

# **JAC: Jena Atomic Calculator**

## **— User Guide, Compendium & Theoretical Background —**

<https://github.com/OpenJAC/JAC.jl>

Reference: S. Fritzsche, *Computer Physics Communications* 240, 1 (2019)

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Sunday 23<sup>rd</sup> June, 2024

### Atomic representations

- Configuration-based expansions
- Restricted active spaces (layer-by-layer)
- CI+perturbation theory; Gamov states
- Approximate Green functions, ...

### Processes & properties

- Transition probabilities
- Excitation, ionization & recombination
- Auger, DR, Rayleigh-Compton, multi- $\gamma$
- Hyperfine & Zeeman splitting; plasma
- Isotope shifts, Lande & form factors

### Atomic cascades

- Average single-configuration approach
- Multiple-configuration approach
- Incorporation of shake-up & shake-off
- Ion & electron distributions, ...

### Symbolic Racah algebra

- Wigner symbols, special values
- Symmetries & recursions
- Symbolic sum rule evaluation
- Spherical harmonics & tensors

Interactive High-Level Language

# JAC

## Jena Atomic Calculator

A Julia implementation for  
atomic computations.

*Open-source applications  
in physics, science and  
technology.*

### Atomic responses

- Field-induced processes & ionization
- High-harmonic generation
- Particle-impact processes
- Charge exchange

### Time evolution

- Liouville equation for statistical tensors  
& atomic density matrices
- Atoms in intense light pulses
- Angle & polarization-dep. observables

### Atomic descriptors

- Feature transform. & machine learning
- Bi-spectra of electronic densities
- Subshell & coupling descriptors
- Atomic fragments & effective charges

### Semi-empirical estimates

- Weak-field ionization rates
- Asymptotic behaviour & formulas
- Stopping powers
- Plasma Stark broadening, ...

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