Technion – Israel Institute of Technology

HW6

Numerical Methods

019003

Alon Spinner	305184335	alonspinner@gmail.com
Oren Elmakis	311265516	orenelmakis@gmail.com

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Question 1

We were asked to solve with five optimization methods three different test functions: Rosenbrock's banana, Easom, and Eggholder. In each method we required to implement five runs with different initial conditions.

1.

The first problem we will discuss about is the Rosenbrock's banana function:

$$f(x, y) = (2 - x)^2 + 10(y - x^2)^2$$

Rosenbrock's banana function is a known non-convex test function in optimization. The challenging part of this function, besides being non-convex, is the narrow valley that consists of the function's optimum.

$$x = 2$$

$$y = 4$$

The following tables summarize the results for five runs of each method.

Steepest Descent:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	602	5415	3.23e-04	1.67e-05	0.6686	-0.31	1.96	3.85	0.0013
2	328	2949	2.98e-04	1.46e-05	0.0385	-0.29	1.97	3.91	4.88e-04
3	407	3663	2.96e-04	1.44e-05	-0.126	1.27	1.97	3.9	5.35e-04
4	452	4077	2.92e-04	1.41e-05	2.92	1.22	1.97	3.9	5.14e-04
5	595	5427	3.21e-04	1.66e-05	-2.33	2.83	1.96	3.85	0.0013

Quasi-Newton:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	16	63	9.99e-06	6.315e-09	2.49	2.16	2	4	2.48e-10
2	28	108	6.58e-04	3.04e-08	-2.79	5.48	2	4	1.58e-10
3	20	78	3.196e-04	2.55e-08	2.58	0.59	2	4	4.35e-11
4	19	72	9.41e-04	1.28e-06	2.83	1.59	2	4	8.05e-11
5	19	72	1.04e-04	4.04e-08	-0.08	1.58	2	4	5.65e-11

Downhill-simplex:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	67	129	7.124e-05	1.74e-10	0.30	0.53	2	4	2.65e-11

2	50	92	8.24e-05	3.11e-11	1.20	1.56	2	4	1.02e-10
3	79	144	3.355e-05	8.93e-11	-1.72	-0.08	2	4	1.58e-10
4	57	111	7.60e-05	3.19e-10	1.87	1.94	2	4	1.1e-11
5	48	93	8.62e-05	3.56e-10	2.79	3.76	2	4	6.97e-11

Genetic algorithm:

	Generations	Evaluations	Change of the mean function value in the last iteration	Change of best function value in the last iteration	X Init best	Y Init best	X Min	Y Min	Function minimum
1	86	4095	2.7e6	0	-7.9	-6.1	1.36	2.19	1.5148
2	97	4612	1.53e06	0	2.36	-3.39	1.78	3.22	0.050
3	82	3907	4.9e4	0	-9.92	-2.92	2.95	8.61	1.00
4	83	3954	4.44e5	0	-7.96	-6.07	1.61	2.56	0.179
5	73	3484	6.41e6	0	6.50	9.57	2.79	7.83	0.636

Simulated Annealing:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	1329	1338	0.201	0	-2.30	5.09	2.21	5.11	0.5108
2	1450	1459	2.03e-05	0	-1.77	3.61	2.35	5.52	0.123
3	1202	1211	0	0	2.98	-0.86	1.85	3.43	0.0213
4	4373	4400	0	0	0.7178	4.373	4.16	17.68	5.996
5	1153	1160	5.73e-05	0	1.413	4.08	1.03	0.99	0.9908

Performance analysis:

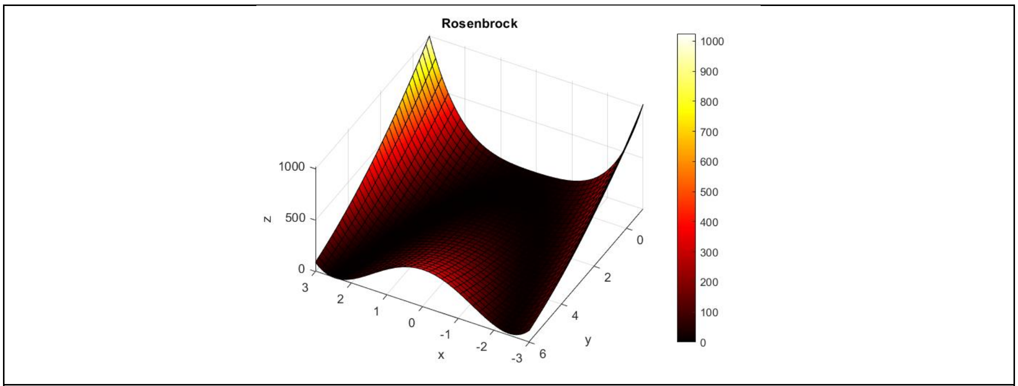

Fig. 1 - Rosenbrock's banana

we tested each method five times from random initial locations. The newton method got the best results in convergence time and value from the five methods we examined.

- The convergence zone of Rosenbrock's function is a narrow valley with a shallow gradient shown in Fig. 1 as a black area. The quasi-newton that uses the second derivative presents fast convergence to the minima of the function. In contrast, the steepest descent algorithm shows a slow convergence rate. This is because of the shallow and in-directed gradient of the function.

- The downhill-simplex uses local search based on three points to infer the "motion" of the simplex. This local relative gradient search allows the approach to reach the minima in batter performance than the steepest-decent, which uses a point-based gradient. However, the Quasi-newton still presents a better convergence rate using the second derivative.
- The Genetic algorithm provides poor performance in relative the gradient approaches and the direct search Downhill-simplex. The narrow valley of Rosenbrock's function is causing difficulty finding the minima by searching globally in a stochastic approach without utilizing any gradient data.
- The Simulated Annealing uses the stochastic search of random points and adapts the step size based on the temperature state. The search for the minima with random step size leads the algorithm to miss the narrow "valley" of Rosenbrock's function and the minima point.

2.

The second problem we will discuss is the Easom function:

$$f(x,y) = -\cos(x)\cdot\cos(y)\cdot e^{-\left((x-\pi)^2+(y-\pi)^2\right)}$$

The Easom function is flat with a small convex area close to the minima point.

The following tables summarize the results for five runs of each method.

Steepest Descent:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	3	36	0.0016	4.02e-06	2.05	4.36	3.14	3.14	-1
2	4	33	1.02e-05	1.58e-10	1.72	3.73	3.14	3.14	-1
3	15	165	5.45e-04	3.07e-09	0.316	4.77	1.30	4.98	-8.1e-05
4	4	27	1.25e-06	2.37e-12	4.43	3.87	3.14	3.14	-1
5	33	300	2.45e-04	6.21e-10	4.93	1.54	4.97	1.30	-8.11e-05

Quasi-Newton:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	5	33	3.46e-05	1.78e-09	2.30	2.07	3.141	3.141	-1
2	26	165	1.788e-05	1.6699e-10	0.747	4.074	2.4364	2.8807	-0.4180
3	14	84	4.96e-05	3.326-09	4.86	2.46	3.141	3.141	-1
4	6	21	3.136e-04	1.151e-07	3.32	3.88	3.141	3.141	-1
5	6	36	2.99e-06	1.31e-11	1.80	2.96	3.141	3.141	-1

Downhill-simplex:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	34	65	1.23e-04	7.344e-10	3.32	3.71	3.141	3.141	-1
2	51	99	0	0	2.47	0.56	1.3	1.3	-8.11e-05
3	30	61	6.129e-04	6.11e-10	2..77	3.76	3.141	3.141	-1
4	58	112	0	0	0.034	0.88	1.30	1.30	-8.11e-05
5	29	58	5.80e-05	6.62e-10	2.8	3.33	3.141	3.141	-1

Genetic algorithm:

	Generations	Evaluations	Change of the mean function value in the last iteration	Change of best function value in the last iteration	X Init best	Y Init best	X Min	Y Min	Function minimum
1	94	4471	0.0591	0	1.028	3.49	3.05	3.183	-0.989
2	127	6022	0.0196	0	-5.61	-5.22	3.25	3.14	-0.979
3	129	6116	0.0049	0	4.64	-9.68	3.17	3.29	-0.964
4	69	3296	0.0212	0	6.27	8.71	3.128	3.07	-0.9921
5	102	4847	0.0718	0	2.24	-4.72	3.13	3.09	-0.9965

Simulated Annealing:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	1000	1011	0	0	1.85	1.97	1.85	1.97	-0.0056
2	1001	1010	0.0340	0	0.96	2.52	38.79	95.08	0
3	1001	1010	0	0	4.00	1.37	84.69	60.44	0
4	1000	1011	0	0	0.0092	4.60	-0.80	-19.34	-8e-02
5	1000	1011	21.78	0	1.1177	4.64	-85.67	-45.01	0

Performance analysis:

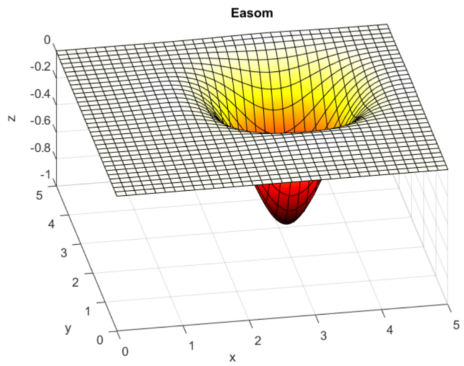

Fig. 2 - Easom function

we tested each method five times from random initial locations. From the five methods, we examined, in convergence success, the best results using the Genetic algorithm. However, the genetic algorithm requires more time to converge relative to the gradients approaches.

- The Easom function shape characteristics are a flat search area with shallow gradients and a small narrow convex area close to the minima. When the initial guess is found outside the proximity of the minima area, the steepest descent and the newton method fail to follow the gradients and locate the minima.
- Similar to the gradient approaches, the simplex uses local gradients and fails to follow the relative gradients when the initial guess is out of the minima proximity.
- The genetic algorithm uses a stochastic global search to find the minima. The Easom function difficulty is finding the minima proximity, while the convergence area is a simple convex zone. Therefore the genetic algorithm successfully located the minima of the function by searching globally with a population of fifty genes that scan simultaneously fifty different points each iteration.
- The Simulated Annealing failed to reach the convergence area and find the minima. The reason is again the random search of the function, which is not dependent on the function shape.

3.

The third problem we will discuss is the Eggholder function:

$$f(x,y) = -(y+10)\cdot\sin\left(\sqrt{\left|\frac{x}{2}+(y+10)\right|}\right) - x\cdot\sin\left(\sqrt{|x-(y+10)|}\right)$$

The Eggholder is a test function used to examine optimization approaches in problems with many local minima.

The following tables summarize the results for five runs of each method.

Steepest Descent:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	21	84	0.00137	1.58e-04	407.36	48.77	442.92	12,66	-443.49
2	13	60	0.0084	5.93e-05	452.89	139.24	566.86	138.80	-705.407

3	24	81	0.0154	2.139e-04	63.493	273.44	150.177	338.288	-493.96
4	21	102	0.0358	0.0018	456.68	478.75	439.64	491.14	-935.33
5	27	87	0.0379	0.0013	316.17	482.44	347.21	536.32	-888.9469

Quasi-Newton:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	9	33	0.030	8.82e-05	481.43	424.82	522.16	450.31	-976.911
2	5	21	0.0031	9.38e-07	407.90	461.15	439.481	4990.97	-935.338
3	6	36	0.0042	2.4e-06	262.25	263.56	284.95	272.47	-562.127
4	6	39	0.0027	1.32e-05	75.92	46.76	47.99	35.44	-88.30
5	9	42	0.006	2.34e-05	202.22	422.24	242.03	293.379	-540.426

Downhill-simplex:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	47	88	1.16e-04	7.95e-11	355.60	16.75	442.18	12.78	-443.497
2	51	98	1.06e-04	2.1e-10	340.22	201.51	418.56	208.04	-629.633
3	61	118	6.1e-05	2.88e-10	74.35	0.22	91.40	18.56	-111.7888
4	45	86	6.58e-05	1.25e-11	407.952	226.154	418.56	208.04	-629.633
5	52	99	8.89e-05	5.68e-10	242.84	441.29	347.32	536.4154	-888.9491

Genetic algorithm:

	Generations	Evaluations	Change of the mean function value in the last iteration	Change of best function value in the last iteration	X Init best	Y Init best	X Min	Y Min	Function minimum
1	86	4095	4.481	0	1.22	8.95	122.38	-93.51	-203.15
2	73	3484	3.256	0	3.49	-4.60	90.209	17.23	-111.569
3	126	5975	8.609	0	9.73	8.28	181.57	109.41	-293.2017
4	123	5834	4.112	0	-9.31	-7.23	181.32	109.05	-293.2619
5	58	2779	3.075	0	7.22	0.45	-46.68	75	-126.4128

Simulated Annealing:

	Iterations	Evaluations	Last step size	Change of function value in the last	X Init	Y Init	X Min	Y Min	Function minimum
1	4280	4317	0	0	219.28	410.52	522.16	450.31	-976.911
2	2466	2493	5.415	0	222.15	72.30	150.22	338.29	-493.96
3	1679	1696	0.0074	0	102.00	491.65	150.23	338.31	-493.96
4	1147	1160	0.877	0	94.00	422.09	-105.88	460.15	-565.997
5	3149	3180	0	0	52.09	61.91	180.83	108.66	-293.29

Performance analysis:

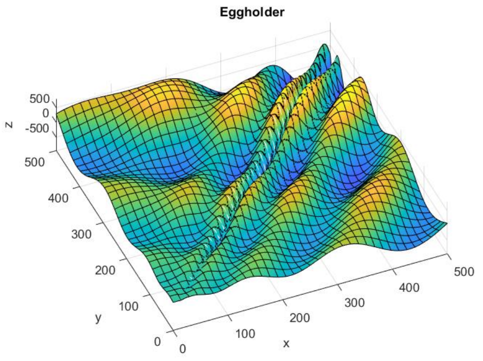

Fig. 3 - Easom function

we tested each method five times from random initial locations.

- As can be seen in the results, the algorithms could not avoid falling to local minima, and the convergence point was dependent on the initial position.
- Similarly, as with the gradient-based approaches, the algorithm got to the close minima and depended on the initial position.
- Because of the multiple minima points found in the function, the genetic algorithm is falling to arbitrary one of the minima points.
- The simulated annealing falls to arbitrary minima points because of the function's random search and multiple minima points.

To conclude, in the Eggholder function all the algorithms couldn't ensure convergence to the global minima point. Because of the multiple minima points found in the function and without prior knowledge, the optimization algorithms are fail to reach the global minima.