

CSC 433/33

Computer Graphics

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Matrix Math

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What is a Matrix?

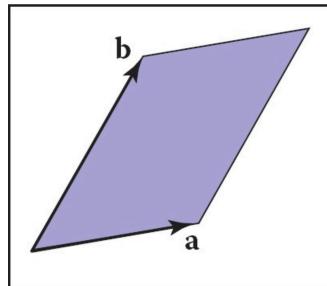
- A **matrix** is any rectangular array of numbers
 - Typically described by how many rows and columns it has, e.g. an m -by- n matrix has m rows and n columns
- In computer graphics, matrices are frequently used to represent:
 - Geometric transforms, in particular of shapes represented as vectors
 - Systems of linear equations
 - Sets of partial derivatives, Convolution kernels, ...
- Compare with both vectors and scalars?

Recall: Vector Multiplication

- Given two 3D vectors:
$$\mathbf{a} = (x_a, y_a, z_a) \quad \mathbf{b} = (x_b, y_b, z_b)$$
- So far, we've learned two forms for "multiplication":
 - Dot (inner) product (2 vectors in, 1 scalar out)
$$\mathbf{a} \cdot \mathbf{b} = x_a x_b + y_a y_b + z_a z_b$$
 - Cross product (2 vectors in, 1 vector out)
$$\mathbf{a} \times \mathbf{b} = (y_a z_b - z_a y_b, z_a x_b - x_a z_b, x_a y_b - y_a x_b)$$

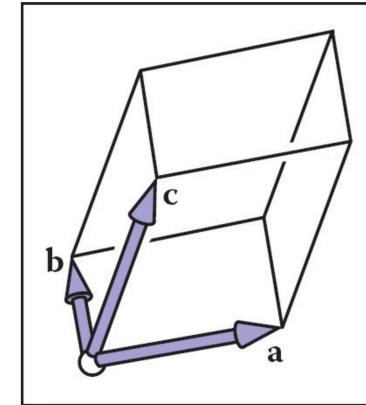
Determinants as Vector Multiplication

- Usually thought of as an operation on a matrix (similar to vector norms) that produces a scalar, but they can also be considered a multiplication of vectors:
- For 2d vectors \mathbf{a} and \mathbf{b} , the **determinant**, $\det(\mathbf{ab})$, is equal to the *signed* area of the parallelogram formed by \mathbf{a} and \mathbf{b}
 - Signed** here means that $\det(\mathbf{ab}) = -\det(\mathbf{ba})$,
 - It is positive iff \mathbf{b} is CCW to \mathbf{a}
 - Related: $\|\mathbf{a} \times \mathbf{b}\|$
 - Conclusion $\det(\mathbf{ab})=0$ iff \mathbf{ab} are collinear .



Determinants as Vector Multiplication

- For 3d vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} , the determinant, $|\mathbf{abc}|$, is the *signed* volume of the parallelepiped formed by \mathbf{a} , \mathbf{b} , and \mathbf{c}
- Sign refers to left-handed or right-handed coordinate system
- What could we say about $\mathbf{a}, \mathbf{b}, \mathbf{c}$ if $|\mathbf{abc}|=0$?



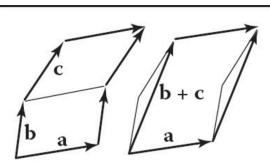
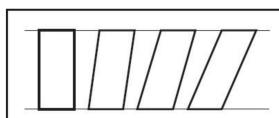
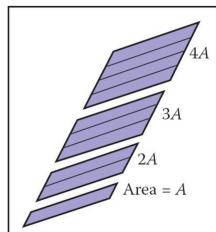
Properties of Determinants

- Scale one side, scale determinant:

$$|(k\mathbf{a})\mathbf{b}| = |\mathbf{a}(k\mathbf{b})| = k|\mathbf{ab}|$$
- Shearing does not change area:

$$|(\mathbf{a} + k\mathbf{b})\mathbf{b}| = |\mathbf{a}(\mathbf{b} + k\mathbf{a})| = |\mathbf{ab}|$$
- Distribution:

$$|\mathbf{a}(\mathbf{b} + \mathbf{c})| = |\mathbf{ab}| + |\mathbf{ac}|$$



Determinants Defined

- Let \mathbf{x} be unit vectors in the directions of the x-axis. Similarly define \mathbf{y} and \mathbf{z} .
- For 2, 2D vectors

$$\begin{aligned} |\mathbf{ab}| &= |(x_a\mathbf{x} + y_a\mathbf{y})(x_b\mathbf{x} + y_b\mathbf{y})| \\ &= x_a x_b |\mathbf{xx}| + x_a y_b |\mathbf{xy}| + y_a x_b |\mathbf{yx}| + y_a y_b |\mathbf{yy}| \\ &= x_a x_b (0) + x_a y_b (+1) + y_a x_b (-1) + y_a y_b (0) \\ &= x_a y_b - y_a x_b \end{aligned}$$
- For 3, 3D vectors

$$\begin{aligned} |\mathbf{abc}| &= |(x_a\mathbf{x} + y_a\mathbf{y} + z_a\mathbf{z})(x_b\mathbf{x} + y_b\mathbf{y} + z_b\mathbf{z})(x_c\mathbf{x} + y_c\mathbf{y} + z_c\mathbf{z})| \\ &= x_a y_b z_c - x_a z_b y_c - y_a x_b z_c + y_a z_b x_c + z_a x_b y_c - z_a y_b x_c \end{aligned}$$

Matrix Operations

Matrix Arithmetic

- Multiplication by a scalar, element-wise

$$2 \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ 6 & 4 \end{bmatrix}$$

- Matrix-Matrix addition, element-wise

$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$$

Matrix-Matrix Multiplication

- Can multiply any r -by- m matrix with any m -by- c matrix

$$\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ \boxed{a_{i1} & \dots & a_{im}} \\ \vdots & & \vdots \\ a_{r1} & \dots & a_{rm} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & \boxed{b_{1j}} & \dots & b_{1c} \\ \vdots & & \vdots & & \vdots \\ b_{m1} & \dots & b_{mj} & \dots & b_{mc} \end{bmatrix} = \begin{bmatrix} p_{11} & \dots & p_{1j} & \dots & p_{1c} \\ \vdots & & \vdots & & \vdots \\ p_{i1} & \dots & \boxed{p_{ij}} & \dots & p_{ic} \\ \vdots & & \vdots & & \vdots \\ p_{r1} & \dots & p_{rj} & \dots & p_{rc} \end{bmatrix}$$

- Each new term is defined as

$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{im}b_{mj}$$

Matrix-Matrix Multiplication

- Multiplication is not commutative, $\mathbf{AB} \neq \mathbf{BA}$
- And if $\mathbf{AB} = \mathbf{AC}$, it does not necessarily follow that $\mathbf{B} = \mathbf{C}$
- But, multiplication is associative and distributive:

$$\begin{aligned} (\mathbf{AB})\mathbf{C} &= \mathbf{A}(\mathbf{BC}), \\ \mathbf{A}(\mathbf{B} + \mathbf{C}) &= \mathbf{AB} + \mathbf{AC}, \\ (\mathbf{A} + \mathbf{B})\mathbf{C} &= \mathbf{AC} + \mathbf{BC}. \end{aligned}$$

Matrix Inverses

- For square (n -by- n) matrices, we can define the action of undoing multiplication, called the **matrix inverse**
 - To do so, we need a matrix that behaves like “1”, called the **identity matrix**, \mathbf{I} , e.g.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The **inverse matrix** of \mathbf{A} , written \mathbf{A}^{-1} , is the matrix that ensures $\mathbf{AA}^{-1} = \mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} \text{ because } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Matrix Inverses

- If the inverse of \mathbf{A} is \mathbf{A}^{-1} , then the inverse of \mathbf{A}^{-1} is \mathbf{A} , thus
$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$
- Inverse of the product of two matrices, \mathbf{A} and \mathbf{B} , is the product of the inverses, with the order reversed
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Matrix Transpose

- The **transpose** of a matrix \mathbf{A} , written \mathbf{A}^T , swaps the rows and columns, e.g. $a_{ij} = a_{ji}^T$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

- Transpose of the product of two matrices is similar to inverse of product:

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Matrix Determinant

- If the matrix is **square** (has the same number of rows/columns), it can be computed by taking the **columns** of the matrix as vectors and computing their determinant

Rules for Determinants and Matrix Operations

- Determinant and Matrix-Matrix Multiplication:

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$$

- Determinant and Matrix Inverse:

$$|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$$

- Determinant and Matrix Transpose:

$$|\mathbf{A}^T| = |\mathbf{A}|$$

Matrix-Vector Multiplication

- Can consider a vector of length m as just an m -by-1 matrix
 - Convention is that vectors are columns
- Multiplying a vector by a matrix produces another vector, in doing so, transforms the vector, e.g.

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} -y_a \\ x_a \end{bmatrix}$$

- This matrix has the effect of rotating the vector by 90 degrees

Vector Operations as Matrix Operations

- Can rewrite dot product, outputting a 1-by-1 matrix

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} \quad [x_a \ y_a \ z_a] \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = [x_a x_b + y_a y_b + z_a z_b]$$

- Likewise, one can also define the **outer product** of a pair of vectors, which takes two vectors and produces a matrix

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} [x_b \ y_b \ z_b] = \begin{bmatrix} x_a x_b & x_a y_b & x_a z_b \\ y_a x_b & y_a y_b & y_a z_b \\ z_a x_b & z_a y_b & z_a z_b \end{bmatrix}$$

Matrix Multiplication as Vector Operations

- We can think of matrix-matrix multiplication as a collection of vector operations

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \end{bmatrix} = \begin{bmatrix} -\mathbf{r}_1 & - \\ -\mathbf{r}_2 & - \\ -\mathbf{r}_3 & - \end{bmatrix} \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix}$$

- Alternate interpretation is a weighted sum

$$\begin{bmatrix} | \\ \mathbf{y} \\ | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix};$$

$$\mathbf{y} = x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + x_3 \mathbf{c}_3.$$

- More generally, the matrix-matrix product \mathbf{AB} is an array containing all pairwise dot products

Special Matrix Types for Matrix Operations

- **Diagonal matrices** only have entries on the diagonal, thus $a_{ij} = 0$ when $i \neq j$, e.g. the identity matrix
- Diagonal matrices are a special instance of **symmetric matrices**, matrices that equal their transpose $\mathbf{A} = \mathbf{A}^T$
- Matrices whose columns are orthogonal vectors of *unit length* are called **orthogonal matrices**.
 - Determinant of an orthogonal matrix is always 1 or -1
 - Orthogonal matrices are almost their own inverse, for an orthogonal matrix \mathbf{R} , $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$.

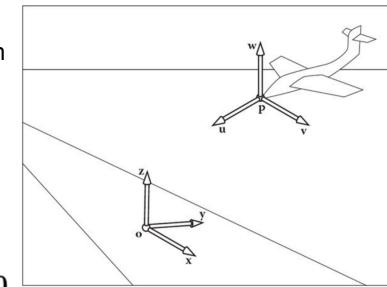
Orthonormal Bases

- The columns of an orthogonal matrix define an orthonormal basis!

- Vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} define an orthonormal basis if

$$\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$$



Basis

- We say that vectors \mathbf{a}, \mathbf{b} form a basis in 2D if any other vector \mathbf{v} could be written as a linear combination of \mathbf{a}, \mathbf{b} $\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b}$ where α, β are scalars.
- A set of vectors $\mathbf{v}_1 \dots \mathbf{v}_n$ are **linearly dependent** if we can find scalars $\alpha_1 \dots \alpha_n$ such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_n\mathbf{v}_n = 0$
- A basis is any set of linearly independent vectors.
- Examples: any two non-collinear vectors. They define a plane that they span. Any other vector \mathbf{v} on this plane is a linear combination of these vectors.
- Examples, any 3 linearly independent vectors in 3D. Furthermore, we can create any other vector \mathbf{v} as a linear combination of these vectors:
$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \dots + \alpha_n\mathbf{v}_n$$

Examples of Matrix Types

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

- Diagonal, Symmetric, but not Orthogonal

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 7 \\ 2 & 7 & 1 \end{bmatrix}$$

- Symmetric, but not Diagonal or Orthogonal

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Orthogonal, but not Symmetric or Diagonal

Transformations

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_2 \end{bmatrix}$$

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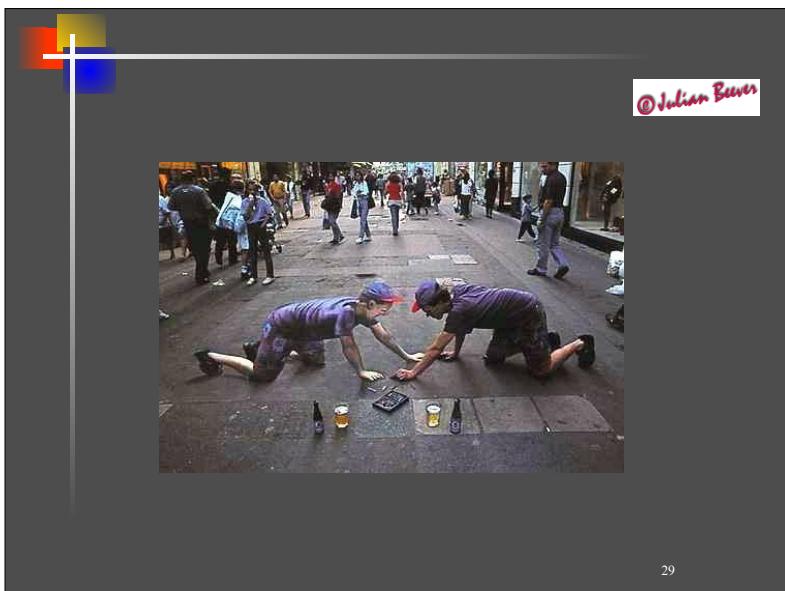
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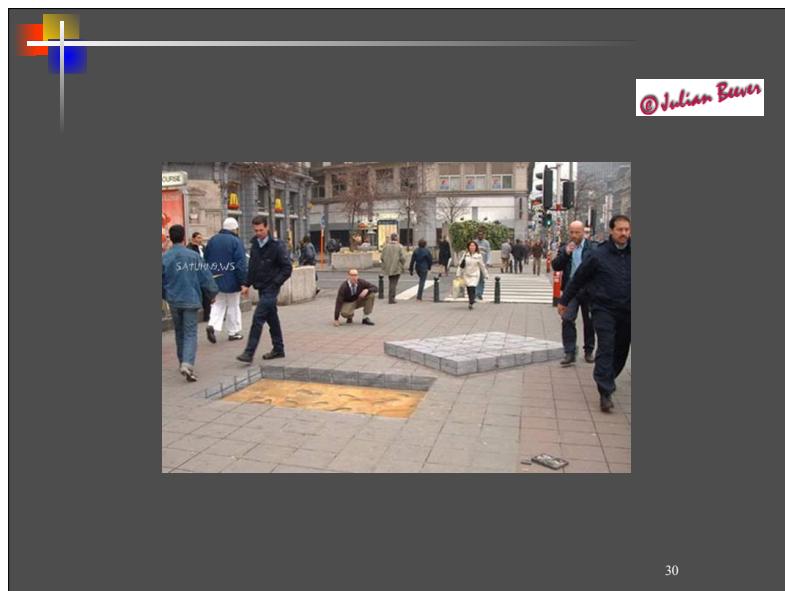


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Transformations

- ❑ A transformation τ is a mapping of R^n to itself
 - (not necessarily one-to-one. Many points might be transformed to the same point.)
 - Linear transformation:
 $\tau(\alpha\mathbf{u} + \beta\mathbf{b}) = \alpha \cdot \tau(\mathbf{u}) + \beta \cdot \tau(\mathbf{v})$
 for any \mathbf{u}, \mathbf{v} vectors, and α, β scalars.
 In particular, zero is transformed to zero.
- ❑ Affine transformation – $\tau(V) = AV+b$
 - A – matrix
 - b – vector
- ❑ Relevant Example: Holography (project a billboard to the image plane). More about homographs later in the course.

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Transformations

- ❑ Transforming an object = transforming all its points
- ❑ For a polygonal model = transforming its vertices

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Scaling

- $V = (v_x, v_y)$ – vector in XY plane
- Scaling operator S with parameters (s_x, s_y) :

$$S^{(s_x, s_y)}(V) = (v_x s_x, v_y s_y)$$



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Scaling

- Matrix form:

$$S^{(s_x, s_y)}(V) = \begin{pmatrix} v_x & v_y \end{pmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} = \begin{pmatrix} v_x s_x & v_y s_y \end{pmatrix}$$

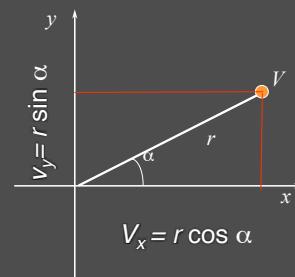
- Independent in x and y
- Non-uniform scaling is allowed
- What is the meaning of scaling by zero?

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Polar form of a point

- Polar form:

$$V = (v_x, v_y) = (r \cos \alpha, r \sin \alpha)$$



r is the distance of v from the origin $(0,0)$

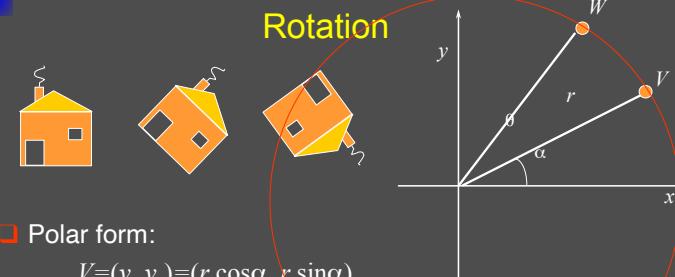
$$r = \sqrt{v_x^2 + v_y^2}$$

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Rotation

- Polar form:

$$V = (v_x, v_y) = (r \cos \alpha, r \sin \alpha)$$
- Rotate V anti-clockwise by θ to W :



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Rotation CCW by θ

- ❑ Polar form:
 $V = (v_x, v_y) = (r \cos\alpha, r \sin\alpha)$
- ❑ Rotate V counter-clockwise by θ to W

$$W = (r \cos\alpha, r \sin\alpha) \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= V \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$W = (w_x, w_y)$$

$$= (r \cos(\alpha + \theta), r \sin(\alpha + \theta))$$

$$= (r \cos\alpha \cos\theta - r \sin\alpha \sin\theta, r \cos\alpha \sin\theta + r \sin\alpha \cos\theta)$$

$$R^\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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Rotation Properties

- ❑ R^θ is orthogonal

$$R^{-1} = R^{-\theta} = (R^\theta)^T$$

- ❑ Rotation by $-\theta$ is

$$R^{-\theta}(V) = (v_x, v_y) \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = (R^\theta)^{-1}$$

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Translation

- ❑ Translation operator T with parameters (t_x, t_y) :

$$T^{(t_x, t_y)}(V) = (v_x + t_x, v_y + t_y)$$

- ❑ Can we express T in a matrix form?

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Translation - Homogeneous Coordinates

- ❑ To represent T in matrix form – use homogeneous coordinates:

$$V^h = (v_x^h, v_y^h, v_w^h) = (v_x^h, v_y^h, 1)$$

- ❑ Conversion (projection) from homogeneous to Euclidean:

$$V = (v_x, v_y) = \left(\frac{v_x^h}{v_w^h}, \frac{v_y^h}{v_w^h} \right)$$

- ❑ In homogeneous coordinates:

$$(2, 2, 1) = (4, 4, 2) = (1, 1, 0.5)$$

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Translation

- Using homogeneous coordinates:

$$T^{(t_x, t_y)}(V^h) = (v_x, v_y, 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} = (v_x + t_x, v_y + t_y, 1)$$

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Matrix Form

- Why is it useful to use **Matrix** form to represent the **transformations** ?

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Transformation Composition

- What operation rotates by θ around $P = (p_x, p_y)$?

- Translate P to origin
- Rotate around origin by θ
- Translate back



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Transformation Composition

$$T^{(-p_x, -p_y)} R^\theta T^{(p_x, p_y)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p_x & -p_y & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_x & p_y & 1 \end{bmatrix}$$

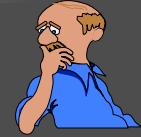
Let Q denote this matrix (computed once).

For every point p of the many points of the “house”, we apply $p' = pQ$

(read: The old corner p is transformed to the new corner p')

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Transformations Quiz



- What do these transformations do?
- And these homogeneous ones?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

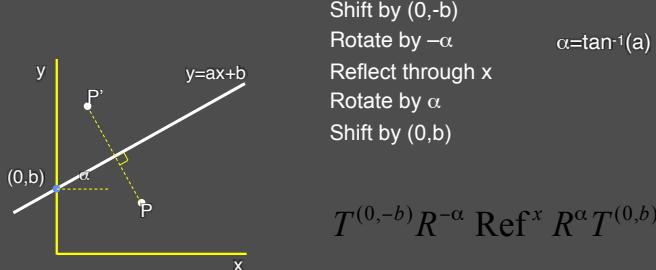
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- How can one reflect a planar object through an arbitrary line in the plane?

- Can one rotate a planar object in the plane by reflection?

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Arbitrary Reflection



Shift by $(0, -b)$
Rotate by $-\alpha$
 $\alpha = \tan^{-1}(a)$

Reflect through x
Rotate by α
Shift by $(0, b)$

$$T^{(0, -b)} R^{-\alpha} \text{Ref}^x R^\alpha T^{(0, b)}$$

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