Autonomous Navigation and Perception (086762)
Homework #3

Submission guidelines:

- Submission in pairs by: 19 April 2021, 14:30.
- Email submission as a *single zip or pdf file* to `anpsubmission@gmail.com`.
- Make sure to name the submitted file ID1-ID2.zip (or .pdf) with ID1,ID2 - the students' ids.
- For hands-on parts, your source code should be submitted as well (include it in the above zip file).

Theoretical questions:

1. Consider two distributions $\mathbb{P}(X, Y)$ and $\mathbb{Q}(X, Y)$ over random variables X and Y. Prove that the KL divergence between these two distributions can be factorized as

$$\text{KL}[\mathbb{P}(X,Y)||\mathbb{Q}(X,Y)] = \text{KL}[\mathbb{P}(Y)||\mathbb{Q}(Y)] + \text{KL}[\mathbb{P}(X \mid Y)||\mathbb{Q}(X \mid Y)]$$

2. Suppose you have *discrete* state, action, and observation spaces, $\mathcal{X}$, $\mathcal{A}$ and $\mathcal{Z}$, respectively. Consider the transition and observation models $\mathbb{P}_T(X' \mid X, a)$ and $\mathbb{P}_Z(z \mid X)$ are given for any $X', X \in \mathcal{X}$, $a \in \mathcal{A}$ and $z \in \mathcal{Z}$. Denote number of elements in some set $\mathcal{S}$ by $|\mathcal{S}|$ (also known as cardinality). For example, the state and observation at any time i can only assume a value from $\mathcal{X} = \{x_1, \ldots x_n\}$ and $\mathcal{Z} = \{z_1, \ldots, z_m\}$, respectively, with $|\mathcal{X}| = n$ and $|\mathcal{Z}| = m$ (similarly for action space $\mathcal{A}$).

   In this question we consider a single look-ahead step (myopic) planning, and we are interested in understanding the computational complexity of calculating the objective function for a single candidate action. Suppose the current belief is $b_k \doteq b[X_k]$, consider some candidate action $a_k \in \mathcal{A}$ and assume a general reward function $r(b, a)$.

   (a) Write the objective function $J(b_k, a_k)$ for a single look-ahead step case. Write the expectation operator explicitly. Describe the steps to calculate $J(b_k, a_k)$ for the given problem setting.

   (b) Derive an expression for the probability of acquiring a future observation $z_{k+1} \in \mathcal{Z}$ while considering candidate action a_k and current belief b_k. Express your answer only in terms of quantities given in the question.

   (c) What is the computational complexity of performing the calculations in clause 2b? Denote it by $\mathcal{O}_z$, and express your answer in terms of cardinality of the action, observation and state spaces.

   (d) Write explicit equations for calculating the posterior future belief[1] $b_{k+1} \doteq b[X_{k+1}]$, given current belief is $b_k = b[X_k]$ while considering action $a_k \in \mathcal{A}$ and future observation $z_{k+1} \in \mathcal{Z}$.

   (e) What is the computational complexity of performing the calculations in clause 2d? Denote it by $\mathcal{O}_b$ and express your answer in terms of cardinality of the action, observation and state spaces.

   (f) Consider now the reward function is entropy, i.e. $r(b[X], a) \doteq \mathcal{H}(b[X])$. Write down the corresponding expression for a given belief $b[X]$ and indicate the computational complexity to calculate it. Denote it by $\mathcal{O}_r$ and express your answer in terms of cardinality of the action, observation and state spaces.

   (g) What is the computational complexity of calculating exactly the objective function $J(b_k, a_k)$ (from clause 2a)? Express your answer in terms $\mathcal{O}_r$, $\mathcal{O}_b$, $\mathcal{O}_z$ and cardinality of the action, observation and state spaces.

[1] Assume recursive formulation.

1

$p(x,y)$; $q(x,y)$

$$KL\left(p(x,y)\,\|\,q(x,y)\right) =$$

$$= \int_y \int_x p(x,y) \log \frac{p(x,y)}{q(x,y)}\, dx\, dy =$$

$$= \int_y \int_x p(x,y) \log \left(\frac{p(x|y)\, p(y)}{q(x|y)\, q(y)}\right) dx\, dy$$

$$= \int_y \int_x p(x,y) \left[\log \left(\frac{p(y)}{q(y)}\right) + \log \left(\frac{p(x|y)}{q(x|y)}\right)\right] dx\, dy =$$

$$= \int_y \int_x p(x,y) \log \left(\frac{p(y)}{q(y)}\right) dx\, dy +$$

$$+ \int_y \int_x p(x,y) \log \left(\frac{p(x|y)}{q(x|y)}\right) dx\, dy =$$

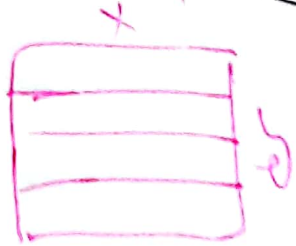
$$= \int_y p(y)\log\frac{p(y)}{q(y)}\,dy + \iint_{y\,x} p(x,y)\log\left(\frac{p(x|y)}{q(x|y)}\right)dx\,dy =$$

$$= KL\left(p(y)\,\|\,q(y)\right) + \iint_{y\,x} p(x|y)\,p(y)\log\left(\frac{p(x|y)}{q(x|y)}\right)dx\,dy =$$

$$= KL\left(p(y)\,\|\,q(y)\right) + \int_y p(y)\int_x p(x|y)\log\left(\frac{p(x|y)}{q(x|y)}\right)dx\,dy =$$

$$\downarrow ?$$

$$= KL\left(p(y)\,\|\,q(y)\right) + KL\left(p(x|y)\,\|\,q(x|y)\right)$$



2. Discrete state, action and observation spaces.

$P_T(x'|x,a)$ – motion model

$P_Z(z|x)$ – observation model

$X = \{x_1 \_\_ x_n\}$ $|X|$ elements

$Z = \{z_1 \_\_ z_m\}$ $|Z|$ elements

$A = \{a_1 \_\_ a_s\}$ $|A|$ elements

myopic planning with $r(b,a)$

(a) $J(b_k, a_k) = \mathbb{E}_{z_{k+1}}\left(r(b_k, a_k) + r_T(b_{k+1})\right) =$

$= r(b_k, a_k) + \mathbb{E}_{z_{k+1}} r_T(b_{k+1}) =$

$$= r(b_k, a_k) + \sum_{z_{k+1}} p(z_{k+1} | b_k, a_k) r_T(b_{k+1} | b_k, a_k, z_{k+1})$$

Steps:

(i) calculate $r(b_k, a_k)$

(ii) calculate $\mathbb{E}_{z_{k+1}} r_T(b_{k+1})$:

- for each $z_i \in Z$ compute $p(z_i | b_k, a_k) = w_i$
- for each z_i, w_i calculate $r_T^i(b_{k+1} | b_k, a_k, z_i)$
- Sum: $\mathbb{E} r_T(b_{k+1}) \approx \sum_i w_i r_T^i$

bayes filter:

General knowledge

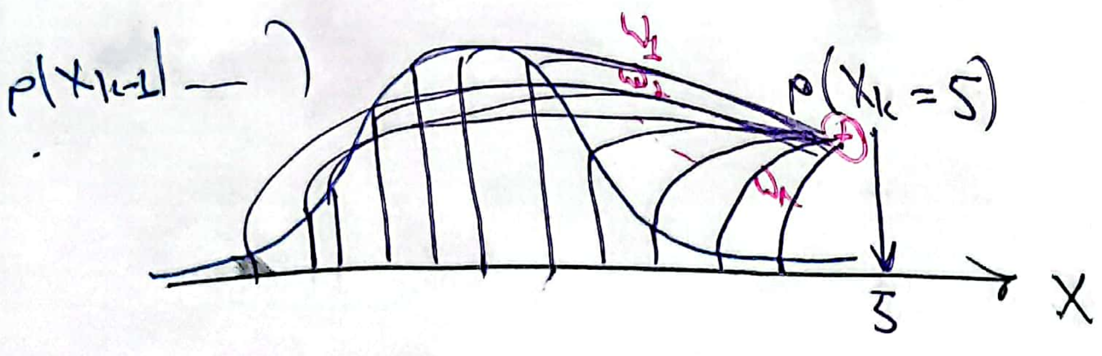
$$p(x_k | z_{1:k}, a_{0:k-1}) =$$

$$= \frac{p(z_k | z_{1:k-1}, a_{0:k-1}, x_k)\, p(x_k | z_{1:k-1}, a_{0:k-1})}{p(z_k | z_{1:k-1}, a_{0:k-1})} =$$

$$= \eta\, p(z_k | x_k)\, p(x_k | z_{1:k-1}, a_{0:k-1}) =$$

$$= \eta\, p(z_k | x_k) \int p(x_k, x_{k-1} | z_{1:k-1}, a_{0:k-1})\, dx_{k-1} =$$

$$= \eta\, p(z_k | x_k) \int p(x_k | x_{k-1}, a_{k-1})\, p(x_{k-1} | z_{1:k-1}, a_{0:k-2})\, dx_{k-1} =$$



(b) $p(z_{k+1} | b_k, a_k) =$

computing for $z_i = z^*$ $\forall z^*$

$$\overset{MR}{=} \int_{b_{k+1}} p(z_{k+1}, b_{k+1} | b_k, a_k) \, db_{k+1} =$$

$$\overset{CR}{=} \int_{b_{k+1}} p(z_{k+1} | b_{k+1}, b_k, a_k) \, p(b_{k+1} | b_k, a_k) \, db_{k+1} =$$

$$\overset{indep.}{=} \int_{b_{k+1}} p(z_{k+1} | b_{k+1}) \, p(b_{k+1} | b_k, a_k) \, db_{k+1}$$

<u>plugging for some value</u>: $p(z_{k+1} = z^* | b_k, a_k)$

$$p(z_{k+1} = z^* | b_k, a_k) = \sum_{x_i} p(z_{k+1} = z^* | b_{k+1} = x_i) \, p(b_{k+1} = x_i | b_k, a_k) =$$

$$= p(z_{k+1} = z^* | b_k, a_k) = \sum_{x_i} \left[p(z_{k+1} = z^* | b_{k+1} = x_i) \sum_{x_j} \left[p(b_{k+1} = x_i | b_k = x_j, a_k) \cdot p(b_k = x_j) \right] \right]$$

why not shown by MR?

(c) The algorithm for (b):

init $p(z_{k+1} = z^* | b_k, a_k) = 0$

for each x_i in X do:

init $p(b_{k+1} = x_i | b_k, a_k) = 0$

for each x_j in X do:

$p(b_{k+1} = x_i | b_k, a_k) \mathrel{+}= p(b_{k+1} = x_i | b_k = x_j, a_k) \cdot p(b_k = x_j)$

$p(z_{k+1} = z^* | b_k, a_k) \mathrel{+}= p(z_{k+1} = z^* | b_{k+1} = x_i) \cdot p(b_{k+1} = x_i | b_k, a_k)$

$\Rightarrow O_z = O(|X|^2)$

(d) $p(b_{k+1} | b_k, a_k, z_{k+1}) =$ in the way to compute σ_t^i

$$\overset{\text{bayes}}{=} \frac{p(z_{k+1} | b_{k+1}, b_k, a_k)\, p(b_{k+1} | b_k, a_k)}{p(z_{k+1} | b_k, a_k)} =$$

$$\overset{\text{indep.}}{=} \frac{p(z_{k+1} | b_{k+1})\, p(b_{k+1} | b_k, a_k)}{p(z_{k+1} | b_k, a_k)} =$$

$$\overset{(d\text{-}b)}{=} \frac{p(z_{k+1} | b_{k+1})\, p(b_{k+1} | b_k, a_k)}{\int_{b_{k+1}} p(z_{k+1} | b_{k+1})\, p(b_{k+1}, b_k, a_k)\, db_{k+1}}$$

for some value $b_{k+1} = b^*$

$p(b_{k+1} = b^* | b_k, a_k, z_{k+1}) =$

$$= \frac{p(z_{k+1} | b_{k+1} = b^*) \sum_{x_j} p(b_{k+1} = b^* | b_k = x_j, a_k)}{\sum_{x_i} \left[p(z_{k+1} | b_{k+1} = x_i) \sum_{x_j} \left[p(b_{k+1} = x_i | b_k = x_j, a_k)\, p(b_k = x_j) \right] \right]}$$

(e) $p(z_{k+1} | b_{k+1} = b^*) \sim O(1)$

$p(b_{k+1} = b^* | b_k, a_k) \sim O(|X|)$

$p(z_{k+1} | b_k, a_k) \sim O(|X|^2)$

$\Rightarrow O_b = O(|X|^2)$

(f) $r(b[x], \lambda) = H(b[x])$

$$H(b) = -\sum_x b[x] \log(b[x])$$

$$\Rightarrow O_r = O(|X|)$$

(g) $J(b_k, a_k) = \mathbb{E}[r(b_k, a_k) + \tilde{J}(b_{k+1})] =$

{(myopic setting)}

$$= r(b_k, a_k) + \int_{z_{k+1}} p(z_{k+1} | b_k, a_k) \tilde{J}(b_{k+1} | b_k, a_k, z_{k+1}) dz_{k+1}$$

$$O(J(b_k, a_k)) = O_r + |Z|(O_z + (O_r + O_b)) =$$

$$= O(|Z||X|^2)$$