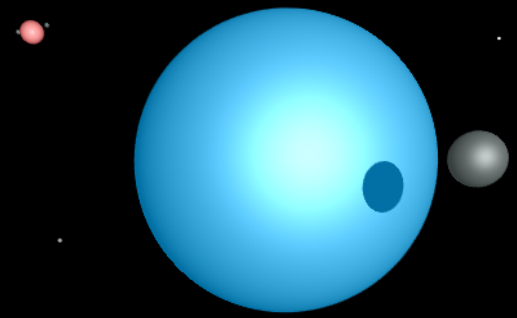
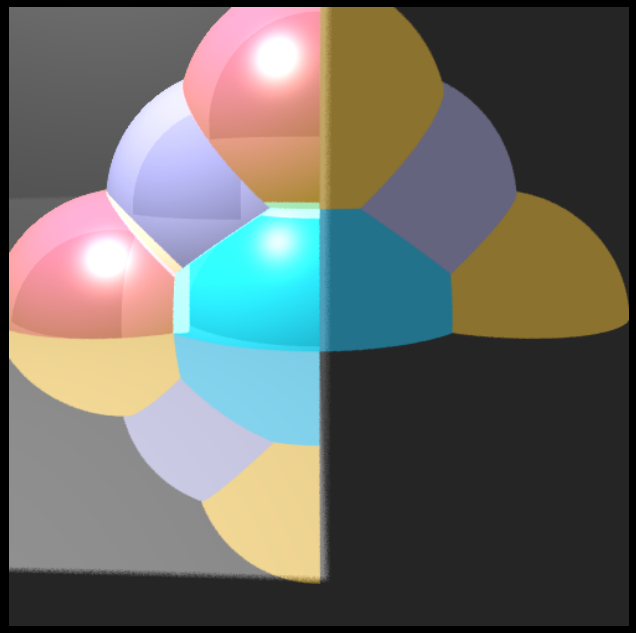
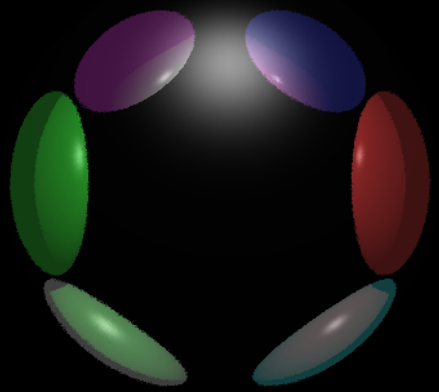
CSC 433/533
Computer Graphics

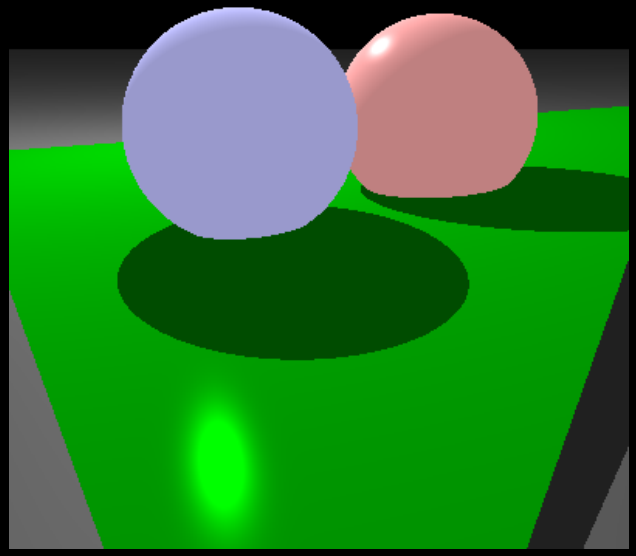
Alon Efrat
Credit: Joshua Levine
November 10

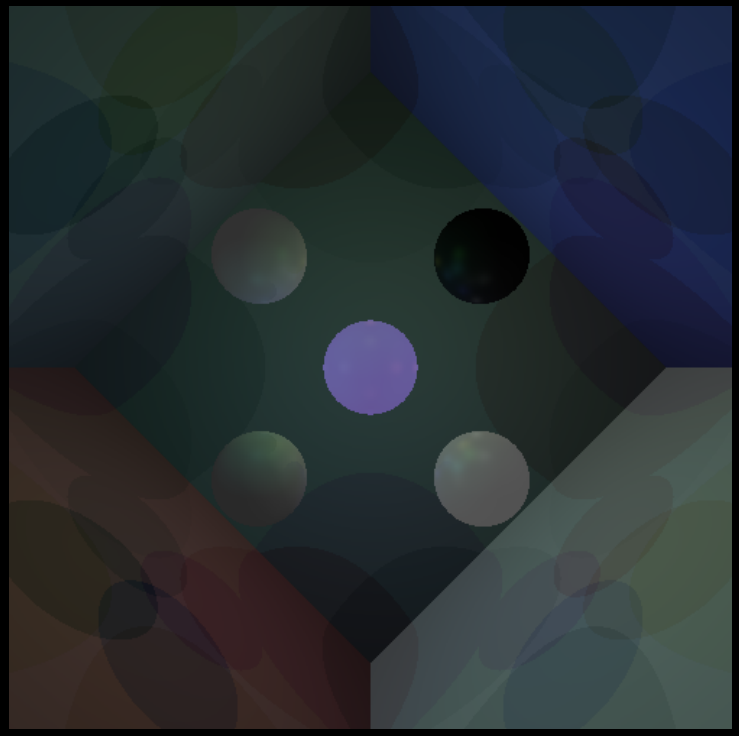
Today: More transformations and viewing

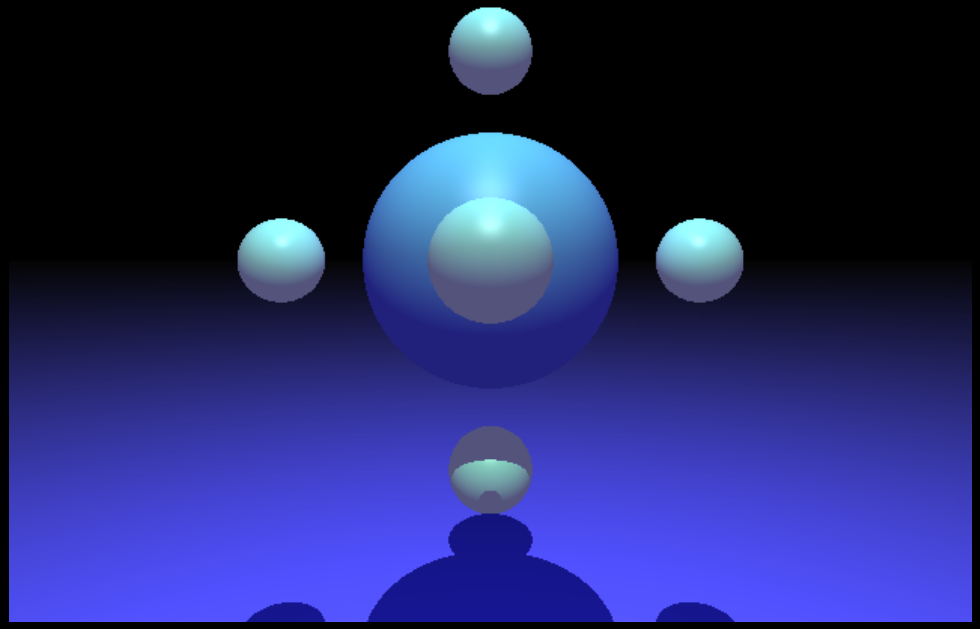


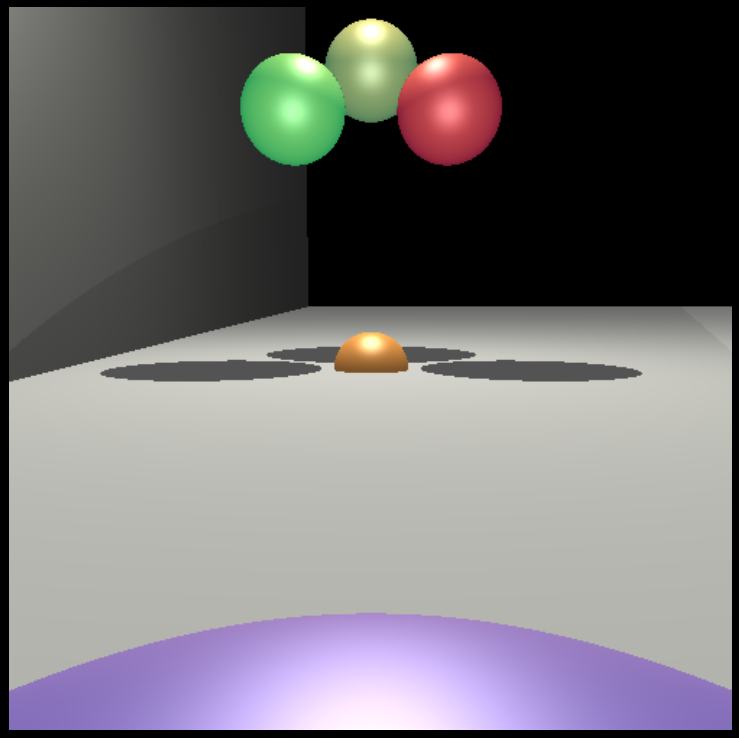


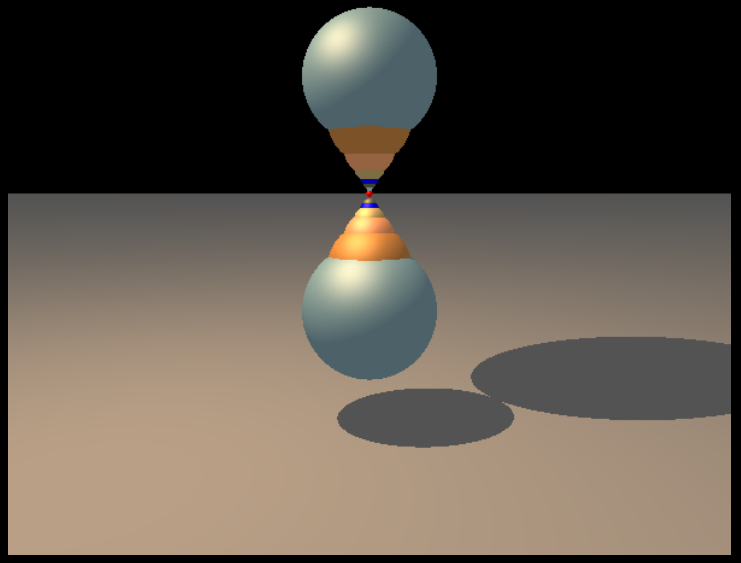


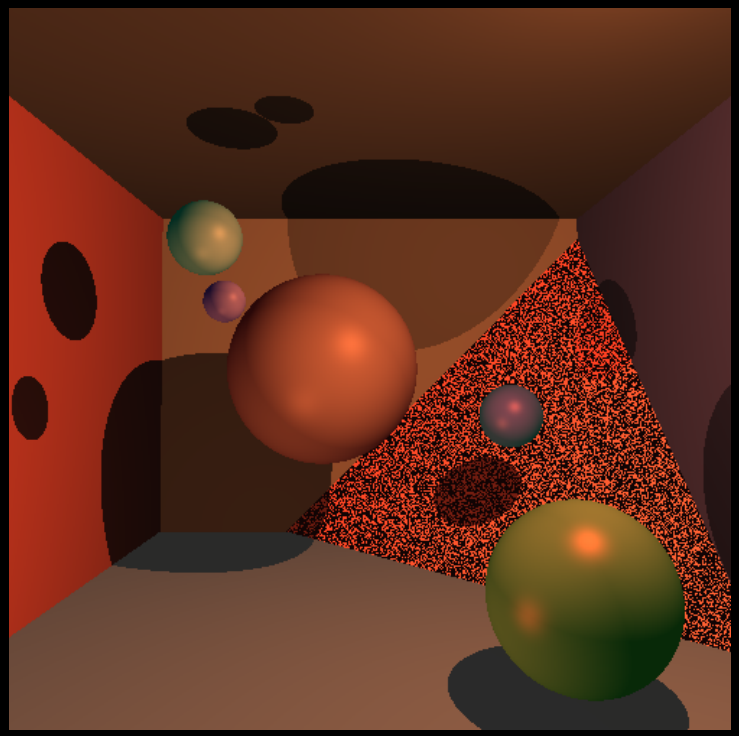


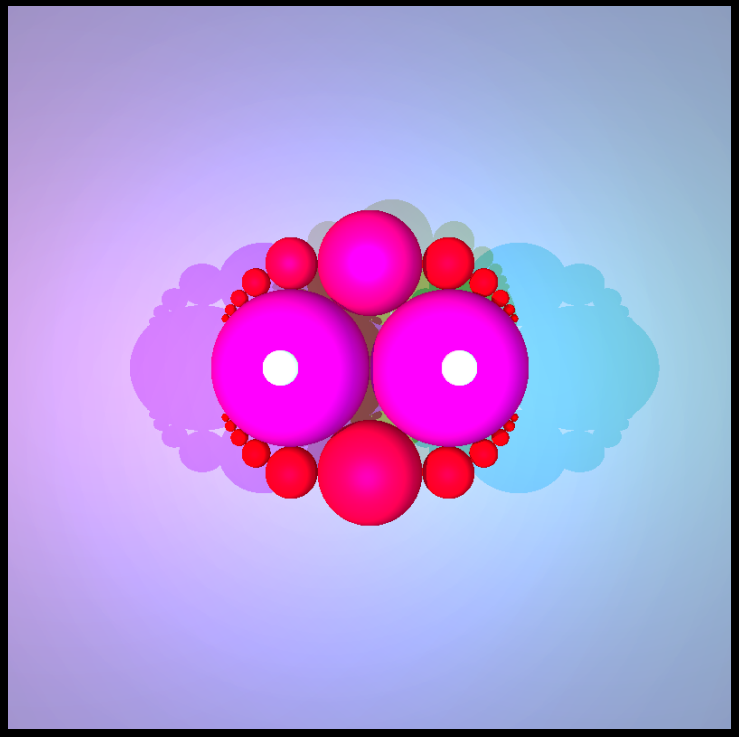


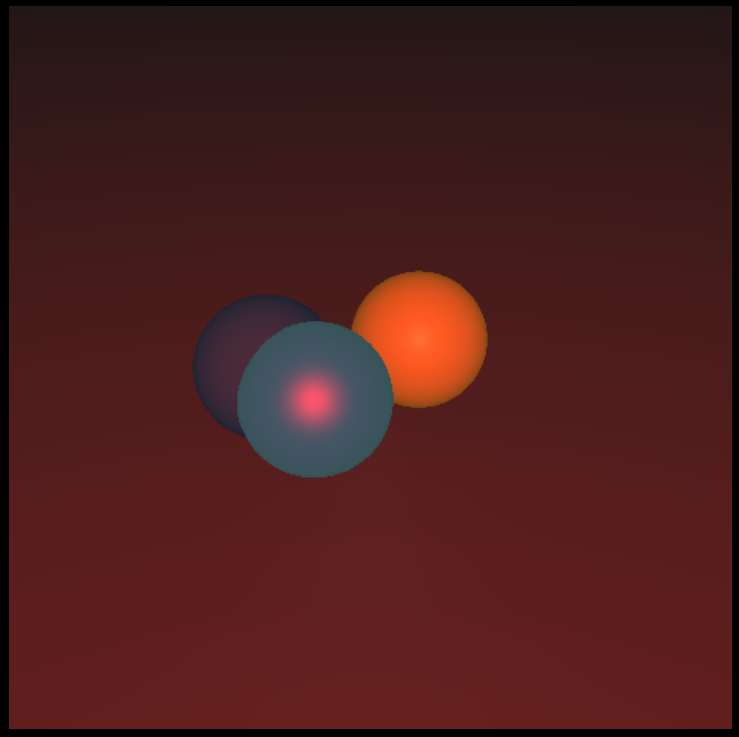


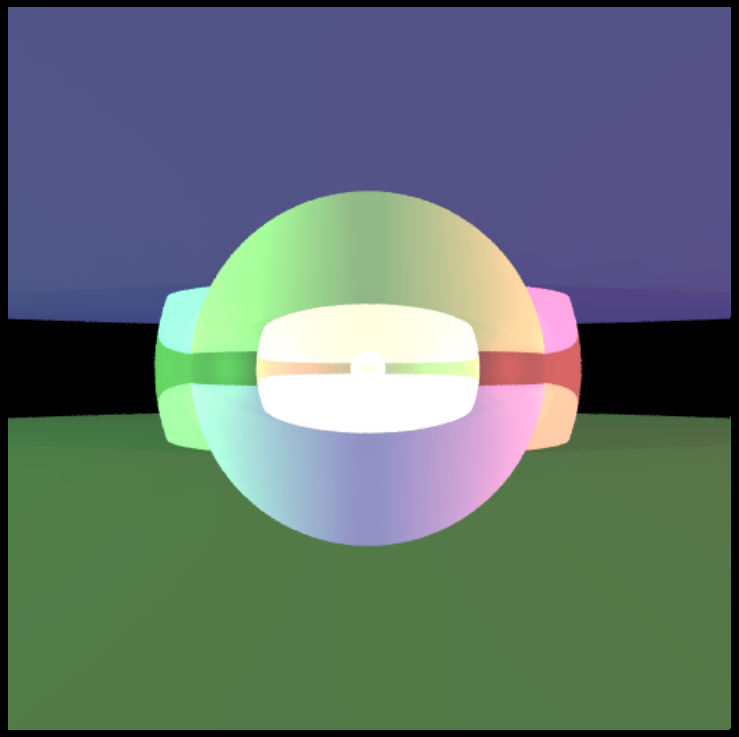


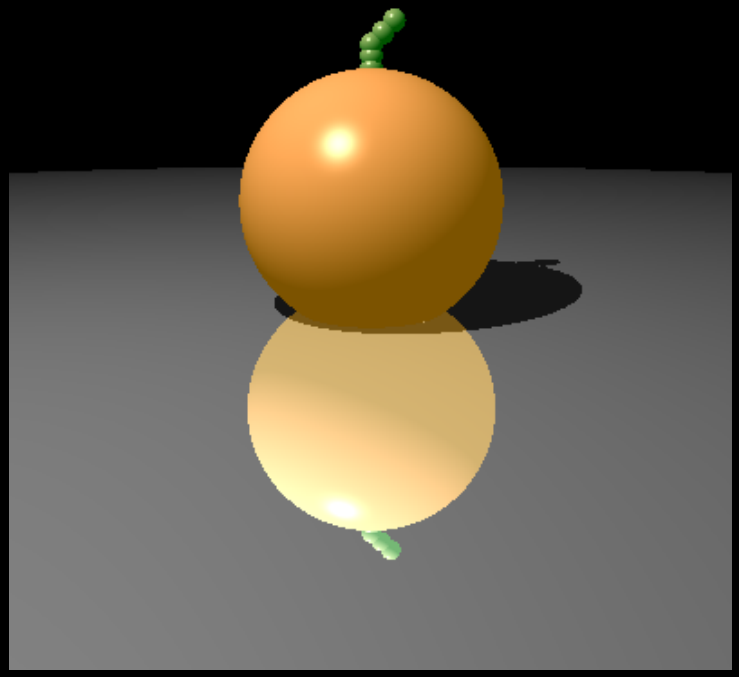


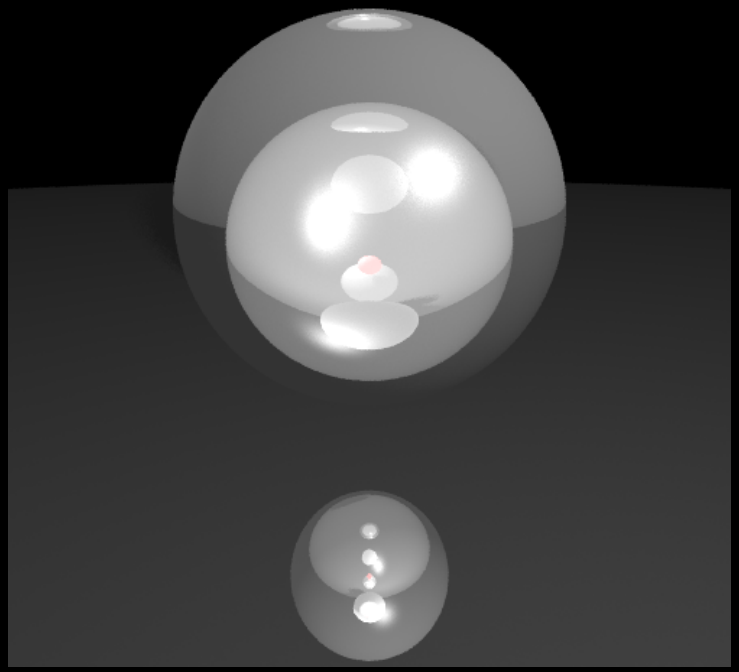


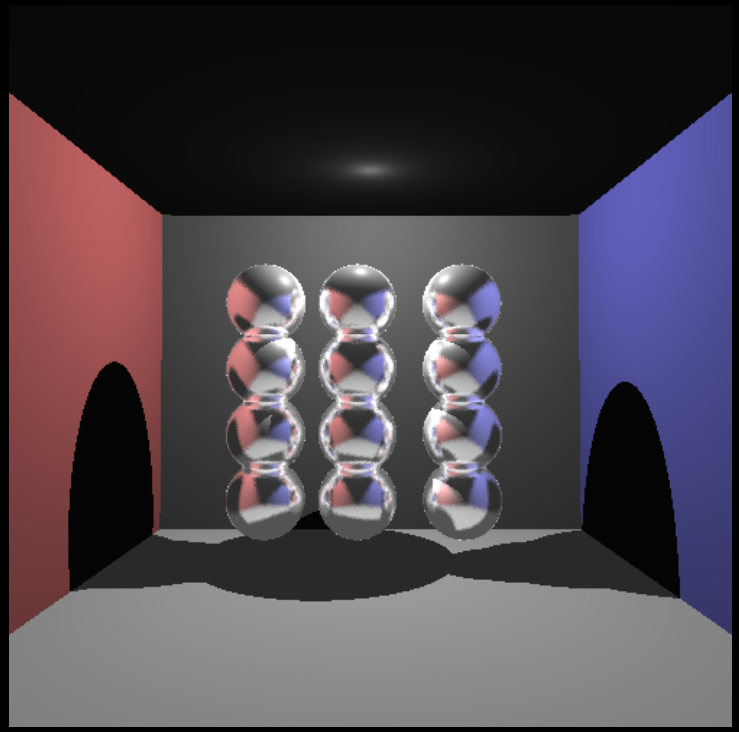


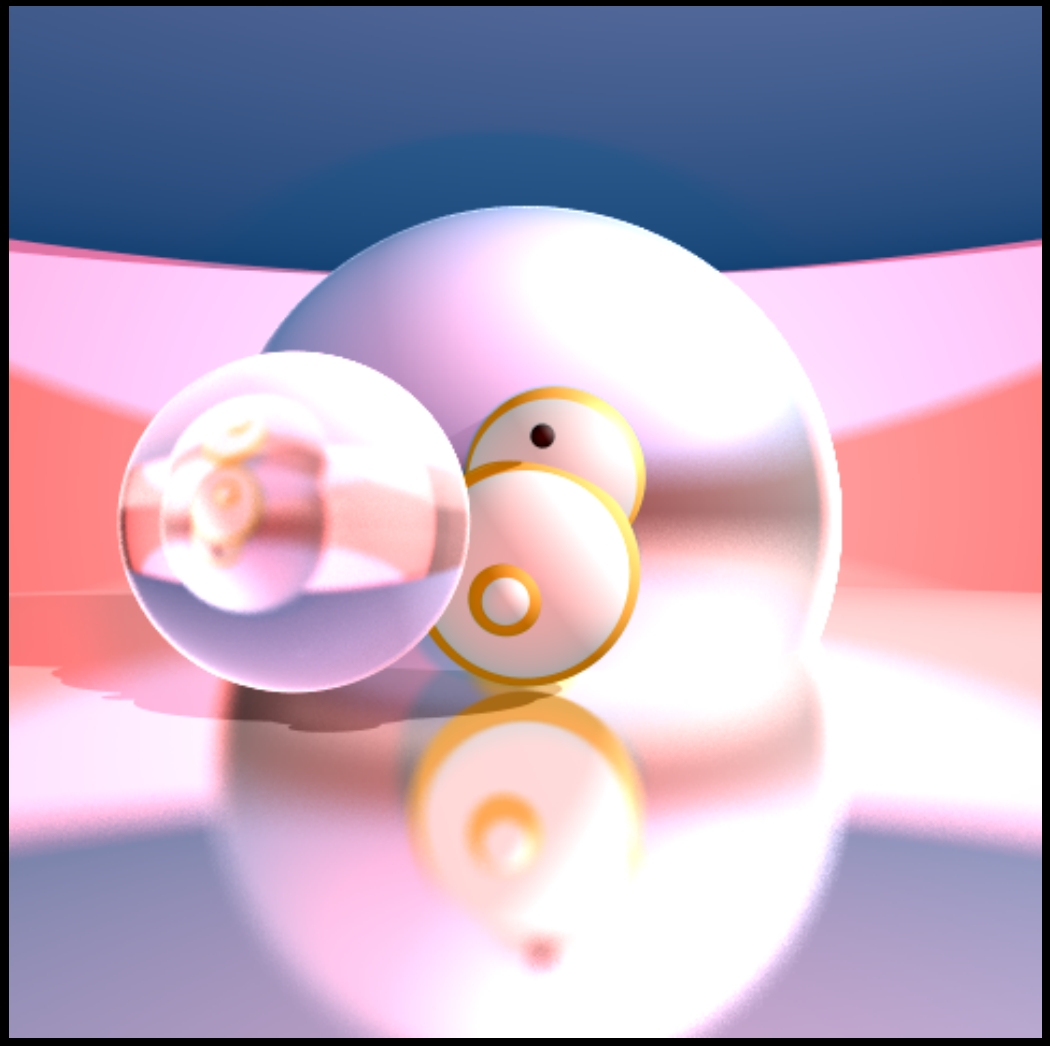


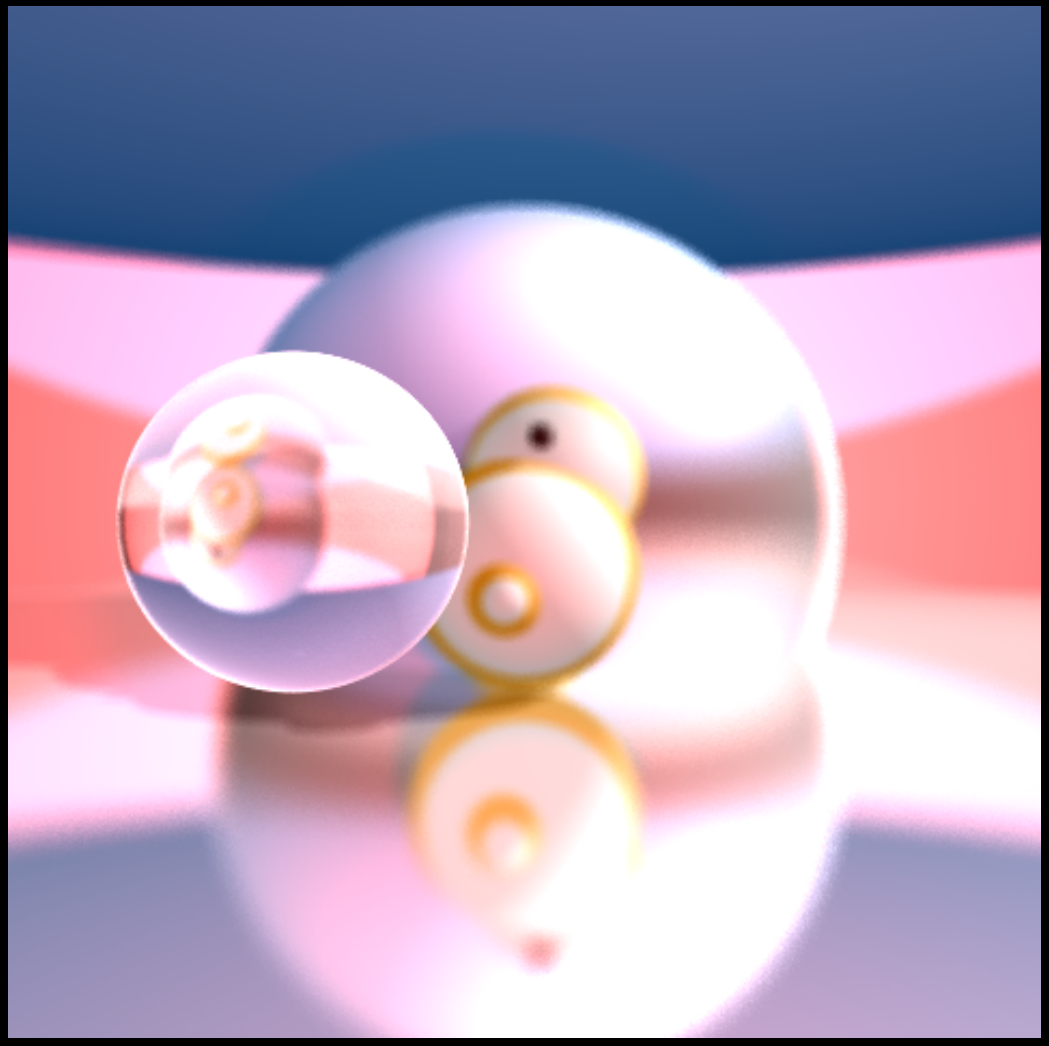


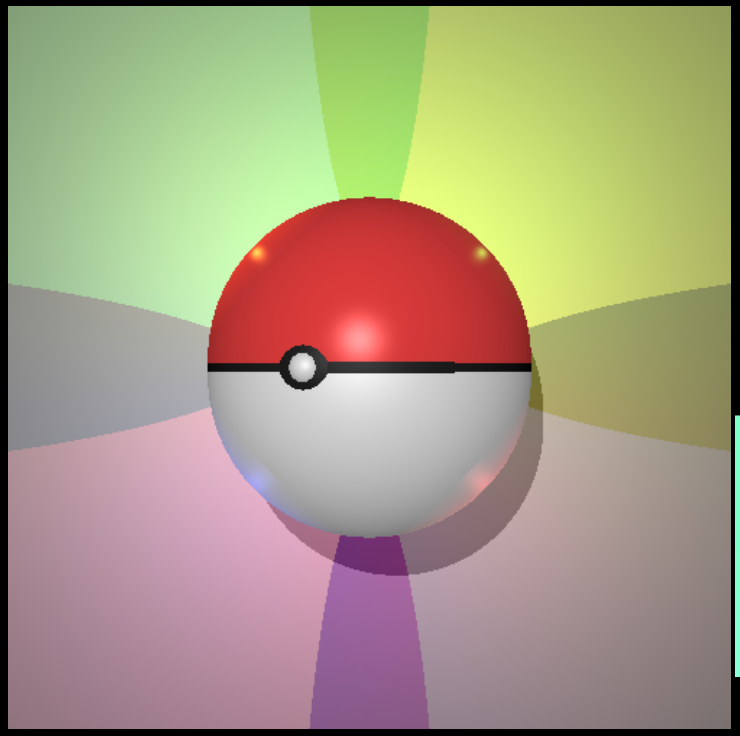


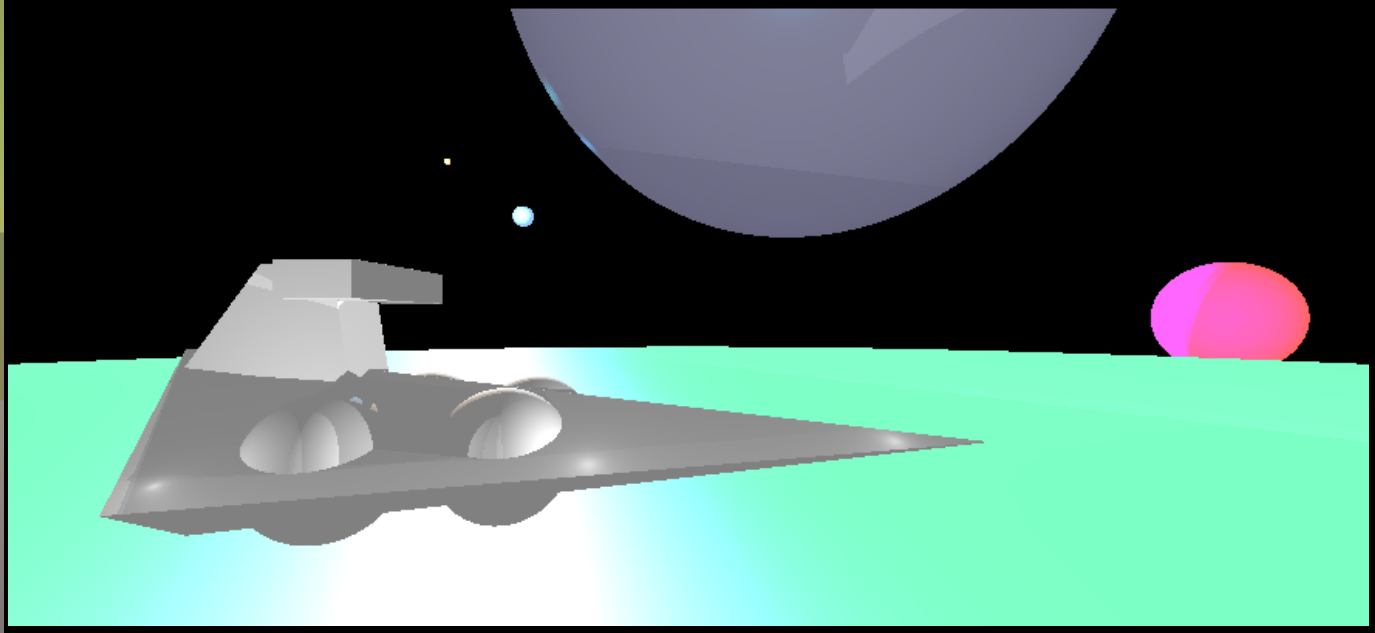


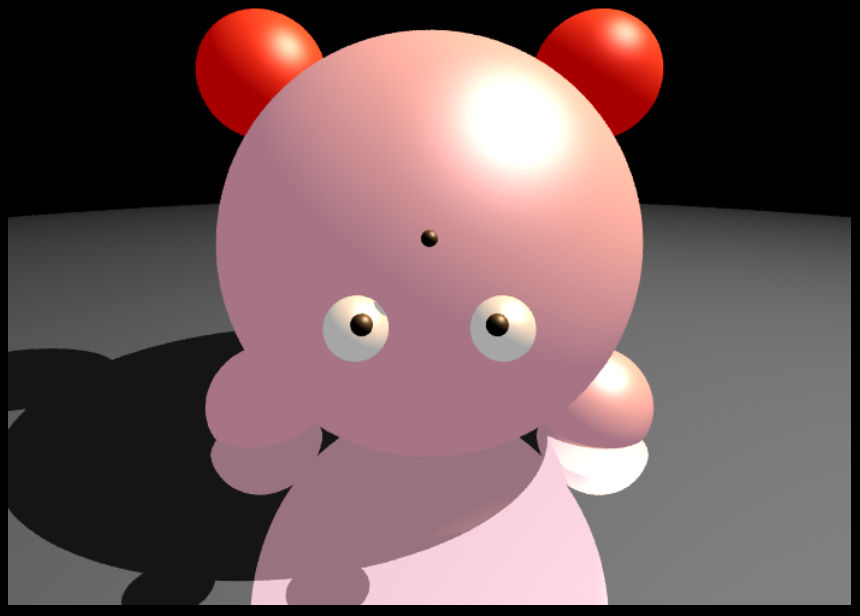












3D Linear Transformations

- We can transform points in a 3D coordinate system by multiplying the point (a vector) by a matrix (the transformation), just like in 2D!

- The only difference is we will use 3x3 matrices A by **x** = (x,y,z), or A**x**, e.g. for scale and shear:

$$\text{scale}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

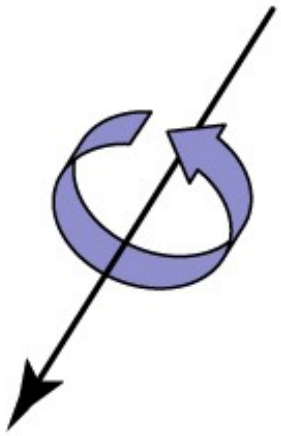
$$\text{shear-x}(d_y, d_z) = \begin{bmatrix} 1 & d_y & d_z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotations in 3D

- In 2D, a rotation is about a point
- In 3D, a rotation is about an axis

convention: positive rotation is CCW

2D



3D

convention: positive rotation is CCW when axis vector is pointing at you

Rotations about 3D Axes

- In 3D, we need to pick an axis to rotate about

$$\text{rotate-z}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

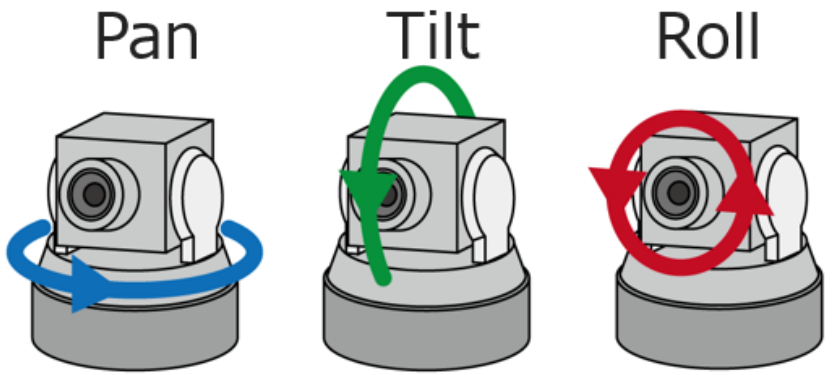
- And we can pick any of the three axes

$$\text{rotate-x}(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

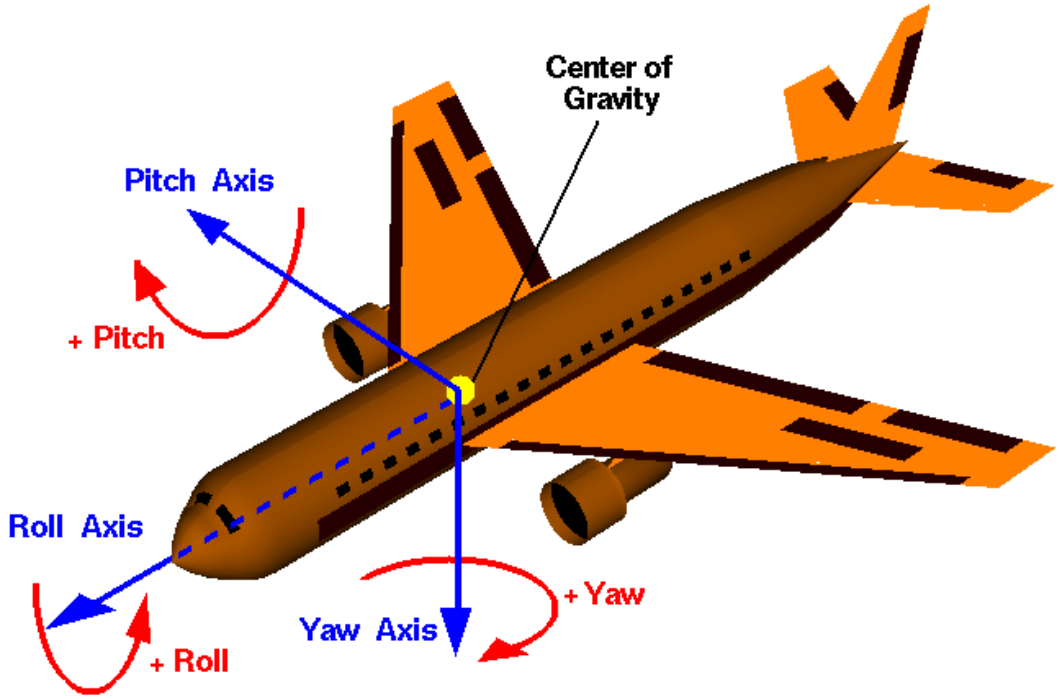
$$\text{rotate-y}(\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

Building Complex Rotations from Axis-Aligned Rotations

- Rotations about **x**, **y**, **z** are sometimes called **Euler angles**
- Build a combined rotation using matrix composition



Ishikawa Watanabe Laboratory



Wikipedia

Arbitrary Rotations

- To rotate about any axis: we change the coordinate space we are working in, using orthogonal matrices.

- Consider orthogonal matrix R_{uvw}, form by taking three orthogonal vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$:

Property of orthogonal vectors:

$$\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{w} = 1$$
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$$

$$\mathbf{R}_{uvw} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{bmatrix}$$

Arbitrary Rotations

- What happens when we apply R_{uvw} to any of the basis vectors, e.g.:

$$\mathbf{R}_{uvw}\mathbf{u} = \begin{bmatrix} \mathbf{u}\cdot\mathbf{u} \\ \mathbf{v}\cdot\mathbf{u} \\ \mathbf{w}\cdot\mathbf{u} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{x}$$

- But this means that if we apply R_{uvw}^{T} to the Cartesian coordinate vectors, e.g.:

$$\mathbf{R}_{uvw}^{\mathrm{T}}\mathbf{y} = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} = \mathbf{v}$$

Arbitrary Rotations

- This means that if we want to rotation around an arbitrary axis, we need only to use a change of coordinates

- E.g. to rotate around a direction $\mathbf{w}$, we
  - Compute orthogonal directions $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. All unit vectors.
  - Change the $\mathbf{uvw}$ axes to be **xyz** (R_{uvw})
  - Apply a rotate-z()
  - Finally, change the axes back to $\mathbf{uvw}$ ($R_{uvw}^{\mathbf{T}}$)

$$\begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{bmatrix}$$

$R_{uvw}^{\mathbf{T}}$ rotate-z() R_{uvw}

Notation: x_u is the x-coordinate of the vector **u**

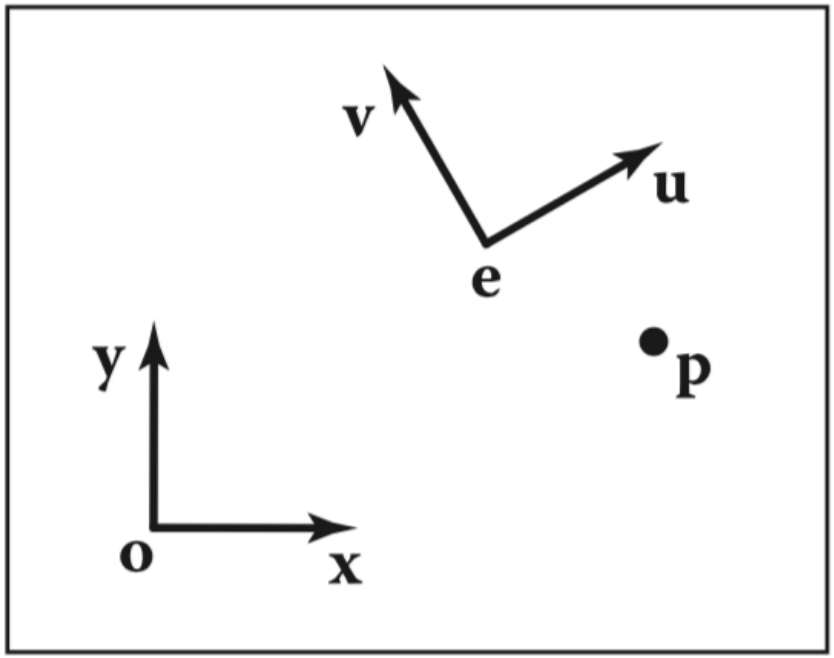
Coordinate Transformations

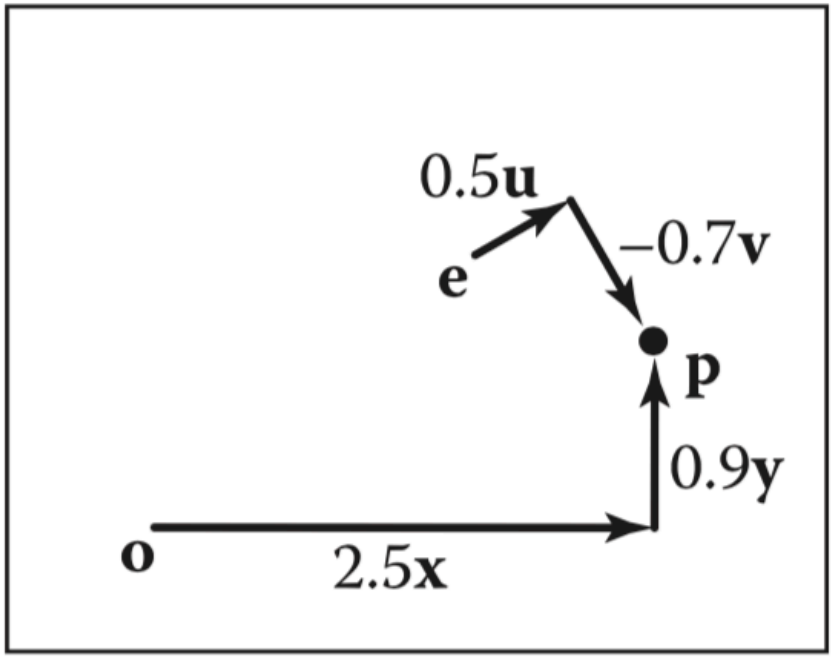
Coordinate Systems

- Points in space can be represented using an origin position and a set of orthogonal basis vectors:

$$\mathbf{p} = (x_p, y_p) \equiv \mathbf{0} + x_p\mathbf{x} + y_p\mathbf{y} \qquad \mathbf{p} = (u_p, v_p) \equiv \mathbf{e} + u_p\mathbf{u} + v_p\mathbf{v}$$

- Any point can be described in either coordinate system





Matrices for Converting Coordinate Systems

- (Remember: Rotating the world CCW is equivalent to rotating the coordinate system CW)

- Using homogenous coordinates and affine transformations, we can convert between coordinate systems:

$$\begin{bmatrix} x_p \\ y_p \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_e \\ 0 & 1 & y_e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_u & x_v & 0 \\ y_u & y_v & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_p \\ v_p \\ 1 \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_e \\ y_u & y_v & y_e \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_p \\ v_p \\ 1 \end{bmatrix}$$

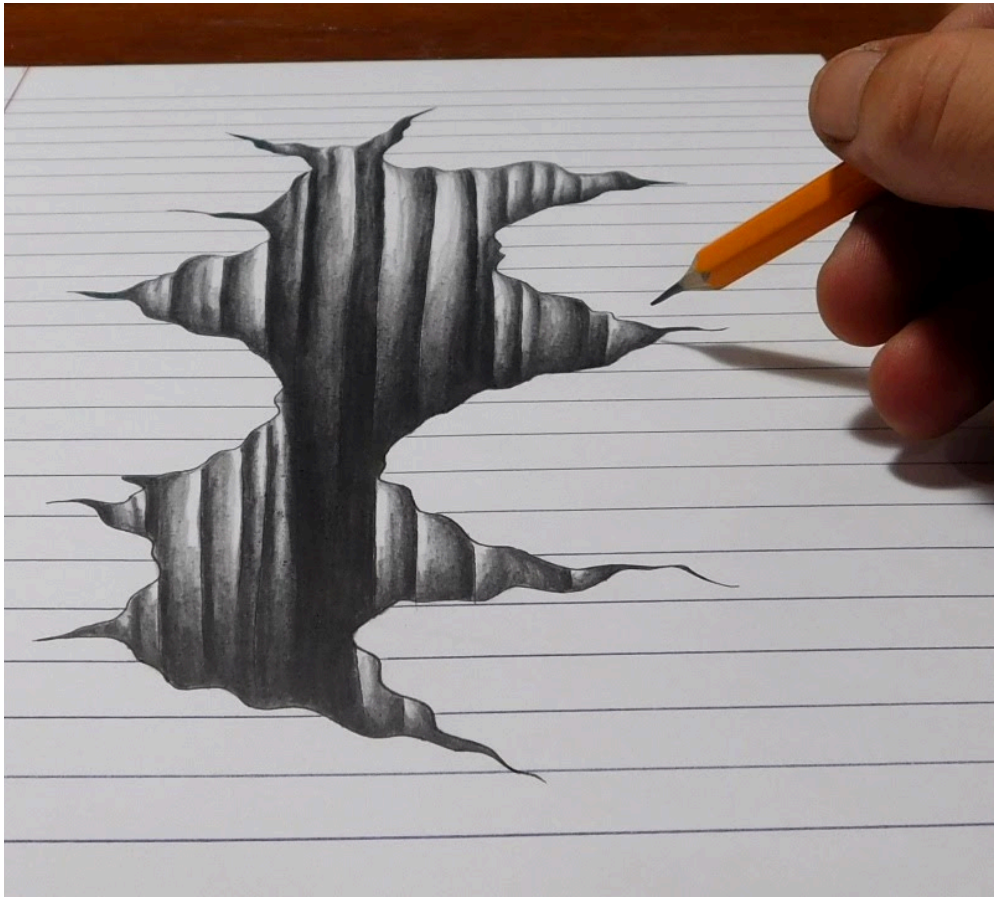
- More generally, any arbitrary coordinate system transform:

$$\mathbf{P}_{uv} = \begin{bmatrix} \mathbf{x}_{uv} & \mathbf{y}_{uv} & \mathbf{o}_{uv} \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_{xy}$$

Viewing

Recall: Two Ways to Think About How We Make Images

- Drawing



- Photography

Recall: Two Ways to Think About Rendering

- Object-Ordered
- Decide, for every object in the scene, its contribution to the image

TODAY

- Image-Ordered
- Decide, for every pixel in the image, its contribution from every object

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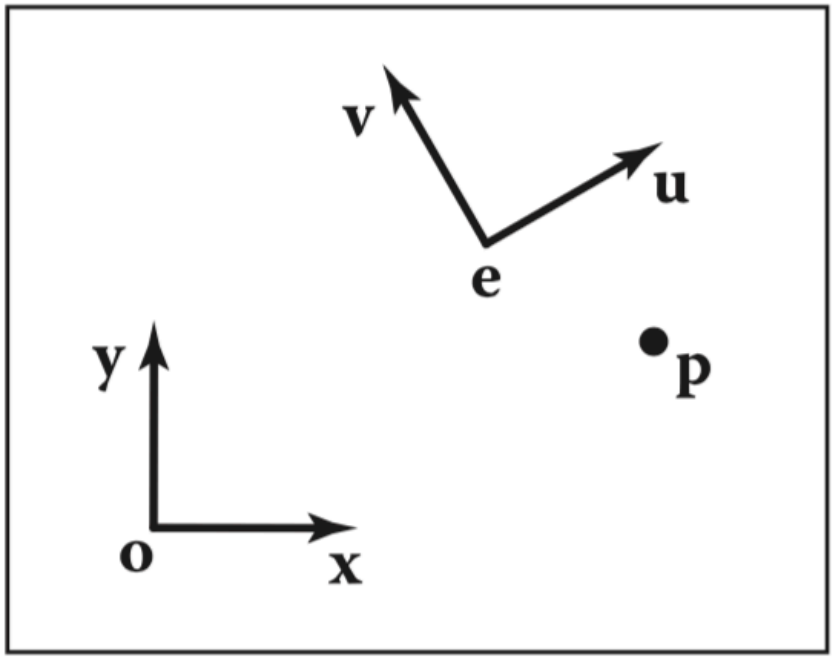
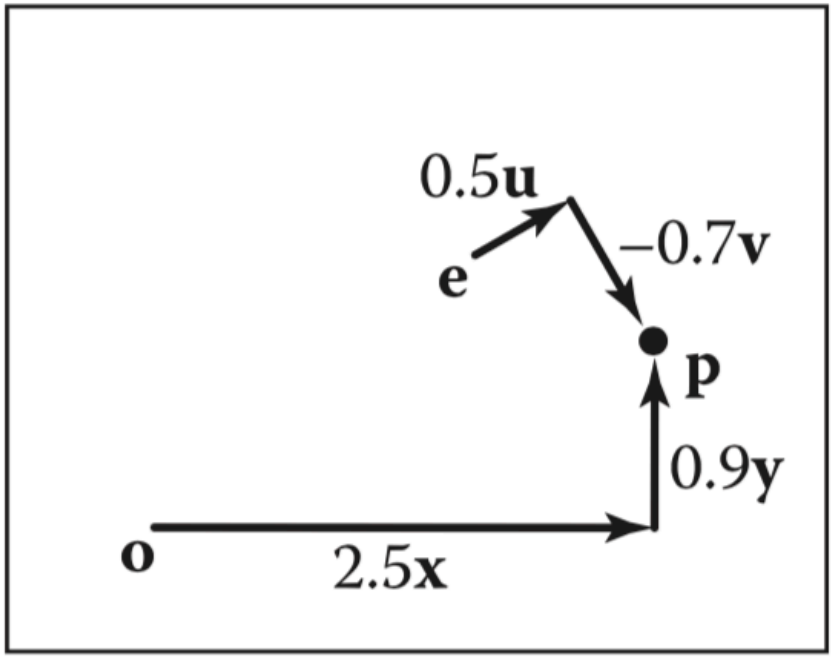
- Any point can be described in either coordinate system





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- More generally, any arbitrary coordinate system transform:

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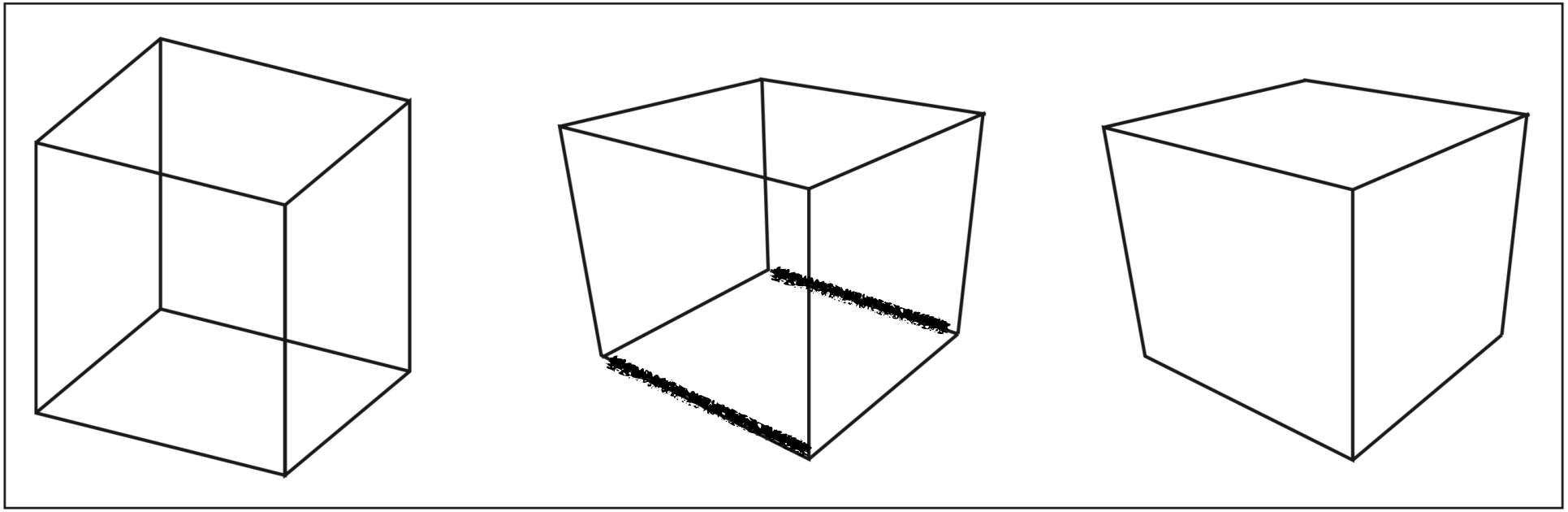
View Transformations

Using Transformations for Rendering

- Idea for today: Matrices can be used to move objects from 3D spaces to the 2D space of an image
  - Broadly, this reduction of dimensions is called **viewing transformation**
  - We will compose multiple matrix-based transformations to rethink cameras

Drawing by Transformation

- For now, we will consider drawing wireframe objects (collections of 3D line segments)



Orthographic

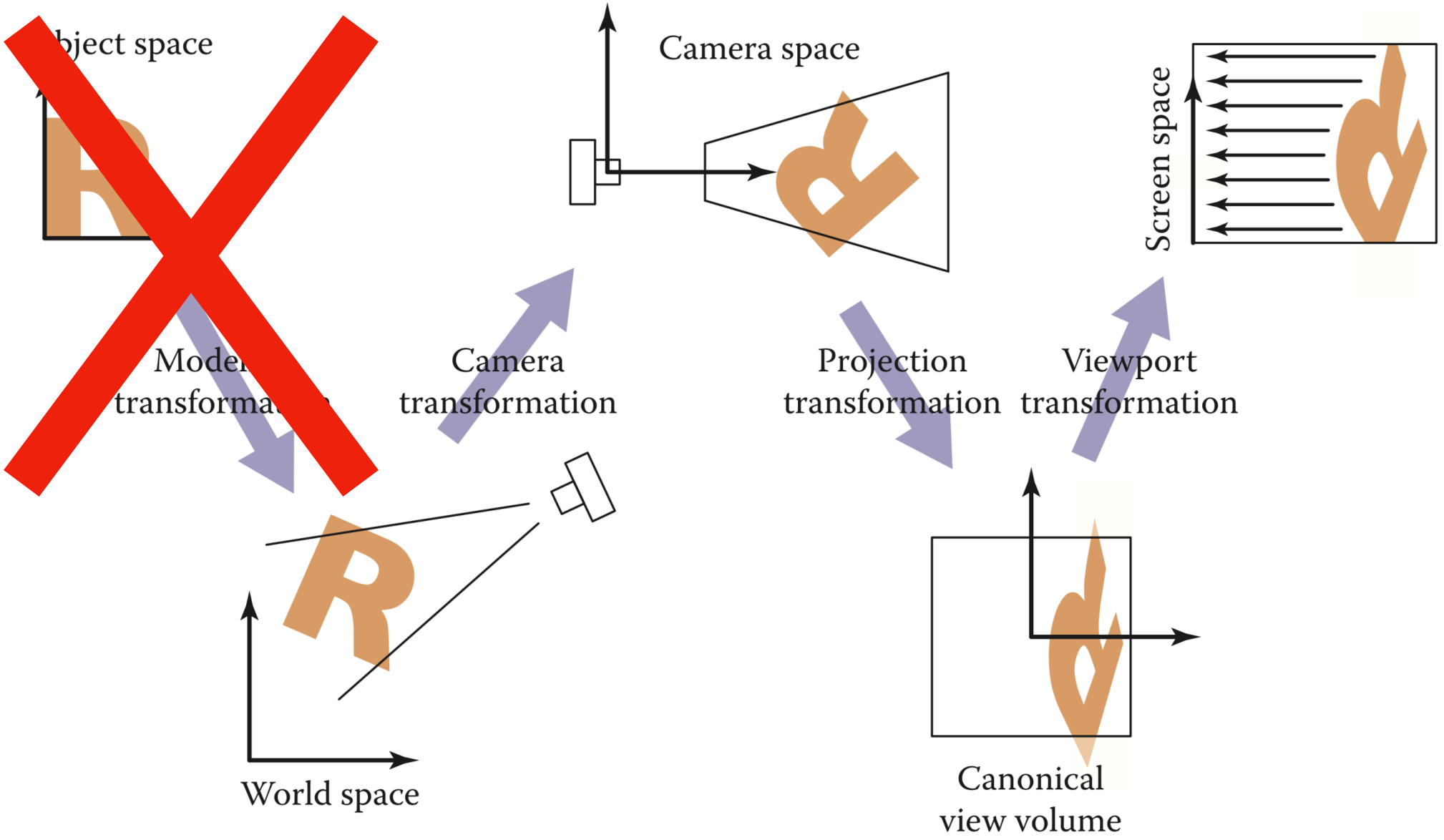
Perspective
farther objects looks smaller
Parallel lines might meet

Perspective +
Hidden Line Removal

Step-by-Step Viewing Transformations

(Each arrow is a matrix)

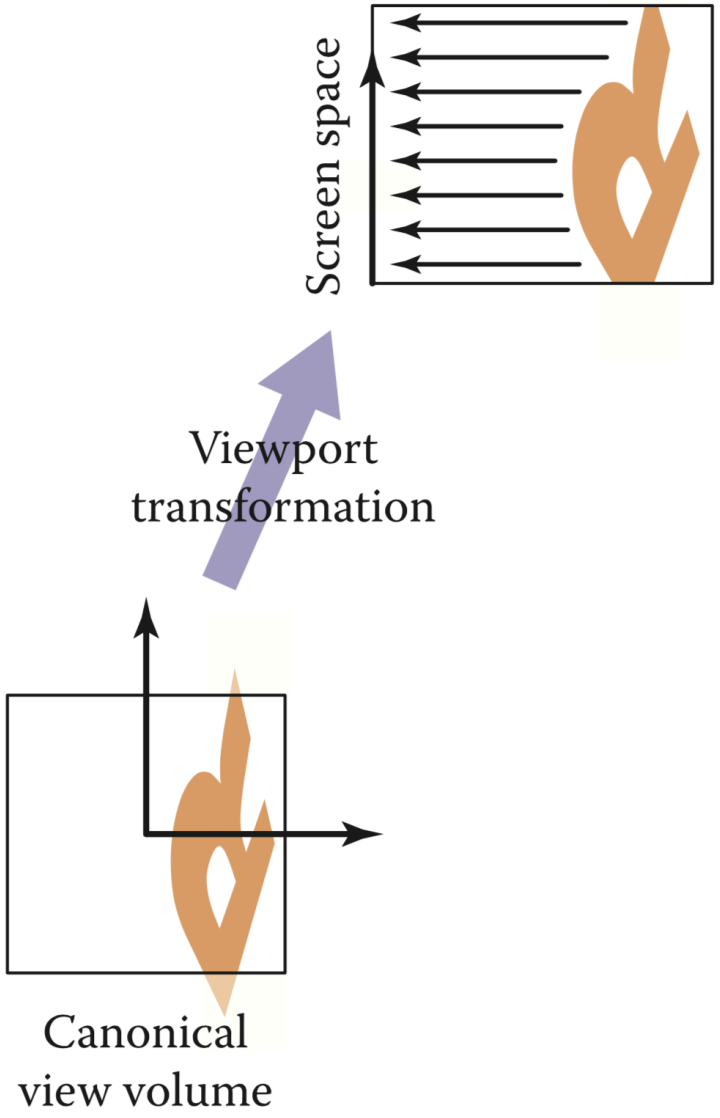
We'll Discuss Later



Viewport Transformation

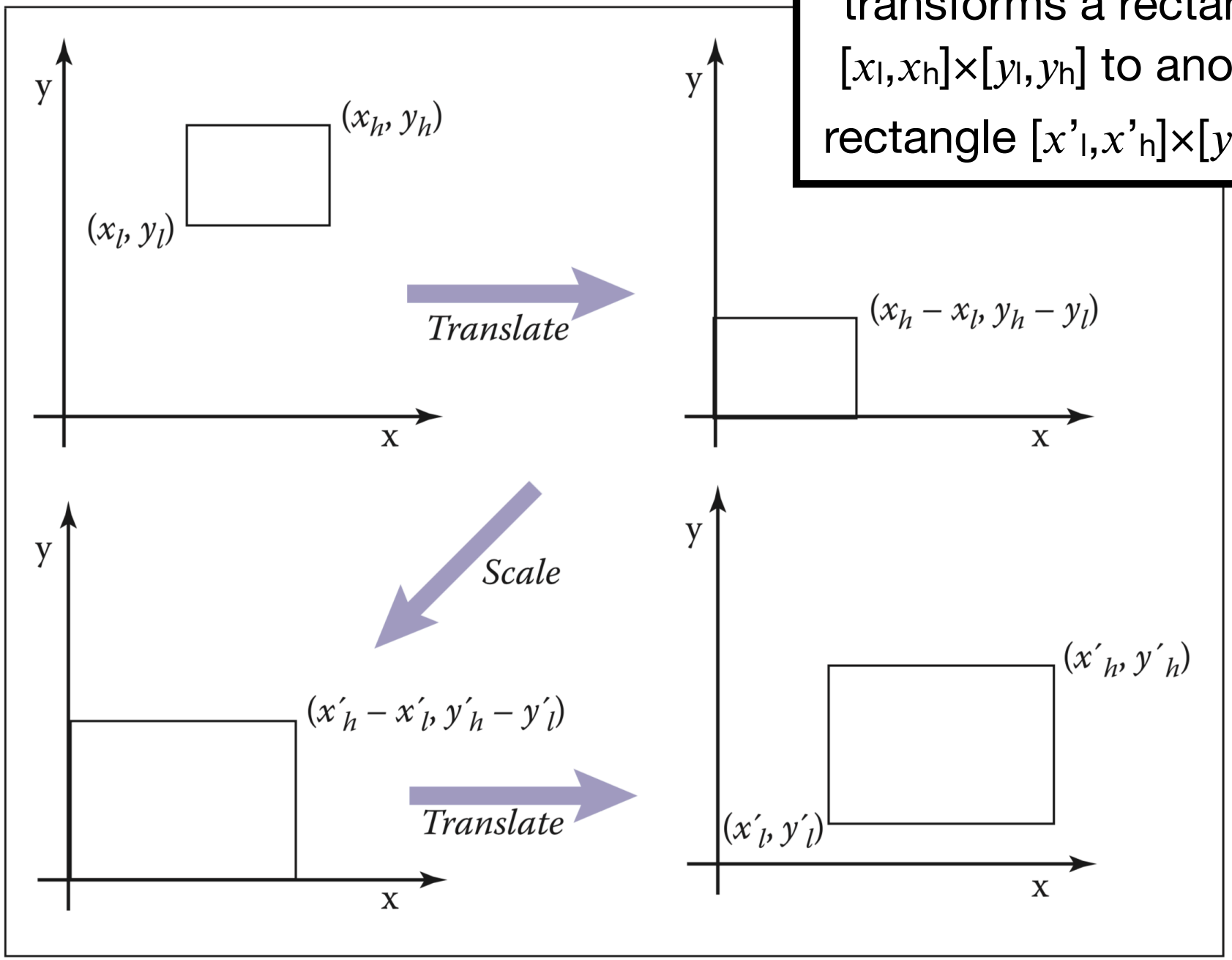
- Goal: Transform from a canonical 2D space to pixel coordinates
  - Canonical space:
    $(x_{canonical}, y_{canonical}) \in [-1,1] \times [-1,1]$
  - Pixel space: Integers
    $(x_{screen}, y_{screen}) \in [0\ ,\ n_x\ ] \times [0\ ,\ n_y]$
  - n_x, n_y. are the number of columns and rows
  - To be precise,

    $(x_{screen}, y_{screen}) \in [0.5, n_x - 0.5] \times [0.5, n_y - 0.5]$
- Initially, we will think of this as transformation of a 2D to 2D space



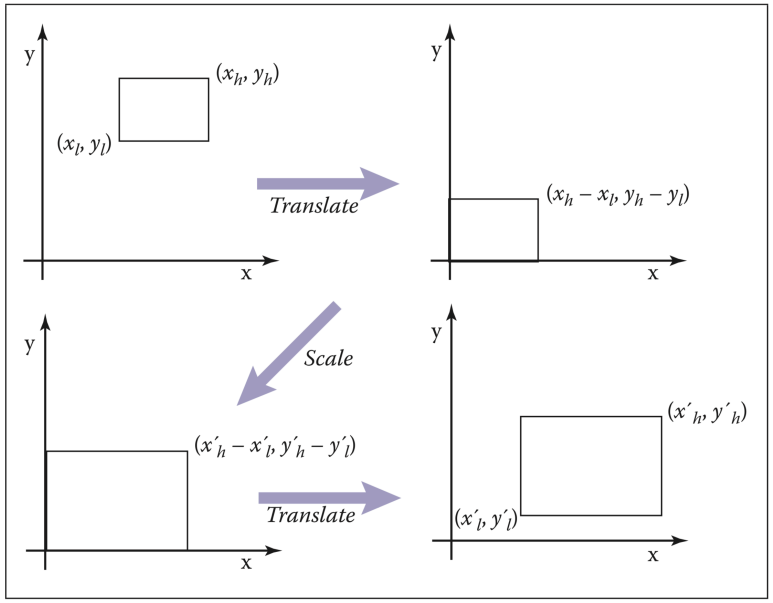
Viewports as Windowing

A **windowing operation** transforms a rectangle $[x_l, x_h] \times [y_l, y_h]$ to another rectangle $[x'_l, x'_h] \times [y'_l, y'_h]$



Viewports as Windowing

- Decompose windowing into three steps



$$\text{translate}(x'_l,\ y'_l)\ \text{scale}\left(\frac{x'_h - x'_l}{x_h - x_l}\ ,\ \frac{y'_h - y'_l}{y_h - y_l}\right)\ \ \text{translate}(-x_l, -y_l)$$

Viewports as Windowing

$$\begin{bmatrix} 1 & 0 & {x'}_l \\ 0 & 1 & {y'}_l \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{{x'}_h - {x'}_l}{x_h - x_l} & 0 & 0 \\ 0 & \frac{{y'}_h - {y'}_l}{y_h - y_l} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_l \\ 0 & 1 & -y_l \\ 0 & 0 & 1 \end{bmatrix}$$

- Multiplying together:

$$\begin{bmatrix} \frac{{x'}_h - {x'}_l}{x_h - x_l} & 0 & \frac{{x'}_l x_h - {x'}_h x_l}{x_h - x_l} \\ 0 & \frac{{y'}_h - {y'}_l}{y_h - y_l} & \frac{{y'}_l y_h - {y'}_h y_l}{y_h - y_l} \\ 0 & 0 & 1 \end{bmatrix}$$

Sidebar: Combining a 3x3 Linear Matrix Followed by a Translation

- Translation *after* the linear transformation can always be read off separately.

- Often useful for debugging.

$$\begin{bmatrix} 1 & 0 & 0 & x_t \\ 0 & 1 & 0 & y_t \\ 0 & 0 & 1 & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & x_t \\ a_{21} & a_{22} & a_{23} & y_t \\ a_{31} & a_{32} & a_{33} & z_t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using Windowing to define the Viewport Transformation

- Plugging in with our known constants:

- (recall - canonical are between [-1,1]. Screen in pixels)

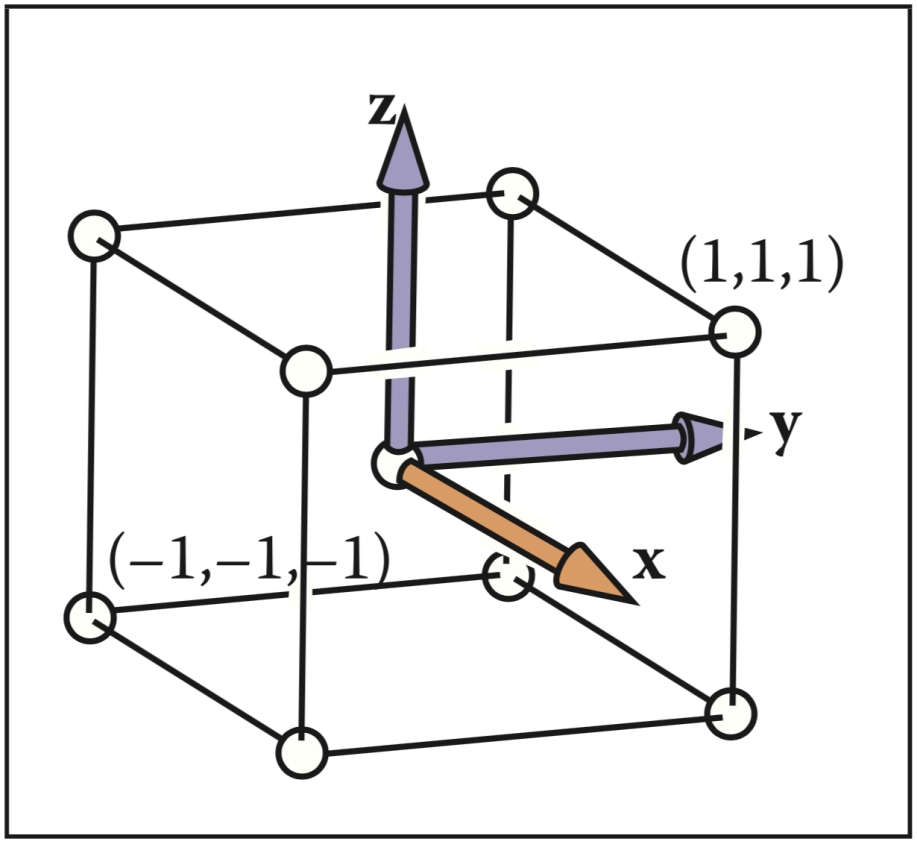
$$\begin{bmatrix} x_{\text{screen}} \\ y_{\text{screen}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{n_x}{2} & 0 & \frac{n_x-1}{2} \\ 0 & \frac{n_y}{2} & \frac{n_y-1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\text{canonical}} \\ y_{\text{canonical}} \\ 1 \end{bmatrix}$$

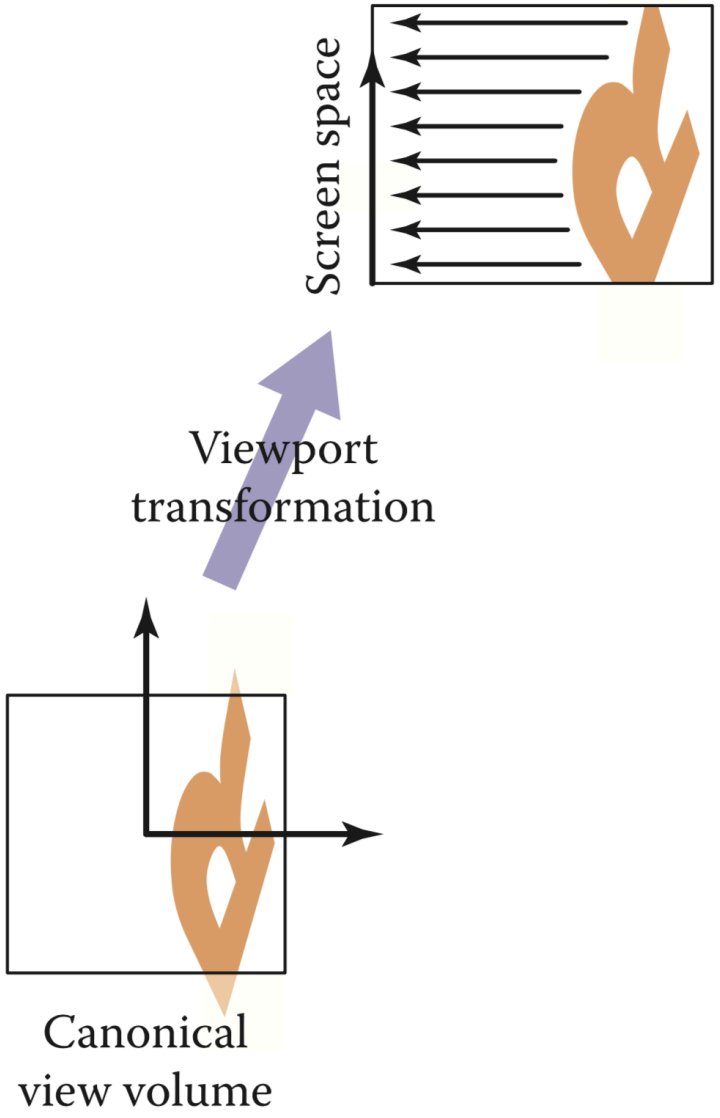
- Right now, we do not need z-values, but eventually we will need to carry them through with no changes:

$$M_{\text{vp}} = \begin{bmatrix} \frac{n_x}{2} & 0 & 0 & \frac{n_x-1}{2} \\ 0 & \frac{n_y}{2} & 0 & \frac{n_y-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Canonical View Volume

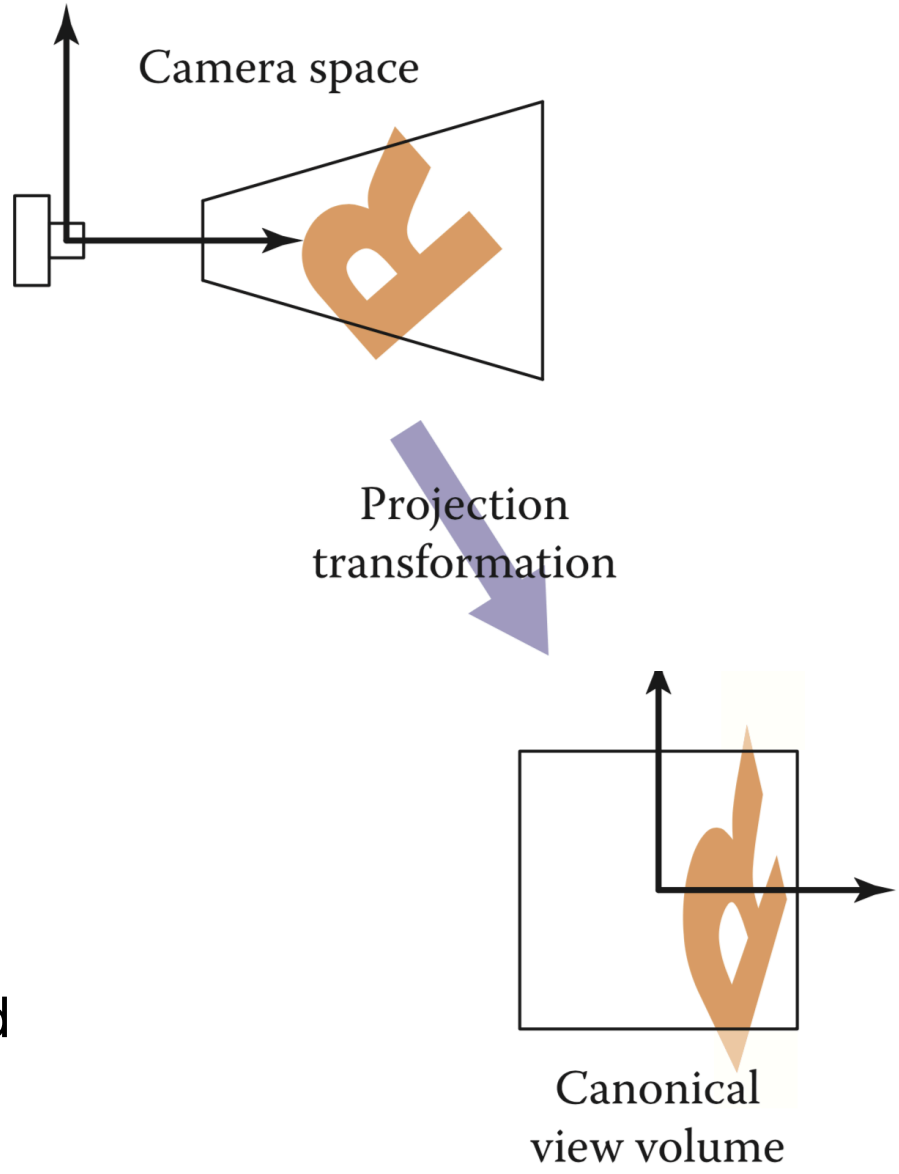
- In actuality, our viewport transformation will work with the **canonical view volume**





Orthographic Projection

- Goal: Convert objects from 3D representation to canonical view volume
- We will start by modeling this 3D space as an axis-aligned boxe
  - View volume: $[l,r]\times[b,t]\times[f,n]$
  - Canonical view volume: [-1,1]×[-1,1]×[-1,1]
- Reshapes the view volume as defined by the camera

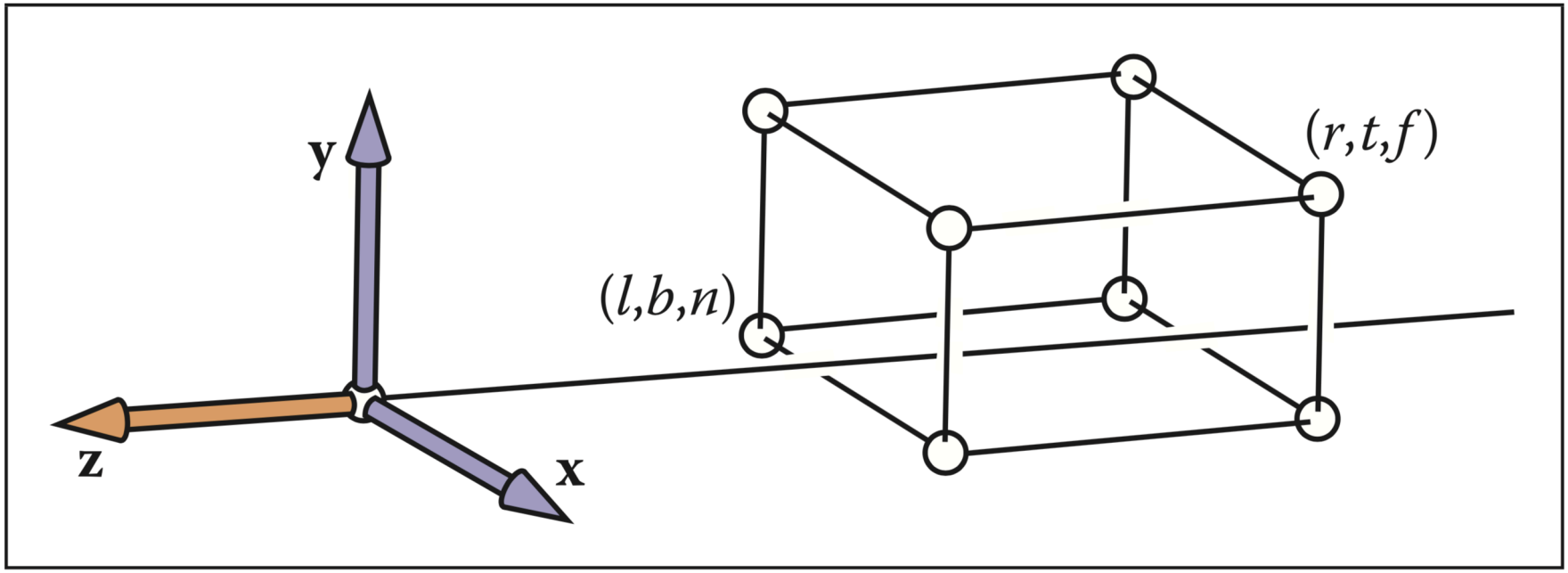


Orthographic Projection

- **Orthographic view volume** defined by six scalars:

- Convention: $n > f$, but note that both are *negative*

$$x = l \equiv \text{left plane},$$
$$x = r \equiv \text{right plane},$$
$$y = b \equiv \text{bottom plane},$$
$$y = t \equiv \text{top plane},$$
$$z = n \equiv \text{near plane},$$
$$z = f \equiv \text{far plane}.$$



Orthographic Projection

- Just a 3D windowing transformation!
- The left-lower-near point is transformed into (-1,-1,-1)
- The right-upper-top is transformed into (-1,-1,-1)

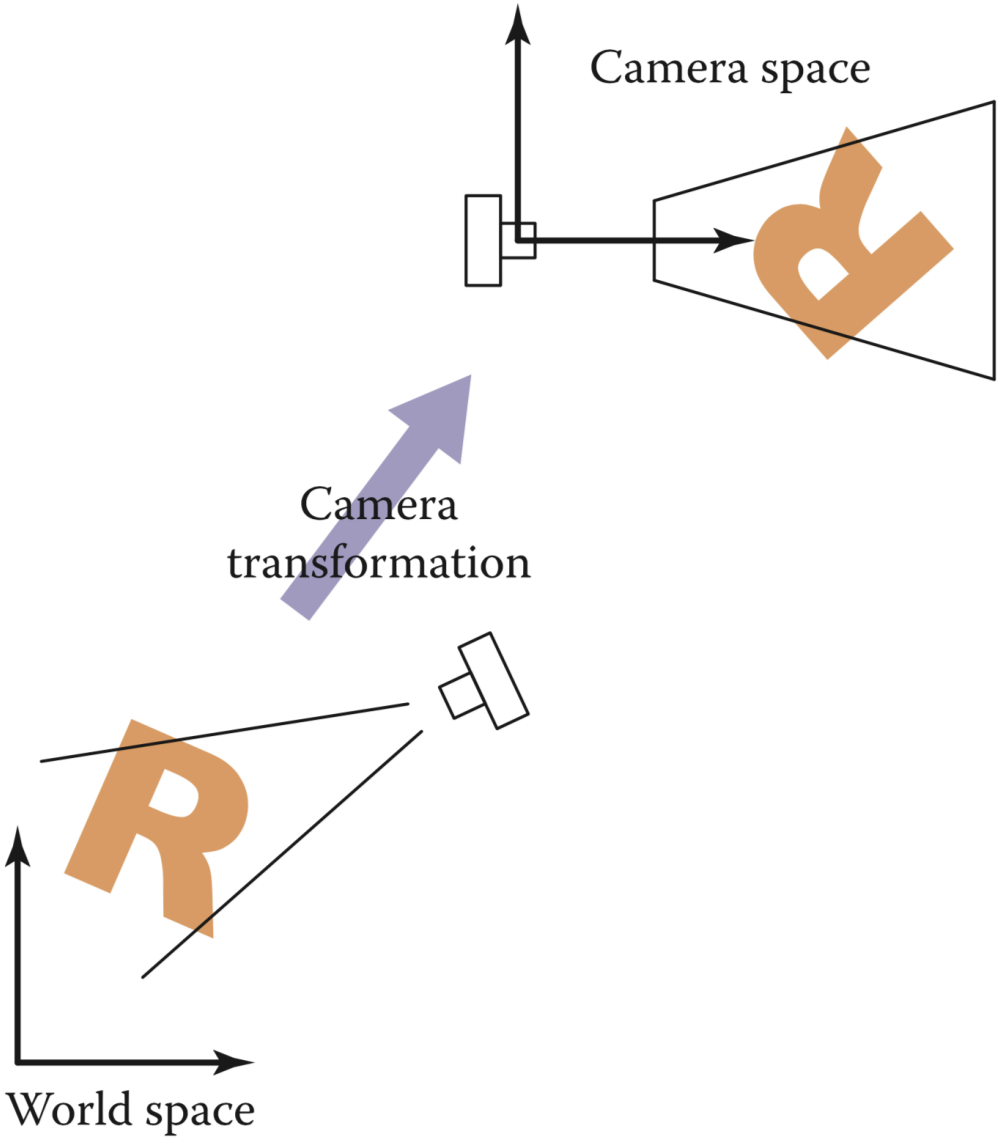
$$\begin{bmatrix} \frac{x'_h - x'_l}{x_h - x_l} & 0 & 0 & \frac{x'_l x_h - x'_h x_l}{x_h - x_l} \\ 0 & \frac{y'_h - y'_l}{y_h - y_l} & 0 & \frac{y'_l y_h - y'_h y_l}{y_h - y_l} \\ 0 & 0 & \frac{z'_h, z'_l}{z_h - z_l} & \frac{z'_l z_h - z'_h z_l}{z_h - z_l} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_{\text{orth}} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Camera Transformations

- Goal: Transform 3D space to arbitrary camera parameters

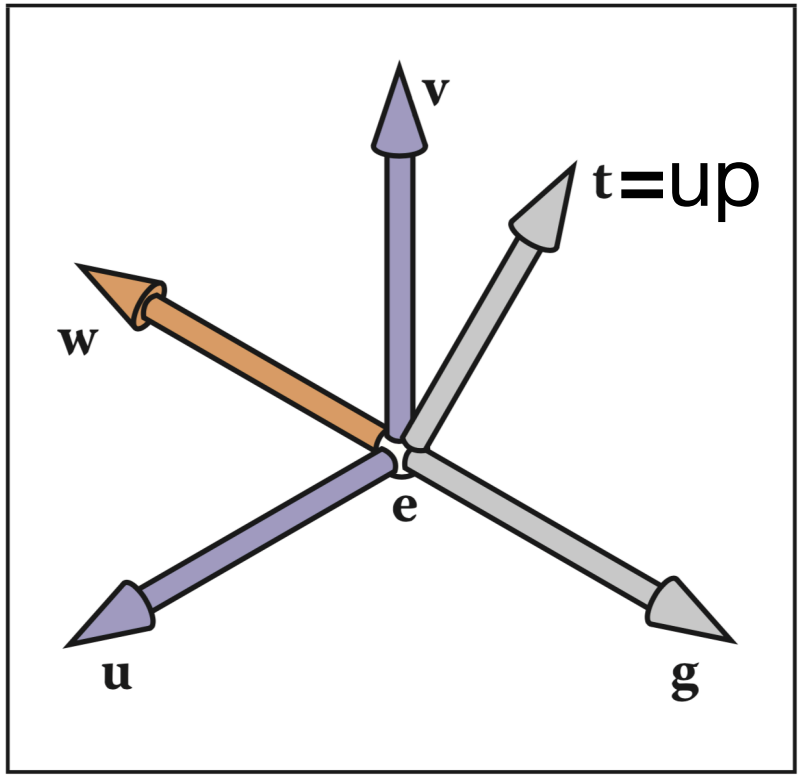
- Camera modeled with three vectors:
  - **e**, the eye position
  - **g**, the gaze direction
  - **t**, the view **up** direction



Camera Coordinates

- We will convert to a camera coordinate system with origin, $\mathbf{e}$, and orthogonal basis vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

$\mathbf{e}$, the eye position
$\mathbf{g}$, the gaze direction
$\mathbf{t}$, the view up direction

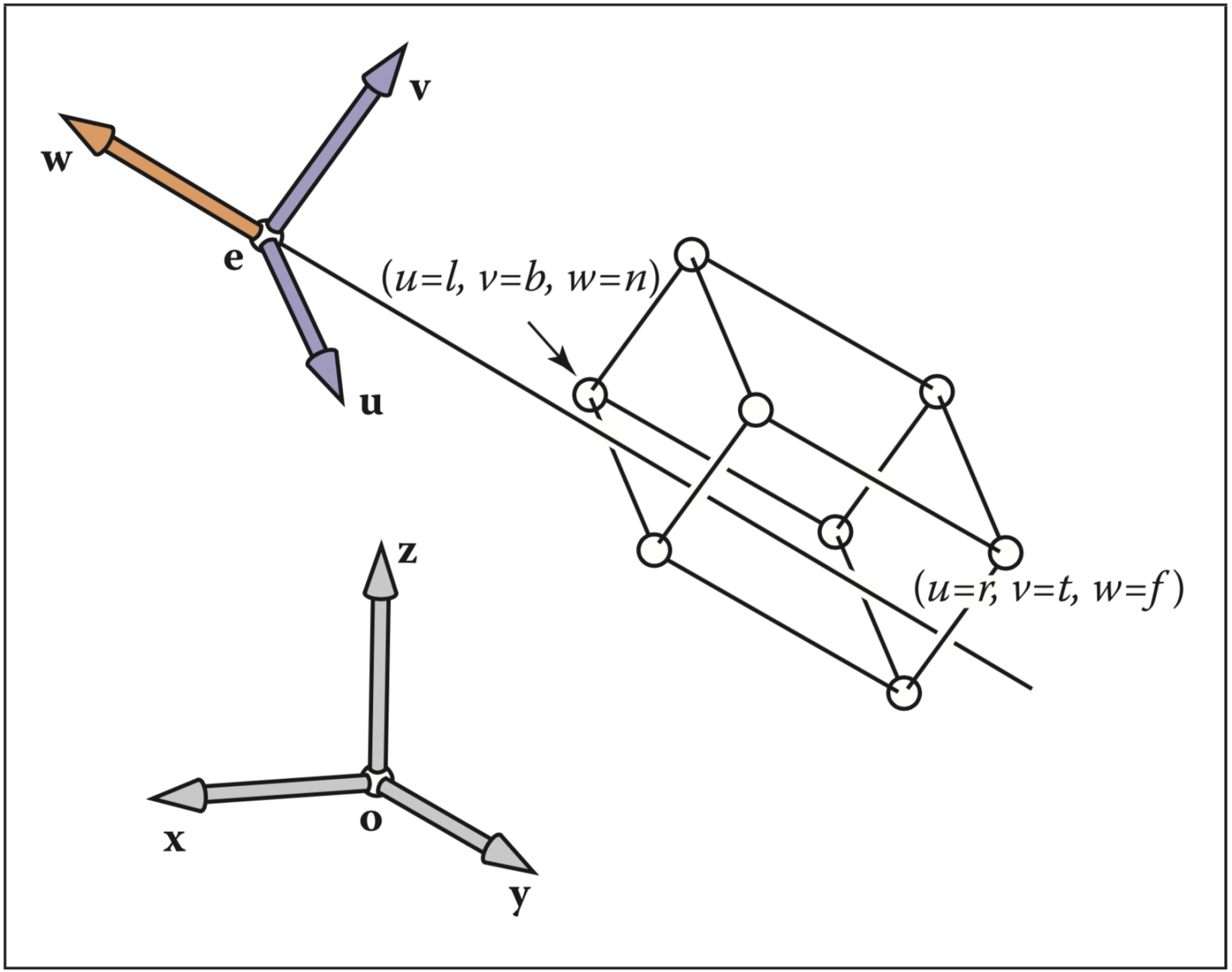


$$\mathbf{w} = -\frac{\mathbf{g}}{\|\mathbf{g}\|},$$

$$\mathbf{u} = -\frac{\mathbf{t}\times\mathbf{w}}{\|\mathbf{t}\times\mathbf{w}\|},$$

$$\mathbf{v} = \mathbf{w}\times\mathbf{u}.$$

Camera Coordinates



Changing Coordinates

- We need to both translate the origin and change coordinate systems

$$\mathbf{M}_{\text{cam}} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} x_u & y_u & z_u & 0 \\ x_v & y_v & z_v & 0 \\ x_w & y_w & z_w & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_e \\ 0 & 1 & 0 & -y_e \\ 0 & 0 & 1 & -z_e \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So this matrix moves the 'eye' to the origin, are rotates, such that points on the w-axis, are rotates into points on the z-axis
points on the v-axis, are rotated into points on the y-axis
points on the u-axis, are rotated into points on the x-axis

Viewing Algorithm

```
construct M_vp //viewport
construct M_orth
construct M_cam
M = M_vp * M_orth * M_cam
for each 3D object O {
  O_screen = M * O
  draw(O_screen)
}
```

For example, if O is a triangle, with vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, transform all 3 vertices $\mathbf{Ma}$, $\mathbf{Mb}$, and $\mathbf{Mc}$

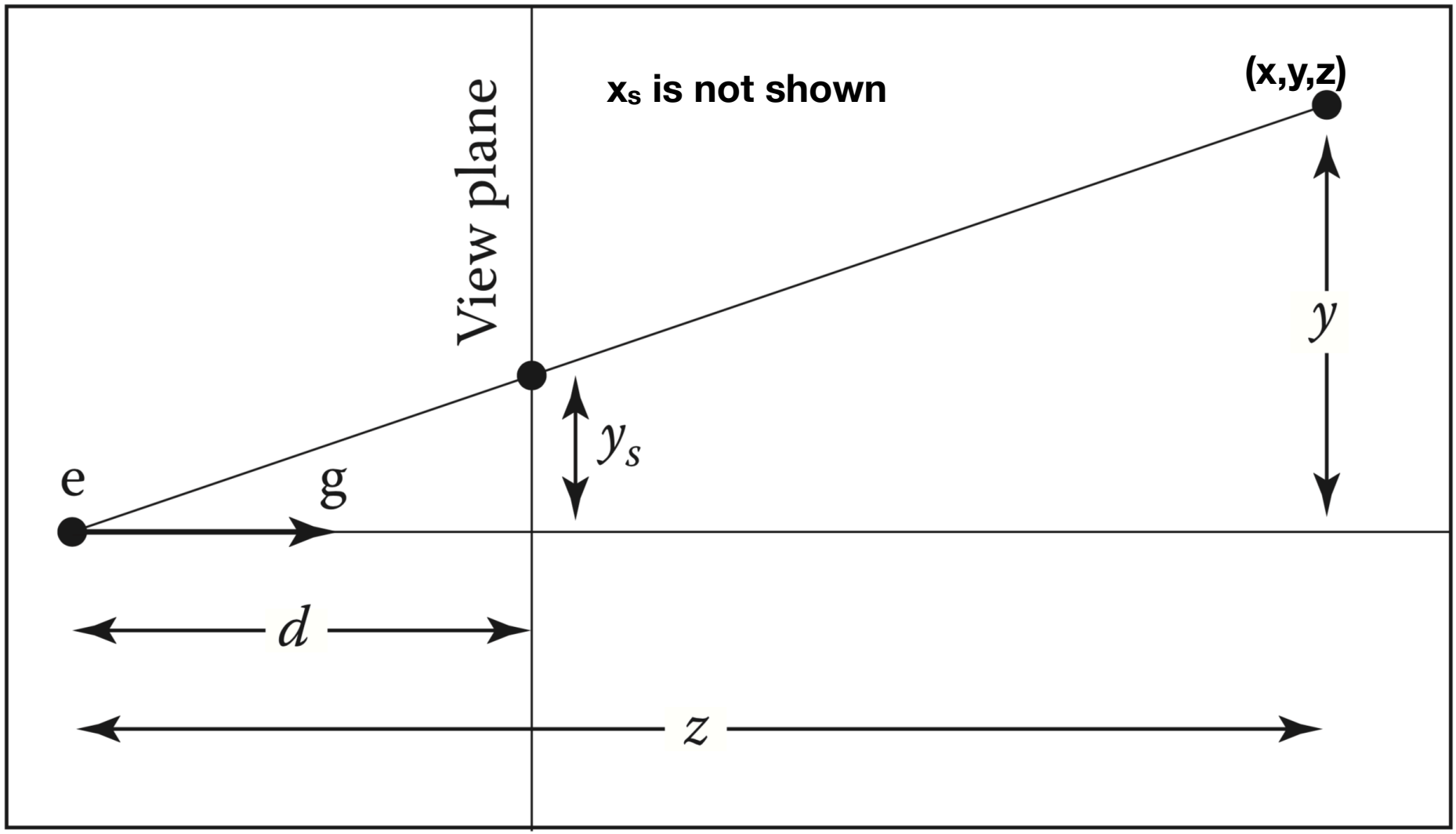
Projective Transformations

Relative Size Based on Distance

- Key idea of perspective: the size of an object on the screen is proportional to 1/z

$$y_s = \frac{d}{z} y$$

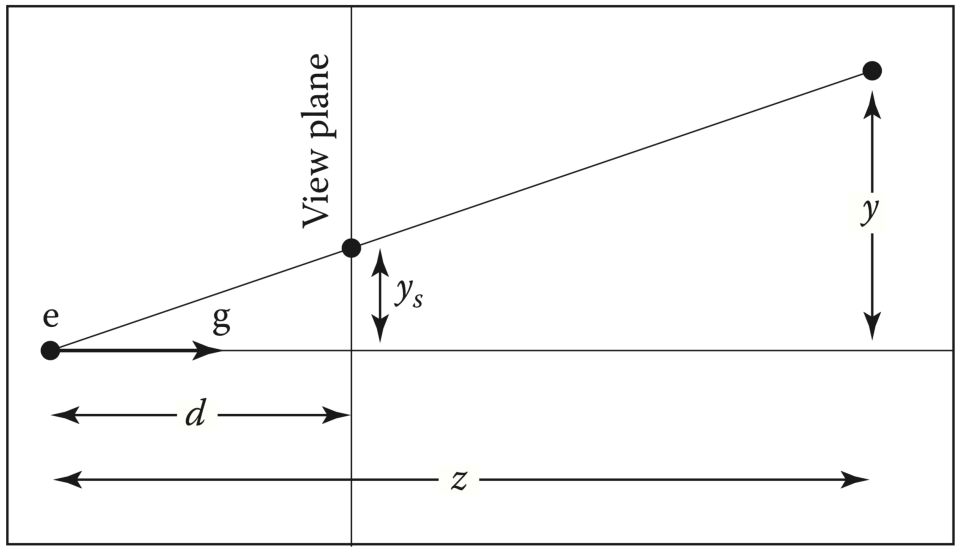
(x_s,y_s) are coordinates of the point (x,y,z) on the screen



Using Homographies for Perspective

- We can now replace:

$$y_s = \frac{d}{z}\, y$$



- With:

$$\begin{bmatrix} y_s \\ 1 \end{bmatrix} \sim \begin{bmatrix} d & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \begin{pmatrix} \frac{d}{z} y \\ 1 \end{pmatrix}$$

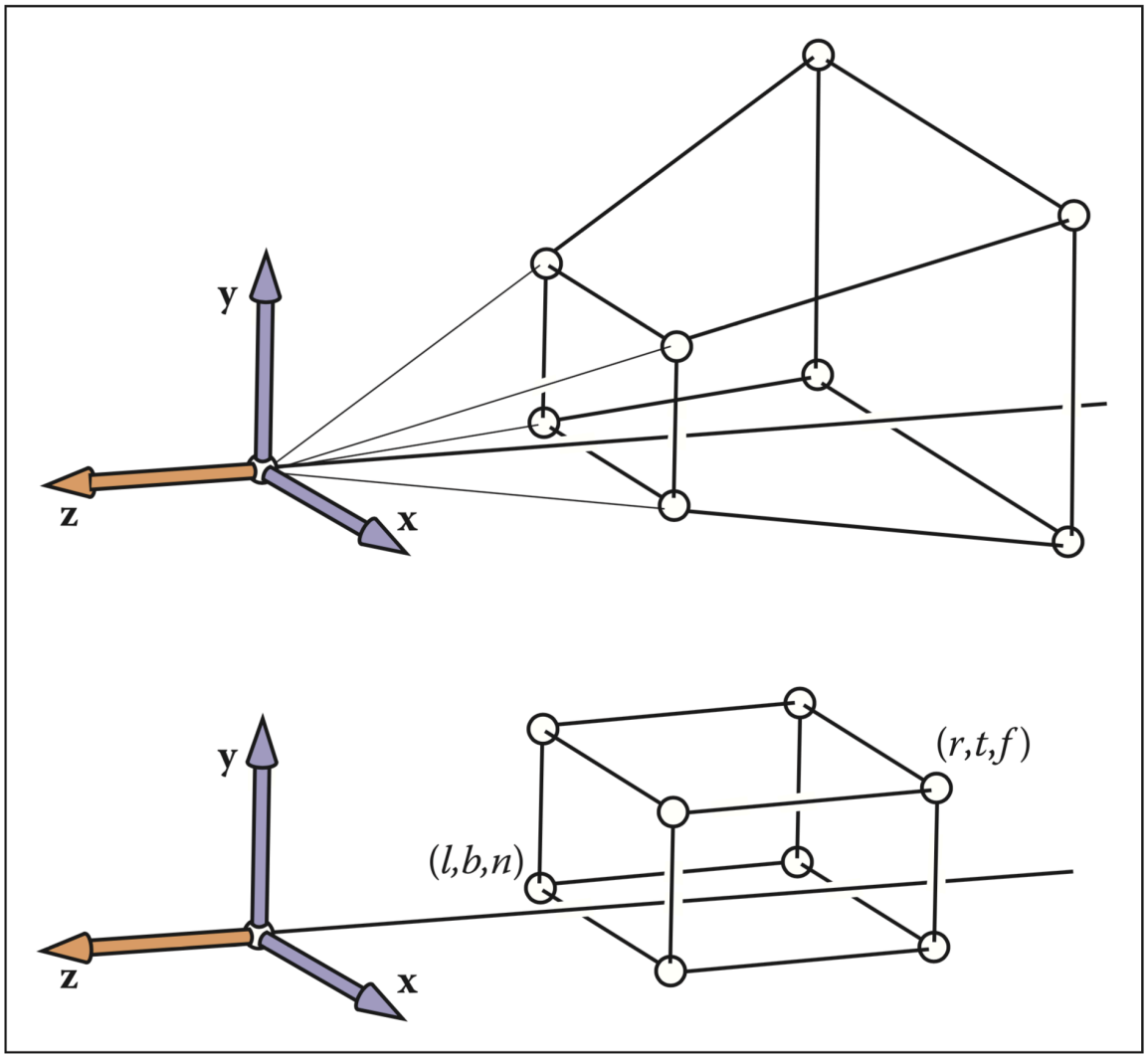
The ~ sign means “equal in Homogeneous coordinates”

Perspective Matrix

- Our matrix:

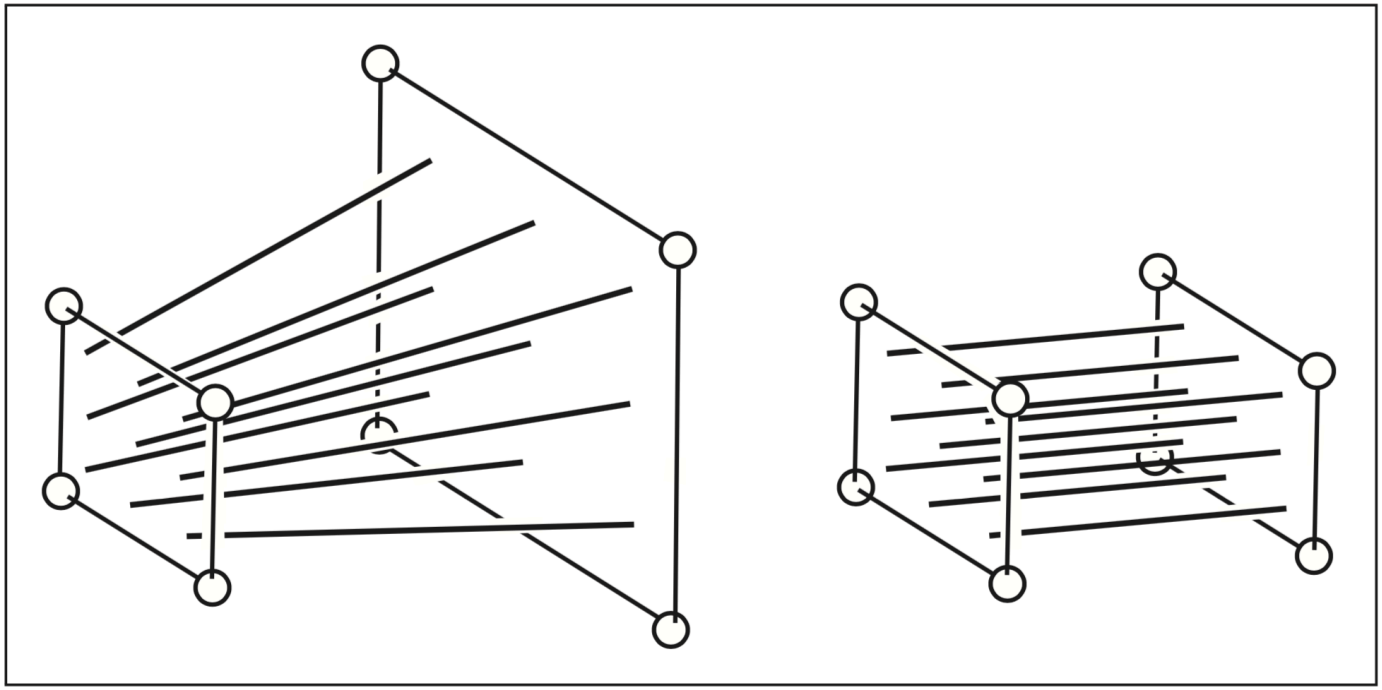
$$\mathbf{P} = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Keeps near plane fixed, maps far plane to back of the box



Perspective Distortion

- Effect on view rays / lines:



- Note that affine transformation cannot do this because it keeps parallel lines parallel

Perspective Distortion

- Perspective matrix effect on coordinates is nonlinear distortion in z:

$$\begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- But it does, however, preserve order in the z-coordinate (which will become useful very soon)

Perspective Projection Matrix

- Concatenating the perspective matrix with the orthographic projection provides the perspective projection matrix:

$$\mathbf{M}_{\text{per}} = \mathbf{M}_{\text{orth}}\mathbf{P} \qquad \mathbf{M}_{\text{per}} = \begin{bmatrix} \frac{2n}{r-l} & 0 & \frac{l+r}{l-r} & 0 \\ 0 & \frac{2n}{t-b} & \frac{b+t}{b-t} & 0 \\ 0 & 0 & \frac{f+n}{n-f} & \frac{2fn}{f-n} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- We can define l, r, b, and t relative to the near plane, since we keep it fixed

Putting it all together

```
construct M_vp
construct M_per
construct M_cam
M = M_vp * M_per * M_cam
for each 3D object O {
  O_screen = M * O
  draw(O_screen)
}
```

Equivalently:

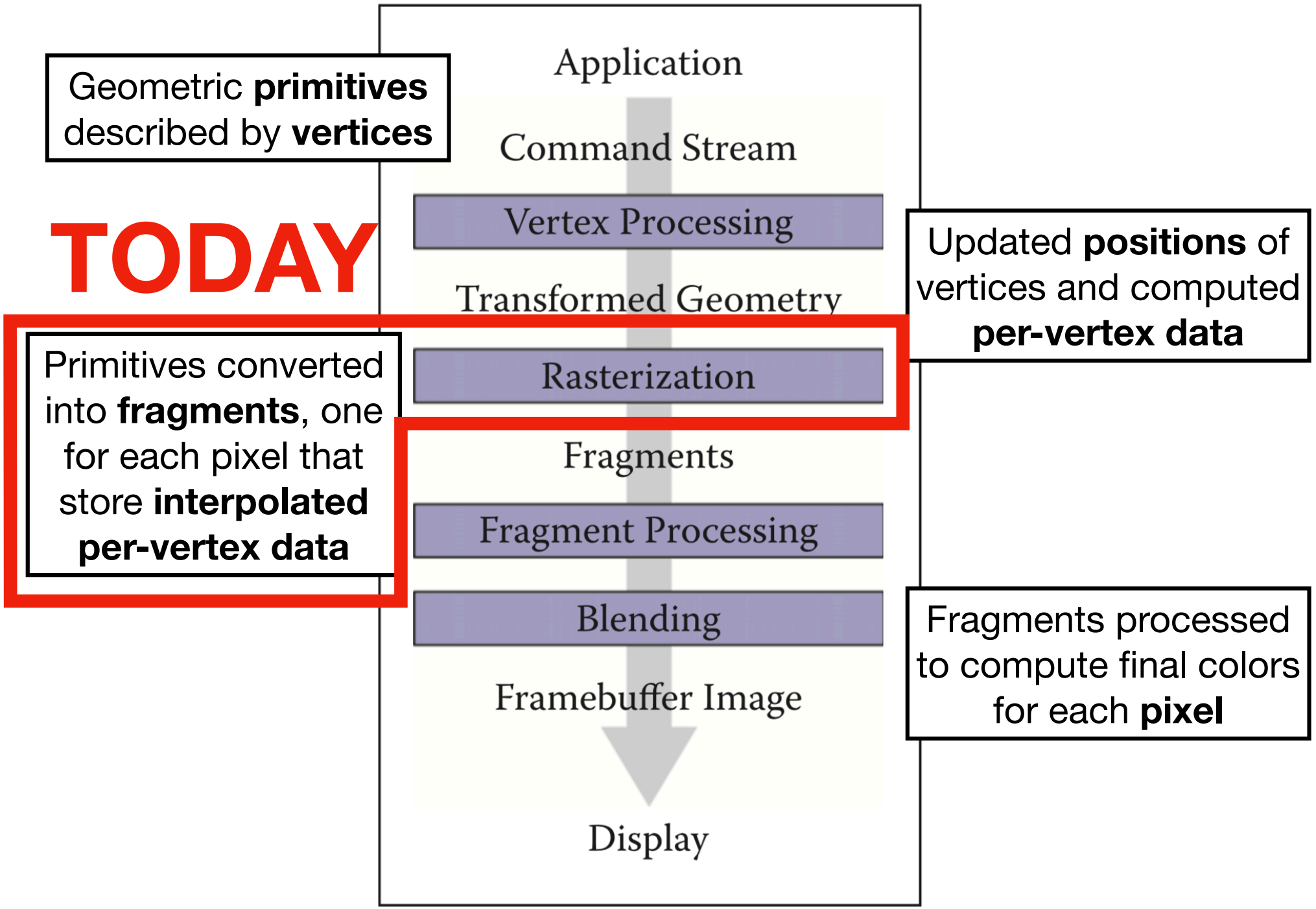
$$\mathbf{M} = \mathbf{M}_{\text{vp}}\mathbf{M}_{\text{orth}}\mathbf{P}\mathbf{M}_{\text{cam}}$$

For a given vertex $\mathbf{a} = (x, y, z)$, $\mathbf{p} = \mathbf{M}\mathbf{a}$ should result in drawing $(x_\mathbf{p}/w_\mathbf{p}, y_\mathbf{p}/w_\mathbf{p}, z_\mathbf{p}/w_\mathbf{p})$ on the screen

The Graphics Pipeline

The Graphics Pipeline

- The sequence of operations that we'll perform to transform from objects in 3D to color pixels on the screen.
  - A "pipeline" because there are many stages
- All variants of object-ordered rendering follow a similar sequence to what we'll discuss
- Generally, there are plenty of opportunities for optimization in this and it can be much faster than raytracing with the right hardware


Geometric **primitives** described by **vertices**
TODAY
Application
Command Stream
Vertex Processing
Transformed Geometry
Rasterization
Fragments
Fragment Processing
Blending
Framebuffer Image
Display
Updated **positions** of vertices and computed **per-vertex data**
Primitives converted into **fragments**, one for each pixel that store **interpolated per-vertex data**
Fragments processed to compute final colors for each **pixel**

Geometric Primitives

- Points, lines, and triangles **only**.
- What about?
  - Curves/Text? Approximate them with chains of line segments
  - Polygons? Break them up into triangles
  - Curved surfaces? Approximate them with triangles
- Using minimal primitives keeps the pipeline simple and uniform for efficiency.

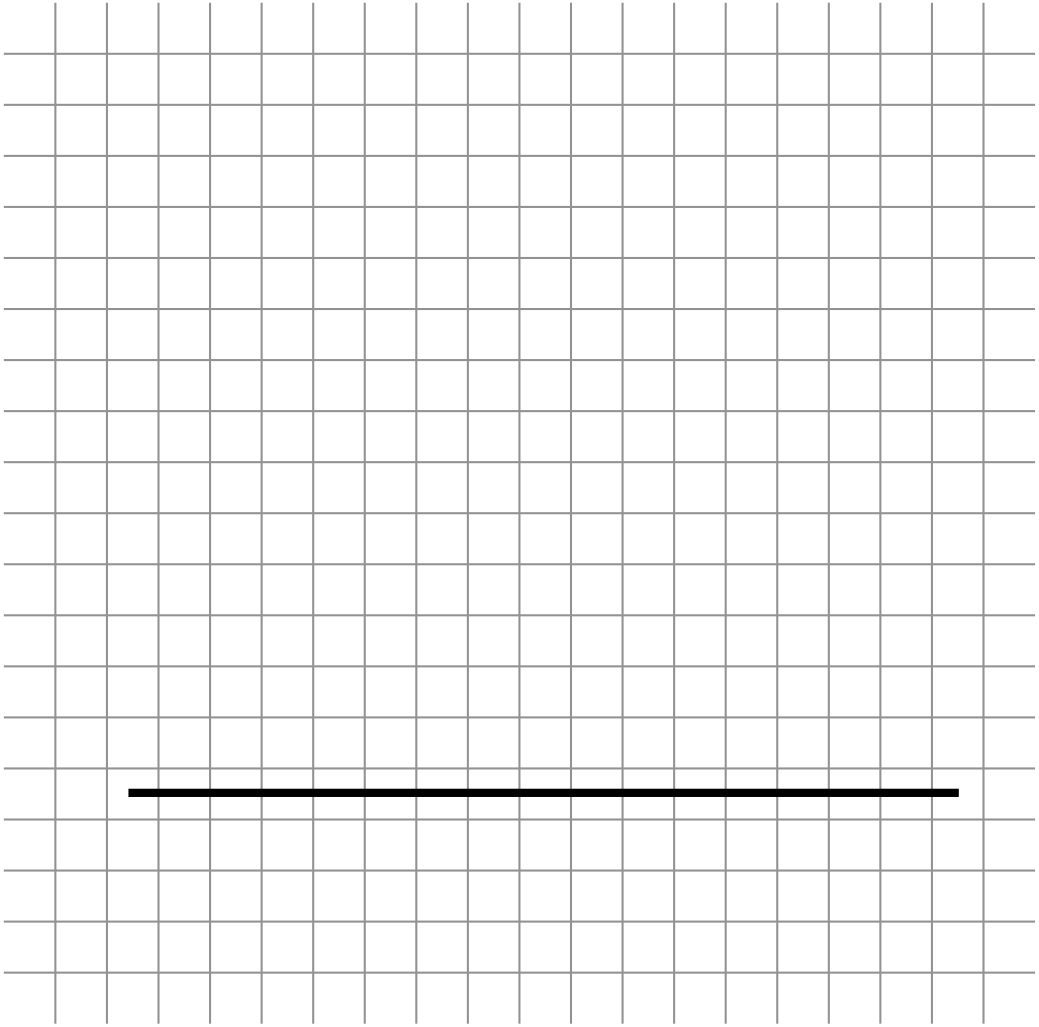
Rasterization

Two Jobs of a Rasterizer

- To *enumerate* the pixels that are covered by each primitive
- To *interpolate* the values/attributes across the primitives, e.g. normals, colors, texture coordinates
- Output: a set of **fragments**, one for each pixel.
  - Fragments live at a pixel and store the interpolated attributes.

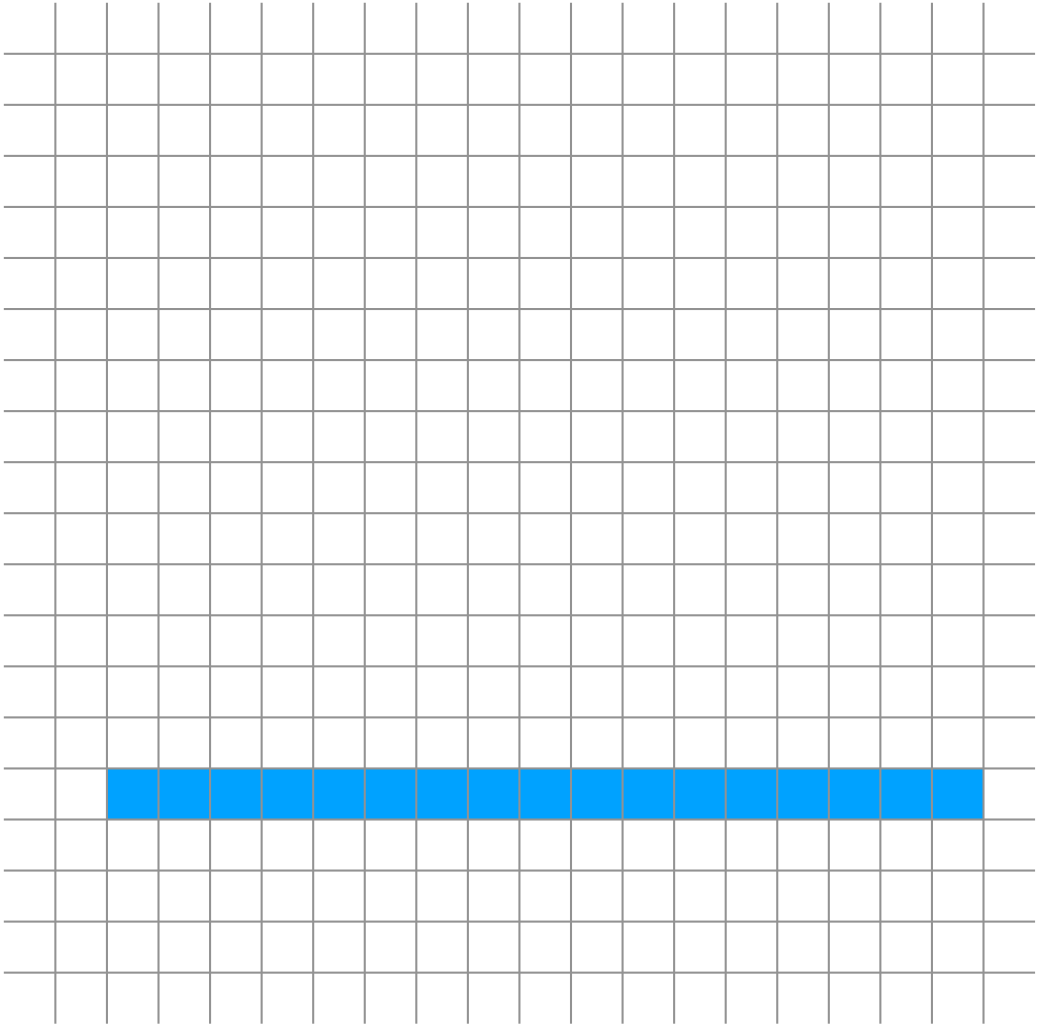
Rasterizing Lines to Fragments w/ Point Sampling

- Option 1: Fatten the line and decide which pixels hit the rectangle



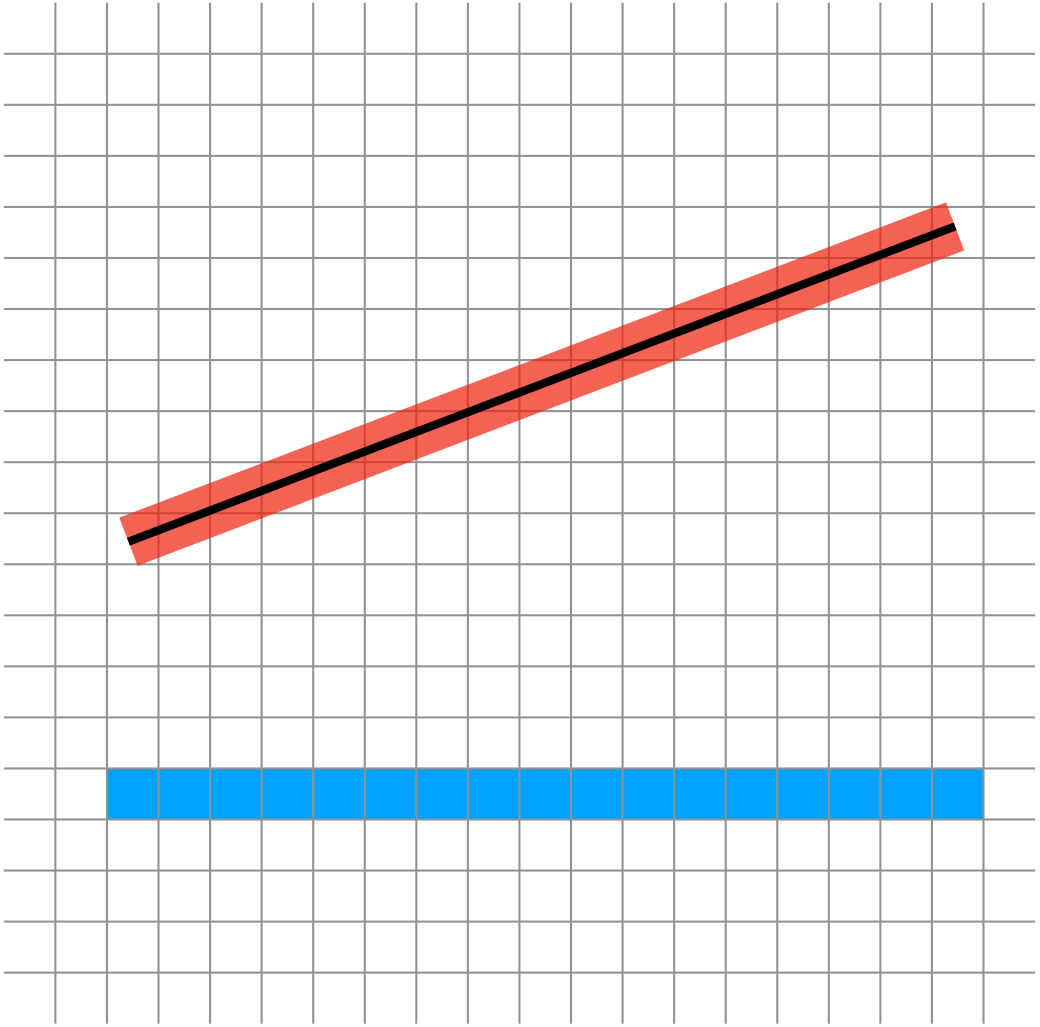
Rasterizing Lines to Fragments w/ Point Sampling

- Option 1: Fatten the line and decide which pixels hit the rectangle



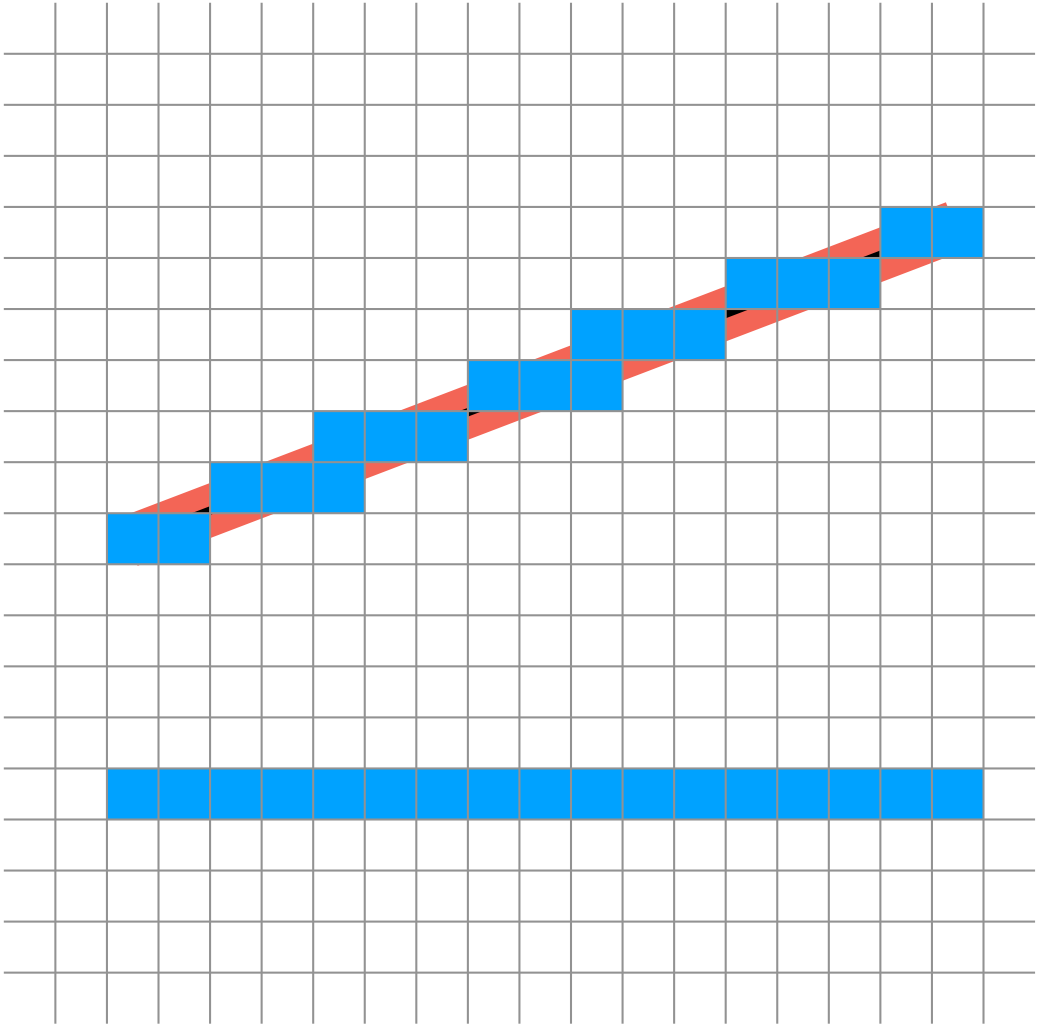
Rasterizing Lines to Fragments w/ Point Sampling

- Option 1: Fatten the line and decide which pixels hit the rectangle
- Decide which pixels are turned on if their center is inside the line



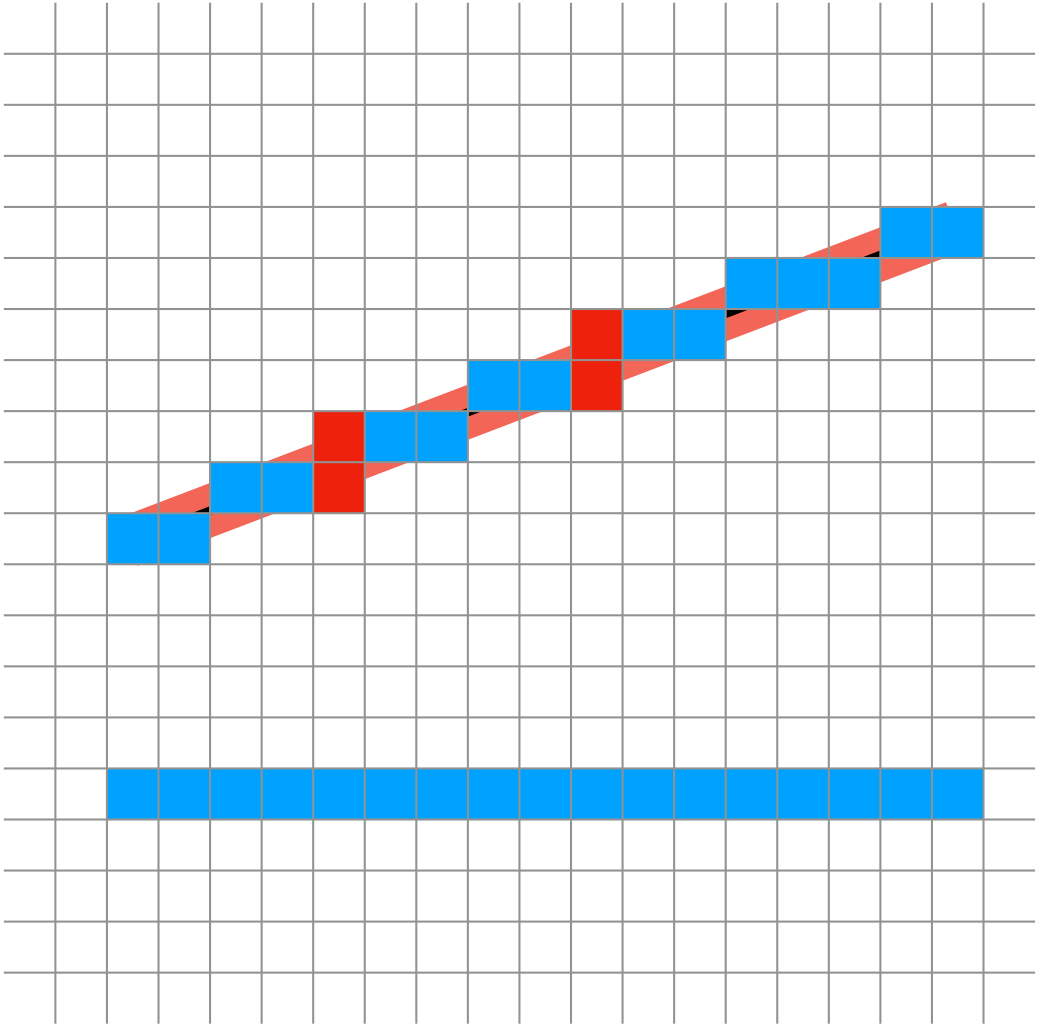
Rasterizing Lines to Fragments w/ Point Sampling

- Option 1: Fatten the line and decide which pixels hit the rectangle
- Decide which pixels are turned on if their center is inside the line

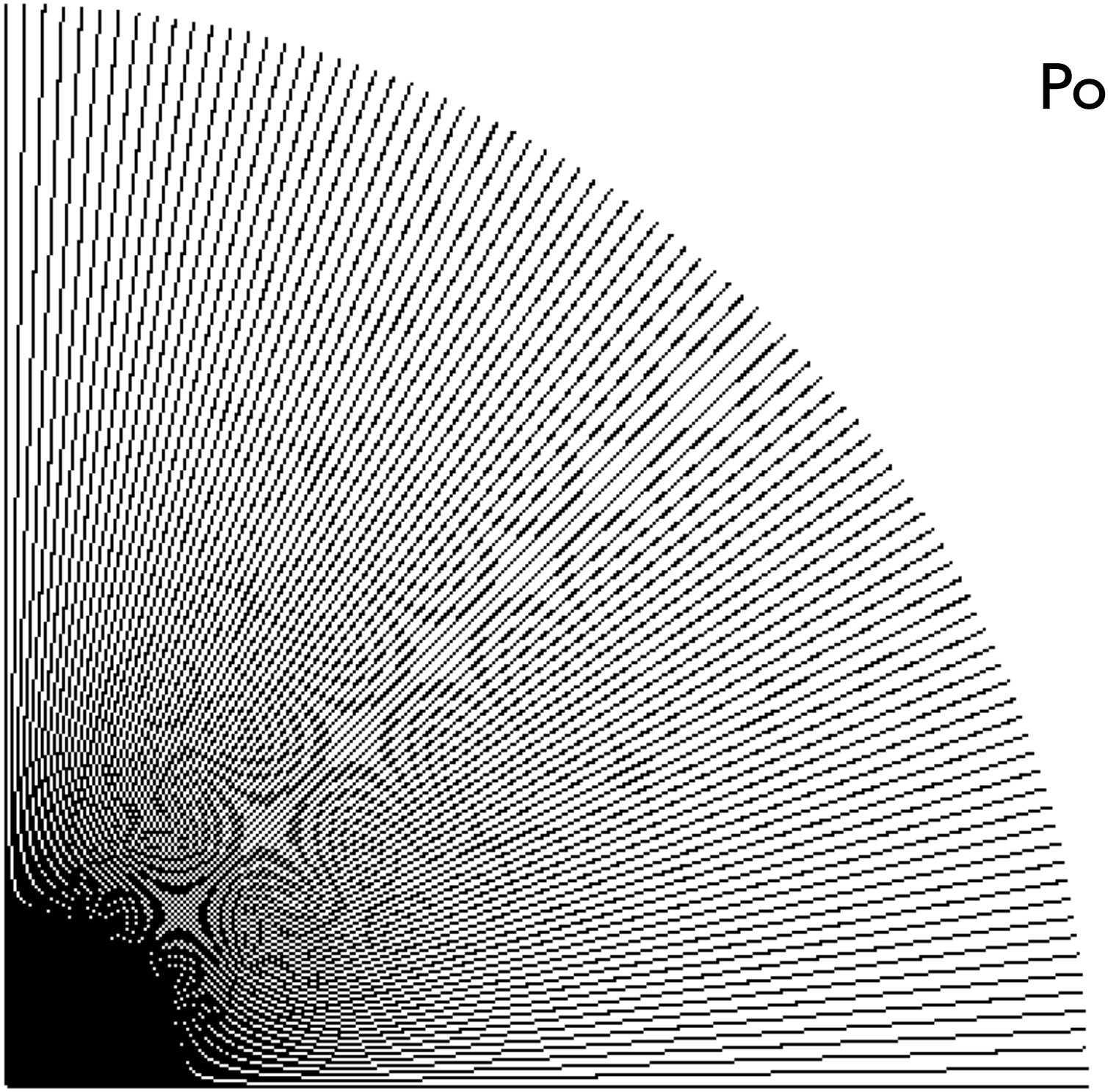


Rasterizing Lines to Fragments w/ Point Sampling

- Option 1: Fatten the line and decide which pixels hit the rectangle
- Decide which pixels are turned on if their center is inside the line
- Problem: line width is not kept constant



Point sampling in action



Bresenham's Algorithm

- Foundational algorithm in computer graphics while at IBM from
- Published in 1965

An algorithm is given for computer control of a digital plotter.

The algorithm may be programmed without multiplication or division instructions and is efficient with respect to speed of execution and memory utilization.

Algorithm for computer control of a digital plotter

by J. E. Bresenham

This paper describes an algorithm for computer control of a type of digital plotter that is now in common use with digital computers.[1]

The plotter under consideration is capable of executing, in response to an appropriate pulse, any one of the eight linear movements shown in Figure 1. Thus, the plotter can move linearly from a point on a mesh to any adjacent point on the mesh. A typical mesh size is 1/100th of an inch.

The data to be plotted are expressed in an (x, y) rectangular coordinate system which has been scaled with respect to the mesh; i.e., the data points lie on mesh points and consequently have integral coordinates.

It is assumed that the data include a sufficient number of appropriately selected points to produce a satisfactory representation of the curve by connecting the points with line segments, as illustrated in Figure 2. In Figure 3, the line segment connecting

Figure 1 Plotter movements

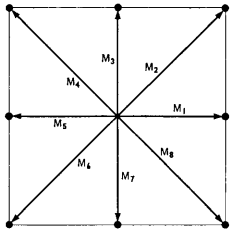


Figure 2 Curve defined by linear segments joining data points

Equations for a line

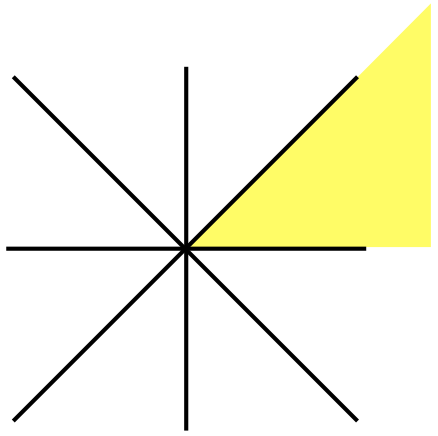
- Given points (x_0, y_0) and (x_1, y_1), the implicit equation for the line is:

$$f(x, y) \equiv (y_0 - y_1)x + (x_1 - x_0)y + x_0 y_1 - x_1 y_0 = 0$$

- We'll consider lines only defined in one octant: $x_0 \leq x_1$ (otherwise, swap the points) and that the slope of the line, $m \in (0,1]$ where

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

- But the same technique can be used for any octant

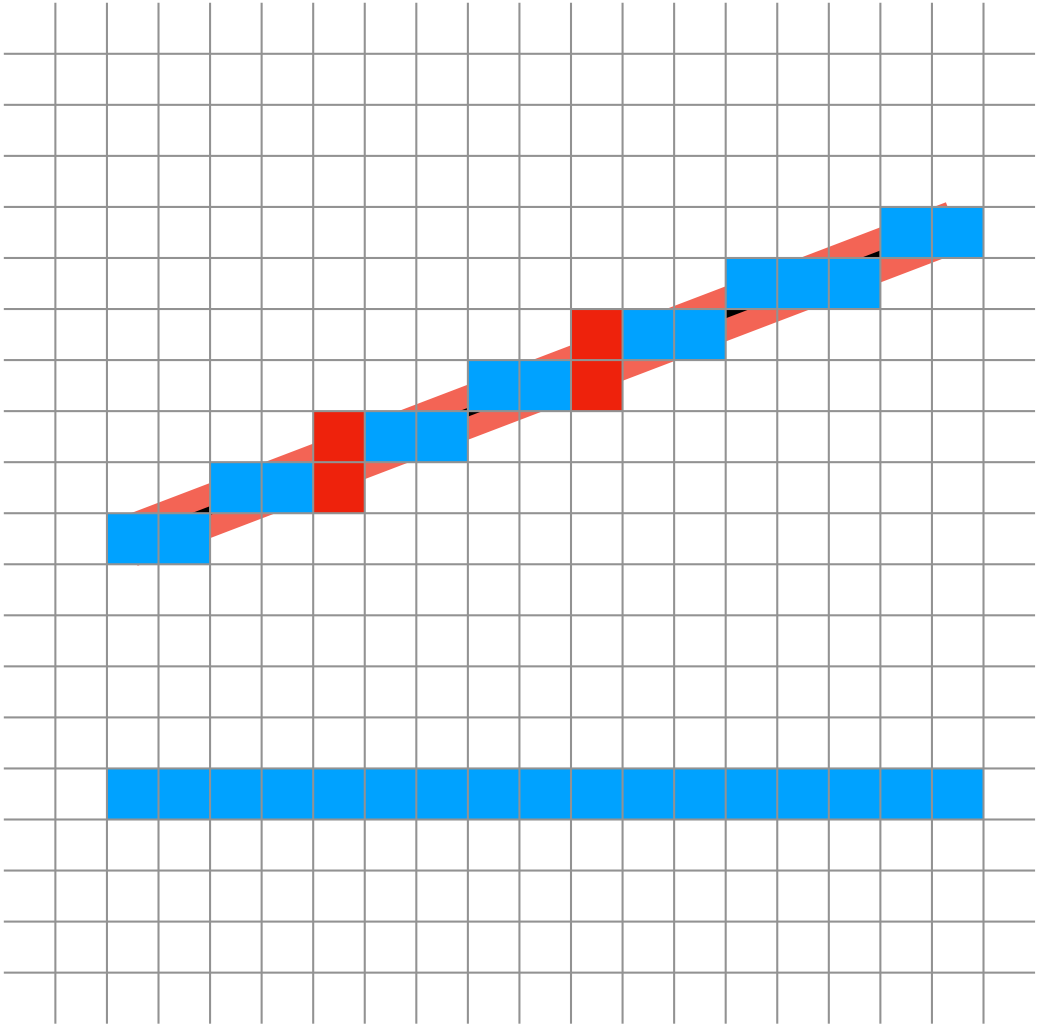


Bresenham's Approach

- Key idea: only turn on one pixel per column
- Multiple variants of this algorithm, we'll discuss the **midpoint algorithm**

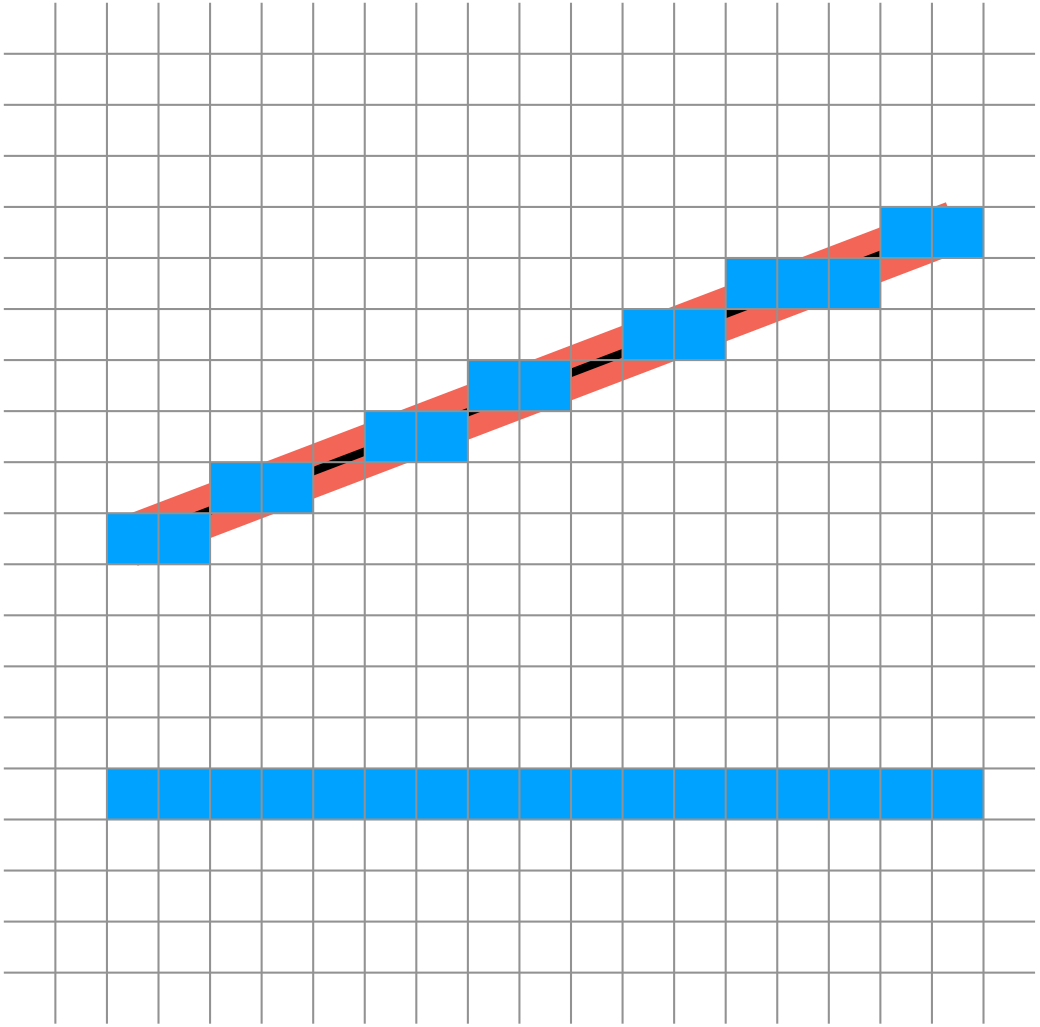
Resolving Sampling Issues with the Midpoint Algorithm

- Turn on only the pixel closest to the line in each column

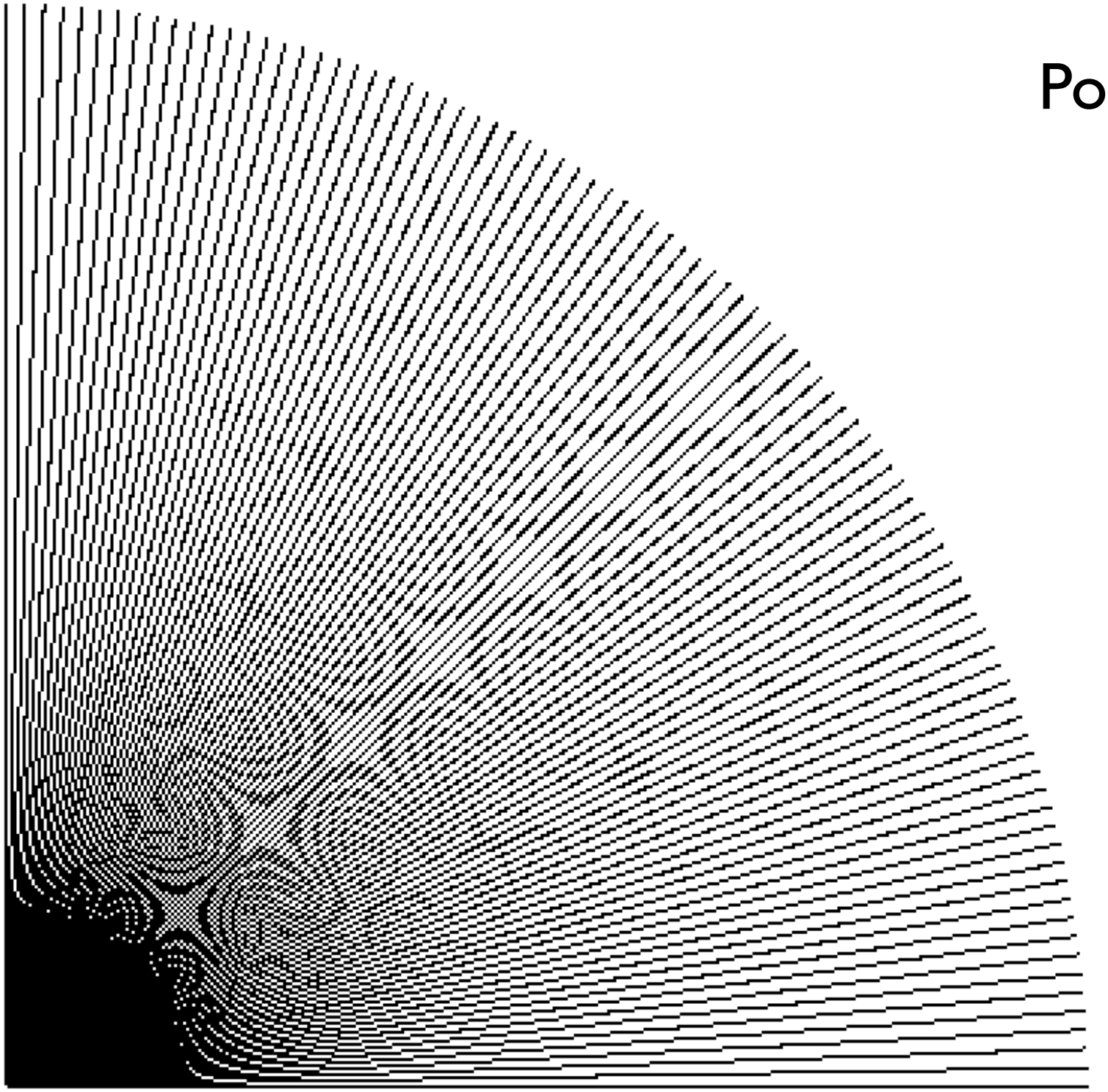


Resolving Sampling Issues with the Midpoint Algorithm

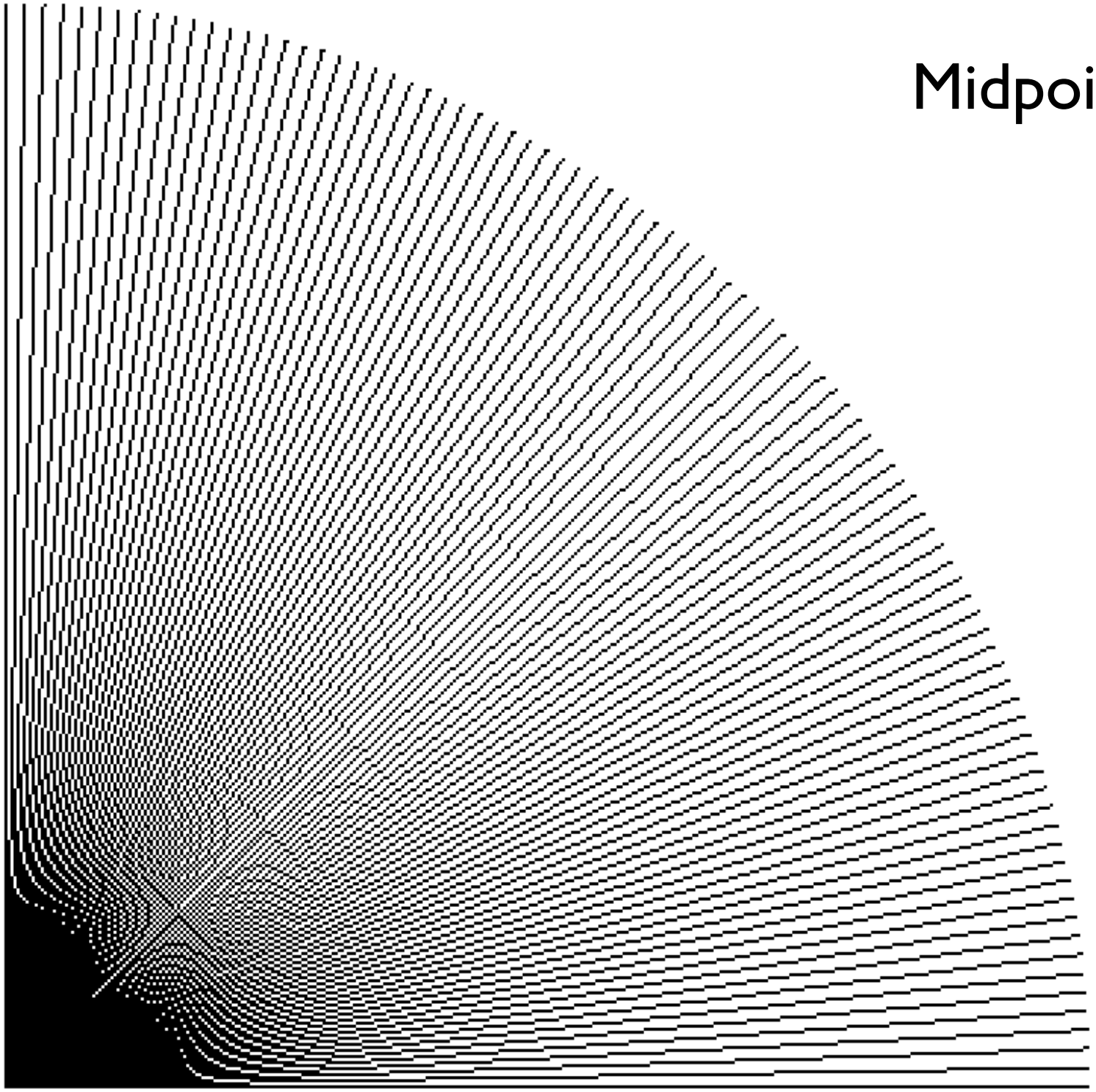
- Turn on only the pixel closest to the line in each column



Point sampling in action

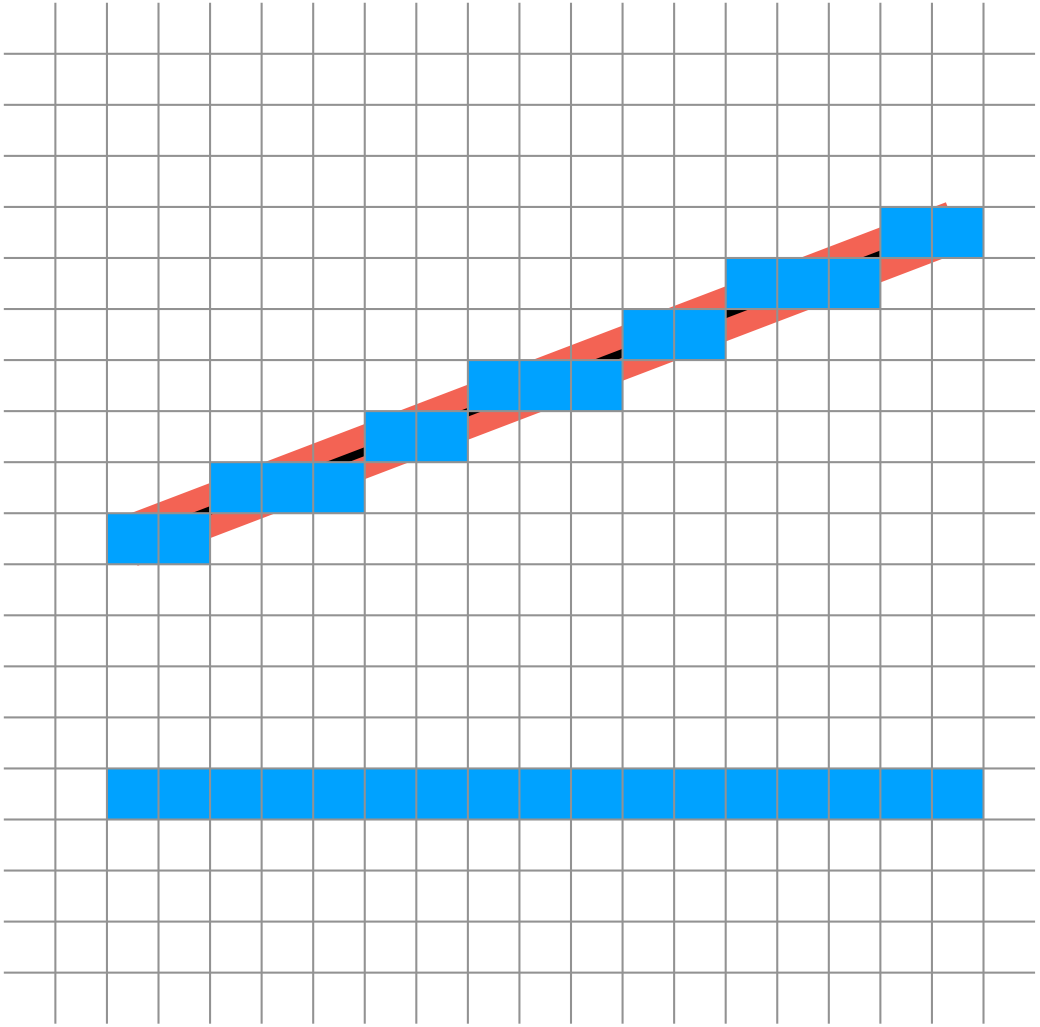


Midpoint algorithm in action

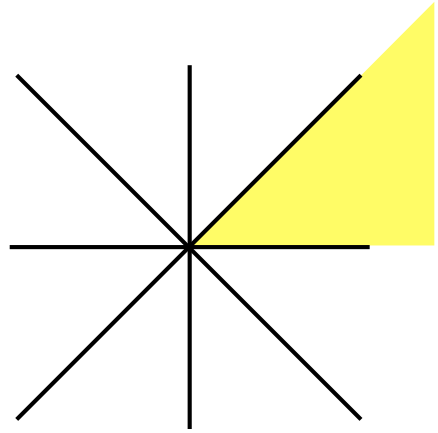


Resolving Sampling Issues with the Midpoint Algorithm

- Since 0<m<1, the line is more horizontal than vertical, so we turn on only **one pixel per column**
- Turn on only the pixel closest to the line in each column



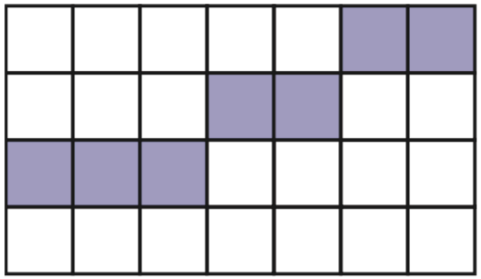
The Midpoint Algorithm

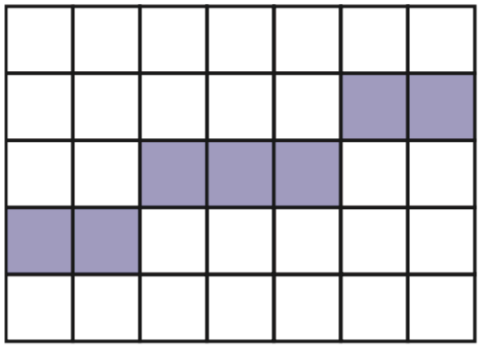


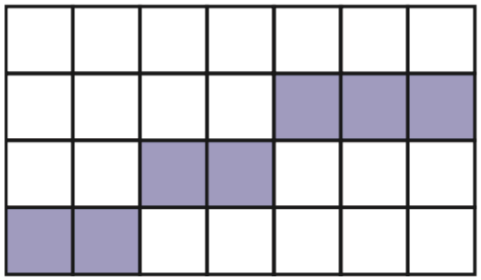
- For the case $m \in (0,1]$ there is more "run" (horizontal) than "rise" (vertical)

- We can draw any reasonable line that is 1 pixel thick by stepping horizontally for a while, and then deciding if we should move up to a new row

- Suggests that we should do:



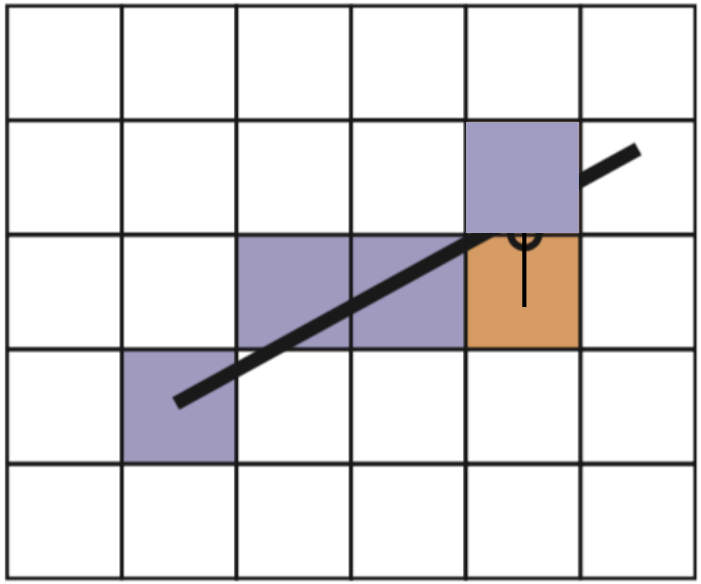




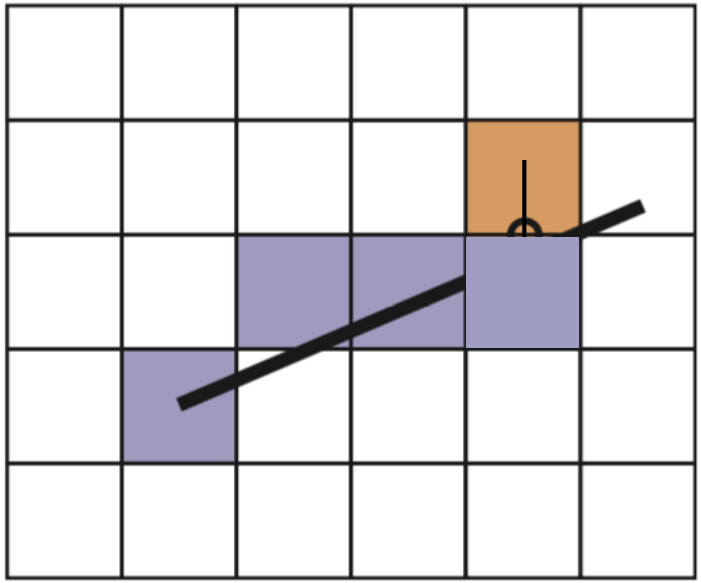
```
y = y0;
for (x in [x0,x1]) {
  draw(x,y);
  if (some condition) {
    y = y+1;
  }
}
```

The Midpoint Algorithm

- Idea: Compare the midpoint of the line segment connecting the next two pixels to the line
  - If line is below the midpoint, then we have moved up a row
  - Otherwise, we stay on the same row:



```
y = y0;
for (x in [x0,x1]) {
  draw(x,y);
  if (f(x+1,y+0.5) < 0) {
    y = y+1;
  }
}
```



Considerations w/ Rasterizing Triangles to Fragments

- Want to be able to interpolate values like colors, texture coordinates
- Have to handle the situation where two triangles share an edge
- A region-based approach based on deciding if pixel centers are inside triangles works well.

Rasterizing Triangles to Fragments

- How to determine if a pixel center is inside of the triangle?
- This is straightforward with barycentric coordinates

```
for all x {
  for all y {
  compute (α, β, γ) for (x,y)
  if (α ∈ [0,1] and β ∈ [0,1] and γ ∈ [0,1]) {
    draw(x,y);
  }
}
```

Recall: Barycentric Coordinates

- A coordinate system to write all points $\mathbf{p}$ as a weighted sum of the vertices

  $\mathbf{p} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$

  $\alpha + \beta + \gamma = 1$

- Equivalently, α, β, γ are the proportions of area of subtriangles relative total area, A

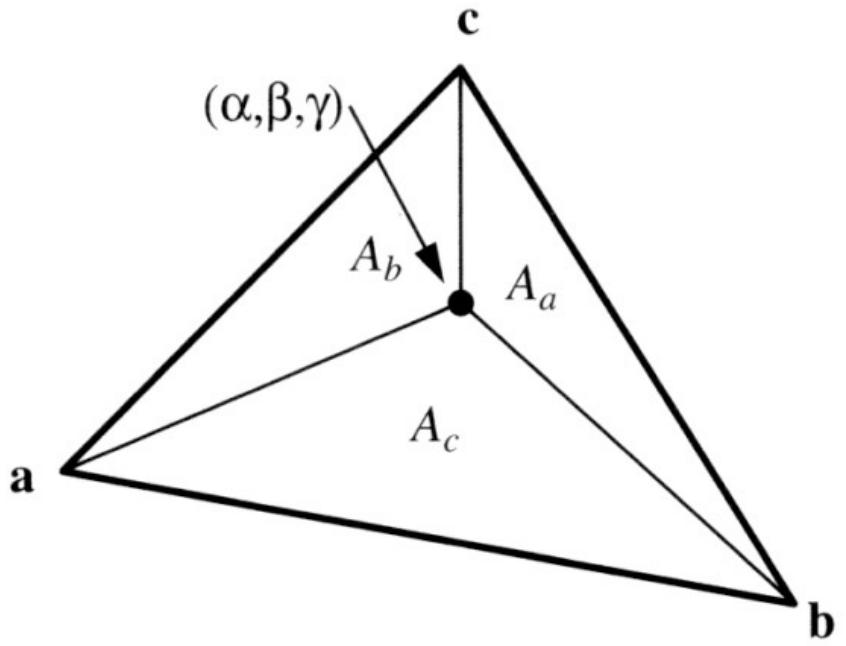
  $A_a / A = \alpha$

  $A_b / A = \beta$

  $A_c / A = \gamma$

- Triangle interior test:

  $\alpha > 0$, $\beta > 0$, and $\gamma > 0$



[Shirley 2000]

Computing Barycentric Coordinates for Pixels

- We can rely on the equations for each edge of the triangle. Given points (x_0, y_0), (x_1, y_1), and (x_2, y_2) on a triangle, we know:

$$f_{01}(x,y) = (y_o - y_1)x + (x_1 - x_0)y + x_0 y_1 - x_1 y_0$$
$$f_{12}(x,y) = (y_1 - y_2)x + (x_2 - x_1)y + x_1 y_2 - x_2 y_1$$
$$f_{20}(x,y) = (y_2 - y_0)x + (x_0 - x_2)y + x_2 y_0 - x_0 y_2$$

- And we can write the barycentric coordinate as:

$$\alpha = f_{12}(x,y) \,/\, f_{12}(x_0, y_0)$$
$$\beta = f_{20}(x,y) \,/\, f_{20}(x_1, y_1)$$
$$\gamma = f_{01}(x,y) \,/\, f_{01}(x_2, y_2)$$

Acceleration: #1

- Only check x,y that are in the bounding box of the triangle

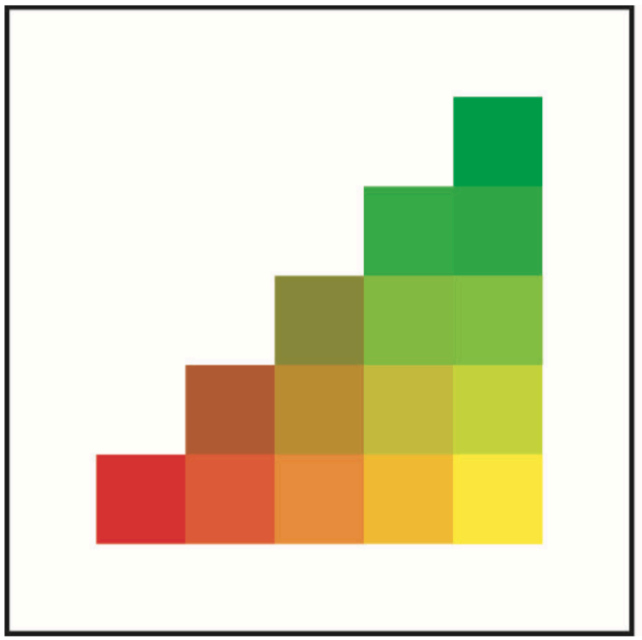
```
xmin = floor(min({x_0,x_1,x_2}))
xmax = ceiling(max({x_0,x_1,x_2}))
ymin = floor(min({y_0,y_1,y_2}))
ymax = ceiling(max({y_0,y_1,y_2}))

for all x in [xmin,xmax] {
  for all y in [ymin,ymax] {
  compute (α, β, γ) for (x,y)
  if (α ∈ [0,1] and β ∈ [0,1] and γ ∈ [0,1]) {
    draw(x,y);
  }
}
```

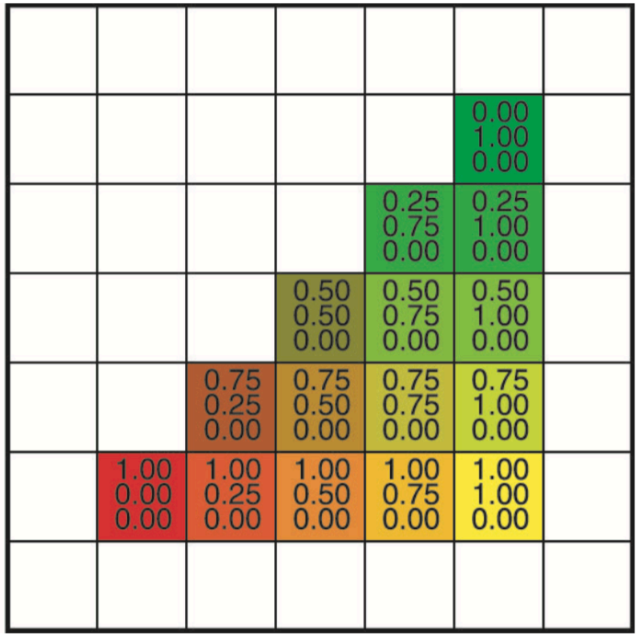
Interpolating Color w/ Gouraud Shading

- Having the barycentric coordinates means that we can interpolate color:

$$\mathbf{c}=\alpha\mathbf{c}_0+\beta\mathbf{c}_1+\gamma\mathbf{c}_2$$

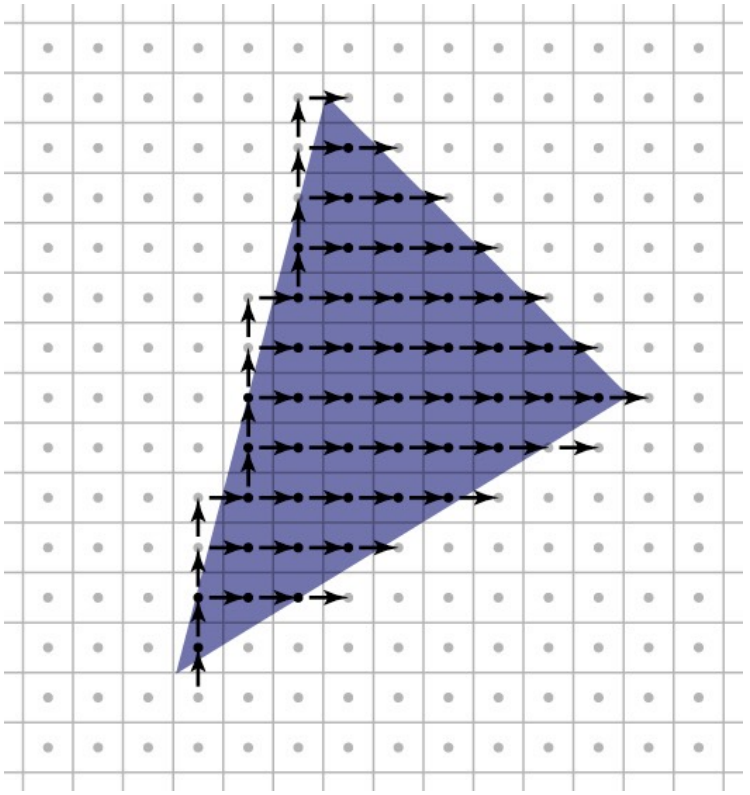
```
for all x in [xmin,xmax] {
  for all y in [ymin,ymax] {
  compute (α, β, γ) for (x,y)
  if (α,β,γ ∈ [0,1]) {
    c = α*c0 + β*c1 + γ*c2
    draw(x,y);
  }
}
```

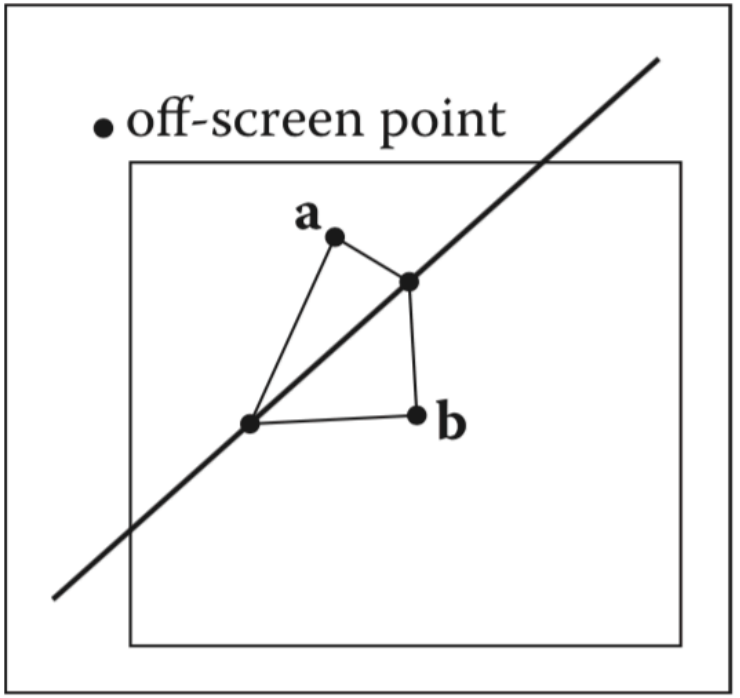




Some Additional Considerations

- Can we accelerate rendering using some variant of incremental rendering?
  - Yes: But we will still need to interpolate colors
- How to break the tie when a pixel center falls on a shared edge?
  - Lots of schemes, but one is use an offscreen point and prefer the triangle whose opposite corner is closer





Clipping

Should We Rasterize All Transformed Primitives?

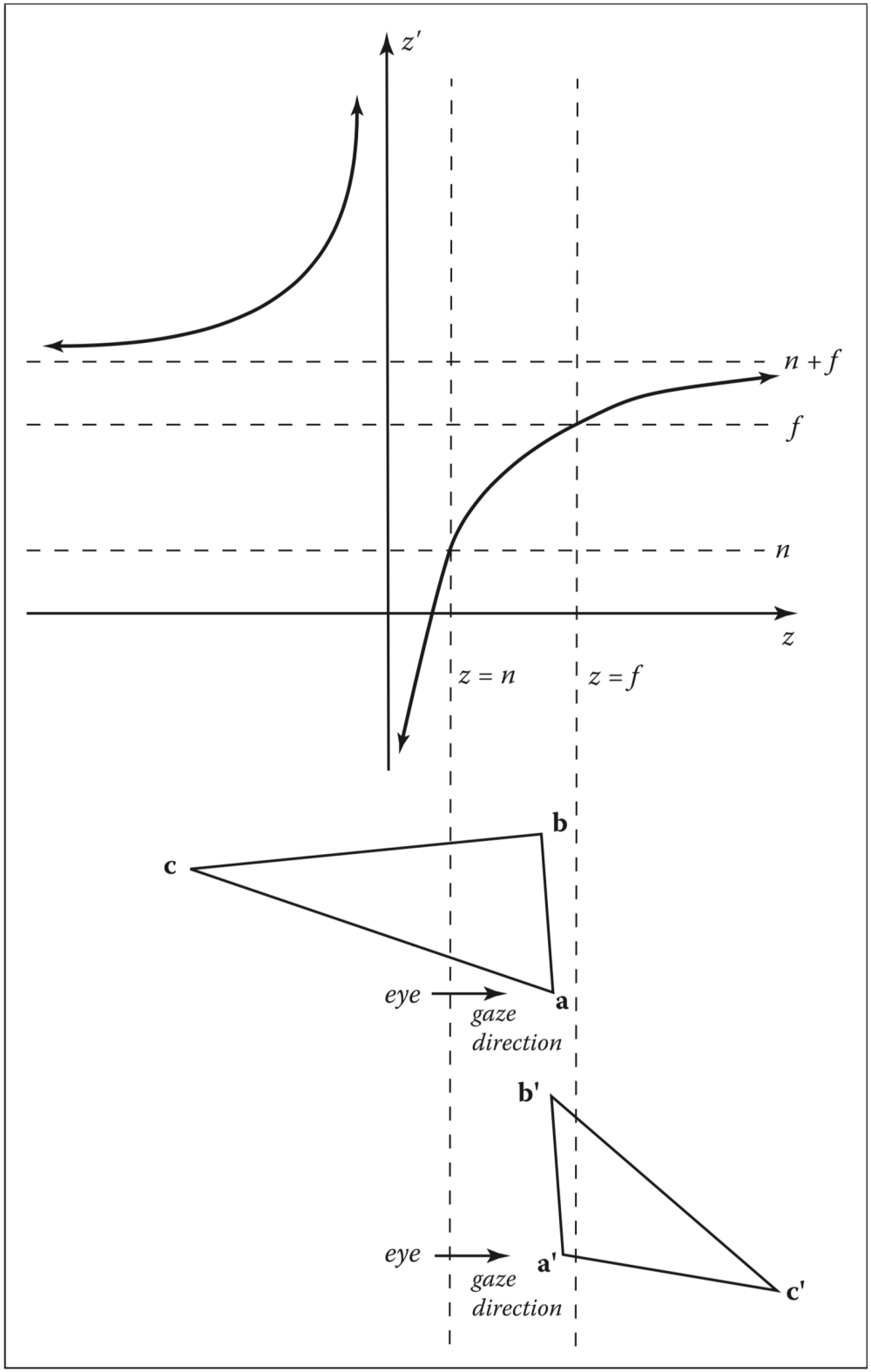
- Primitives outside of the view volume, particularly those behind the eye, will be projected by the projection transformation to nonsensical locations

$$\begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{bmatrix} \sim \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n+f-\frac{fn}{z} \\ 1 \end{bmatrix}$$

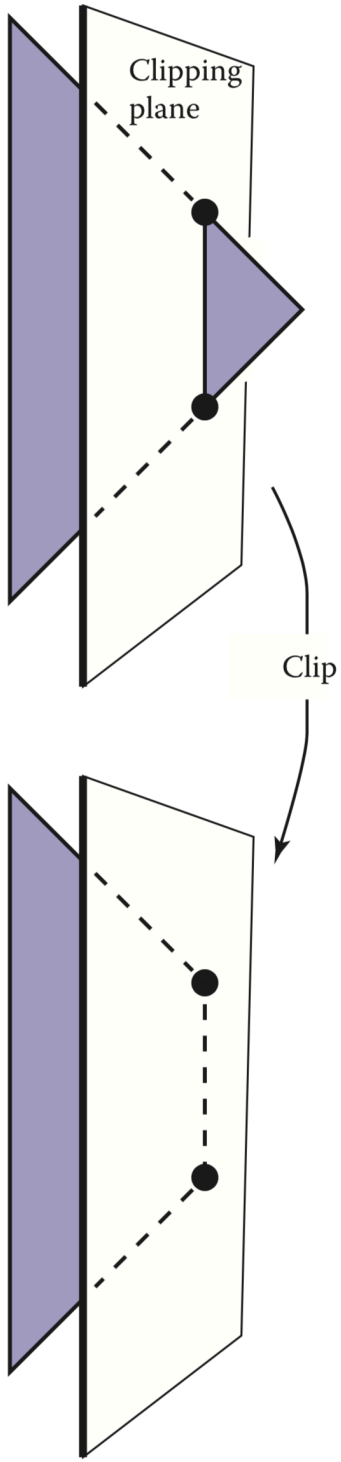
The homogeneous divide makes this a hyperbolic equation

Nonsensical Transformations

- z values are transformed to z' values by projective transform
- Effect is to move vertex $\mathbf{c}$ from behind eye to vertex $\mathbf{c}'$ in front of eye



Solution: Clip Triangles Against View Volume

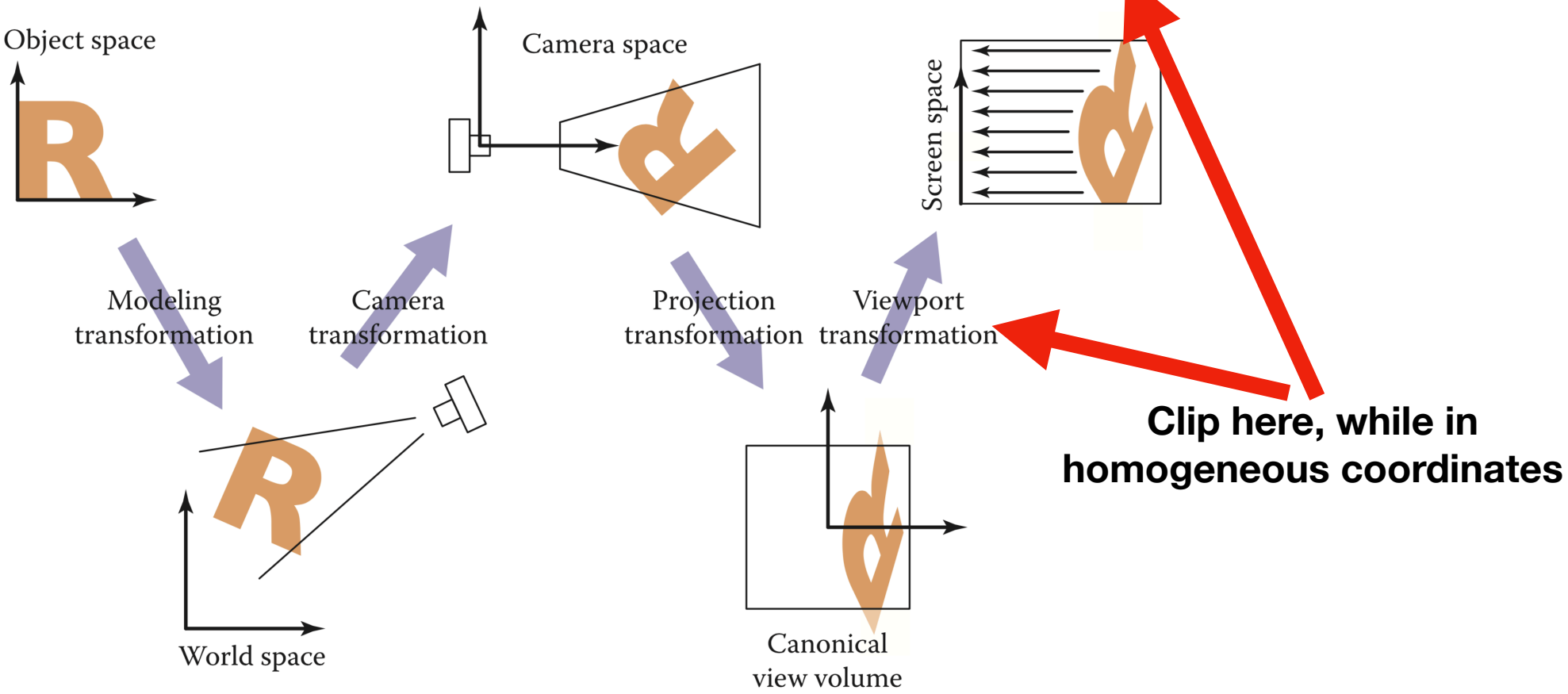


- Before rasterization, we explicitly check that the triangle is contained with the view volume
- View volume is defined by six planes, so we check for intersections with all of them, retaining the portion of the triangle that is inside

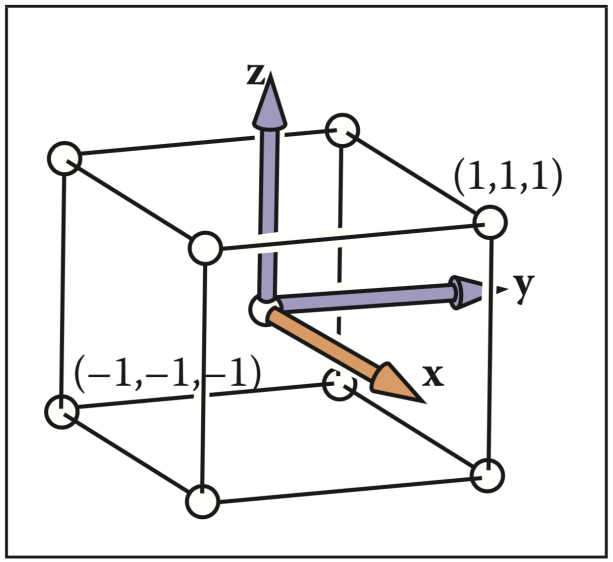
When to Clip?

- Technically, any time before the homogeneous divide is reasonable, but it helps to use the simplest planes.
- Frequently, this is done with the canonical view volume

$$\mathbf{M} = \mathbf{M}_{\text{vp}}\mathbf{M}_{\text{orth}}\mathbf{P}\mathbf{M}_{\text{cam}}$$



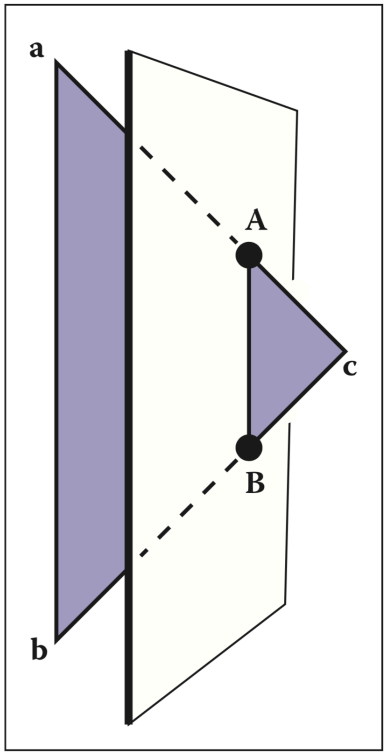
Clipping Planes



- Because we have not done the homogeneous divide, the view volume is actually a four dimensional object
- Math is a little bit tedious, but these equations use $l = b = n =$ -1 and $r = t = f = 1$
- Each equation defines a side of the box in the form $f(\mathbf{p}) = (\mathbf{p} - \mathbf{a}) \cdot \mathbf{n} = 0$, or equivalently: $f(\mathbf{p}) = \mathbf{n} \cdot \mathbf{p} + \mathbf{D} = 0$

$$
\begin{aligned}
-x + lw &= 0 \\
x - rw &= 0 \\
-y + bw &= 0 \\
y - tw &= 0 \\
-z + nw &= 0 \\
z - fw &= 0
\end{aligned}
$$

Clipping a Triangle Against a Plane



- Basic idea: Check if any of three line segments cross the plane.

- How? Solve for intersection points:

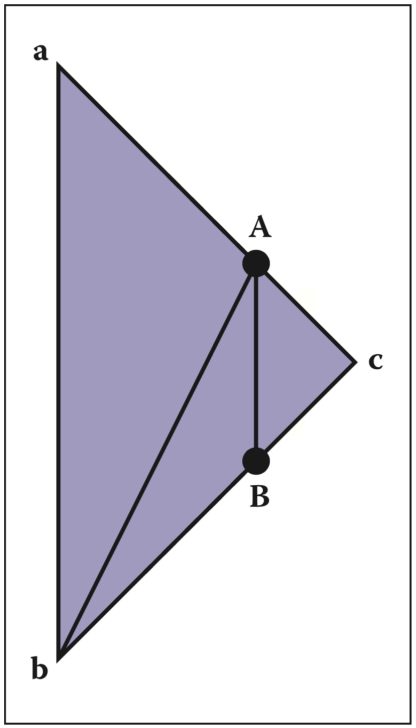
$$f(\mathbf{p}) = \mathbf{n} \cdot \mathbf{p} + \mathbf{D} = 0$$

- By plugging in:

$$\mathbf{p} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

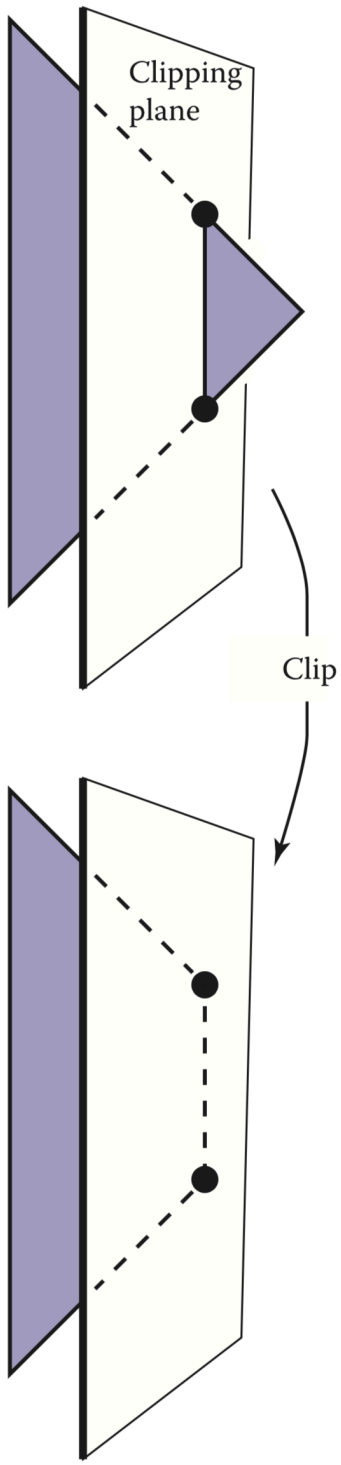
$$t = \frac{\mathbf{n} \cdot \mathbf{a} + D}{\mathbf{n} \cdot (\mathbf{a} - \mathbf{b})}$$

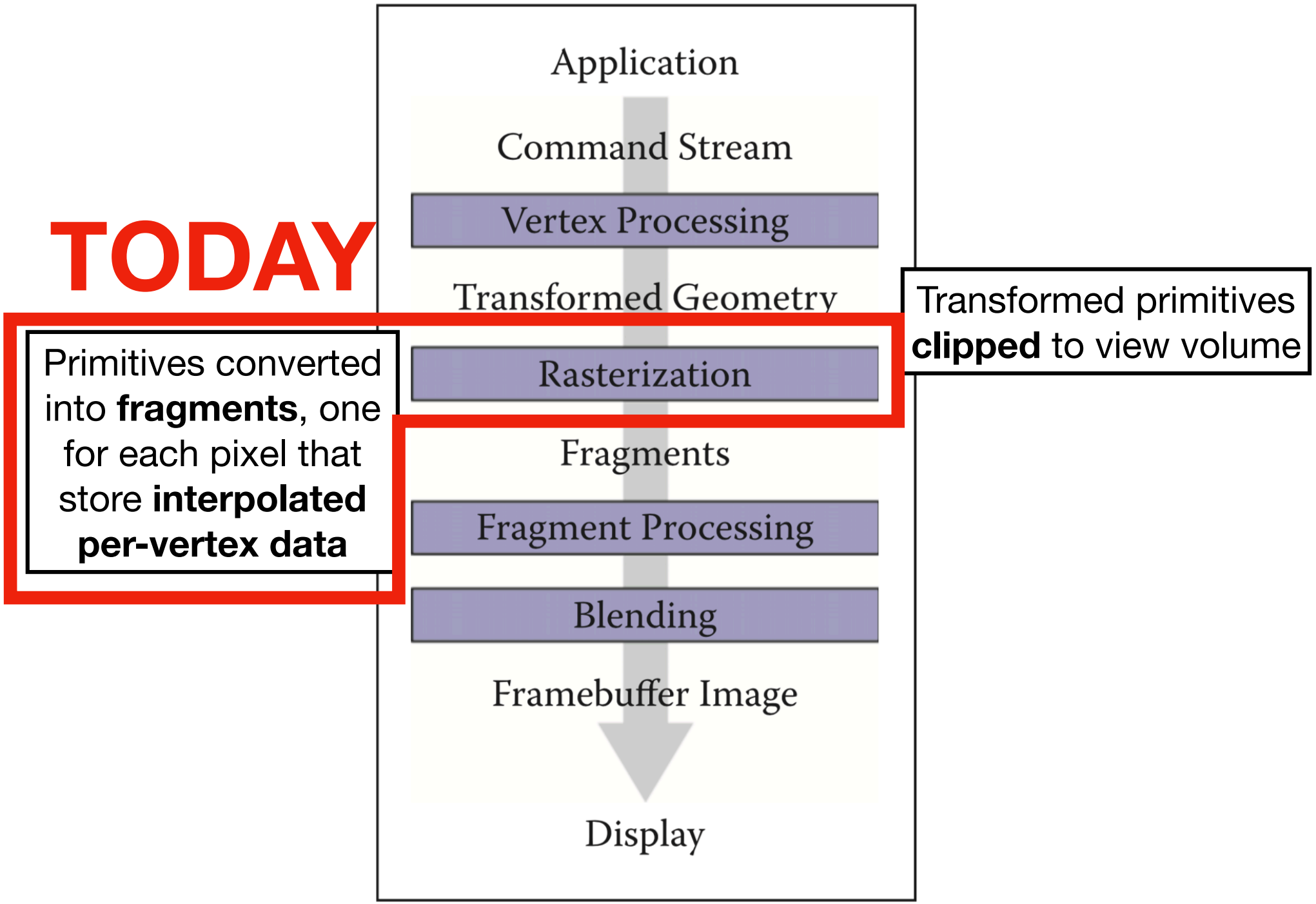
Note: $\mathbf{a}$, $\mathbf{b}$ are both 4D homogeneous points and $\mathbf{n}$ is a 4D vector



Clipping Algorithm

```
for (each triangle t) {
  for (each plane p of view volume) {
    if (t entirely outside of p) {
      discard t.
    } else if (t spans p) {
      q = clip(t).
      if (q is a quadrilateral) {
        replace t with two triangles.
      } else {
        replace t with q.
      }
    }
  }
}
```




TODAY
Application
Command Stream
Vertex Processing
Transformed Geometry
Rasterization
Fragments
Fragment Processing
Blending
Framebuffer Image
Display
Transformed primitives **clipped** to view volume
Primitives converted into **fragments**, one for each pixel that store **interpolated per-vertex data**

Lec21 Required Reading

- FOCG, Ch. 8.2