Numerical Methods for Engineering – Graduate Course 019003

Prep HW 1: Linear systems

Due date: before next lecture (on Moodle)

1. Solve (manually by hand) the linear system defined by this augmented matrix. Use the Gauss method with row pivoting. Why is the row pivoting important?

$$(A|b)=\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 2 & 3 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & -2 & 3 & 2 & 3 \end{array}\right)$$

2. Solve the next system of equations (manually by hand) using the LU decomposition method (Doolittle):

$$\left(\begin{array}{ccc|c} 1 & 7 & -4 & -51 \\ 4 & -4 & 9 & 62 \\ 12 & -1 & 3 & 8 \end{array}\right)$$

3. Solve the next system of equations (manually by hand) using the Thomas (LU for tri-diagonal matrices):

$$\begin{pmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 124 \\ 4 \\ 14 \end{pmatrix}$$

Solution inside

Alon Spinner

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$$(A|b) = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 2 & 3 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ -1 & -2 & 3 & 2 & 3 \end{array}\right] \rightarrow$$

$$\begin{array}{l} r_3 : r_3 - r_1 \\ r_4 : r_4 + r_1 \end{array} \quad \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & -3 & 3 & 2 & 4 \\ 0 & 0 & 6 & 6 & 6 \end{array}\right] \longrightarrow$$

$$\xrightarrow{r_3 : r_3 + 3 \cdot r_2} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 9 & 11 & 7 \\ 0 & 0 & 6 & 6 & 6 \end{array}\right] \rightarrow$$

$$\xrightarrow{r_4 : r_4 - \frac{2}{3} r_3} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 9 & 11 & 7 \\ 0 & 0 & 0 & -\frac{4}{3} & \frac{4}{3} \end{array}\right]$$

$$\Rightarrow x_4 = -1$$

$$x_3 = \frac{7 - 11 \cdot x_4}{9} = 2$$

$$x_2 = \frac{1 - 3x_4 - 2x_3}{1} = 0$$

$$x_1 = \frac{3 - 4x_4 - 3x_3 \cancel{- 2x_2}}{1} = 1$$

$$x = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$$

doolittle

2. $\left[\begin{array}{ccc|c} 1 & 7 & -4 & 51 \\ 4 & -4 & 9 & 62 \\ 12 & -1 & 3 & 8 \end{array}\right]$

$\forall j \quad i=1 \Rightarrow U_{ij} = A_{ij}$

$\quad\; i>1 \quad U_{ij} = A_{ij} - \sum_{k=1}^{i-1} L_{ik} U_{kj}$

$\forall i \quad j=1 \quad L_{ij} = A_{ij}$

$\quad\; j>1 \quad L_{ij} = \dfrac{A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj}}{U_{jj}}$

by construction:

$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 12 & ? & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 7 & -4 \\ 0 & ? & ? \\ 0 & 0 & ? \end{bmatrix}$

$U_{22} = A_{22} - L_{21} \cdot U_{12} = -4 - 4 \cdot 7 = -32$

$U_{23} = A_{23} - L_{21} \cdot U_{13} = 9 - (-16) = 25$

$$L_{22} = \frac{A_{22} - L_{21} \cdot U_{12}}{U_{22}} = \frac{-4 - 4 \cdot 7}{-32} = 1$$ by construction

$$L_{32} = \frac{A_{32} - L_{31} \cdot U_{12}}{U_{22}} = \frac{-1 - 12 \cdot 7}{-32} = 2.6562$$

$$U_{33} = A_{33} - (L_{31} \cdot U_{13} + L_{32} \cdot U_{23}) = -15.4$$

$$L_{33} = \frac{A_{33} - (L_{31} \cdot U_{13} + L_{32} \cdot U_{23})}{U_{33}} = 1$$ by construction

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 12 & 2.6562 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 7 & -4 \\ 0 & -32 & 25 \\ 0 & 0 & -15.4 \end{bmatrix}$$

solve: $Ly = b$

$\longrightarrow$ solve: $Ux = y$

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$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -51 \\ 4 & 1 & 0 & 62 \\ 12 & 2.6562 & 1 & 8 \end{array}\right] \xrightarrow{\text{Matlab}} y = \begin{bmatrix} -51 \\ 266 \\ -86.5492 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 7 & -4 & -51 \\ 0 & -32 & 25 & 266 \\ 0 & 0 & -15.4 & -86.5492 \end{array}\right] \xrightarrow{\text{Matlab}} X = \begin{bmatrix} 1.0670 \\ -3.9218 \\ 5.6201 \end{bmatrix}$$

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3.

$$\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 4 \\ 14 \end{bmatrix}$$

THoMS:

$$\begin{bmatrix} d_1 & c_1 & & \\ a_2 & d_2 & c_2 & \\ & & \ddots & \\ & & a_n & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

forward:

$$\begin{bmatrix} 1 & c'_1 & & \\ & \ddots & \ddots & \\ & & & c'_{n-1} \\ & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b'_1 \\ \vdots \\ b'_n \end{bmatrix}$$

backward:

$$x_n = b'_n$$

$$x_{n-1} = b'_{n-1} - c'_{n-1} x_n$$

$$\vdots$$

$$x_1 = b'_1 - c'_1 x_2$$

Forward Math:

$$c_1' = \frac{c_1}{d_1} \; ; \; b_1' = \frac{b_1}{d_1}$$

$$c_i' = \frac{c_i}{d_i - a_i c_{i-1}'} \qquad b_i' = \frac{b_i - a_i \cdot b_{i-1}'}{d_i - a_i c_{i-1}'}$$

$$b_n' = \frac{b_n - a_n \cdot b_{n-1}'}{d_n - a_n c_{n-1}'}$$

Implement:

$$\Longrightarrow \begin{bmatrix} \overset{d_1}{2} & \overset{c_1}{-1} & \\ \overset{a_2}{-1} & \overset{d_2}{2} & \overset{c_2}{-1} \\ & \overset{a_3}{-1} & \overset{d_3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \overset{b_1}{124} \\ \overset{b_2}{4} \\ \overset{b_3}{14} \end{bmatrix}$$

$$c_1' = \frac{c_1}{d_1} = -\frac{1}{2} \qquad b_1' = \frac{b_1}{d_1} = 62$$

$$c_2' = \frac{c_2}{d_2 - a_2 \cdot c_1'} = -\frac{2}{3} \qquad b_2' = \frac{b_2 - a_2 \cdot b_1'}{d_2 - a_2 c_1'} = 44$$

$$b_3' = \frac{b_3 - a_3 \cdot b_2'}{d_3 - a_3 \cdot c_2'} = 43.5$$

$$x_3 = b_3' = 43.5$$

$$x_2 = b_2' - c_2' \cdot x_3 = 44 + \frac{2}{3} \cdot 43.5 = 73$$

$$x_1 = b_1' - c_1' x_2 = 62 + \frac{1}{2} \cdot 73 = 98.5$$

$$x = \begin{bmatrix} 98.5 \\ 72 \\ 43.5 \end{bmatrix}$$