Technion – Israel Institute of Technology

HW4

Vision Aided Navigation

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Pose SLAM

Question 1: Assume you are given a prior on initial pose $p(x_0) = N(\hat{x}_0, \Sigma_0)$. Additionally, suppose the robot obtains noisy odometry (relative pose) measurements $z_k = x_{k+1} \ominus x_k + v_k$, with $v_k \sim N(0, \Sigma_v)$. Express the joint posterior $p(x_{0:k+1}|z_{0:k})$ as a product of the form $p(z_k|x_k, x_{k+1})$ and the prior $p(x_0)$ probabilistic terms. We shall call the probabilistic terms comprising the joint posterior factors.

$$p(x_{0:k+1}|z_{0:k}) \underbrace{=}_{Bayes} \frac{p(z_k|x_{0:k+1}, z_{0:k-1}) \cdot p(x_{0:k+1}|z_{0:k-1})}{p(z_k|z_{0:k-1})} \underbrace{=}_{CR} \frac{p(z_k|x_{0:k+1}, z_{0:k-1}) \cdot p(x_{k+1}|x_{0:k}, z_{0:k-1}) \cdot p(x_{0:k}|z_{0:k-1})}{p(z_k|z_{0:k-1})}$$

$Markov\ assumption$:

$$p(x_{k+1}|x_{0:k}, z_{0:k-1}) = p(x_{k+1}|x_k)$$

$$p(z_k|x_{0:k}, z_{0:k}) = p(z_k|x_{k+1}, x_k)$$

$Define\ Normalizer$:

$$p(z_k|z_{0:k-1}) = \frac{1}{\eta_k}$$

$$p(x_{0:k+1}|z_{0:k-1}) = \eta_k \cdot p(z_k|x_k, x_{k+1}) \cdot p(x_{k+1}|x_k) \cdot p(x_{0:k}|z_{0:k-1})$$

$Given\ no\ motion\ model\ we\ will\ assume\ p(x_{i+1}|x_i)\ is\ distributed\ uniformaly\ across\ the\ space$:

$$p(x_{i+1}|x_i) = const$$

$define\ \ \eta'_k = \eta_k \cdot const$:

$$p(x_{0:k+1}|z_{0:k-1}) = \eta'_k \cdot const \cdot p(x_{0:k}|z_{0:k-1}) \cdot p(z_k|x_k, x_{k+1})$$

$Similarly\ we\ will\ develop\ the\ expression\ for\ p(x_{0:k}|z_{0:k-1})$:

$$p(x_{0:k}|z_{0:k-1}) = \eta'_{k-1} \cdot p(x_{0:k-1}|z_{0:k-2}) \cdot p(z_{k-1}|x_{k-1}, x_k)$$

$$p(x_{0:k+1}|z_{0:k}) = \eta'_k \eta'_{k-1} \cdot p(x_{0:k-1}|z_{0:k-2}) \cdot p(z_k|x_k, x_{k+1}) \cdot p(z_{k-1}|x_{k-1}, x_k)$$

$And\ after\ k\ iterations\ we\ get\ the\ following\ expression$:

$$p(x_{0:k+1}|z_{0:k}) = p(x_0) \prod_{i=0}^{k} \eta'_i \cdot p(z_i|x_i, x_{i+1})$$

$define \prod_{i=0}^{k} \eta'_i = \eta$:

$$\boxed{p(x_{0:k+1}|z_{0:k}) = \eta p(x_0) \prod_{i=0}^{k} p(z_i|x_i, x_{i+1})}$$

Question 2: Formulate the smoothing optimization problem. Describe an iterative process to obtain the MAP estimate.

$$z_k = x_{k+1} \ominus x_k + v_k = h(x_{k+1}, x_k) + v_k$$

$$x^*_{0:k+1} = \arg\max\big(p(x_{0:k+1}|z_{0:k})\big) = \arg\max\left(\eta p(x_0)\prod_{i=0}^{k} p(x_{i+1}|x_i)\cdot p(z_i|x_i, x_{i+1})\right)$$

$$= \arg\min\left(-\log\left(p(x_0)\prod_{i=0}^{k} p(x_{i+1}|x_i)\cdot p(z_i|x_i, x_{i+1})\right)\right)$$

$$= \arg\min\left(||x_0 - \hat{x}_0||^2_{\Sigma_0} + \sum_{i=0}^{k}||z_i - h(x_{i+1}, x_i)||^2_{\Sigma_v}\right)$$

Above is the formulation of the optimization problem.
We provide an optimal solution via LMS, pseudo-inverse below:

We define:

$$J(x_{0:k+1}) = ||x_0 - \hat{x}_0||^2_{\Sigma_0} + \sum_{i=0}^{k}||z_i - h(x_{i+1}, x_i)||^2_{\Sigma_v}$$

Linearizing $h(x_i, x_{i+1})$ around $(\bar{x}_i, \bar{x}_{i+1})$:

We Define $\Delta x_i = x_i - \bar{x}_i$

$$h(x_i, x_{i+1}) \approx h(\bar{x}_i, \bar{x}_{i+1}) + \frac{\partial h}{x_i}\bigg|_{\bar{x}_i,\bar{x}_{i+1}}\Delta x_i + \frac{\partial h}{x_{i+1}}\bigg|_{\bar{x}_i,\bar{x}_{i+1}}\Delta x_{i+1} = h(\bar{x}_i, \bar{x}_{i+1}) + H_i^1\Delta x_i + H_i^2\Delta x_{i+1}$$

$$J(x_{0:k+1}) \approx ||x_0 - \hat{x}_0||^2_{\Sigma_0} + \sum_{i=0}^{k}||z_i - h(\bar{x}_i, \bar{x}_{i+1}) + H_i^1\Delta x_i + H_i^2\Delta x_{i+1}||^2_{\Sigma_v} =$$

$$= \left|\left|\Sigma_0^{-\frac{1}{2}}(x - \hat{x}_0)\right|\right|^2 + \sum_{i=0}^{k}\left|\left|\Sigma_v^{-\frac{1}{2}}(z_i - h(\bar{x}_i, \bar{x}_{i+1}) + H_i^1\Delta x_i + H_i^2\Delta x_{i+1})\right|\right|^2 =$$

$$= ||A\theta - b||^2$$

$$\theta = \begin{bmatrix} x_0 - \bar{x}_0 \\ x_1 - \bar{x}_1 \\ \vdots \\ \vdots \\ x_{k+1} - \bar{x}_{k+1} \end{bmatrix} = \begin{bmatrix} x_0 - \hat{x}_0 \\ x_1 - \bar{x}_1 \\ \vdots \\ \vdots \\ x_{k+1} - \bar{x}_{k+1} \end{bmatrix}$$

$$A = \begin{bmatrix} \Sigma_0^{-\frac{1}{2}} - \Sigma_v^{-\frac{1}{2}} H_0^1 & -\Sigma_v^{-\frac{1}{2}} H_0^2 & 0 & \cdots & 0 \\ 0 & -\Sigma_v^{-\frac{1}{2}} H_1^1 & -\Sigma_v^{-\frac{1}{2}} H_1^2 & & \\ \vdots & & -\Sigma_v^{-\frac{1}{2}} H_2^1 & -\Sigma_v^{-\frac{1}{2}} H_2^2 & \\ \vdots & & & \ddots & \\ 0 & & & -\Sigma_v^{-\frac{1}{2}} H_k^1 & -\Sigma_v^{-\frac{1}{2}} H_k^2 \end{bmatrix}$$

$$b = \begin{bmatrix} \Sigma_v^{-\frac{1}{2}} \left(-z_0 + h(\hat{x}_0, \bar{x}_1)\right) \\ \Sigma_v^{-\frac{1}{2}} \left(-z_1 + h(\bar{x}_1, \bar{x}_2)\right) \\ \vdots \\ \vdots \\ \Sigma_v^{-\frac{1}{2}} \left(-z_k + h(\bar{x}_k, \bar{x}_{k+1})\right) \end{bmatrix}$$

The vector θ^* which minimizes the squared residuals $||A\theta - b||^2$ is given by minimizing $||A\theta - b||$ with pseudo-inverse:

$$\theta^* = (A^T A)^{-1} A^T b$$

As such, the solution for our states will be given by:

$$x_{0:k+1} = \bar{x}_{0:k+1} + \theta^*$$

The combined covariance on all states:

$$\Sigma = I^{-1} = (A^T A)^{-1} = \left(\Sigma_0^{-1} + \sum_{i=0}^{k} \left({H_i^1}^T \Sigma_v^{-1} {H_i^1}^T + {H_i^2}^T \Sigma_v^{-1} {H_i^2}^T \right) \right)^{-1}$$