CSC 433/33
Computer Graphics

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Lecture 17
Matrix Math + Coding

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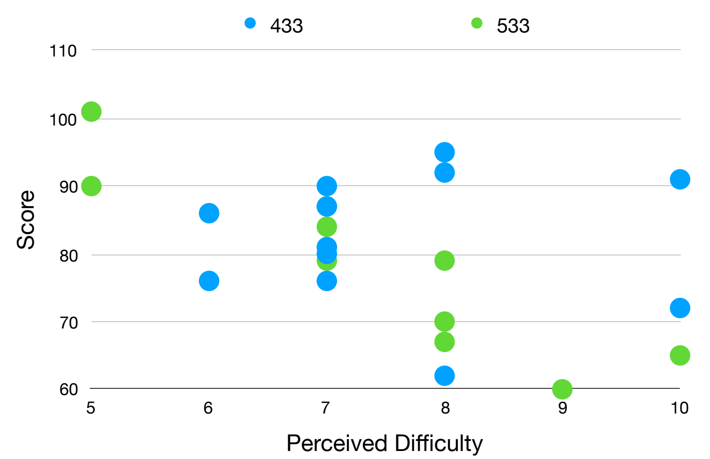
Today's Agenda

- Goals for today: Review/Introduce linear algebra concepts, connect them to Javascript implementation

Perceived Difficulty Vs. Score



Matrices

What is a Matrix?

- A **matrix** is any rectangular array of numbers
  - Typically described by how many rows and columns it has, e.g. an *m-by-n* matrix has *m* rows an *n* columns
- In computer graphics, matrices are frequently used to represent:
  - Geometric transforms, in particular of shapes represented as vectors
  - Systems of linear equations
  - Sets of partial derivatives, Convolution kernels, …
- Compare with both vectors and scalars?

Recall: Vector Multiplication

- Given two 3D vectors:

$$\mathbf{a} = (x_a, y_a, z_a) \quad \mathbf{b} = (x_b, y_b, z_b)$$

- So far, we've learned two forms for "multiplication":
  - Dot (inner) product (2 vectors in, 1 scalar out)

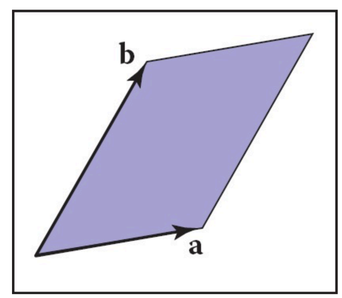
$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b$$

  - Cross product (2 vectors in, 1 vector out)

$$\mathbf{a} \times \mathbf{b} = (y_a z_b - z_a y_b, z_a x_b - x_a z_b, x_a y_b - y_a x_b)$$

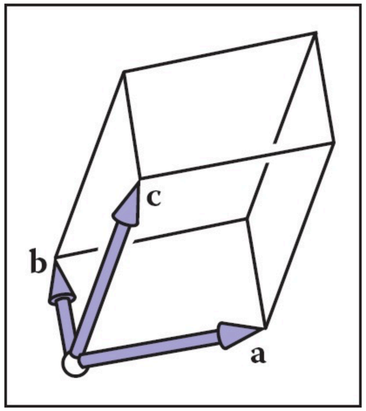
Determinants as Vector Multiplication

- Usually thought of as an operation on a matrix (similar to vector norms) that produces a scalar, but they can also be considered a multiplication of vectors:
- For 2d vectors **a** and **b**, the **determinant**, |**ab**|, is equal to the *signed* area of the parallelogram formed by **a** and **b**
  - **Signed** here means that |**ab**| = -|**ba**|
  - Related: ||**a** × **b**||

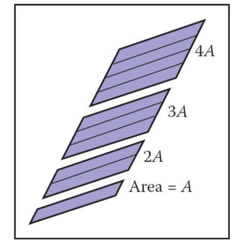


Determinants as Vector Multiplication

- For 3d vectors **a**, **b**, and **c**, the determinant, $|\mathbf{abc}|$, is the *signed* volume of the parallelepiped formed by **a**, **b**, and **c**
- Sign refers to left-handed or right-handed coordinate system

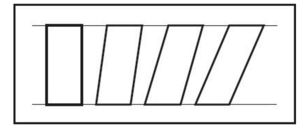


Properties of Determinants



- Scale one side, scale determinant:

$$|(k\mathbf{a})\mathbf{b}| = |\mathbf{a}(k\mathbf{b})| = k|\mathbf{ab}|$$

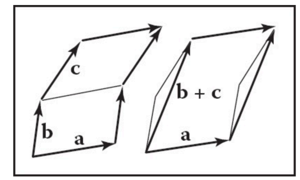


- Shearing does not change area:

$$|(\mathbf{a} + k\mathbf{b})\mathbf{b}| = |\mathbf{a}(\mathbf{b} + k\mathbf{a})| = |\mathbf{ab}|$$

- Distribution:

$$|\mathbf{a}(\mathbf{b} + \mathbf{c})| = |\mathbf{ab}| + |\mathbf{ac}|$$



Determinants Defined

- For 2, 2D vectors

$$|\mathbf{ab}| = |(x_a\mathbf{x} + y_a\mathbf{y})(x_b\mathbf{x} + y_b\mathbf{y})|$$
$$= x_a x_b|\mathbf{xx}| + x_a y_b|\mathbf{xy}| + y_a x_b|\mathbf{yx}| + y_a y_b|\mathbf{yy}|$$
$$= x_a x_b(0) + x_a y_b(+1) + y_a x_b(-1) + y_a y_b(0)$$
$$= x_a y_b - y_a x_b$$

- For 3, 3D vectors

$$|\mathbf{abc}| = |(x_a\mathbf{x}+y_a\mathbf{y}+z_a\mathbf{z})(x_b\mathbf{x}+y_b\mathbf{y}+z_b\mathbf{z})(x_c\mathbf{x}+y_c\mathbf{y}+z_c\mathbf{z})|$$
$$= x_a y_b z_c - x_a z_b y_c - y_a x_b z_c + y_a z_b x_c + z_a x_b y_c - z_a y_b x_c$$

Matrix Operations

Matrix Arithmetic

- Multiplication by a scalar, element-wise

$$2\begin{bmatrix}1 & -4\\ 3 & 2\end{bmatrix} = \begin{bmatrix}2 & -8\\ 6 & 4\end{bmatrix}$$

- Matrix-Matrix addition, element-wise

$$\begin{bmatrix}1 & -4\\ 3 & 2\end{bmatrix} + \begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix} = \begin{bmatrix}3 & -2\\ 5 & 4\end{bmatrix}$$

Matrix-Matrix Multiplication

- Can multiply any *r-by-m* matrix with any *m-by-c* matrix

$$\begin{bmatrix}a_{11} & \cdots & a_{1m}\\ \vdots & & \vdots\\ \boxed{a_{i1} \quad \cdots \quad a_{im}} & & \\ \vdots & & \vdots\\ a_{r1} & \cdots & a_{rm}\end{bmatrix} \begin{bmatrix}b_{11} & \cdots & \boxed{\begin{matrix}b_{1j}\\ \vdots\\ b_{mj}\end{matrix}} & \cdots & b_{1c}\\ \vdots & & & & \vdots\\ b_{m1} & \cdots & & \cdots & b_{mc}\end{bmatrix} = \begin{bmatrix}p_{11} & \cdots & p_{1j} & \cdots & p_{1c}\\ \vdots & & \vdots & & \vdots\\ p_{i1} & \cdots & \boxed{p_{ij}} & \cdots & p_{ic}\\ \vdots & & \vdots & & \vdots\\ p_{r1} & \cdots & p_{rj} & \cdots & p_{rc}\end{bmatrix}$$

- Each new term is defined as

$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

Matrix-Matrix Multiplication

- Multiplication is not commutative, **AB** ≠ **BA**
- And if **AB** = **AC**, it does not necessarily follow that **B** = **C**
- But, multiplication is associative and distributive:

$$\begin{aligned}(\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}),\\ \mathbf{A}(\mathbf{B}+\mathbf{C}) &= \mathbf{AB}+\mathbf{AC},\\ (\mathbf{A}+\mathbf{B})\mathbf{C} &= \mathbf{AC}+\mathbf{BC}.\end{aligned}$$

Matrix Inverses

- For square (*n-by-n*) matrices, we can define the action of undoing multiplication, called the **matrix inverse**
  - To do so, we need a matrix that behaves like “1”, called the **identity matrix**, $\mathbf{I}$, e.g.

$$\mathbf{I} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

- The **inverse matrix** of **A**, written $\mathbf{A}^{-1}$, is the matrix that ensures $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}^{-1} = \begin{bmatrix}-2.0 & 1.0\\ 1.5 & -0.5\end{bmatrix} \text{ because } \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}-2.0 & 1.0\\ 1.5 & -0.5\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Matrix Inverses

- If the inverse of **A** is $\mathbf{A}^{-1}$, then the inverse of $\mathbf{A}^{-1}$ is **A**, thus

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

- Inverse of the product of two matrices, **A** and **B**, is the product of the inverses, with the order reversed

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Matrix Transpose

- The **transpose** of a matrix **A**, written $\mathbf{A}^\mathbf{T}$, swaps the rows and columns, e.g. $a_{ij} = a^\mathbf{T}_{ji}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^\mathrm{T} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

- Transpose of the product of two matrices is similar to inverse of product:

$$(\mathbf{AB})^\mathbf{T} = \mathbf{B}^\mathbf{T}\mathbf{A}^\mathbf{T}$$

Matrix Determinant

- If the matrix is **square** (has the same number of rows/columns), it can be computed by taking the columns of the matrix as vectors and computing their determinant

Rules for Determinants and Matrix Operations

- Determinant and Matrix-Matrix Multiplication:

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

- Determinant and Matrix Inverse:

$$|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$$

- Determinant and Matrix Transpose:

$$|\mathbf{A}^\mathbf{T}| = |\mathbf{A}|$$

Matrix-Vector Multiplication

- Can consider a vector of length *m* as just an *m-by-1* matrix
  - Convention is that vectors are columns
- Multiplying a vector by a matrix produces another vector, in doing so, transforms the vector, e.g.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} -y_a \\ x_a \end{bmatrix}$$

  - This matrix has the effect of rotating the vector by 90 degrees

Vector Operations as Matrix Operations

- Can rewrite dot product, outputting a 1-by-1 matrix

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} \qquad \begin{bmatrix} x_a & y_a & z_a \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} x_a x_b + y_a y_b + z_a z_b \end{bmatrix}$$

- Likewise, one can also define the **outer product** of a pair of vectors, which takes two vectors and produces a matrix

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} \begin{bmatrix} x_b & y_b & z_b \end{bmatrix} = \begin{bmatrix} x_a x_b & x_a y_b & x_a z_b \\ y_a x_b & y_a y_b & y_a z_b \\ z_a z_b & z_a y_b & z_a z_b \end{bmatrix}$$

Matrix Multiplication as Vector Operations

- We can think of matrix-matrix multiplication as a collection of vector operations

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \end{bmatrix} = \begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ - & \mathbf{r}_3 & - \end{bmatrix} \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix}$$

- Alternate interpretation is a weighted sum

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};$$

$$\mathbf{y} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3.$$

- More generally, the matrix-matrix product **AB** is an array containing all pairwise dot products

Special Matrix Types for Matrix Operations

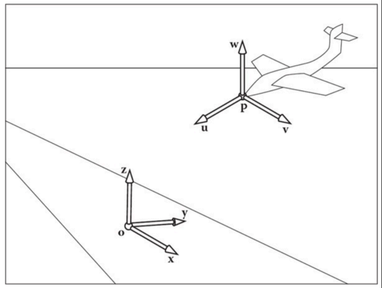
- **Diagonal matrices** only have entries on the diagonal, thus $a_{ij} = 0$ when $i \neq j$, e.g. the identify matrix
- Diagonal matrices are a special instance of **symmetric matrices**, matrices that equal their transpose $\mathbf{A} = \mathbf{A}^\mathrm{T}$
- Matrices whose columns are orthogonal vectors *of unit length* are called **orthogonal matrices**.
  - Determinant of an orthogonal matrix is always 1 or -1
  - Orthogonal matrices are almost their own inverse, for an orthogonal matrix **R**, $\mathbf{R}^\mathrm{T}\mathbf{R} = \mathbf{R}\mathbf{R}^\mathrm{T} = \mathbf{I}$.

Recall: Orthonormal Bases

- The columns of an orthogonal matrix define an orthonormal basis!
- Vectors **u**, **v**, and **w** define an orthonormal basis if

$$||\mathbf{u}|| = ||\mathbf{v}|| = ||\mathbf{w}|| = 1$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$$



Examples of Matrix Types

- Diagonal, Symmetric, but not Orthogonal $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$
- Symmetric, but not Diagonal or Orthogonal $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 7 \\ 2 & 7 & 1 \end{bmatrix}$
- Orthogonal, but not Symmetric or Diagonal $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Computing with Matrices

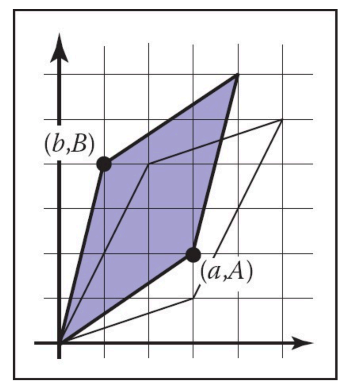
Computing Determinants

- Instead of thinking about the determinant of a group of vectors, one can instead write it in matrix form.

$$\left| \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \right| \equiv \begin{vmatrix} a & A \\ b & B \end{vmatrix} = aB - Ab$$

- This gives algebraic support to why the determinant of a matrix is equal to the determinant of its transpose

$$\begin{vmatrix} a & A \\ b & B \end{vmatrix} = \begin{vmatrix} a & b \\ A & B \end{vmatrix} = aB - Ab$$



Computing Determinants

- For any square matrix, one computes them recursively using the determinants of submatrices, called **cofactors**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\boxed{a^c_{11} = \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}} \quad \boxed{a^c_{12} = -\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}}$$

$$\boxed{a^c_{13} = \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}} \quad \boxed{a^c_{14} = -\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}}$$

$$|\mathbf{A}| = \boxed{a_{11}a^c_{11}} + \boxed{a_{12}a^c_{12}} + \boxed{a_{13}a^c_{13}} + \boxed{a_{14}a^c_{14}}$$

Example Determinant Computation

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = 0\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}$$
$$= 0(32 - 35) - 1(24 - 30) + 2(21 - 24)$$
$$= 0.$$

- Note that since the determinant is zero, the volume of the parallelepiped is also zero
- This is the result of the columns (or rows) not being linearly independent

Computing Inverses

- An inefficient (for large matrices) way to compute inverses, uses the transpose of the matrix replacing terms with cofactors

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} a^c_{11} & a^c_{21} & a^c_{31} & a^c_{41} \\ a^c_{12} & a^c_{22} & a^c_{32} & a^c_{42} \\ a^c_{13} & a^c_{23} & a^c_{33} & a^c_{43} \\ a^c_{14} & a^c_{24} & a^c_{34} & a^c_{44} \end{bmatrix}$$

Computing Inverses

- Our formulation works because

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} a^c_{11} & \cdot & \cdot & \cdot \\ a^c_{12} & \cdot & \cdot & \cdot \\ a^c_{13} & \cdot & \cdot & \cdot \\ a^c_{14} & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} |\mathbf{A}| & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

- And for the off-diagonal rows we can show

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} a^c_{11} & \cdot & \cdot & \cdot \\ a^c_{12} & \cdot & \cdot & \cdot \\ a^c_{13} & \cdot & \cdot & \cdot \\ a^c_{14} & \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

  - Off-diagonals are zero because this dot product is equivalent to the determinant of a matrix where two rows are identical

A Few Other Facts About Matrices

Solving Linear Systems with Matrices

- Given n equations with n unknowns

$$\begin{aligned} 3x + 7y + 2z &= 4, \\ 2x - 4y - 3z &= -1, \\ 5x + 2y + z &= 1. \end{aligned}$$

- We can write this as a matrix of coefficients times a vector of variables

$$\begin{bmatrix} 3 & 7 & 2 \\ 2 & -4 & -3 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

- This can be solved using a matrix inverse, but for small matrices a closed form known as Cramer's rule can be used

Representing Planes with Matrices

- The plane through points (x_i,y_i,z_i) is defined by

$$\begin{vmatrix} x - x_0 & x - x_1 & x - x_2 \\ y - y_0 & y - y_1 & y - y_2 \\ z - z_0 & z - z_1 & z - z_2 \end{vmatrix} = 0$$

- This works because each column is a vector from the point (x,y,z) to (x_i,y_i,z_i), the volume of the parallelepiped with those vectors as sides is zero only if (x,y,z) is coplanar with all three points

Coding Demo with Matrices

Lec18 Required Reading

- FOCG, Ch. 6