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Friedman numbers have density 1

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Abstract

A Friedman number is a positive integer which is the result of an expression combining all of its own digits by use of the four basic operations, exponentiation and digit concatenation. One of the fundamental questions regarding Friedman numbers, first raised by Erich Friedman in August 2000, is how common they are among the integers. In this paper, we prove that Friedman numbers have density 1 by drawing on the number-theoretical tools of indirect self-reference, famously used in Gödel's incompleteness theorem and in Turing's halting theorem. We further prove that the density of Friedman numbers remains 1 regardless of the base of representation.

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1. Introduction

Friedman numbers are numbers that can be computed from their own digits, each digit used exactly once, by use of the four basic arithmetic operations, exponentiation and digit concatenation (as long as digit concatenation is not the only operation used). Parentheses can be used at will. An example of a Friedman number is 25, which can be represented as 5^2. An example of a non-Friedman number is any power of 10, because no power of 10 can be expressed as the result of a computation using only arithmetic operations and exponentiation if the initial arguments in the computation are a smaller power of 10 and several zeros.

Friedman numbers are sequence A036057 of the Encyclopedia of Integer Sequences [1]. They were first introduced by Erich Friedman in August 2000 [2], and the first question to be asked about them was what their density is,

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inside the population of integers. That is, if $F(n)$ is the number of Friedman numbers in the range $[1,n]$, what is $\lim_{n\to\infty} F(n)/n$?

This question has remained open for the decade that has passed since then. However, much progress was made on other aspects of the problem. The following review summarizes some of these advances, as detailed in [2].

Mike Reid, Ulrich Schimke and Philippe Fondanaiche calculated the exhaustive list of all Friedman numbers up to 10,000 (there are 72 of them), and these were later supplemented by Erich Friedman to an exhaustive list that ranges until 100,000. In total, there are 842 Friedman numbers smaller than 100,000, or 0.842%. This is not much higher than the 0.72% among the first 10,000 numbers, and does not suggest the density 1 proved here.

Ron Kaminsky proved the existence of infinitely many prime Friedman numbers. Brendan Owen and Mike Reid showed (independently) that any string of digits can be the prefix to a Friedman number by appending to it a fixed suffix. (For example, 12588304 forms one such suffix. Any number N followed by the digits 12588304 is a Friedman number and can be calculated from its own digits as $N*10^8+3548^2$.) Erich Friedman augmented this result by showing that any string of digits can be the suffix to a Friedman number by preceding it with a prefix that is dependent only on the length of the suffix. (For example, if XY is a two-digit number, then $2500XY = 500^2 + XY$ is a Friedman number. More generally, $25 * 10^k$ is a prefix that can precede any k-digit suffix to make a Friedman number.)

While these results, and many like them, were able to show that the density of Friedman numbers is greater than 0, only less than one percent of the integers were previously known to be Friedman numbers, whereas in this paper we prove that the density of Friedman numbers within the integers is 1.

2. Background

Friedman numbers have not received much attention among number theorists since their introduction a decade ago. One reason for this may be that their definition is dependent on their base of representation, making it a less "number-theoretical" definition. Their main mentions are in recreational mathematics, as one of many subsets of numbers exhibiting narcissism, which is the general ability to compute a number from its digits under various constraints. (See, e.g., [3].) G.H. Hardy [4] summarized the opposition towards the study of narcissism as follows: "These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals much to a mathematician."

In this paper, I challenge this view and reintroduce Friedman numbers to the mainstream of number theory. The crux of seminal proofs such as Gödel's incompleteness theorem [5] and Turing's halting problem theorem [6] is in the ability to use an integer (or string of characters from a known alphabet) as instructions for the production of another integer (or another string of characters from a known alphabet) and in the use of indirect self-reference to elicit a nontrivial connection between the input and output. Friedman numbers extend on this platform by allowing additional variables other than the input integer to affect the final result. This is done in our choice of operations over the integer's digits. Friedman numbers are therefore an extension over the "numbers as instructions" theme by adding nondeterminism. In this sense, Friedman numbers can be taken as a variation of Gödel numbering more suitable to the study of nondeterministic computational processes.

Accordingly, our proof that Friedman numbers are dense invokes the two basic techniques used in the classic proofs of Gödel and Turing: encoding and indirect self-reference.

2.1. Encoding

A fundamental building-block used by both Turing and Gödel is the ability to encode tuples of integers into a single integer in a way that later allows unambiguous decoding.

For the present proof, we do not require the tuple to be composed of arbitrary integers. A sufficiently rich set of integers is enough. However, it is important that the encoding be compact, in the sense that computing the encoded integer from the tuple being encoded should require minimal use of extraneous digits not originally from the tuple. The encoding chosen for the proof requires no extraneous digits in the computation. However, it does require that the tuple be composed only of radical-free integers, that is, of integers greater than one that cannot be represented as a nontrivial integer power (an integer taken to an integer power greater than one).

Specifically, the encoding used for $\{x_i\}_{i=1}^t$, where all x_i are radical-free integers (not necessarily distinct), is by the number $\mathrm{enc}(\{x_i\}_{i=1}^t) = x_1^{x_2^{x_3^{\cdots}}}$. It can easily be seen that $\mathrm{enc}(\{x_i\}_{i=1}^t)$ contains all information necessary to unambiguously reconstruct the entire tuple (including its length, t).

2.2. Indirect self-reference

Indirect self reference is the ability of an entity (in this case a mathematical entity) to refer to itself without use of an explicit self-reference. The technique for doing so is sometimes referred to as "Quining" after Willard van Orman Quine [7], and was popularized in recreational mathematics / recreational computer science by Bratley and Milo [8] who were the first to use it in order to create a computer program that outputs its own code, a type of program that has since then received the name "quine".

Typically, Quining requires several parts of a computer program to be repetitions of the same information, encoded in different ways. A good example of this is in Quine's original use of the technique, where he utilized it to devise what is now known as Quine's paradox:

> "Yields a falsehood when appended to its own quotation" yields a falsehood when appended to its own quotation.

The sentence composing Quine's paradox accomplishes self-reference by being composed of two repetitions of the same phrase ("yields a falsehood when appended to its own quotation") that receive different interpretations because one is inside quotes and the other not. In a sense, the first repetition can be viewed as "data" and the second repetition, not in quotes, can be viewed as "instructions" showing how to manipulate the data. (Specifically, the instructions here are to append the data to itself, enclosing the first repetition in quotation marks.)

An example for a Friedman number that works like a quine is $10411041 = 1041 * (10^4 + 1)$. Here, too, repetition is used, with one repetition ("1041") used as data to be copied, and another repetition ("$*(10^4+1)$") giving specific instructions regarding how to manipulate the data to form the repetitions of the original number. Note, however, that, unlike other forms of quines, Friedman-number quines do not require the various repetitions to be encoded differently in order to be interpreted differently. There is no need to place one repetition "in quotes" and the other not, in order to separate the code from the data. The nondeterminism is enough, because it allows us to make a different choice regarding which mathematical operations to use in conjunction with each repetition.

In this paper, we require a Quining process that generates a large number of repetitions from the original data. If the data is s and the base of representation is 10, this can be done by

$$\frac{10^{L(s)r}-1}{10^{L(s)}-1}s,$$

where $L(s)$ is the digit length of s and r is the number of desired repetitions. More generally, in base b we can use

$$\frac{b^{L(s)r}-1}{b^{L(s)}-1}s.$$

3. Outline of the main proof

We wish to prove the following claim:

Theorem 1. *If $F(n)$ is the number of Friedman numbers in the range $[1, n]$, then $\lim_{n\to\infty} F(n)/n = 1$.*

Let us first introduce some notation.

Let x and y be integers. We use the notation $L(x)$ to denote the digit length of x, $x.y$ to be the concatenation of the digits of x and of y, and $[x]^k$ to be the concatenation of k copies of x.

The proof itself can be divided into three steps.

In the first step, we combine the results of Owen and Reid with the results of Friedman. Owen and Reid showed that there exists a number m such that for all b, $b.m$ is a Friedman number. That is, they found a suffix that, when appended to any string of digits, creates a Friedman number. Friedman, on the other hand, showed that there exists a number m and a number k such that the number $m * 10^k + b$ is a Friedman number for any b satisfying $L(b) \leq k$. Here, m is not a suffix, but rather a prefix placed at a specific digit position. We say that m is placed here "at position k". Friedman furthermore showed that such (m, k)-pairs can be found with arbitrarily large k.

We extend on these results by showing pairs of (m, k) values where m is not necessarily a prefix or a suffix, but is rather a middle-marker. Specifically, we claim the existence of (m, k)-pairs such that any number, n, is a Friedman number if the substring of n of length $L(m)$ starting at digit-position k of n is m. (That is, if m is a substring of n such that the least significant digit of m occupies the k'th least significant digit of n, then n is a Friedman number.)

In the second step, we show how for a given marker length $l = L(m)$ we can construct a large number of such (m, k)-pairs with distinct k values. In fact, for each value of l we utilize only a single m, and even so we demonstrate that the number of (m, k)-pairs increases exponentially with l.

In the third step, we show that the rate at which the number of (m, k)-pairs increases with l is sufficient to prove that $F(n)/n$ converges to 1, completing the proof of the theorem.

Once completing the main proof, that base-10 Friedman numbers have density 1, we augment the proof for use with the Friedman numbers of any other base of representation.

4. Constructing a basic (m, k)-pair

Let the **span** of a nonnegative integer, x, be the set of nonnegative integers that can be produced from x's digits by use of the four basic arithmetic operations, exponentiation and digit concatenation. (It is tempting, at this point, to redefine Friedman numbers as the set of integers, x, for which $x \in \text{span}(x)$. However, such a redefinition would not be correct: $\text{span}(x)$ includes x for every x, because x can be produced from itself by concatenating all of its own digits in order. In the definition of Friedman numbers, such concatenation-only calculations are disallowed explicitly.)

Theorem 2. *There exist numbers m and k, such that for any a and any $b < 10^k$, $x = a.m.[0]^t.b$ is a Friedman number if $t = k - L(b)$. Moreover, for any $s > 2$ there exists a C, such that for any sufficiently large r', the number $m = [s]^{r's}$, in combination with any $k \in \text{span}\big([s]^{(s-2)r'-C}\big)$, forms an (m, k)-pair that satisfies this condition.*

We name such an (m, k)-pair a "middle marker", and $L(m)$ the "marker length". The reason that we are looking at middle markers that are of the form $m = [s]^r$ is that, as discussed above, Friedman numbers are by definition self-referential, and Quining, the general tool that can be used to form such numbers, requires repetitions in the basic data.

Consider the span of such repetitions:

Lemma 2.1. *For any positive integer s and any nonnegative i, there exists a number, c_i, such that $i \in \text{span}([s]^{c_i})$.*

Proof. If $i = 0$, $i = s - s$ provides a solution with $c_i = 2$. For positive values of i, one can choose $c_i = 2i$ by representing i as

$$\underbrace{\frac{s}{s} + \cdots + \frac{s}{s}}_{i \text{ times}}.$$

This is not necessarily the smallest possible c_i for any particular i. $\square$

With this lemma, it is possible to prove Theorem 2 as follows:

Proof. Let s be some constant value, $s > 2$. $0, 1, 10$ and $L(s)$ are all constants, and therefore, by Lemma 2.1, there exist c_i values c_0, c_1, c_{10} and $c_{L(s)}$ for which $i \in \text{span}([s]^{c_i})$.

Let $r = r's$. For such a value of r, $r \in \text{span}\big([s]^{r'}\big)$ holds, because r can be represented as the summation of r' copies of s: $r = \underbrace{s + \cdots + s}_{r' \ times}$.

Let c_k be such that $k \in \text{span}([s]^{c_k})$.

We can now use the numerical technique for Quining introduced in Section 2.2, to create a "middle marker" containing r copies of s. This is done with the calculation

$$\left(a * (0+10)^{L(s)r} + \frac{10^{L(s)r} - 1}{10^{L(s)} - 1} s\right) * 10^k + \underbrace{0 + \cdots + 0}_{t \ times} + b,$$

which can be computed by use of the digits of a, of b, of t zeros and by use of $c_0 + 2c_1 + 4c_{10} + 3c_{L(s)} + 1 + 2r' + c_k$ copies of s. If $c_0 + 2c_1 + 4c_{10} + 3c_{L(s)} + 1 + 2r' + c_k = r$, then this computation yields x, proving that x is a Friedman number.

The reason we require the form $(0+10)^{L(s)r}$ in the expression is in order to enable its replacement with $0 * 10^{L(s)r}$ for the case $a = 0$.

Let $C = c_0 + 2c_1 + 4c_{10} + 3c_{L(s)} + 1$. To prove that x is a Friedman number, we want to choose c_k such that $C + 2r' + c_k = r$. Utilizing the fact that $r = r's$, we can rearrange the equation to get $c_k = (s-2)r' - C$. For any value of r' greater than $\frac{C}{s-2}$, any choice of $k \in \text{span}\big([s]^{(s-2)r'-C}\big)$ yields a Friedman number. $\square$

5. Finding a large number of Friedman numbers with a given marker length

So far, we have considered the ability to generate a number from a number, but, technically, what we needed was to generate, for example, from $[s]^r$ three copies of $L(s)$, four copies of 10, etc.. It is therefore natural to also consider the ability to generate a tuple of numbers from a number. We say that $(L(s), L(s), L(s), 10, 10, \ldots) \in \text{tuplespan}([s]^r)$.

Formally, let the **tuplespan** of an integer, s, be the set of tuples $(x_1, \ldots, x_n)$ such that $\forall i : 1 \le i \le n, x_i \in \text{span}(s_i)$, where $\{s_i\}_{i=1}^n$ are numbers that result from a partitioning of the digits of s.

Let $N(s)$ be the size of the largest subset of tuplespan(s) in which each tuple is composed only of radical-free numbers and no two tuples are prefixes of each other, in the sense that if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are both elements of the subset, with $n < m$, then $\exists i : 1 \le i \le n$, such that $x_i \ne y_i$. We say, borrowing terminology from coding theory, that the subset is a "prefix code".

Theorem 3. *If $m = [s]^r$, with $s > 3$, then the number of k values that are divisible by $L(m)$ for which (m, k) is a pair that forms a marker for a Friedman number is in $\Omega(g^r)$, where $g = N(s)^{\frac{s-3}{s}}$.*

Proof. Theorem 2 shows that any $k \in \text{span}\big([s]^{(s-2)r'-C}\big)$ yields a Friedman number for $m = [s]^{r's}$. Here, we restrict ourselves to k values of the form $k'l$, where l, the marker length, is $l = L(m) = L(s)r's$. The value $L(s)$ can be computed by $c_{L(s)}$ copies of s and $r's$ can be computed using r' copies of s, so $l \in \text{span}\big([s]^{r'+c_{L(s)}}\big)$.

Accordingly, choosing $k' \in \text{span}\big([s]^{(s-3)r'-C-c_{L(s)}}\big)$ yields a Friedman number. For convenience, we define $C' = C + c_{L(s)}$. Our aim is to prove that

$$\lim_{r \to \infty} \left|\text{span}\left([s]^{(s-3)r'-C'}\right)\right| g^{-r} > 0.$$

To show this, consider once again the tuple-encoding procedure enc$(\cdot)$ introduced in Subsection 2.1. It ensures that any tuple of radical-free numbers can be encoded uniquely. Specifically, tuplespan(s) includes $N(s)$ distinct tuples composed of radical-free integers that form a prefix code, and consequently tuplespan$\big(s^{(s-3)r'-C'}\big)$ includes the concatenation of any $(s-3)r'-C'$ of these tuples, creating a total of $N(s)^{(s-3)r'-C'}$ tuples, all of which are distinct due to the prefix-code constraint. Each of these tuples can now be encoded by enc$(\cdot)$ to form a distinct value of k.

The total number of possible k values is therefore at least

$$N(s)^{(s-3)r'-C'} \propto N(s)^{(s-3)r'} = g^{r's} = g^r,$$

proving the claim.

This completes the proof for $m = [s]^{r's}$. If r is not a multiple of s, the number of possible k values is at least as many as there are for $m' = [s]^{\lfloor r/s \rfloor s}$, because m' is a substring of m. This bounds the number of possible k values at

$$N(s)^{(s-3)\lfloor r/s \rfloor - C'} > N(s)^{(s-3)\left(\frac{r}{s}-1\right)-C'},$$

which is still in $\Omega(g^r)$. $\square$

6. Bounding the density of Friedman numbers

We return now to proving Theorem 1, for which we require an additional lemma.

Lemma 3.1. *There exists a value of s, for which $g = N(s)^{\frac{s-3}{s}} > 10^{L(s)}$.*

Proof. There are 5 radical-free integers that are 1 digit long, 82 radical-free integers that are 2 digits long and 872 radical-free integers that are 3 digits long. Let us define the constant s to be the digit concatenation of 13 copies of each of these radical-free integers. The digit length of s, $L(s)$, is $2785 * 13 = 36205$.

The specific s we have chosen is composed of 13 copies of each of the 959 radical-free integers below 1000. If we partition s into these radical-free integers and form all tuples of length 959 that can be created by distinct permutations of these numbers, then we have $G = (959 * 13)!/13!^{959}$ distinct tuples. This number is a lower bound for $N(s)$, because the tuples, being of constant length, necessarily form a prefix code.

Either direct calculation or use of Stirling's formula can be used to show that G is a value with 36258 digits, whereas $10^{L(s)\frac{s}{s-3}}$ has only $L(s) + 1 =$ 36206 digits. Therefore, $G > 10^{L(s)\frac{s}{s-3}}$, and $g \geq G^{\frac{s-3}{s}} \Rightarrow g > 10^{L(s)}$. $\square$

As a side note, for this particular s it is possible to choose $c_0 = c_1 = c_{10} = c_{L(s)} = 1$, and even more complicated expressions such as $10^{L(s)} - 1$ are still within the span of a single copy of s. This should come as no surprise, because s contains over 2000 copies of each of the ten digits. However, the present proof does not rely on the specific value of any c_i.

With Lemma 3.1 it is possible to prove Theorem 1 as follows:

Proof. In order to bound the density of Friedman numbers within the integers, consider all (m, k)-pairs constructable using a specific marker length $l = L(s)r$, where $k = k'l$, and let K be the maximum value of k among them. Furthermore, let M be a number greater or equal to $l + K$, and let x be an integer chosen randomly and uniformly in $[0, 10^M)$. We wish to bound from above the probability of x **not** being any of the Friedman numbers corresponding to any of the (m, k)-pairs in the set.

For any given (m, k), this probability is $1 - 10^{-l}$, because exactly l digits are restricted to a specific value. However, the probabilities relating to any two (m, k)-pairs with the same l are independent, because the restricted digits do not overlap (hence our requirement that k be divisible by l). This means that the total probability is bounded by $(1 - 10^{-l})^{C_0 g^r}$, where C_0 is the multiplicative constant in the $\Omega(g^r)$ bound from Theorem 3.

Our goal is to prove that this probability drops to zero as r tends to infinity. Equivalently, we wish to prove that $10^{-l} C_0 g^r = C_0 \left(g/10^{L(s)}\right)^r$ tends to infinity with r. However, Lemma 3.1 already showed that $g > 10^{L(s)}$.

This proves that as n increases, the probability of an integer uniformly sampled in $[1, n]$ being a Friedman number approaches 1, which is what we set out to prove. $\square$

7. Other bases of representation

The operations defining Friedman numbers are operations on digits, and hence the definition of a Friedman number is dependent on the base of representation. What is a Friedman number in decimal notation may not be a Friedman number in binary, and vice versa. This may be the reason why Friedman numbers have not been studied much within the context of number theory.

However, Gödel numbering can be said to exhibit similar properties: different representation models can be used to encode tuples (or claims, or computer programs) as integers. A quine program in a specific programming language is not necessarily a quine in another programming language. The important fact, from a theoretical perspective, is that any general-purpose computer language allows the construction of a quine.

In light of this, it may be said that the true test of the number-theoretic nature of this type of study is in whether the mechanism of the proof and the ultimate conclusions are agnostic to the representation model, rather than whether the representation model itself is dependent on external factors.

In this section, we pursue this line of reasoning by showing that the proof given above can be augmented to any representation base.

Theorem 4. *For any b, if $F(n, b)$ is the number of base-b Friedman numbers in the range $[1, n]$, then $\lim_{n\to\infty} F(n, b)/n = 1$.*

The base 10 proof relies on a specific choice of s: s was chosen to be the concatenation of 13 copies of the radical-free integers up to 3 digits long. In base b we replace the maximum number of digits by a variable, t, and the number of copies by a variable, d. We show that for any b, suitable t and d can be found.

First, we note that the constants needed (0, 1, b and $L(s)$) can always be calculated using enough copies of s, and that the proof is not reliant on the exact number of any of these c_i. Any t and d will do here.

Choosing t and d arbitrarily allows, in fact, most of the base-10 proof to remain intact. The only difficulty is in finding a (t, d)-pair that satisfies the final inequality:

$$g > b^{L(s)}.$$

Let k_i be the number of radical-free integers with i digits, and let $k = \sum_{i=1}^{t} k_i$, meaning that the total number of radical-free integers in s is kd.

We begin with a lemma:

Lemma 4.1. *There exists a value of t for which $k \log_b k > \sum_{i=1}^{t} ik_i$.*

Proof. The number of integers in the range $[1, x]$ that are not radical-free is at most

$$\sum_{\substack{p \leq \log_2 x \\ p \text{ prime}}} \lfloor \sqrt[p]{x} - 1 \rfloor.$$

The number of summands can be bounded by $\log_2 x$ and the largest one among them is smaller than $\sqrt{x}$. Therefore, the total number of integers in the range $[1, x]$ that are not radical-free is smaller than $\log_2 x \sqrt{x}$. The number of radical-free integers is greater than $x - \log_2 x \sqrt{x}$.

If we take x to be b^t, we see that the number, k, of radical-free integers with up to t digits satisfies $k > b^t - b^{t/2} t \log_2 b$ (noting that b^t itself is not radical-free).

We claim:

$$\lim_{t\to\infty} \left(t - \frac{k \log_b k}{b^t} \right) = 0 < \frac{1}{b} \leq \lim_{t\to\infty} \left(t - \frac{\sum_{i=1}^{t} ik_i}{b^t} \right),$$

A direct corollary of which is that for a sufficiently large t the inequality of Lemma 4.1 holds.

The first equality in the claim is given by

$$\lim_{t\to\infty}\left(t-\frac{k\log_b k}{b^t}\right)\le\lim_{t\to\infty}\left(t-\frac{(b^t-b^{t/2}t\log_2 b)\log_b(b^t-b^{t/2}t\log_2 b)}{b^t}\right)$$

$$=\lim_{t\to\infty} t-(1-b^{-t/2}t\log_2 b)(t+\log_b(1-b^{-t/2}t\log_2 b))$$

$$=\lim_{t\to\infty} b^{-t/2}t^2\log_2 b+b^{-t/2}t\log_2 b\log_b(1-b^{-t/2}t\log_2 b)-\log_b(1-b^{-t/2}t\log_2 b)$$

$$=\lim_{t\to\infty}-\log_b(1-b^{-t/2}t\log_2 b)=0,$$

whereas $t-\frac{k\log_b k}{b^t}\ge 0$ is guaranteed by $k\le b^t$, the total number of integers up to b^t.

On the other hand, k_t cannot be larger than $b^{t-1}(b-1)$ because this is the total number of t-digit numbers, whereas the rest of the k_i together cannot accumulate to more than b^{t-1} for the same reason. Therefore:

$$\lim_{t\to\infty}\left(t-\frac{\sum_{i=1}^t ik_i}{b^t}\right)\ge\lim_{t\to\infty}\left(t-\frac{(t-1)b^{t-1}+tb^{t-1}(b-1)}{b^t}\right)$$

$$=\lim_{t\to\infty}\left(t-\frac{tb^t-b^{t-1}}{b^t}\right)=\frac{1}{b}.$$

□

We claim that an $s=s(t,d)$ calculated with the value of t as provided by Lemma 4.1 and a sufficiently large d satisfies $\log_b g>L(s)$.

Specifically, we claim

Lemma 4.2. *For an $s=s(t,d)$ calculated with a value of t satisfying $k\log_b k>\sum_{i=1}^t ik_i$,*

$$\lim_{d\to\infty}\frac{\log_b g-L(s)}{d}>0.$$

Proof. g can be bounded from below by $G^{\frac{s-3}{s}}$, where

$$G=\frac{(kd)!}{d!^k}.$$

Substituting Stirling's formula

$$\lim_{d\to\infty}\sqrt{2\pi d}\left(\frac{d}{e}\right)^d(d!)^{-1}=1$$

we get

$$\lim_{d\to\infty}(2\pi d)^{-(k-1)/2}k^{dk+1/2}G^{-1}=1$$

and

$$\lim_{d\to\infty}\log_b G-(dk+1/2)\log_b k+\frac{k-1}{2}\log_b(2\pi d)=0.$$

Recall that by definition $L(s)=d\sum_{i=1}^{t}ik_i$. Substituting this in, we get

$$\lim_{d\to\infty}\frac{\log_b g-L(s)}{d}\geq\lim_{d\to\infty}\frac{\frac{s-3}{s}\log_b G-L(s)}{d}$$

$$=\lim_{d\to\infty}\frac{s-3}{s}k\log_b k-\sum_{i=1}^{t}ik_i=k\log_b k-\sum_{i=1}^{t}ik_i>0.$$

□

Using the values of d and t guaranteed by the two lemmas to exist, we are now able to construct a value of s that satisfies the inequality $\log_b g>L(s)$, completing the proof of Theorem 4, and demonstrating that the Friedman numbers of any base b have density 1.

8. Conclusions and further research

We have shown, using techniques borrowed from classical number theory, that Friedman numbers have density 1, and outlined a possibility for the use of Friedman numbers (or a similarly constructed set) as a platform in which to study the effects of nondeterministic computation on Gödel numbering.

We have also shown that, as in the case of Gödel numbering, though the details of the construction may vary with the representation model used, the method is, at heart, independent of the specifics of the representation, and utilized this fact to prove that Friedman numbers are of density 1 in every representation base.

The extent to which Friedman numbers (or similar sequences) can be exploited in the exploration of nondeterministic computational models is a question that remains open for further exploration. Closer to the subject matter at hand, questions regarding the density of several commonly-mentioned subsets of Friedman numbers also remain open. Among these are vampire numbers [9], first introduced by Clifford A. Pickover, and "nice" Friedman numbers [10], introduced by Mike Reid. Both sets restrict in some meaningful way the computation generating the numbers in the set. A vampire number utilizes no exponentiation, whereas a nice Friedman number requires the computation to retain the number's original digit order. Both restrictions cause the middle-marker method presented here not to function. The density of both subsets is not currently known.

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