(1) נניח שקיים $u\in V$ כך ש $\langle u,e_k\rangle=0$

נראה שההיטל שלו הוא אפס.

$$\hat{u}=\sum_{1}^{k}\langle u,e_k\rangle e_k=0$$

מכאן אפשר להסיק ש $u=\vec{0}$

וזו סתירה לנתון.

$$\hat{0}=\sum_{1}^{n}\langle 0,e_k\rangle e_k=0$$

② $f(x) = e^x$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x = \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)$$

$$= \frac{e^{\pi}}{\pi} - \frac{1}{\pi e^{\pi}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cdot \cos(nx)\,dx = \frac{1}{\pi}\left[e^x \cdot \cos(nx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^x \cdot \frac{\sin(nx)}{n}\right]$$

$$= \frac{1}{\pi}\left[e^x \cdot \cos(nx)\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} e^x \cdot \sin(nx)\right]$$

$$= \frac{1}{\pi}\left[e^{\pi}\cos(\pi n) - e^{-\pi}\cos(n\pi) - \frac{1}{n}\left(e^x \cdot \sin(nx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^x \frac{-\cos(nx)}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[e^{\pi}(-1)^n - \frac{(-1)^n}{e^{\pi}} - \frac{1}{n^2}\int_{-\pi}^{\pi} e^x \cos(nx)\right]$$

$$= \frac{e^{\pi}(-1)^n}{\pi} - \frac{(-1)^n}{\pi e^{\pi}} - \frac{1}{n^2\pi}\underbrace{\int_{-\pi}^{\pi} e^x \cdot \cos(nx)\,dx}_{T}$$

$$\frac{1}{\pi}T = \frac{e^{\pi}(-1)^n}{\pi} - \frac{(-1)^n}{\pi e^{\pi}} - \frac{1}{n^2\pi}T \quad / + \frac{1}{n^2\pi}T$$

לשני
הצדדים

$$\frac{1}{\pi}T + \frac{1}{n^2\pi}T = \frac{e^{\pi}(-1)^n}{\pi} - \frac{(-1)^n}{\pi e^{\pi}}$$

$$\frac{1}{\pi} T\left(1+\frac{1}{n^2}\right) = \frac{e^{\pi}(-1)^n}{\pi} - \frac{(-1)^n}{\pi e^{\pi}}$$

$$\frac{1}{\pi} T\left(\frac{n^2+1}{n^2}\right) = \frac{e^{\pi}(-1)^n}{\pi} - \frac{(-1)^n}{\pi e^{\pi}} \quad \Big/ : \frac{n^2+1}{n^2}$$

$$\frac{1}{\pi} T = \left(\frac{e^{\pi}(-1)^n}{\pi} - \frac{(-1)^n}{\pi e^{\pi}}\right)\frac{n^2}{n^2+1}$$

$$\frac{1}{\pi} T = \frac{n^2 e^{\pi}(-1)^n}{(n^2+1)\pi} - \frac{n^2(-1)^n}{(n^2+1)\pi}$$

$$= \frac{(e^{2\pi}-1)(-1)^n}{(n^2+1)\pi e^{\pi}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cdot \sin(nx)$$

$$\int \sin(nx) e^x dx = \sin(nx) e^x - \int e^x \cos(nx) \cdot n \, dx$$

$$= \sin(nx) e^x - n \int \cos(nx) e^x dx =$$

$$\sin(nx) e^x - n\left(\cos(nx) e^x - \int e^x \cdot (-\sin(nx)\, n\right)$$

$$\sin(nx) e^x - n(\cos(nx) e^x + n \int e^x \sin(nx) dx$$

נעבור לצורת משוואה

$$\int e^{x}\sin(ax) = \sin(nx)e^{x} - n\cos(ax)e^{x} \underline{- n^{2}\int e^{x}\sin(nx)\,dx}$$

נעביר את הימני אגף

$$\int e^{x}\sin(ax) + n^{2}\int e^{x}\sin(ax)\,dx = \sin(nx)e^{x} - n\cos(ax)e^{x}$$

$$\int e^{x}\sin(nx)\left(1+n^{2}\right) = \sin(nx)\cdot e^{x} - n\cos(ax)e^{x}$$

נחלק ב $(1+n^{2})$

$$\int e^{x}\sin(nx) = \frac{\sin(nx)\cdot e^{x} - n\cos(ax)e^{x}}{n^{2}+1}\Bigg|_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\sin(nx) = \frac{(-e^{2\pi}+1)(-1)^{n}\cdot n}{(n^{2}+1)\pi e^{\pi}}$$

$$e^{x} \cong \frac{\frac{e^{\pi}}{\pi} - \frac{1}{\pi e^{\pi}}}{2} + \sum_{1}^{\infty}\frac{(e^{2\pi}-1)(-1)^{n}}{(n^{2}+1)\pi e^{\pi}}\cos(ax) + \sum_{1}^{\infty}\frac{(-e^{2\pi}+1)(-1)^{n}\cdot n}{(n^{2}+1)\pi e^{\pi}}\sin(ax)$$

3

$$f(x) = x - 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x-1)\cos(nx)\,dx = \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} x\cos(nx)}_{0} - \frac{1}{\pi}\underbrace{\int_{-\pi}^{\pi} \cos(nx)}_{0}$$

$$a_n = 0$$

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{1}^{\infty} 0^2 = 0$$

$$(4)\quad \langle x, y\rangle = \sum_{n=0}^{N-1}\left(x(n)\cdot\overline{y(n)}\right)$$

$$\langle u_k, u_k + 2u_j\rangle = \langle u_k, u_k\rangle + 2\langle u_k, u_j\rangle$$

$$= \sum_{0}^{N-1} u_k\cdot\overline{u_k} + 2\sum_{0}^{N-1} u_k\cdot\overline{u_j}$$

$$= \sum_{n=0}^{N-1} e^{\frac{i2\pi\cdot n}{N}\cdot k}\cdot e^{-\frac{i2\pi\cdot n}{N}\cdot k} + 2\cdot\sum_{n=0}^{N-1} e^{\frac{i2\pi\cdot n}{N}\cdot k}\cdot e^{-\frac{i2\pi\cdot n}{N}\cdot j}$$

$$= \sum_{n=0}^{N-1} e^{0} + 2\sum_{n=0}^{N-1} e^{\frac{2i\pi n(k-j)}{N}}$$

$$= N + 2\sum_{n=0}^{N-1} e^{\frac{2i\pi n(k-j)}{N}}$$

$$= N + 2\sum_{n=0}^{N-1}\left(e^{\frac{2i\pi(k-j)}{N}}\right)^{n} \longrightarrow S_N = \frac{\left(e^{\frac{2i\pi(k-j)}{\cancel{N}}}\right)^{\cancel{N}} - 1}{\left(e^{\frac{2i\pi(k-j)}{N}}\right) - 1}$$

$$= \frac{\left(e^{2i\pi(k-j)}\right) - 1}{\left(e^{\frac{2i\pi(k-j)}{N}}\right) - 1} = \frac{1-1}{e^{\frac{2i\pi(k-j)}{N}} - 1} = \frac{0}{e^{\frac{2i\pi(k-j)}{N}} - 1}$$

$$= N + 0 = N$$

(5)

נניח $g(x)$ פונקציה זוגית

$h(x)$ פונקציה אי זוגית

1c)

$$f(x) = g(x) + h(x)$$

$$f(-x) = g(x) - h(x)$$

$$f(x) + f(-x) = g(x) + h(x) + g(x) - h(x)$$

$$f(x) + f(-x) = 2 \cdot g(x)$$

(1) $$g(x) = \frac{f(x) + f(-x)}{2}$$

(2) $$h(x) = f(x) - g(x)$$

$$= f(x) - \frac{f(x) + f(-x)}{2}$$

$$= \frac{f(x) - f(-x)}{2}$$

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

$$= \frac{2f(x)}{2} = f(x) \checkmark$$

2)

$$h(x) \overset{5}{=} \frac{\ell}{2}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) - \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(-nx) + b_n \sin(-nx)\right]$$

$$= \frac{\ell}{2}\left[\sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) - \sum_{n=1}^{\infty} a_n \cos(nx) - b_n \sin(nx)\right]$$

$$= \frac{\ell}{2}\sum_{n=1}^{\infty} b_n \cdot \sin(nx) - \sum_{n=1}^{\infty} -b_n \cdot \sin(nx)$$

$$= \frac{\ell}{2}\sum_{n=1}^{\infty} b_n \cdot \sin(nx) + b_n \cdot \sin(nx)$$

$$= \sum_{n=1}^{\infty} b_n \cdot \sin(nx)$$

$$g(x) \overset{5}{=} \frac{\ell}{2}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(-nx) + b_n \sin(-nx)\right]$$

$$\overset{5}{=} \frac{\ell}{2}\left[a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) + \sum_{n=1}^{\infty} a_n \cos(nx) - b_n \sin(nx)\right]$$

$$\overset{5}{=} \frac{\ell}{2}\left[a_0 + \sum_{n=1}^{\infty} 2a_n \cos(nx)\right] = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$