New Bounds for Largest Planar Graphs With Fixed Maximum Degree and Diameter

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Abstract

Let $p(\Delta, D)$ be the largest number of vertices in a planar graph with maximum degree Δ and diameter D. Let $pr(\Delta, D)$ be the largest number of vertices in a planar Δ-regular graph with diameter D. We improve many of the best-known upper bounds for $p(\Delta, D)$ and $pr(\Delta, D)$ and most of the lower bounds for $pr(\Delta, D)$. In particular, we prove that $p(3, D) \leq 2^{D+1} - 1$, $p(4, D) \leq (5 + 23 \cdot 3^{D-3})/2$, and $p(5, D) \leq (8 + 46 \cdot 4^{D-2})/3$.

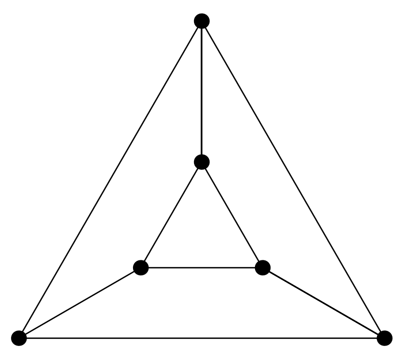

Figure 1: The planar 3-regular graph with diameter 2 and 6 vertices

1 Preliminaries

Lately, there has been considerable progress and interest in the degree-diameter problem: to find the largest graph with specified maximum degree and diameter [1, 2, 3, 4, 6, 8, 10]. We consider the same problem restricted to planar graphs. Let $p(\Delta, D)$ be the largest number of vertices in a planar graph with maximum degree Δ and diameter D. Let $pr(\Delta, D)$ be the largest number of vertices in a planar Δ-regular graph with diameter D. Note that $pr(\Delta, D) \leq p(\Delta, D)$ whenever pr is defined.

The only graphs with diameter 1 are (by definition) complete graphs, which are planar only for 4 vertices or less. Therefore $p(\Delta, 1) = pr(\Delta, 1) = \Delta + 1$ for $\Delta \leq 3$ and $p(\Delta, 1)$ and $pr(\Delta, 1)$ are undefined for $\Delta \geq 4$. The only connected graph with maximum degree 1 (equivalently, 1-regular) is the complete graph on 2 vertices, which has diameter 1, so $p(1, D)$ and $pr(1, D)$ are undefined for $D \geq 2$. Since the only connected graphs with maximum degree 2 are paths and cycles, it is easy to see that $p(2, D) = pr(2, D) = 2D + 1$. From now on, we consider only $\Delta \geq 3$ and $D \geq 2$.

It is fairly simple to show $pr(3, 2) = 6$. The corresponding graph is unique, and is shown in Figure 1. We have found a planar 4-regular graph with diameter 2 and 9 vertices, shown in Figure 2. We will show that $pr(4, 2) = 9$. The second author has recently shown $pr(3, 3) = 12$ [11]. There are two such graphs, and they are shown in Figure 3. The first of these graphs is the graph

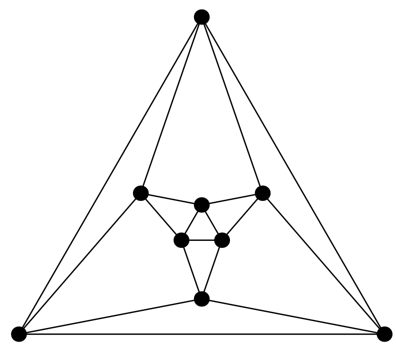

Figure 2: A planar 4-regular graph with diameter 2 and 9 vertices

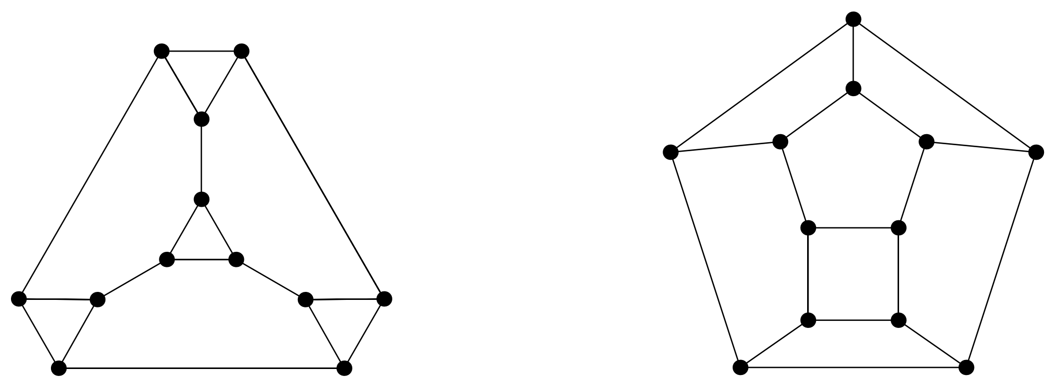

Figure 3: The planar 3-regular graphs with diameter 3 and 12 vertices

of a truncated tetrahedron. Göbel and Kern showed that there is no planar 5-regular graph with diameter 2 [7]. Euler's formula implies that there are no planar Δ-regular graphs for $\Delta \geq 6$. Hell and Seyffarth have computed $p(\Delta, 2)$ for $\Delta \geq 8$ [9].

All other values of $p(\Delta, D)$ and $pr(\Delta, D)$ exist, but are not known exactly. We reduce many of the upper bounds on $p(\Delta, D)$ and $pr(\Delta, D)$, some of which were close to the well-known Moore upper bound for (not necessarily planar) graphs:

$$1+\Delta+\Delta(\Delta-1)+\Delta(\Delta-1)^2+\cdots+\Delta(\Delta-1)^{D-1} = \begin{cases} \dfrac{\Delta(\Delta-1)^D-2}{\Delta-2} & \text{if } \Delta \neq 2, \\ 2D+1 & \text{if } \Delta = 2. \end{cases}$$

We also raise most of the lower bounds on $pr(\Delta, D)$ by constructing larger planar regular graphs. Table 1 shows the current lower and upper bounds for $pr(\Delta, D)$. Table 2 shows the current upper bounds for $p(\Delta, D)$. In each table, the newest bounds (shown in this paper) are in bold, and the bounds known to be optimal are marked with an asterisk. Fellows, Hell, and Seyffarth [5, 9] have shown that

$$\begin{aligned} p(\Delta, 2) &= \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1 && \text{for } \Delta \geq 8, \\ p(\Delta, 3) &\leq 8\Delta + 12, \\ p(\Delta, D) &\leq (6D+3)(2\Delta^{\lfloor D/2 \rfloor}+1) && \text{for } \Delta \geq 4. \end{aligned}$$

The bounds not appearing in bold are covered by these three cases. The most recent bounds can be found (at least through 1999) at

`http://www.math.unc.edu/Grads/rpratt/degdiam.html`.

We remark that $p(\Delta, D) < p(\Delta+1, D+1)$ since adding a leaf to a vertex of degree Δ preserves planarity and increases the diameter by at most 1. Note also that $p(\Delta, D) < p(\Delta, D+1)$ since replacing any edge with a path of length 2 preserves planarity and maximum degree and increases the diameter by at most 1. Hence the entries in Table 2 increase along diagonals and to the right. It appears that $p(\Delta, D) < p(\Delta+1, D)$ as well, but we have not proved this.

$\Delta \backslash D$	1	2	3	4	5	6	7	8
1	2*							
2	3*	5*	7*	9*	11*	13*	15*	17*
3	4*	6*	12*	18, **30**	28, **62**	**36**, **122**	**52**, **244**	**76**, **488**
4		**9***	16, **33**	**27**, **96**	**44**, **291**	**81**, **867**	**134**, **2595**	**243**, **7779**
5			16, 52	**28**, **248**	**62**, **984**	**124**, **3928**	**254**, 11294	**500**, **62808**

Table 1: Best-known lower and upper bounds for $pr(\Delta, D)$

$\Delta \backslash D$	2	3	4	5	6	7	8	9	10
3	**7***	**15**	**31**	**63**	**127**	**255**	**511**	**1023**	**2047**
4	**11**	**35**	**104**	**313**	**934**	**2797**	**8386**	**25153**	**75454**
5	**13**	52	**248**	**984**	**3928**	11295	**62808**	71307	393813
6	**13**	60	**521**	2409	**12971**	19485	132243	147801	979839
7	**13**	68	**938**	3267	26793	30915	244953	273771	2117745
8	13*	76	**1529**	4257	39975	46125	417843	467001	4128831
9	14*	84	**2324**	5379	56901	65655	669273	748011	7440237
10	16*	92	**3353**	6633	78039	90045	1020051	1140057	12600063
11	17*	100	**4646**	8019	103857	119835	1493433	1669131	20292489
12	19*	108	**6233**	9537	134823	155565	2115123	2363961	31352895
13	20*	116	**8144**	11187	171405	197775	2913273	3256011	46782981
14	22*	124	**10409**	12969	214071	247005	3918483	4379481	67765887
15	23*	132	12177	14883	263289	303795	5163801	5771307	95681313
16	25*	140	13851	16929	319527	368685	6684723	7471161	132120639

Table 2: Best-known upper bounds for $p(\Delta, D)$

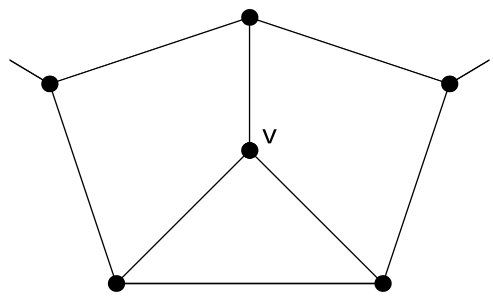

Figure 4: Part of a planar 3-regular graph

Our method for improving the upper bounds is explained in Section 2. The details of the upper bounds for specific Δ and D are given in Section 3. These bounds were first worked out by hand, and then verified by computer. The proof that $pr(4,2) = 9$ is contained in Section 4. Our lower bounds are in Section 5.

2 Upper Bounds for $p(\Delta, D)$ and $pr(\Delta, D)$

Throughout, we use F for the number of faces of a planar graph (including the one on the boundary), V for the number of vertices, and E for the number of edges. Let $V(f)$ be the number of vertices in face f, and call f a $V(f)$-*gon*. For a vertex v, we call the multiset of $\deg(v)$ numbers $\{V(f) : v \in f\}$ the *face numbers* of v. Let

$$R(v) = \sum_{v \in f} \frac{1}{V(f)}$$

be the sum of reciprocals of the face numbers of v. For example, the vertex v in Figure 4 has face numbers $\{3, 4, 4\}$, and $R(v) = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{5}{6}$.

Theorem 1 will guarantee the existence of a vertex in the graph where the face numbers are relatively small. Theorem 2 will then use such a vertex to bound V. Theorem 3 will let us limit the number of cases we consider.

Theorem 1. Every planar graph on V vertices has some vertex w with $R(w) \geq \dfrac{\deg(w)-2}{2} + \dfrac{2}{V}$.

Proof. Euler's formula for connected planar graphs tells us $V-E+F=2$. If we sum the degrees of the vertices of any graph, every edge is counted twice, so we have $\sum_v \deg(v) = 2E$. Similarly, if we sum $R(v)$ over all vertices v, then each summand $\frac{1}{V(f)}$ gets counted $V(f)$ times, so $\sum_v R(v) = F$. Now define $S(v) = \dfrac{\deg(v)}{2} - R(v)$. Then summing over all vertices v, and applying Euler's formula, we have

$$\sum_v S(v) = \sum_v \frac{\deg(v)}{2} - \sum_v R(v) = E - F = V - 2.$$

Hence the average value of $S(v)$ over all vertices is $\dfrac{1}{V}\sum_v S(v) = 1 - \dfrac{2}{V}$. Therefore, there must be some vertex w with $S(w) \leq 1 - \dfrac{2}{V}$. That is, $R(w) \geq \dfrac{\deg(w)-2}{2} + \dfrac{2}{V}$. ■

Fix a vertex w. We define *level* i to be the set of vertices that are distance i from the vertex w. Let $N(i)$ be the number of vertices in level i, and let $N(i,j)$ be the number of edges from level i to level j. Observe that $N(i,j)=0$ if $|i-j|>1$.

For each face f containing w, we define the *pseudo face number* of f as follows. Let k be the smallest number such that either

(a) two edges of f from level $k-1$ meet at the same vertex in level k, or

(b) an edge of f joins two vertices in level k

If (a) holds, define the pseudo face number of f to be $2k$. Otherwise, define the pseudo face number of f to be $2k+1$. In either case, the pseudo face number of f is at most the face number of f.

Now let $a_i(w)$ be the number of occurrences of i in the pseudo face numbers of the $\deg(w)$ faces containing w. Define a new sequence by $b_0(w) = 1$, $b_1(w) = \deg(w)$, and

$$b_{i+1}(w) = (\Delta - 1)b_i(w) - a_{2i}(w) - 2a_{2i+1}(w) - a_{2i+2}(w). \tag{1}$$

Note that if all the $a_i(w) = 0$ and $\deg(w) = \Delta$, then $\sum_{i=0}^{D} b_i(w)$ is the Moore bound.

Theorem 2. A planar graph with maximum degree Δ and diameter D has no more than $\sum_{i=0}^{D} b_i(w)$ vertices, where w is any vertex.

Proof. Fix a vertex w, and define $N(i)$ and $N(i,j)$ as above. Since a graph with diameter D has $N(i) = 0$ for all $i > D$, it suffices to show that $N(i) \leq b_i(w)$. We will do this by induction on i. It is clearly true for $i = 0$ and $i = 1$ since $N(0) = 1$ and $N(1) = \deg(w)$. We now assume

$$N(i) \leq b_i(w) \tag{2}$$

and show that $N(i+1) \leq b_{i+1}(w)$.

Since we have at most $\Delta N(i)$ edges coming out of the vertices on level i, we have

$$N(i-1,i) + 2N(i,i) + N(i,i+1) \leq \Delta N(i). \tag{3}$$

Each face containing w with pseudo face number $2i$ gives two edges from level $i-1$ that meet at the same vertex in level i, so we have

$$N(i-1,i) \geq N(i) + a_{2i}(w). \tag{4}$$

Each face containing w with pseudo face number $2i+1$ gives an edge from level i to level i, so

$$N(i,i) \geq a_{2i+1}(w) \tag{5}$$

Each face containing w with pseudo face number $2i+2$ gives two edges from level i that meet at the same vertex in level $i+1$. Therefore

$$N(i,i+1) \geq N(i+1) + a_{2i+2}(w). \tag{6}$$

Now

$$\begin{aligned}
N(i+1) &\le N(i,i+1) - a_{2i+2}(w) && \text{by (6)}\\
&\le \Delta N(i) - N(i-1,i) - 2N(i,i) - a_{2i+2}(w) && \text{by (3)}\\
&\le \Delta N(i) - N(i) - a_{2i}(w) - 2a_{2i+1}(w) - a_{2i+2}(w) && \text{by (4) and (5)}\\
&\le (\Delta-1)b_i(w) - a_{2i}(w) - 2a_{2i+1}(w) - a_{2i+2}(w) && \text{by (2)}\\
&= b_{i+1}(w) && \text{by (1).} \quad \blacksquare
\end{aligned}$$

Theorem 3. Lowering a pseudo face number of w never raises the upper bound $\sum_{i=0}^{D} b_i(w)$.

Proof. Note that $\sum_{i=3}^{\infty} a_i(w) = \deg(w)$. This means that lowering a pseudo face number does not change the sum of numbers subtracted from the $(\Delta-1)b_i(w)$; it only subtracts them sooner. Let

$$c_i(w) = a_{2i}(w) + 2a_{2i+1}(w) + a_{2i+2}(w),$$

so that $b_{i+1}(w) = (\Delta-1)b_i(w) - c_i(w)$. It then suffices to show that subtracting 1 from $c_i(w)$ and adding 1 to $c_{i-1}(w)$ does not increase any $b_j(w)$. This is true since

$$b_{i+1}(w) = (\Delta-1)b_i(w) - c_i(w) > (\Delta-1)(b_i(w)-1) - (c_i(w)-1)$$

if $\Delta > 2$. $\blacksquare$

Our method for lowering the upper bounds on $p(\Delta, D)$ and $pr(\Delta, D)$ is to start with the Moore bound $V \le M$. Theorem 1 gives us a vertex w for which

$$R(w) \ge \frac{\deg(w)-2}{2} + \frac{2}{V} \ge \frac{\deg(w)-2}{2} + \frac{2}{M}.$$

By Theorem 3, we can use the actual face numbers in place of the pseudo face numbers when computing the upper bound $\sum_{i=0}^{D} b_i(w)$, if we redefine $a_i(w)$ as follows. Let $a_2(w) = 0$, and for $i \ge 3$ let $a_i(w)$ be the number of i-gons containing w (that is, the number of occurrences of i in the face numbers of w). By the same argument used in the proof of Theorem 3, we need only consider "maximal" sets of face numbers. For each possible value of $\deg(w) \leq \Delta$ and each possible set of face numbers for w, we use Theorem 2 to get an upper bound on V, and V must be no bigger than the largest upper bound produced. We iterate this whole procedure until the bound on V no longer improves. We also make use of the fact that a Δ-regular graph for odd Δ must have an even number of vertices.

3 Details of Upper Bounds

When computing maximal face numbers, we must consider all possibilities for $\deg(w) \leq \Delta$. The cases $\deg(w) = 1$, $\deg(w) = 2$, and $\deg(w) \geq 6$ can be dealt with simultaneously for all Δ and D.

If $\deg(w) = 1$, the fact that $V \geq 4$ implies that any face numbers will satisfy the bound given in Theorem 1 since $R(w) > 0 \geq \dfrac{1-2}{2} + \dfrac{2}{V} = \dfrac{4-V}{2V}$. So $\{(2D+1)+\}$ are the only maximal face numbers. (A face number followed by a plus sign, such as $n+$, means that any face number at least as large as n will do. Note that $\sum_{i=0}^{D} b_i(w)$ is unaffected by the values of $a_i(w)$ when $i \geq 2D+1$.) Since all $a_i(w) = 0$ in this case, we have $b_0(w) = 1$, $b_1(w) = \deg(w) = 1$, $b_2(w) = \Delta - 1$, $b_3(w) = (\Delta - 1)^2$, $\ldots$, $b_D(w) = (\Delta - 1)^{D-1}$. Then

$$\sum_{i=0}^{D} b_i(w) = 1 + 1 + (\Delta - 1) + (\Delta - 1)^2 + \cdots + (\Delta - 1)^{D-1} = 1 + \frac{(\Delta - 1)^D - 1}{\Delta - 2}.$$

If $\deg(w) = 2$, the fact that $\Delta \geq 3$ implies that $V \geq 2D + 1$. Then $\{(2D+1)+, (2D+1)+\}$ are the only maximal face numbers, because

$$R(w) = \frac{2}{2D+1} \geq \frac{2-2}{2} + \frac{2}{V}.$$

Since all $a_i(w) = 0$ in this case, we have $b_0(w) = 1$, $b_1(w) = \deg(w) = 2$,

$b_2(w) = 2(\Delta - 1)$, $b_3(w) = 2(\Delta - 1)^2$, ... , $b_D(w) = 2(\Delta - 1)^{D-1}$. Then

$$\sum_{i=0}^{D} b_i(w) = 1+2+2(\Delta-1)+2(\Delta-1)^2+\cdots+2(\Delta-1)^{D-1} = 1+2\cdot\frac{(\Delta-1)^D - 1}{\Delta - 2}.$$

Note that this upper bound is always larger than the degree-1 upper bound derived above.

If $\deg(w) \geq 6$, no face numbers satisfy the bound given in Theorem 1, as we now demonstrate. For a fixed value of $\deg(w)$, the maximum value of $R(w)$ is $\deg(w)/3$, corresponding to the face numbers $\{3, 3, \ldots, 3\}$. But $\deg(w) \geq 6$ implies

$$\frac{\deg(w)}{3} \leq \frac{\deg(w) - 2}{2} < \frac{\deg(w) - 2}{2} + \frac{2}{V},$$

so we need not consider the cases when $\deg(w) \geq 6$.

If $3 \leq \deg(w) \leq 5$, the lists of maximal face numbers depend on Δ and D, so we will compute them separately for each specific case. For each value of Δ, we give a list of sets of maximal face numbers that can be easily verified to be "dominating" in the following sense. Any set of face numbers satisfying $R(w) \geq \dfrac{\deg(w) - 2}{2} + \dfrac{2}{V}$ (with $\deg(w) \leq \Delta$) either appears in the list or can be obtained from one of the sets of face numbers appearing in the list by reducing some of the face numbers.

We now give some details for specific upper bounds, starting with graphs with maximum degree 3.

$p(3, D)$:

We have the following dominating list of maximal face numbers.

Face Numbers			$\sum b_i(w)$
$(2D+1)+$			2^D
$(2D+1)+$	$(2D+1)+$		$2^{D+1}-1$
5	6	7	$4+15\cdot 2^{D-3}$
5	5	9	$4+15\cdot 2^{D-3}$
4	7	9	$4+15\cdot 2^{D-3}$
4	6	11	$4+29\cdot 2^{D-4}$
4	5	19	$4+447\cdot 2^{D-8}$
4	4	$(2D+1)+$	$2+3\cdot 2^{D-1}$
3	11	13	$4+61\cdot 2^{D-5}$
3	10	14	$4+241\cdot 2^{D-7}$
3	9	17	$4+239\cdot 2^{D-7}$
3	8	23	$4+1855\cdot 2^{D-10}$
3	7	41	$4+917503\cdot 2^{D-19}$
3	6	$(2D+1)+$	$2+13\cdot 2^{D-3}$

For $D \geq 6$, the largest of these $\sum_{i=0}^{D} b_i(w)$ is $2^{D+1}-1$. By examining the cases $2 \leq D \leq 5$ separately, we also show that $p(3,2) \leq 7 = 2^{2+1}-1$, $p(3,3) \leq 15 = 2^{3+1}-1$, $p(3,4) \leq 31 = 2^{4+1}-1$, and $p(3,5) \leq 63 = 2^{5+1}-1$. Therefore $p(3,D) \leq 2^{D+1}-1$.

$p(3, 2)$:

The Moore bound is $V \leq 10$. The maximal face numbers are in the list below.

Face Numbers			$\sum b_i(w)$
5+			$1+1+2=4$
5+	5+		$1+2+4=7$
4	4	5+	$1+3+4=8$
3	5+	5+	$1+3+4=8$

Note that $\sum_{i=0}^{2} b_i(w)$ is unaffected by the values of $a_i(w)$ when $i \geq 5$. For each possible set of face numbers, $\sum_{i=0}^{2} b_i(w)$ is calculated. The largest is 8, so $p(3,2) \leq 8$. Iterating with $V \leq 8$ gives the following maximal face numbers.

Face Numbers			$\sum b_i(w)$
5+			$1+1+2=4$
5+	5+		$1+2+4=7$
4	4	4	$1+3+3=7$
3	4	5+	$1+3+3=7$

Therefore $p(3,2) \leq 7$.

$p(3,3)$:

We have $V \leq 22$.

Face Numbers			$\sum b_i(w)$
7+			$1+1+2+4=8$
7+	7+		$1+2+4+8=15$
5	5	5	$1+3+6+6=16$
4	5	7+	$1+3+5+7=16$
3	7+	7+	$1+3+4+8=16$

Iterating with $V \leq 16$ gives the following maximal face numbers.

Face Numbers			$\sum b_i(w)$
7+			$1+1+2+4=8$
7+	7+		$1+2+4+8=15$
4	5	5	$1+3+5+5=14$
4	4	7+	$1+3+4+6=14$
3	6	7+	$1+3+4+7=15$

Therefore $p(3,3) \leq 15$.

$p(3,4)$:

We have $V \leq 46$.

Face Numbers			$\sum b_i(w)$
9+			$1+1+2+4+8=16$
9+	9+		$1+2+4+8+16=31$
5	5	6	$1+3+6+7+13=30$
4	6	7	$1+3+5+8+13=30$
4	5	9+	$1+3+5+7+14=30$
3	9+	9+	$1+3+4+8+16=32$

Iterating with $V \leq 32$ gives the following maximal face numbers.

Face Numbers			$\sum b_i(w)$
9+			$1+1+2+4+8=16$
9+	9+		$1+2+4+8+16=31$
5	5	6	$1+3+6+7+13=30$
4	6	6	$1+3+5+7+12=28$
4	5	8	$1+3+5+7+13=29$
4	4	9+	$1+3+4+6+12=26$
3	8	9+	$1+3+4+8+15=31$

Therefore $p(3,4) \leq 31$.

$p(3,5)$:

We have $V \leq 94$.

Face Numbers			$\sum b_i(w)$
11+			$1+1+2+4+8+16=32$
11+	11+		$1+2+4+8+16+32=63$
5	6	6	$1+3+6+8+14+28=60$
5	5	8	$1+3+6+8+15+29=62$
4	7	7	$1+3+5+9+14+28=60$
4	6	9	$1+3+5+8+15+28=60$
4	5	11+	$1+3+5+7+14+28=58$
3	10	11+	$1+3+4+8+16+31=63$

Therefore $p(3,5) \leq 63$.

Hence we have shown that, in all cases, $p(3, D) \leq 2^{D+1} - 1$.

$pr(3, D)$:

For 3-regular graphs, we can improve this upper bound slightly. A simple improvement is to subtract 1 since the upper bound is odd and any 3-regular graph must have an even number of vertices. That is, $pr(3, D) \leq p(3, D) - 1 \leq 2^{D+1} - 2$. But we can often reduce this bound further by considering only those face numbers that arise from a vertex w with $\deg(w) = \Delta = 3$. The smallest value of D for which we obtain any improvement is $D = 6$.

$pr(3, 6)$:

Since $V \leq 2^{6+1} - 2 = 126$, we must have a vertex w with $R(w) \geq \frac{3-2}{2} + \frac{2}{126} \approx .516$. The maximal face numbers for w are in the list below.

Face Numbers			$\sum b_i(w)$
5	6	6	$1+3+6+8+14+28+56 = 116$
5	5	8	$1+3+6+8+15+29+58 = 120$
4	7	8	$1+3+5+9+15+29+58 = 120$
4	6	10	$1+3+5+8+15+29+57 = 118$
4	5	13+	$1+3+5+7+14+28+56 = 114$
3	10	12	$1+3+4+8+16+31+60 = 123$
3	9	13+	$1+3+4+8+16+30+60 = 122$

Because 3-regular graphs have an even number of vertices, we have $pr(3, 6) \leq 122$.

$pr(3,7)$:

We have $V \leq 2^{7+1} - 2 = 254$ and $R(w) \geq .507$.

Face Numbers			$\sum b_i(w)$
5	6	7	$1+3+6+9+15+30+60+120 = 244$
5	5	9	$1+3+6+8+16+30+60+120 = 244$
4	7	8	$1+3+5+9+15+29+58+116 = 236$
4	6	10	$1+3+5+8+15+29+57+114 = 232$
4	5	15+	$1+3+5+7+14+28+56+112 = 226$
3	11	11	$1+3+4+8+16+32+60+120 = 244$
3	10	13	$1+3+4+8+16+31+61+120 = 244$
3	9	15+	$1+3+4+8+16+30+60+120 = 242$

Therefore $pr(3,7) \leq 244$.

$pr(3,8)$:

We have $V \leq 2^{8+1} - 2 = 510$ and $R(w) \geq .503$.

Face Numbers			$\sum b_i(w)$
5	6	7	$1+3+6+9+15+30+60+120+240 = 484$
5	5	9	$1+3+6+8+16+30+60+120+240 = 484$
4	7	9	$1+3+5+9+16+30+60+120+240 = 484$
4	6	11	$1+3+5+8+15+30+58+116+232 = 468$
4	5	17+	$1+3+5+7+14+28+56+112+224 = 450$
3	11	12	$1+3+4+8+16+32+61+121+242 = 488$
3	10	14	$1+3+4+8+16+31+61+121+241 = 486$
3	9	16	$1+3+4+8+16+30+60+120+239 = 481$
3	8	17+	$1+3+4+8+15+29+58+116+232 = 466$

Therefore $pr(3,8) \leq 488$.

Next we examine graphs with maximum degree 4.

$p(4, D)$:

We have the following dominating list of maximal face numbers.

Face Numbers				$\sum b_i(w)$
$(2D+1)+$				$(1+3^D)/2$
$(2D+1)+$	$(2D+1)+$			3^D
5	6	7		$(5+23\cdot 3^{D-2})/2$
5	5	9		$(5+205\cdot 3^{D-4})/2$
4	7	9		$(5+199\cdot 3^{D-4})/2$
4	6	11		$(5+583\cdot 3^{D-5})/2$
4	5	19		$(5+45925\cdot 3^{D-9})/2$
4	4	$(2D+1)+$		$(3+19\cdot 3^{D-2})/2$
3	11	13		$(5+1693\cdot 3^{D-6})/2$
3	10	14		$(5+5063\cdot 3^{D-7})/2$
3	9	17		$(5+15145\cdot 3^{D-8})/2$
3	8	23		$(5+404593\cdot 3^{D-11})/2$
3	7	41		$(5+7877549941\cdot 3^{D-20})/2$
3	6	$(2D+1)+$		$(3+59\cdot 3^{D-3})/2$
3	4	4	5	$3+10\cdot 3^{D-2}$
3	3	5	7	$3+32\cdot 3^{D-3}$
3	3	4	11	$3+269\cdot 3^{D-5}$
3	3	3	$(2D+1)+$	$2+3^D$

The largest of these $\sum_{i=0}^{D} b_i(w)$ is $(5+23\cdot 3^{D-2})/2$. But by examining the cases $2 \le D \le 4$ separately, we also show that $p(4,2) \le 11$, $p(4,3) \le 35$, and $p(4,4) \le 104$.

$p(4,2)$:

We have $V \leq (5 + 23 \cdot 3^{2-2})/2 = 14$.

Face Numbers				$\sum b_i(w)$
5+				$1+1+3=5$
5+	5+			$1+2+6=9$
4	5+	5+		$1+3+8=12$
3	3	4	4	$1+4+6=11$
3	3	3	5+	$1+4+6=11$

Iterating with $V \leq 12$ gives the following maximal face numbers.

Face Numbers				$\sum b_i(w)$
5+				$1+1+3=5$
5+	5+			$1+2+6=9$
4	4	5+		$1+3+7=11$
3	3	4	4	$1+4+6=11$
3	3	3	5+	$1+4+6=11$

Therefore $p(4,2) \leq 11$.

$p(4,3)$:

We have $V \leq (5 + 23 \cdot 3^{3-2})/2 = 37$.

Face Numbers				$\sum b_i(w)$
7+				$1+1+3+9=14$
7+	7+			$1+2+6+18=27$
5	5	6		$1+3+9+22=35$
4	6	7+		$1+3+8+22=34$
3	7+	7+		$1+3+7+21=32$
3	4	4	4	$1+4+7+18=30$
3	3	5	5	$1+4+8+20=33$
3	3	4	7+	$1+4+7+20=32$

Therefore $p(4,3) \leq 35$.

$p(4,4)$:

We have $V \leq (5 + 23 \cdot 3^{4-2})/2 = 106$.

Face Numbers				$\sum b_i(w)$
9+				$1+1+3+9+27=41$
9+	9+			$1+2+6+18+54=81$
5	6	6		$1+3+9+23+67=103$
5	5	8		$1+3+9+23+68=104$
4	7	7		$1+3+8+23+65=100$
4	6	9+		$1+3+8+22+65=99$
3	9+	9+		$1+3+7+21+63=95$
3	4	4	5	$1+4+8+20+60=93$
3	3	5	6	$1+4+8+21+62=96$
3	3	4	9+	$1+4+7+20+60=92$

Therefore $p(4,4) \leq 104$.

We now show the details for 4-regular graphs. By considering only those face numbers that arise from a vertex w with $\deg(w) = \Delta = 4$, we see that $pr(4,3) \leq \max\{30, 33, 32\} = 33$ and $pr(4,4) \leq \max\{93, 96, 92\} = 96$.

$pr(4,5)$:

We have $V \leq (5 + 23 \cdot 3^{5-2})/2 = 313$ and $R(w) \geq 1.006$.

Face Numbers				$\sum b_i(w)$
3	4	4	5	$1+4+8+20+60+180=273$
3	3	5	7	$1+4+8+22+64+192=291$
3	3	4	11+	$1+4+7+20+60+180=272$

Therefore $pr(4,5) \leq 291$.

$pr(4,6+)$**:**

We have $R(w) > 1$.

Face Numbers				$\sum b_i(w)$
3	4	4	5	$1+4+8+20(1+3+\cdots+3^{D-3}) = 3+10\cdot 3^{D-2}$
3	3	5	7	$1+4+8+22+64(1+3+\cdots+3^{D-4}) = 3+32\cdot 3^{D-3}$
3	3	4	11	$1+4+7+20+60+180+538(1+\cdots+3^{D-6}) = 3+269\cdot 3^{D-5}$
3	3	3	$(2D+1)+$	$1+4+6+6\cdot 3+\cdots+6\cdot 3^{D-2} = 2+3^D$

The largest of these $\sum_{i=0}^{D} b_i(w)$ is $3+32\cdot 3^{D-3}$. Therefore, $pr(4,6+) \le 3+32\cdot 3^{D-3}$.

Next we examine graphs with maximum degree 5.

$p(5, D)$:

We have the following dominating list of maximal face numbers.

Face Numbers					$\sum b_i(w)$
$(2D+1)+$					$(2+4^D)/3$
$(2D+1)+$	$(2D+1)+$				$(1+2\cdot 4^D)/3$
5	6	7			$2+59\cdot 4^{D-3}$
5	5	9			$2+234\cdot 4^{D-4}$
4	7	9			$2+226\cdot 4^{D-4}$
4	6	11			$2+890\cdot 4^{D-5}$
4	5	19			$2+223914\cdot 4^{D-9}$
4	4	$(2D+1)+$			$(4+38\cdot 4^{D-2})/3$
3	11	13			$2+3410\cdot 4^{D-6}$
3	10	14			$2+13625\cdot 4^{D-7}$
3	9	17			$2+54442\cdot 4^{D-8}$
3	8	23			$2+3467946\cdot 4^{D-11}$
3	7	41			$2+904806443690\cdot 4^{D-20}$
3	6	$(2D+1)+$			$(4+155\cdot 4^{D-3})/3$
3	4	4	5		$(7+11\cdot 4^{D-1})/3$
3	3	5	7		$(7+182\cdot 4^{D-3})/3$
3	3	4	11		$(7+2750\cdot 4^{D-5})/3$
3	3	3	$(2D+1)+$		$(5+10\cdot 4^{D-1})/3$
3	3	3	3	5	$(8+46\cdot 4^{D-2})/3$

The largest of these $\sum_{i=0}^{D} b_i(w)$ is $(8+46\cdot 4^{D-2})/3$. Therefore $p(5, D) \leq (8+46\cdot 4^{D-2})/3$. But by examining the case $D = 2$ separately, we also show that $p(5, 2) \leq 13$.

$p(5+, 3+)$:

As argued at the beginning of this section, we need not consider the cases $\deg(w) \geq 6$. So when $\Delta > 5$ we have the same dominating list of maximal face numbers as for $\Delta = 5$. But the corresponding sums depend on Δ, and general formulas in terms of Δ are tedious to compute. One of the tamer formulas arises from the case $\deg(w) = 5$, where the only maximal face numbers are $\{3,3,3,3,5\}$. We have

$$\sum_{i=0}^{D} b_i(w) = 1+5+(5\Delta-13)+(5\Delta^2-18\Delta+11)(1+(\Delta-1)+\cdots+(\Delta-1)^{D-3})$$
$$= 5\Delta - 7 + (5\Delta^2-18\Delta+11)\cdot\frac{(\Delta-1)^{D-2}-1}{\Delta-2}.$$

Although it remains to examine the other possibilities for $\deg(w)$, the authors conjecture that this is in fact the upper bound produced by our iterative method.

Finally, we examine 5-regular graphs.

$pr(5,3+)$:

We have already seen that $pr(5,D) \le p(5,D) \le (8+46\cdot 4^{D-2})/3$. This improves the best-known upper bound for $pr(5,D)$ when $D=4,5,6,8$.

$p(\Delta,2)$:

Recall that $p(\Delta,2) = \lfloor\frac{3}{2}\Delta\rfloor + 1$ for $\Delta \ge 8$. We now show that $p(\Delta,2) \le 13$ when $5 \le \Delta \le 7$.

Suppose $p(7,2) > 13$. Let G be a planar graph on $p(7,2)$ vertices with the largest number of edges among all such graphs with maximum degree 7 and diameter 2. If G were not triangulated, then we could add an edge to get a planar graph with maximum degree 8 and diameter 2, contradicting $p(8,2)=13$. Hence G is triangulated. By Theorem 2 in [14], G has a vertex of degree 4 or 3.

Suppose G has a vertex of degree 4. Since G is triangulated and has diameter 2 and maximum degree 7, it has no more than 17 vertices, as shown in Figure 5. Now we must triangulate the outer face f (a 12-gon) in such a way that the resulting graph has diameter 2. Some set of 3 consecutive vertices along f forms a triangle. Let x be the middle vertex (of these 3). If x has degree 3 in Figure 5, one of the internal (not in f) vertices of G has no path of length at most 2 to x. Hence x must have degree 4 in Figure 5. Figure 6 shows f with its 12 vertices in the same positions as in Figure 5. It is

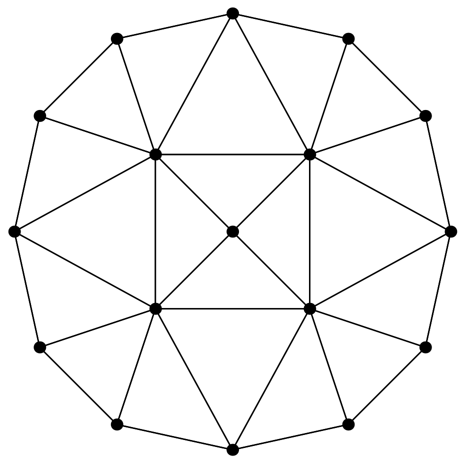

Figure 5: Failed attempt at a triangulated (planar) graph with maximum degree 7, diameter 2, and 17 vertices

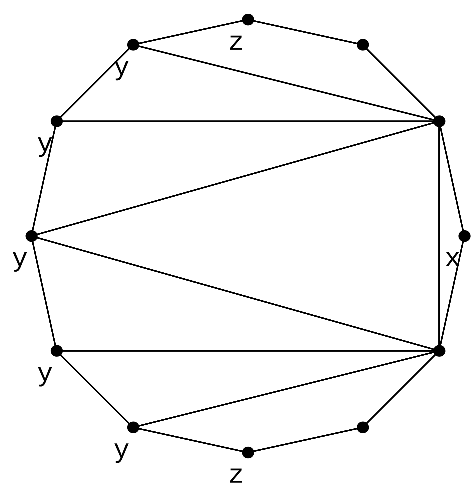

Figure 6: The face f

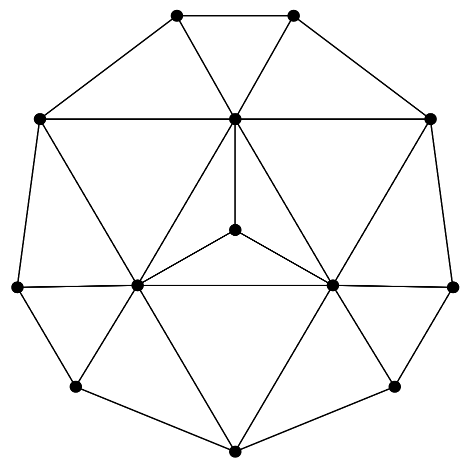

Figure 7: A triangulated (planar) graph with maximum degree 7, diameter 2, and a vertex of degree 3 has no more than 13 vertices

conceptually easier (but equivalent) to think of triangulating the finite region bounded by this 12-cycle, rather than the infinite outer region. In order for G to have diameter 2, each of the vertices labeled y in Figure 6 must be adjacent to one of the two neighbors of x in f. But then there cannot be a path of length 2 between the two vertices labeled z. Hence, G cannot have 17 vertices. Completely analogous arguments (with several cases) show that G cannot have 16 vertices (there are 2 cases to check), 15 vertices (5 cases to check), or 14 vertices (8 cases to check), contradicting the choice of G.

Suppose G has a vertex of degree 3. Then the graph shown in Figure 7 demonstrates that G has no more than 13 vertices, contradicting the choice of G. Therefore $p(7,2) \leq 13$.

Suppose $p(6,2) > 13$. Let G be a planar graph on $p(6,2)$ vertices with the largest number of edges among all such graphs with maximum degree 6 and diameter 2. As before, G must be triangulated, since otherwise we could add an edge to get a planar graph with maximum degree 7 and diameter 2, contradicting $p(7,2) \leq 13$. Again applying Theorem 2 in [14], we know that G has a vertex of degree 4 or 3. The graphs shown in Figure 8 demonstrate

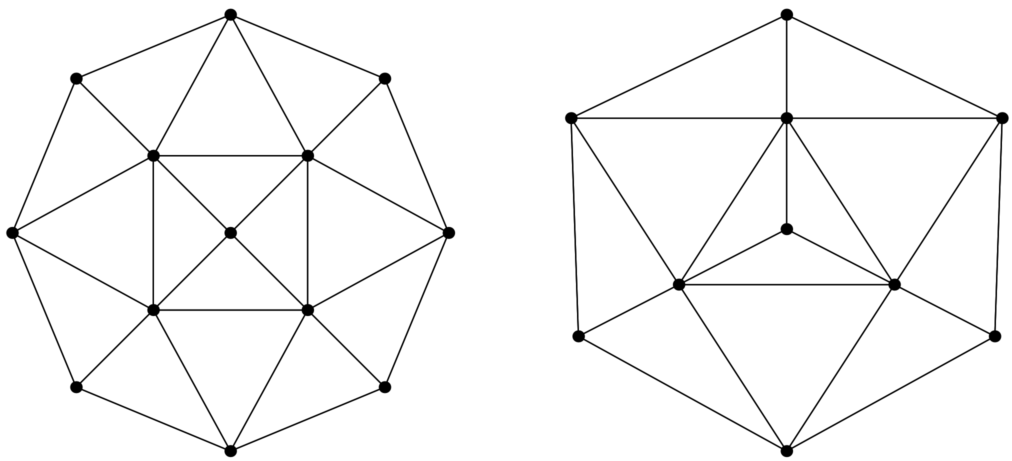

Figure 8: A triangulated (planar) graph with maximum degree 6 and diameter 2 has no more than 13 vertices

that G has no more than 13 vertices, contradicting the choice of G. Therefore $p(6,2) \leq 13$.

Suppose $p(5,2) > 13$. Let G be a planar graph on $p(5,2)$ vertices with the largest number of edges among all such graphs with maximum degree 5 and diameter 2. As before, G must be triangulated, since otherwise we could add an edge to get a planar graph with maximum degree 6 and diameter 2, contradicting $p(6,2) \leq 13$. Again applying Theorem 2 in [14], we know that G has a vertex of degree 4 or 3. The graphs shown in Figure 9 demonstrate that G has no more than 9 vertices, contradicting the choice of G. Therefore $p(5,2) \leq 13$.

4 Proof of $pr(4,2) = 9$

Because of Figure 2, we need only show $pr(4,2) \leq 9$. In the last section, we showed that $pr(4,2) \leq p(4,2) \leq 11$. Iterating with $V \leq 11$ and $R(w) \geq 1.181$ gives only the face numbers $\{3,3,3,5+\}$. This case is pictured in Figure 10. The two vertices labeled x must be distance 2 from each other, forcing the dotted line. The two vertices labeled y cannot be distance 2 from each other without the graph being non-planar. So no planar 4-regular graph with

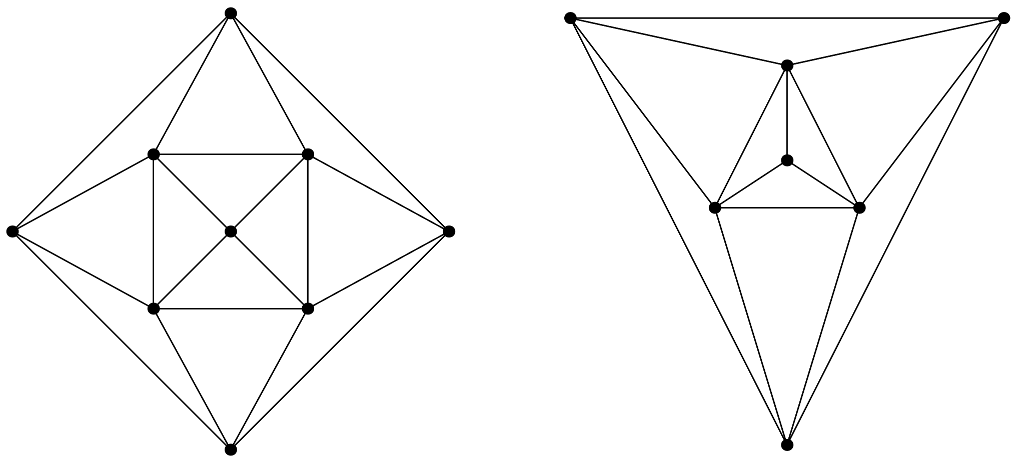

Figure 9: A triangulated (planar) graph with maximum degree 5 and diameter 2 has no more than 9 vertices

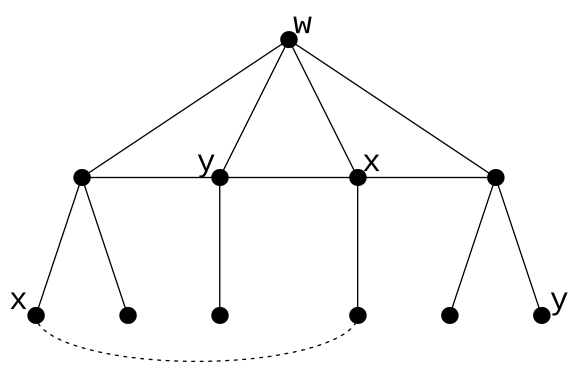

Figure 10: Failed attempt at a planar 4-regular graph with diameter 2 and 11 vertices

diameter 2 and 11 vertices exists.

There are three possibilities for such a graph with 10 vertices. These are shown in Figure 11. As before, the two vertices labeled x must be distance 2 from each other, forcing the dotted lines. The two vertices labeled y cannot be distance 2 from each other without the graph being non-planar. So no planar 4-regular graph with diameter 2 and 10 vertices exists. Therefore $pr(4,2) = 9$. ■

5 Lower Bounds

Although we construct only planar regular graphs, the inequality $pr(\Delta, D) \leq p(\Delta, D)$ implies that any lower bound for $pr(\Delta, D)$ is also a lower bound for $p(\Delta, D)$. It turns out that the best-known lower bounds for $pr(4, D)$ and $p(4, D)$ coincide.

A graph showing $pr(3,4) \geq 18$ is shown in Figure 12. Gordon Royle [13] has shown that there are exactly 3 such graphs, and that there are no such graphs with 20 vertices. Graphs showing $pr(4,3) \geq 16$ and $pr(5,3) \geq 16$ are shown in Figures 13 and 14. All three of these graphs are from [7], and are likely the largest possible.

A graph showing $pr(3,5) \geq 28$, due to Charles Delorme, is shown in Figure 15.

We now exhibit infinite families of graphs which improve the best-known lower bounds for $pr(\Delta, D)$ for larger values of D.

When $\Delta = 3$ and $D \geq 6$ is even, there is a graph with $5 \cdot 2^{D/2} - 4$ vertices, as in Figure 16. The lower and upper halves are trees with degree 3 and depth $D/2$. On the left branch, we connect corresponding endpoints, and on the two right branches, we identify corresponding endpoints. We then add horizontal edges so that every endpoint has degree 3. The left branch contains $2(1 + 2 + 4 + \cdots + 2^{(D-2)/2}) = 2^{(D+2)/2} - 2$ vertices. The other two branches contain $2^{(D-2)/2}$ fewer vertices each, for a total of $5 \cdot 2^{D/2} - 4$ vertices.

When $\Delta = 3$ and $D \geq 7$ is odd, there is a graph with $7 \cdot 2^{(D-1)/2} - 4$ vertices, as in Figure 17. The lower and upper halves are trees with the left branch with depth $(D+1)/2$ and the two right branches with depth

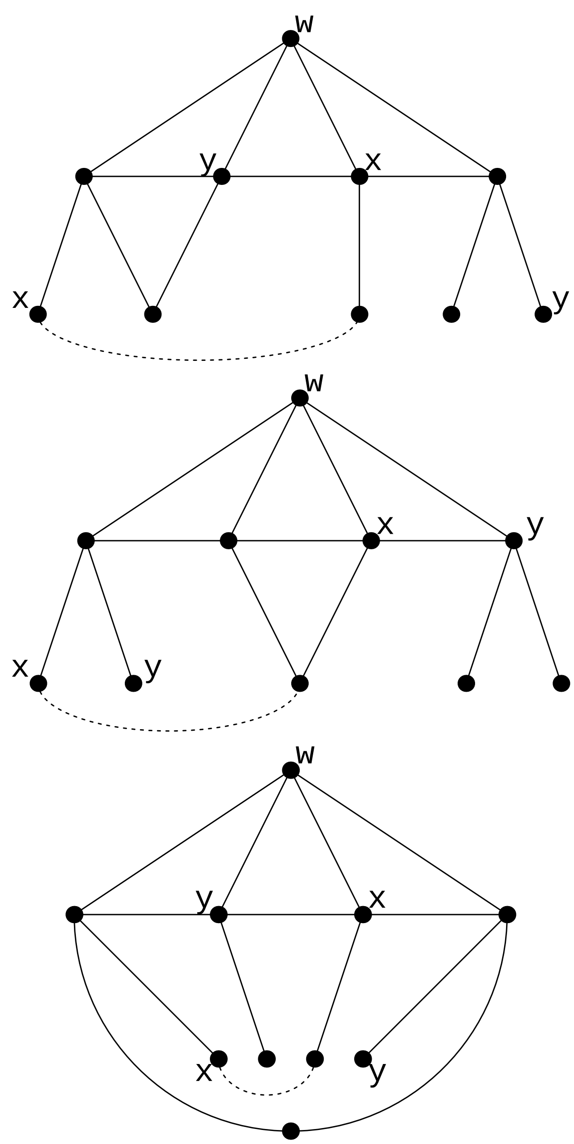

Figure 11: Failed attempts at a planar 4-regular graph with diameter 2 and 10 vertices

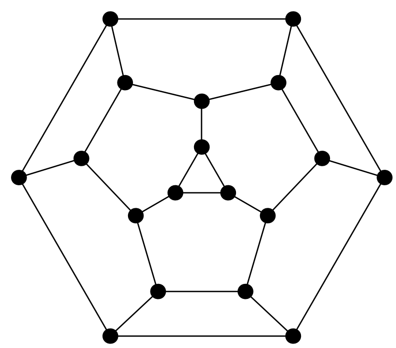

Figure 12: A planar 3-regular graph with diameter 4 and 18 vertices

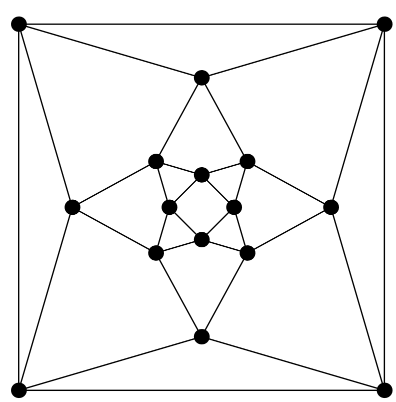

Figure 13: A planar 4-regular graph with diameter 3 and 16 vertices

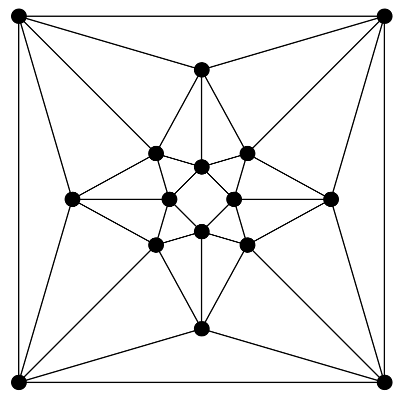

Figure 14: A planar 5-regular graph with diameter 3 and 16 vertices

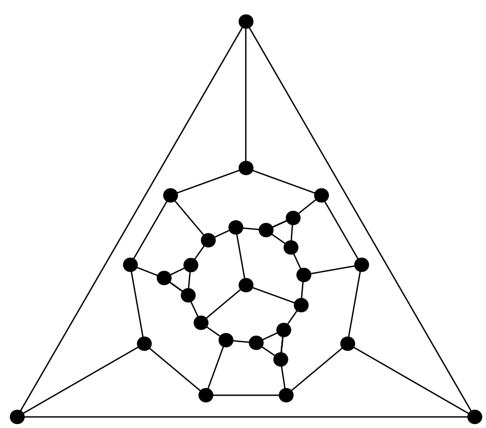

Figure 15: A planar 3-regular graph with diameter 5 and 28 vertices

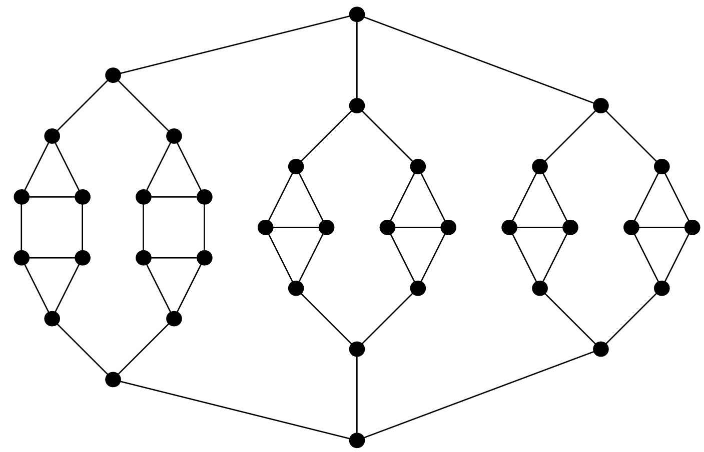

Figure 16: A planar 3-regular graph with diameter 6 and 36 vertices

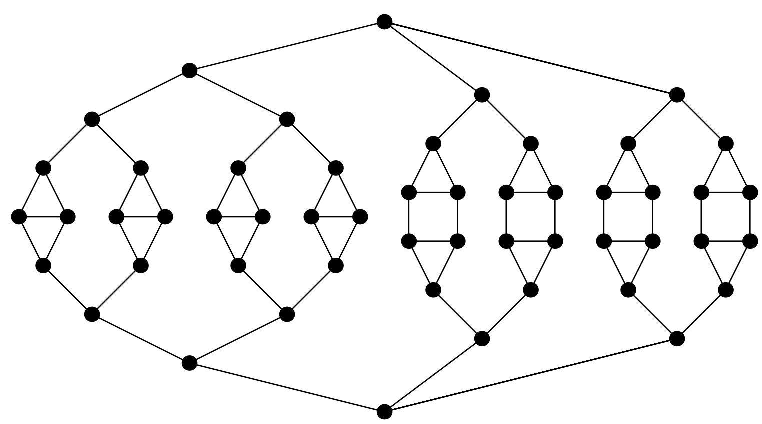

Figure 17: A planar 3-regular graph with diameter 7 and 52 vertices

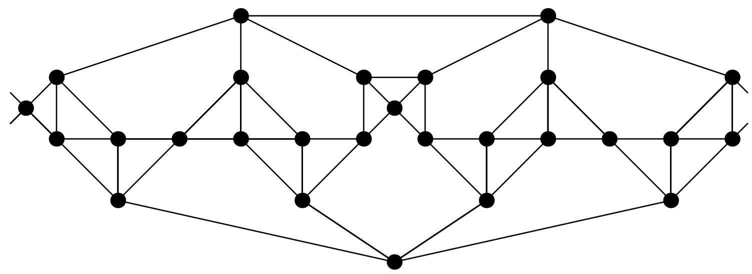

Figure 18: A planar 4-regular graph with diameter 4 and 27 vertices

$(D-1)/2$. We identify the endpoints at a depth of $(D+1)/2$, and connect the other endpoints, adding horizontal edges where necessary. The two right branches contain $2^{(D+1)/2}-2$ vertices each, and the left branch contains $2+2(2^{(D+1)/2}-2-2^{(D-3)/2})$ vertices, for a total of $7\cdot 2^{(D-1)/2}-4$ vertices.

When $\Delta = 4$ and $D \geq 4$ is even, there is a graph with $3^{(D+2)/2}$ vertices. The first such graph was constructed by James Preen [12], but the one in Figure 18 was constructed by the authors. The lower half is a tree with depth $D/2$. The upper half is a tree with two vertices at the top level with depth $(D-2)/2$. There are also two additional **special vertices**, located between the two branches of the upper tree. These special vertices are connected to two vertices above and two vertices below. We connect endpoints and add horizontal edges where needed, being careful to connect every endpoint in the upper tree to the lower tree and vice versa. The upper half contains $2+6+6\cdot 3+\cdots+6\cdot 3^{(D-4)/2} = 3^{D/2}-1$ vertices. The lower half contains $1+4+4\cdot 3+\cdots+4\cdot 3^{(D-2)/2} = 2\cdot 3^{D/2}-1$ vertices. The total number of vertices is therefore $3^{(D+2)/2}$.

For the rest of the constructions, we utilize what we will call an **almost tree**: a graph with only one cycle of length 3 at the top layer.

When $\Delta = 4$ and $D \geq 5$ is odd, there is a graph with $5\cdot 3^{(D-1)/2}-1$ vertices, as in Figure 19. The lower half is a tree with depth $(D-1)/2$. The upper half is an almost tree with depth $(D+1)/2$. We then connect endpoints, adding horizontal edges where necessary. The upper half contains $3^{(D+1)/2}$ vertices. The lower half contains $2\cdot 3^{(D-1)/2}-1$ vertices. The total

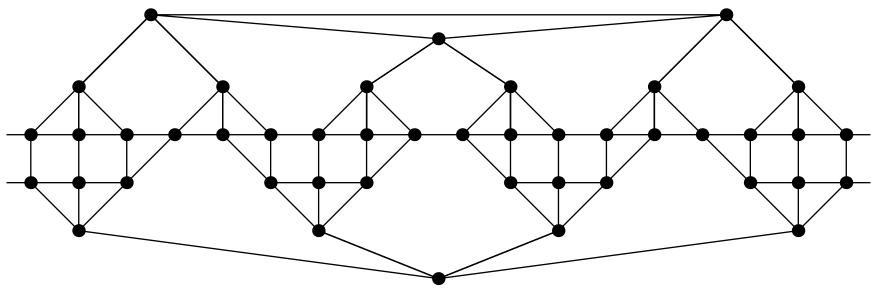

Figure 19: A planar 4-regular graph with diameter 5 and 44 vertices

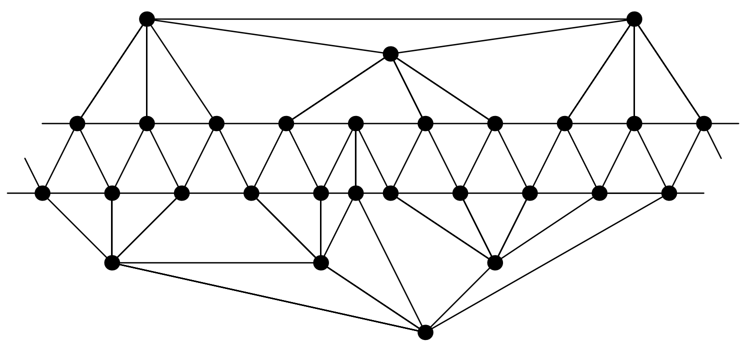

Figure 20: A planar 5-regular graph with diameter 4 and 28 vertices

number of vertices is therefore $5 \cdot 3^{(D-1)/2} - 1$.

When $\Delta = 5$ and $D = 4$, there is a graph with 28 vertices. This graph was constructed by James Preen [12], and is shown in Figure 20.

When $\Delta = 5$ and $D \geq 6$ is even, there is a similar graph with $(94 \cdot 4^{(D-4)/2} - 4)/3$ vertices, as in Figure 21. The upper half is an almost tree with depth $(D-2)/2$, with k additional special vertices. (We will find k in a moment.) These vertices are all connected horizontally. The $(9 \cdot 4^{(D-4)/2} + 2k)$ vertices in the next layer (called the **maximal layer**) are connected to the ones above in a zig-zag pattern, and are connected to themselves horizontally. Vertices directly below a special vertex have an edge leading 2 layers down. With the exception of one more horizontal edge, the rest of the bottom is a

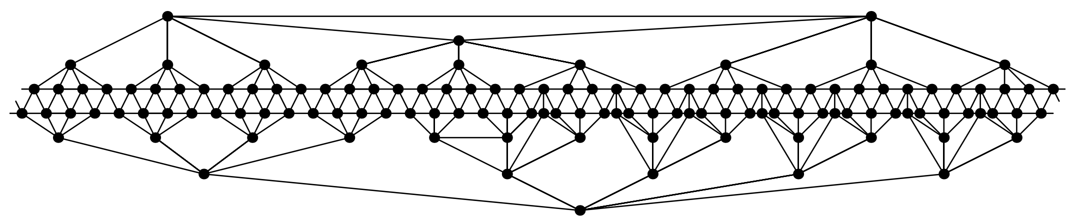

Figure 21: A planar 5-regular graph with diameter 6 and 124 vertices

tree.

There are $9 \cdot 4^{(D-4)/2}$ endpoints in the upper tree. There could be a maximum of $5 \cdot 4^{(D-2)/2}$ vertices in the maximal layer, if the bottom were a tree. Every special vertex adds 2 vertices to the maximal layer, but decreases the maximum by 4 vertices due to the edges leading 2 layers down. Therefore the largest possible value of k is

$$k = \left\lfloor \frac{5 \cdot 4^{(D-2)/2} - 9 \cdot 4^{(D-4)/2}}{6} \right\rfloor = \frac{11 \cdot 4^{(D-4)/2} - 2}{6}.$$

There are $3+9+9\cdot 4+\cdots+(9\cdot 4^{(D-4)/2}+k) = 3\cdot 4^{(D-2)/2}+k$ vertices in the upper half, and $1+5+5\cdot 4+\cdots+(5\cdot 4^{(D-4)/2}-k)+(9\cdot 4^{(D-4)/2}+2k) = \frac{47}{3}\cdot 4^{(D-4)/2}-\frac{2}{3}+k$ vertices in the lower half. Using the value of k above, there are a total of $(94\cdot 4^{(D-4)/2}-4)/3$ vertices.

When $\Delta = 5$ and $D \geq 5$ is odd, there is a graph with $4^{(D+1)/2}-2$ vertices, as in Figure 22. The lower half is a tree with depth $(D-1)/2$ with an additional $k = (4^{(D-1)/2}-4)/6$ special vertices. The upper half is an almost tree with depth $(D+1)/2$ and two horizontal edges. The lower half contains $1+5+5\cdot 4+\cdots+5\cdot 4^{(D-3)/2} = \frac{5}{3}\cdot 4^{(D-1)/2}-\frac{2}{3}+k$ vertices, and the upper half contains $3+9+9\cdot 4+\cdots 9\cdot 4^{(D-5)/2}+5\cdot 4^{(D-3)/2}+k = 2\cdot 4^{(D-1)/2}+k$ vertices. Using the above value for k, we have a total of $4^{(D+1)/2}-2$ vertices.

It is easy to check that the above graphs have the claimed diameter. Paths of length at most D always exist within the same branch or through either the top or bottom layers.

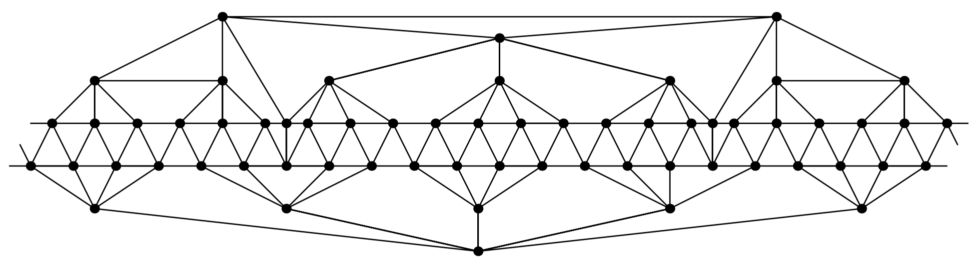

Figure 22: A planar 5-regular graph with diameter 5 and 62 vertices

6 Conclusion

While we have reduced many of the upper bounds and raised a few of the lower bounds, the gaps between them remain large for most cases. It seems likely that the actual values are close to the current lower bounds. In particular, a proof that $p(\Delta, D) < p(\Delta + 1, D)$ would immediately imply that the current lower bounds for $p(4,2)$, $p(5,2)$, $p(6,2)$, and $p(7,2)$ are exact. A possible direction for future research would be to reduce the upper bounds even further, perhaps by using our iterative approach together with some new graph invariant in place of $\sum S(v)$.

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