1

א)

$$f(x) = \cos^5(x) \qquad \cos^5(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^5$$

$$\frac{e^{5ix} + 5e^{3ix} + 10e^{ix} + 10e^{-ix} + 5e^{-3ix} + e^{-5ix}}{2^5}$$

לפי הבינום
של ניוטון

$$\frac{1}{32}e^{5ix} + \frac{5}{32}e^{3ix} + \frac{10}{32}e^{ix} + \frac{10}{32}e^{-ix} + \frac{5}{32}e^{-3x} + \frac{1}{32}e^{-5ix}$$

$$C_1 = C_{-1} = \frac{10}{32}$$

$$C_3 = C_{-3} = \frac{5}{32}$$

$$C_5 = C_{-5} = \frac{1}{32}$$

זהו טור פורייה אקספוננציאלי
מרוכב.

אלו הם מקדמים:

נרשום בצורה טריגונומטרית:

$$\cos(5x) + i\cancel{\sin(5x)} + \cos(5x) - i\cancel{\sin(5x)} \qquad e^{5ix} + e^{-5ix}$$

$$5(\cos(3x) + i\cancel{\sin(3x)} + \cos(3x) - i\cancel{\sin(3x)}) \qquad 5(e^{3ix} + e^{-3ix})$$

$$10(\cos(x) + i\cancel{\sin(x)} + \cos(x) - i\cancel{\sin(x)}) \qquad 10(e^{ix} + e^{-ix})$$

$$f(x) = \frac{2\cos(5x) + 10\cdot\cos(3x) + 20\cos(x)}{32}$$

$$\boxed{f(x) = \frac{1}{16}\cos(5x) + \frac{5}{16}\cos(3x) + \frac{10}{16}\cos(x)}$$

2) $f(x) = \begin{cases} x-\pi & 0 \le x \le \pi \\ x+\pi & -\pi < x < 0 \end{cases}$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi}(x-\pi)dx + \int_{-\pi}^{0}(x+\pi)dx\right] = \frac{1}{\pi}\left[\left(\frac{x^2}{2}-\pi x\right)\Big|_0^{\pi} + \left(\frac{x^2}{2}+\pi x\right)\Big|_{-\pi}^{0}\right] =$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \pi^2 - \frac{\pi^2}{2} + \pi^2\right] = \underline{0}$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi}(x-\pi)\cos(nx) + \int_{-\pi}^{0}(x+\pi)\cos(nx)\right]$$

$$= \frac{1}{\pi}\left[\underbrace{\int_0^{\pi} x\cos(nx)}_{1} - \pi\underbrace{\int_0^{\pi}\cos(nx)}_{2} + \underbrace{\int_{-\pi}^{0} x\cos(nx)}_{3} + \pi\underbrace{\int_{-\pi}^{0}\cos(nx)}_{4}\right]$$

1 $\displaystyle\int_0^{\pi} x\cos(nx) = \frac{x\cdot\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n} = \frac{\cos(\pi n)-\cos(0)}{n^2}$

$$= \boxed{\frac{(-1)^n-1}{n^2}}$$

2 $\displaystyle\int_0^{\pi}\cos(nx) = \frac{\sin nx}{n}\Big|_0^{\pi} = \boxed{0}$

3 Like 1 but $\displaystyle\int_{-\pi}^{0} = \frac{-\cos(0)+\cos(-\pi n)}{n^2} = \boxed{\frac{1-(-1)^n}{n^2}}$

4 $\displaystyle\int_{-\pi}^{0}\cos(nx) = \boxed{0}$

$$a_n = \frac{(-1)^n-1 \quad 1-(-1)^n}{n^2\pi} = \underline{0}$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi}(x-\pi)\sin(nx) + \int_{-\pi}^{0}(x+\pi)\sin(nx)\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin(nx) - \pi\int_0^{\pi}\sin(nx) + \int_{-\pi}^{0} x\cdot\sin(nx) + \pi\int_{-\pi}^{0}\sin(nx)\right]$$
$$\quad ① \qquad ② \qquad ③ \qquad ④$$

① $$\int_0^{\pi} x\cdot\sin(nx) = \frac{x(-\cos nx)}{n}\Big|_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n} = \boxed{\frac{-(-1)^n\pi}{n}}$$

② $$\int_0^{\pi}\sin(nx) = \frac{\cos nx}{n}\Big|_0^{\pi} = \boxed{\frac{1-(-1)^n}{n}}$$

③ $$\int_{-\pi}^{0} x\cdot\sin(nx) = \frac{x(-\cos nx)}{n}\Big|_{-\pi}^{0} + \int_{-\pi}^{0}\frac{\cos nx}{n} = \boxed{\frac{-\pi(-1)^n}{n}}$$

④ $$\int_{-\pi}^{0}\sin(nx) = \frac{\cos nx}{n}\Big|_{-\pi}^{0} = \boxed{\frac{(-1)^n-1}{n}}$$

$$b_n = \frac{1}{\pi}\left[\cancel{-\frac{\pi(-1)^n}{n}} + \cancel{\frac{\pi(-1)^n}{n}} - \frac{\pi}{n} - \cancel{\frac{\pi(-1)^n}{n}} + \cancel{\frac{\pi(-1)^n}{n}} - \frac{\pi}{n}\right]$$

$$= \frac{1}{\cancel{\pi}}\cdot\frac{-2\cancel{\pi}}{n} = \underline{\frac{-2}{n}}$$

$$\boxed{\text{טריגונומטרי} \quad f(x) = \sum_{1}^{\infty}\frac{-2}{n}\sin(nx)}$$

$$C_n = \frac{a_n}{2} - i\frac{b_n}{2} = 0 - i\left(-\frac{2}{2n}\right) = \frac{i}{n}$$

$$C_{-n} = \frac{a_n}{2} + i\frac{b_n}{2} = 0 + i\left(\frac{-2}{-2n}\right) = \frac{i}{n}$$

$$\boxed{\text{מרוכב} \quad f(x) = \sum_{-\infty}^{\infty}\frac{i}{n}\cdot e^{inx}}$$

2

$$\int_{-\pi}^{\pi} f(x)\cdot\sin^3(nx)\,dx \qquad \underline{1c}$$

$$\sin^3(nx) = \left(\frac{e^{inx} - e^{-inx}}{2i}\right)^3 = \frac{e^{3inx} - 3e^{inx} + 3e^{-inx} - e^{-3inx}}{-8i}$$

$$= \frac{-1}{8i}\left(e^{3inx} - e^{-3inx}\right) - \frac{3}{8i}\left(-e^{inx} + e^{-inx}\right)$$

$$= \frac{-1}{8i}\left(\cancel{\cos(3nx)} + i\sin(3nx) - \cancel{\cos(3nx)} + i\sin(3nx)\right) -$$

$$\frac{3}{8i}\left(-\cancel{\cos(nx)} - i\sin(nx) + \cancel{\cos(nx)} - i\sin(nx)\right)$$

$$= \frac{-1}{8i}\left(2i\sin(3nx)\right) - \frac{3}{8i}\left(-2i\sin(nx)\right)$$

$$= \frac{-1}{4}\sin(3nx) + \frac{3}{4}\sin(nx) = \frac{-\sin(3nx) + 3\sin(nx)}{4}$$

$$\frac{1}{4}\int_{-\pi}^{\pi} f(x)\left(-\sin(3nx) + 3\sin(nx)\right)dx =$$

$$\frac{-1}{4}\int_{-\pi}^{\pi} f(x)\sin(3nx)\,dx + \frac{3}{4}\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx$$

$$= \frac{-1}{4}\cdot b_{3n} + \frac{3}{4}b_n = \boxed{\frac{\pi}{4}\left(-b_{3n} + 3b_n\right)}$$

(2)

$$\int_{-\pi}^{\pi} f(x)\cdot\cos^2(nx)\,dx =$$

$$\int_{-\pi}^{\pi} f(x)\left(\frac{1+\cos(2nx)}{2}\right)dx =$$

$$\frac{1}{2}\int_{-\pi}^{\pi} f(x) + \frac{1}{2}\int_{-\pi}^{\pi} f(x)\cdot\cos(2nx)\,dx =$$

$$\boxed{\frac{\pi}{2}\left(a_0 + a_{2n}\right)}$$

③

$$C_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{inx}\,dx$$

$$\overline{C_n} = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{-inx}\,dx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} -f(x)\cdot e^{inx}$$

$$= \frac{-1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{inx} \quad /\cdot(-1)$$

$$\boxed{-\overline{C_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot e^{inx}\,dx = C_{-n}}$$

II)

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot\left(\cos(nx) + i\sin(nx)\right)dx$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx + i\int_{-\pi}^{\pi} f(x)\cdot\sin(nx)\,dx\right)$$

זוגית x אי זוגית

0

אי זוגית x אי זוגית

ניתן לראות כי אם $f(x)$ אי זוגית

אז החלק הממשי מתאפס!

לכן אם $f(x)$ אי זוגית וממשית אז C_n מדומה.

④ $f(x) = x^2$ $\qquad [0,2]$ $\qquad$ k

$$a_0 = 1\int_0^2 x^2 = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3} - 0 = \boxed{\frac{8}{3}}$$

$$a_n = \frac{2}{2-0}\int_0^2 x^2\cdot\cos(\pi n x)\,dx =$$

$$= \left(\frac{x^2\cdot\sin(\pi n x)}{\pi n} - \int_0^2 \frac{2x\cdot\sin(\pi n x)}{\pi n}\,dx\right)$$

$$= \left(\frac{x^2\cdot\sin(\pi n x)}{\pi n}\Big|_0^2 - 2\left(\frac{x\cdot(-\cos(\pi n x))}{\pi^2 n^2}\Big|_0^2 + \int_0^2 \frac{\overset{0}{\cos(\pi n x)}\,dx}{\pi^2 n^2}\right)\right)$$

$$= \left(\frac{4\overset{0}{\sin(4\pi n)}}{\pi n} - 0\right) - 2\left(\frac{-2\cdot\overset{1}{\cos(2\pi n)}}{\pi^2 n^2} - 0 + \frac{\overset{1}{\cos(2\pi n)} - 1}{\pi^2 n^2}\right)$$

$$= -2\left(\frac{-2}{\pi^2 n^2}\right) = \boxed{\frac{4}{\pi^2 n^2}}$$

$$b_n = \int_0^2 x^2\cdot\sin(\pi n x) = \frac{-x^2\cdot\overset{1}{\cos(\pi n x)}}{\pi n}\Big|_0^2 - 2\int_0^2 x\cdot\frac{\cos\pi n x}{\pi n}\,dx$$

$$= \frac{-4}{\pi n} - 2\left(\frac{x\cdot(-\overset{0}{\sin(\pi n x)})}{\pi^2 n^2}\Big|_0^2 + \int_0^2 \frac{\sin\pi n x}{\pi^2 n^2}\,dx\right)$$

$$= \frac{-4}{\pi n} - 2\int_0^2 \frac{\sin\pi n x}{\pi^2 n^2}\,dx = \frac{-4}{\pi n} - 2\left(\frac{\overset{0}{\cos(\pi n x)}}{\pi^3 n^3}\right)\Big|_0^2 = \boxed{\frac{-4}{\pi n}}$$

$$\boxed{f(x) \cong \frac{4}{3} + \frac{4}{\pi}\sum_1^\infty \frac{\cos(\pi n x)}{\pi n^2} - \frac{\sin(\pi n x)}{n}}$$

④

$$f(x)=\begin{cases} e^{x}, & \frac{1}{2}<x\leq 1 \\ 0, & x=\frac{1}{2} \\ -e^{-x}, & 0\leq x<\frac{1}{2} \end{cases}$$

$$f(x)=\underbrace{\int_{\frac{1}{2}}^{1} e^{x}\cdot e^{-2\pi i n x}\,dx}_{1}-\underbrace{\int_{0}^{\frac{1}{2}} e^{-x}\cdot e^{-2\pi i n x}\,dx}_{2}$$

1 $$\int_{\frac{1}{2}}^{1} e^{x}\cdot e^{-2\pi i n x}\,dx=\int_{\frac{1}{2}}^{1} e^{x(1-2\pi i n x)}\,dx=\left.\frac{e^{x(1-2\pi i n x)}}{1-2\pi i n}\right|_{\frac{1}{2}}^{1}$$

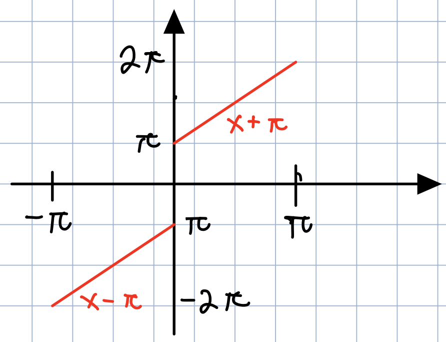
$$=\frac{e^{1-2\pi i n}-e^{\frac{1}{2}(1-2\pi i n)}}{1-2\pi i n}=\boxed{\frac{e^{1}\cdot 1-e^{\frac{1}{2}}\cdot(-1)^{n}}{1-2\pi i n}}$$

2 $$-\int_{0}^{\frac{1}{2}} e^{-x}\cdot e^{-2\pi i n x}\,dx=-\int_{0}^{\frac{1}{2}} e^{-x(2i\pi n+1)}\,dx=\left.\frac{e^{-x(2i\pi n+1)}}{2i\pi n+1}\right|_{0}^{\frac{1}{2}}$$

$$=\frac{e^{-\frac{1}{2}(2i\pi n+1)}-1}{2i\pi n+1}=\frac{e^{-\frac{1}{2}}\cdot e^{-i\pi n}-1}{2i\pi n+1}=\boxed{\frac{e^{-\frac{1}{2}}\cdot(-1)^{n}-1}{2i\pi n+1}}$$

$$\boxed{f(x)\simeq\sum_{-\infty}^{\infty}\left(\frac{e-e^{\frac{1}{2}}\cdot(-1)^{n}}{1-2\pi i n}+\frac{e^{-\frac{1}{2}}\cdot(-1)^{n}-1}{1+2i\pi n}\right)\cdot e^{2i\pi n x}}$$

④



$\underline{d}$

$$f(x)=\begin{cases} x+\pi, & 0\le x\le \pi \\ x-\pi, & -\pi\le x<0 \end{cases}$$

$$f(x)'=\begin{cases} 1, & 0\le x\le \pi \\ -1, & -\pi\le x<0 \end{cases}$$

$$\tilde{f}(x)=\frac{f(x^-)+f(x^+)}{2}$$

תנאי דריכלה מתקיימים כאן.

$f\in E_1$

$$f(-\pi)=\frac{f(-\pi^-)+f(-\pi^+)}{2}=\frac{2\pi-2\pi}{2}=0$$

$$f(\pi)=\frac{f(\pi^-)+f(\pi^+)}{2}=\frac{2\pi-2\pi}{2}=0$$

$$f(\pi/2)=\frac{f(\pi/2^-)+f(\pi/2^+)}{2}=\frac{\frac{3}{2}\pi+\frac{3}{2}\pi}{2}=\frac{3}{2}\pi$$

$$f(0)=\frac{f(0^-)+f(0^+)}{2}=\frac{-\pi+\pi}{2}=0$$

$$f(-1)=\frac{f(-1^-)+f(-1^+)}{2}=\frac{(-1-\pi)+(-1-\pi)}{2}=-1-\pi$$

(5) $\int_0^2 x \cdot e^{-x}\,dx \qquad n=100$

מלבנים:

$$h=\frac{2}{100}=\frac{1}{50}$$

$$\int_0^2 x\cdot e^{-x}\,dx=\frac{1}{50}\sum_{1}^{100} f\left(\frac{n}{50}\right)=\frac{1}{50}\sum_{1}^{100}\frac{n}{50}\cdot e^{\frac{-n}{50}}$$

$$=0.5966630121$$

טרפזים:

$$h=\frac{b-a}{n}=\frac{1}{50} \qquad x_i=\frac{i}{50} \qquad i=1,2,\dots,100$$

$$S=\int_a^b f(x)\,dx=\int_{x_0}^{x_1} f(x)\,dx+\dots+\int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$=\frac{h}{2}\Big(f(x_0)+f(x_1)+f(x_1)+\dots+f(x_{n-1})+f(x_{n-1})+f(x_n)\Big)$$

$$=\frac{h}{2}\Big(f_0+2\left(f_1+f_2+\dots+f_{n-1}\right)+f_n\Big)$$

$$=\frac{1}{100}\left(0+\left(2\sum_{1}^{99}\frac{n}{50}\cdot e^{\frac{-n}{50}}\right)+2\cdot e^{-2}\right)$$

$$=0.5939563064$$

סימפסון:

$$h=\frac{1}{50} \qquad f_i=\frac{i}{50}\cdot e^{\frac{-i}{50}}$$

$$f \cong \int_a^b f(x)\,dx = \int_a^{a+2h} f(x)dx + \int_{a+2h}^{a+4h} f(x)dx + \ldots + \int_{a+(n-2)h}^{b} f(x)\,dx$$

$$\cong \frac{h}{3}\left(f_0+4f_1+f_2\right)+\frac{h}{3}\left(f_2+4f_3+f_4\right)+\ldots+\frac{h}{3}\left(f_{2k-2}+4f_{2k-1}+f_{2k}\right)$$

$$=\frac{h}{3}\left(f_0+4f_1+2f_2+4f_3+\ldots+2f_{2k-2}+4f_{2k-1}+f_{2k}\right)$$

$$f(x)\cong\frac{1}{150}\left(0+4\sum_{1}^{50}\frac{2n-1}{50}\cdot e^{\frac{-(2n-1)}{50}}+2\sum_{1}^{50}\frac{2n}{50}\cdot e^{\frac{-2n}{50}}\right)$$

$$=\underline{0.5939941477}$$

ישיר:

$$\int_0^2 x e^{-x} = -x\cdot e^{-x}\Big|_0^2+\int_0^2 e^{-x}\,dx = -x\cdot e^{-x}-e^{-x}\Big|_0^2$$

$$=-2e^{-2}-e^{-2}+1=\underline{0.5939941503}$$

	סימפסון	טרפז	מלבנים
שיערוך	$\frac{1}{90}\cdot\frac{1}{50^4}\cdot\frac{2}{e^2}\cdot\frac{4}{e^2}$	$\frac{1}{3}\cdot\frac{1}{50^2}=\frac{1}{7500}$	0.02
שגיאה	0.0000000026	0.00023388359	0.0026688617