x, j_y, j_z)$ be an angular momentum operator with the associated ladder operators $j_\pm = j_x \pm ij_y$

a) Compute the commutators $[j_+, j_-]$, $[j_z, j_\pm]$ and $\left[\boldsymbol{j}^2, j_\pm\right]$

b) In the lecture it was shown, that

$$j_\pm |j, m\rangle = c_\pm |j, m \pm 1\rangle$$

Derive the expression for the coefficient $c_\pm$.

Task 4: Angular Momentum Coupling (2 Points)

a) Let $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$ be angular momentum operators. Show that $\boldsymbol{J} = \boldsymbol{j}_1 + \boldsymbol{j}_2$ is also an angular momentum operator.

b) Proof that $\left[\boldsymbol{J}^2, \boldsymbol{j}_{1,2}^2\right] = \left[J_z, \boldsymbol{j}_{1,2}^2\right] = 0$

Task 5: Vector Model of the Angular Momentum (2 Points)

Let $\boldsymbol{j}$ be an angular momentum operator and $|j, m\rangle$ an eigenvector of $\boldsymbol{j}^2$ and $\boldsymbol{j}_z$. Show, without using an explicit representation of $\boldsymbol{j}$, that

- $\langle j, m \,|\, j_x \,|\, j, m\rangle = 0$
- $\langle j, m \,|\, j_y \,|\, j, m\rangle = 0$