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— Advanced Quantum Theory —

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Winter Term

Homework (3)

Task 1: Scattering by a Central Potential

In first-Born approximation, the scattering amplitude $f(\mathbf{k}, \mathbf{k}')$ is given by

$$f(\mathbf{k}, \mathbf{k}') = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\mathbf{r}\, V(\mathbf{r}) \mathrm{e}^{i\mathbf{q}\mathbf{r}} \,,$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ denotes the momentum transfer.

a) Calculate the scattering amplitude for a square well

$$\begin{aligned} V(r) &= -V_0 \;\; \text{for } r \leq b \quad (V_0 > 0) \\ &= 0 \qquad \text{for } r > b \end{aligned}$$

b) Calculate the scattering amplitude for a Gaussian potential

$$V\,(r) = A\mathrm{e}^{-\kappa r^2}$$

c) Calculate the scattering amplitude for the Yukawa potential

$$V(r) = V_0 \frac{\mathrm{e}^{-\kappa r}}{r} \,.$$

d) In this task, we want to assume elastic scattering and perform a suitable limit of the Yukawa potential to obtain the Coulomb result. Show that the well-known Rutherford scattering is recovered

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{q_1^2 q_2^2}{16E^2 \sin^4(\theta/2)} \,.$$

Task 2: Spherical Bessel and Neumann Functions

a) The spherical Bessel functions are defined by the series expansion

$$j_l(x) = 2^l x^l \sum_{s=0}^{\infty} \frac{(-1)^s (s+l)!}{s!(2s+2l+1)!} x^{2s} \,,$$

and the Neumann functions are also defined by the series expansion

$$n_l(x) = \frac{(-1)^{l+1}}{2^l x^{l+1}} \sum_{s=0}^{\infty} \frac{(-1)^s (s-l)!}{s!(2s-2l)!} x^{2s} \,,$$

Using these expansions, find $j_0(x)$ and $n_0(x)$. Show that the following expressions for spherical Hankel functions

$$h_0^{(1)}(x) = \frac{-i}{x}e^{ix}, \quad h_0^{(2)}(x) = \frac{i}{x}e^{-ix}\,.$$

are fulfilled.
Hint: $h_l^{(1)}(x) = j_l(x) + in_l(x)$ and $h_l^{(2)}(x) = j_l(x) - in_l(x)$.
b) Apart from the spherical Bessel functions, there are "normal" Bessel functions of the first kind $J_l(r)$ which are solutions of the Bessel's equation

$$r^2\frac{\mathrm{d}^2 J_l}{\mathrm{d}r^2} + r\frac{\mathrm{d}J_l}{\mathrm{d}r} + (r^2 - l^2)J_l = 0\,.$$

The relation between the Bessel functions and spherical Bessel functions reads

$$j_l(r) = \sqrt{\frac{\pi}{2r}}J_{l+1/2}(r)\,.$$

Use Bessel's equation to find the differential equation whose solution are the spherical Bessel functions $j_l(r)$.

Task 3: Scattering by a Hard-Sphere Potential
In the scattering of particles by a hard-sphere with potential

$$\begin{aligned} V(r) &= \infty \;\text{ for } r < R_0 \\ &= 0 \quad \text{otherwise}\,. \end{aligned}$$

for low energies, $kR_0 \ll 1$, it is known that only s-waves contribute to the scattering. Use the following form for the scattering wave function

$$\psi(\boldsymbol{r}) = \frac{u_l(r)}{r}Y_{lm}(\theta,\phi)$$

to find the radial function $u_0(r)$ with the help of the boundary condition at $r = R_0$. Also obtain the scattering phase δ_0. Find the total scattering cross section and compare it with the result from classical mechanics.