MA5233 Assignment 1

Approximating Poisson's Equation in two dimensions

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1 Introduction

The objective of this assignment is to approximate the solution of the following boundary value problem featuring Poisson's equation in two dimensions.

$$-\Delta u = f, \;\; \Omega = (0,1) \times (0,1)$$
$$f(x,y) = 400(x^4 - y^4)\sin(20xy)$$
$$u(x,0) = 0, \;\; x \in (0,1)$$
$$u(0,y) = 0, \;\; y \in (0,1)$$
$$u_y(x,1) = 20x(x^2-1)\cos(20x) - 2\sin(20x), \;\; x \in (0,1)$$
$$u_x(1,y) = 20y(1-y^2)\cos(20y) + 2\sin(20y), \;\; y \in (0,1)$$

2 Finite Differences

The finite difference method uses direct approximations of the derivatives appearing on the left-hand side to attain a system of linear equations. We begin by discretizing the domain. Since we have mixed boundary conditions, we will have to allow for a half-interval at the "top and right" ends of the grid, where the Neumann boundary conditions apply. This leads to the following parameters.

N - the number of grid cells along one dimension,

h - the size of the interval between grid-lines (equal in both directions).

Here,

$$h = \frac{1}{N + 1/2}$$

We derive the numerical scheme beginning with the Poisson equation.

$$-\Delta u = f$$

$$-\partial_{xx}u - \partial_{yy}u = f$$

Let i denote the index for the vertical grid lines. $i = 0$ denotes the line where $x = 0$, and $i = N$ denotes the line where $x = 1 - h/2$. Similarly, let j be the index for the horizontal grid lines, where $j = 0$ is the line $y = 0$ and $j = N$ is the line $y = 1 - h/2$. Then we can approximate one order of differentiation as follows.

$$-\frac{\partial_x u(x_{i+1/2}, y_j) - \partial_x u(x_{i-1/2}, y_j)}{h} - \frac{\partial_y u(x_i, y_{j+1/2}) - \partial_y u(x_i, y_{j-1/2})}{h} = f(x_i, y_j)$$

Applying finite differences again, we reach the following equation.

$$-\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_{i-1}, y_{j-1})}{h^2} = f(x_i, y_j)$$

The above equation holds for $i, j = 1, ...N-1$, giving us $(N-1)^2$ equations. Note that if i or j is 1, then we can simply plug in the Dirichlet boundary conditions $u = 0$. If i or j is equal to N, then we have to make use of the Neumann boundary conditions as follows. Whenever $i = N$, we have

$$-\frac{\partial_x u(1, y_j) - \frac{1}{h}(u(x_i, y_j) - u(x_{i-1}, y_j))}{h} - \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_{i-1}, y_{j-1})}{h^2} = f(x_i, y_j),$$

where $\partial_x u(1, y_j)$ is a given boundary condition. Similarly, when $j = N$, we have

$$-\frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{\partial_y u(x_i, 1) - \frac{1}{h}(u(x_i, y_j) + u(x_{i-1}, y_{j-1}))}{h} = f(x_i, y_j)$$

And, of course, if both $i = j = N$, then we have to adjust both terms. This concludes the numerical scheme with N^2 equations and N^2 unknowns.

3 Finite Volumes

The finite volume method seeks to approximate the solution by averages over grid-cells. To arrive at the numerical scheme, we will integrate both sides of the equation over a cell and divide by the cell-area. We again begin by discretizing the domain into a grid. To accompany the different types of boundary conditions, we again like to employ a half-interval at one end of the domain. This time, we have the half-intervals at the "bottom and left", where the Dirichlet boundary conditions are given. The parameters N, and h stay the same as in the fininite difference method.

When discussing the finite differences, we were not interested in giving an index to the grid line that followed from the half-step. Similarly, this time we are not interested in indexing the grid lines coinciding with the coordinate axes. We instead set $i = 0$ for $x = h/2$, $i = N$ for $x = 1$, and similarly $j = 0$ for $y = h/2$ and $j = N$ for $y = 1$. We denote the cell contained by $i-1, i, j-1, j$ by indices i, j and hence denote that cell's sought after average-cell value by $u_{i,j}$. Taking the integral of Poisson's equation over a given such cell and dividing by its area we obtain

$$-\frac{1}{h^2}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} u_{xx}\,dxdy - \frac{1}{h^2}\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} u_{yy}\,dydx = -\frac{1}{h^2}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} f\,dxdy$$

which, by taking integrals of derivatives, simplifies to

$$-\frac{1}{h^2}\int_{y_{j-1}}^{y_j}(u_x(x_i,y)-u_x(x_{i-1},y))\,dy - \frac{1}{h^2}\int_{x_{i-1}}^{x_i}(u_y(x,y_j)-u_y(x,y_{j-1}))\,dx = -\frac{1}{h^2}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} f\,dxdy$$

Next, we make the simplifying assumption, that along the small interval of length h in one direction, the partial derivative in the other direction stays constant and is given by the difference of the average of the adjacent cells as follows.

$$\begin{aligned}
u_x(x_i, y) &= \frac{u_{i+1,j} - u_{i,j}}{h} \\
u_x(x_{i-1}, y) &= \frac{u_{i,j} - u_{i-1,j}}{h} \\
u_y(x, y_j) &= \frac{u_{i,j+1} - u_{i,j}}{h} \\
u_y(x, y_{j-1}) &= \frac{u_{i,j} - u_{i,j-1}}{h}
\end{aligned}$$

This assumption seems reasonable, since all we have to work with is the average values of the cells. Taking integrals along the intervals then amounts to multiplication by a factor h.

$$-\frac{u_x(x_i,y)-u_x(x_{i-1},y)}{h} - \frac{u_y(x,y_j)-u_y(x,y_{j-1})}{h} = -\frac{1}{h^2}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} f\,dxdy$$

Plugging in, we get the following numerical scheme.

$$-\frac{-4u_{i,j} + u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}}{h^2} = -\frac{1}{h^2}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} f\,dxdy$$

Of course, the division by h^2 could be cancelled on both sides. Finally, we have to discuss the boundary conditions. The above equation holds for $i, j = 1, ..., N-1$. Note, note that whenever i or j are zero, we make use of the Dirichlet boundary conditions to obtain the "ghost cell" averages $u_{0,j}, u_{i,0}$ and so on, which we assume to be 0. Finally, if $i = N$ or $j = N$, then we have to make use of the Neumann conditions again, obtaining for $i = N$

$$-\frac{-3u_{i,j} + u_{i-1,j} + hu_x(1, y_{j-1/2}) + u_{i,j-1} + u_{i,j+1}}{h^2} = -\frac{1}{h^2}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} f\,dxdy,$$

and for $j = N$

$$-\frac{-3u_{i,j} + u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + hu_y(x_{i-1/2}, 1)}{h^2} = -\frac{1}{h^2}\int_{y_{j-1}}^{y_j}\int_{x_{i-1}}^{x_i} f\,dxdy.$$

Of course, we adjust appropriately, when both $i, j = N$, in which case the coefficient of $u_{i,j}$ will become -2. This numerical scheme again features N^2 unknowns, excluding the ghost cells, and N^2 equations.

4 Finite Elements

The finite element method aims to approximate the solution as a linear combination of basis functions. The resulting approximation will only be once weakly differentiable, and so we have to derive the weak form of Poisson's equation. We integrate both sides multiplied by a test function over the domain.

$$-\int_\Omega \Delta u \varphi \, dA = \int_\Omega f \varphi \, dA$$

$$-\int_\Omega (\nabla \cdot \nabla u) \varphi \, dA = \int_\Omega f \varphi \, dA$$

We use integration by parts to obtain the following.

$$-\int_{\partial\Omega} \varphi(\nabla u \cdot n) \, ds + \int_\Omega (\nabla u \cdot \nabla \varphi) \, dA = \int_\Omega f \varphi \, dA,$$

where n is the unit normal pointing out of the domain. We then assume that φ vanishes on the part of the boundary where Dirichlet conditions have been given, and only has support on the other part of the boundary, which we shall call Γ_2. Note that $(\nabla u \cdot n)$ is exactly the Neumann boundary condition. Rearranging slightly, we get.

$$\int_\Omega (\nabla u \cdot \nabla \varphi) \, dA = \int_\Omega f \varphi \, dA + \int_{\Gamma_2} \varphi(\nabla u \cdot n) \, ds,$$

This time, I have decided on a discretization where $h = 1/N$ exactly, and the indices i, j run appropriately from 0 to N. The φ above is a test function, but we will use the same function as our basis functions. We define them as follows.

$$\varphi_{i,j} = \begin{cases} \frac{1}{h^2}(x - x_{i-1})(y - y_{j-1}) & \text{if } (x, y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \\ \frac{1}{h^2}(x_{i+1} - x)(y - y_{j-1}) & \text{if } (x, y) \in [x_i, x_{i+1}] \times [y_{j-1}, y_j] \\ \frac{1}{h^2}(x - x_{i-1})(y_{j+1} - y) & \text{if } (x, y) \in [x_{i-1}, x_i] \times [y_j, y_{j+1}] \\ \frac{1}{h^2}(x_{i+1} - x)(y_{j+1} - y) & \text{if } (x, y) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}] \end{cases}$$

These are hat functions that have support over 4 adjacent grid cells. Whenever i and/or j are such that one of the mentioned cells lies outside of Ω, we just ignore this part, and don't define it. Again, whenever i or j are zero, then so is φ.

Now, we can approximate the solution as follows.

$$u = \sum_{i=1}^{N} \sum_{j=1}^{N} u_{i,j} \varphi_{i,j}$$

When we substitute this equation into the derived weak form of Poisson's equation, we need to take note that the two φ's represent two different purposes, and so they must receive two different sets of indices. We let r and s take identical roles to i and j respectively. Then we get

$$\sum_{i=1}^{N} \sum_{j=1}^{N} u_{ij} \int_\Omega \nabla \varphi_{ij} \cdot \nabla \varphi_{rs} \, dA = \int_\Omega f \varphi_{rs} \, dA + \int_{\Gamma_2} \varphi_{rs}(\nabla u \cdot n) \, ds.$$

The hat functions and their derivatives have overlapping support only if they are adjacent. The equation thus reduces to

$$\sum_{i=r-1}^{r+1} \sum_{j=s-1}^{s+1} u_{ij} \int_\Omega \nabla \varphi_{ij} \cdot \nabla \varphi_{rs} \, dA = \int_\Omega f \varphi_{rs} \, dA + \int_{\Gamma_2} \varphi_{rs}(\nabla u \cdot n) \, ds.$$

To find the various values of the coefficients on the left hand side, we recognize there are only three different kinds of overlapping that can occur. Corner-to-corner diagonal overlaps produce a value of $-1/3$. Cells in which the overlap agrees in one direction, but not the other usually occur twice (except on the Neumann boundary), and each cell produces a value of $-1/6$, which usually translates to $-1/3$ for non-diagonal overlaps. Finally, when we are dealing with a full overlap, we have in each of the four cells a value of $+2/3$, which amounts to $+8/3$ for regular cells, and $+4/3$, or even $+2/3$ on the Neumann boundary (The function is strictly speaking not defined there, but we have extended it so).

The term integrating over $f\varphi_{rs}$ is computed using numerical integration. Finally, we have the term integrating over the boundary Γ_2. This term takes one of the following three forms depending on where on the boundary we are.

$$\int_{y_{s-1}}^{y_s} u_x(1,y)\cdot\frac{y-y_{s-1}}{h}\,dy+\int_{y_s}^{y_{s+1}} u_x(1,y)\cdot\frac{y_{s+1}-y}{h}\,dy$$

$$\int_{x_{r-1}}^{x_r} u_y(x,1)\cdot\frac{x-x_{r-1}}{h}\,dx+\int_{x_r}^{x_{r+1}} u_y(x,1)\cdot\frac{x_{r+1}-x}{h}\,dy$$

$$\int_{x_{N-1}}^{x_N} u_y(x,1)\cdot\frac{x-x_{N-1}}{h}\,dx+\int_{y_N}^{y_{N+1}} u_x(1,y)\cdot\frac{y_{N+1}-y}{h}\,dy$$

Otherwise it vanishes. This concludes the numerical scheme, which features N^2 unknowns and N^2 equations.

5 Results

We begin with the results of the finite difference method. To start with, we set $N = 10$ to obtain the following approximation.

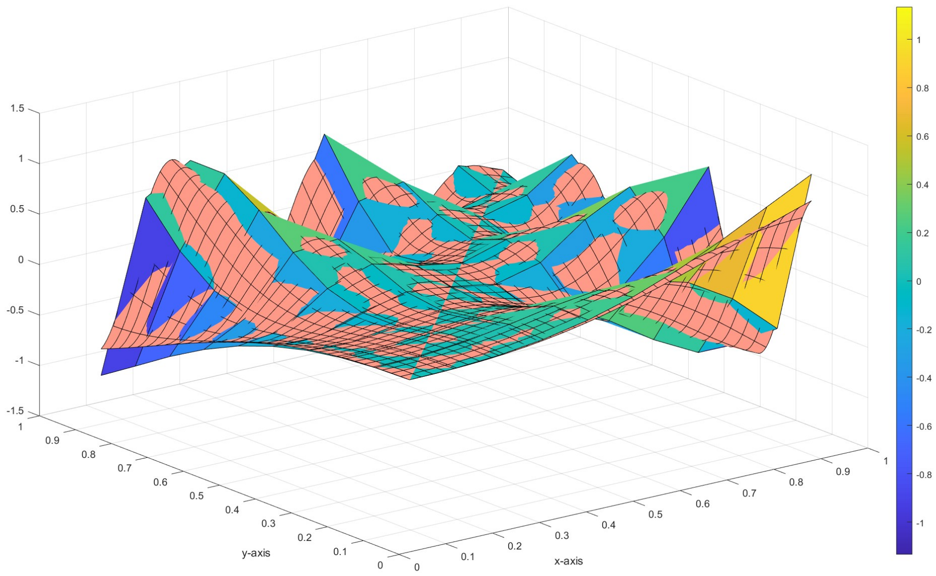

Figure 1: Finite difference method, $N = 10$

The pink surface represents the exact solution. As expected, this discretization is not yet fine enough to obtain a satisfactory approximation. Let's look at $N = 80$.

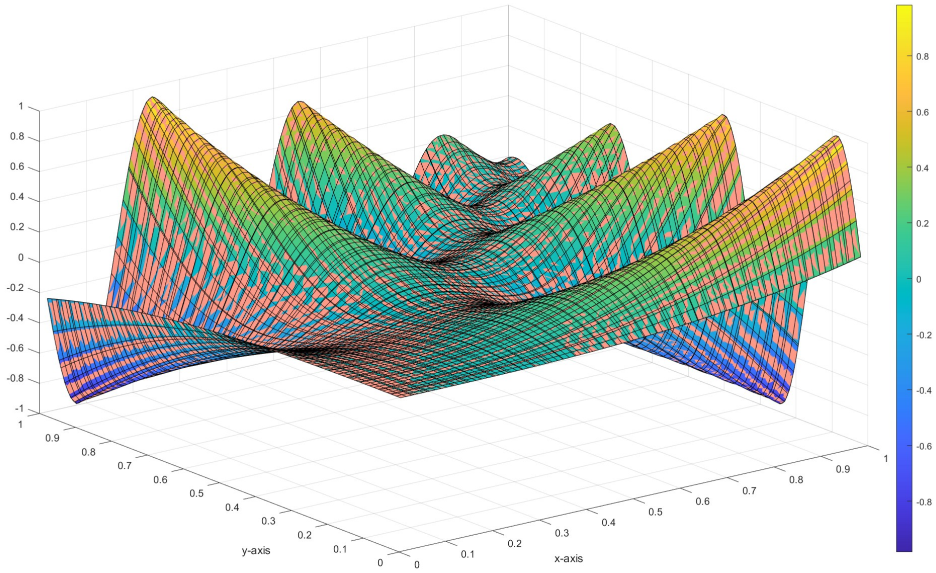

Figure 2: Finite difference method, $N = 80$

This seems a much better approximation.

Next up, is the finite volume method. I chose to present $N = 20$ to see the progression from $N = 10$, albeit using a different method.

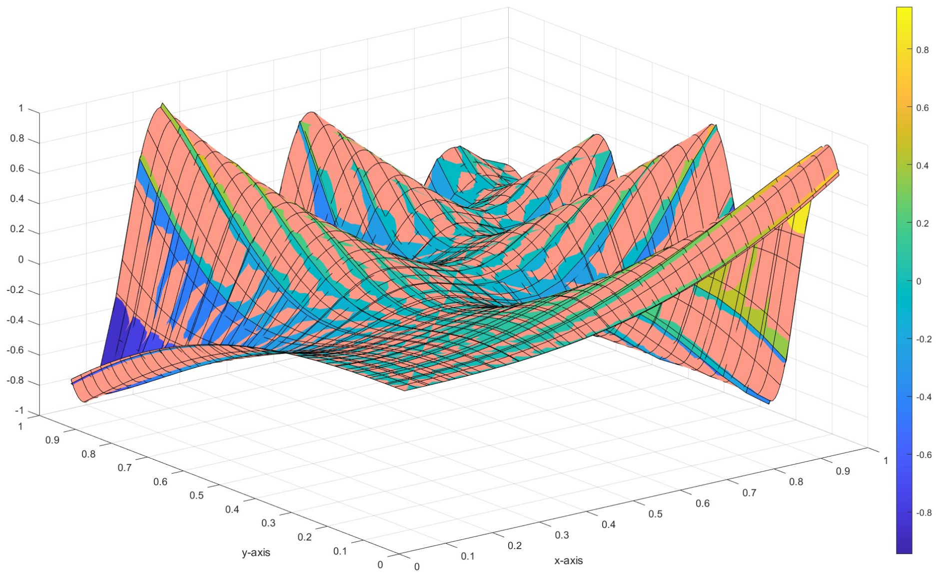

Figure 3: Finite volume method, $N = 20$

Again, we show $N = 80$ for this method, as well.

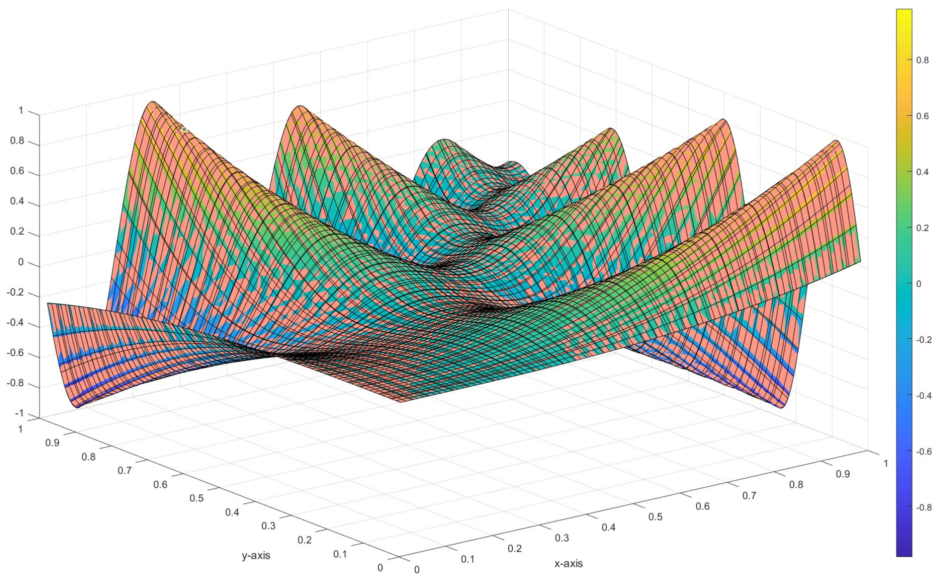

Figure 4: Finite volume method, $N = 80$

Finally, $N = 40$ for the finite element method, which shows quite a good approximation.

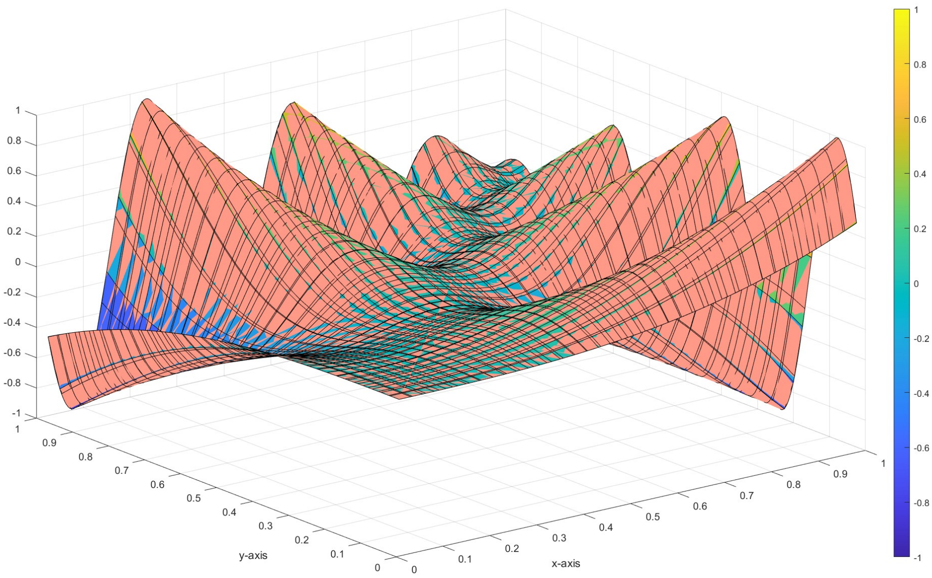

Figure 5: Finite element method, $N = 40$

And again, $N = 80$ to show that the finite element method also converges.

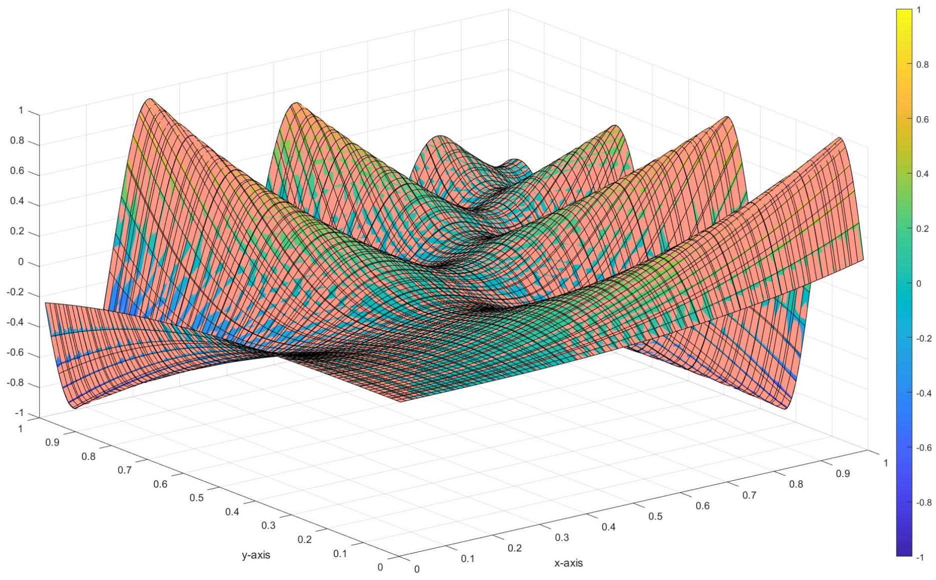

Figure 6: Finite element method, $N = 80$

6 Error Analysis

We start by presenting the raw data of the error's calculated.

Finite Differences	
N	Maximum error
10	0.2627
20	0.6453
40	0.01715
80	0.004332

Table 1: FD error analysis

Finite Volumes	
N	L1 error
10	0.04175
20	0.01132
40	0.002915
80	0.000739

Table 2: FV error analysis

Finite Volumes	
N	L2 error
10	0.1525
20	0.09844
40	0.06747
80	0.04729

Table 3: FE error analysis

We can see that all methods indeed converge. As in the 1-dimensional case, the error of the FD and the FV methods decreases quadratically by $1/4$ for every doubling of N. However, it seems, if not for some error in computation, the error of the finite element method seems to decrease only by a factor of roughly $2/3$ for the same refinement of the grid. Visually, the approximation seems to converge at about the same rate as for the other two methods, which leaves error in error-computation as one of the possibilities, and simply the fact that this error measures something different as the other possibility.

To conclude, we present the log-plots of the error analysis.

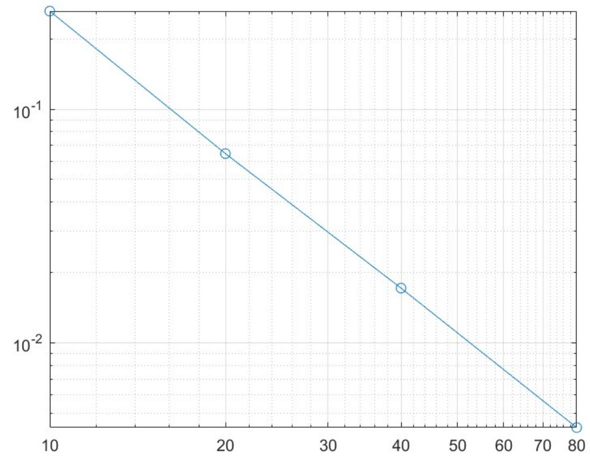

Figure 7: FE maximum error

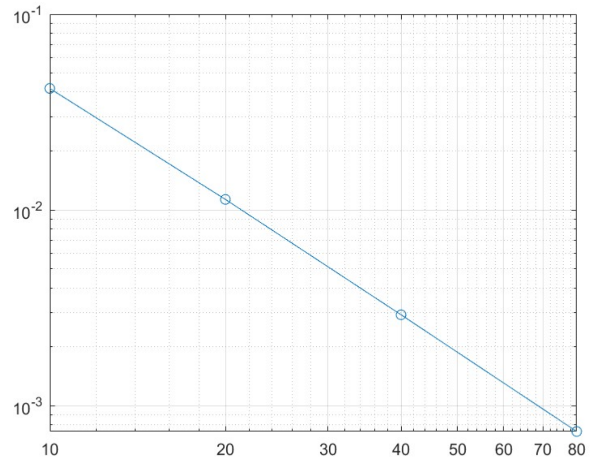

Figure 8: FV L1 error

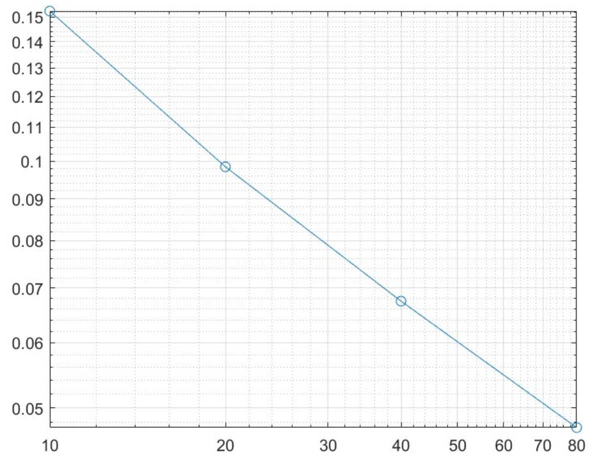

Figure 9: FE L2 error