Autonomous Navigation and Perception (086762)
Homework #3

Submission guidelines:

- Submission in pairs by: 19 April 2021, 14:30.
- Email submission as a *single zip or pdf file* to `anpsubmission@gmail.com`.
- Make sure to name the submitted file ID1-ID2.zip (or .pdf) with ID1,ID2 - the students' ids.
- For hands-on parts, your source code should be submitted as well (include it in the above zip file).

Theoretical questions:

1. Consider two distributions $\mathbb{P}(X, Y)$ and $\mathbb{Q}(X, Y)$ over random variables X and Y. Prove that the KL divergence between these two distributions can be factorized as

$$\texttt{KL}[\mathbb{P}(X,Y)||\mathbb{Q}(X,Y)] = \texttt{KL}[\mathbb{P}(Y)||\mathbb{Q}(Y)] + \texttt{KL}[\mathbb{P}(X \mid Y)||\mathbb{Q}(X \mid Y)]$$

2. Suppose you have *discrete* state, action, and observation spaces, $\mathcal{X}$, $\mathcal{A}$ and $\mathcal{Z}$, respectively. Consider the transition and observation models $\mathbb{P}_T(X' \mid X, a)$ and $\mathbb{P}_Z(z \mid X)$ are given for any $X', X' \in \mathcal{X}$, $a \in \mathcal{A}$ and $z \in \mathcal{Z}$. Denote number of elements in some set $\mathcal{S}$ by $|\mathcal{S}|$ (also known as cardinality). For example, the state and observation at any time i can only assume a value from $\mathcal{X} = \{x_1, \ldots x_n\}$ and $\mathcal{Z} = \{z_1, \ldots, z_m\}$, respectively, with $|\mathcal{X}| = n$ and $|\mathcal{Z}| = m$ (similarly for action space $\mathcal{A}$).

   In this question we consider a single look-ahead step (myopic) planning, and we are interested in understanding the computational complexity of calculating the objective function for a single candidate action. Suppose the current belief is $b_k \doteq b[X_k]$, consider some candidate action $a_k \in \mathcal{A}$ and assume a general reward function $r(b, a)$.

   (a) Write the objective function $J(b_k, a_k)$ for a single look-ahead step case. Write the expectation operator explicitly. Describe the steps to calculate $J(b_k, a_k)$ for the given problem setting.

   (b) Derive an expression for the probability of acquiring a future observation $z_{k+1} \in \mathcal{Z}$ while considering candidate action a_k and current belief b_k. Express your answer only in terms of quantities given in the question.

   (c) What is the computational complexity of performing the calculations in clause 2b? Denote it by $\mathcal{O}_z$, and express your answer in terms of cardinality of the action, observation and state spaces.

   (d) Write explicit equations for calculating the posterior future belief[1] $b_{k+1} \doteq b[X_{k+1}]$, given current belief is $b_k = b[X_k]$ while considering action $a_k \in \mathcal{A}$ and future observation $z_{k+1} \in \mathcal{Z}$.

   (e) What is the computational complexity of performing the calculations in clause 2d? Denote it by $\mathcal{O}_b$ and express your answer in terms of cardinality of the action, observation and state spaces.

   (f) Consider now the reward function is entropy, i.e. $r(b[X], a) \doteq \mathcal{H}(b[X])$. Write down the corresponding expression for a given belief $b[X]$ and indicate the computational complexity to calculate it. Denote it by $\mathcal{O}_r$ and express your answer in terms of cardinality of the action, observation and state spaces.

   (g) What is the computational complexity of calculating exactly the objective function $J(b_k, a_k)$ (from clause 2a)? Express your answer in terms $\mathcal{O}_r$, $\mathcal{O}_b$, $\mathcal{O}_z$ and cardinality of the action, observation and state spaces.

[1] Assume recursive formulation.