Prof. Stephan Fritzsche
TPI and HI Jena
— Advanced Quantum Theory —

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Winter Term

Home Work (9)

Task 1: Oscillator nuclear potential (4 Points)

The nuclear shell model can be obtained from the Schrödinger equation with a spherical oscillator $V(r) = \frac{1}{2} M\omega_0^2 r$. After some transformations, this equation can be written as

$$z \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + \left(\ell + \frac{3}{2} - z\right) \frac{\mathrm{d}f}{\mathrm{d}z} + \frac{1}{2}\left(\frac{E}{\hbar\omega_0} - \ell - \frac{3}{2}\right) f = 0 \,.$$

Assume a power series expansion $f(z) = \sum_{k=0}^{\infty} a_k z^k$. This series terminates at some finite $k = n$, i.e. $a_n \neq 0$ but $a_{n+1} = 0$.

a) Find the expression for the energy E.

b) Find all allowed states $(n+1)\,\ell$ corresponding to the first three lowest energies E. Use here the spectrosopic notation $\ell = 0, 1, 2, ... \rightarrow s, p, d, ...$ For example, the $(n+1)\,\ell$ state with $n = 0$ and $\ell = 0$ is written as $1s$ in this notation .

c) Show that this harmonic oscillator potential can explain the lowest three magic numbers (2, 8, 20) of particularly stable nuclei.

Hint: You might want to look at the three lowest energies and their degeneracies.

Task 2: Hydrogen molecular ion (2 Points)

The hydrogen molecular ion H_2^+ is composed of two protons and one electron. The electronic Hamiltonian is given in this case by

$$H = -\frac{1}{2}\nabla^2 - \frac{1}{r_A} - \frac{1}{r_B} + \frac{1}{R}\,,$$

where R is the internuclear separation, and $\mathbf{r}_A = \mathbf{r} - \mathbf{R}/2$, $\mathbf{r}_B = \mathbf{r} + \mathbf{R}/2$ and $\mathbf{r}$ are the position vectors of the electron with respect to the protons A and B, and to the midpoint of the internuclear line, respectively. The electronic wave function is assumed to be

$$|\Phi_{g,u}\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle \pm |B\rangle\right)$$

Here $|A\rangle = \psi_{1s}(r_A)$ and $|B\rangle = \psi_{1s}(r_B)$, while $\psi_{1s}(r)$ is the normalised ground state wave function for atomic hydrogen. Use $|\Phi_{g,u}\rangle$ as trial functions in the variational method to express the energy $E_{g,u}(R)$ in terms of the integrals

$$I = \langle A|B\rangle\,, \quad J = \langle A|\frac{1}{r_B}|A\rangle\,, \quad K = \langle A|\frac{1}{r_B}|B\rangle\ .$$