

# SETS OF POINTS THAT ARE THE VERTICES OF K NON-OVERLAPPING N-GONS

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## Abstract

We consider collections of convex, non-overlapping  $n$ -gons in the plane with the property that any point that is a vertex of one of the  $n$ -gons is the vertex of exactly  $k$  of them.

## Triangles

We call a finite, non-empty, non-overlapping collection of triangles in the plane  **$k$ -contained** if any point that is a vertex of one of the triangles is the vertex of exactly  $k$  triangles. Thus any non-overlapping collection of triangles which do not share vertices is 1-contained. The smallest known  $k$ -contained sets of triangles for  $2 \leq k \leq 5$  are shown in Figures 1 and 2.

Are these the smallest  $k$ -contained collections possible? Certainly for  $k = 2$  and  $k = 3$  they are, as can be easily verified by checking small cases. We suspect they are optimal for  $k = 4$  and  $k = 5$  as well, but do not have a proof.

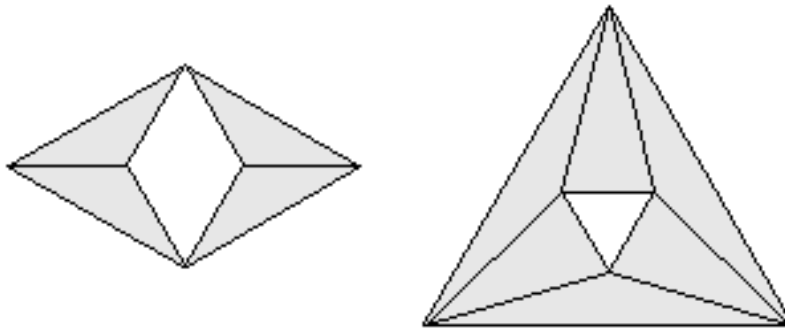


Figure 1. 2-contained (left) and 3-contained (right) triangles

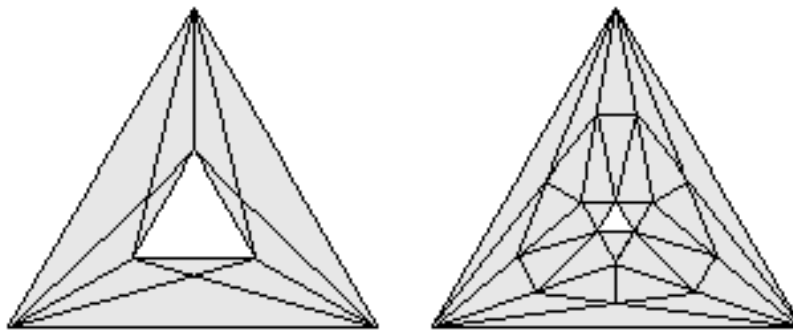


Figure 2. 4-contained (left) and 5-contained (right) triangles

There do not exist  $k$ -contained collections of triangles with  $k$  larger than 5. If there were, then the set of vertices and edges of the triangles would form a planar graph with every vertex having degree at least 6, which is impossible from Euler's Formula.

The triangles in the 2-contained set in Figure 1 are congruent. The smallest known 3-contained set of congruent triangles is shown in Figure 3. It is not known whether there is a 4-contained set of congruent triangles.

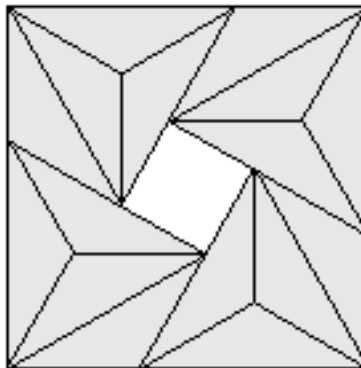


Figure 3. A 3-contained set of 12 congruent triangles

## Other Polygons

In the same spirit, we can define a set of arbitrary convex polygons to be **k-contained** if any point that is a vertex of one of them is the vertex of exactly  $k$  of them. The smallest 2-contained and smallest known 3-contained sets of quadrilaterals are shown in Figure 4. The quadrilaterals in the 2-contained set in Figure 4 are congruent. It is unknown whether there is a 3-contained set of congruent quadrilaterals.

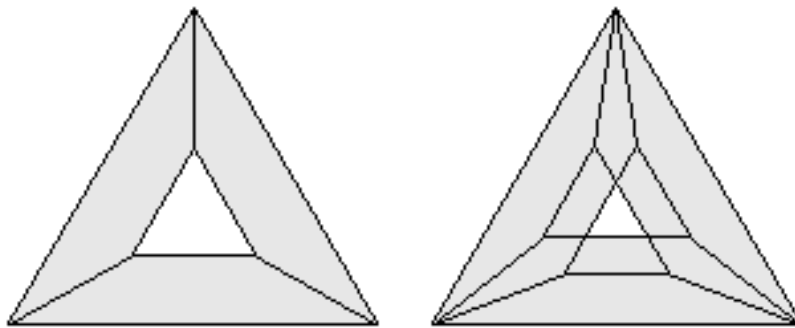


Figure 4. 2-contained (left) and 3-contained (right) quadrilaterals

The smallest 2-contained and 3-contained sets of pentagons are shown in Figure 5. Euler's Formula implies there are no 4-contained sets of quadrilaterals or pentagons.

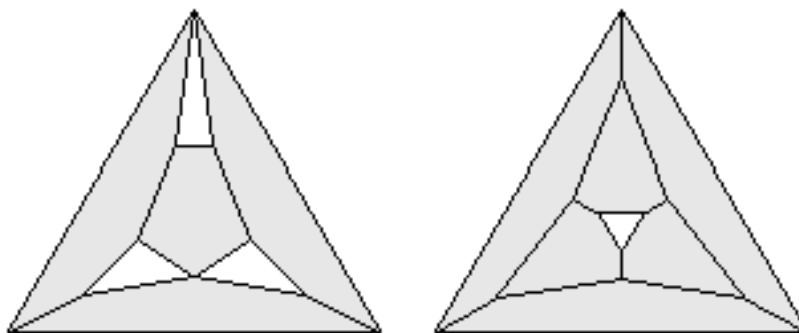


Figure 5. 2-contained (left) and 3-contained (right) pentagons

Euler's Formula also implies the only other  $k$ -contained sets of  $n$ -gons are when  $k$

$= 2$  and  $6 \leq n \leq 10$ . The planar realizations of a truncated tetrahedron, truncated cube, and truncated dodecahedron give 2-contained sets of  $n$ -gons for  $n = 6, 8$ , and  $10$  respectively. To get 2-contained sets of 7-gons or 9-gons, we start with 2-contained sets of 8-gons or 10-gons respectively, and contract to points edges that connect pairs of polygons. It is not known whether these give the smallest possible 2-contained sets.

## Open Questions

1. Are the  $k$ -contained sets of polygons presented here the smallest possible?
2. Does there exist a 4-contained set of congruent triangles, a 3-contained set of congruent quadrilaterals, or a 2-contained set of congruent pentagons?
3. How do these results change if we do not require the polygons to be convex?
4. What are the corresponding results in 3 dimensions?