CSC 433/533
Computer Graphics

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Lecture 08
Vector Math + Coding

Sept. 23, 2019

Today's Agenda

- Reminders:
  - A02 questions?
- Goals for today:
  - Introduce some mathematics and connect it to code

Vectors

What is a Vector?

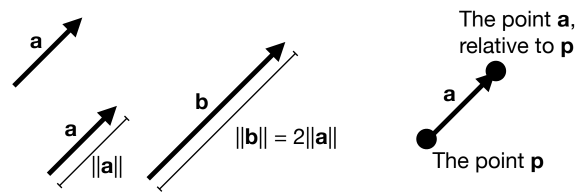
- A **vector** describes a length and a direction
- A vector is also a tuple of numbers
  - But, it often makes more sense to think in terms of the length/direction than the coordinates/numbers
  - And, especially in code, we want to manipulate vectors as objects and abstract the low-level operations
  - Compare with a **scalar**, or just a single number

Properties

- Two vectors, **a** and **b**, are the same (written **a** = **b**) if they have the same length and direction.
- A vector's **length** is denoted with || ||,
  - e.g. the length of **a** is ||**a**||
- A **unit vector** has length one
- The **zero vector** has length zero, and undefined direction

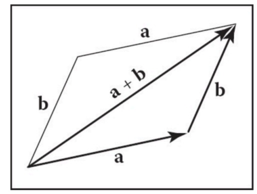
Vectors in Pictures

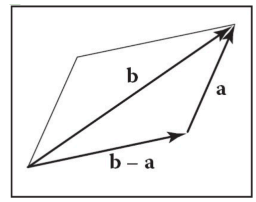
- We often use an arrow to represent a vector
  - The length of the arrow indicates the length of the vector, the direction of the arrow indicates the direction of the vector.
- The position of the arrow is irrelevant!
  - However, we can use vectors to represent positions by describing displacements from a common point



Vector Operations

- Vectors can be added, e.g. for vectors **a**,**b**, there exists a vector **c** = **a**+**b**
- Defined using the parallelogram rule: idea is to trace out the displacements and produced the combined effect
- Vectors can be negated (flip tail and head), and thus can be subtracted
- Vectors can be multiplied by a scalar, which scales the length but not the direction

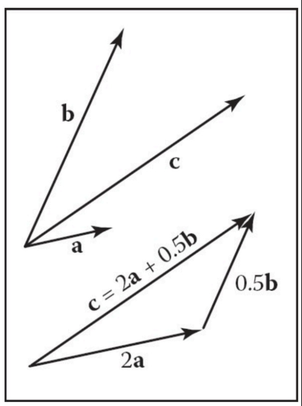




Vectors Decomposition

- By linear independence, any 2D vector can be written as a combination of any two nonzero, nonparallel vectors
- Such a pair of vectors is called a **2D basis**

$$\mathbf{c} = a_c\mathbf{a} + b_c\mathbf{b}$$



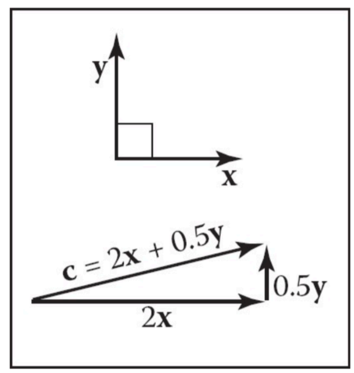
Canonical (Cartesian) Basis

- Often, we pick two perpendicular vectors, **x** and **y**, to define a common basis
- Notationally the same,

$$\mathbf{a} = x_a\mathbf{x} + y_a\mathbf{y}$$

- But we often don't bother to mention the basis vectors, and write the vector as **a** = (x_a,y_a), or

$$\mathbf{a} = \begin{bmatrix} x_a \\ y_a \end{bmatrix}$$



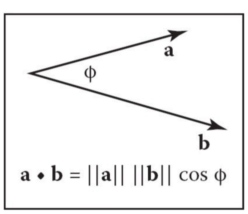
Vector Multiplication: Dot Products

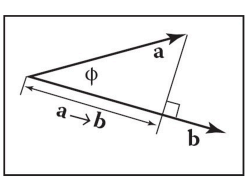
- Given two vectors **a** and **b**, the **dot product**, relates the lengths of **a** and **b** with the angle ϕ between them:

$$\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||\ ||\mathbf{b}|| \cos \phi$$

  - Sometimes called the scalar product, as it produces a scalar value
- Also can be used to produce the **projection**, **a**→**b**, of **a** onto **b**

$$\mathbf{a} \to \mathbf{b} = ||\mathbf{a}\,|| \quad \cos\phi = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{b}||}$$





Dot Products are Associative and Distributive

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a},$$
$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c},$$
$$(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k\mathbf{a} \cdot \mathbf{b}$$

- And, we can also define them directly if **a** and **b** are expressed in Cartesian coordinates:

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b$$

3D Vectors

- Same idea as 2D, except these vectors are defined typically with a basis of three vectors
  - Still just a direction and a magnitude
  - But, useful for describing objects in three-dimensional space
- Most operations exactly the same, e.g. dot products:

$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b$$

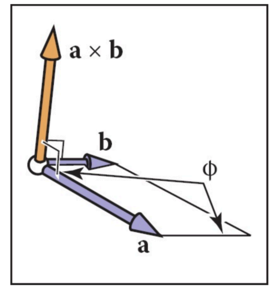
Cross Products

- In 3D, another way to "multiply" two vectors is the **cross product**, **a** × **b**:

  $||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||\,||\mathbf{b}|| \sin \phi$

- $||\mathbf{a} \times \mathbf{b}||$ is always the area of the parallelogram formed by **a** and **b**, and **a** × **b** is always in the direction perpendicular
- Cross products distribute, but order matters:

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$
$$\mathbf{a} \times (k\mathbf{b}) = k(\mathbf{a} \times \mathbf{b})$$
$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$



Cross Products

- Since the cross product is always orthogonal to the pair of vectors, we can define our 3D Cartesian coordinate space with it:

$$\begin{aligned} \mathbf{x} &= (1,0,0) \\ \mathbf{y} &= (0,1,0) \\ \mathbf{z} &= (0,0,1) \end{aligned} \qquad \begin{aligned} \mathbf{x} \times \mathbf{y} &= +\mathbf{z}, \\ \mathbf{y} \times \mathbf{x} &= -\mathbf{z}, \\ \mathbf{y} \times \mathbf{z} &= +\mathbf{x}, \\ \mathbf{z} \times \mathbf{y} &= -\mathbf{x}, \\ \mathbf{z} \times \mathbf{x} &= +\mathbf{y}, \\ \mathbf{x} \times \mathbf{z} &= -\mathbf{y}. \end{aligned}$$

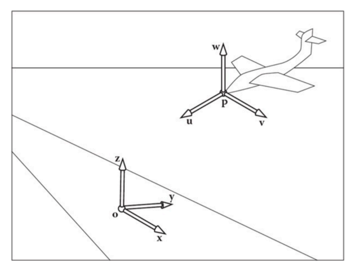
- In practice though (and the book derives this), we use the following to compute cross products:

$$\mathbf{a} \times \mathbf{b} = (y_a z_b - z_a yb,\; z_a x_b - x_a z_b, x_a yb - y_a x_b)$$

Orthonormal Bases

- Frequently, we will use three arbitrary vectors to define a coordinate space relative to an object.
- Vectors **u**, **v**, and **w** define an orthonormal basis if

$$||\mathbf{u}|| = ||\mathbf{v}|| = ||\mathbf{w}|| = 1$$
$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$$



Constructing Orthonormal Bases from a Single Vector

- Say you have a single direction **a** and want an orthonormal basis that aligns with it
- First, make **a** into a unit vector **w**

$$\mathbf{w} = \frac{\mathbf{a}}{||\mathbf{a}||}$$

- Then, choose any vector **t** that is not collinear with **w**

$$\mathbf{u} = \frac{\mathbf{t} \times \mathbf{w}}{||\mathbf{t} \times \mathbf{w}||}$$

- Once **w** and **u** are defined, **v** is simply: $\mathbf{v} = \mathbf{w} \times \mathbf{u}$

Constructing Orthonormal Bases from a Pair of Vectors

- Given two vectors **a** and **b**, which might not be orthonormal to begin with:

$$\begin{aligned} \mathbf{w} &= \frac{\mathbf{a}}{||\mathbf{a}||}, \\ \mathbf{u} &= \frac{\mathbf{b} \times \mathbf{w}}{||\mathbf{b} \times \mathbf{w}||}, \\ \mathbf{v} &= \mathbf{w} \times \mathbf{u}. \end{aligned}$$

- In this case, **w** will align with **a** and **v** will be the closest vector to **b** that is perpendicular to **w**

Lec09 Required Reading

- FOCG, Ch. 4.1-4.4