Autonomous Navigation and Perception (086762)
Homework #4

Submission guidelines:

- Submission in pairs by: 27 April 2022.
- Email submission as a *single zip or pdf file* to `anpsubmission@gmail.com`.
- Make sure to name the submitted file ID1-ID2.zip (or .pdf) with ID1,ID2 - the students' ids.
- For hands-on parts, your source code should be submitted as well (include it in the above zip file).

Theoretical questions:

1. (Graph search) Consider the graph $G = (V, E)$ shown in Figure 1. Find the shortest path between the denoted start and goal nodes using the following algorithms. In each case, provide a detailed explanation of the algorithm run, and state the order in which states are expanded.
   (a) Breadth-first search (BFS) algorithm, considering unweighted edges, as in Figure 1(left).
   (b) Depth-first search (DFS) algorithm, considering unweighted edges, as in Figure 1(left).
   (c) Best-first search (Dijkstra) algorithm, considering weighted edges, as in Figure 1(right).
   (d) A* algorithm, considering weighted edges and the heuristic function h, as in Figure 1(right).
   (e) Is A* more efficient than Dijkstra in the considered setting? Explain your answer.

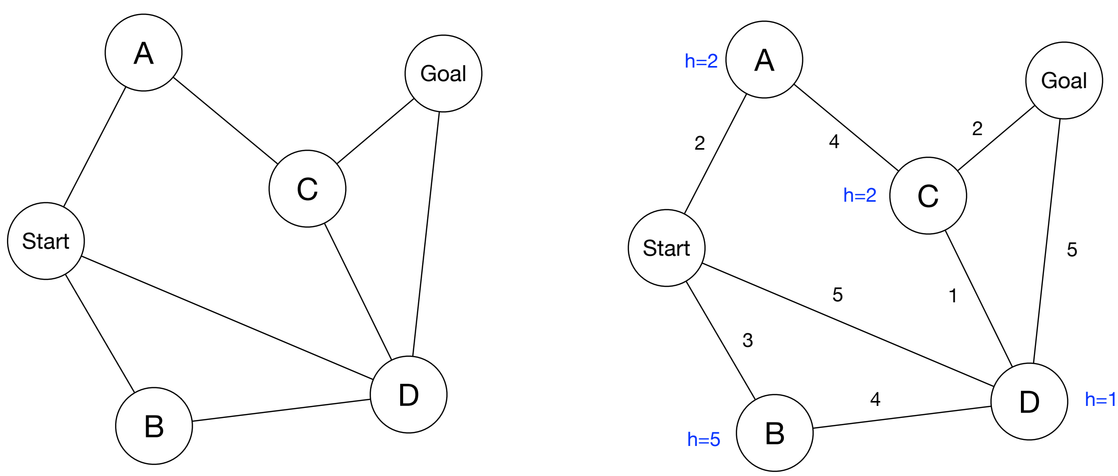


Figure 1

2. (A^*) In this question we focus on the properties of heuristic functions. Reminder:
   - A heuristic function is **admissible** if $h(x_i) \leq \mathbf{dist}(x_i, x_\tau)$ for every node x_i with coordinates x_i in graph G, where $\mathbf{dist}(x_i, x_\tau)$ is the shortest distance from x_i to x_τ.
   - A heuristic function is **consistent** if
     - $h(x_\tau) = 0$ for the goal node τ with coordinates x_τ.

– $h(x_i) \leq c(x_i, x_j) + h(x_j)$ for every node i with coordinates x_i and its children j with coordinates x_j.

- A heuristic function is $\epsilon-$**consistent**, with $\epsilon \geq 1$, if
  – $h(x_\tau) = 0$ for the goal node τ with coordinates x_τ.
  – $h(x_i) \leq \epsilon c(x_i, x_j) + h(x_j)$ for every node i with coordinates x_i and its children j with coordinates x_j.

(a) Prove that if $h^{(1)}$ and $h^{(2)}$ are consistent heuristics, then $h \doteq \max(h^{(1)}, h^{(2)})$ is also consistent.

(b) Prove that if $h^{(1)}$ and $h^{(2)}$ are consistent heuristics, then $h \doteq h^{(1)} + h^{(2)}$ is $\epsilon-$consistent.

(c) Provide an example of an admissible heuristic h that is not consistent. Show explicitly why h does not satisfy the consistency property.

Hands-on tasks:

1. (Probabilistic Roadmap) Consider a 2D environment of size $N \times N$ and a known set C_{obs} of scattered obstacles. For simplicity, consider all obstacles are represented by rectangular polygons of size $n_x^{obs} \times n_y^{obs}$ aligned with x and y axes (i.e. not rotated).

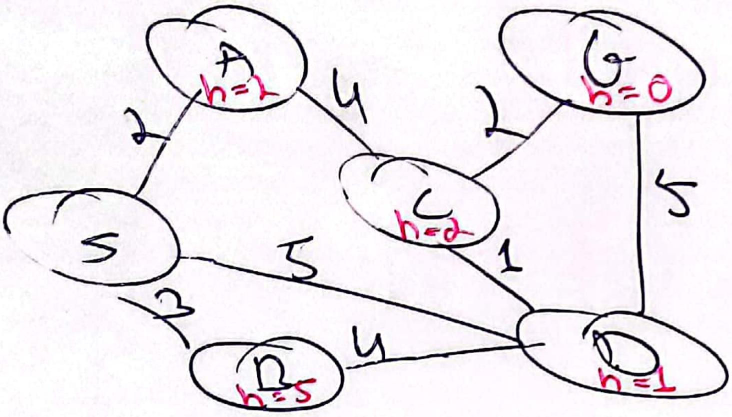
   (a) Implement *on your own* a function `GeneratePRM` which creates a simplified 2D probabilistic roadmap (PRM) $G = (V, E)$. In this simplified version, consider any two (sampled) points v_1 and v_2 in the 2D space can be connected by a straight line if the Euclidean distance $\mathbf{dist}(v_1, v_2)$ between them is less than a distance threshold th_d, and the path from v_1 to v_2 does not intersect with an obstacle (i.e. in C_{free}). Assign the weight of the corresponding edge (v_1, v_2) to be $\mathbf{dist}(v_1, v_2)$. Assume sampling of new candidate positions is done from a uniform distribution

   - Input: distance threshold th_d, number of nodes N_{nodes}, set of obstacles C_{obs}.
   - Output: a PRM $G = (V, E)$.

   (b) Let $N = 100$. Randomly scatter $n_{obs} = 15$ identical obstacles in the 2D environment, with $n_x^{obs} = 15$ and $n_y^{obs} = 10$. Given this environment (in terms of obstacles), draw on separate plots the obstacles and the corresponding probabilistic maps considering all permutations $N_{nodes} \in \{100, 500\}$ and $th_d \in \{20, 50\}$. For each configuration, indicate in the plot the number of edges and an average node degree in the constructed PRM G.

2. (Graph search) Consider the generated environment and the constructed PRM with $N_{nodes} = 100$ and $th_d = 50$ from clause 1b. Consider start and goal positions to be, respectively, at the bottom left and top right corners of the 2D $N \times N$ considered environment. For simplicity, denote by x_{start} and x_{goal} the closest nodes in G to these positions. Implement *on your own* the Dijkstra algorithm, or/and an A* with Euclidean distance heuristic, and use it to find the shortest path from x_{start} to x_{goal}. Plot the corresponding shortest path on top of the PRM, while indicating the obstacles.

1.

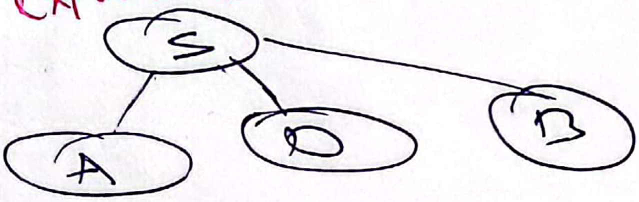


(a) BFS:

I. init

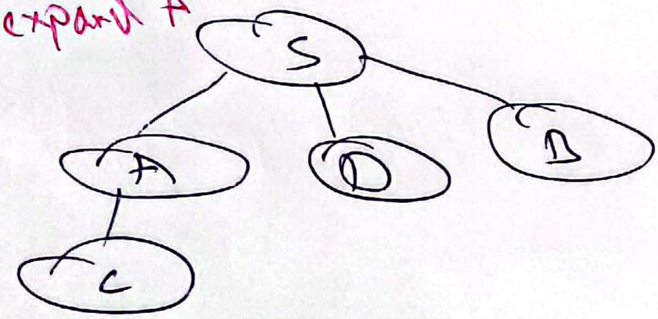
open = {S}

II. expand S



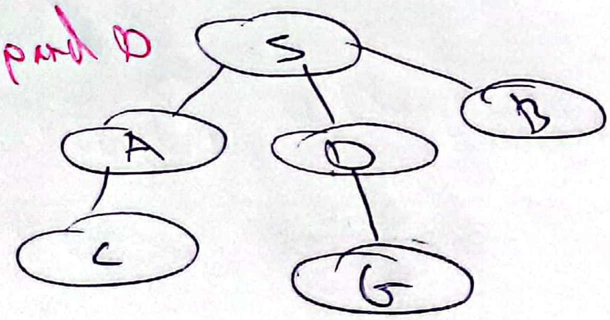
open = {A, D, B}

III. expand A



open = {C, D, B}

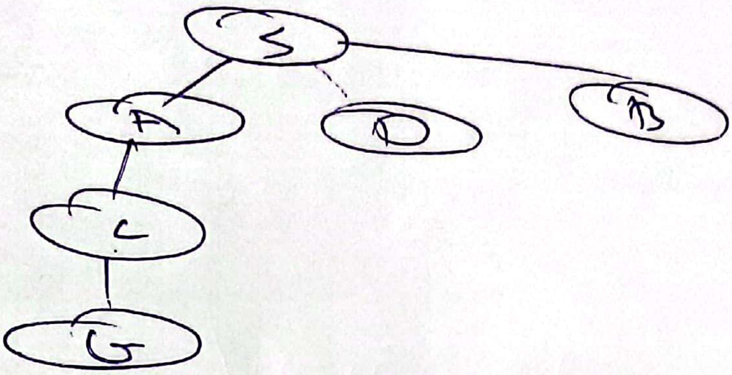
IV. expand D



open = {C, G, B}

IV expand C

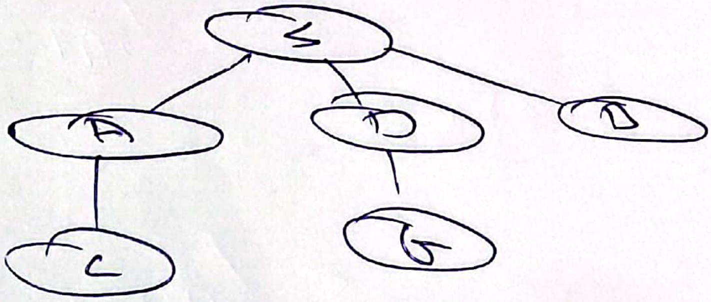
open = {D, B, G}



V expand G ⇒ open = {D, B}

VI expand D

open = {B, G}



VII expand G → open = {B}

VIII expand D ⇒ open = {}

shortest path: S → D → G

$d_G = 2$

V. expand B ⇒ open = {C, G}

VI. expand C ⇒ open = {G}

VII. expand G ⇒ open = {}

shortest path: S → D → G

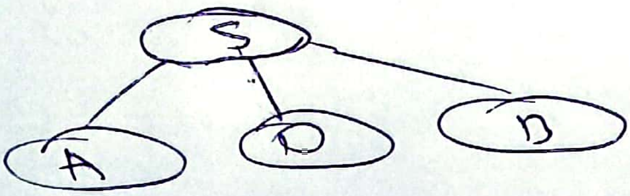
$d_G = 2$

(b) DFS:

I. init

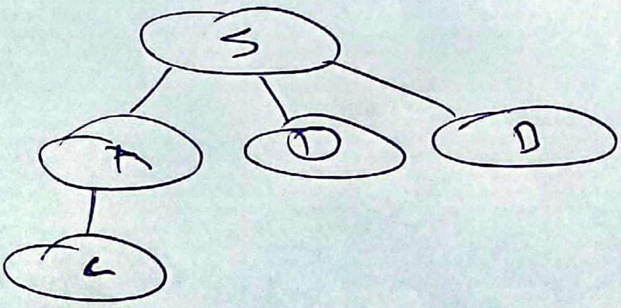
open = {S}

II. expand S



open = {A, D, B}

III expand A



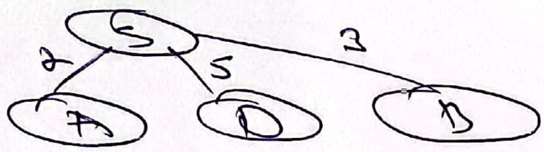
open = {C, D, B}

(c) Dijakstra:

I. init

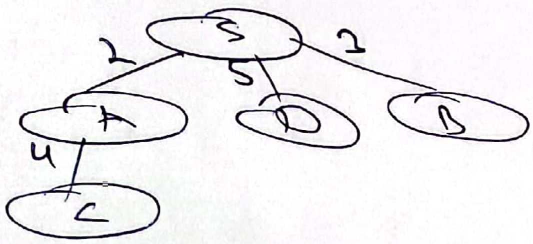
open = {S}

II. expand S



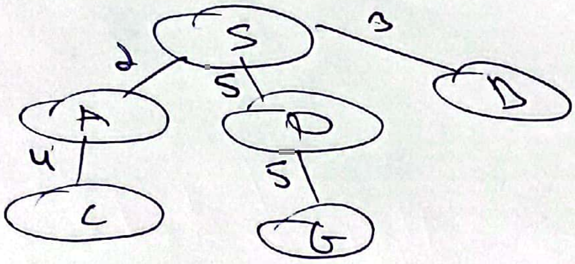
open = {A, D, B}

III. expand A



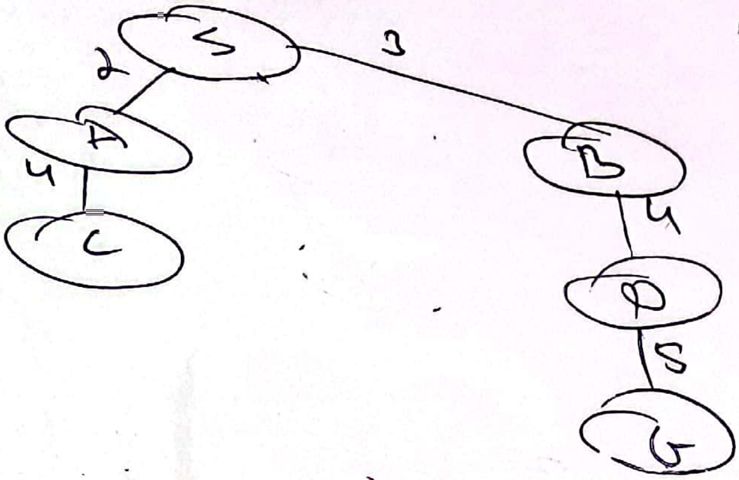
open = {C, D, B}

IV expand D



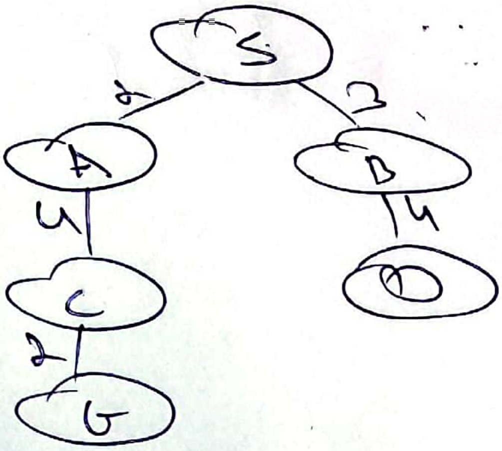
open = {C, G, B}

IV expand B



open = {C, G, D}

V expand C



open = {G, D}

VI expand D $\Rightarrow$ open = {G}

VI expand G $\Rightarrow$ open = {}

shortest path

S → A → C → G

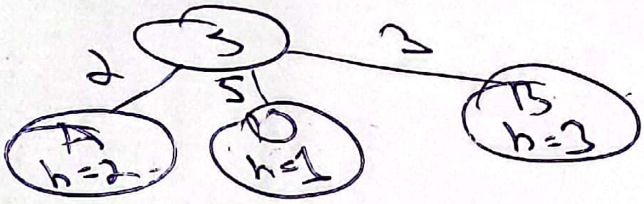
$g_G = 5$

(d) A^* :

I. init

open = {S}

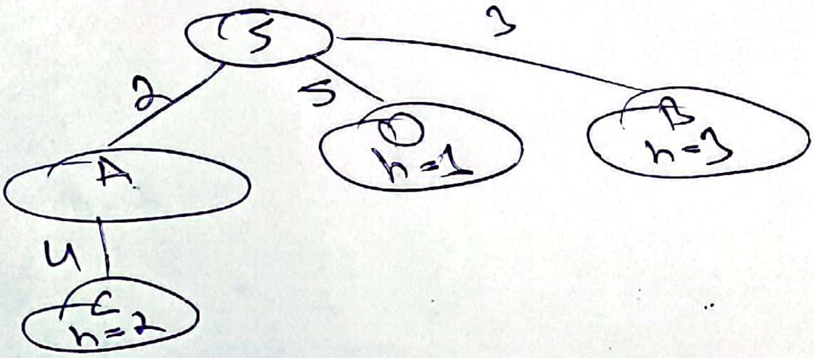
II. expand S



open = {A, D, B}

closed = {S}

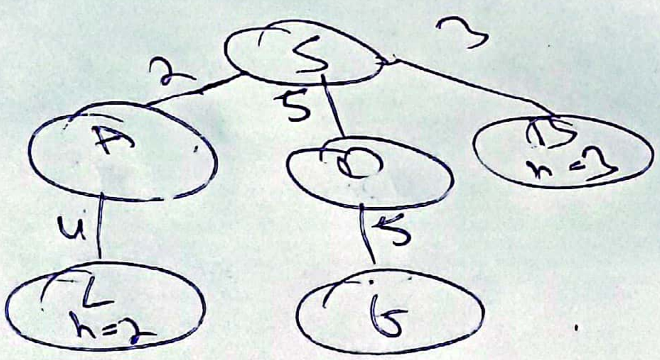
III. expand A



open = {D, B, C}

closed = {S, A}

IV. expand D



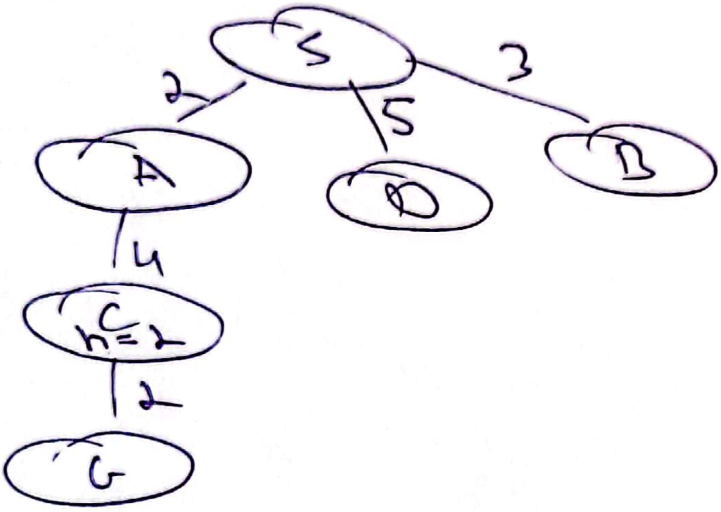
open = {B, C}

closed = {S, A, D}

V expand B ⇒ open = {C}

closed = {S, A, D, B}

VI expand C



open = {G}

closed = {S, A, D, B, C}

VII expand G ⇒ terminate

shortest path: S → A → C → G

$g_G = 8$

c. A* was actually less efficient as it required 1 extra step which is due to a bad heuristic:

$h(C) = 2 \quad w(G, C) = 2$

$h(D) = 1 \quad w(G, D) = 5$

2.

(a) $h^{(1)}, h^{(2)}$ are consistent

show that $h = \max(h^{(1)}, h^{(2)})$ is also consistent

- $h(T) = \max(\underbrace{h^{(1)}(T)}_{0}, \underbrace{h^{(2)}(T)}_{0}) = 0$

- $c(x_i, x_j) + h(x_j) =$

$= c(x_i, x_j) + \max(h^{(1)}(x_j), h^{(2)}(x_j)) =$

$\left\{ \text{assume } h^{(1)}(x_j) \geq h^{(2)}(x_j) \text{ without loss of generality} \right\}$

$= c(x_i, x_j) + h^{(1)}(x_j) \geq c(x_i, x_j) + h^{(2)}(x_j) \geq$

$\geq h^{(2)}(x_i)$

↑

consistenty of $h^{(2)}$

also: $c(x_i, x_j) + h^{(1)}(x_j) \geq h^{(1)}(x_i)$

↑

consistenty of $h^{(1)}$

hence:

$$c(x_i, x_j) + h(x_j) \geq h^{(1)}(x_i)$$

$$c(x_i, x_j) + h(x_j) \geq h^{(2)}(x_j)$$

$$\Rightarrow c(x_i, x_j) + h(x_j) \geq \max\left(h^{(1)}(x_i), h^{(2)}(x_i)\right) =$$

$$= h(x_i)$$

(b) $h^{(1)}, h^{(2)}$ are consistent.

Show that $h = h^{(1)} + h^{(2)}$ is ε-consistent

- $h(T) = h^{(1)}(T) + h^{(2)}(T) = 0$

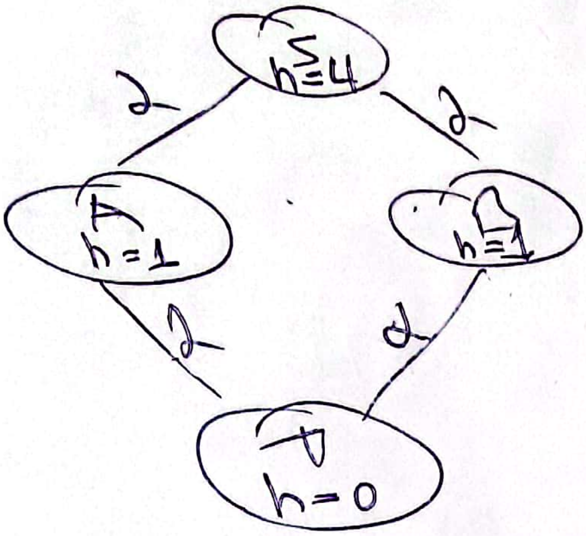
- $$(+)\begin{cases} h^{(1)}(x_j) + c(x_i, x_j) \geq h^{(1)}(x_i) \\ h^{(2)}(x_j) + c(x_i, x_j) \geq h^{(2)}(x_i) \end{cases}$$

$$h^{(1)}(x_j) + h^{(2)}(x_j) + 2c(x_i, x_j) \geq h^{(1)}(x_i) + h^{(2)}(x_i)$$

$$h(x_j) + 2c(x_i, x_j) \geq h(x_i)$$

$\Rightarrow \varepsilon = 2$ consistent

(c) example 2



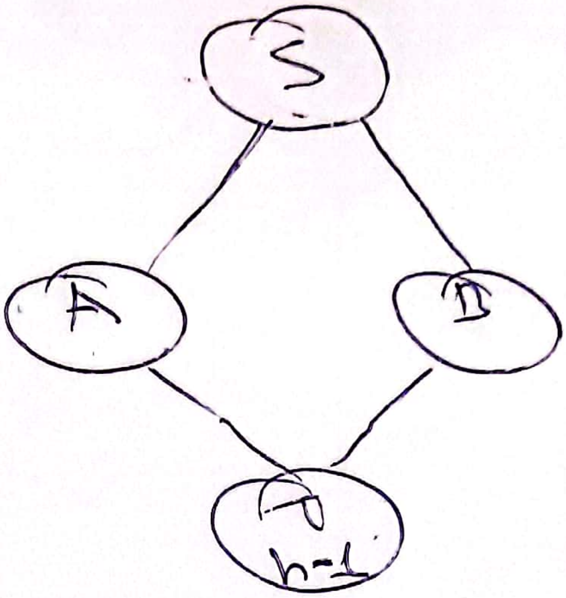
$h(S) ? h(B) + w(S,B)$

$4 > 3$

$\Downarrow$

$h(S) \not\leq h(B) + w(S,B)$

example 1



$h(T) = 1$