Prof. Stephan Fritzsche
TPI and HI Jena
— Theoretical Atomic Physics —

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Winter Term

Home Work (2)

Task 1: Greens function

In the lecture, it was shown that the Greens function of the Schrödinger equation can be computed by solving the integral

$$G(\boldsymbol{r}, k) = \frac{m}{2ir\pi^2\hbar^2} \int_{-\infty}^{\infty} \mathrm{d}q \, q \frac{\mathrm{e}^{\mathrm{i}qr}}{k^2 - q^2} \, .$$

This integral can be solved by applying the residue theorem. Show that the Greens function is given by

$$G(\boldsymbol{r}, k) = -\frac{m}{2\pi\hbar^2} \frac{\mathrm{e}^{ikr}}{r}$$

Task 2: Probability Current Density

Compute the probability current density that is generated by the outgoing part of the wave function

$$\psi_{\boldsymbol{k}}^{(out)} = \frac{\mathrm{e}^{\mathrm{i}kr}}{r} f(\boldsymbol{k}, \boldsymbol{k}') \, .$$

Task 3: Asymptotic Solution

Show that the scattering solution

$$\Psi_{\boldsymbol{k}}^{(out)} = \frac{\mathrm{e}^{\mathrm{i}kr}}{r} f(\boldsymbol{k}, \boldsymbol{k}')$$

is for large r also a solution of the free Schrödinger equation .

Task 4: Born Approximation

In the lecture, we found the scattering amplitude

$$f(\boldsymbol{k}, \boldsymbol{k}') = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\boldsymbol{r}' \mathrm{e}^{-i\boldsymbol{k}'\cdot\boldsymbol{r}'} V(\boldsymbol{r}') \, \Psi_{\boldsymbol{k}}(\boldsymbol{r}') \qquad \boldsymbol{k}' = k\frac{\boldsymbol{r}}{r}$$

The first Born approximation is obtained by inserting $\Psi_{\boldsymbol{k}} = \mathrm{e}^{i\boldsymbol{k}\boldsymbol{r}}$.

a) Prove that in first Born approximation

$$f(\boldsymbol{k}, \boldsymbol{k}') = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\boldsymbol{r}' \mathrm{e}^{-i\boldsymbol{q}\boldsymbol{r}'} V(\boldsymbol{r}') \, ,$$

where the momentum transfer $\boldsymbol{q} = \boldsymbol{k}' - \boldsymbol{k}$ was defined.

b) The potential of a general charge distribution $Q\rho(\boldsymbol{r})$ is given by

$$V(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0} \int \mathrm{d}^3\boldsymbol{r}' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

Show that the scattering amplitude in first Born approximation factorizes into

$$f\left(\boldsymbol{k}, \boldsymbol{k}'\right) = F(\boldsymbol{q})\, \tilde{f}\left(\boldsymbol{k}, \boldsymbol{k}'\right)$$

where $\tilde{f}(\boldsymbol{k}, \boldsymbol{k}')$ is the scattering amplitude of a point charge

$$\tilde{f}\left(\boldsymbol{k}, \boldsymbol{k}'\right) = -\frac{m}{2\pi\hbar^2} \int \mathrm{d}^3\boldsymbol{r}' \frac{Q}{4\pi\epsilon_0 r'} \mathrm{e}^{-i\boldsymbol{q}\boldsymbol{r}'}$$

and the form factor $F(\boldsymbol{q})$ is given by

$$F(\boldsymbol{q}) = \int \mathrm{d}^3\boldsymbol{r}' \mathrm{e}^{-i\boldsymbol{q}\boldsymbol{r}'} \rho\left(\boldsymbol{r}'\right)$$

c) Show that for a spherically symmetric potential $V(\boldsymbol{r}) = V(r)$, the scattering amplitude can be expressed as

$$f\left(\boldsymbol{k}, \boldsymbol{k}'\right) = -\frac{2m}{\hbar^2} \int_{r=0}^{\infty} \mathrm{d}r' V\left(r'\right) \frac{r' \sin\left(qr'\right)}{q}$$