Irreducible unitary representations of the infinite symmetric group $S(\infty)$

Rapport de stage de recherche

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Abstract

Following the approach proposed by Olshanki in [O] and generalized by Tsankov in [T1] and [T2], the objective of this work is to expose the remarkable theorem by Lieberman [L] which asserts that every irreducible unitary representation of the infinite symmetric group $S(\infty)$, is induced by an irreducible representation of a finite quotient of some specific subgroups of $S(\infty)$. Moreover, every irreducible unitary representation of $S(\infty)$ is the sum of those induced representations.

Résumé

Suivant l'approche proposée par Olshanski dans [O] et généralisé par Tsankov dans [T1] et [T2], le but de ce travail est d'exposer le remarquable théorèm par Lieberman [L] qui affirme que chaque représentation unitaire irréductible du groupe symétrique infini $S(\infty)$, est induite para une repésentation irreductible d'un quotient fini de certains sous-groupes de $S(\infty)$. De plus, chaque représentation unitaire irréductible de $S(\infty)$ est la somme de cés représentations induites.

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Contents

1 Preliminaries

1.1 The unitary group $\mathcal{U}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$

In this section we recall the basic notions and concepts required for the rest of the work.

Definition 1.1. A group G is said to be a topological group if G is a topological space and the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are both continuous.

Throughout our exposition, we will consider Hilbert spaces, all of them separable complex. We denote the inner product of two points ξ, η in a Hilbert space $\mathcal{H}$ by $\langle \xi, \eta \rangle$.

Definition 1.2. An operator $U : \mathcal{H} \to \mathcal{H}$ is said to be unitary if its inverse is the adjoint operator, i.e. $UU^* = U^*U = I$.

We denote the set of all unitary operators of a Hilbert space by $\mathcal{H}$ by $\mathcal{U}(\mathcal{H})$.

Proposition 1.3. *Let U be an operator on $\mathcal{H}$. Then $U \in \mathcal{U}(\mathcal{H})$ if and only if $\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle$, for all $\xi, \eta \in \mathcal{H}$, and U is onto.*

Proof. If U is unitary, by definition $UU^* = U^*U = I$, then U invertible and hence U is onto. Now, $\langle U\xi, U\eta \rangle = \langle \xi, U^*U\eta \rangle = \langle \xi, \eta \rangle$, for all $\xi, \eta \in \mathcal{H}$.

Now, let U onto such that U preserves the inner product. We need to show that U is one to one. Suppose that $U\xi = U\eta$. Thus $\langle U\xi - U\eta, U\xi - U\eta \rangle = 0$. Since $\langle U\xi - U\eta, U\xi - U\eta \rangle = \langle \xi - \eta, \xi - \eta \rangle = 0$, then $\xi = \eta$.

□

An easy but necessary remark is the following.

Remark 1.4. $\mathcal{U}(\mathcal{H})$ is a group. Indeed, if $U, T \in \mathcal{U}(\mathcal{H})$ and $\xi, \eta \in \mathcal{H}$, then $\langle UT\xi, UT\eta \rangle = \langle T\xi, U^*UT\eta \rangle = \langle \xi, \eta \rangle$, as $U, T \in \mathcal{U}(\mathcal{H})$. Now, since $U^{-1} = U^*$, it is immediate that $U^{-1} \in \mathcal{U}(\mathcal{H})$.

In the following section we will see that, respect to a certain topology, $\mathcal{U}(\mathcal{H})$ is in fact a topological group.

1.2 Topologies on $\mathcal{U}(\mathcal{H})$

We are interested in defining two equivalent topologies in $\mathcal{U}(\mathcal{H})$. The first topology we introduce is called the **strong operator topology**, in which an element $U_0 \in \mathcal{U}(\mathcal{H})$ has a base of neighborhoods consisting of all the sets of the form

$$\{U \in \mathcal{U}(\mathcal{H}) : ||(U - U_0)\xi_i|| < \epsilon, 1 \leq i \leq k\}$$

where $\xi_i \in \mathcal{H}$ for all $i = 1, \ldots, k$, and $\epsilon > 0$.

Definition 1.5. We say that a sequence $\{U_n : n \in \mathbb{N}\}$ of operators in $\mathcal{U}(\mathcal{H})$ converges to U in the strong operator topology, abbreviated by $U_n \xrightarrow{s.o.t.} U$, if $U_n\xi \to U\xi$ for every $\xi \in \mathcal{H}$, i.e $||U_n\xi - U\xi|| \to 0$, for every $\xi \in \mathcal{H}$.

The second topology is called the **weak operator topology**, in which an element $U_0 \in \mathcal{U}(\mathcal{H})$ has a base of neighborhoods consisting of all the sets of the form

$$\{U \in \mathcal{U}(\mathcal{H}) : |\langle (U - U_0)\xi_i, \eta_i\rangle| < \epsilon, 1 \leq i \leq k\}$$

where $\xi_i, \eta_i \in \mathcal{H}$ for all $i = 1, \ldots, k$ and $\epsilon > 0$.

Definition 1.6. We say that a sequence $\{U_n : n \in \mathbb{N}\}$ of operators in $\mathcal{U}(\mathcal{H})$ converges to U in the weak operator topology, abbreviated by $U_n \xrightarrow{w.o.t.} U$, if $\langle U_n\xi, \eta\rangle \to \langle U\xi, \eta\rangle$, for every $\xi, \eta \in \mathcal{H}$.

Proposition 1.7. *The strong operator topology and the weak operator topology coincide on $\mathcal{U}(\mathcal{H})$.*

Proof. Let $\{U_n\}$ be a sequence of operators in $\mathcal{U}(\mathcal{H})$ and let $\epsilon > 0$. Suppose that $U_n \xrightarrow{s.o.t.} U$, i.e there exists $N > 0$ such that $||U_n\xi - U\xi|| < \frac{\epsilon}{||\eta||}$, for all $n \geq N$. Since $|\langle U_n\xi, \eta\rangle - \langle U\xi, \eta\rangle| = |\langle (U_n - U)\xi, \eta\rangle|$, for all $\xi, \eta \in \mathcal{H}$, then by the Cauchy-Schwartz inequality we have:

$$\begin{array}{rcl} |\langle U_n\xi, \eta\rangle - \langle U\xi, \eta\rangle| & = & |\langle (U_n - U)\xi, \eta\rangle| \\ & \leq & ||(U_n - U)\xi|| \cdot ||\eta|| < \epsilon, \text{ for all } \xi, \eta \in \mathcal{H} \end{array},$$

which proves that $U_n \xrightarrow{s.o.t.} U$ implies $U_n \xrightarrow{w.o.t.} U$.

Suppose now that $U_n \xrightarrow{w.o.t.} U$, i.e $\langle U_n\xi, \eta\rangle \to \langle U\xi, \eta\rangle$, for all $\xi, \eta \in \mathcal{H}$. In particular, $\langle U_n\xi, \xi\rangle \to \langle U\xi, \xi\rangle$, for all $\xi \in \mathcal{H}$. Then

$$\begin{array}{rcl} ||U_n\xi - U\xi||^2 & = & ||(U_n - U)\xi||^2 \\ & = & \langle U_n\xi - U\xi, U_n\xi - U\xi\rangle \\ & = & \langle U_n\xi, U_n\xi\rangle + \langle U_n\xi, U_\xi\rangle - \overline{\langle U\xi, U_n\xi\rangle} + \langle U\xi, U\xi\rangle \\ & = & \langle U_n\xi, U_n\xi\rangle - (\langle U_n\xi, U\xi + \overline{\langle U_n\xi, U_\xi\rangle}) + \langle U\xi, U\xi\rangle \\ & = & 2||\xi||^2 - 2\,\mathrm{Re}\langle U_n\xi, U\xi\rangle \to 0, \end{array}$$

which proves that $U_n \xrightarrow{w.o.t.} U$ implies $U_n \xrightarrow{s.o.t.} U$. ☐

Proposition 1.8. *$\mathcal{U}(\mathcal{H})$ is a topological group respect to the strong operator topology.*

Proof. Let us prove that the map $(U, T) \mapsto UT$ is continuous. Let $(U_0, T_0) \in \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$. Let $\epsilon > 0$. Then a basic neighborhood of U_0T_0 is of the form

$$V = \{R \in \mathcal{U}(\mathcal{H}) : ||(R - U_0T_0)\xi|| < \epsilon\}, \text{ where } \xi \in \mathcal{H}$$

Let $W = \{(U, T) \in \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) : ||(U - U_0)T_0\xi|| < \frac{\epsilon}{2}, ||(T - T_0)\xi|| < \frac{\epsilon}{2}\}$. Hence W is a neighborhood of $(U_0, T_0) \in \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$ and for all $(U, T) \in W$, we have that

$$\begin{aligned} ||(UT - U_0T_0)\xi|| &= ||((UT - UT_0)\xi) + (UT_0 - U_0T_0)\xi|| \\ &\leq ||(UT - U_0T_0)\xi|| + ||(U - U_0)T_0\xi|| \\ &\leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}$$

In other words, the image of W under the map $(U, T) \mapsto UT$ is contained in V, as required.

By the Proposition 1.7, we can use the weak operator topology to show that the map $U \mapsto U^{-1}$ is continuous. Let $U_0 \in \mathcal{U}(\mathcal{H})$. A basic neighborhood of U_0^{-1} is of the form

$$V = \{T \in \mathcal{U}(\mathcal{H}) : |\langle (T - U_0^{-1})\xi, \eta\rangle| < \epsilon\}, \text{ where } \xi, \eta \in \mathcal{H}.$$

Then the neighborhood $W = \{U \in \mathcal{U}(\mathcal{H}) : |\langle (U - U_0)\xi, \eta\rangle| < \epsilon\}$, where $\xi, \eta \in \mathcal{H}$, is such that if $U \in W$, then, since $|\langle (U^{-1} - U_0^{-1})\xi, \eta\rangle| = |\langle (U - U_0)\eta, \xi\rangle| < \epsilon$, for all $\xi, \eta \in \mathcal{H}$, we have that the image of W under the map $U \mapsto U^{-1}$ is contained in V, as required.

□

1.3 Unitary representations of a group G

We introduce now the main notion of this work.

Definition 1.9. A unitary representation of a topological group G is a strongly continuous homomorphism from G to the unitary group of some Hilbert space $\mathcal{H}$.

$$\pi : G \to \mathcal{U}(\mathcal{H})$$

By strongly continuous we mean that the map $G \to \mathcal{H}$, $g \to \pi(g)\xi$ is continuous for every $\xi \in \mathcal{H}$.

According to the Proposition 1.8, we are only interested in homomorphisms between topological groups. It will be common to use the phrase π is a representation of G in $\mathcal{H}$ instead of the symbol $\pi : G \to \mathcal{U}(\mathcal{H})$.

In what follows π is always a unitary representation of a topological group G and $\mathcal{H}(\pi)$ denotes its Hilbert space.

Definition 1.10. A subspace $\mathcal{H}_1$ of $\mathcal{H}(\pi)$ is said to be π-invariant if the following holds:

$$\xi \in \mathcal{H}_1 \text{ implies } \pi(g)\xi \in \mathcal{H}_1, \text{ for every } g \in G.$$

Every representation has at least two π-invariant subspaces: the null space $\{0\}$ and $\mathcal{H}(\pi)$ itself. These π-invariant subspaces are said to be trivial.

Definition 1.11. A unitary representation π is said to be irreducible if $\mathcal{H}(\pi)$ has no non-trivial π-invariant subspaces.

As we show below, the orthogonal complement of π-invariant subspaces are π-invariant as well.

Proposition 1.12. *Let $\mathcal{H}_1$ be a π-invariant subspace of $\mathcal{H}(\pi)$, then $\mathcal{H}_1^\perp$ is a π-invariant subspace of $\mathcal{H}(\pi)$.*

Proof. Let $\xi \in \mathcal{H}_1^\perp$. We want to show that $\pi(g)\xi \in \mathcal{H}_1^\perp$ for all $g \in G$. Indeed, if $g \in G$ and $\eta \in \mathcal{H}_1$, then $\langle \pi(g), \eta \rangle = \langle \xi, \pi(g)^* \eta \rangle = \langle \xi, \pi(g^{-1})\eta \rangle = 0$, as $\xi \in \mathcal{H}_1^\perp$ and $\mathcal{H}_1$ is π-invariant.

□

Definition 1.13. A vector $\xi \in \mathcal{H}(\pi)$ is said to be cyclic if the linear span of $\{\pi(g)\xi : g \in G\}$ is dense in $\mathcal{H}(\pi)$.

The following statement tells us that every non-zero point of $\mathcal{H}(\pi)$ is cyclic provided π is irreducible.

Proposition 1.14. *Let ξ be a non-zero point in $\mathcal{H}(\pi)$. If H_ξ denotes the linear span of $\{\pi(g)\xi : g \in G\}$, then we have.*

1. *The closure of H_ξ, denoted by $\operatorname{cl} H_\xi$, is π-invariant.*
2. *If π is irreducible, then ξ is cyclic.*

Proof. (1). Let $\xi_0 \in \operatorname{cl} H_\xi$. Then there exists a sequence $\{\xi_n\}_{n\in\mathbb{N}} \subseteq H_\xi$ such that $\xi_n \to \xi_0$. Clearly $\pi(g)\xi_n \in H_\xi$. Since $\pi(g)$ is continuous, we have that $\pi(g)\xi_n \to \pi(g)\xi_0$. Hence, $\pi(g)\xi_0 \in \operatorname{cl} H_\xi$ and consequently, $\operatorname{cl} H_\xi$ is π-invariant.

(2). If π is irreducible, then the only proper subspaces of $\mathcal{H}(\pi)$ are $\{0\}$ and $\mathcal{H}(\pi)$. Thus, by (1), $\operatorname{cl} H_\xi$ must be equal to $\mathcal{H}(\pi)$, proving that ξ is cyclic.

□

Definition 1.15. Let π be a representation of G. We say that π is cyclic if there exists a point ξ in $\mathcal{H}(\pi)$ such that ξ is cyclic.

The following statement gives us a description of all the unitary representation of a group G. In this result, the Hilbert space of the representation will be denoted simply by $\mathcal{H}$.

Theorem 1.16. *Every unitary representation of G is a direct sum of cyclic subrepresentations.*

Proof. Let $\xi_1 \in \mathcal{H}\backslash\{0\}$. By the Proposition 1.14, $\operatorname{cl} H_{\xi_0}$ is a π-invariant subspace of $\mathcal{H}$. If $\operatorname{cl} H_{\xi_0} = \mathcal{H}$, the proof is completed. If not, we consider $(\operatorname{cl} H_{\xi_0})^{\perp}$. Let ξ_2 a non zero point of $(\operatorname{cl} H_{\xi_0})^{\perp}$ and let H_{ξ_2} be the linear span of $\{\pi(g)\xi_2 : g \in G\}$. Then, by the Proposition 1.14, $\operatorname{cl} H_{\xi_2}$ is π-invariant and by construction $\operatorname{cl} H_{\xi_1} \perp \operatorname{cl} H_{\xi_2}$, and so on.

Let I be an index and let $\mathcal{F}$ be the family of all $\{\operatorname{cl} H_{\xi_i} : i \in I\}$, each composed of a sequence of mutually orthogonal, invariant an cyclic subspaces. We order $\mathcal{F}$ by inclusion. Thus $\mathcal{F}$ is a partial order. By Zorn's lemma, $\mathcal{F}$ has a maximal element. By the separability of $\mathcal{H}$, this maximal element contains at most a countable number of subspaces and their direct sum, by maximality, must coincide with $\mathcal{H}$.

$\square$

We introduce now the notions of equivalent representations.

Definition 1.17. Let π and τ be two representations of G in $\mathcal{H}(\pi)$ and $\mathcal{H}(\tau)$, respectively. We say that the representations π and τ are equivalent if there exists a unitary isomorphism $I : \mathcal{H}(\pi) \to \mathcal{H}(\tau)$ such that

$$I\pi(g) = \tau(g)I \text{ for all } g \in G.$$

Proposition 1.18. *Let π and τ be two representations of G in $\mathcal{H}(\pi)$ and $\mathcal{H}(\tau)$, respectively. If there exist cyclic vectors $\xi_0 \in \mathcal{H}(\pi)$ and $\eta_0 \in \mathcal{H}(\tau)$ such that*

$$\langle \pi(g)\xi_0, \xi_0\rangle = \langle \tau(g)\eta_0, \eta_o\rangle \text{ for all } g \in G,$$

then π and τ are equivalent.

Proof. Let H_{ξ_0} be the linear span of $\{\pi(g)\xi_0 : g \in G\}$ and let H_{η_0} be the linear span of $\{\tau(g)\eta_0 : g \in G\}$. Then every ξ is of the form $\xi = \sum_{i=0}^{n} \alpha_i \pi(g_i)\xi_0$ and every $\eta \in H_{\eta_0}$ can be written as $\eta = \sum_{i=0}^{n} \beta_i \tau(g_i)\eta_0$. Hence we define the map $I : H_{\xi_0} \to H_{\eta_0}$ as follows:

$$\xi = \sum_{i=0}^{n} \alpha_i \pi(g_i)\xi_0 \mapsto \sum_{i=0}^{n} \alpha_i \tau(g_i)\eta_0.$$

Then we have

$$\begin{aligned}
||I\xi||^2 &= \langle \sum_{i=1}^{n} \alpha_i \tau(g_i)\eta_0, \sum_{i=1}^{n} \alpha_i \tau(g_i)\eta_0\rangle \\
&= \sum_{i,j=1}^{\infty} \alpha_i \bar{\alpha_i} \langle \tau(g_i)\eta_0, \tau(g_j)\eta_0\rangle \\
&= \sum_{i,j=1}^{\infty} \alpha_i \bar{\alpha_i} \langle \tau(g_j^{-1} g_i)\eta_0, \eta_0\rangle \\
&= \sum_{i,j=1}^{\infty} \alpha_i \bar{\alpha_i} \langle \pi(g_j^{-1} g_i)\xi_0, \xi_0\rangle, \text{by hypothesis} \\
&= \langle \sum_{i=0}^{n} \alpha_i \pi(g_i)\xi_0, \sum_{i=0}^{n} \alpha_i \pi(g_i)\xi_0\rangle \\
&= ||\xi||^2.
\end{aligned}$$

Thus, I is an isometric linear map such that $I\pi(g)\xi = \tau(g)I\xi$ for all ξ in $\mathcal{H}(\pi)$. Hence, I can be extended uniquely to a unitary map $J : \mathcal{H}(\pi) \to \mathcal{H}(\tau)$ such that $J\pi = \tau J$. In other words, π is equivalent to τ.

□

2 $S(\infty)$ and their subgroups

By $S(\infty)$ we will denote the group of all bijections from $\mathbb{N}$ to $\mathbb{N}$. A bijection $\sigma \in S(\infty)$ is said to be finite if the set $\{i \in \mathbb{N} : \sigma i \neq i\}$ is finite. We denote the group of all finite bijections by S. It is easy to see that S is a countable normal subgroup of $S(\infty)$.

Given $n = 1, 2, \dots$, we set the following subgroups of $S(\infty)$:

- $S_n(\infty) = \{\sigma \in S(\infty)\} : \sigma 1 = 1, \dots, \sigma n = n\}$
- $\tilde{S_n}(\infty) = \{\sigma \in S(\infty) : \sigma\{1, \dots, n\} = \{1, \dots, n\}\}$

In words, $S_n(\infty)$ is the set of bijections that fixes pointwise the first n numbers while $\tilde{S}_n(\infty)$ fixes setwise the first n numbers.

Proposition 2.1. *$S_n(\infty)$ is a normal subgroup of $\tilde{S}_n(\infty)$.*

Proof. Let $\sigma_0 \in S_n(\infty)$ and let σ be any element of $\tilde{S}_n(\infty)$. We want to show that $\sigma\sigma_0\sigma^{-1} \in S_n(\infty)$. Indeed, if $i \in \{1, \dots, n\}$, then $\sigma^{-1} i \in \{1, \dots, n\}$. Since σ_0 fixes any element of the set $\{1, \dots, n\}$, then $\sigma_0\sigma^{-1} i = \sigma^{-1} i$. Thus, $\sigma\sigma_0\sigma^{-1} i = \sigma\sigma^{-1} i = i$, as required.

□

According to the last proposition, we can consider the quotient $\tilde{S}_n(\infty)/S_n(\infty)$, which is precisely the finite symmetric group. We denote $\tilde{S}_n(\infty)/S_n(\infty)$ by $S(n)$.

In $S(\infty)$, there is a natural structure of topological group which topology is induced by the pointwise convergence, i.e. given $\{\sigma_n\}$ a sequence of bijections in $S(\infty)$, $\sigma_n \to g$ if and only if $\sigma_n i \to \sigma i$ for all $i \in \mathbb{N}$. It is important to say that we are considering $\mathbb{N}$ with the discrete topology. Hence, the subgroups $S_n(\infty)$ form a family of basic neighborhoods of the identity. Moreover, the subgroups $S_n(\infty)$ and $\tilde{S}_n(\infty)$ are open in $S(\infty)$.

3 Representations of $S(\infty)$

3.1 The canonical representation of $S(\infty)$

At the moment we have not given any example of a representation. It is time to do it. We set $G = S(\infty)$ and $\mathcal{H} = \ell^2(\mathbb{N})$, i.e. $\mathcal{H}$ is the Hilbert space consisting of all functions $f : \mathbb{N} \to \mathbb{C}$ such that $\sum\limits_{i\in\mathbb{N}} |f(i)|^2 < \infty$ and for which the inner product is defined by

$$\langle f, g\rangle = \sum_{i\in\mathbb{N}} f(i)\overline{g(i)}$$

Proposition 3.1. *The map $\pi : G \to \mathcal{H}$ defined by*

$$\pi(\sigma)f(i) = f(\sigma^{-1}i)$$

is a unitary representation of G.

Proof. The map π is a homomorphism. Indeed, if $\sigma_0, \sigma_1 \in G$, $f \in \mathcal{H}$ and $i \in \mathbb{N}$, then we have

$$\begin{array}{rcl} \pi(\sigma_0)\pi(\sigma_1)f(i) & = & \pi(\sigma_0)(\pi(\sigma_1))f(i) \\ & = & \pi(\sigma_1)f(\sigma_0^{-1}i) \\ & = & f(\sigma_1^{-1}\sigma_0^{-1}i) \\ & = & f((\sigma_0\sigma_1)^{-1}i) \\ & = & \pi(\sigma_0\sigma_1)f(i) \end{array} .$$

Therefore $\pi(\sigma_0)\pi(\sigma_1) = \pi(\sigma_0\sigma_1)$.

Let us show that π is unitary. Indeed, for any $f, g \in \mathcal{H}$ and $i \in \mathbb{N}$, we have

$$\begin{aligned}
\langle \pi(\sigma)f(i), \pi(\sigma)g(i)\rangle &= \langle f(\sigma^{-1}i), g(\sigma^{-1}i)\rangle \\
&= \sum_{i\in\mathbb{N}} f(\sigma^{-1}i)\overline{g(\sigma^{-1}i)} \\
&= \sum_{i\in\mathbb{N}} f(i)\overline{g(i)} \\
&= \langle f(i), g(i)\rangle.
\end{aligned}$$

Therefore $\langle \pi(\sigma)f, \pi(\sigma)g\rangle = \langle f, g\rangle$.

□

The unitary representation π of G constructed above is known as the canonical representation of G.

3.2 The induced representation

We are interested now in giving a representation of $S(\infty)$ using a representation for the subgroup $S(n)$. For convenience we set $G = S(\infty)$, $H = \tilde{S}_n(\infty)$ and $V = S_n(\infty)$.

Let τ be a unitary irreducible representation of $S(n)$. We can consider τ as a representation of H such that τ is trivial on V.

Definition 3.2. T is said to be a complete system of left cosets representatives of H in G if it satisfies the following properties:

1. $\bigcup_{g\in T} gH = G$.
2. $gH \cap g'H = \emptyset$, for all $g, g' \in T$ with $g \neq g'$.

Let T be a complete system of left cosets representatives. Let M be the space of all functions $f : G \to \mathcal{H}(\tau)$ for which:

$$f(gh) = \pi(h^{-1})f(g) \text{ for all } g \in G, h \in H.$$

Remark 3.3. For every $f \in M$, the norm of f is constant on left cosets of H in G. Indeed, for all $g \in G, h \in H$, we have

$$\begin{aligned}
||f(gh)||^2 &= ||\pi(h^{-1})f(g)||^2 \\
&= \langle \pi(h^{-1})f(g), \pi(h^{-1})f(g)\rangle \\
&= \langle f(g), f(g)\rangle \\
&= ||f(g)||^2
\end{aligned}$$

For $f \in M$, we define the norm of f as follows.

$$||f|| = (\sum_{g\in T} ||f(g)||^2)^{1/2}$$

Remark 3.4. By the previous remark, $||f||$ does not depend on the choice of T.

We will consider the Hilbert space $\mathcal{H} = \{f \in M : ||f|| < \infty\}$.

Definition 3.5. The induced representation $\mathrm{Ind}_H^G(\tau)$ in the Hilbert space $\mathcal{H}$ is defined by

$$(\mathrm{Ind}_H^G(\tau)(g))f(x) = f(g^{-1}x)$$

We have given the name of representation to $\mathrm{Ind}_H^G(\tau)$ without proving this fact. Let's fix this abuse.

Proposition 3.6. $\mathrm{Ind}_H^G(\tau)$ *is a unitary representation of* G *in* $\mathcal{H}$.

Proof. For brevity we set $\pi = \mathrm{Ind}_H^G(\tau)$. To show that π is unitary it's enough to mimic the computation done in the Proposition 3.1.

Let us verify that π is norm preserving. Indeed, given $g \in G$ we have that $||\pi(g)f||^2 = \sum_{y\in T} ||\pi(g)f(y)||^2 = \sum_{y\in T} ||f(g^{-1}y)||^2$. We note that the map $y \to g^{-1}y$ is an isomorphism from T to $g^{-1}T$. Thus, since $||f||$ does not depend on the choice of T, we have that

$$||\pi(g)f||^2 = \sum_{y\in T} ||\pi(g)f(y)||^2 = \sum_{y\in T} ||f(g^{-1}y)||^2 = \sum_{y\in T} ||f(y)||^2.$$

Therefore $||\pi(g)f|| = ||f||$.

□

As expected, the induced representation will be an irreducible representation for G provided τ is an irreducible representation for H. To prove this, we need the following lemma. We set $\pi = \mathrm{Ind}_H^G(\tau)$.

Let $\mathcal{H}_V(\pi)$ be the subspace of $\mathcal{H}(\pi)$ consisting of all the V-invariants functions, i.e. $\mathcal{H}_V(\pi) = \{f \in \mathcal{H}(\pi) : \pi(g)f = f$, for $g \in V\}$.

Lemma 3.7. $\mathcal{H}_V(\pi) = \{f \in \mathcal{H}(\pi) : f(x) = 0,$ *for* $x \notin H\}$.

Proof. ($\supseteq$). Let $f \in \mathcal{H}(\pi)$ such that $f(x) = 0$ for $x \notin H$. We want to show that f is V-invariant. Let $g \in V$ and let $y \in G$. Then there are two cases.

(i). If $y \notin H$, we observe first that $g^{-1}y \notin H$. Indeed, if $g^{-1}y \in H$, then $g^{-1}y\{1, \ldots, n\} = \{1, \ldots, n\}$. Thus, $y\{1, \ldots, n\} = g\{1, \ldots, n\}$ and since $g \in V$

and V is normal in H, it follows that $y\{1,\dots,n\} = \{1,\dots,n\}$, which means that $y \in H$. Thus, $0 = f(y) = f(g^{-1}y) = \pi(g)f(y)$ (by definition of π).

(ii). If $y \in H$, then $\pi(g)f(y) = f(g^{-1}y) = f(yy^{-1}g^{-1}y)$. Since V is normal in H, we have that $y^{-1}g^{-1}y \in V$. Thus $f(yy^{-1}g^{-1}y) = \tau(y^{-1}gy)f(y) = f(y)$, as τ is trivial in V. Hence $\pi(g)f(y) = f(y)$.

($\subseteq$) Towards a contradiction, suppose that $f \in \mathcal{H}(\pi)$ is V-invariant but $f(x) \neq 0$ for some $x \in G \setminus H$.

First, let us show that HxH contains infinitely many left cosets of H. For this, it enough to show that there exist $h_1, h_2, \dots$ in H such that $h_i xH \neq h_j xH$ for all $i \neq j$. Since $x \notin H$, then $x\{1,\dots,n\} \neq \{1,\dots,n\}$. This means that there exists $i \in \{1,\dots,n\}$ such that $xi > n$. Rearranging if necessary, we can suppose that $x1 = n+1$. Hence, for each $i \in \mathbb{N}$ we define $h_i \in G$ as follows:

$$h_i(j) = \begin{cases} j, & \text{if } 1 \leq j \leq n, \\ x1 + i, & \text{if } j \geq n+1. \end{cases}$$

By definition, $h_i \in H$ for all $i \in \mathbb{N}$. We note that $h_i \neq h_j$ for all $i \neq j$. With this in mind, we can see that $h_i xH \neq h_i xH$ for all $i \neq j$. In other words, there are infinitely many left cosets of H contained in HxH.

Now, since $[H : V] < \infty$, VxH also contains infinitely many left cosets of H. Now, as f is V-invariant, we have that $f(v^{-1}x) = f(x)$ for every $v \in V$. Moreover, we know that $||f||$ is constant on left cosets of H. In particular, $||f||$ is constant on the left cosets of the form vxH, which are infinitely many. Then $||f(x)||$ must be infinite, contradicting that $f \in \mathcal{H}(\pi)$.

□

The following statement asserts that we can give a representation equivalent to τ.

Proposition 3.8. *The representations $\pi(H)|_{\mathcal{H}_V(\pi)}$ and τ are equivalent.*

Proof. Let $I : \mathcal{H}_V(\pi) \to \mathcal{H}(\tau)$ defined as follows:

$$If(x) = f(1) \text{ for every } x \in G.$$

Let us show that I is a unitary isomorphism. The linearity of I is immediate.

To show that I is norm preserving we observe the following: setting $g = 1$ in the Remark 3.3, we get that $||f(h)|| = ||f(1)||$, for all $h \in H$. In other words, $||f(h)|| = ||If(x)||$ for all $x \in G, h \in H$. Now, according to the Lemma 3.7, f vanishes out of H, as $f \in \mathcal{H}_V(\pi)$. Hence we can conclude that $||If|| = ||f||$.

Let $h \in H$ and show that $\pi(h)I = I\tau(h)$. Indeed, if $f \in \mathcal{H}_V(\pi)$ and $x \in G$, then

$$\begin{aligned} \pi(h)If(x) &= \pi(h)If(x) \\ &= \pi(h)f(1) \\ &= f(h^{-1} \cdot 1) \\ &= f(1 \cdot h^{-1}) \\ &= \tau(h)f(1) \\ &= \tau(h)If(x) \end{aligned}$$

□

Theorem 3.9. $\pi = \mathrm{Ind}_H^G(\tau)$ *is irreducible if and only if* τ *is irreducible.*

Proof. The first direction is immediate. Indeed, if $\mathcal{H}(\tau)$ contains a non-trivial τ-invariant subspace $\mathcal{H}_0$ and τ_0 denotes the subrepresentation for τ in $\mathcal{H}(\tau_0)$, then $\mathrm{Ind}_H^G(\tau_0)$ is a non-trivial subrepresentation of $\mathrm{Ind}_H^G(\tau)$.

Let us prove the direction of main interest. We will prove that π is irreducible showing that if $\mathcal{H}(\pi) = \mathcal{K} \oplus \mathcal{K}^\perp$ for some $\mathcal{K}$ π-invariant subspace of $\mathcal{H}(\pi)$, then $\mathcal{K} = \mathcal{H}(\pi)$ or $\mathcal{K}^\perp = \mathcal{H}(\pi)$. Let $\mathcal{K}$ be such subspace. As the projection onto $\mathcal{K}$ commutes with $\pi(V)$, we have that

$$\mathcal{H}_V(\pi) = (\mathcal{H}_V(\pi) \cap \mathcal{K}) \oplus (\mathcal{H}_V(\pi) \cap \mathcal{K}^\perp),$$

where $\mathcal{H}_V(\pi) \cap \mathcal{K}$ and $\mathcal{H}_V(\pi) \cap \mathcal{K}^\perp$ are $\pi(H)$-invariant. According to the Proposition 3.8, $\pi(H)|_{\mathcal{H}_V(\pi)}$ and τ are equivalent and τ is irreducible by hypothesis, then we have that either $\mathcal{H}_V(\pi) \subseteq \mathcal{K}$ or $\mathcal{H}_V(\pi) \subseteq \mathcal{K}^\perp$. Now, we can check that for $f \in \mathcal{H}_V(\pi)$, if H_f denotes the linear span of $\{\pi(g)f : g \in G\}$, then H_f is dense in $\mathcal{H}(\pi)$, i.e. $\mathrm{cl}\, H_f = \mathcal{H}(\pi)$. By the Proposition 1.14, $\mathrm{cl}\, H_f$ is π-invariant, then $\mathrm{cl}\, H_f \subseteq \mathcal{H}_V(\pi)$. Hence, $\mathcal{H}(\pi) \subseteq \mathcal{K}$ or $\mathcal{H}(\pi) \subseteq \mathcal{K}^\perp$ and by assumption $\mathcal{K}, \mathcal{K}^\perp \subseteq \mathcal{H}(\pi)$. Then $\mathcal{H}(\pi) = \mathcal{K}$ or $\mathcal{H}(\pi) = \mathcal{K}^\perp$, proving that π is irreducible.

□

In what follows π is any unitary representation of G, $A \subseteq \mathbb{N}$, $H_A = \{g \in G : gA = A\}$ and $V_A = \{g \in G : gi = i \text{ for all } i \in A\}$.

Lemma 3.10. *There exist* $A_0 \subseteq \mathbb{N}$ *and* $\xi_0 \in \mathcal{H}(\pi)$ *such that* ξ_0 *is* V_{A_0}*-invariant.*

Proof. It suffices to show that $\bigcup_{A \subseteq \mathbb{N}} \mathcal{H}_{V_A}$ is dense in $\mathcal{H}(\pi)$. Let $\xi \in \mathcal{H}(\pi)$ and let $\epsilon > 0$. As $\{V_A : A \subseteq \mathbb{N}\}$ forms a basis at identity of G and π is continuous, there exists $A_0 \subseteq \mathbb{N}$ such that $\{\pi(v)\xi_0 : v \in V_{A_0}\}$ is contained in the ball of radio ϵ and centered in ξ_0. Let X be the closure of the convex hull of $\{\pi(v)\xi_0 : v \in V_A\}$ and let ξ_0 be the element of least norm of X. Then $||\xi_0 - \xi|| \leq \epsilon$ and ξ_0 is fixed by V_A.

$\square$

We call the set given by the previous lemma the set A_0 of minimum size such that $\mathcal{H}_{V_{A_0}} \neq \{0\}$.

Lemma 3.11. *Let A_0 be a subset of $\mathbb{N}$ of minimum size such that $\mathcal{H}_{V_{A_0}} \neq \{0\}$. Let $\xi_0 \in \mathcal{H}_{V_{A_0}}$. Then $\langle \pi(g)\xi_0, \xi_0 \rangle = 0$ for all $g \notin H_{A_0}$.*

Proof. Towards a contradiction, suppose that $\langle \pi(g)\xi_0, \xi_0 \rangle = c$, with $|c| > 0$ for some $g \notin H_{A_0}$. Then $gA_0 \neq A_0$. Let $A_1 = gA_0$ and let $B = A_0 \cap A_1$. Hence B is a proper subset of A_0. We choose a collection of $v_1, v_2, \ldots$ such that:

(i) $v_i \in V_{A_0}$ for all $i = 1, 2, \ldots$

(ii) $(v_i A_1) \cap (v_j A_1) = B$, for all $i \neq j$.

If we set $A_j = v_j A_1$ and $\xi_i = \pi(v_i g)\xi_0$. As the unit ball in any Hilbert space is weakly compact, then the sequence of ξ_i contains a weakly convergent subsequence, so passing to a subsequence we can assume that ξ_i converges weakly. Let η be the weak limit of the $\xi_i' s$. We observe that

$$
\begin{aligned}
\langle \xi_i, \xi_0 \rangle &= \langle \pi(v_i g)\xi_0, \xi_0 \rangle \\
&= \langle \pi(g)\xi_0, \pi(v_i^{-1})\xi_0 \rangle \\
&= \langle \pi(g)\xi_0, \xi_0 \rangle, \text{as } v_i \in A_0 \text{ and } \xi_0 \in \mathcal{H}_{V_{A_0}}(\pi)
\end{aligned}
$$

Therefore $\langle \xi_i, \xi_0 \rangle = \langle \pi(g)\xi_0, \xi_0 \rangle \neq 0$. Consequently $\eta \neq 0$.

Now, we fix $h \in V_B$. We want to show that $\pi(h)\eta = \eta$. We observe that for every finite subset C of $\mathbb{N}$, there exists $j \in \mathbb{N}$ and there exists $k_j \in V_{A_j}$ such that $k_j(c) = h(c)$ for all $c \in C$. For this, we choose $j \in \mathbb{N}$ such that $A_j \cap C \subseteq B$ and we define k_j as follows:

$$
k_j(a) = \begin{cases} a, & \text{if } a \in A_j, \\ h(a), & \text{if } a \in C \end{cases}
$$

We observe first that $k_j v_j g(a) = v_j g(a)$ for all $a \in A_0$. Indeed, if $a \in A_0$, then $g(a) \in A_1$, as $g \notin H_{A_0}$. Now, since $A_j = v_j A_1$, then $v_j g(a) \in A_j$. Thus, by definition of k_j, it follows that $k_j v_j g(a) = v_j g(a)$. In other words, we have that $(v_j g)^{-1} k_j v_j g(a) = a$ for all $a \in A_0$, i.e $(v_j g)^{-1} k_j v_j g \in V_{A_0}$. Hence $\pi((v_j g)^{-1} k_j v_j g)\xi_0 = \xi_0$, as ξ_0 is V_{A_0}-invariant, or equivalently:

$$
\pi(k_j v_j g)\xi_0 = \pi(v_j g)\xi_0
$$

Then

$$\begin{array}{rcl}\pi(h)\eta &=& \lim_{j\to\infty}\pi(k_j)\xi_j\\ &=& \lim_{j\to\infty}\pi(k_j)\pi(v_jg)\xi_0\\ &=& \lim_{j\to\infty}\pi(k_jv_jg)\xi_0\\ &=& \lim_{j\to\infty}\pi(v_jg)\xi_0=\eta\end{array}$$

proving that η is V_B-invariant. This contradicts the choice of A_0.

□

3.3 The main theorem

We are in conditions to prove the main theorem. For this we set $A = A_0$, $H = H_{A_0}$ and $V = V_{A_0}$ and $\xi_0 \in \mathcal{H}_V(\pi)$.

Theorem 3.12. *Every irreducible unitary representation of G is a sum of irreducible irreducible representations of the form $\mathrm{Ind}_H^G(\tau)$, where τ is an irreducible representation of H that factors through the finite quotient H/V.*

Proof. Let π be an irreducible unitary representation of G. Let $\mathcal{K}$ be the linear span of $\{\pi(h)\xi_0 : h \in H_A\}$. We observe two relevant facts:

1. If $g_1H \neq g_2H$ then $\pi(g_1)\mathcal{K} \perp \pi(g_2)\mathcal{K}$. Indeed, if $h_1, h_2 \in H$ then we have that

$$\langle \pi(g_1h_1)\xi_0, \pi(g_2h_2)\xi_0\rangle = \langle \pi(h_2^{-1}g_2^{-1}g_1h_1)\xi_0, \xi_0\rangle.$$

   Now, we observe that $h_2^{-1}g_2^{-1}g_1h_1 \notin H$. If not, $h_2^{-1}g_2^{-1}g_1h_1 \in H$ implies that $g_2^{-1}g_1 \in H_A$, which is not possible since $g_1H \neq g_2H$ by hypothesis. Hence, setting $g = h_2^{-1}g_2^{-1}g_1h_1$ in the Lemma 3.11, it follows that

$$\langle \pi(g_1h_1)\xi_0, \pi(g_2h_2)\xi_0\rangle = \langle \pi(h_2^{-1}g_2^{-1}g_1h_1)\xi_0, \xi_0\rangle = 0,$$

   as required.

2. $\mathcal{K}$ is V-invariant, i.e $\pi(v)\pi(h)\xi_0 = \pi(h)\xi_0$ for all $v \in V$. Indeed, we have that V is normal in H, i.e $h^{-1}vh \in V$ for all $h \in H, v \in V$. Then $\pi(h^{-1}vh)\xi_0 = \xi_0$, as ξ_0 is V-invariant. Hence $\pi(v)\pi(h)\xi_0 = \pi(h)\xi_0$, as required.

We note that the last assertion shows that we have obtained a representation for H in $\mathcal{K}$ which is trivial in V. Let τ be this representation and consider the representations induced by it . For brevity we set $\pi_0 = \mathrm{Ind}_H^G(\tau)$. Our purpose now is to show that π_0 and π are equivalent. For this, let T be a complete system of left cosets of H. We define $I : \mathcal{H}(\pi_0) \to \mathcal{H}(\pi)$ as follows:

$$I(f) = \bigoplus_{x\in T} \pi(x)f(x)$$

Let us check that I satisfies the properties required. By definition, I is linear.

The linear map I preserves the inner product. Indeed, if $f_1, f_2 \in \mathcal{H}(\pi_0)$, then

$$\begin{aligned}
\langle If_1, If_2\rangle &= \langle \bigoplus_{t\in T} \pi(t)f_1(t), \bigoplus_{t\in T} \pi(t)f_2(t)\rangle \\
&= \sum_{t\in T}\langle \pi(t)f_1(t), \pi(t)f_2(t)\rangle \\
&= \sum_{t\in T}\langle f_1(t), f_2(t)\rangle \\
&= \langle f_1, f_2\rangle
\end{aligned}$$

We check now that $I\pi_0 = \pi I$. Let $f \in \mathcal{H}(\pi_0)$ and $g \in G$. Thus

$$I(\pi_0(g)f) = \bigoplus_{t\in T} \pi(t)\pi_0(g)f(t) = \bigoplus_{t\in T} \pi(t)f(g^{-1}t), \text{ by definition of } \pi_0.$$

On the other hand

$$\pi(g)If = \pi(g) \bigoplus_{s\in T} \pi(s)f(s) = \bigoplus_{s\in T} \pi(gs)f(s).$$

To guarantee the equality we look for, it is enough to show that if t and gs are in the same left coset of H, then $\pi(gs)f(s) = \pi(t)f(g^{-1}t)$. Indeed, if $gs \in tH$, then $gs = th$ for some $h \in H$. Thus we have:

$$\pi(gs)f(s) = \pi(th)f(s) = \pi(t)\tau(h)f(s).$$

Now, we recall that $f \in \mathcal{H}(\pi_0)$, where $\pi_0 = \mathrm{Ind}_H^G(\tau)$. Hence f satisfies the property $f(xy) = \tau(y^{-1})f(x)$ for all $x \in G, y \in H$. Consequently:

$$\pi(gs)f(s) = \pi(th)f(s) = \pi(t)\tau(h)f(s) = \pi(t)f(sh^{-1}) = \pi(t)f(g^{-1}t).$$

Therefore $I(\pi_0(g)f) = \bigoplus_{t\in T} \pi(t)f(g^{-1}t) = \bigoplus_{s\in T} \pi(gs)f(s) = \pi(g)If$, as required.

The representation π is irreducible by hypothesis, then the Theorem 3.9 implies τ is irreducible.

At the moment we have shown that π contains an equivalent subrepresentation which is of the form $\mathrm{Ind}_H^G(\tau)$, where τ is an irreducible representation that factors through H/V. Zorn's lemma allows us to conclude that π is actually a sum of the induced representations constructed above.

□

4 Final comment

The proof of the main theorem of this work is intended to collect the ideas proposed by Professor Tsankov in [T1] and [T2], in which generalizes the result by Lieberman [L], obtaining a complete classification of the irreducible unitary representations of groups much more general than those considered here.

Commentaire final

La preuve du théorème principal de ce travail est destiné à recueillir les idées proposées par le Professeur Tsankov dans [T1] et [T2], dans lesquels généralise le résultat par Lieberman [L], obtenant une classification complète des représentations unitaires irréductibles de groupes beaucoup plus généraux que les ici traités.

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