Technion – Israel Institute of Technology

HW5

Vision Aided Navigation

086761

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Question 1: Factor graph, variable elimination and Bayes net.
Consider a SAM problem where a robot travels through an unknown environment and captures observations using its onboard sensors. Assume the robot starts at time t_0, with a known prior $p(x)$ and consider motion and observation models $p(x_k|x_{k-1}, u_{k-1})$ and $p(z_{k,i}|x_k, l_i)$, respectively, where l_i denotes the i^{th} landmark. The robot moves according to given controls and observes a single landmark at time instances t_1 and t_2.

a: Write the joint pdf corresponding to the above scenario until time t_4: $p(x_{0:4}, l|u_{0:3}, z_1, z_2)$

$$p(x_k|x_{k-1}, u_{k-1}) \sim motion\ model$$

$$p(z_k|x_k, l_i) \sim measurement\ model$$

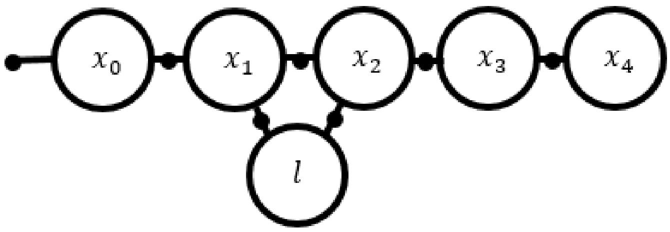
$$p(x_{0:4}, l|u_{0:3}, z_1, z_2) \underbrace{=}_{\substack{cond. \\ +indep.}} p(x_4|x_3u_3)p(x_3|x_2, u_2) \cdot \underbrace{p(x_{0:2}, l|u_{0:1}, z_{1:2})}_{known\ structure} =$$

$$= p(x_4|x_3u_3)p(x_3|x_2, u_2) \cdot p(x_0) \prod_{i=1}^{2} \eta_i p(z_i|x_i, l)p(x_i|u_{i-1}, x_{i-1}) =$$

$$\left\{ \eta = \prod_{i=1}^{2} \eta_i : \ not\ a\ function\ of\ x\ or\ l, the\ varibles\ we\ optimize\ on \right\}$$

$$= \eta \cdot p(x_4|x_3u_3)p(x_3|x_2, u_2) \cdot p(x_0) \prod_{i=1}^{2} p(z_i|x_i, l)p(x_i|u_{i-1}, x_{i-1})$$

b: Draw the corresponding factor graph.



c: Eliminate the factor graph into a Bayes net, assuming elimination order:
$x_0, x_1, x_2, x_3, x_4, l$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) = \eta \cdot p(x_0)p(x_1|x_0,u_0)p(x_2|x_1,u_1)p(x_3|x_2,u_2)p(x_4|x_3,u_3)p(z_1|x_1,l)p(z_2|x_2,l)$$

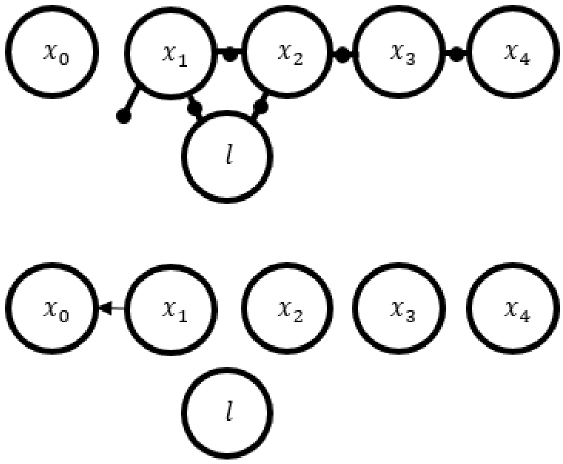
$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto f_0(x_0)f_1(x_0,x_1)f_2(x_1,x_2)f_3(x_2,x_3)f_4(x_3,x_4)f_{l1}(x_1,l)f_{l2}(x_2,l)$$

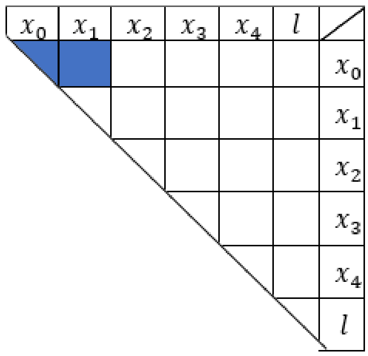
$\underline{Elinination\ of\ x_0:}$

$$f_{joint}(x_0,x_1) = f_0(x_0)f_1(x_0,x_1) \propto p(x_0|x_1) \cdot f_{1-new}(x_1)$$

$After\ the\ elimination\ of\ x_0\ we\ get:$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) \cdot f_{1-new}(x_1)f_2(x_1,x_2)f_3(x_2,x_3)f_4(x_3,x_4)f_{l1}(x_1,l)f_{l2}(x_2,l)$$



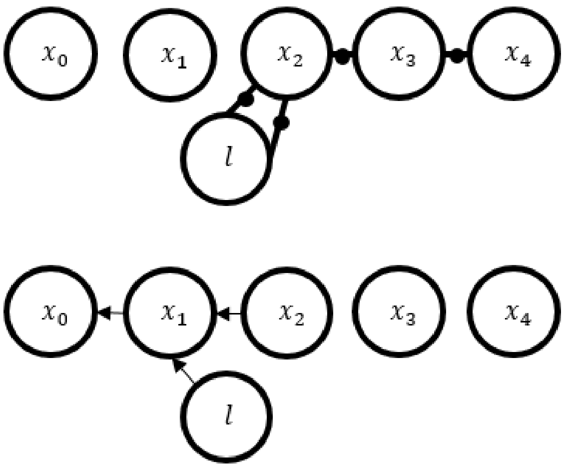


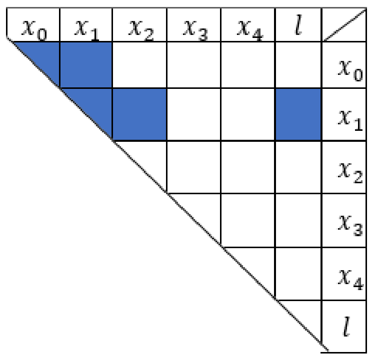
$\underline{Elinination\ of\ x_1:}$

$$f_{joint}(x_1,x_2,l) = f_{1-new}(x_1)f_2(x_1,x_2)f_{l1}(x_1,l) \propto p(x_1|x_2,l) \cdot f_{2-new}(x_2,l)$$

$After\ the\ elimination\ of\ x_1\ we\ get:$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2,l) \cdot f_{2-new}(x_2,l)f_3(x_2,x_3)f_4(x_3,x_4)f_{l2}(x_2,l)$$



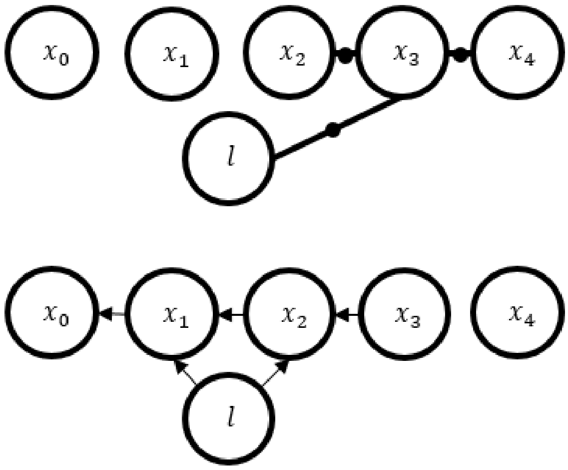


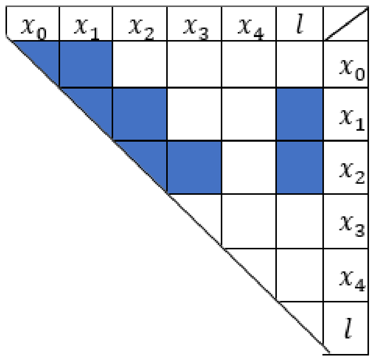
Elinination of x_2:

$$f_{joint}(x_2, x_3, l) = f_{2-new}(x_2, l) f_3(x_2, x_3) f_{l2}(x_2, l) \propto p(x_2|x_3, l) \cdot f_{3-new}(x_3, l)$$

After the elimination of x_2 we get:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) p(x_1|x_2, l) p(x_2|x_3, l) \cdot f_{3-new}(x_3, l) f_4(x_3, x_4)$$



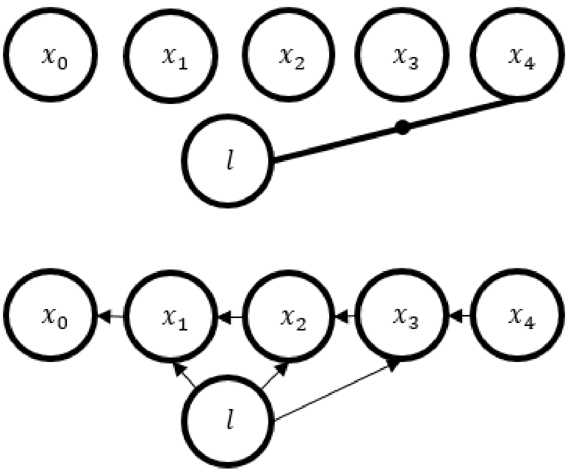


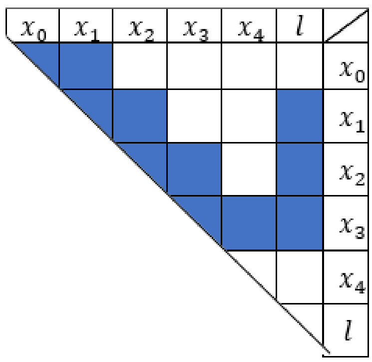
Elinination of x_3:

$$f_{joint}(x_3, x_4, l) = f_{3-new}(x_3, l) f_4(x_3, x_4) \propto p(x_3|x_4, l) \cdot f_{4-new}(x_4, l)$$

After the elimination of x_3 we get:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) p(x_1|x_2, l) p(x_3|x_4, l) \cdot f_{4-new}(x_4, l)$$



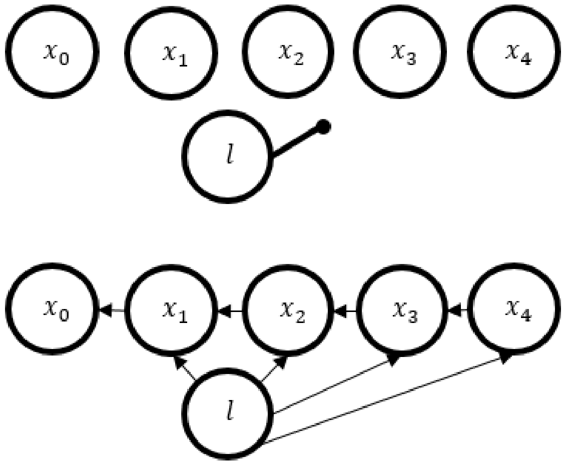


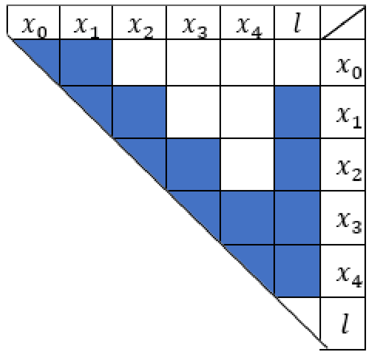
Elinination of x_4:

$$f_{joint}(x_4, l) = f_{4-new}(x_4, l) \propto p(x_4|l) \cdot f_{l-new}(l)$$

After the elimination of x_4 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_2|x_3, l)p(x_4|l) \cdot f_{l-new}(l)$$



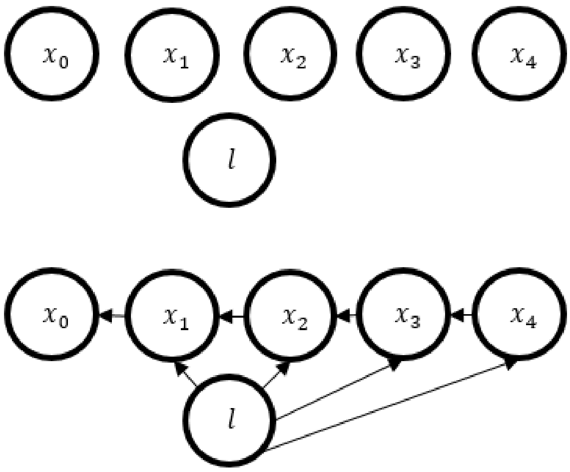


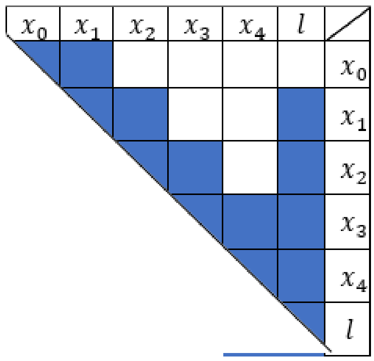
Elinination of l:

$$f_{joint}(l) = f_{l-new}(l) \propto p(l)$$

After the elimination of l *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_3|x_4, l)p(l)$$





d: Repeat the previous clause using a different variable elimination order:

$x_4, x_3, x_2, l, x_1, x_0$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) = \eta \cdot p(x_0)p(x_1|x_0, u_0)p(x_2|x_1, u_1)p(x_3|x_2, u_2)p(x_4|x_3, u_3)p(z_1|x_1, l)p(z_2|x_2, l)$$

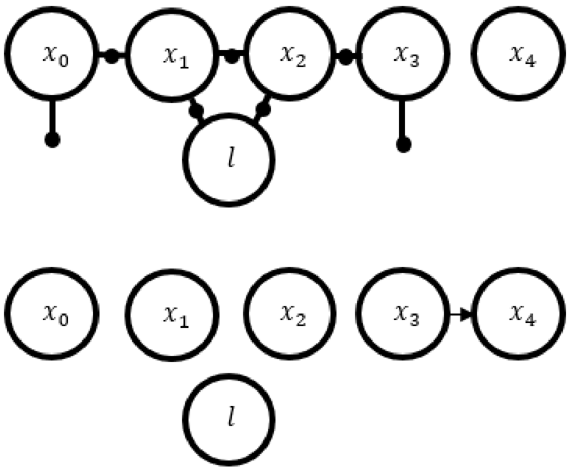
$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_{l1}(x_1, l)f_{l2}(x_2, l)$$

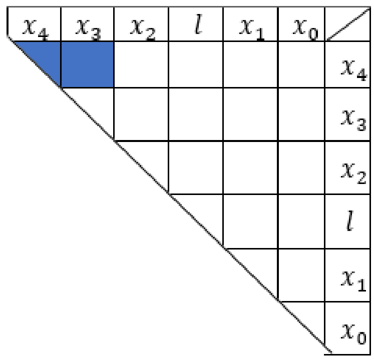
$\underline{Elinination\ of\ x_4:}$

$$f_{joint}(x_3, x_4) = f_4(x_3, x_4) \propto p(x_4|x_3) \cdot f_{3-new}(x_3)$$

$After\ the\ elimination\ of\ x_4\ we\ get:$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3) \cdot f_{3-new}(x_3)f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_{l1}(x_1, l)f_{l2}(x_2, l)$$



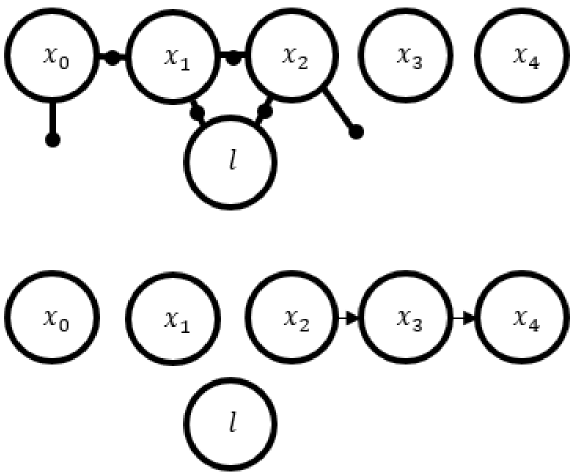


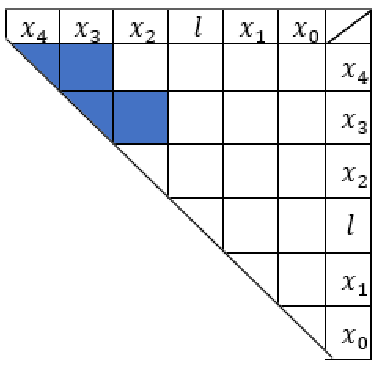
$\underline{Elinination\ of\ x_3:}$

$$f_{joint}(x_2, x_3) = f_{3-new}(x_3)f_3(x_2, x_3) \propto p(x_3|x_2) \cdot f_{2-new}(x_2)$$

$After\ the\ elimination\ of\ x_3\ we\ get:$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2) \cdot f_{2-new}(x_2)f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_{l1}(x_1, l)f_{l2}(x_2, l)$$



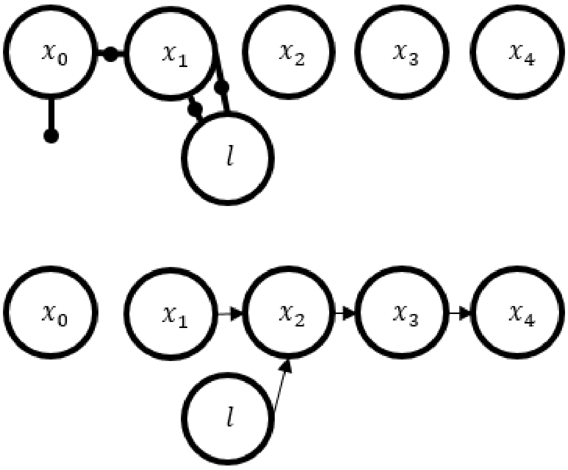


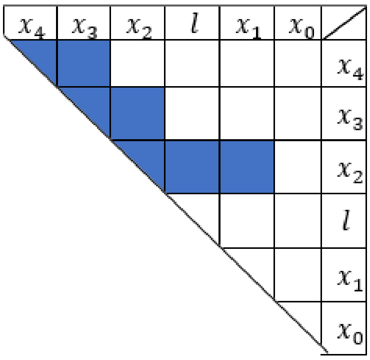
Elinination of x_2:

$$f_{joint}(x_1, x_2, l) = f_{2-new}(x_2)f_2(x_1, x_2)f_{l2}(x_2, l) \propto p(x_2|x_1, l) \cdot f_{l-new}(x_1, l)$$

After the elimination of x_2 we get:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2)p(x_2|x_1, l) \cdot f_{l-new}(x_1, l)f_0(x_0)f_1(x_0, x_1)f_{l1}(x_1, l)$$



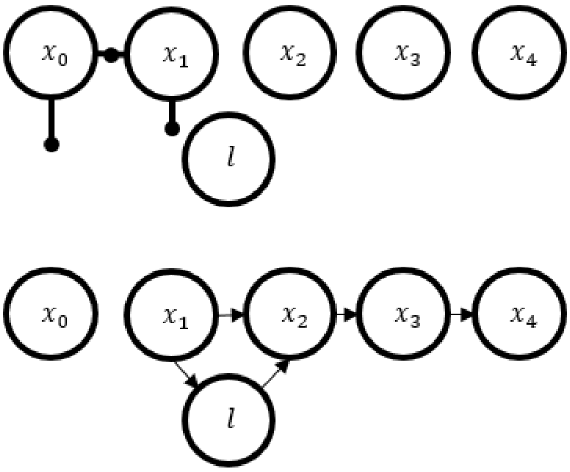


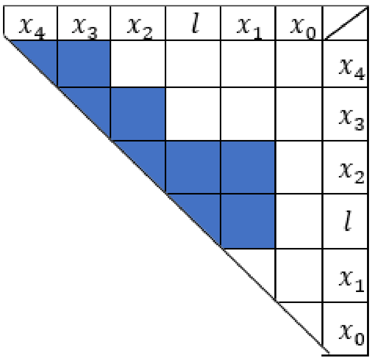
Elinination of l:

$$f_{joint}(x_1, l) = f_{l-new}(x_1, l)f_{l1}(x_1, l) \propto p(l|x_1) \cdot f_{1-new}(x_1)$$

After the elimination of x_3 we get:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2)p(l|x_1) \cdot f_{1-new}(x_1)f_0(x_0)f_1(x_0, x_1)$$



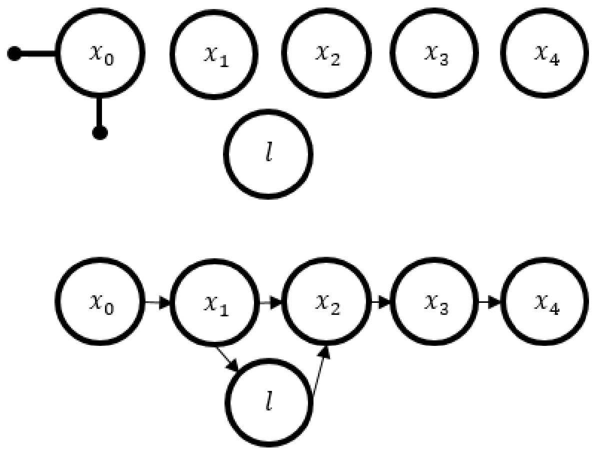


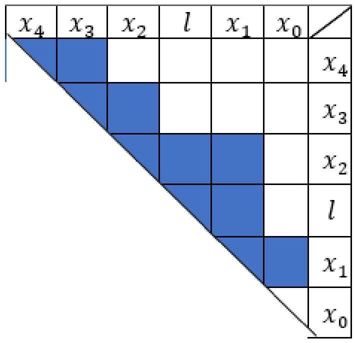
Elinination of x_1:

$$f_{joint}(x_0, x_1) = f_{1-new}(x_1) f_1(x_0, x_1) \propto p(x_1|x_0) \cdot f_{0-new}(x_0)$$

$After\ the\ elimination\ of\ x_1\ we\ get:$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2)p(l|x_1)p(x_1|x_0) \cdot f_{0-new}(x_0) f_0(x_0)$$



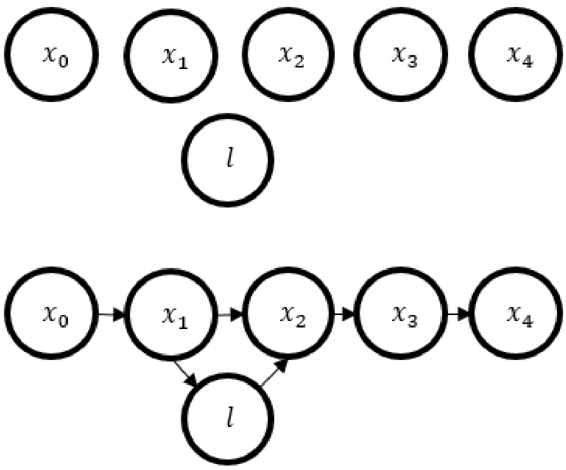


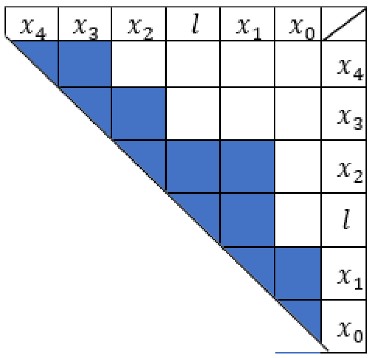
Elinination of x_0:

$$f_{joint}(x_0) = f_{0-new}(x_0) f_0(x_0) \propto p(x_0)$$

$After\ the\ elimination\ of\ x_0\ we\ get:$

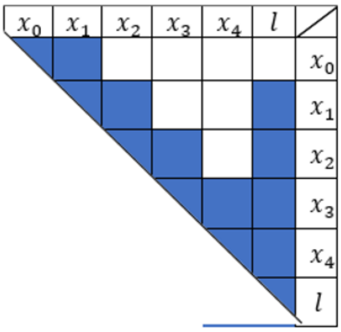
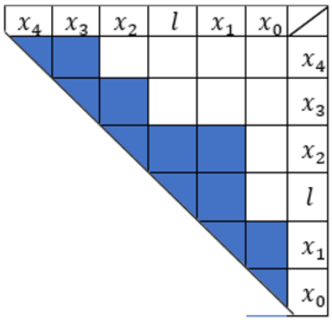
$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2)p(l|x_1)p(x_1|x_0)p(x_0)$$





e: Which of the two elimination orders you would prefer in terms of estimation accuracy and computational aspects?

We show the elimination order and resulting R matrix for each section below.

{c} Elimination Order: $\{x_0, x_1, x_2, x_3, x_4, l\}$	{d} Elimination Order: $\{x_4, x_3, x_2, l, x_1, x_0\}$
	

Regarding computation efficiency: We would prefer the elimination order in {d}, as it produces a sparser R matrix (12 non-zero elements vs 14), and with more structure – all rows but one contains two elements at the start of the row.

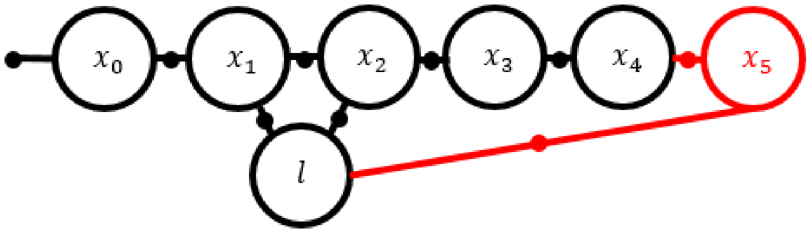
Regarding Accuracy: Both matrices contain the same information. As such, the solution of the LMS problem is independent of the elimination order, or which R matrix we choose to use.

$$R_d^T R_d = R_c^T R_c = A^T A$$

Question 2: Incremental factorization.

Consider now the robot, from question 1, executes command u_4 and moves to a new location; denote its new pose by x_5. Assume the robot observes again the landmark l from the new location.

a: Draw the factor graph of the problem and indicate the new factors and variable nodes.



b: Consider the Bayes net from question 1(c) with elimination order $x_0, x_1, x_2, x_3, x_4, l$. Perform incremental factorization by updating this Bayes net with the new information using the elimination order:

$$x_0, x_1, x_2, x_3, x_4, l, x_5$$

$$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) = \eta \cdot p(x_0)p(x_1|x_0,u_0)p(x_2|x_1,u_1)p(x_3|x_2,u_2)p(x_4|x_3,u_3)p(x_5|x_4,u_4)p(z_1|x_1,l)p(z_2|x_2,l)\ p(z_3|x_5,l)$$

$$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) \propto f_0(x_0)f_1(x_0,x_1)f_2(x_1,x_2)f_3(x_2,x_3)f_4(x_3,x_4)f_5(x_4,x_5)f_{l1}(x_1,l)f_{l2}(x_2,l)f_{l3}(x_5,l)$$

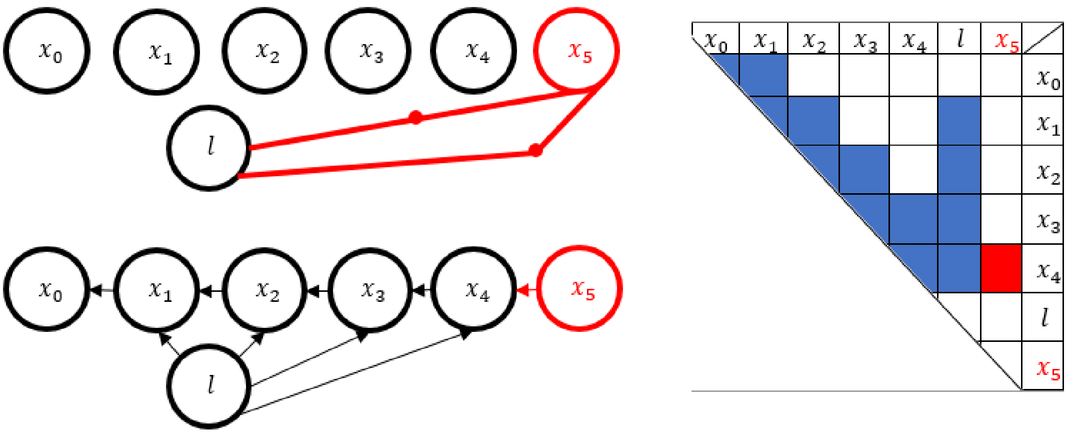
We need to reeliminat the factors x_4, l beacose they involve the changes.

<u>*Back to Elimination of x_4:*</u>

$$f_{joint}(x_4, x_5, l) = f_{4-new}(x_4, l)f_5(x_4, x_5) \propto p(x_4|x_5, l) \cdot f_{l-new}(x_5, l)$$

After the elimination of x_4 we get:

$$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2,l)p(x_2|x_3,l)p(x_4|x_5,l) \cdot f_{l3}(x_5,l)f_{l-new}(x_5,l)$$

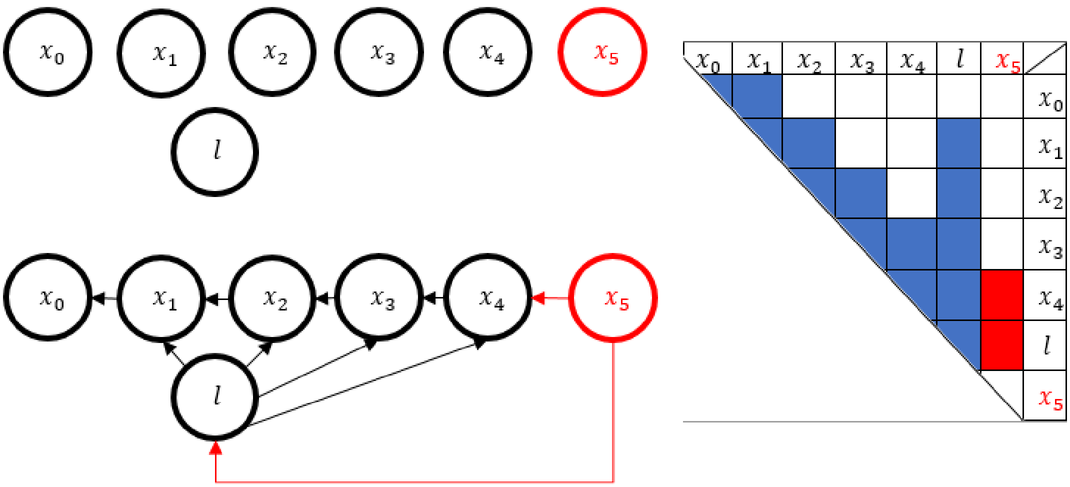


<u>*Elinination of l:*</u>

$$f_{joint}(x_5, l) = f_{l-new}(x_5, l)f_{l3}(x_5, l) \propto p(l|x_5) \cdot f_{5-new}(x_5)$$

After the elimination of l we get:

$$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2,l)p(x_3|x_4,l)p(l|x_5) \cdot f_{5-new}(x_5)$$

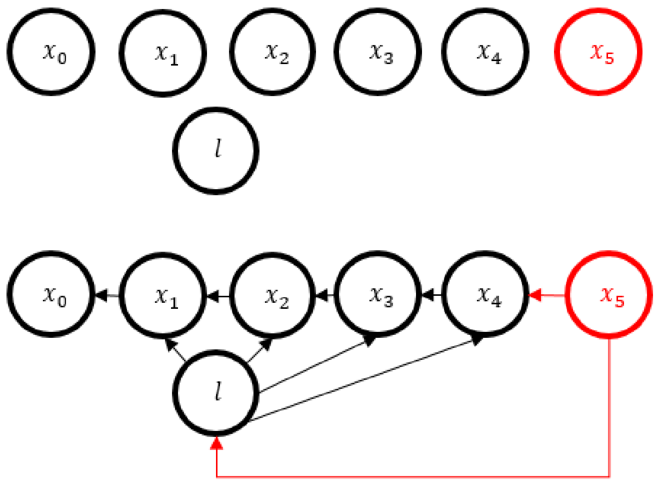


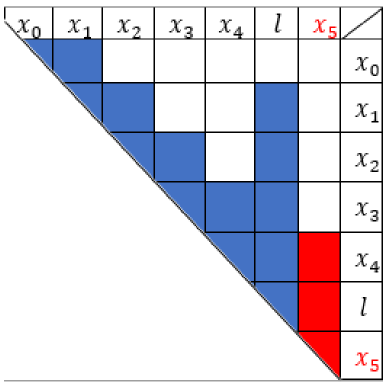
Elinination of x_5:

$$f_{joint}(x_5) = f_{5-new}(x_5) \propto p(x_5)$$

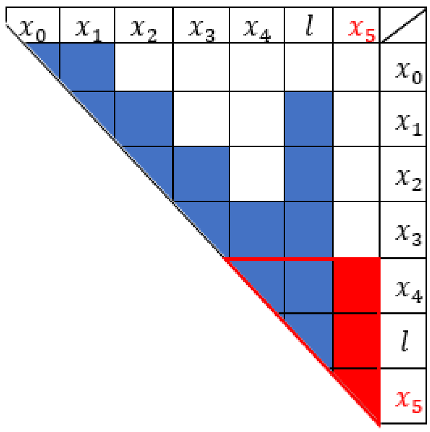
After the elimination of x_5 we get:

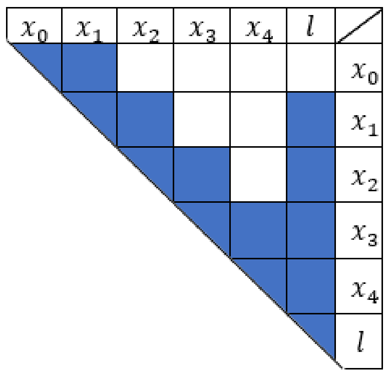
$$p(x_{0:4}, x_5, l | u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_3|x_4, l)p(l|x_5)p(x_5)$$





c: Show the corresponding updated square root information matrix R



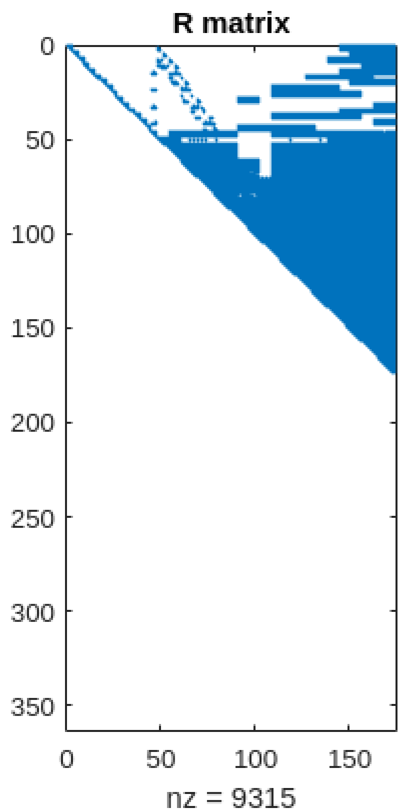


Q3 - Variable Ordering

```
load('hw5_A.mat'); %produces A matrix in the workspace
```

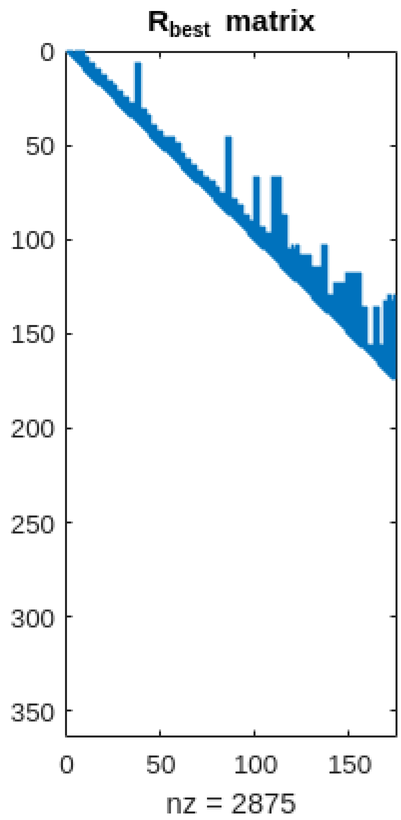
(a) Obtain R with QR fatorization

```
[~,R] = qr(A);
spy(R);
title('R matrix');
```



(b) best R

```
p = colamd(A);
[~,Rbest] = qr(A(:,p));
spy(Rbest);
title('R_{best} matrix');
```



R_{best} has ~30% values of the initial R matrix.

Hence will require to do only ~30% of the computations when sovling back subsitution with R_{best} when compared to R.