Technion – Israel Institute of Technology

HW5

Vision Aided Navigation

086761

Alon Spinner	305184335	alonspinner@gmail.com
Sher Hazan	308026467	sherhazan@campus.technion.ac.il

December 22, 2021

Question 1: Factor graph, variable elimination and Bayes net.
Consider a SAM problem where a robot travels through an unknown environment and captures observations using its onboard sensors. Assume the robot starts at time t_0, with a known prior $p(x)$ and consider motion and observation models $p(x_k|x_{k-1},u_{k-1})$ and $p(z_{k,i}|x_k,l_i)$, respectively, where l_i denotes the i^{th} landmark. The robot moves according to given controls and observes a single landmark at time instances t_1 and t_2.

a: Write the joint pdf corresponding to the above scenario until time t_4: $p(x_{0:4},l|u_{0:3},z_1,z_2)$

$$p(x_k|x_{k-1},u_{k-1}) \sim motion\ model$$

$$p(z_k|x_k,l_i) \sim measurement\ model$$

$$p(x_{0:4},l|u_{0:3},z_1,z_2) \underbrace{=}_{\substack{cond.\\+indep.}} p(x_4|x_3u_3)p(x_3|x_2,u_2)\cdot \underbrace{p(x_{0:2},l|u_{0:1},z_{1:2})}_{known\ structure} =$$

$$= p(x_4|x_3u_3)p(x_3|x_2,u_2)\cdot p(x_0)\prod_{i=1}^{2}\eta_i p(z_i|x_i,l)p(x_i|u_{i-1},x_{i-1}) =$$

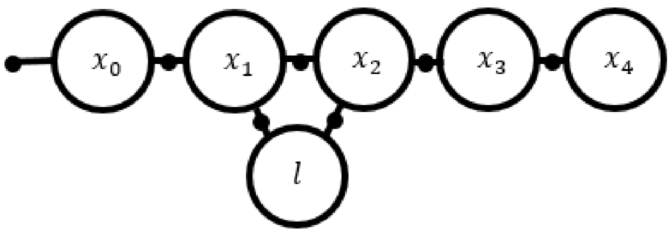
$$\left\{\eta = \prod_{i=1}^{2}\eta_i:\ not\ a\ function\ of\ x\ or\ l, the\ varibles\ we\ optimize\ on\right\}$$

$$= \eta\cdot p(x_4|x_3u_3)p(x_3|x_2,u_2)\cdot p(x_0)\prod_{i=1}^{2}p(z_i|x_i,l)p(x_i|u_{i-1},x_{i-1})$$

b: Draw the corresponding factor graph.

$$p(x_{0:4},l|u_{0:3},z_{1:2}) = \eta\cdot p(x_0)p(x_1|x_0,u_0)p(x_2|x_1,u_1)p(x_3|x_2,u_2)p(x_4|x_3,u_3)p(z_1|x_1,l)p(z_2|x_2,l)$$

$$\propto f_0(x_0)f_1(x_0,x_1)f_2(x_1,x_2)f_3(x_2,x_3)f_4(x_3,x_4)f_{l1}(x_1,l)f_{l2}(x_2,l)$$



c: Eliminate the factor graph into a Bayes net, assuming elimination order:

$x_0, x_1, x_2, x_3, x_4, l$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) = \eta \cdot p(x_0)p(x_1|x_0, u_0)p(x_2|x_1, u_1)p(x_3|x_2, u_2)p(x_4|x_3, u_3)p(z_1|x_1, l)p(z_2|x_2, l)$$

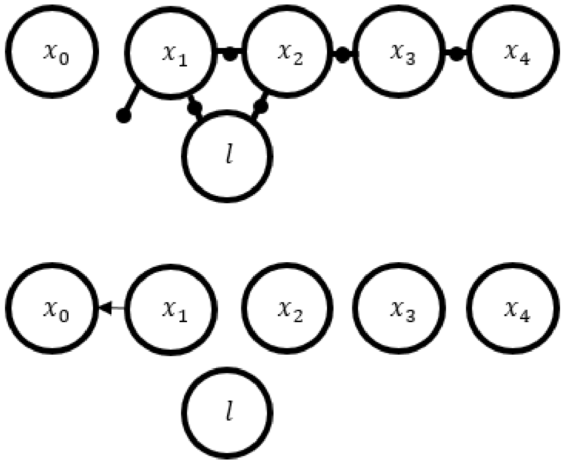
$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_{l1}(x_1, l)f_{l2}(x_2, l)$$

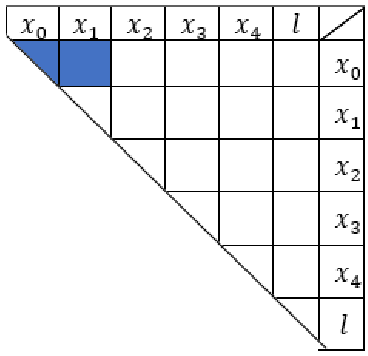
$\underline{Elinination\ of\ x_0}$:

$$f_{joint}(x_0, x_1) = f_0(x_0)f_1(x_0, x_1) \propto p(x_0|x_1) \cdot f_{1-new}(x_1)$$

$After\ the\ elimination\ of\ x_0\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) \cdot f_{1-new}(x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_{l1}(x_1, l)f_{l2}(x_2, l)$$



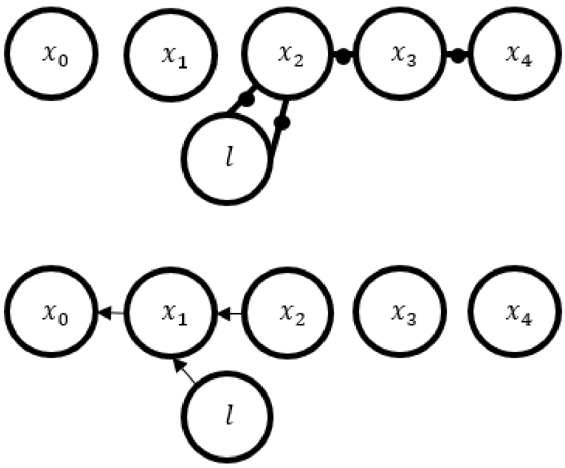


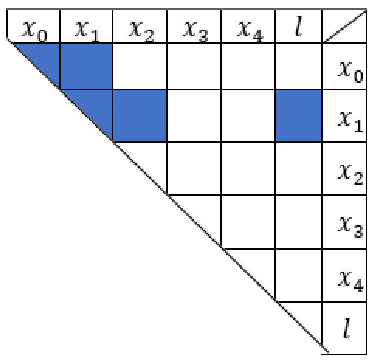
$\underline{Elinination\ of\ x_1}$:

$$f_{joint}(x_1, x_2, l) = f_{1-new}(x_1)f_2(x_1, x_2)f_{l1}(x_1, l) \propto p(x_1|x_2, l) \cdot f_{2-new}(x_2, l)$$

$After\ the\ elimination\ of\ x_1\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2, l) \cdot f_{2-new}(x_2, l)f_3(x_2, x_3)f_4(x_3, x_4)f_{l2}(x_2, l)$$



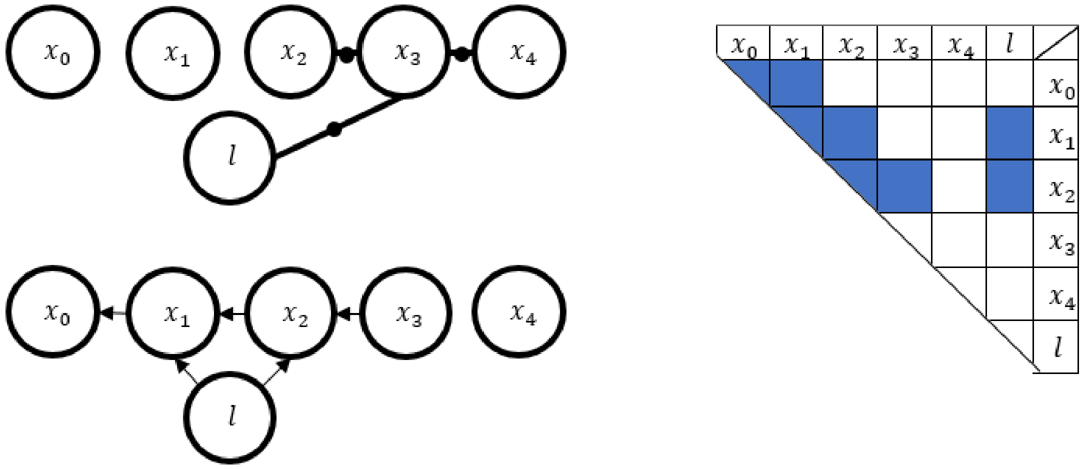


$\underline{Elinination\ of\ x_2}$:

$$f_{joint}(x_2, x_3, l) = f_{2-new}(x_2, l) f_3(x_2, x_3) f_{l2}(x_2, l) \propto p(x_2|x_3, l) \cdot f_{3-new}(x_3, l)$$

$After\ the\ elimination\ of\ x_2\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) p(x_1|x_2, l) p(x_2|x_3, l) \cdot f_{3-new}(x_3, l) f_4(x_3, x_4)$$

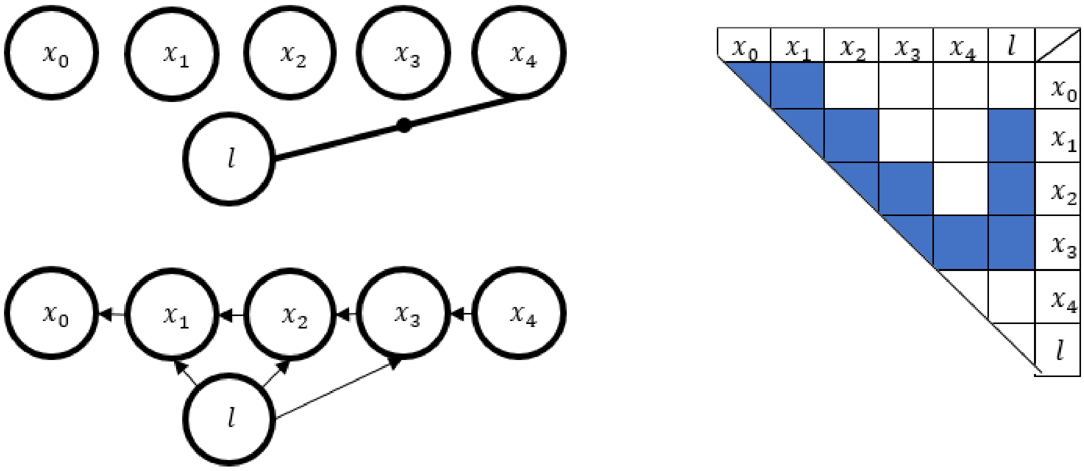


$\underline{Elinination\ of\ x_3}$:

$$f_{joint}(x_3, x_4, l) = f_{3-new}(x_3, l) f_4(x_3, x_4) \propto p(x_3|x_4, l) \cdot f_{4-new}(x_4, l)$$

$After\ the\ elimination\ of\ x_3\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) p(x_1|x_2, l) p(x_3|x_4, l) \cdot f_{4-new}(x_4, l)$$

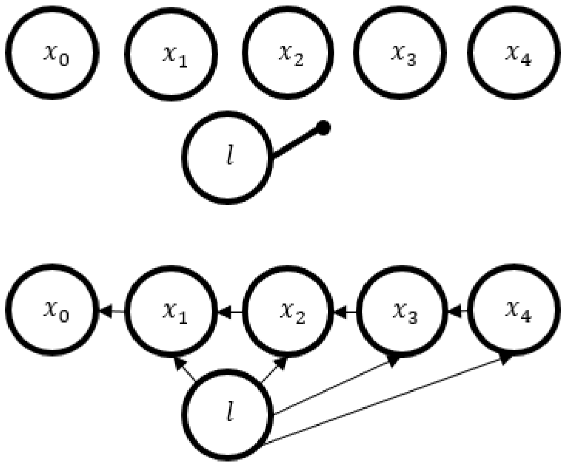


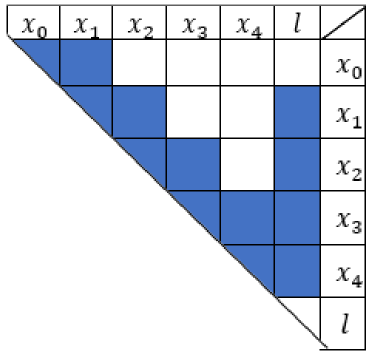
$\underline{Elinination\ of\ x_4}$:

$$f_{joint}(x_4, l) = f_{4-new}(x_4, l) \propto p(x_4|l) \cdot f_{l-new}(l)$$

$After\ the\ elimination\ of\ x_4\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_2|x_3, l)p(x_4|l) \cdot f_{l-new}(l)$$



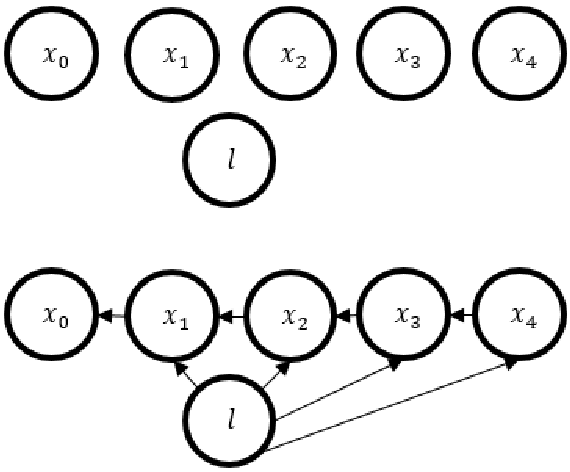


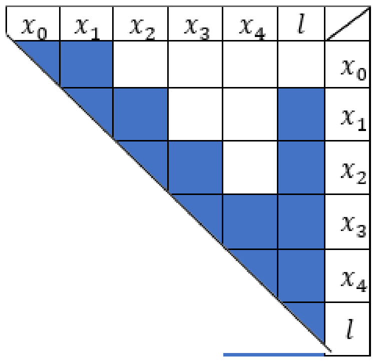
$\underline{Elinination\ of\ l}$:

$$f_{joint}(l) = f_{l-new}(l) \propto p(l)$$

$After\ the\ elimination\ of\ l\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_3|x_4, l)p(l)$$





d: Repeat the previous clause using a different variable elimination order: $x_4, x_3, x_2, l, x_1, x_0$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) = \eta \cdot p(x_0)p(x_1|x_0, u_0)p(x_2|x_1, u_1)p(x_3|x_2, u_2)p(x_4|x_3, u_3)p(z_1|x_1, l)p(z_2|x_2, l)$$

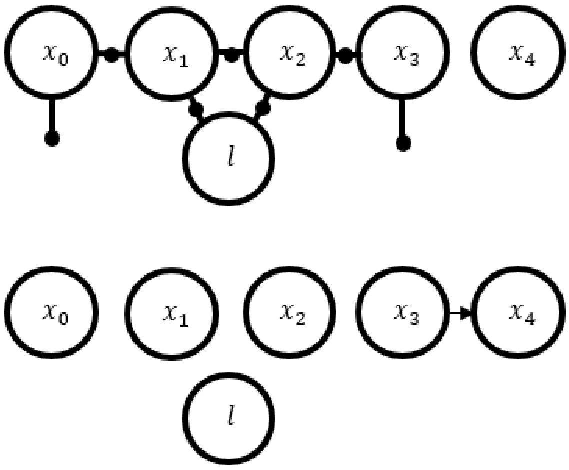
$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_{l1}(x_1, l)f_{l2}(x_2, l)$$

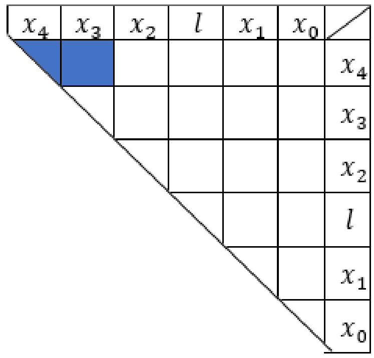
$\underline{Elinination\ of\ x_4}$:

$$f_{joint}(x_3, x_4) = f_4(x_3, x_4) \propto p(x_4|x_3) \cdot f_{3-new}(x_3)$$

$After\ the\ elimination\ of\ x_4\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3) \cdot f_{3-new}(x_3)f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_{l1}(x_1, l)f_{l2}(x_2, l)$$



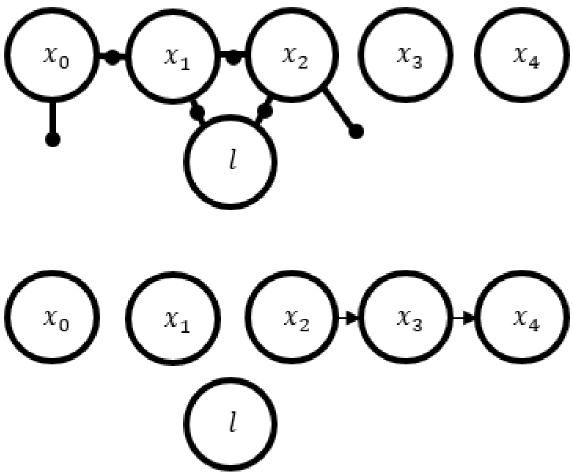


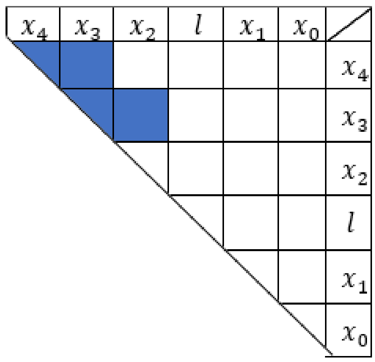
$\underline{Elinination\ of\ x_3}$:

$$f_{joint}(x_2, x_3) = f_{3-new}(x_3)f_3(x_2, x_3) \propto p(x_3|x_2) \cdot f_{2-new}(x_2)$$

$After\ the\ elimination\ of\ x_3\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2) \cdot f_{2-new}(x_2)f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_{l1}(x_1, l)f_{l2}(x_2, l)$$



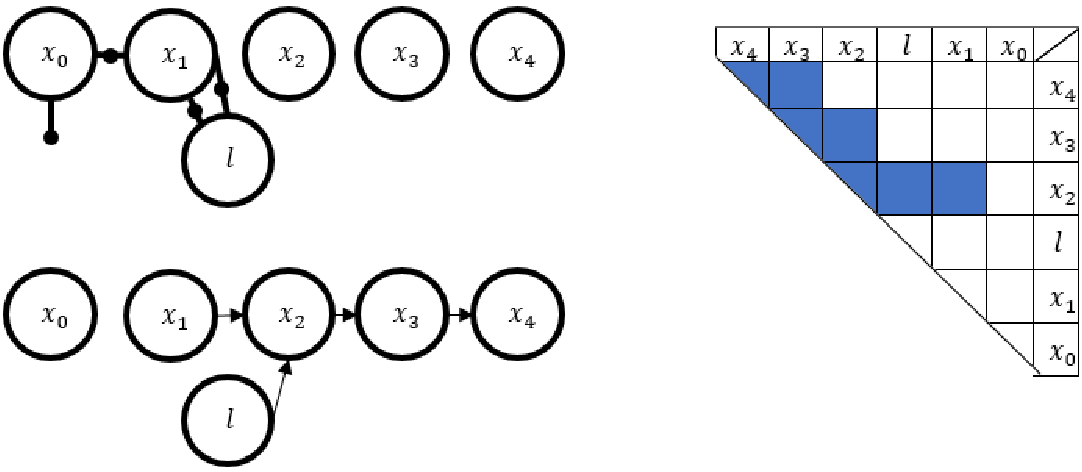


Elinination of x_2:

$$f_{joint}(x_1, x_2, l) = f_{2-new}(x_2) f_2(x_1, x_2) f_{l2}(x_2, l) \propto p(x_2|x_1, l) \cdot f_{l-new}(x_1, l)$$

After the elimination of x_2 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3) p(x_3|x_2) p(x_2|x_1, l) \cdot f_{l-new}(x_1, l) f_0(x_0) f_1(x_0, x_1) f_{l1}(x_1, l)$$

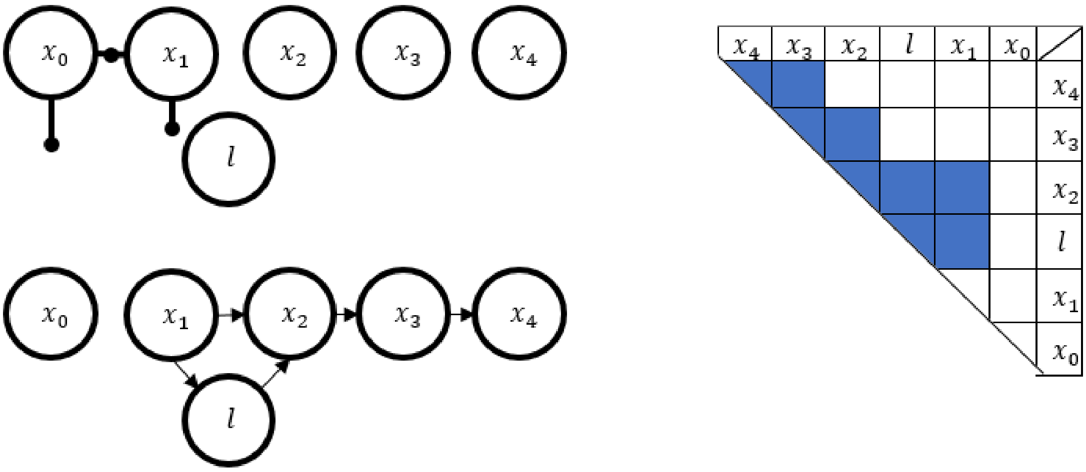


Elinination of l:

$$f_{joint}(x_1, l) = f_{l-new}(x_1, l) f_{l1}(x_1, l) \propto p(l|x_1) \cdot f_{1-new}(x_1)$$

After the elimination of x_3 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3) p(x_3|x_2) p(l|x_1) \cdot f_{1-new}(x_1) f_0(x_0) f_1(x_0, x_1)$$

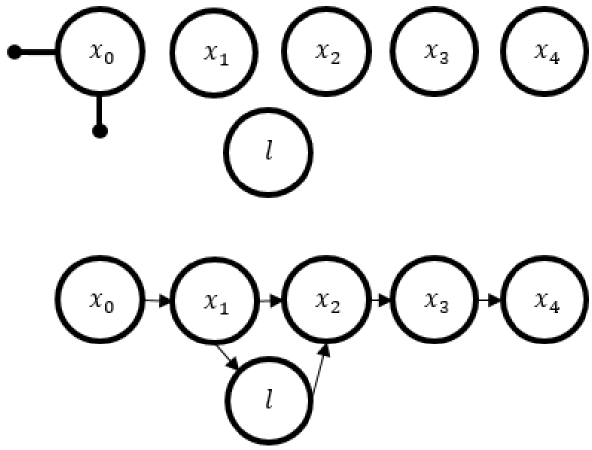


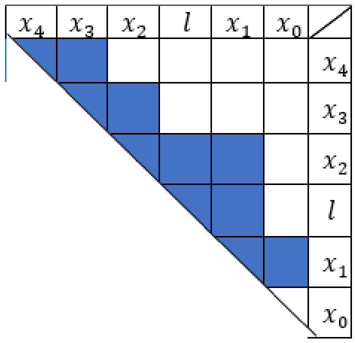
Elinination of x_1:

$$f_{joint}(x_0, x_1) = f_{1-new}(x_1) f_1(x_0, x_1) \propto p(x_1|x_0) \cdot f_{0-new}(x_0)$$

After the elimination of x_1 *we get*:

$$p(x_{0:4}, l | u_{0:3}, z_{1:2}) \propto p(x_4|x_3) p(x_3|x_2) p(l|x_1) p(x_1|x_0) \cdot f_{0-new}(x_0) f_0(x_0)$$



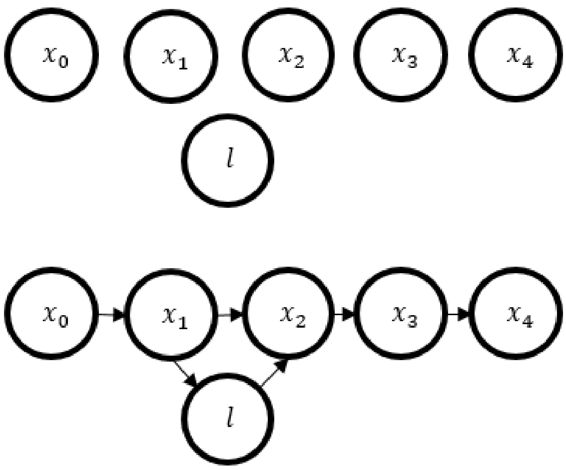


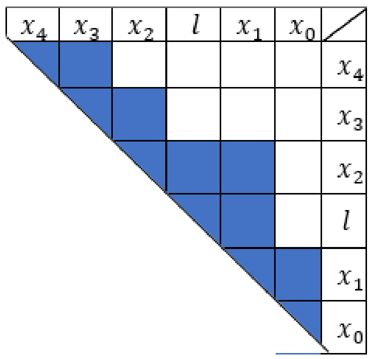
Elinination of x_0:

$$f_{joint}(x_0) = f_{0-new}(x_0) f_0(x_0) \propto p(x_0)$$

After the elimination of x_0 *we get*:

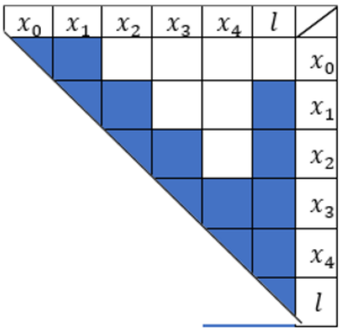
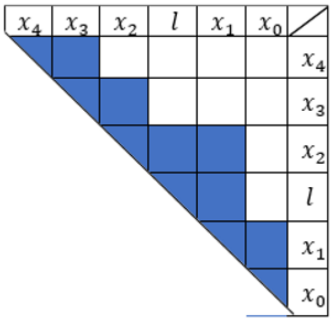
$$p(x_{0:4}, l | u_{0:3}, z_{1:2}) \propto p(x_4|x_3) p(x_3|x_2) p(l|x_1) p(x_1|x_0) p(x_0)$$





e: Which of the two elimination orders you would prefer in terms of estimation accuracy and computational aspects?

We show the elimination order and resulting R matrix for each section below.

{c} Elimination Order: $\{x_0, x_1, x_2, x_3, x_4, l\}$	{d} Elimination Order: $\{x_4, x_3, x_2, l, x_1, x_0\}$
	

Regarding computation efficiency: We would prefer the elimination order in {d}, as it produces a sparser R matrix (12 non-zero elements vs 14), and with more structure – all rows but one contains two elements at the start of the row.

Regarding Accuracy: Both matrices contain the same information. As such, the solution of the LMS problem is independent of the elimination order, or which R matrix we choose to use.

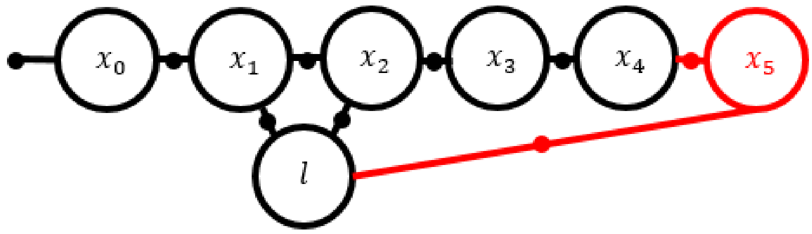
$$R_d^T R_d = R_c^T R_c = A^T A$$

Question 2: Incremental factorization.

Consider now the robot, from question 1, executes command u_4 and moves to a new location; denote its new pose by x_5. Assume the robot observes again the landmark l from the new location.

> a: Draw the factor graph of the problem and indicate the new factors and variable nodes.

We add one node for pose x_5, and two additional nodes for the factors that need to be computed: one for the motion model, and one for the measurement model.



b: Consider the Bayes net from question 1(c) with elimination order $x_0, x_1, x_2, x_3, x_4, l$. Perform incremental factorization by updating this Bayes net with the new information using the elimination order:

$$x_0, x_1, x_2, x_3, x_4, l, x_5$$

$p(x_{0:4}, x_5, l | u_{0:3}, z_{1:2}, z_3) = \eta \cdot p(x_0)p(x_1|x_0, u_0)p(x_2|x_1, u_1)p(x_3|x_2, u_2)p(x_4|x_3, u_3)p(x_5|x_4, u_4)p(z_1|x_1, l)p(z_2|x_2, l)\ p(z_3|x_5, l)$

$p(x_{0:4}, x_5, l | u_{0:3}, z_{1:2}, z_3) \propto f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_5(x_4, x_5)f_{l1}(x_1, l)f_{l2}(x_2, l)f_{l3}(x_5, l)$

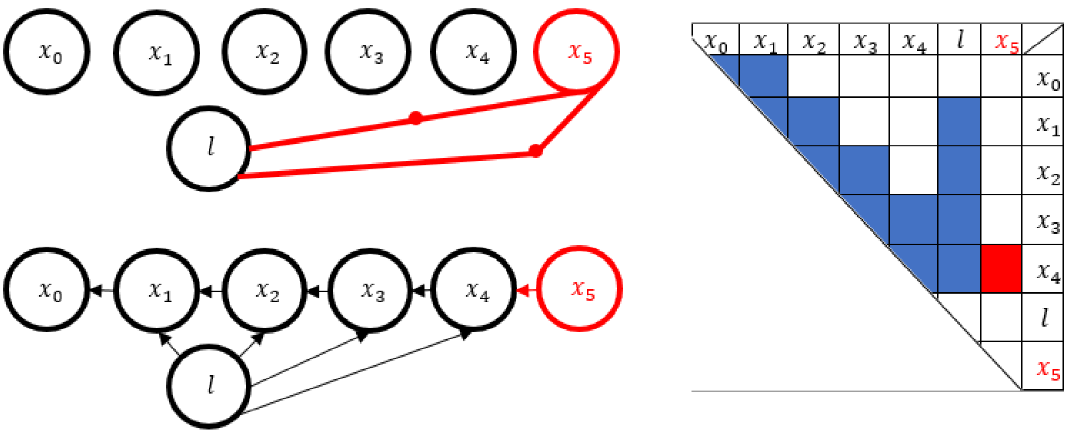
We need to reeliminate the factors x_4 and l because they involve the changes.

<u>*Back to Elimination of x_4:*</u>

$$f_{joint}(x_4, x_5, l) = f_{4-new}(x_4, l)f_5(x_4, x_5) \propto p(x_4|x_5, l) \cdot f_{l-new}(x_5, l)$$

After the elimination of x_4 we get:

$p(x_{0:4}, x_5, l | u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_2|x_3, l)p(x_4|x_5, l) \cdot f_{l3}(x_5, l)f_{l-new}(x_5, l)$

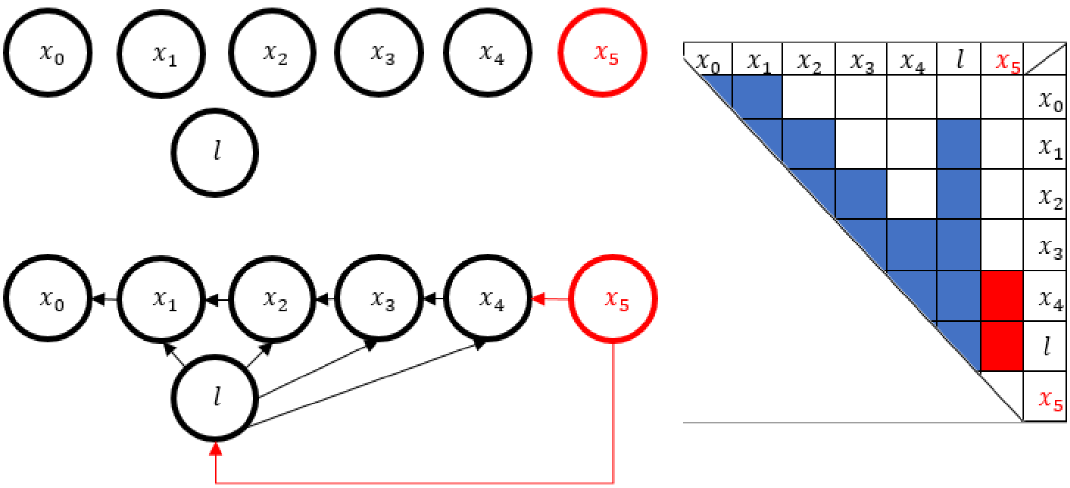


<u>*Back to Elinination of l:*</u>

$$f_{joint}(x_5, l) = f_{l-new}(x_5, l)f_{l3}(x_5, l) \propto p(l|x_5) \cdot f_{5-new}(x_5)$$

After the elimination of l we get:

$$p(x_{0:4}, x_5, l | u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_3|x_4, l)p(l|x_5) \cdot f_{5-new}(x_5)$$

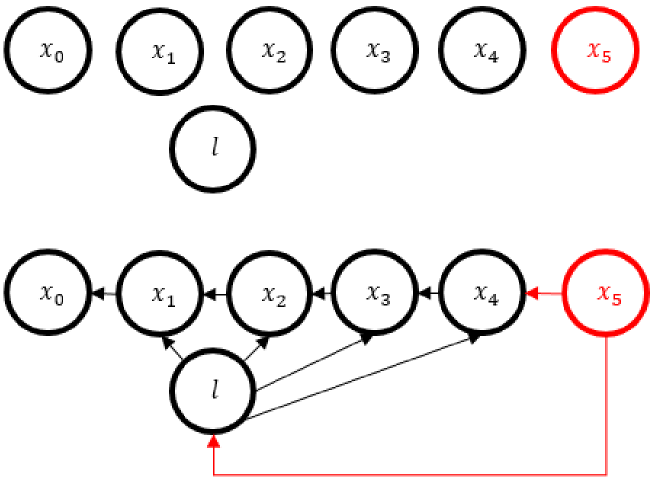


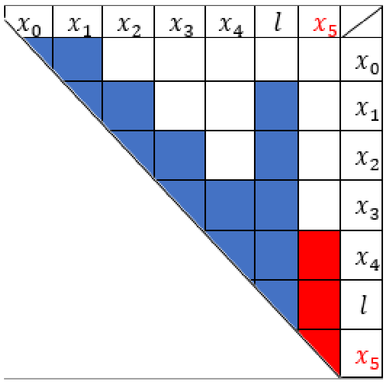
Elinination of x_5:

$$f_{joint}(x_5) = f_{5-new}(x_5) \propto p(x_5)$$

After the elimination of x_5 we get:

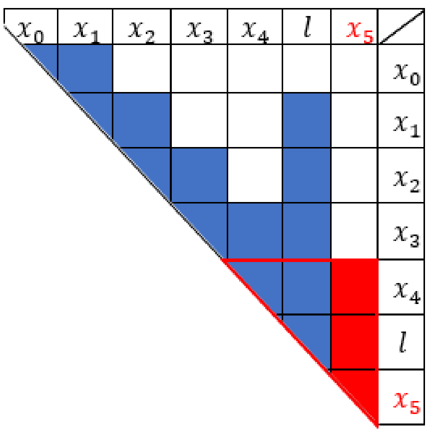
$$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_3|x_4, l)p(l|x_5)p(x_5)$$

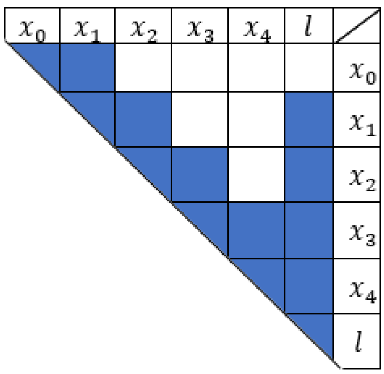




c: Show the corresponding updated square root information matrix R

The new non-zero elements, the boxes colored red in the new R matrix, describe states that depend on x_5 in the bayes-net graph.
In general, values in the red triangle are in the "update impact zone" and are subject to change when computing the new R matrix.



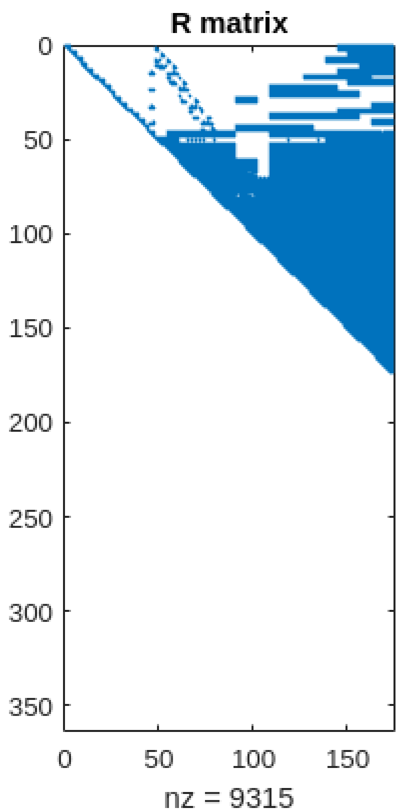


Q3 - Variable Ordering

```
load('hw5_A.mat'); %produces A matrix in the workspace
```

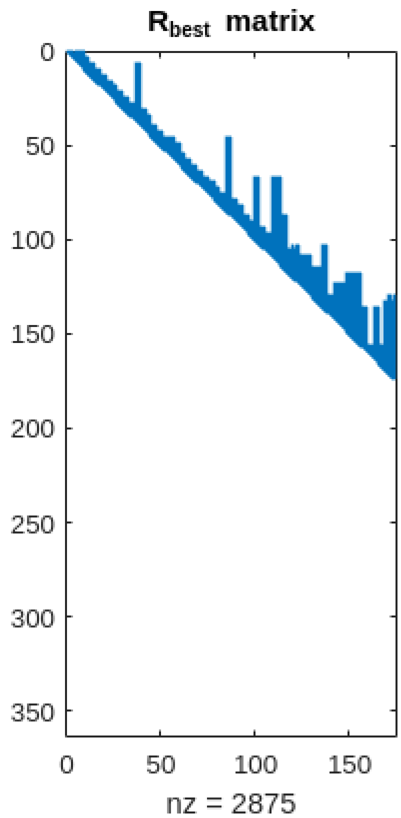
(a) Obtain R with QR fatorization

```
[~,R] = qr(A);
spy(R);
title('R matrix');
```



(b) best R

```
p = colamd(A);
[~,Rbest] = qr(A(:,p));
spy(Rbest);
title('R_{best} matrix');
```



R_{best} has ~30% values of the initial R matrix.

Hence will require to do only ~30% of the computations when sovling back subsitution with R_{best} when compared to R.