Prof. Stephan Fritzsche
TPI and HI Jena
— Advanced Quantum Theory —

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Winter Term

Home Work (4)

Task 1: Born Series (1 Point)

Show that the Born series

$$\psi_k(\boldsymbol{r}) = \sum_{n=0}^{\infty} U^n \phi_k = \sum_{n=0}^{\infty} \psi_k^{(n)}(\boldsymbol{r})$$

with

$$\psi_k^{(0)} = \mathrm{e}^{i\boldsymbol{kr}}$$
$$\psi_k^{(n)} = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3 r \frac{\mathrm{e}^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} V(\boldsymbol{r}')\,\psi_k^{(n-1)}(\boldsymbol{r}')$$

is a solution of the Lippman-Schwinger equation.

Task 2: Asymptotic form of Bessel functions (4 Points)

Let us consider the integral in the complex plane

$$I(r) = \int_C f(z)\mathrm{e}^{rS(z)}\mathrm{d}z$$

along some path C. A saddle point z_0 of the function S defines usually the relation $S'(z_0) = 0$. In the limit $r \to \infty$, the integral $I(r)$ has the asymptotic (method of steepest descents)

$$I(r \to \infty) \approx \sqrt{\frac{2\pi}{r|S''(z_0)|}} f(z_0)\mathrm{e}^{rS(z_0)+i\alpha},$$

with

$$\alpha = \frac{\pi}{2} - \frac{1}{2}\arg S''(z_0), \text{ arg referes to the phase of complex number } z = |z|\mathrm{e}^{i\arg z}.$$

The Hankel functions $H_\nu^\pm(r)$ are defined by

$$H_\nu^\pm(r) = \frac{1}{\pi i}\int_{C_\pm} \mathrm{e}^{(r/2)(z-1/z)}\frac{dz}{z^{\nu+1}}.$$

a) Use the asymptotic of $I(r)$ to prove the asymptotic form of the Hankel functions $H_\nu^\pm(r)$

$$H_\nu^+(r\to\infty) \approx \sqrt{\frac{2}{\pi r}}\mathrm{e}^{i(r-\nu\frac{\pi}{2}-\frac{\pi}{4})} \text{ and } H_\nu^-(r\to\infty) \approx \sqrt{\frac{2}{\pi r}}\mathrm{e}^{-i(r-\nu\frac{\pi}{2}-\frac{\pi}{4})}.$$

Hint: the saddle point z_0 lies in the interval $\text{Im}(z_0) > 0$ for H_ν^+. On the other hand, the saddle point z_0 lies in the interval $\text{Im}(z_0) < 0$ for H_ν^-.

b) Find the asymptotic form of the Bessel $J_\nu(r)$ and Neumann $N_\nu(r)$ functions when $r \to \infty$.

Hint: $H_\nu^+ = J_\nu + iN_\nu$ and $H_\nu^- = J_\nu - iN_\nu$.

c) Find the asymptotic form of the spherical Bessel function $j_\nu(r)$ when $r \to \infty$.

Hint: $j_\nu(r) = \sqrt{\frac{\pi}{2r}} J_{\nu+1/2}(r)$.

Task 3: Expansion of Plane Waves (3 Points)

In the lecture, it was shown that plane waves can be expanded in terms of spherical harmonics

$$\mathrm{e}^{ikr\cos\vartheta} = \sum_{\ell=0}^{\infty} a_{\ell 0} j_\ell(kr) Y_{\ell 0}(\vartheta, \varphi) = \sum_{l=0}^{\infty} \left(\frac{2\ell+1}{4\pi}\right)^{1/2} a_{\ell 0} j_\ell(kr) P_\ell(\cos\vartheta)$$

Show that the expansion coefficients are given by

$$a_{\ell 0} = i^\ell \left(4\pi(2\ell+1)\right)^{1/2}$$

Hint: Use the asymptotic behavior of the spherical Bessel functions:

$$j_\ell(kr) \xrightarrow[kr\to\infty]{} \frac{1}{kr} \sin\left(kr - \ell\frac{\pi}{2}\right)$$

Task 4: Asymptotic Evaluation of Integrals

Method of Steepest Descents