Technion – Israel Institute of Technology

HW5

Vision Aided Navigation

086761

Alon Spinner	305184335	alonspinner@gmail.com
Sher Hazan	308026467	sherhazan@campus.technion.ac.il

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Question 1: Factor graph, variable elimination and Bayes net.
Consider a SAM problem where a robot travels through an unknown environment and captures observations using its onboard sensors. Assume the robot starts at time t_0, with a known prior $p(x)$ and consider motion and observation models $p(x_k|x_{k-1},u_{k-1})$ and $p(z_{k,i}|x_k,l_i)$, respectively, where l_i denotes the i^{th} landmark. The robot moves according to given controls and observes a single landmark at time instances t_1 and t_2.

a: Write the joint pdf corresponding to the above scenario until time t_4: $p(x_{0:4},l|u_{0:3},z_1,z_2)$

$$p(x_k|x_{k-1},u_{k-1}) \sim motion\ model$$

$$p(z_k|x_k,l_i) \sim measurement\ model$$

$$p(x_{0:4},l|u_{0:3},z_1,z_2) \underbrace{=}_{\substack{cond.\\+indep.}} p(x_4|x_3u_3)p(x_3|x_2,u_2)\cdot \underbrace{p(x_{0:2},l|u_{0:1},z_{1:2})}_{known\ structure} =$$

$$= p(x_4|x_3u_3)p(x_3|x_2,u_2)\cdot p(x_0)\prod_{i=1}^{2}\eta_i p(z_i|x_i,l)p(x_i|u_{i-1},x_{i-1}) =$$

$$\left\{\eta = \prod_{i=1}^{2}\eta_i:\ not\ a\ function\ of\ x\ or\ l, the\ varibles\ we\ optimize\ on\right\}$$

$$= \eta\cdot p(x_4|x_3u_3)p(x_3|x_2,u_2)\cdot p(x_0)\prod_{i=1}^{2}p(z_i|x_i,l)p(x_i|u_{i-1},x_{i-1})$$

b: Draw the corresponding factor graph.

Factor graphs are a graphical way of representing some function $g(x_1,\dots,x_n)$ where g can be written as a product of functions (factors).

$$g(x_1,\dots,x_n) = \prod_i f_i\,(S_i)$$

where $S_i \subseteq (x_i,\dots x_n)$
This manipulation enables us to encode the independence relationships.

Factor graphs are a bipartite graph: contain two types of nodes, and one type of edge.
Big nodes represent variables, and the smaller ones are represented factors.
Each factor node is connected to one or more variable nodes through edges according to the variables it acts upon.

In our case, the big nodes will represent poses and landmarks (states).
The small nodes will represent factors born from motion and measurement models.

For example, given some motional model $x_i = h(x_{i-1},u_{i-1}) + v$, where v represents gaussian noise with variance Σ, each motion's model conditional probability pdf

$p(x_i|x_{i-1}, u_{i-1})$ has two states involved, and with the assumption of a gaussian distribution, corresponds to the following factor:

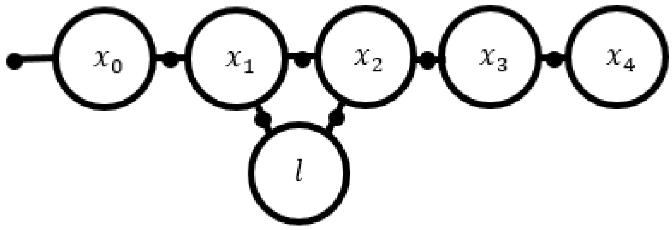
$$p(x_i|x_{i-1}, u_{i-1}) \propto f_{motion_i}(x_i, x_{i-1}) = e^{-\frac{1}{2}||x_i - h(x_{i-1}, u_{i-1})||_{\Sigma}^{2}}$$

Note that the function h itself is not explicit in the factor's arguments.

In our SLAM problem, a joint distribution over a state (poses and landmarks) is proportional to a multiplication of factors.
Below is the factor graph representation for our SLAM problem.

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) = \eta \cdot p(x_0)p(x_1|x_0, u_0)p(x_2|x_1, u_1)p(x_3|x_2, u_2)p(x_4|x_3, u_3)p(z_1|x_1, l)p(z_2|x_2, l)$$

$$\propto f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_{l1}(x_1, l)f_{l2}(x_2, l).$$



We would like to note that the advantage of factor graphs over Bayes Nets is that they can specify any factored function, and not just probability densities. This makes them better suited for inference (or so the documentation for GTSAM says…)

c: Eliminate the factor graph into a Bayes net, assuming elimination order:
$x_0, x_1, x_2, x_3, x_4, l$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) = \eta \cdot p(x_0)p(x_1|x_0, u_0)p(x_2|x_1, u_1)p(x_3|x_2, u_2)p(x_4|x_3, u_3)p(z_1|x_1, l)p(z_2|x_2, l)$$

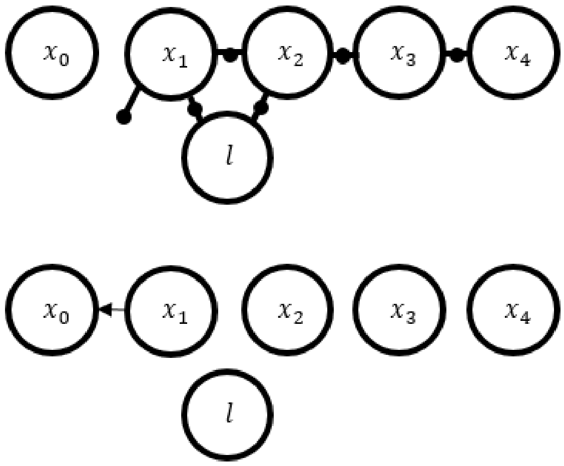
$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_{l1}(x_1, l)f_{l2}(x_2, l)$$

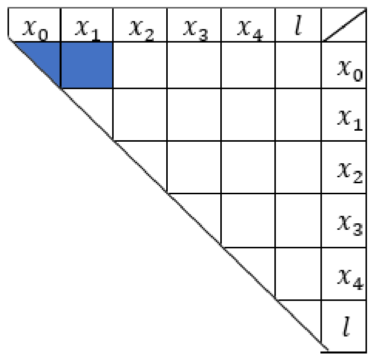
$\underline{Elinination\ of\ x_0}$:

$$f_{joint}(x_0, x_1) = f_0(x_0)f_1(x_0, x_1) \propto p(x_0|x_1) \cdot f_{1-new}(x_1)$$

$After\ the\ elimination\ of\ x_0\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) \cdot f_{1-new}(x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_{l1}(x_1, l)f_{l2}(x_2, l)$$



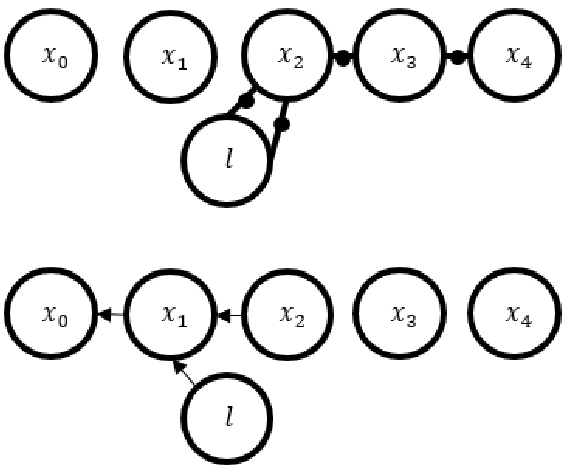


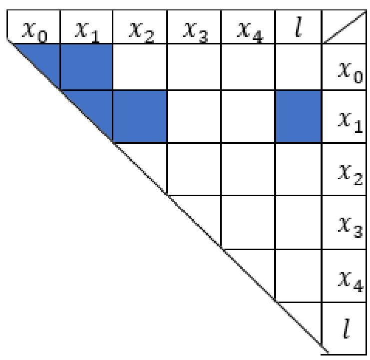
$\underline{Elinination\ of\ x_1}$:

$$f_{joint}(x_1, x_2, l) = f_{1-new}(x_1)f_2(x_1, x_2)f_{l1}(x_1, l) \propto p(x_1|x_2, l) \cdot f_{2-new}(x_2, l)$$

$After\ the\ elimination\ of\ x_1\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2, l) \cdot f_{2-new}(x_2, l)f_3(x_2, x_3)f_4(x_3, x_4)f_{l2}(x_2, l)$$



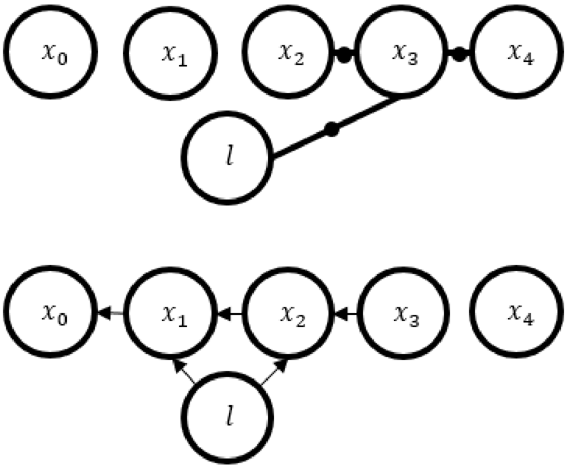


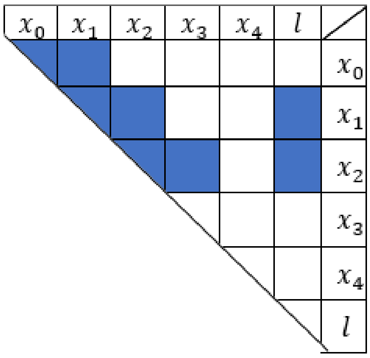
Elinination of x_2:

$$f_{joint}(x_2, x_3, l) = f_{2-new}(x_2, l) f_3(x_2, x_3) f_{l2}(x_2, l) \propto p(x_2|x_3, l) \cdot f_{3-new}(x_3, l)$$

$After\ the\ elimination\ of\ x_2\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) p(x_1|x_2, l) p(x_2|x_3, l) \cdot f_{3-new}(x_3, l) f_4(x_3, x_4)$$



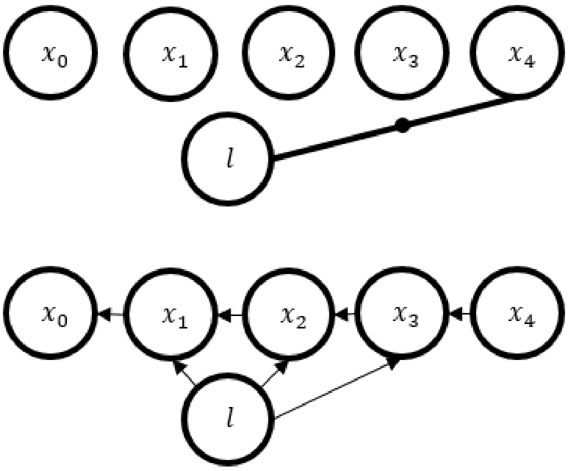


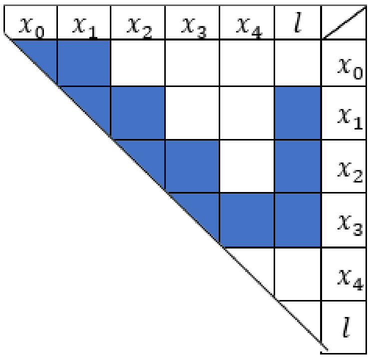
Elinination of x_3:

$$f_{joint}(x_3, x_4, l) = f_{3-new}(x_3, l) f_4(x_3, x_4) \propto p(x_3|x_4, l) \cdot f_{4-new}(x_4, l)$$

$After\ the\ elimination\ of\ x_3\ we\ get$:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1) p(x_1|x_2, l) p(x_3|x_4, l) \cdot f_{4-new}(x_4, l)$$



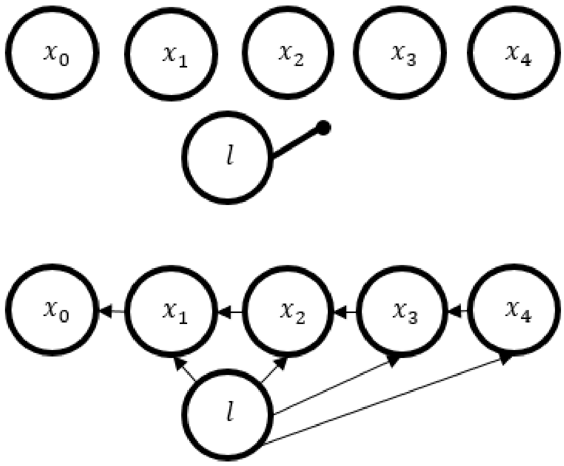


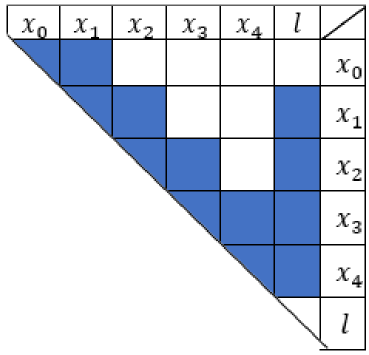
Elinination of x_4:

$$f_{joint}(x_4, l) = f_{4-new}(x_4, l) \propto p(x_4|l) \cdot f_{l-new}(l)$$

After the elimination of x_4 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_2|x_3, l)p(x_4|l) \cdot f_{l-new}(l)$$



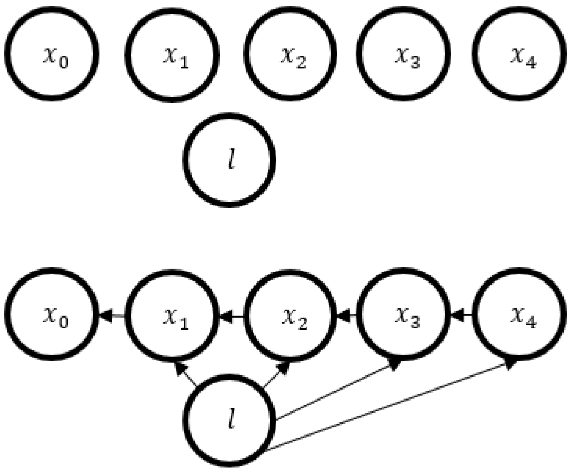


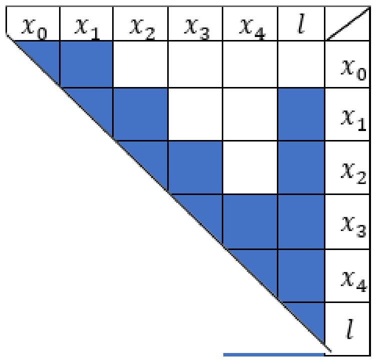
Elinination of l:

$$f_{joint}(l) = f_{l-new}(l) \propto p(l)$$

After the elimination of l *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_3|x_4, l)p(l)$$





d: Repeat the previous clause using a different variable elimination order:

$$x_4, x_3, x_2, l, x_1, x_0$$

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) = \eta \cdot p(x_0)p(x_1|x_0, u_0)p(x_2|x_1, u_1)p(x_3|x_2, u_2)p(x_4|x_3, u_3)p(z_1|x_1, l)p(z_2|x_2, l)$$

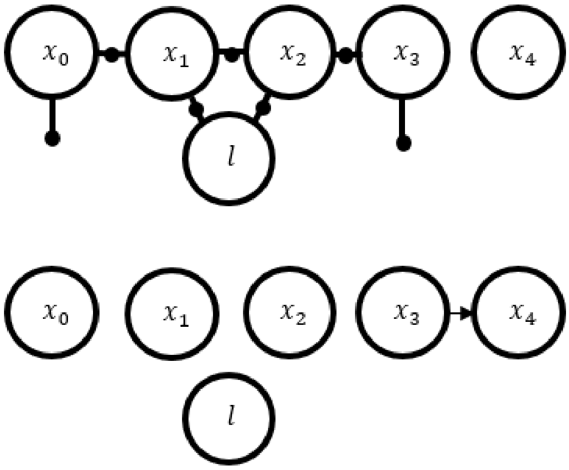
$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_4(x_3, x_4)f_{l1}(x_1, l)f_{l2}(x_2, l)$$

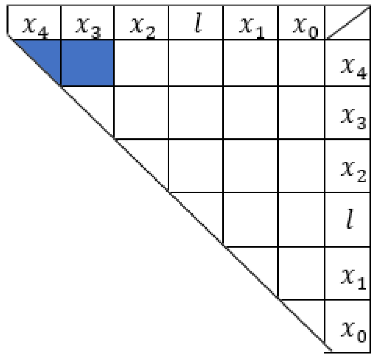
<u>*Elinination of* x_4:</u>

$$f_{joint}(x_3, x_4) = f_4(x_3, x_4) \propto p(x_4|x_3) \cdot f_{3-new}(x_3)$$

After the elimination of x_4 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3) \cdot f_{3-new}(x_3)f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_3(x_2, x_3)f_{l1}(x_1, l)f_{l2}(x_2, l)$$



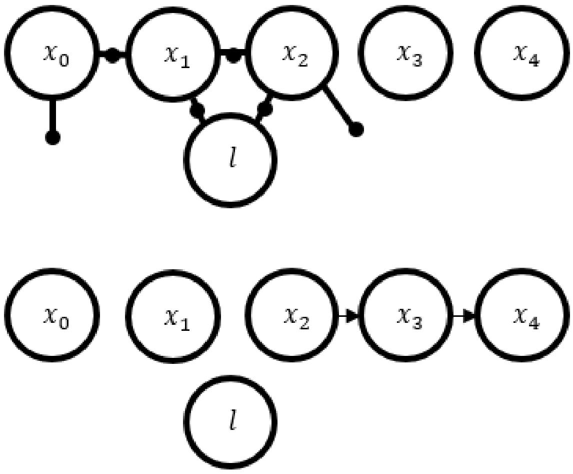


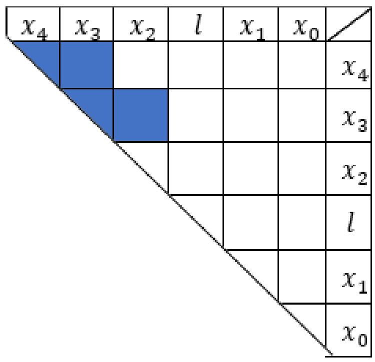
<u>*Elinination of* x_3:</u>

$$f_{joint}(x_2, x_3) = f_{3-new}(x_3)f_3(x_2, x_3) \propto p(x_3|x_2) \cdot f_{2-new}(x_2)$$

After the elimination of x_3 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2) \cdot f_{2-new}(x_2)f_0(x_0)f_1(x_0, x_1)f_2(x_1, x_2)f_{l1}(x_1, l)f_{l2}(x_2, l)$$



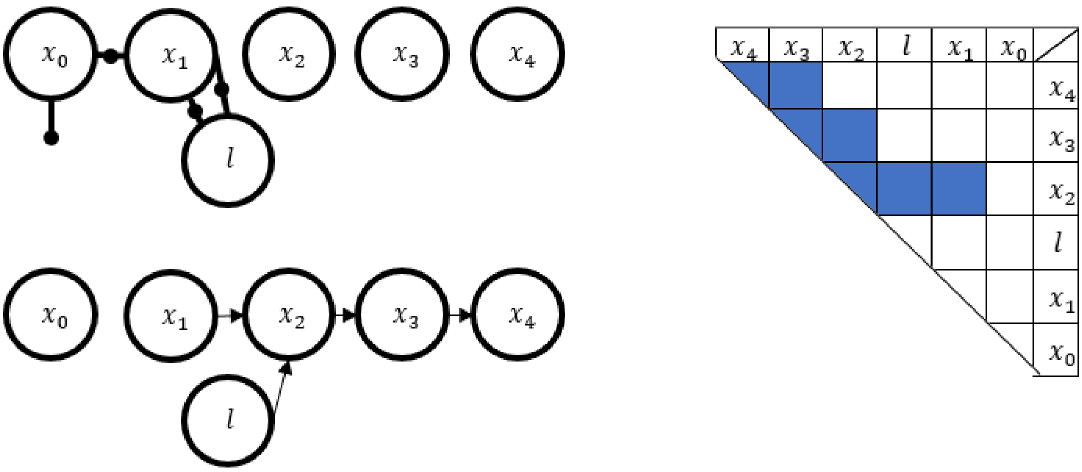


Elinination of x_2:

$$f_{joint}(x_1, x_2, l) = f_{2-new}(x_2) f_2(x_1, x_2) f_{l2}(x_2, l) \propto p(x_2|x_1, l) \cdot f_{l-new}(x_1, l)$$

After the elimination of x_2 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2)p(x_2|x_1, l) \cdot f_{l-new}(x_1, l) f_0(x_0) f_1(x_0, x_1) f_{l1}(x_1, l)$$

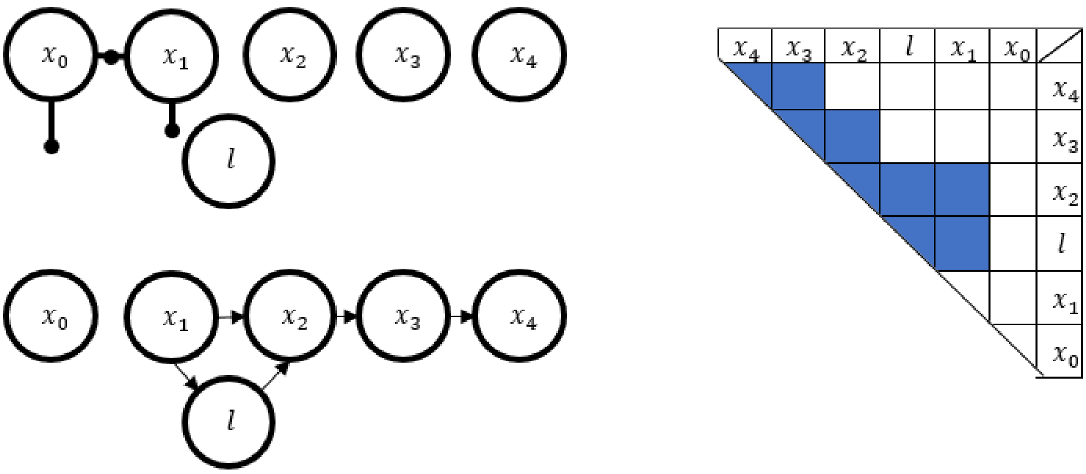


Elinination of l:

$$f_{joint}(x_1, l) = f_{l-new}(x_1, l) f_{l1}(x_1, l) \propto p(l|x_1) \cdot f_{1-new}(x_1)$$

After the elimination of x_3 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2)p(l|x_1) \cdot f_{1-new}(x_1) f_0(x_0) f_1(x_0, x_1)$$

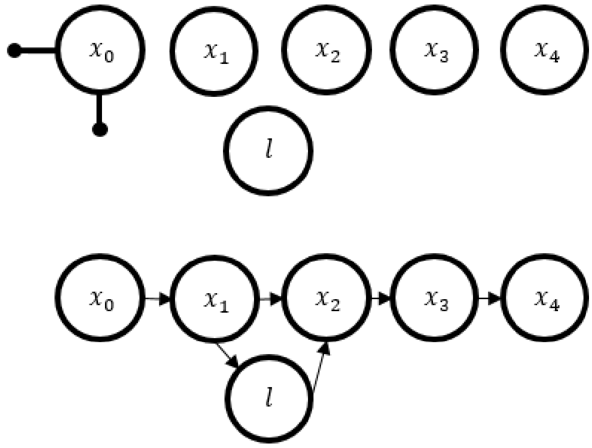


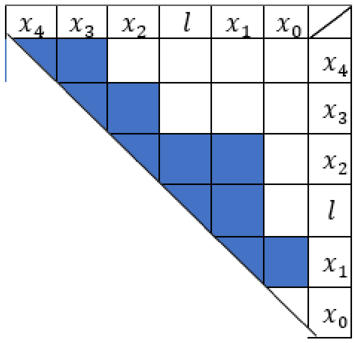
Elinination of x_1:

$$f_{joint}(x_0, x_1) = f_{1-new}(x_1) f_1(x_0, x_1) \propto p(x_1|x_0) \cdot f_{0-new}(x_0)$$

After the elimination of x_1 *we get*:

$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2)p(l|x_1)p(x_1|x_0) \cdot f_{0-new}(x_0) f_0(x_0)$$



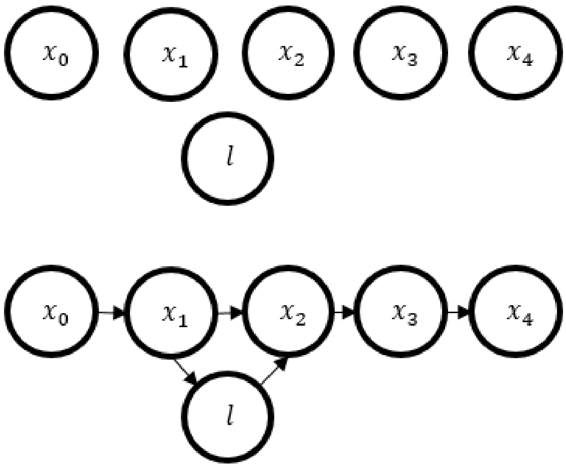


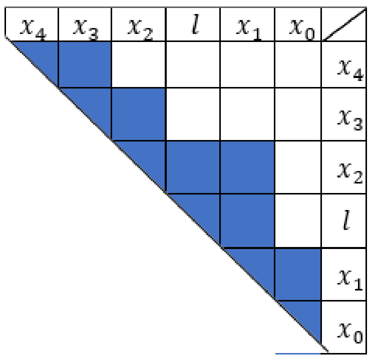
Elinination of x_0:

$$f_{joint}(x_0) = f_{0-new}(x_0) f_0(x_0) \propto p(x_0)$$

After the elimination of x_0 *we get*:

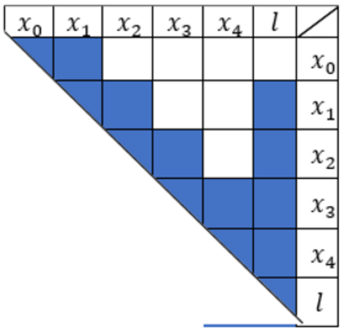
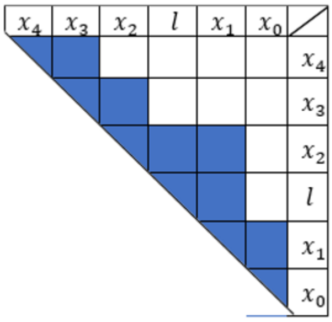
$$p(x_{0:4}, l|u_{0:3}, z_{1:2}) \propto p(x_4|x_3)p(x_3|x_2)p(l|x_1)p(x_1|x_0)p(x_0)$$





e: Which of the two elimination orders you would prefer in terms of estimation accuracy and computational aspects?

We show the elimination order and resulting R matrix for each section below.

{c} Elimination Order: $\{x_0, x_1, x_2, x_3, x_4, l\}$	{d} Elimination Order: $\{x_4, x_3, x_2, l, x_1, x_0\}$
	

Regarding computation efficiency: We would prefer the elimination order in {d}, as it produces a sparser R matrix (12 non-zero elements vs 14), and with more structure – all rows but one contains two elements at the start of the row.

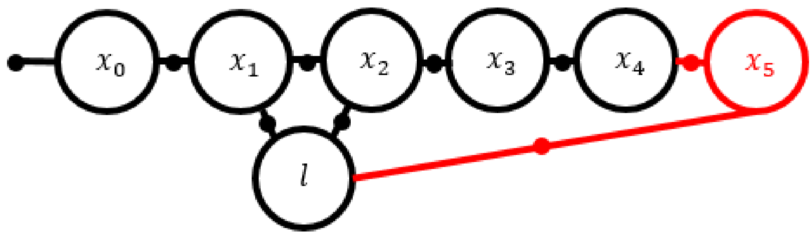
Regarding Accuracy: Both matrices contain the same information. As such, the solution of the LMS problem is independent of the elimination order, or which R matrix we choose to use.

$$R_d^T R_d = R_c^T R_c = A^T A$$

Question 2: Incremental factorization.
Consider now the robot, from question 1, executes command u_4 and moves to a new location; denote its new pose by x_5. Assume the robot observes again the landmark l from the new location.

a: Draw the factor graph of the problem and indicate the new factors and variable nodes.

We add one node for pose x_5, and two additional nodes for the factors that need to be computed: one for the motion model, and one for the measurement model.



b: Consider the Bayes net from question 1(c) with elimination order $x_0, x_1, x_2, x_3, x_4, l$. Perform incremental factorization by updating this Bayes net with the new information using the elimination order:

$$x_0, x_1, x_2, x_3, x_4, l, x_5$$

$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) = \eta \cdot p(x_0)p(x_1|x_0,u_0)p(x_2|x_1,u_1)p(x_3|x_2,u_2)p(x_4|x_3,u_3)p(x_5|x_4,u_4)p(z_1|x_1,l)p(z_2|x_2,l)\ p(z_3|x_5,l)$

$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) \propto f_0(x_0)f_1(x_0,x_1)f_2(x_1,x_2)f_3(x_2,x_3)f_4(x_3,x_4)f_5(x_4,x_5)f_{l1}(x_1,l)f_{l2}(x_2,l)f_{l3}(x_5,l)$

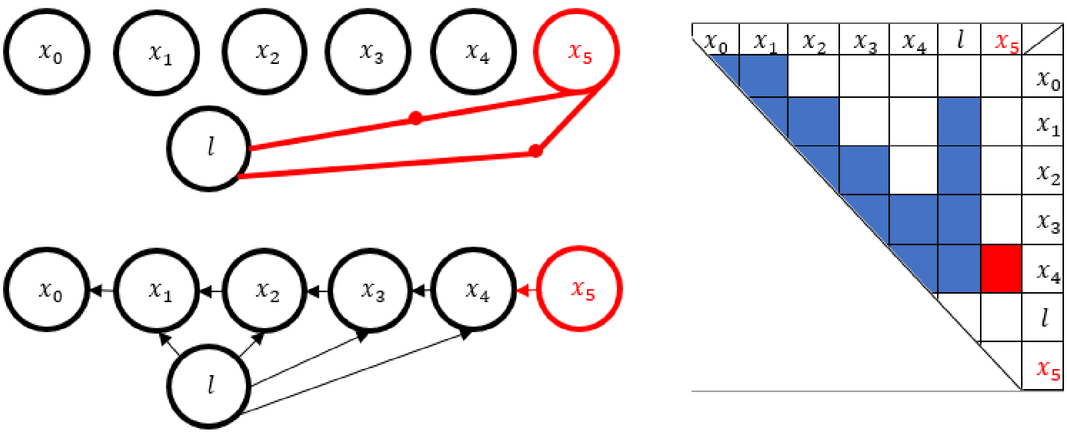
$We\ need\ to\ reeliminate\ the\ factors\ x_4\ and\ l\ because\ they\ involve\ the\ changes.$

<u>$Back\ to\ Elimination\ of\ x_4$:</u>

$$f_{joint}(x_4, x_5, l) = f_{4-new}(x_4, l)f_5(x_4, x_5) \propto p(x_4|x_5, l) \cdot f_{l-new}(x_5, l)$$

$After\ the\ elimination\ of\ x_4\ we\ get$:

$$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2,l)p(x_2|x_3,l)p(x_4|x_5,l) \cdot f_{l3}(x_5,l)f_{l-new}(x_5,l)$$

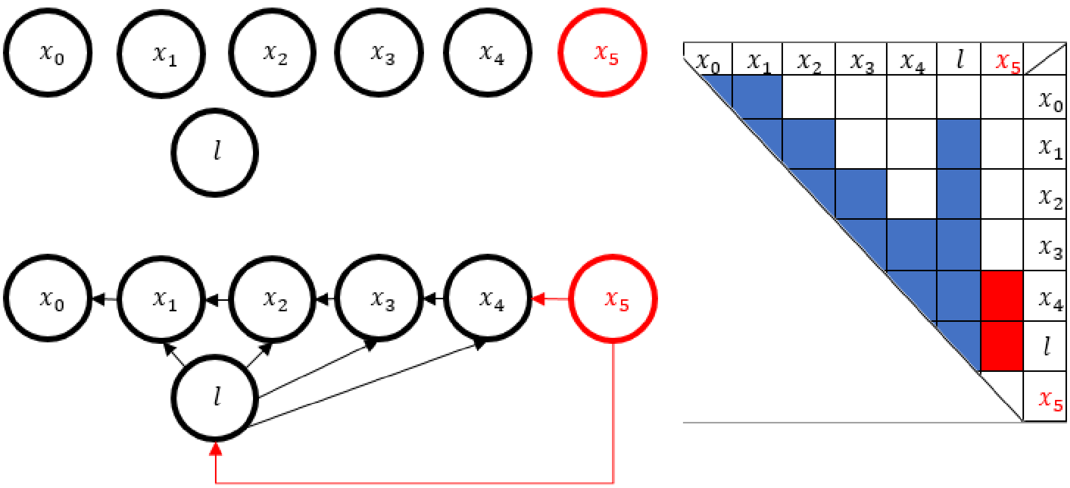


<u>$Back\ to\ Elinination\ of\ l$:</u>

$$f_{joint}(x_5, l) = f_{l-new}(x_5, l)f_{l3}(x_5, l) \propto p(l|x_5) \cdot f_{5-new}(x_5)$$

$After\ the\ elimination\ of\ l\ we\ get$:

$$p(x_{0:4}, x_5, l|u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2,l)p(x_3|x_4,l)p(l|x_5) \cdot f_{5-new}(x_5)$$

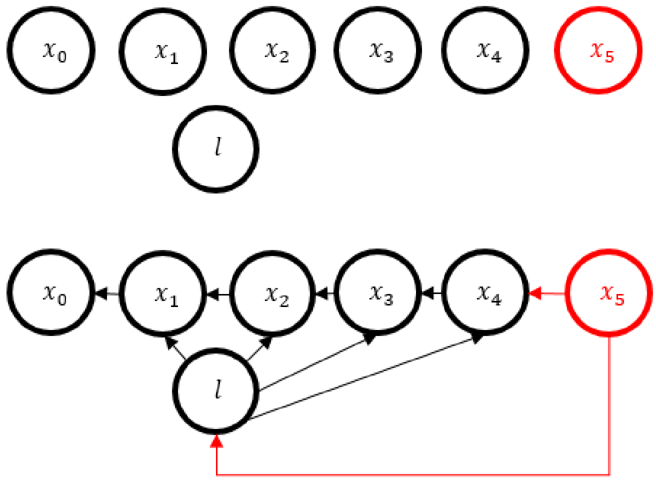


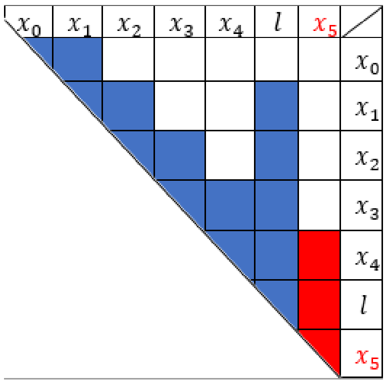
Elinination of x_5:

$$f_{joint}(x_5) = f_{5-new}(x_5) \propto p(x_5)$$

After the elimination of x_5 *we get*:

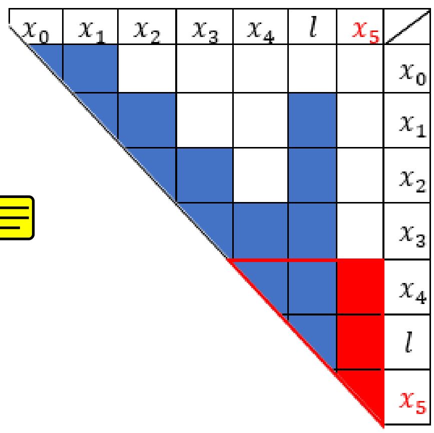
$$p(x_{0:4}, x_5, l | u_{0:3}, z_{1:2}, z_3) \propto p(x_0|x_1)p(x_1|x_2, l)p(x_3|x_4, l)p(l|x_5)p(x_5)$$

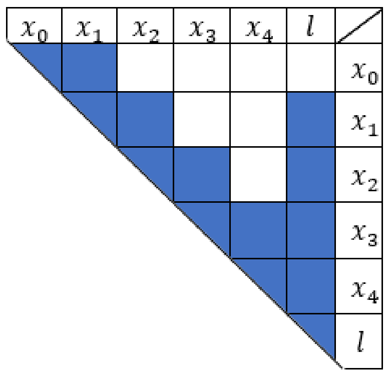




c: Show the corresponding updated square root information matrix R

The new non-zero elements, the boxes colored red in the new R matrix, describe states that depend on x_5 in the bayes-net graph.
In general, values in the red triangle are in the "update impact zone" and are subject to change when computing the new R matrix.



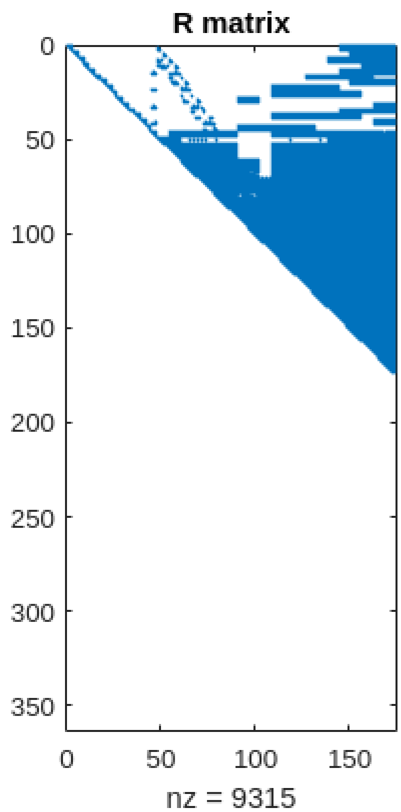


Q3 - Variable Ordering

```
load('hw5_A.mat'); %produces A matrix in the workspace
```

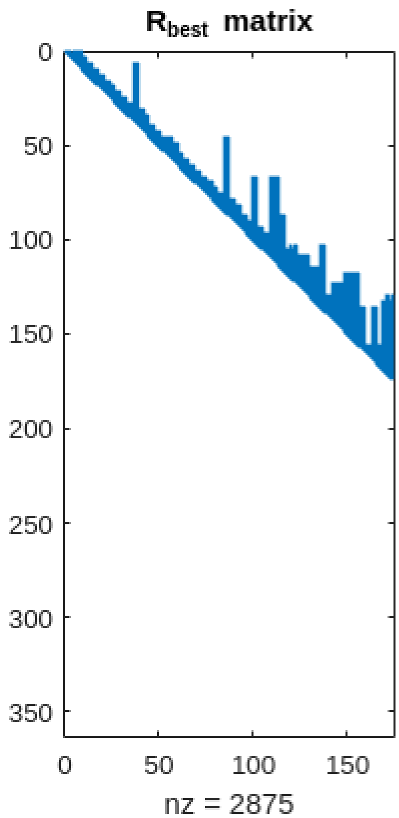
(a) Obtain R with QR fatorization

```
[~,R] = qr(A);
spy(R);
title('R matrix');
```



(b) best R

```
p = colamd(A);
[~,Rbest] = qr(A(:,p));
spy(Rbest);
title('R_{best} matrix');
```



R_{best} has ~30% values of the initial R matrix.

Hence will require to do only ~30% of the computations when sovling back subsitution with R_{best} when compared to R.