Technion – Israel Institute of Technology

HW5

Numerical Methods

019003

Alon Spinner	305184335	alonspinner@gmail.com
Oren Elmakis	311265516	orenelmakis@gmail.com

December 30, 2021

Question 1

Assuming the system is in steady state, we can assume that the power $[Watt]$ input must equal the power output. Hence, we will impose equilibrium between the in and out heat fluxes on our control volumes.

$$\dot{Q} = \dot{Q}_{in} - \dot{Q}_{out} = 0$$

We will denote q as heat flux, with units of $[Watt]$.

q' will be power per unit length $\left[\frac{Watt}{m}\right]$

q'' will be power per unit length $\left[\frac{Watt}{m^2}\right]$

From 'Heat '

Newton's convection law:

$$q'' = -h_c(T_a - T_{boundry})$$

Fourier's Convection law:

$$q'' = -k\frac{\partial T}{\partial x}$$

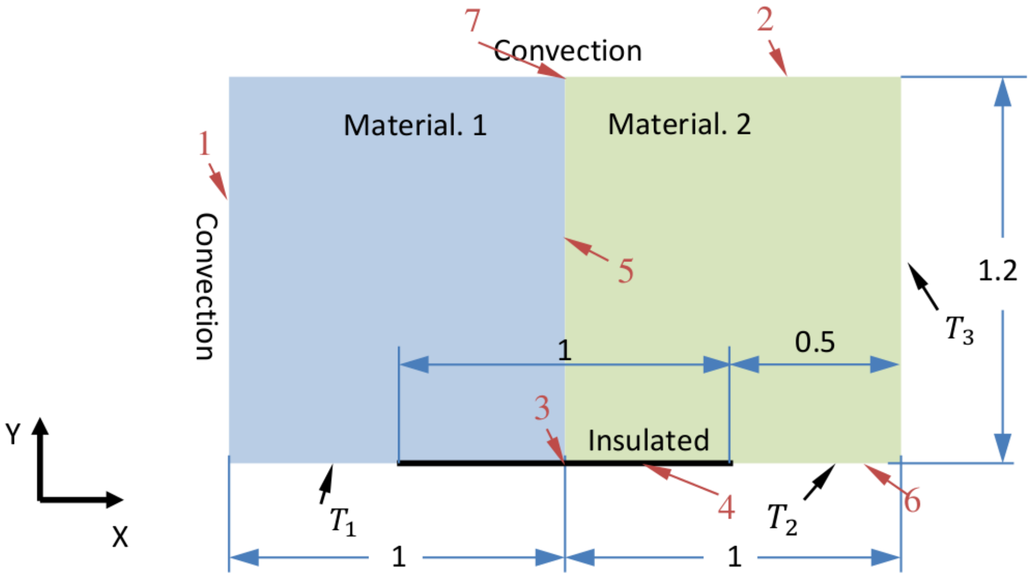


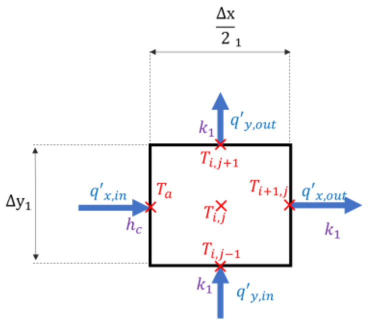
Figure 1 A plate consisting of two materials with different properties

Section a

We were asked to use the control volume approach for calculating the boundary equations on points 1,2,3,4 boundary points.

Point 1

We created a control volume of size $\Delta y_1, \Delta x_1$.



$$\dot{Q}_{in} = q'_{x,in}\Delta y_1 + q'_{y,in}\frac{\Delta x_1}{2} = h_c(T_a - T_{i,j})\Delta y_1 + k_1\frac{T_{i,j} - T_{i,j-1}}{\Delta y_1}\frac{\Delta x_1}{2}$$

$$\dot{Q}_{out} = q'_{y,out}\frac{\Delta x_1}{2} + q'_{x,out}\Delta y_1 = k_1\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_1}{2} + k_1\frac{T_{i+1,j} - T_{i,j}}{\Delta x_1}\Delta y_1$$

Hence the balance equation is:

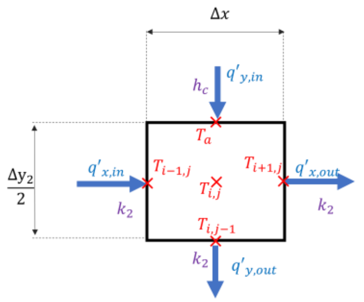
$$0 = \underbrace{h_c(T_a - T_{i,j})\Delta y_1}_{q'_{x,in}} + \underbrace{k_1\frac{T_{i+1,j} - T_{i,j}}{\Delta x_1}\Delta y_1}_{q'_{x,out}} - \underbrace{k_1\frac{T_{i,j} - T_{i,j-1}}{\Delta y_1}\frac{\Delta x_1}{2}}_{q'_{y,in}} + \underbrace{k_1\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_1}{2}}_{q'_{y,out}}$$

$$0 = h_c(T_a - T_{i,j})\Delta y_1 + k_1(T_{i+1,j} - T_{i,j}) - k_1\frac{T_{i,j} - T_{i,j-1}}{2} + k_1\frac{T_{i,j} - T_{i,j-1}}{2}$$

$$T_{i,j}(h_c\Delta y_1 + 2k_1) = T_a h_c \Delta y_1 + \frac{k_1}{2}T_{i,j-1} + \frac{k_1}{2}T_{i,j+1} + k_1 T_{i+1,j}$$

Point 2 –

We created a control volume of size $\Delta y_2, \Delta x_2$.



$$\dot{Q}_{in} = q'_{y,in}\Delta x + q'_{x,in}\frac{\Delta y_2}{2} = h_c(T_a - T_{i,j})\Delta x_2 + k_2\frac{T_{i,j} - T_{i,j-1}}{\Delta x_2}\frac{\Delta y_2}{2}$$

$$\dot{Q}_{out} = q'_{y,out}\Delta x_2 + q'_{x,out}\frac{\Delta y_2}{2} = k_2\frac{T_{i,j-1} - T_{i,j}}{\Delta y_2}\Delta x_2 + k_2\frac{T_{i+1,j} - T_{i,j}}{\Delta x_2}\frac{\Delta y_2}{2}$$

Hence the balance equation is:

$$0 = \underbrace{h_c(T_a - T_{i,j})\Delta x_2}_{q'_{y,in}} + \underbrace{k_2 \frac{T_{i,j-1} - T_{i,j}}{\Delta y_2}\Delta x_2}_{q'_{y,out}} - \underbrace{k_2 \frac{T_{i,j} - T_{i-1,j}}{\Delta x_2}\frac{\Delta y_2}{2}}_{q'_{x,in}} + \underbrace{k_2 \frac{T_{i+1,j} - T_{i,j}}{\Delta x_2}\frac{\Delta y_2}{2}}_{q'_{x,out}}$$

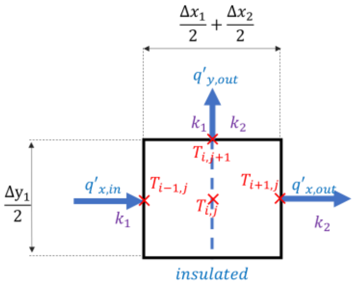
according to the second material grid $\Delta x_2 = \Delta x$ and $\Delta y_2 = 2\Delta x$, we get

$$0 = h_c(T_a - T_{i,j})\Delta x_2 + \frac{k_2}{2}(T_{i,j-1} - T_{i,j}) - k_2(T_{i,j} - T_{i-1,j}) + k_2(T_{i+1,j} - T_{i,j})$$

$$T_{i,j}\left(h_c\Delta x_2 + \frac{5}{2}k_2\right) = T_a h_c \Delta x_2 + \frac{k_2}{2}T_{i,j-1} + k_2 T_{i+1,j} + k_2 T_{i-1,j}$$

Point 3 –

We created a control volume of size $\Delta y_1, \frac{\Delta x_1}{2} + \frac{\Delta x_2}{2}$.



$$\dot{Q}_{in} = q'_{x,in}\frac{\Delta y_1}{2} = k_1 \frac{T_{i,j} - T_{i-1,j}}{\Delta x_1}\frac{\Delta y_1}{2}$$

$$\dot{Q}_{out} = q'_{y,out}\frac{\Delta x_2}{2} + q'_{y,out}\frac{\Delta x_1}{2} + q'_{x,out}\frac{\Delta y_1}{2} = k_2\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_2}{2} + k_1\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_1}{2} + k_2\frac{T_{i+1,j} - T_{i,j}}{\Delta x_2}\frac{\Delta y_1}{2}$$

Hence the balance equation is:

$$0 = \underbrace{k_1\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_1}{2} + k_2\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_2}{2}}_{q'_{y,out}} - \underbrace{k_1\frac{T_{i,j} - T_{i-1,j}}{\Delta x_1}\frac{\Delta y_1}{2}}_{q'_{x,in}} + \underbrace{k_2\frac{T_{i+1,j} - T_{i,j}}{\Delta x_2}\frac{\Delta y_1}{2}}_{q'_{x,out}}$$

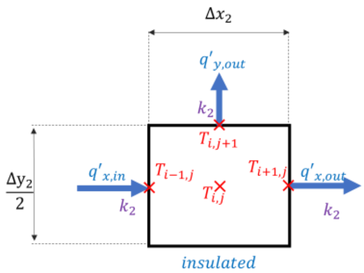
The grid is $\frac{\Delta x_2}{2} + \frac{\Delta x_1}{2} = \Delta x,\ \Delta y_1 = \Delta x$

$$0 = \frac{1}{2}k_1(T_{i,j+1} - T_{i,j}) + \frac{1}{2}k_2(T_{i,j+1} - T_{i,j}) - \frac{1}{2}k_1(T_{i,j} - T_{i-1,j}) + \frac{1}{2}k_2(T_{i+1,j} - T_{i,j})$$

$$T_{i,j}(k_1 + k_2) = \frac{1}{2}T_{i,j+1}(k_1 + k_2) + \frac{k_1}{2}T_{i-1,j} + \frac{k_2}{2}T_{i+1,j}$$

Point 4 –

We created a control volume of size $\Delta x_2, \Delta y_2$.



$$\dot{Q}_{in} = q'_{x,in}\frac{\Delta y_2}{2} = k_2\frac{T_{i,j} - T_{i-1,j}}{\Delta x_2}\frac{\Delta y_2}{2}$$

$$\dot{Q}_{out} = q'_{x,out}\frac{\Delta y_2}{2} + q'_{y,out}\Delta x_2 = k_2\frac{T_{i+1,j} - T_{i,j}}{\Delta x_2}\frac{\Delta y_2}{2} + k_2\frac{T_{i,j+1} - T_{i,j}}{\Delta y_2}\Delta \mathrm{x}_2$$

Hence the balance equation is:

$$0 = \underbrace{-k_2\frac{T_{i,j} - T_{i-1,j}}{\Delta x_2}\frac{\Delta y_2}{2}}_{q'_{x,in}} + \underbrace{k_2\frac{T_{i+1,j} - T_{i,j}}{\Delta x_2}\frac{\Delta y_2}{2}}_{q'_{x,out}} + \underbrace{k_2\frac{T_{i,j+1} - T_{i,j}}{\Delta y_2}\Delta \mathrm{x}_2}_{q'_{y,out}}$$

According to the second material given grid $\Delta x_2 = \Delta x$ and $\Delta y_2 = 2\Delta x$

$$0 = -(T_{i,j} - T_{i-1,j}) + (T_{i+1,j} - T_{i,j}) + \frac{1}{2}(T_{i,j+1} - T_{i,j})$$

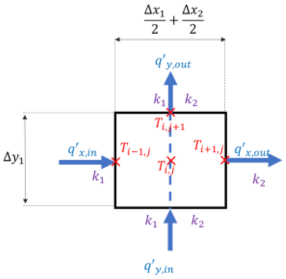
$$T_{i,j} = \frac{2T_{i-1,j} + 2T_{i+1,j} + T_{i,j+1}}{5}$$

Section b

We were asked to formulate the equations of points 5,6,7:

Point 5 –

We created a control volume of size $\Delta y_1, \frac{\Delta x_1}{2} + \frac{\Delta x_2}{2}$.



$$\dot{Q}_{in} = q'_{x,in}\Delta y_1 + q'_{y,in}\frac{\Delta x_1}{2} + q'_{y,in}\frac{\Delta x_2}{2} = k_1\frac{T_{i,j} - T_{i-1,j}}{\Delta x_1}\Delta y_1 + k_1\frac{T_{i,j} - T_{i,j-1}}{\Delta y_1}\frac{\Delta x_1}{2} + k_2\frac{T_{i,j} - T_{i,j-1}}{\Delta y_1}\frac{\Delta x_2}{2}$$

$$\dot{Q}_{out} = q'_{x,out}\Delta y_1 + q'_{y,out}\frac{\Delta x_1}{2} + q'_{y,out}\frac{\Delta x_2}{2} = k_2\frac{T_{i,j} - T_{i-1,j}}{\Delta x_2}\Delta y_1 + k_1\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_1}{2} + k_2\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_2}{2}$$

Hence the balance equation is:

$$0 = \underbrace{k_1\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_1}{2} + k_2\frac{T_{i,j+1} - T_{i,j}}{\Delta y_1}\frac{\Delta x_2}{2}}_{q'_{y,out}} - \underbrace{(k_1\frac{T_{i,j} - T_{i,j-1}}{\Delta y_1}\frac{\Delta x_1}{2} + k_2\frac{T_{i,j} - T_{i,j-1}}{\Delta y_1}\frac{\Delta x_2}{2})}_{q'_{y,in}}$$
$$-\underbrace{k_1\frac{T_{i,j} - T_{i-1,j}}{\Delta x_1}\Delta y_1}_{q'_{x,in}} + \underbrace{k_2\frac{T_{i+1,j} - T_{i,j}}{\Delta x_2}\Delta y_1}_{q'_{x,out}}$$

The grid size is $\frac{\Delta x_2}{2} + \frac{\Delta x_1}{2} = \frac{\Delta y_1}{2} = \Delta x$

$$0 = \frac{1}{2}k_1(T_{i,j+1} - T_{i,j}) + \frac{1}{2}k_2(T_{i,j+1} - T_{i,j}) - \frac{1}{2}k_1(T_{i,j} - T_{i,j-1}) - \frac{1}{2}k_2(T_{i,j} - T_{i,j-1})$$
$$-k_1(T_{i,j} - T_{i-1,j}) + k_2(T_{i+1,j} - T_{i,j})$$

$$T_{i,j}(k_1+k_2) = \frac{T_{i,j+1}(k_1+k_2) + T_{i,j-1}(k_1+k_2) + 2k_1T_{i-1,j} + 2k_2T_{i+1,j}}{4}$$

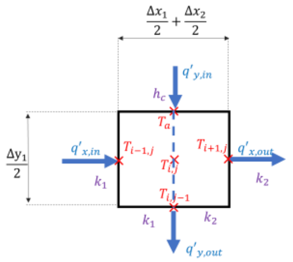
Point 6 –

According to the figure of the plate

$$T_{ij} = T_2$$

Point 7 –

We created a control volume of size $\Delta y_1, \frac{\Delta x_1}{2} + \frac{\Delta x_2}{2}$.



$$\dot{Q}_{in} = q'_{x,in}\frac{\Delta y_1}{2} + q'_{y,in}\left(\frac{\Delta x_1}{2}+\frac{\Delta x_2}{2}\right) = \frac{k_1\frac{T_{i,j}-T_{i-1,j}}{\Delta x_1}\Delta y_1}{2} + h_c\left(T_{i,j}-T_a\right)\left(\frac{\Delta x_1}{2}+\frac{\Delta x_2}{2}\right)$$

$$\dot{Q}_{out} = q'_{x,out}\frac{\Delta y_1}{2} + q'_{y,out}\frac{\Delta x_1}{2} + q'_{y,out}\frac{\Delta x_2}{2} = k_2\frac{T_{i,j}-T_{i-1,j}}{\Delta x_2}\frac{\Delta y_1}{2} + k_1\frac{T_{i,j}-T_{i,j-1}}{\Delta y_1}\frac{\Delta x_1}{2} + k_2\frac{T_{i,j+1}-T_{i,j-1}}{\Delta y_1}\frac{\Delta x_2}{2}$$

Hence the balance equation is:

$$\begin{aligned} 0 = \underbrace{h_c\left(T_a - T_{i,j}\right)\left(\frac{\Delta x_1}{2}+\frac{\Delta x_2}{2}\right)}_{q'_{y,in}} - \underbrace{\left(k_1\frac{T_{i,j}-T_{i,j-1}}{\Delta y_1}\frac{\Delta x_1}{2} + k_2\frac{T_{i,j}-T_{i,j-1}}{\Delta y_1}\frac{\Delta x_2}{2}\right)}_{q'_{y,out}} \\ -\underbrace{k_1\frac{T_{i,j}-T_{i-1,j}}{\Delta x_1}\Delta y_1}_{q'_{x,in}} + \underbrace{k_2\frac{T_{i+1,j}-T_{i,j}}{\Delta x_2}\Delta y_1}_{q'_{x,out}} \end{aligned} \tag{1}$$

The grid size is $\frac{\Delta x_2}{2} + \frac{\Delta x_1}{2} = \frac{\Delta y_1}{2} = \Delta x$

$$\begin{aligned} 0 = h_c\left(T_a - T_{i,j}\right)\frac{\Delta x_1}{2} + h_c\left(T_a - T_{i,j}\right)\frac{\Delta x_2}{2} - \frac{k_1}{2}\left(T_{i,j}-T_{i,j-1}\right) - \frac{k_2}{2}\left(T_{i,j}-T_{i,j-1}\right) \\ -k_1\left(T_{i,j}-T_{i-1,j}\right) + k_2\left(T_{i+1,j}-T_{i,j}\right) \end{aligned} \tag{2}$$

$$T_{i,j}\left(h_c\left(\frac{\Delta x_1}{2}+\frac{\Delta x_2}{2}\right) + \frac{3}{2}(k_1+k_2)\right) = h_cT_a\left(\frac{\Delta x_1}{2}+\frac{\Delta x_2}{2}\right) + \frac{T_{i,j-1}}{2}(k_1+k_2) + T_{i-1,j}k_1 + T_{i+1,j}k_2$$

Section c

We will open the equation using the finite difference approach with second order central difference.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \rightarrow \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta y^2} = 0$$

The steps of Δx and Δy are equal, thus

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \rightarrow T_{i+1,j} - 2T_{i,j} + T_{i-1,j} + T_{i+1,j} - 2T_{i,j} + T_{i-1,j} = 0$$

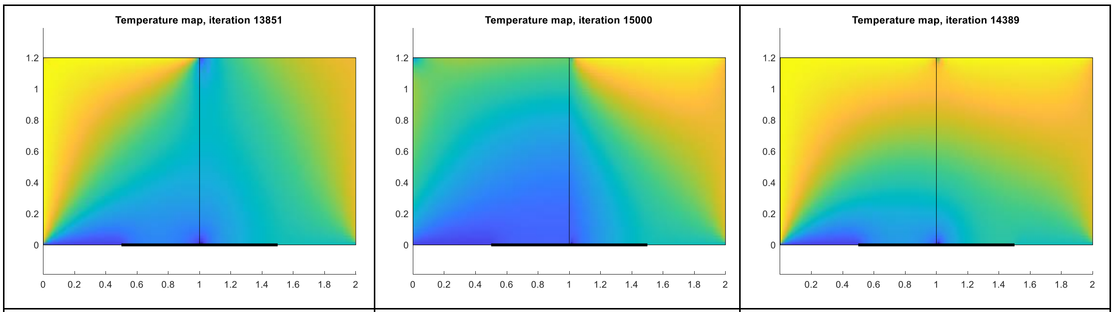
We can arrange the equation as follows (keep remember that $\Delta x = \Delta y$) in material 1)

$$\frac{(T_{i+1,j} - T_{i,j})}{\Delta x} \Delta y - \frac{(T_{i,j} - T_{i-1,j})}{\Delta x} \Delta y + \frac{(T_{i,j+1} - T_{i,j})}{\Delta y} \Delta x + \frac{(T_{i,j} - T_{i,j-1})}{\Delta y} \Delta x = 0$$

Multiply by k_1, and we get the control volume.

$$\underbrace{k_1 \frac{(T_{i+1,j} - T_{i,j})}{\Delta x} \Delta y}_{right\ side\ flux} - \underbrace{k_1 \frac{(T_{i,j} - T_{i-1,j})}{\Delta x} \Delta y}_{left\ side\ flux} + \underbrace{k_1 \frac{(T_{i,j+1} - T_{i,j})}{\Delta y} \Delta x}_{upward\ flux} + \underbrace{k_1 \frac{(T_{i,j} - T_{i,j-1})}{\Delta y} \Delta x}_{from\ downward\ flux} = 0$$

Section d



a	b	c
$k_1 = 1, k_2 = 100, h = 2$	$k_1 = 100, k_2 = 1, h = 2$	$k_1 = 1, k_2 = 1, h = 100$

✓

The change in the Thermal conduction value in the materials causes difference in the heat distribution.

- Comparing figures (a) and (b), It shows that the material with the higher thermal conductivity will have a more even temperature distribution. The reason figures (a) and (b) aren't symmetric are because of the bottom boundary conditions.

- Figure (c) presents a scenario of high convection, which causes high heat flux that gets into materials one and two, causing temperature increase. Having the same thermal conductivity for both materials, the difference between the system's sides are only due to the bottom boundary conditions.

Question 2 –

Section a

The equation is given as

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + v(x,t)$$

The first derivative is based on time forward differentiation

$$\frac{\partial u}{\partial t} = \frac{u_i^{t+1} - u_i^t}{\Delta t},$$

Accordingly, the explicit form of the partial second derivative is an upward triangle, with time propagation on the top, for index i

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{\Delta x^2}$$

$$\frac{u_i^{t+1} - u_i^t}{\Delta t} = k\frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{\Delta x^2} + \nu(x_i, t_i)$$

The implicit representation form is based on an inverted triangle,

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{\Delta x^2}$$

$$\frac{u_i^{t+1} - u_i^t}{\Delta t} = k\frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{\Delta x^2} + \nu(x_i, t_i)$$

Crank Nicolson method uses a dummy point that is in the middle of the time step and represented as a combination of the time propagation and the previous state:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left(\frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{\Delta x^2} + \frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{\Delta x^2}\right)$$

$$\frac{u_i^{t+1} - u_i^t}{\Delta t} = \frac{1}{2}\left(\frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{\Delta x^2} + \frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{\Delta x^2}\right) + \nu(x_i, t_i)$$

Section b

We were asked to solve the heat transfer function using the explicit method.

First, we have to formulate the boundary conditions of both sides:

In $x = 0$ according to the conservation of heat equation:

$$\frac{q(x) - q(x + \Delta x)}{\Delta x} = \rho C\frac{T^{i+1} - T^i}{\Delta t}$$

From Fourier's law

$$\frac{h_c\left(T_a - T_0^{i+1}\right) + \frac{k\left(T_{j+1}^{i+1} - T_0^{i+1}\right)}{\Delta x} + \int\left(v\left(x_j, t_{i+1}\right) - v\left(x_{j+1}, t_{i+1}\right)\right)dxdt}{\Delta x} = \frac{T_0^{i+1} - T_0^i}{\Delta t}$$

$$T_j^{i+1}\left(\Delta x + h_c\Delta t + \frac{k\Delta t}{\Delta x}\right) = \left(T_a h_C + \frac{k}{dx}T_{j+1}^{i+1} + \left(\left(v\left(x_j, t_{i+1}\right) - v\left(x_j, t_i\right)\right) - \left(v\left(x_{j+1}, t_{i+1}\right) - v\left(x_{j+1}, t_i\right)\right)\right)dxdt\right)\Delta t + T_j^i\Delta x$$

In $x = 1$, the outward flux is known $q = 1$

$$\frac{-\frac{k\left(T_{j+1}^{i+1} - T_j^{i+1}\right)}{\Delta x} + \int\left(v\left(x_j, t_{i+1}\right) - v\left(x_{j+1}, t_{i+1}\right)\right)dxdt - q_1}{\Delta x} = \frac{T_{j+1}^{i+1} - T_{j+1}^i}{\Delta t}$$

$$T_{j+1}^{i+1}\left(\Delta x + \frac{k\Delta t}{\Delta x}\right) = \left(\frac{k}{\Delta x}T_j^{i+1} + \left(\left(v\left(x_j, t_{i+1}\right) - v\left(x_j, t_i\right)\right) - \left(v\left(x_{j+1}, t_{i+1}\right) - v\left(x_{j+1}, t_i\right)\right)\right)dxdt - q_1\right)\Delta t + T_{j+1}^i\Delta x$$

And the central nodes equations for temperature distribution

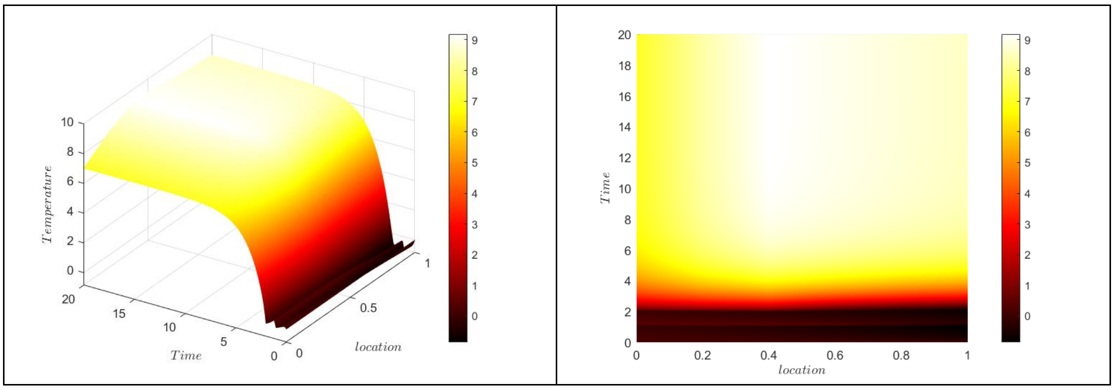
$$T_i^{t+1} = T_i^t + k\Delta t\frac{T_{i+1}^t - 2T_i^t + T_{i-1}^t}{\Delta x^2} + \nu(x_i, t_i)\Delta t$$

The convergence and stability of the explicit method is

$$\Delta t \le \frac{1}{2}\frac{\Delta x^2}{k}$$

As a result, of using $\Delta x = 0.1$, the Δt which causes to diverge is 0.01.

The following figure demonstrates the time-location distribution of the 1D heated rod with $\Delta x = 0.2$ and $\Delta t = 0.01$:



The code is attached rod_explicit.m

Section c

The boundary conditions are the same as in section a:

$$T_j^{i+1}\left(\Delta x + h_c \Delta t + \frac{k\Delta t}{\Delta x}\right) = \left(T_a h_C + \frac{k}{dx} T_{j+1}^{i+1} + \int \left(v(x_j, t_{i+1}) - v(x_{j+1}, t_{i+1})\right) dxdt\right)\Delta t + T_j^i \Delta x$$

However, now we don't have the T_{j+1}^{i+1}

$$T_j^{i+1}\left(\Delta x + h_c \Delta t + \frac{k\Delta t}{\Delta x}\right) - \frac{k\Delta t}{\Delta x} T_{j+1}^{i+1} = \left(T_a h_C + \left(\left(v(x_j, t_{i+1}) - v(x_j, t_i)\right) - \left(v(x_{j+1}, t_{i+1}) - v(x_{j+1}, t_i)\right)\right) dxdt\right)\Delta t + T_j^i \Delta x$$

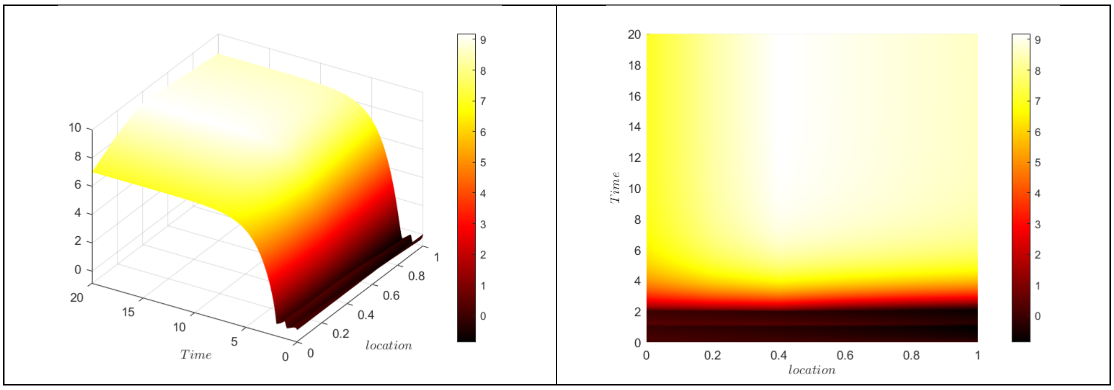
And the boundary condition of the end of the rod:

$$T_{j+1}^{i+1}\left(\Delta x + \frac{k\Delta t}{\Delta x}\right) - \frac{k\Delta t}{\Delta x} T_j^{i+1} = \left(\left(\left(v(x_j, t_{i+1}) - v(x_j, t_i)\right) - \left(v(x_{j+1}, t_{i+1}) - v(x_{j+1}, t_i)\right)\right) dxdt - q_1\right)\Delta t + T_j^i \Delta x$$

The central nodes equation is:

$$u_i^{t+1} = u_i^t + \frac{\Delta t k}{2}\left(\frac{u_{i+1}^{t+1} - 2u_i^{t+1} + u_{i-1}^{t+1}}{\Delta x^2} + \frac{u_{i+1}^t - 2u_i^t + u_{i-1}^t}{\Delta x^2}\right) + v(x_i, t_i)$$

The following figure demonstrates the time-location distribution of the 1D heated rod with $\Delta x = 0.2$ and $\Delta t = 0.01$, which is equal to the results of the explicit:



The code is attached rod_CN.m

Section d

In this question, we were asked to implement the methods of explicit and crank Nicolson for solving the 1D heated rod PDE. The main difference between the methods is the performing complexity and the stability. In comparison to the Crank Nicolson, which requires a tridiagonal algebraic solution, the explicit method implementation is straightforward. However, in contrast to the Crank Nicolson method that has unconditional stability, the explicit approach is restricted to stability term of $\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{k}$.

The temperature distribution over the rod is increases via the heating from $v(x,t)$. The left boundary $(x = 0)$ dissipates heat to the environment via convection, while the right boundary $(x = 1)$ has a constant heat flux out. As can be noted from the temperature difference in the figure above, the convection dissipation is more effective than the constant heat transfer on the right side.

Question 3 –

Section a

Given that $dx = dy$ we have to prove the ADI scheme for inner points, the PDE equation for plate $2x2$

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + f(x,y)T$$

The first step is forward differentiation of $\frac{\Delta t}{2}$

$$\frac{\partial T}{\partial t} = \frac{T_{i,j}^{l+\frac{1}{2}} - T_{i,j}^{l}}{\frac{\Delta t}{2}}$$

time propagation of $\frac{\Delta t}{2}$ in the $\hat{y}$ direction

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1}^{l+\frac{1}{2}} - 2T_{i,j}^{l+\frac{1}{2}} + T_{i,j-1}^{l+\frac{1}{2}}}{\Delta y^2}$$

And using second order central difference OF $\hat{x}$ explicitly

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l}}{\Delta x^2}$$

Consequently, with $\Delta x = \Delta y$ and define $\lambda = \frac{\Delta t k}{\Delta x^2}$ the results

$$2T_{i,j}^{l+\frac{1}{2}} - 2T_{i,j}^{l} = \lambda\left(T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l} + T_{i,j+1}^{l+\frac{1}{2}} - 2T_{i,j}^{l+\frac{1}{2}} + T_{i,j-1}^{l+\frac{1}{2}}\right) + \Delta t f(x,y)T_{i,j}^{l}$$

after some order

$$-\lambda T_{i,j-1}^{l+\frac{1}{2}} + 2(1+\lambda)T_{i,j}^{l+\frac{1}{2}} - \lambda T_{i,j+1}^{l+\frac{1}{2}} = \lambda T_{i-1,j}^{l} + 2(1-\lambda)T_{i,j}^{l} + \lambda T_{i+1,j}^{l} + \Delta t f(x,y)T_{i,j}^{l}$$

The second step from $l + \frac{\Delta t}{2} \rightarrow l + \Delta t$

$$\frac{\partial T}{\partial t}=\frac{T_{i,j}^{l+1}-T_{i,j}^{l+\frac{1}{2}}}{\frac{\Delta t}{2}}$$

time propagation of Δt in the $\hat{x}$ direction

$$\frac{\partial^2 T}{\partial x^2}=\frac{T_{i,j+1}^{l+1}-2T_{i,j}^{l+1}+T_{i,j-1}^{l+1}}{\Delta x^2}$$

With $\hat{y}$ remain the same.

$$\frac{\partial^2 T}{\partial y^2}=\frac{T_{i,j+1}^{l+\frac{1}{2}}-2T_{i,j}^{l+\frac{1}{2}}+T_{i,j-1}^{l+\frac{1}{2}}}{\Delta y^2}$$

Consequently, with $\Delta x=\Delta y$ and define $\lambda=\frac{\Delta t k}{\Delta x^2}$ the results

$$2T_{i,j}^{l+1}-2T_{i,j}^{l+\frac{1}{2}}=\lambda\left(T_{i+1,j}^{l+1}-2T_{i,j}^{l+1}+T_{i-1,j}^{l+1}+T_{i,j+1}^{l+\frac{1}{2}}-2T_{i,j}^{l+\frac{1}{2}}+T_{i,j-1}^{l+\frac{1}{2}}\right)+\Delta t f(x,y)T_{i,j}^{l+0.5}$$

And after ordering

$$-\lambda T_{i-1,j}^{l+1}+2(1+\lambda)T_{i,j}^{l+1}-\lambda T_{i+1,j}^{l+1}=\lambda T_{i,j-1}^{l+0.5}+2(1-\lambda)T_{i,j}^{l+0.5}+\lambda T_{i,j+1}^{l+0.5}+\Delta t f(x,y)T_{i,j}^{l+0.5}$$

Section b

The initial condition is

$$T(x,y,0)=0$$

and we got three trivial boundary conditions

$$T(x,0,t)=x(2-x)$$

$$T(2,y,t)=2-y$$

$$T(x,2,t)=1$$

And another Neumann condition

$$\frac{\partial T(0,y,t)}{\partial x}=0$$

We will use dummy point outside the domain (to keep the accuracy of the method, using central derivative)

$$\frac{T_{-1,y}^{t}-T_{1,y}^{t}}{2\Delta \mathrm{x}}=0$$

$$T_{-1,y}^{t}=T_{1,y}^{t}$$

Now, we can substitute the result to the ADI equation for the first step over $\hat{y}$ direction.

$$-\lambda T_{0,j-1}^{l+\frac{1}{2}}+2(1+\lambda)T_{0,j}^{l+\frac{1}{2}}-\lambda T_{0,j+1}^{l+\frac{1}{2}}=\lambda T_{1,j}^{l}+2(1-\lambda)T_{0,j}^{l}+\lambda T_{1,j}^{l}+\Delta t f(0,y)T_{0,j}^{l}$$

$$-\lambda T_{0,j-1}^{l+\frac{1}{2}} + 2(1+\lambda)T_{0,j}^{l+\frac{1}{2}} - \lambda T_{0,j+1}^{l+\frac{1}{2}} = 2(1-\lambda)T_{0,j}^{l} + 2\lambda T_{1,j}^{l} + \Delta t f(0,y) T_{0,j}^{l}$$

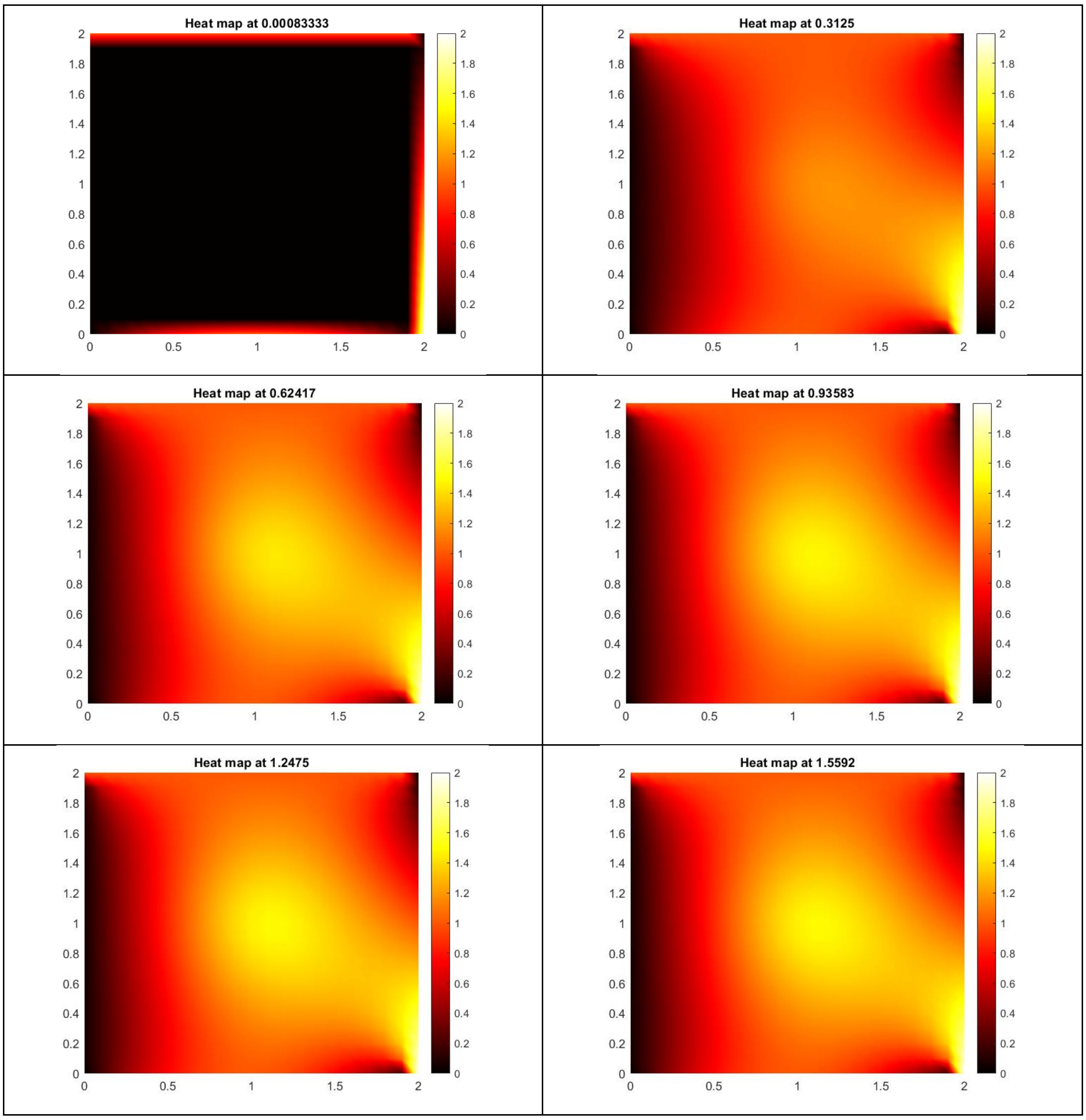
The second direction $\hat{x}$ equation.

$$-\lambda T_{1,j}^{l+1} + 2(1+\lambda)T_{0,j}^{l+1} - \lambda T_{1,j}^{l+1} = \lambda T_{0,j-1}^{l+0.5} + 2(1-\lambda)T_{0,j}^{l+0.5} + \lambda T_{0,j+1}^{l+0.5} + \Delta t f(x,y) T_{0,j}^{l+0.5}$$

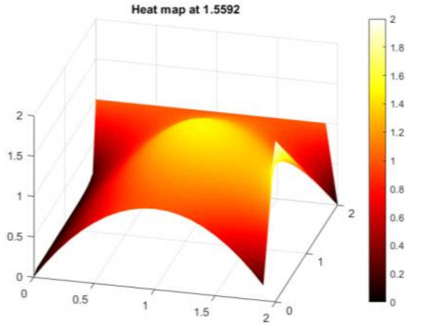
$$2(1+\lambda)T_{0,j}^{l+1} - 2\lambda T_{1,j}^{l+1} = \lambda T_{0,j-1}^{l+0.5} + 2(1-\lambda)T_{0,j}^{l+0.5} + \lambda T_{0,j+1}^{l+0.5} + \Delta t f(x,y) T_{0,j}^{l+0.5}$$

Section d

According to the value of $\lambda = \frac{1}{6}$, $\Delta x = 0.1$, and $k = 2$ the value of $\Delta t = \lambda \cdot \frac{(\Delta x)^2}{k} = 8.33 \cdot 10^{-4}$



The heat distribution can be seen initially starting in the low right corner ($x = 2, y = 0$), where the temperature of the boundary condition is $T = 2\ degrees$. The function added to the equation of the heat distribution is a heat source that amplifies the temperature in the center of the plate and causes a local warming peak, as shown in the following figure.



Moreover, the dissipation direction is towards the edges, where the temperature is low.

The code is attached ADI.m