Numerical Methods for Engineering – Graduate Course 019003

Prep HW 5: Ordinary differential equations with boundary conditions

Due date: next recitation

1. Solve the heat transfer equation $\frac{d^2T}{dx^2} + (T_0 - T) = 0$ in the domain $x \in [0\ 1]$ with the boundary conditions $T(x=0) = 0$, $T(x=1) = 1$ using the shooting and Euler method with the step size $h = 0.5$, $T_0 = 1$.
2. Solve the previous equation using the finite difference method.

$$\frac{\partial^2 T}{\partial x^2} + (T_0 - T) = 0 \qquad x \in [0,1]$$

$T(x=0) = 0$ $\quad T_0 = 1$

$T(x=1) = 1$ $\quad h = \frac{1}{2}$ ← Euler shooting

$$y(x_{i+1}) = y(x_i) + f(x_i, y_i) \cdot h$$

$$\frac{\partial T}{\partial x}(x=0) = 0 \quad \text{n.N.}$$

(1) $\frac{\partial^2 T}{\partial x^2}(0) = T(0) - T_0 = -1$

$$T'\left(\frac{1}{2}\right) = T'(0) + \frac{\partial^2 T}{\partial x}(0) \cdot h = T'(0) - \frac{1}{2}$$

$-\frac{1}{2}$

$$T\left(\frac{1}{2}\right) = T(0) + T'(0) \cdot h = T'(0) \cdot \frac{1}{2}$$

(2) $T''\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right) - T_0 = T'(0) \frac{1}{2} - 1$

$$T'(1) = T'\left(\frac{1}{2}\right) + T''\left(\frac{1}{2}\right)\cdot h =$$

$$= T'(0)\cdot\frac{1}{2} + \left(T'(0)\cdot\frac{1}{2} - 1\right)\cdot\frac{1}{2} =$$

$$= \frac{5}{4}T'(0) - 1$$

$$T(1) = T\left(\frac{1}{2}\right) + T'\left(\frac{1}{2}\right)\cdot h =$$

$$= T'(0)\cdot\frac{1}{2} + \left(T'(0)\cdot\frac{1}{2} - 1\right)\cdot\frac{1}{2} =$$

$$= T'(0) - \frac{1}{4} = 1$$

$$\Rightarrow T'(0) = \frac{5}{4}$$

	$x=0$	$x=\frac{1}{2}$	$x=1$
T	0	2.5/4	1
T'	$\frac{5}{4}$	$\frac{2}{3}$	0.5625
T''	-1		

$T'' = T - T_0$

$T(0) = 0 \quad T_0 = 1$

$T(1) = 1 \quad h = \frac{1}{2}$ · finite difference

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$\Rightarrow$ for $x = \frac{1}{2}$:

$$\frac{1 \cancel{T(1)} - 2 \cdot T\left(\frac{1}{2}\right) + \cancel{T(0)}^{0}}{\frac{1}{4} \; h^2} = T\left(\frac{1}{2}\right) - \cancel{T_0}^{1}$$

$$1 - 2T\left(\frac{1}{2}\right) = \frac{1}{4} T\left(\frac{1}{2}\right) - \frac{1}{4}$$

$$\frac{9}{4} T\left(\frac{1}{2}\right) = \frac{5}{4} \Rightarrow T\left(\frac{1}{2}\right) = \frac{5}{9}$$