Technion – Israel Institute of Technology

# HW3

Vision Aided Navigation

086761


| | | |
|---|---|---|
| Alon Spinner | 305184335 | alonspinner@gmail.com |
| Sher Hazan | 308026467 | sherhazan@campus.technion.ac.il |


December 5, 2021

Question 1 : Consider a prior $p(x) = N(\hat{x}_0, \Sigma_0)$ on random variable $x$ and two measurements from different observation models

$$z_1 = h_1(x) + v_1$$
$$z_2 = h_2(x) + v_2$$

with $v_1 \sim N(0, \Sigma_{v1})$ and $v_2 \sim N(0, \Sigma_{v2})$.

(a) Develop an expression for the posteriori information matrix $I = \Sigma^{-1}$, such that $p(x|z_1, z_2) = N(\mu, \Sigma)$.

$$p(x, z_1, x_2) \underbrace{=}_{CR} p(x|z_1, z_2) \cdot p(z_1, z_2)$$

$$p(x, z_1, x_2) \underbrace{=}_{CR} p(z_1, z_2|x) \cdot p(x)$$

$$\textit{Comparing both right sides give us the bayes rule}$$

$$p(x|z_1, z_2) = \frac{p(z_1, z_2|x)p(x)}{p(z_1, z_2)} \underbrace{=}_{indep.} \frac{p(z_1|x)p(z_2|x)p(x)}{p(z_1, z_2)} = \eta^{-1}\, p(z_1|x)p(z_2|x)p(x)$$

$$\textit{We can write each of the terms explicitly}$$

$$p(x) = \frac{1}{\sqrt{((2\pi)^k|\Sigma_0|}} \exp\,(-\frac{1}{2}(x - \hat{x}_0)^T \Sigma_0^{-1}(x - \hat{x}_0)) \quad x \in \mathbb{R}^k$$

$$p(z_1|x) = \frac{1}{\sqrt{((2\pi)^{m_1}|\Sigma_{v_1}|}} \exp\,(-\frac{1}{2}\big(z_1 - h_1(x)\big)^T \Sigma_{v_1}^{-1}(z_1 - h_1(x)) \quad z_1 \in \mathbb{R}^{m_1}$$

$$p(z_2|x) = \frac{1}{\sqrt{((2\pi)^{m_2}|\Sigma_{v_2}|}} \exp\,(-\frac{1}{2}\big(z_2 - h_2(x)\big)^T \Sigma_{v_2}^{-1}(z_2 - h_2(x)) \quad z_2 \in \mathbb{R}^{m_2}$$

$$\textit{Maximizing likelihood}$$

$$x^* = \underset{x}{argmax}\big(p(x|z_1, z_2)\big) = \underset{x}{argmin}(-\log\big(p(x|z_1, z_2)\big)$$

$$= \underset{x}{argmin}((x - \hat{x}_0)^T \Sigma_0^{-1}(x - \hat{x}_0) + \big(z_1 - h_1(x)\big)^T \Sigma_{v_1}^{-1}(z_1 - h_1(x) + \big(z_2 - h_2(x)\big)^T \Sigma_{v_2}^{-1}\big(z_2 - h_2(x)\big)$$

$$= \underset{x}{argmin}\left(\left\|\Sigma_0^{-\frac{1}{2}}(x - \hat{x}_0)\right\|^2 + \left\|\Sigma_{v_1}^{-\frac{1}{2}}\big(z_1 - h_1(x)\big)\right\|^2 + \left\|\Sigma_{v_2}^{-\frac{1}{2}}\big(z_2 - h_2(x)\big)\right\|^2\right)$$

$$\textit{We linearize } h_2(x), h_1(x) \textit{ around } \hat{x}_0$$

$$h_1(x) \;\approx\; h_1(\hat{x}_0) + \left.\frac{\partial h_1}{\partial x}\right|_{\hat{x}_0} (x - \hat{x}_0) = h_1(\hat{x}_0) + H_1(x - \hat{x}_0)$$

$$h_2(x) \;\approx\; h_2(\hat{x}_0) + \left.\frac{\partial h_2}{\partial x}\right|_{\hat{x}_0} (x - \hat{x}_0) = h_2(\hat{x}_0) + H_2(x - \hat{x}_0)$$

$$\textit{In this way we can write}$$

$$x^* = \underset{x}{argmin}\left(\left\|\Sigma_0^{-\frac{1}{2}}(x-\hat{x}_0)\right\|^2 + \left\|\Sigma_{v_1}^{-\frac{1}{2}}(z_1 - h_1(\hat{x}_0) - H_1(x-\hat{x}_0))\right\|^2 + \left\|\Sigma_{v_2}^{-\frac{1}{2}}(z_2 - h_2(\hat{x}_0) - H_2(x-\hat{x}_0))\right\|^2\right)$$

$$\textit{Denoting } A, b \textit{ as follows}$$

$$A = \begin{bmatrix} \Sigma_0^{-\frac{1}{2}} \\ -\Sigma_{v_1}^{-\frac{1}{2}} H_1 \\ -\Sigma_{v_2}^{-\frac{1}{2}} H_2 \end{bmatrix} \quad b = \begin{bmatrix} \Sigma_0^{-\frac{1}{2}} \hat{x}_0 \\ -\Sigma_{v_1}^{-\frac{1}{2}}\left(z_1 - h_1(\hat{x}_0)\right) - H_1\hat{x}_0 \\ -\Sigma_{v_2}^{-\frac{1}{2}}\left(z_2 - h_2(\hat{x}_0)\right) - H_2\hat{x}_0 \end{bmatrix}$$

$$x^* = \underset{x}{argmin}\left(\left\|Ax-b\right\|^2\right) = \underset{x}{argmin}\left(\left\|Ax-b\right\|\right)$$

$$\textit{We can now apply Pseudo} - \textit{Inverse to obtain } x^* \textit{ that minimizes the linear expression's norm}$$

$$x^* = (A^TA)^{-1}A^Tb$$

$$\textit{We know that the covariance is}$$

$$\Sigma = (A^TA)^{-1}$$

$$\textit{Hence, the information matrix is just}$$

$$I = A^TA = \begin{bmatrix} \Sigma_0^{-\frac{T}{2}} & -H_1^T\Sigma_{v_1}^{-\frac{T}{2}} & -H_2^T\Sigma_{v_2}^{-\frac{T}{2}} \end{bmatrix} \begin{bmatrix} \Sigma_0^{-\frac{1}{2}} \\ -\Sigma_{v_1}^{-\frac{1}{2}} H_1 \\ -\Sigma_{v_2}^{-\frac{1}{2}} H_2 \end{bmatrix} = \boxed{\Sigma_0^{-1} + H_1^T\Sigma_{v_1}^{-1}H_1 + H_2^T\Sigma_{v_2}^{-1}H_2}$$

(b) Consider now a pinhole camera sensor with the corresponding measurement model: $z = \pi(x,l) + v \quad , \quad v \sim N(0,\Sigma_v)$,
An image observation $z$ of a landmark l is obtained. Additionally, a prior $p(l) = N(\hat{l}_0, \Sigma_{l_0})$ on $l$ is available. Indicate what is the initial re-projection error.

$$\textit{We assume this measurement } z \textit{ was provided after } z_1, z_2 \textit{ were incorporated to the prior.}$$

$$e_0 = z - \pi\left(\underset{x}{argmax}\left(p(x|z_1,z_2)\right), \underset{l}{argmax}\left(p(l)\right)\right) = \boxed{z - \pi\left(x^*, \hat{l}_0\right)}$$

(c) Write an expression for the joint pdf $p(x,l|z_1,z_2,z)$ in terms of prior and measurement likelihood terms.

$$p(x,l|z_1,z_2,z) \underbrace{=}_{bayes} \frac{p(z_1,z_2,z|x,l)p(x,l)}{p(z_1,z_2,z)} \underbrace{=}_{indep.} \frac{p(z_1,z_2|x)p(z|x,l)p(x)p(l)}{p(z_1,z_2,z)}$$

$$\underbrace{=}_{bayes} \frac{p(x|z_1,z_2)p(z_1,z_2)p(z|x,l)p(l)}{p(z_1,z_2,z)} \underbrace{=}_{indep.} \boxed{\frac{1}{p(z)}p(x|z_1,z_2)p(z|x,l)p(l)}$$

(d) Develop an expression for the joint information matrix $I'$ over $x$ and $l$ as in $p(x,l|z_1,z_2,z) = N(\times, I'^{-1})$. consider $p(x|z_1,z_2) = N(\mu,\Sigma)$ is given.

$$On\ this\ section\ we\ will\ shorthand\ a\ few\ steps\ that\ have\ been\ fully\ covered\ in\ the\ section\ (a).$$

$$[x^*,l^*] = \underset{x,l}{argmin}\left(-\log\left(p(x,l|z_1z_2,z)\right)\right) =$$

$$= \underset{x,l}{argmin}\left(\left\|\Sigma^{-\frac{1}{2}}(x-\mu)\right\|^2 + \left\|\Sigma_{\mathrm{v}}^{-\frac{1}{2}}\left(z-\pi(x,l)\right)\right\|^2 + \left\|\Sigma_{\mathrm{l}_0}^{-\frac{1}{2}}\left(l-\hat{l}_0\right)\right\|^2\right)$$

$$We\ need\ to\ linearize\ \pi(x,l):$$

$$\pi(x,l) \approx \pi\left(\mu,\hat{l}_0\right) + \nabla_x\pi|_{\mu,\hat{l}_0}(x-\mu) + \nabla_l|_{\mu,\hat{l}_0}\left(l-\hat{l}_0\right) = \pi\left(\mu,\hat{l}_0\right) + J_x(x-\mu) + J_l(l-\hat{l}_0)$$

$$Hence$$

$$[x^*,l^*] = \underset{x,l}{argmin}\left(\left\|\Sigma^{-\frac{1}{2}}(x-\mu)\right\|^2 + \left\|\Sigma_{\mathrm{v}}^{-\frac{1}{2}}\left(z-\pi\left(\mu,\hat{l}_0\right)+J_x(x-\mu)+J_l(l-\hat{l}_0)\right)\right\|^2 + \left\|\Sigma_{\mathrm{l}_0}^{-\frac{1}{2}}\left(l-\hat{l}_0\right)\right\|^2\right)$$

$$Denoting$$

$$\Theta = \begin{bmatrix} x \\ l \end{bmatrix}$$

$$A = \begin{bmatrix} \Sigma^{-\frac{1}{2}} & 0 \\ -\Sigma_{\mathrm{v}}^{-\frac{1}{2}}J_x & -\Sigma_{\mathrm{v}}^{-\frac{1}{2}}J_l \\ 0 & \Sigma_{\hat{l}_0}^{-\frac{1}{2}} \end{bmatrix} \quad b = \begin{bmatrix} \Sigma^{-1/2}\mu \\ -\Sigma_v^{-\frac{1}{2}}\left(z-\pi\left(\mu,\hat{l}_0\right)\right) \\ \Sigma_{l_0}^{-\frac{1}{2}}\hat{l}_0 \end{bmatrix}$$

$$We\ can\ write: [x^*,l^*] = \underset{x,l}{argmin}\left(\left|\left|A\Theta - b\right|\right|\right)$$

$$Least\ squares\ solution\ using\ pseudo\ inverse\ will\ provide\ the\ following\ information\ matrix:$$

$$I' = A^TA = \begin{bmatrix} \Sigma^{-1} + J_x^T\Sigma_v^{-1}J_x & J_x^T\Sigma_v^{-1}J_l \\ J_x^T\Sigma_v^{-1}J_l & \Sigma_0^{-1} + J_l^T\Sigma_v^{-1}J_l \end{bmatrix}$$

Question 2 : Consider two camera poses $x_1 = (R_1, t_1)$ and $x_2 = (R_2, t_2)$, where the following convention is assume.

$$R_i = R_{C_i}^G \quad , \quad t_i = t_{C_i \to G}^G \quad , \quad i = 1,2.$$

Here, the superscript $G$ denotes some global reference frame. Assume each camera captures an image and let $z_1 = (u_1, v_1)^T$ and $z_2 = (u_2, v_2)^T$ be two corresponding image observations from these two images.

(a) Develop the epipolar constraint, expressing all quantities in the second camera frame. Express the constraint in the form $h(x_1, x_2, z_1, z_2) = 0$.

$$In\ general$$

$$z = M_1 X = K[R|t]X$$

$$q_i = K^{-1} z_i = [R|t]X$$

$$Define\ the\ transformation\ from\ 1\ to\ 2$$

$$R_1^2 = R_{C_1}^{C_2} = R_2^T R_1$$

$$t_{2\to 1}^2 = t_{C_2 \to C_1}^2 = R_2^T (t_2 - t_1)$$

$$both\ cameras\ in\ the\ second\ camera\ cordinate$$

$$z_1 = M_1 X = K_1 [R_1^2 | t_{2\to 1}^2] X$$

$$z_2 = M_2 X = K_2 [I|0] X$$

$$Assuming\ cameras\ calibration\ matricies\ are\ known\ K_1\ , K_2$$

$$q_1 = K_1^{-1} z_1 = [R_1^2 | t_{2\to 1}^2] X$$

$$q_2 = K_2^{-1} z_2 = [I|0] X$$

$$Let\ \lambda_1\ , \lambda_2\ be\ some\ unknown\ parameters.\ \left|\left|\lambda_1 q_1\right|\right|\ is\ the\ distance\ to\ X.$$

$$\lambda_2 q_2 = R_1^2 \lambda_1 q_1 + t_{2\to 1}^2$$

$$Eliminate\ \lambda_1, \lambda_2\ algebricaly.$$

$$t_{2\to 1}^2 : [t]_x = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix}$$

$$Multiply\ by\ [t]_x$$

$$\lambda_2 [t]_x q_2 = \lambda_1 [t]_x R_1^2 q_1 + [t]_x t \to\ [t]_x t = 0$$

$$Multiply\ by\ q_2^T\ as\ shown\ at\ the\ lecture$$

$$\lambda_2 q_2^T [t]_x q_2 = \lambda_1 q_2^T [t]_x R_1^2 q_1 \to q_2^T [t]_x q_2 = 0$$

$$\boxed{q_2^T [t]_x R_1^2 q_1 = q_2^T [t]_x (R_2^T R_1) q_1 = 0}$$

$$\text{Essential matrix: } E = [t]_x (R_2^T R_1)$$

$$Fundamental\ matrix:\ F = K_2^{-T} E K_1^{-1}$$

$$Epipolar\ constraint:\ \ z_2^T F z_1 = 0$$

$$\boxed{h(x_1, x_2, z_1, z_2) = z_2^T F(x_1, x_2) z_1 = 0}$$

(b) Assume the true values of the camera poses $x_1$ and $x_2$ are not actually known, and instead we have a prior on each camera

$$p(x_1) = N(\mu_0, \Sigma_{01}) \quad , \quad p(x_2) = N(\mu_0, \Sigma_{02}).$$

Assume the residual error in the epipolar constraint from the previous clause can be modeled as zero-mean Gaussian with covariance $\Sigma_{ep}$, derive a probabilistic expression for MAP $p(x_1, x_2|z_1, z_2)$.

$$\textit{The model we obtained in the previous section with the added Gaussian Noise:}$$

$$h(x_1, x_2, z_1, z_2) = z_2^T F(x_1, x_2) z_1 + v \quad , \quad v \sim N(0, \Sigma_{ep})$$

$$p(x_1, x_2|z_1, z_2) \underset{BR}{=} \frac{p(z_1, z_2|x_1, x_2)p(x_1)p(x_2)}{p(z_1, z_2)} = \eta p(z_1, z_2|x_1, x_2)p(x_1)p(x_2)$$

$$\textit{To find the MAP estimate we'll use}$$

$$\underset{x_1, x_2}{argmax}\left(p(x_1, x_2|z_1, z_2)\right) = \underset{x_1, x_2}{argmin}\left(||h(x_1, x_2, z_1, z_2)||_{\Sigma_{ep}}^2 + ||x_1 - \mu_{01}||_{\Sigma_{01}}^2 + ||x_2 - \mu_{02}||_{\Sigma_{02}}^2\right)$$

## Question 3: Prove the fundamental matrix is singular.

$$\det(F) = \det(K_2^{-T} E K_1^{-1}) = \det(K_2^{-T}) \det(E) \det(K_1^{-1})$$

To prove that $F$ is singular we need to prove that $E$ is singular:

We developed the Essential matrix in the previous question:

$$E = [t]_x (R_2^T R_1)$$

$$\det(E) = \det([t]_x) \cdot det(R_2^T R_1)$$

The determinant of a Rotation matrix cannot be singular as it just transforms representations. It is not allowed to have a kernel space. In-fact, $\left||\det(R)|\right| = 1$.

Hence, we need to prove that $\det([t]_x) = 0$

$$\det([t]_x) = \det\left(\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}\right) = t_z t_x t_y - t_y t_x t_z = 0$$

Blackbox.