MA5250 Project 2

Solving the Euler equations

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1 Introduction

The objective of this assignment is to approximate the solution of the nonlinear Euler equations on a one-dimensional domain, but observing two velocity components. The equations are as follows.

$$\begin{aligned} \partial_t \rho + \partial_x(\rho u) &= 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) &= 0, \\ \partial_t(\rho v) + \partial_x(\rho u v) &= 0, \\ \partial_t e + \partial_x(u(e+p)) &= 0. \end{aligned}$$

The equation of state of an ideal gas is used with adiabatic coefficient $\gamma = 1.4$.

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho(u^2 + v^2).$$

Two problems are to be solved. The first is a standard Riemann problem with initial conditions

$$\begin{aligned} &\rho(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 3 & \text{if } x > 0 \end{cases}, \\ &u(x,0) = 0, \\ &v(x,0) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \\ &p(x,0) = 1. \end{aligned}$$

For this problem all the spatial derivatives vanish, and thus the solution should not evolve. We use this problem as a test to check if our solver can deal with such a scenario.

Second, we solve the classic blast wave problem but with a discontinuity in vertical velocity at $x = 0.5$.

$$\begin{aligned} &\rho(x,0) = 1, \\ &u(x,0) = 0, \\ &v(x,0) = \begin{cases} -10 & \text{if } x \in (0, 0.5) \\ 20 & \text{if } x \in (0.5, 1) \end{cases}, \\ &p(x,0) = \begin{cases} 1000 & \text{if } x \in (0, 0.1) \\ 0.01 & \text{if } x \in (0.1, 0.9) \\ 100 & \text{if } x \in (0.9, 1) \end{cases}. \end{aligned}$$

2 Numerical Solver

The HLLC method was used to solve the equations. The overall structure of the scheme is a conservative one.

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

Here, Q_i^n is supposed to approximate the average of the solution over grid cell i at time step n. Δx represents the uniform width of the grid cells, and Δt the time step of one evolution iteration.

The main difficulty in developing any conservative scheme is presented by approximating the flux at the cell boundaries. $F_{i+1/2}^n$ represents the outgoing flux on the cell boundary between cells i and $i+1$.

Since we are using a finite volume approximation for Q_i^n with piece-wise constant reconstruction, we are presented with discontinuities at the cell boundaries at each time step, which translate into separate Riemann problems, if small enough time steps are considered.

The HLLC method is based on approximating these Riemann problems using the 5-part self-similar structure that arises from a linearized problem of 4 equations. In our special case, two discontinuities will end up having the same speed, and so we approach the problem by assuming a 4-part self-similar solution.

The division of these structures occurs along rays from the origin with slopes characterized by the inverse values of three eigenvalues λ_L, λ^*, and λ_R. To incorporate most (or hopefully all) the waves of the nonlinear solution efficiently, we set

$$\lambda_L = \min\{u_i^n - c_i^n, u_{i+1}^n - c_{i+1}^n\},$$
$$\lambda_R = \max\{u_i^n + c_i^n, u_{i+1}^n + c_{i+1}^n\},$$

where $c = \sqrt{\gamma p/\rho}$ is the speed of sound. The intermediate eigenvalue λ^* can be found by considering the conservation equations across the discontinuities of the 4-part structure.

$$\begin{aligned} F_L^* - f(Q_i^n) &= \lambda_L(Q_L^* - Q_i^n), \\ F_R^* - F_L^* &= \lambda^*(Q_R^* - Q_L^*), \\ f(Q_{i+1}^n) - F_R^* &= \lambda_R(Q_{i+1}^n - Q_L^*). \end{aligned}$$

Here, Q_L^* and F_L^* represent state and flux to the left of the central discontinuity, whereas Q_R^* and F_R^* are the state and flux on the right. Using our knowledge that the central discontinuity is a contact discontinuity, we can use $u_L^* = \lambda^* = u_R^*$ to solve the above system. In doing so we attain the following.

$$\begin{aligned}
\lambda^* &= \frac{\lambda_R \rho_{i+1}^n u_{i+1}^n - \lambda_L \rho_i^n u_i^n - \left[\rho_{i+1}^n (u_{i+1}^n)^2 + p_{i+1}^n - \rho_i^n (u_i^n)^2 - p_i^n\right]}{\lambda_R \rho_{i+1}^n - \lambda_L \rho_i^n - (\rho_{i+1}^n u_{i+1}^n - \rho_i^n u_i^n)}, \\
\rho_L^* &= \frac{\lambda_L - u_i^n}{\lambda_L - \lambda^*}\rho_i^n, \\
\rho_R^* &= \frac{\lambda_R - u_{i+1}^n}{\lambda_R - \lambda^*}\rho_{i+1}^n, \\
e_L^* &= \frac{\lambda_L - u_i^n}{\lambda_L - \lambda^*}e_i^n + \frac{\lambda^* - u_i^n}{\lambda_L - \lambda^*}p_i^n + \frac{\lambda^*(\lambda^* - u_i^n)(\lambda_L - u_i^n)}{\lambda_L - \lambda^*}\rho_i^n, \\
e_R^* &= \frac{\lambda_R - u_{i+1}^n}{\lambda_R - \lambda^*}e_{i+1}^n + \frac{\lambda^* - u_{i+1}^n}{\lambda_R - \lambda^*}p_{i+1}^n + \frac{\lambda^*(\lambda^* - u_{i+1}^n)(\lambda_R - u_{i+1}^n)}{\lambda_R - \lambda^*}\rho_{i+1}^n, \\
Q_L^* &= (\rho_L^*, \rho_L^*\lambda^*, \rho_L^* v_i^n, e_L^*)^T, \\
Q_R^* &= (\rho_R^*, \rho_R^*\lambda^*, \rho_R^* v_{i+1}^n, e_R^*)^T.
\end{aligned}$$

With this approximate solution we can give the flux at the cell boundary.

$$F_{i+1/2}^n = \begin{cases} f(Q_i^n), & \text{if } \lambda_L \geq 0, \\ f(Q_i^n) + \lambda_L(Q_L^* - Q_i^n), & \text{if } \lambda_L < 0 < \lambda^*, \\ f(Q_{i+1}^n) + \lambda_R(Q_R^* - Q_{i+1}^n), & \text{if } \lambda^* \leq 0 < \lambda_R, \\ f(Q_{i+1}^n), & \text{if } \lambda_R \leq 0. \end{cases}$$

Finally, it should be mentioned that the time step needs to be carefully chosen in each iteration so that the solutions to the Riemann problems on the individual cell boundaries do not interact with each other. To be on the safe side, we take

$$\Delta t = \frac{0.1\Delta x}{|\lambda|},$$

where $|\lambda|$ gives the largest eigenvalue observed during the relevant iteration's flux computations across all cells.

3 Results

What follows are the results of the simulations. Note that 400 grid cells have been used to solve the Riemann problem, and 4000 grid cells have been used for the blast wave problem.

We first observe that the test of the Riemann problem has been passed. The solution remains unchanged, except for some tiny errors on the scale of 10^{-15}.

As for the blast wave problem, we reach a similar pattern to that observed in the classical problem without vertical velocity components. However, once the left shock wave hits the vertical velocity discontinuity at $x = 0.5$, it interacts with it, introducing some new disturbances to the upwind side of the wave. The discontinuity in the vertical velocity component itself also starts to propagate along with the first wave.

As a reference, the solution to the classical blast wave problem is included in an Appendix. There, 1000 grid cells have been used.

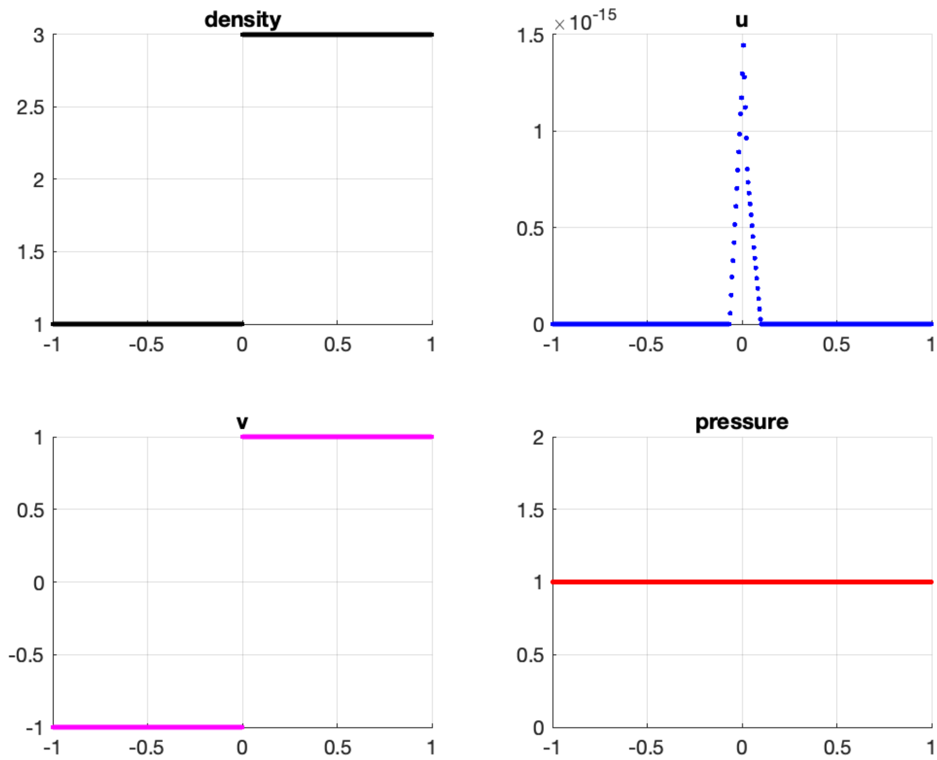

Figure 1: Riemann Problem $t = 1$, note that the scale of u is tiny.

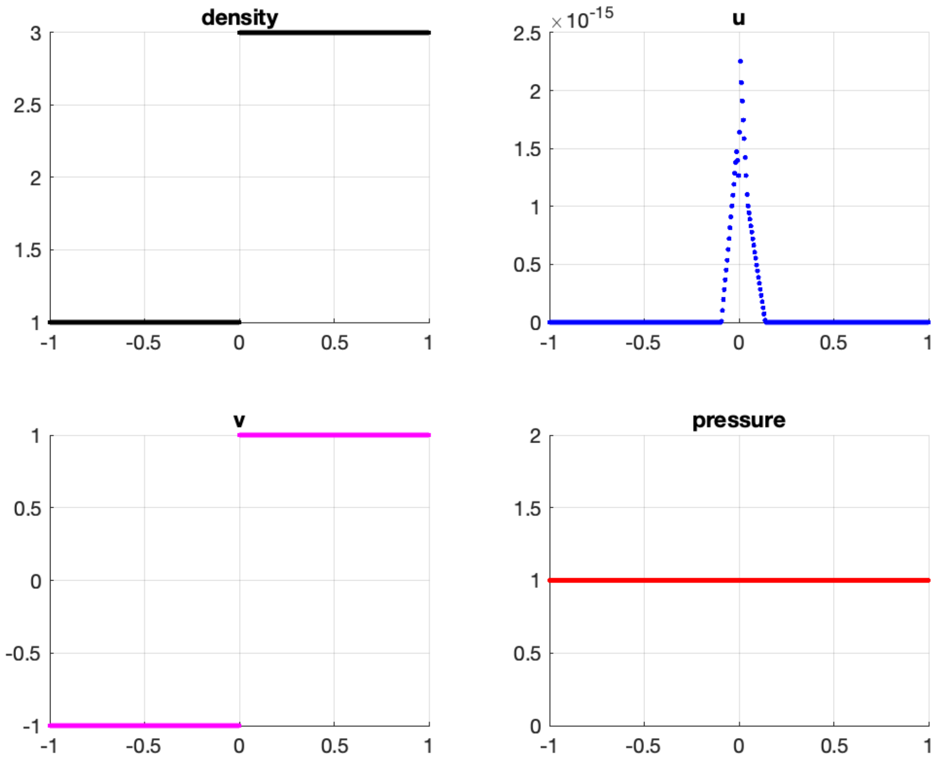

Figure 2: Riemann Problem $t = 2$, note that the scale of u is tiny.

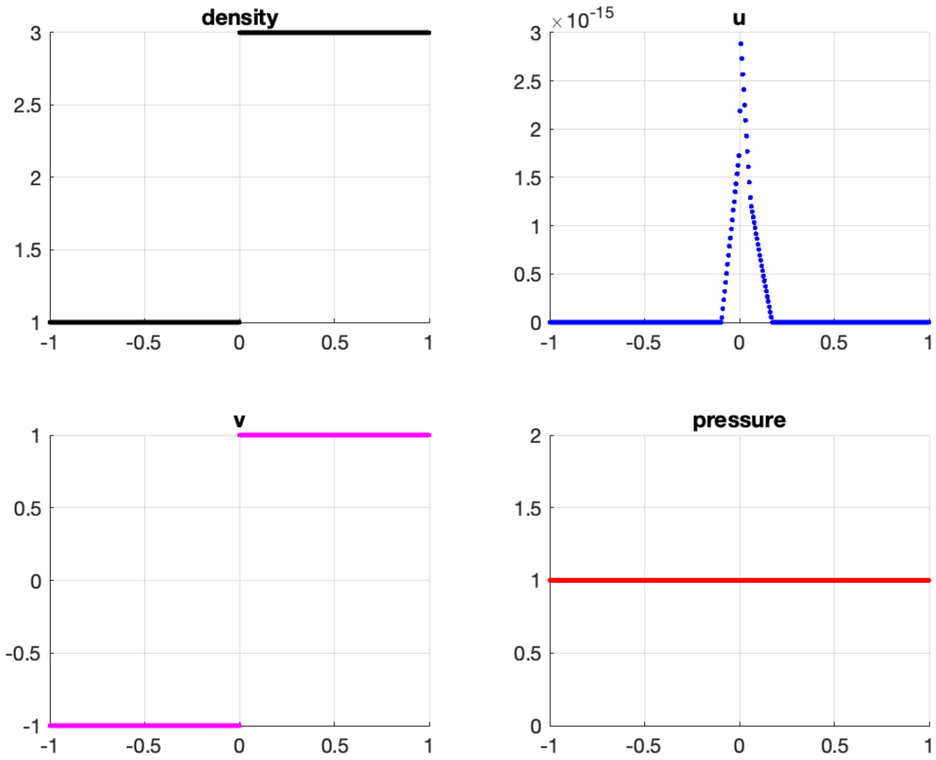

Figure 3: Riemann Problem $t = 3$, note that the scale of u is tiny.

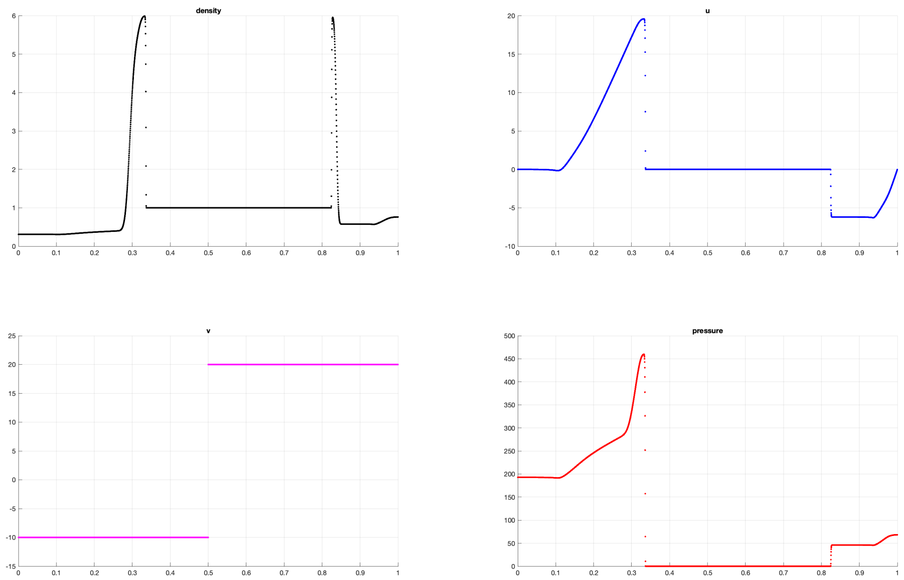

Figure 4: blast wave Problem $t = 0.01$.

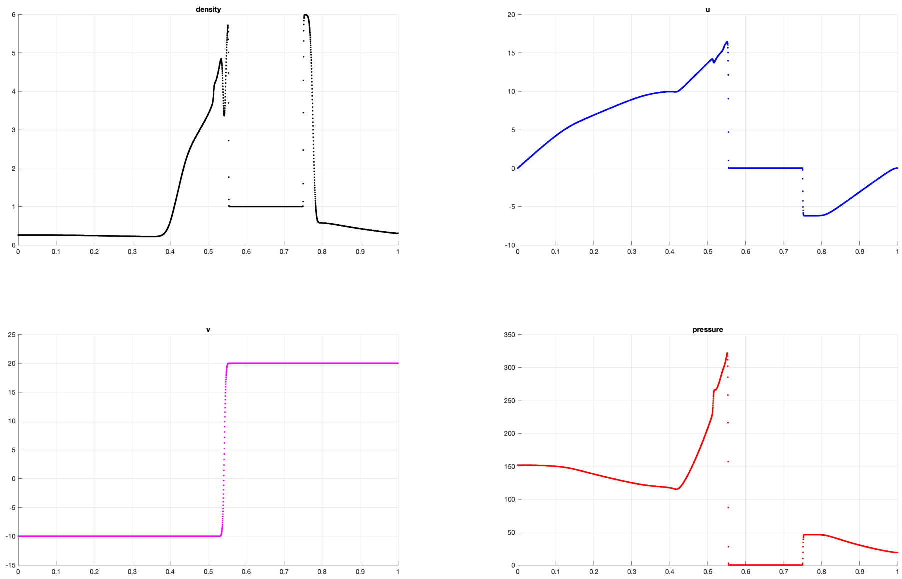

Figure 5: blast wave Problem $t = 0.02$.

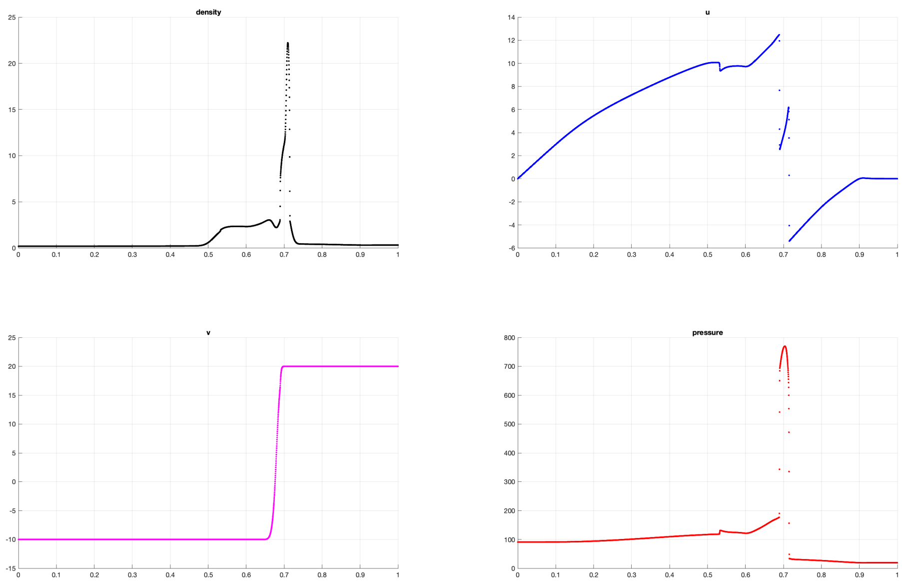

Figure 6: blast wave Problem $t = 0.03$.

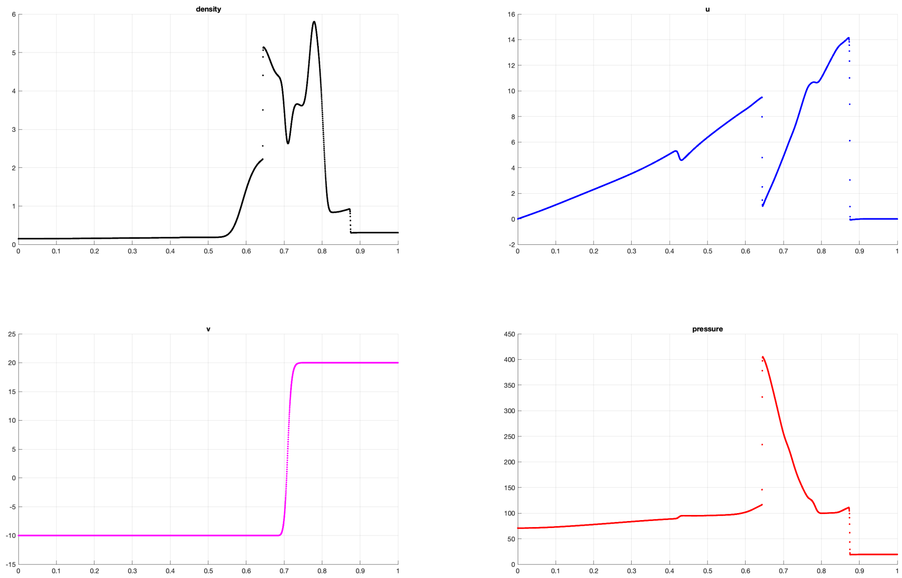

Figure 7: blast wave Problem $t = 0.038$.

4 Appendix

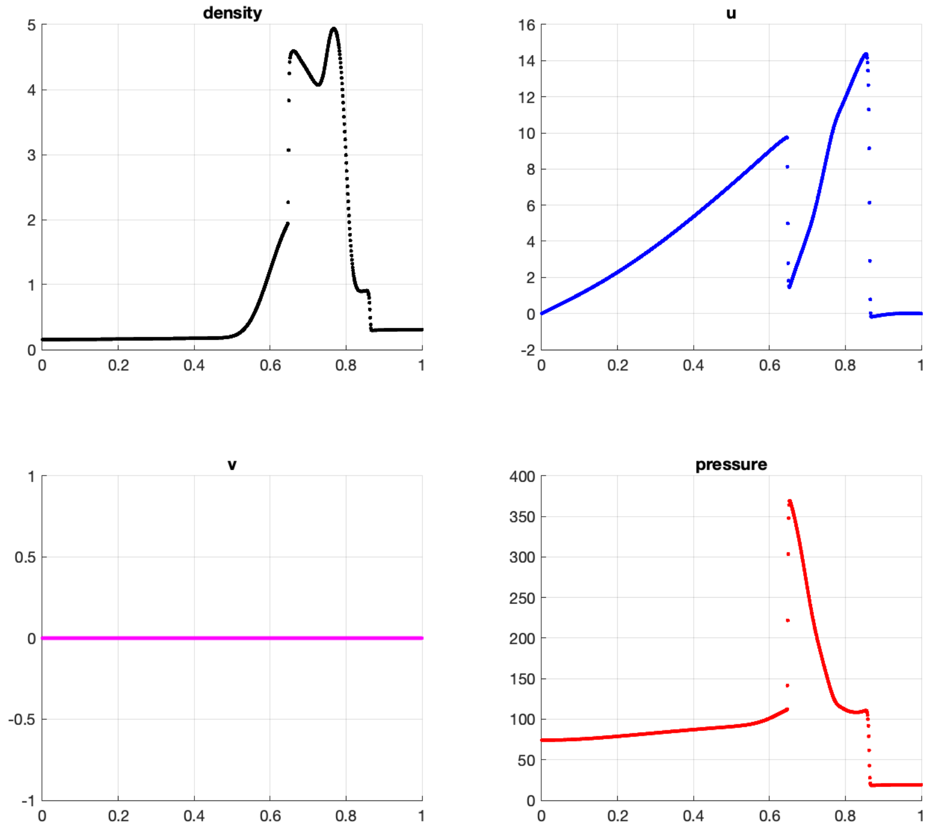

Figure 8: blast wave Problem without vertical velocities $t = 0.038$.